Statistics 191: Introduction to Applied Statistics

Multiple linear regression

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Multiple linear regression

Outline

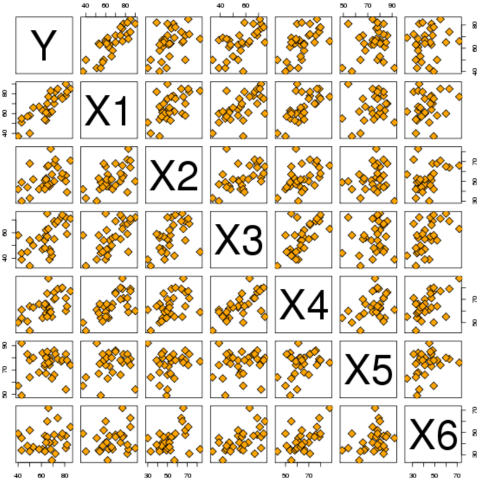
- Specifying the model.
- Fitting the model: least squares.
- Interpretation of the coefficients.
- More on F-statistics.
- Matrix approach to linear regression.
- T-statistics revisited.
- More F statistics.
- Tests involving more than one β.

Job supervisor data

Description

Variable	Description
Y	Overall supervisor job rating
X_1	How well do they handle complaints
X_2	Do they allow special priveleges
X_3	Give opportunity to learn new things
X_4	Raises based on performance
X_5	Too critical of poor performance
X_6	Good rate of advancement

Job supervisor data

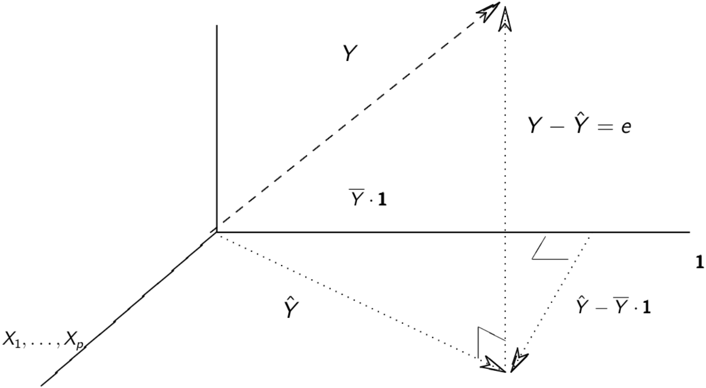


Specifying the model

Multiple linear regression model

- Rather than one predictor, we have $p = 6$ predictors.
- $$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \varepsilon_i$$
- Errors ε are assumed independent $N(0, \sigma^2)$, as in simple linear regression.
- Coefficients are called (partial) regression coefficients because they “allow” for the effect of other variables.

Geometry of Least Squares


Y
$Y - \hat{Y} = e$
$\overline{Y} \cdot \mathbf{1}$
$\mathbf{1}$
$\hat{Y}$
$\hat{Y} - \overline{Y} \cdot \mathbf{1}$
$X_1, \dots, X_p$

Fitting the model

Least squares

- Just as in simple linear regression, model is fit by minimizing

$$\begin{aligned} SSE(\beta_0, \dots, \beta_p) &= \sum_{i=1}^n \left(Y_i - (\beta_0 + \sum_{j=1}^p \beta_j X_{ij})\right)^2 \\ &= \|Y - \widehat{Y}(\beta)\|^2 \end{aligned}$$

- Minimizers: $\widehat{\beta} = (\widehat{\beta}_0, \dots, \widehat{\beta}_p)$ are the "least squares estimates": are also normally distributed as in simple linear regression.

Error component

Estimating σ^2

- As in simple regression

  $$\widehat{\sigma}^2 = \frac{SSE}{n-p-1} \sim \sigma^2 \cdot \frac{\chi^2_{n-p-1}}{n-p-1}$$

  independent of $\widehat{\beta}$.
- Why χ^2_{n-p-1}? Typically, the degrees of freedom in the estimate of σ^2 is
  $n - \#$number of parameters in regression function.

Interpretation of β_j's

Supervisor example

- Take β_1 for example. This is the amount the average job rating increases for one "unit" of "Handles complaints", keeping everything else constant.
- Units of "Handles complaints" are individual favorable responses, so on average for every extra person who rated the supervisor as good at handling complaints (other things being fixed), the average job rating increases by β_1.

Interpretation of β_j's

Why are they *partial* regression coefficients?

- The term *partial* refers to the fact that the coefficient β_j represent the partial effect of $\boldsymbol{X}_j$ on $\boldsymbol{Y}$, i.e. after the effect of all other variables have been removed.
- Specifically,

$$Y_i - \sum_{l=1, l \neq j}^{k} X_{il}\beta_l = \beta_0 + \beta_j X_{ij} + \varepsilon_i.$$

- Let $e_{i,(j)}$ be the residuals from regressing $\boldsymbol{Y}$ onto all $\boldsymbol{X}_{\cdot}$'s except $\boldsymbol{X}_j$, and let $X_{i,(j)}$ be the residuals from regressing $\boldsymbol{X}_j$ onto all $\boldsymbol{X}_{\cdot}$'s except $\boldsymbol{X}_j$, and let $X_{i,(j)}$.
- If we regress $e_{i,(j)}$ against $X_{i,(j)}$, the coefficient is *exactly* the same as in the original model (see R code).

Goodness of fit for multiple regression

Sums of squares

$$SSE = \sum_{i=1}^{n} (Y_i - \widehat{Y}_i)^2$$

$$SSR = \sum_{i=1}^{n} (\overline{Y} - \widehat{Y}_i)^2$$

$$SST = \sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

$$R^2 = \frac{SSR}{SST}$$

R^2 is called the *multiple correlation coefficient* of the model, or the *coefficient of multiple determination*.

Adjusted R^2

Compensating for more variables

- As we add more and more variables to the model – even random ones, R^2 will increase to 1.
- Adjusted R^2 tries to take this into account by replacing sums of squares by *mean squares*

$$R_a^2 = 1 - \frac{SSE/(n-p-1)}{SST/(n-1)} = 1 - \frac{MSE}{MST}.$$

Goodness of fit test

Another F-test

- As in simple linear regression, we measure the goodness of fit of the regression model by

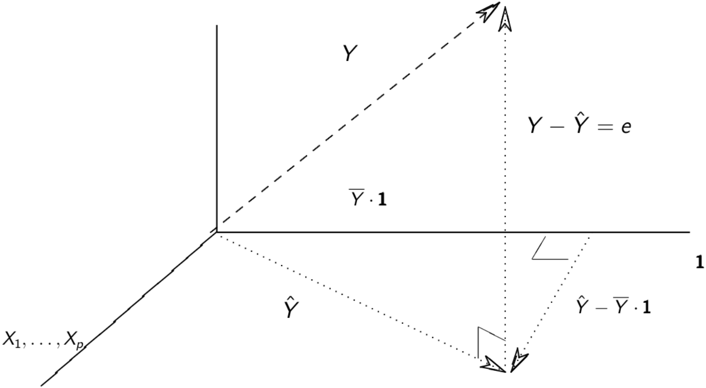
$$F = \frac{MSR}{MSE} = \frac{\|\overline{Y} \cdot \mathbf{1} - \widehat{\boldsymbol{Y}}\|^2/p}{\|Y - \widehat{\boldsymbol{Y}}\|^2/(n-p-1)}.$$

- Under $H_0 : \beta_1 = \dots = \beta_p = 0$,

$$F \sim F_{p,n-p-1}$$

so reject H_0 at level α if $F > F_{p,n-p-1,1-\alpha}$.

Geometry of Least Squares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$X_1, \dots, X_p$

Intuition behind the F test

Measuring lengths

- The F statistic is a ratio of lengths of orthogonal vectors (divided by degrees of freedom).
- We can prove that our model implies

$$\mathbb{E}\left(MSR\right) = \sigma^2 + \underbrace{\|\boldsymbol{\mu} - \overline{\mu}\cdot \mathbf{1}\|^2/p}_{(*)}$$

$$\mathbb{E}\left(MSE\right) = \sigma^2$$

$$\mu_i = \mathbb{E}(Y_i) = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}$$

  so F should be not be too far from 1 if H_0 is true, i.e. $(*) = 0$.
- If F is large, it is evidence that $(*) \neq 0$, i.e. H_0 is false.

F-test revisited

Example in more detail

- *Full (bigger) model :*

$$Y_i = \beta_0 + \beta_1 X_{i1} + \dots \beta_p X_{ip} + \varepsilon_i$$

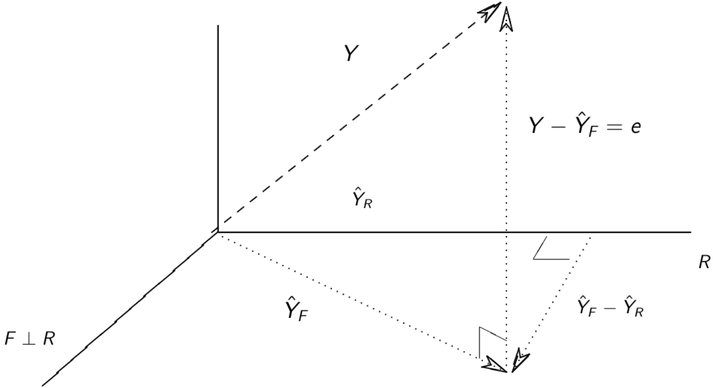
- *Reduced (smaller) model:*

$$Y_i = \beta_0 + \varepsilon_i$$

- The F-statistic has the form

$$F = \frac{(SSE(R) - SSE(F))/(df_R - df_F)}{SSE(F)/df_F}.$$

Geometry of Least Squares



Matrix formulation

Equivalent formulation

$$\boldsymbol{Y}_{n\times 1} = \boldsymbol{X}_{n\times(p+1)}\boldsymbol{\beta}_{(p+1)\times 1} + \boldsymbol{\varepsilon}_{n\times 1}$$

- $\boldsymbol{X}$ is called the *design matrix* of the model
- $\varepsilon \sim N(0, \sigma^2 I_{n\times n})$ is multivariate normal

SSE in matrix form

$$SSE(\beta) = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})$$

Matrix formulation

Design matrix

- The design matrix is the $n \times (p+1)$ matrix with entries

$$\boldsymbol{X} = \begin{pmatrix} 1 & X_{11} & X_{12} & \dots & X_{1,p} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & X_{n1} & X_{n2} & \dots & X_{n,p} \end{pmatrix}$$

Least squares solution

Solving for $\widehat{\beta}$

- Normal equations

$$\frac{\partial}{\partial \beta_j} SSE \bigg|_{\widehat{\beta}} = -2 \left(\boldsymbol{Y} - \boldsymbol{X}\widehat{\beta} \right)^t \boldsymbol{X}_j = 0, \qquad 0 \leq j \leq p.$$

- Equivalent to

$$\begin{aligned} (\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}})^t \boldsymbol{X} &= 0 \\ \widehat{\boldsymbol{\beta}} &= (\boldsymbol{X}^t \boldsymbol{X})^{-1} \boldsymbol{X}^t \boldsymbol{Y} \end{aligned}$$

- Properties:

$$\widehat{\boldsymbol{\beta}} \sim N\left(\boldsymbol{\beta}, \sigma^2 (\boldsymbol{X}^t \boldsymbol{X})^{-1}\right), \text{indep. of } \widehat{\sigma}^2$$

- R code

Inference for multiple regression

Regression function at one point

- One thing one might want to *learn* about the regression function in the supervisor example is something about the regression function at some fixed values of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_6$, i.e. what can be said about

$$\beta_0 + 65 \cdot \beta_1 + 50 \cdot \beta_2 + 55 \cdot \beta_3 + 64 \cdot \beta_4 + 75 \cdot \beta_5 + 40 \cdot \beta_6 \quad (*)$$

  roughly the regression function at “typical” values of the predictors.
- The expression (**??**) is equivalent to

$$\sum_{j=0}^{6} a_j \beta_j, \qquad a = (1, 65, 50, 55, 64, 75, 40).$$

Inference for $\sum_{j=0}^p a_j\beta_j$

Confidence interval for $\sum_{j=0}^p a_j\beta_j$

- Suppose we want a $(1-\alpha)\cdot 100\%$ CI for $\sum_{j=0}^p a_j\beta_j$.
- Just as in simple linear regression:

$$\sum_{j=0}^p a_j\widehat{\beta}_j \pm t_{1-\alpha/2,n-p-1}\cdot SE\left(\sum_{j=0}^p a_j\widehat{\beta}_j\right).$$

Inference for $\sum_{j=0}^{p} a_j\beta_j$

T-statistics revisited

- Suppose we want to test

$$H_0 : \sum_{j=0}^{p} a_j\beta_j = h.$$

As in simple linear regression, it is based on

$$T = \frac{\sum_{j=0}^{p} a_j\widehat{\beta}_j - h}{SE(\sum_{j=0}^{p} a_j\widehat{\beta}_j)}.$$

- If H_0 is true, then $T \sim t_{n-p-1}$, so we reject H_0 at level α if

$$|T| \geq t_{1-\alpha/2,n-p-1}, \qquad \text{OR}$$
$$p-\text{value} = 2 * (1 - \texttt{pt}(|\texttt{T}|, \texttt{n}-\texttt{p}-1)) \leq \alpha.$$

Inference for $\sum_{j=0}^p a_j\beta_j$

One-sided tests

- Suppose, instead, we wanted to test the one-sided hypothesis

$$H_0 : \sum_{j=0}^p a_j\beta_j \leq h, \text{ vs. } H_a : \sum_{j=0}^p a_j\beta_j > h$$

- If H_0 is true, then T is no longer exactly t_{n-p-1} but

$$\mathbb{P}\left(T > t_{1-\alpha,n-p-1}\right) \leq 1-\alpha$$

so we reject H_0 at level α if

$$T \geq t_{1-\alpha,n-p-1}, \qquad \text{OR}$$
$$p - \text{value} = (1 - \texttt{pt}(\texttt{T}, \texttt{n} - \texttt{p} - \texttt{1})) \leq \alpha.$$

Inference for $\sum_{j=0}^{p} a_j\beta_j$

Standard error of $\sum_{j=0}^{p} a_j\widehat{\beta}_j$

- Based on matrix approach to regression

$$SE\left(\sum_{j=0}^{p} a_j\widehat{\beta}_j\right) = \sqrt{\widehat{\sigma}^2 a(X^TX)^{-1}a^T}.$$

- Don't worry too much about implementation – `R` will do this for you in general, R code

Inference for $\sum_{j=0}^{p} a_j \beta_j$

Prediction interval

- "Identical" to simple linear regression.
- Prediction interval at $X_{1,new}, \dots, X_{p,new}$:

$$\widehat{\beta}_0 + \sum_{j=1}^{p} X_{j,new} \widehat{\beta}_j \pm t_{1-\alpha/2,n-p-1}$$

$$\times \sqrt{\widehat{\sigma}^2 + SE\left(\widehat{\beta}_0 + \sum_{j=1}^{p} X_{j,new} \widehat{\beta}_j\right)^2}.$$

Inference for multiple regression

Questions about many (combinations) of β_j's

- In multiple regression we can ask more complicated questions than in simple regression.
- For instance, we could ask whether
  - X_2 : Do they allow special priveleges
  - X_3 : Give opportunity to learn new things

  explains little of the variability in the data, and might be dropped from the regression model.
- These questions can be answered answered by F-statistics.

Inference for more than one β

Dropping one or more variables

- Suppose we wanted to test whether how the supervisor handles special priveleges, or allows employees to try new things explains a significant amount of the variability in the overall job rating. Formally, this is:

$$H_0 : \beta_2 = \beta_3 = 0, \quad \text{vs. } H_a : \text{one of } \beta_2, \beta_3 \neq 0$$

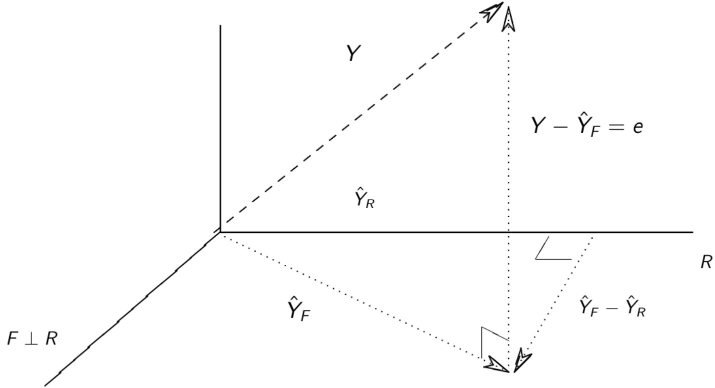
- This test is again an F-test based on two models

$$R : Y_i = \beta_0 + \beta_1 X_{i1} + \beta_4 X_{i4} + \beta_5 X_{i5} + \beta_6 X_{i6} + \varepsilon_i$$

$$F : Y_i = \beta_0 + \sum_{j=1}^{6} \beta_j X_{ij} + \varepsilon_i$$

- **Note:** The reduced model R must be a special case of the full model F to use the F-test.

Geometry of Least Squares



Inference for more than one β

SSE of a model

- In the graphic, a "model", $\mathcal{M}$ is a subspace of $\mathbb{R}^n =$ column space of $\boldsymbol{X}$.
- Least squares fit $=$ projection onto the subspace of $\mathcal{M}$, yielding predicted values $\widehat{Y}_{\mathcal{M}}$
- Error sum of squares:

$$SSE(\mathcal{M}) = \|Y - \widehat{Y}_{\mathcal{M}}\|^2.$$

Inference for more than one β

F-statistic for $H_0 : \beta_2 = \beta_3 = 0$

- $$\begin{aligned} F &= \frac{\frac{SSE(R)-SSE(F)}{2}}{\frac{SSE(F)}{n-1-p}} \qquad (1) \\ &\sim F_{2,n-p-1} \qquad \text{(if } H_0 \text{ is true)} \end{aligned}$$
- Reject H_0 at level α if $F > F_{1-\alpha,2,n-1-p}$.
- Here is an example R code.

Inference for more than one β

Dropping an arbitrary subset

- For an arbitrary model, suppose we want to test

$$\begin{aligned} H_0 :& \beta_{i_1} = \cdots = \beta_{i_j} = 0 \\ H_a :& \text{one of } \beta_{i_1}, \dots \beta_{i_j} \neq 0 \end{aligned}$$

for some subset $\{i_1, \dots, i_j\} \subset \{0, \dots, p\}$.

Inference for more than one β

F-statistic for $H_0 : \beta_{i_1} = \cdots = \beta_{i_j} = 0$

- You guessed it: it is based on the two models:

$$R : Y_i = \sum_{l=0, l \notin \{i_1,\dots,i_j\}}^{p} \beta_j X_{il} + \varepsilon_i$$

$$F : Y_i = \sum_{j=0}^{p} \beta_j X_{il} + \varepsilon_i$$

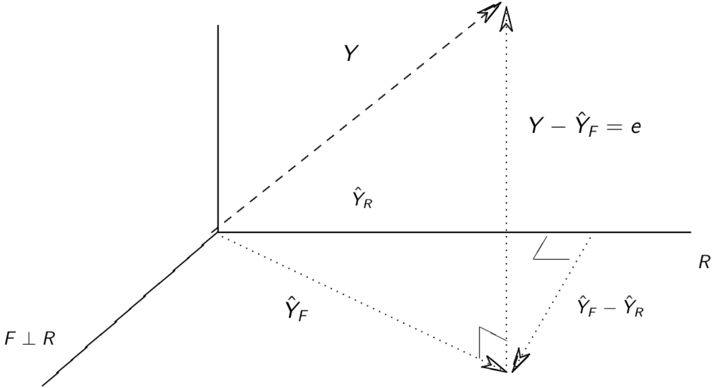
where $X_{i0} = 1$ for all i.

Inference for more than one β

F-statistic for $H_0 : \beta_{i_1} = \cdots = \beta_{i_j} = 0$

- $$F = \frac{\frac{SSE(R) - SSE(F)}{j}}{\frac{SSE(F)}{n-p-1}} \sim F_{j,n-p-1} \qquad \text{(if } H_0 \text{ is true)}$$
- Reject H_0 at level α if $F > F_{1-\alpha,j,n-1-p}$.

Geometry of Least Squares



Inference for more than one β

General F-tests

- Given two models $R \subset F$ (i.e. R is a subspace of F), we can consider testing

$$H_0 : R \text{ is adequate (i.e. } \mathbb{E}(Y) \in R)$$

  vs.

$$H_a : F \text{ is adequate (i.e. } \mathbb{E}(Y) \in F).$$

- The test statistic is

$$F = \frac{(SSE(R) - SSE(F))/(df_R - df_F)}{SSE(F)/df_F}$$

- If H_0 is true, $F \sim F_{df_R - df_F, df_R}$ so we reject H_0 at level α if $F > F_{df_R - df_F, df_R, 1-\alpha}$.

Constraints

Constraining $\beta_1 = \beta_3$ (after deciding $\beta_2 = \beta_4 = \beta_5 = \beta_6 = 0$)

- Full model:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_3 X_{i3} + \varepsilon_i$$

- Reduced model:

$$\begin{aligned} Y_i &= \beta_0 + \tilde{\beta}_1 X_{i1} + \tilde{\beta}_1 X_{i3} + \varepsilon_i \\ &= \beta_0 + \tilde{\beta}_1 (X_{i1} + X_{i3}) + \varepsilon_i \end{aligned}$$

R code.

Constraints

Constraining $\beta_1 + \beta_3 = 1$ (after deciding $\beta_2 = \beta_4 = \beta_5 = \beta_6 = 0$)

- Full model:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_3 X_{i3} + \varepsilon_i$$

- Reduced model:

$$Y_i = \beta_0 + \tilde{\beta}_1 X_{i1} + (1 - \tilde{\beta}_1) X_{i3} + \varepsilon_i$$

$$Y_i - X_{i3} = \beta_0 + \tilde{\beta}_1 (X_{i1} - X_{i3}) + \varepsilon_i$$

R code.