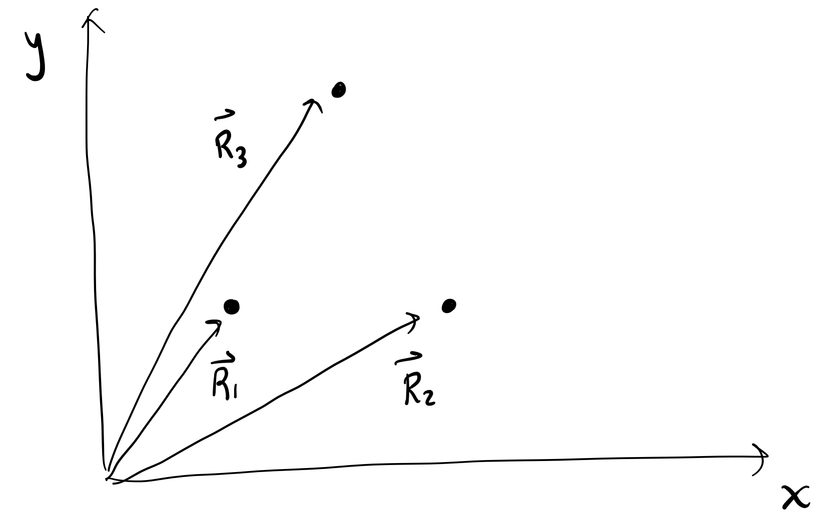
Noise propagation, 2D triangles shape space



$$\Sigma_m^k \triangleq (\mathbb{R}^{km} \setminus \Delta) / (\mathbb{R}_+^* \times SO(m) \ltimes \mathbb{R}^m)$$

$$\Sigma_2^3 = \mathbb{R}^6 / \mathbb{R}_+^* \times SO(2) \ltimes \mathbb{R}^2$$

2D manifold

m-dim normal distribution pdf:

$$f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^m \det(\Sigma)}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right)$$

Covariance matrix

$\vec{x}, \vec{\mu} \in \mathbb{R}^m$ $\qquad$ $\Sigma \in \mathbb{R}^{m \times m}$ symmetric, PSD.

$m = 2$

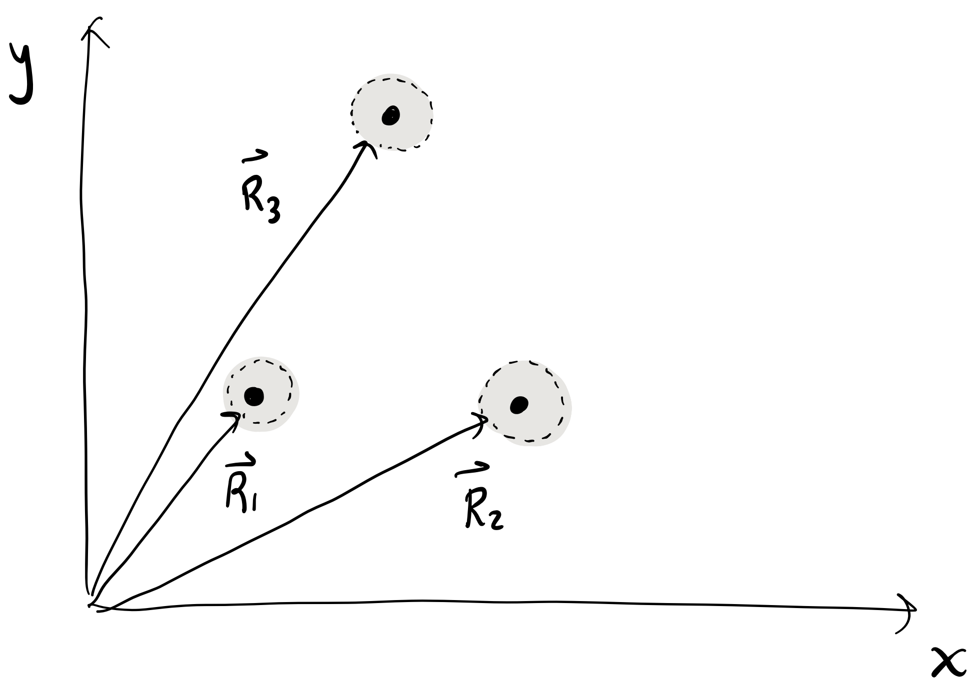
Let's take Σ to be diagonal in $(\hat{x}, \hat{y})$ basis

& isotropic $(\sigma_{x_i}^2 = \sigma_{y_i}^2 = \sigma_i^2)$

$$\therefore \quad \Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \sigma_i^2 \mathbb{1}_{2\times 2}$$

$$f_{\vec{R}_i}(\vec{r}) = \frac{1}{\sqrt{(2\pi)^2 \sigma_i^4}} \exp\left[-\frac{1}{2\sigma_i^2} |\vec{r} - \vec{\mu}_i|^2\right]$$

Also assume $\sigma_i^2 = \sigma^2 \;\forall\, i = 1, 2, 3$



$\vec{R}_1, \vec{R}_2, \vec{R}_3$ are independent

joint pdf :

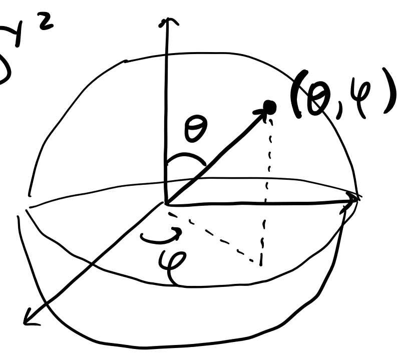
$$f_{\vec{R}_1,\vec{R}_2,\vec{R}_3}(\vec{r}_1,\vec{r}_2,\vec{r}_3) = f_{\vec{R}_1} \cdot f_{\vec{R}_2} \cdot f_{\vec{R}_3}$$

$$= \frac{1}{(2\pi\sigma^2)^3} \exp\left[-\frac{1}{2\sigma^2}\left(|\vec{r}_1-\vec{\mu}_1|^2+|\vec{r}_1-\vec{\mu}_2|^2+|\vec{r}_3-\vec{\mu}_3|^2\right)\right]$$

<u>Given</u>

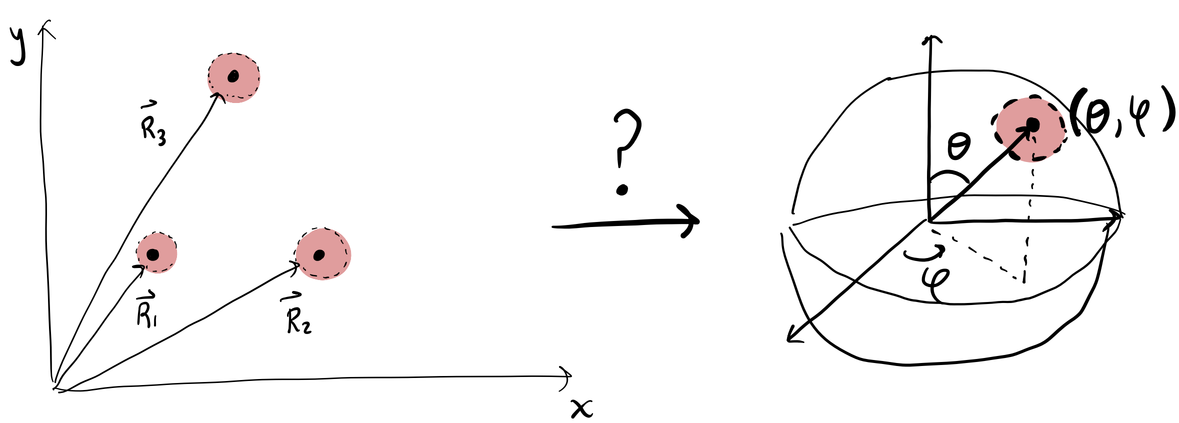
$$f_{\vec{R}_1,\vec{R}_2,\vec{R}_3} : \mathbb{R}^6 \longrightarrow \mathbb{R}_+$$

$\Sigma_2^3 = S^2$



We want to find

$$f_{\Theta,\Phi}(\theta,\varphi) : S^2 \longrightarrow \mathbb{R}^+$$



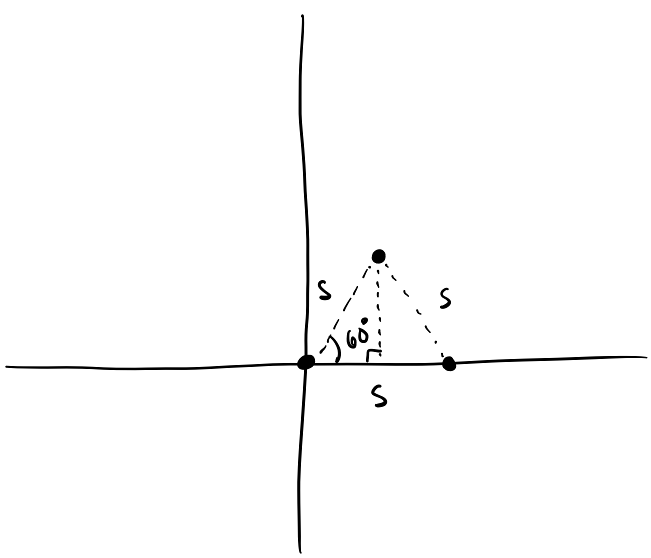
$$\Delta_i \equiv (\vec{R}_{1,i}, \vec{R}_{2,i}, \vec{R}_{3,i})$$

1. Generate N triangles $\{\Delta_i \mid i = 1, 2, \ldots, N\}$ drawn i.i.d. from $f_{\vec{R}_1, \vec{R}_2, \vec{R}_3}$

2. For each Δ_i, find corresponding $\Omega_i \equiv (\Theta_i, \Phi_i)$ in Σ^3_2.

3. Plot $\{\Omega_i, i = 1, 2, \ldots, N\}$ on S^2.

4. Extract $f_{\Theta, \Phi}$?



equilateral : $\left((0,0), (2,0), (1, \sqrt{3})\right)$

30-60-90 : $\left((0,0), (1,0), (1, \sqrt{3})\right)$