Lecture 15: Logistic and Poisson Regression

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Review - Binary responses model

Model: $Y \in \{0, 1\}$,

$$P(Y = 1|X_1, \ldots, X_p) = g^{-1}(\beta_1 X_1 + \beta_2 X_2 + \ldots \beta_p X_p).$$

Where

$$g(\pi) = log\left(\frac{\pi}{1-\pi}\right).$$

The inverse g^{-1} is

$$g^{-1}(z) = \frac{e^z}{1+e^z}.$$

We have no choice but to accept non-constant variance,

$$Var(Y) = \pi(X)[1 - \pi(X)].$$

Review - Model interpretation

An intuitive quantity to assess probabilities:

$$odds = \frac{P(Y = 1|X)}{P(Y = 0|X)}.$$

In the logistic regression model,

$$\log(odds) = \beta X.$$

The parameter β is the contribution of unit increase in X to the increase (decrease) in odds. For example, if X were binary as well,

$$\log\left(\frac{odds(X = 1)}{odds(X = 0)}\right) = \beta.$$

Logit Model for Multinomial Response

If the response Y belong to K categories.

1. Designate one category as the "base" category.
2. $$P(Y = k|X) = \frac{e^{X\beta_k}}{1 + \sum_{l=1}^{K-1} e^{X\beta_l}}$$

   Here, $\beta_k = (\beta_{k1}, \ldots, \beta_{kp})$.

   $$P(Y = K|X) = \frac{1}{1 + \sum_{l=1}^{K-1} e^{X\beta_l}}$$
3. $p \times (K - 1)$ parameters.
4. β_{ki} for k-th category and i-th predictor interpreted as increase in log-odds from base category.

Logit Model for Multinomial Response

Equivalent definition:

$$\log \frac{\pi_k(X)}{\pi_K(X)} = \alpha_k + X\beta_k, \quad k = 1, \ldots, K-1,$$

where

$$\pi_k(X) = P(Y = k|X).$$

Alligator Food Example

Study on the primary food choice of alligoators.

1. Data: 219 alligators captured in four Florida lakes.
2. Response variable: food type, in volume, found in the alligator's stomach. 5 categories:
    1. fish
    2. invertebrate
    3. reptile
    4. bird
    5. other
3. Predictors:
    1. Lake of capture (Hancock, Oklawaha, Trafford, George)
    2. Gender (M, F).
    3. Size ($\leq 2.3m$, $\geq 2.3m$).

Alligator Food Choice Example

TABLE 7.1 Primary Food Choice of Alligators

Lake	Gender	Size (m)	Primary Food Choice: Fish	Invertebrate	Reptile	Bird	Other
Hancock	Male	≤ 2.3	7	1	0	0	5
		> 2.3	4	0	0	1	2
	Female	≤ 2.3	16	3	2	2	3
		> 2.3	3	0	1	2	3
Oklawaha	Male	≤ 2.3	2	2	0	0	1
		> 2.3	13	7	6	0	0
	Female	≤ 2.3	3	9	1	0	2
		> 2.3	0	1	0	1	0
Trafford	Male	≤ 2.3	3	7	1	0	1
		> 2.3	8	6	6	3	5
	Female	≤ 2.3	2	4	1	1	4
		> 2.3	0	1	0	0	0
George	Male	≤ 2.3	13	10	0	2	2
		> 2.3	9	0	0	1	2
	Female	≤ 2.3	3	9	1	0	1
		> 2.3	8	1	0	0	1

Source: Data courtesy of Clint Moore, from an unpublished manuscript by M. F. Delaney and C. T. Moore.

1. Do gender, size, or lake of capture influence food choice?
2. Are there interaction effects?
3. Obtain estimates of $P($ food choice = fish $|$ Gender, Size, Lake $)$.

Functions for multinomial fitting in `R`: `multinom` in library `nnet`.

Fitted probabilities $\hat{\pi}$

If you had data about the size of the alligators (and not just the classification ($\leq$ or ≥ 2.3 *m*), then you can estimate a response curve like this:

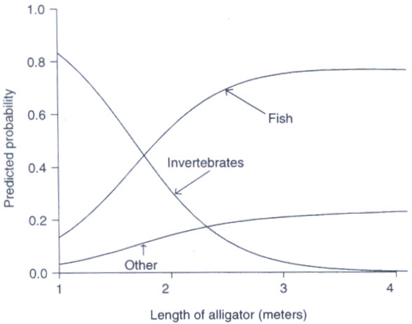

FIGURE 7.1 Estimated probabilities for primary food choice.

From Agresti, Categorical Data Analysis

Count data

1. Men and women were asked whether they believed in the after life (1991 General Social Survey).
2. Results:

	Y	N or U	
M	435	147	582
F	375	134	509
Total	810	281	1091

3. Question: is belief in afterlife independent of gender?

Contingency Tables

	Y	N or U	
M	435	147	582
F	375	134	509
Total	810	281	1091

1. Model: $Y_{ij} \sim Poisson(\lambda_{ij})$.
2. H_0: Independence. i.e. $\lambda_{ij} = \lambda \alpha_i \beta_j$.
3. H_A: λ_{ij} arbitrary.
4. Under independence:

$$\log E(Y_{ij}) = \log \lambda_{ij} = \log \lambda + \log \alpha_i + \log \beta_j.$$

Poisson Regression

1. Model fitting: Newton Raphson.
2. Confidence intervals: same as for Binomial, use local Gaussianity.
3. Assessment of model fit: Deviance residuals.

Loglinear versus Logit Models

	Y	N or U	
M	435	147	582
F	375	134	509
Total	810	281	1091

Model: $Y_{ij} \sim Poisson(\lambda_{ij})$.

If you have two Poissons, $Poiss(\lambda_{i1})$ and $Poiss(\lambda_{i2})$, then conditioned on their sum, each count is a binomial.

$$Y_{i,1} | Y_{i,1} + Y_{i,2} \sim Binomial\left(Y_{i,1} + Y_{i,2}, \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}}\right)$$

Then,

$$\begin{aligned} \text{logit} P(1|\text{ row } = i, \text{row sum} = n_i) &= \log \frac{P(1|\text{ row } = i, \text{row sum} = n_i)}{P(2|\text{ row } = i, \text{row sum} = n_i)} \\ &= \log \frac{\lambda_{i1}}{\lambda_{i2}} \\ &= \log \lambda_{i1} - \log \lambda_{i2}. \end{aligned}$$

2×2 tables

$$\text{logit}P(1|\text{ row } = i, \text{row sum} = n_i) \;=\; \log\lambda_{i1} - \log\lambda_{i2}.$$

Under the null hypothesis:

$$H_0: \;\; \lambda_{ij} = \lambda * \alpha_i * \beta_j,$$

$$\log\lambda_{ij} = \log\lambda + \log\alpha_i + \log\beta_j.$$

$$\text{logit}P(1|\text{ row } = i, \text{row sum} = n_i) = \log\beta_1 - \log\beta_2 \equiv \delta$$

The key is that the above logit does not depend on i. In binomial regression, we are modeling

$$\text{logit}P(1|X) = \beta_0 + \beta_1 X.$$

So testing H_0 is equivalent to testing $\beta_1 = 0$ in logistic regression.

2×2 tables

Thus...

1. Testing the hypothesis $H_0 : \lambda_{ij} = \lambda * \alpha_i * \beta_j$ in the Poisson model is the same as testing independence in the logistic model.
2. To test this hypothesis, you fit the model with λ_{ij} arbitrary, and then use Chi-square test on the difference of deviances.
3. The difference of deviances will be the same as the logit model, but the absolute deviances will be different.

3-way tables: Alcohol Cigarette, and Marijuana Use

Survey asked 2276 students in their final year of high school in a nonurban area near Dayton, Ohio whether they ever used alcohol, cigarettes, or marijuana.

Alcohol Use	Cigarette Use	Marijuana Use Yes	Marijuana Use No
Yes	Yes	911	538
	No	44	456
No	Yes	3	43
	No	2	279

This is example of a $2 \times 2 \times 2$ contingency table. Shorthand: A=alcohol, C=cigarette, M=marijuana.

3-way tables: Types of Interaction

Alcohol Use	Cigarette Use	Marijuana Use Yes	Marijuana Use No
Yes	Yes	911	538
	No	44	456
No	Yes	3	43
	No	2	279

$$Y_{ijk} \sim Poisson(\lambda_{ijk})$$

Conditioned on total (N) $Y_{ijk} \sim Multinom(N, \pi_{ijk})$.
π_{i++} be probability of row $A = i$,
π_{ij+} be probability of $A = i$, $C = j$, etc.

1. A,C, and M **mutually independent**

$$\log \lambda_{ijk} = \lambda + \lambda_i^A + \lambda_j^C + \lambda_k^M$$

$$\pi_{ijk} = \pi_{i++}\pi_{+j+}\pi_{++k}$$

3-way tables: Types of Interaction

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$$Y_{ijk} \sim Poisson(\lambda_{ijk})$$

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π_{i++} be probability of row $A = i$,
π_{ij+} be probability of $A = i$, $C = j$, etc.

1. M is **jointly independent** of A, C

$$\log \lambda_{ijk} = \lambda + \lambda_i^A + \lambda_j^C + \lambda_k^M + \lambda_{ij}^{AC}$$

$$\pi_{ijk} = \pi_{ij+}\pi_{++k}$$

3-way tables: Types of Interaction

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$$Y_{ijk} \sim Poisson(\lambda_{ijk})$$

Conditioned on total (N) $Y_{ijk} \sim Multinom(N, \pi_{ijk})$.
π_{i++} be probability of row $A = i$,
π_{ij+} be probability of $A = i, C = j$, etc.

1. C and M **conditionally independent** given A

$$\log \lambda_{ijk} = \lambda + \lambda_i^A + \lambda_j^C + \lambda_k^M + \lambda_{ij}^{AC} + \lambda_{ik}^{AM}$$

$$\pi_{jk|i} = \pi_{j+|i}\pi_{+k|i}.$$

3-way tables: Types of Interaction

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	No	2	279

$$Y_{ijk} \sim Poisson(\lambda_{ijk})$$

Conditioned on total (N) $Y_{ijk} \sim Multinom(N, \pi_{ijk})$.
π_{i++} be probability of row $A = i$,
π_{ij+} be probability of $A = i, C = j$, etc.

1. Each pair of A,C, and M has **homogeneous association.**

$$\log \lambda_{ijk} = \lambda + \lambda_i^A + \lambda_j^C + \lambda_k^M + \lambda_{ij}^{AC} + \lambda_{ik}^{AM} + \lambda_{ik}^{CM}.$$

e.g. the dependence relationship of A, C does not depend on M.

3-way tables: Types of Interaction

Alcohol Use	Cigarette Use	Marijuana Use Yes	Marijuana Use No
Yes	Yes	911	538
	No	44	456
No	Yes	3	43
	No	2	279

$$Y_{ijk} \sim Poisson(\lambda_{ijk})$$

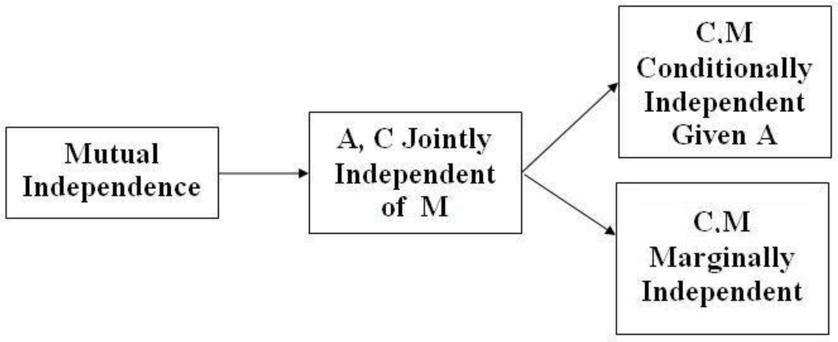
Conditioned on total (N) $Y_{ijk} \sim Multinom(N, \pi_{ijk})$.
π_{i++} be probability of row $A = i$,
π_{ij+} be probability of $A = i, C = j$, etc.

1. **Saturated Model.**

$$\log \lambda_{ijk} = \lambda + \lambda_i^A + \lambda_j^C + \lambda_k^M + \lambda_{ij}^{AC} + \lambda_{ik}^{AM} + \lambda_{ik}^{CM} + \lambda_{ijk}^{ACM}.$$

3-way tables: Types of Interaction

Symbol	Interpretation
(A,C,M)	Mutual Independence
(AC,M)	AC jointly independent of M
(AC,AM)	M, C conditionally independent given A
(AC,AM,CM)	Homogeneous association of each pair.



Marginal independence: fit 2×2 table.

Analysis of 3-way tables

1. Fit log-linear model (Poisson GLM) for each of the models.
   1. Criterion: maximum likelihood.
   2. Fitting method: Newton Raphson.
2. Use a model selection criterion to choose the best one.
   1. AIC, BIC.
   2. Use Deviance χ^2 test to choose between nested models.

`glm(..., family=poisson)`, `loglm(MASS)` in `R`.