Statistics 191: Introduction to Applied Statistics

Qualitative Variables, Interactions & ANOVA

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Qualitative variables + interactions

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Outline

- Qualitative / categorical variables.
- Regression equations differing by group.
- Interactions.
- Analysis of Variance Models

Categorical variables

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Categorical variables

- Most variables we have looked at so far were continuous: height, rating, etc.
- In many situations, we record a categorical variable: sex, state, country, etc.
- How do we include this in our model?

Categorical variables

A simple example

- One example that we have looked at *does* have categorical variables.
- Two sample problem with equal variances: suppose

$$Y = (Z_1, \ldots, Z_m, W_1, \ldots, W_n)$$

with $Z_j \sim N(\mu_1, \sigma^2), 1 \leq j \leq m$ and $W_j \sim N(\mu_2, \sigma^2), 1 \leq j \leq n + m$.

- For $1 \leq i \leq n$, let

$$X_i = \begin{cases} 1 & 1 \leq i \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Categorical variables

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A simple example

- Design matrix

$$X_{(n+m)\times 2} = \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix}$$

Example

IT salary data

- Outcome: S, salaries for IT staff in a corporation.
- Predictors: X, experience (years); E, education (3 levels): 1=Bachelor's, 2=Master's, 3=Ph.D; M, management (2 levels): 1=management, 0=not management.

IT salary

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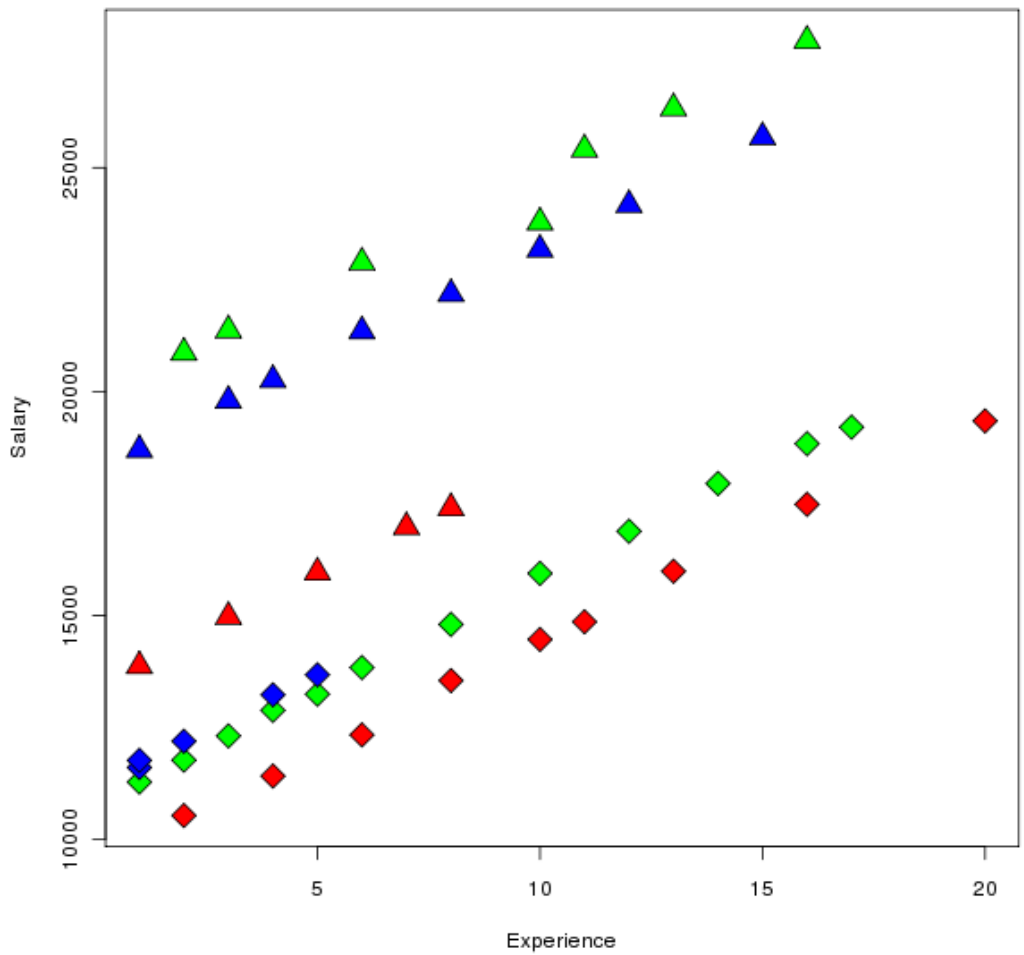

IT salary

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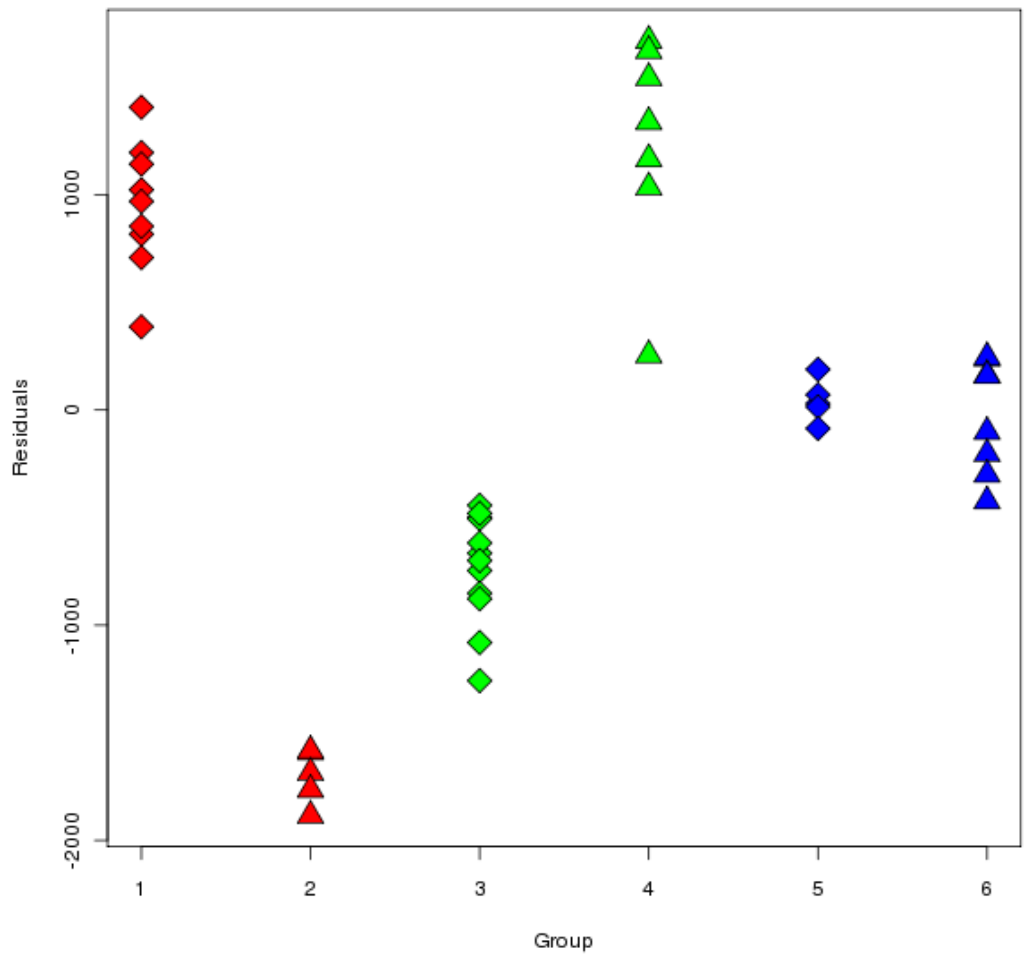

Two solutions

Solution #1: stratification

- One solution is to "stratify" data set by this categorical variable.
- We could break data set up into groups by education and management, and fit fit model

$$S_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

  in each group.
- Problem: this results in smaller samples in each group: lose degrees of freedom for estimating σ^2 within each group.

Two solutions

Solution #2: qualitative predictors

- IF it is reasonable to assume that σ^2 is constant for each observation.
- THEN, we can incorporate all observations into 1 model.

$$S_i = \beta_0 + \beta_1 X_i + \beta_2 E_{i2} + \beta_3 E_{i3} + \beta_4 M_i + \varepsilon_i$$

where

$$E_{i2} = \begin{cases} 1 & \text{if } E_i = 2, \\ 0 & \text{otherwise.} \end{cases}, E_{i3} = \begin{cases} 1 & \text{if } E_i = 3, \\ 0 & \text{otherwise,} \end{cases},$$

Categorical variables: details

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Things to notice

- Although E has 3 levels, we only added 2 variables to the model. In a sense, this is because "intercept" absorbs one level.
- If we added three variables then the columns of design matrix would be linearly dependent.
- Assumes β_1 – effect of experience is the same in all groups, unlike when we fit the model separately. This may or may not be reasonable.

Interactions

Effect of experience

- Our model has enforced the constraint the β_1 is the same within each group.
- Graphically, this seems OK, but how can we "test" this?
- We could fit a model with different slopes in each group, but keeping as many d.f. as we can.
- This model has "interactions" in it: the effect of experience depends on what level of education you have.

Interactions

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Interaction between experience and education

- Model:

$$S_i = \beta_0 + \beta_1 X_i + \beta_2 E_{i2} + \beta_3 E_{i3} + \beta_4 M_i + \beta_5 E_{i2} X_i + \beta_6 E_{i3} X_i + \varepsilon_i.$$

- Note that we took each column corresponding to education and multiplied it by the column for experience to get two new predictors.
- To test whether the slope is the same in each group we would just test $H_0 : \beta_5 = \beta_6 = 0$.
- Based on figure, we expect not to reject H_0.

Interactions

Interaction between management and education

- Based on figure, we expect an interaction effect.
- Fit model

$$S_i = \beta_0 + \beta_1 X_i + \beta_2 E_{i2} + \beta_3 E_{i3} + \beta_4 M_i + \beta_5 E_{i2} M_i + \beta_6 E_{i3} M_i + \varepsilon_i.$$

- Again, testing for interaction is testing $H_0 : \beta_5 = \beta_6 = 0$.

IT salary, outlier removed

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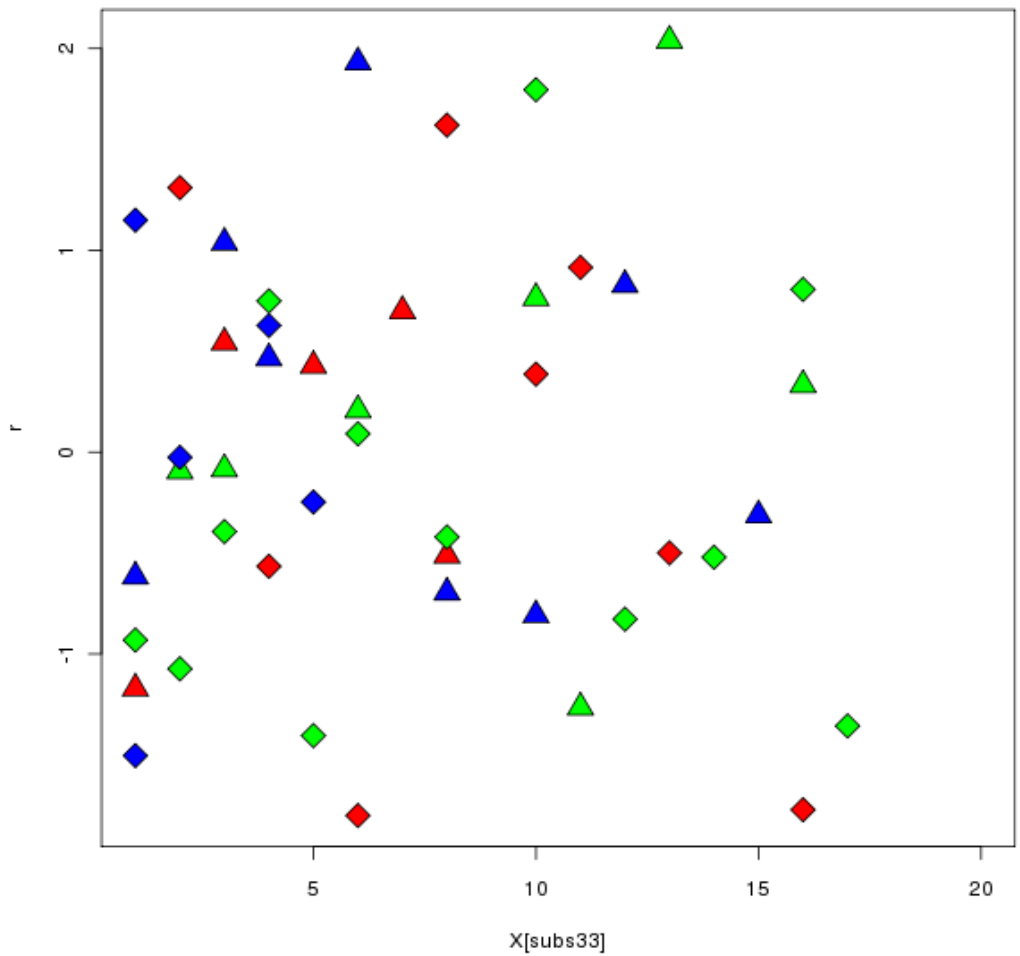

Example

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Minority employment data

TEST	job aptitude test score
RACE	1 if minority, 0 otherwise
JPERF	job performance evaluation

Minority employment data

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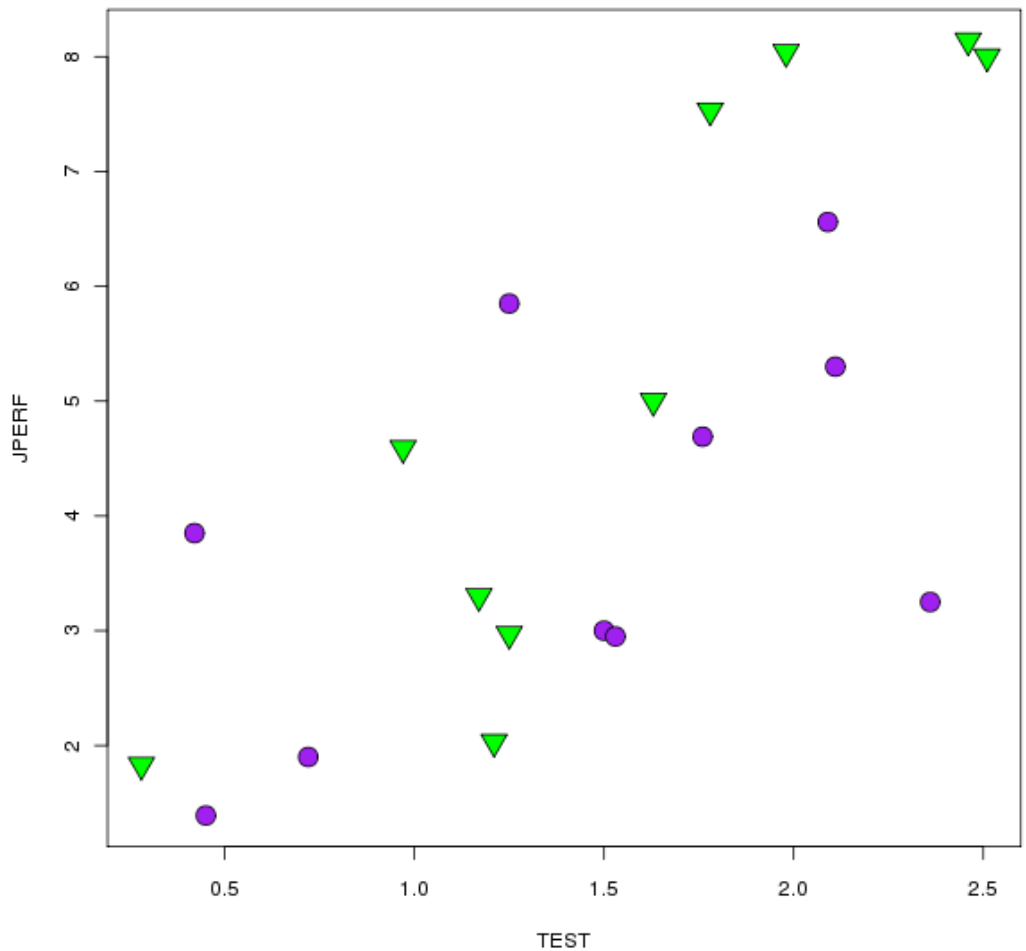

Interactions

General model

- In theory, there may be a linear relationship between *JPERF* and *TEST* but it could be different by group.
- Model:

$$JPERF_i = \beta_0 + \beta_1 TEST_i + \beta_2 RACE_i + \beta_3 RACE_i * TEST_i + \varepsilon_i.$$

- Regression functions:

$$Y_i = \begin{cases} \beta_0 + \beta_1 TEST_i + \varepsilon_i & \text{if } i\text{-th ind. is white} \\ (\beta_0 + \beta_2) + (\beta_1 + \beta_3) TEST_i + \varepsilon_i & \text{if } i\text{-th ind. is white.} \end{cases}$$

No difference

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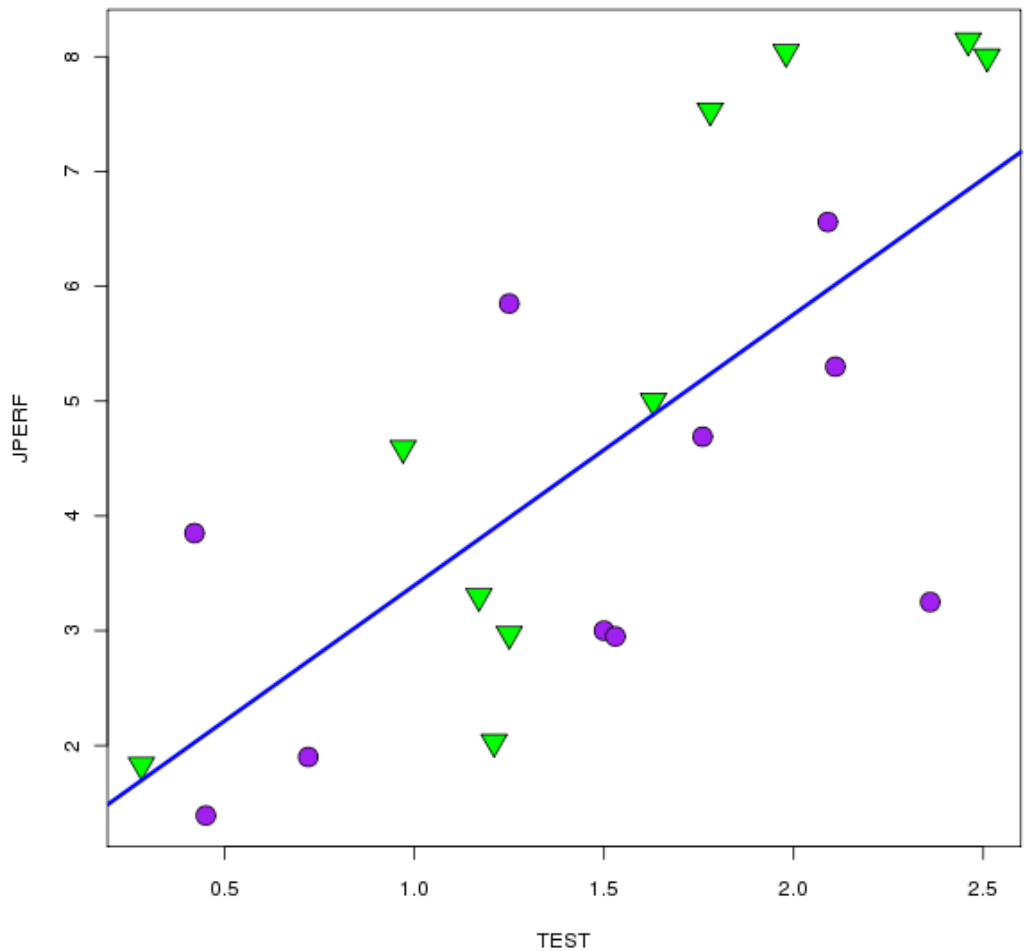

Different slopes, same intercept

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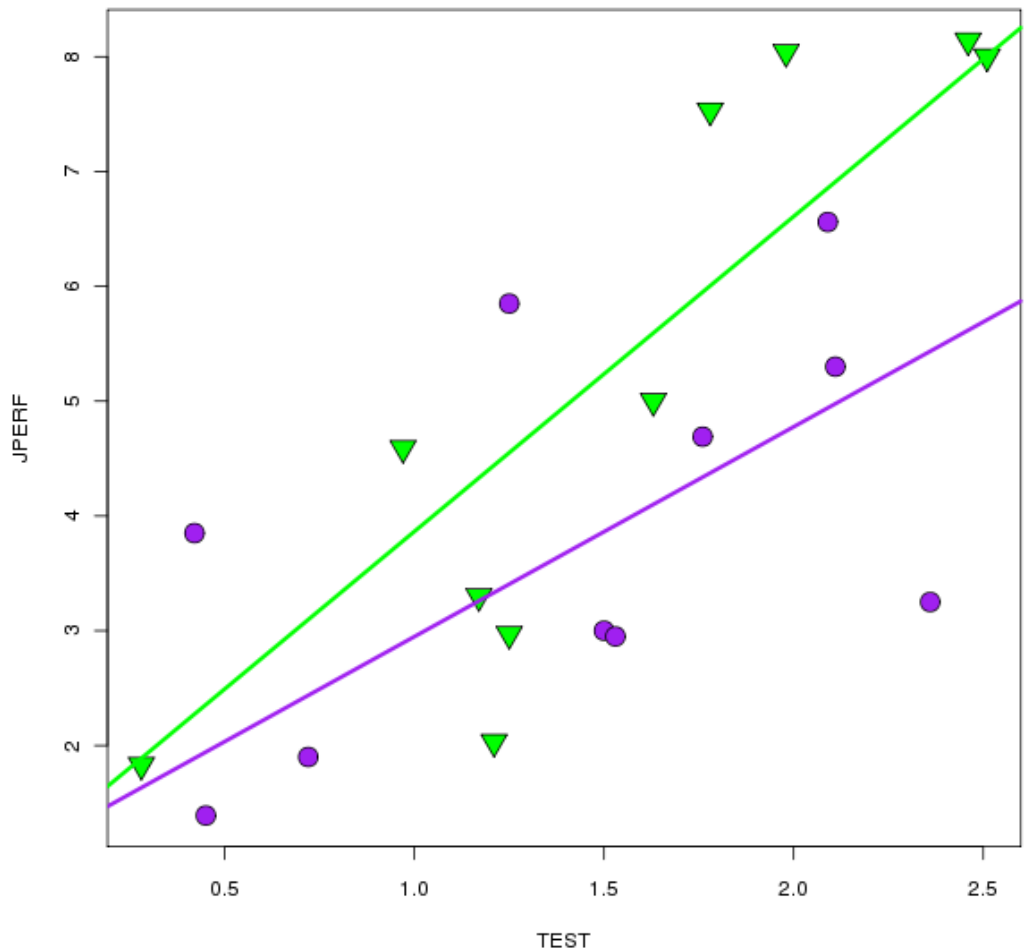

Different intercepts, same slope

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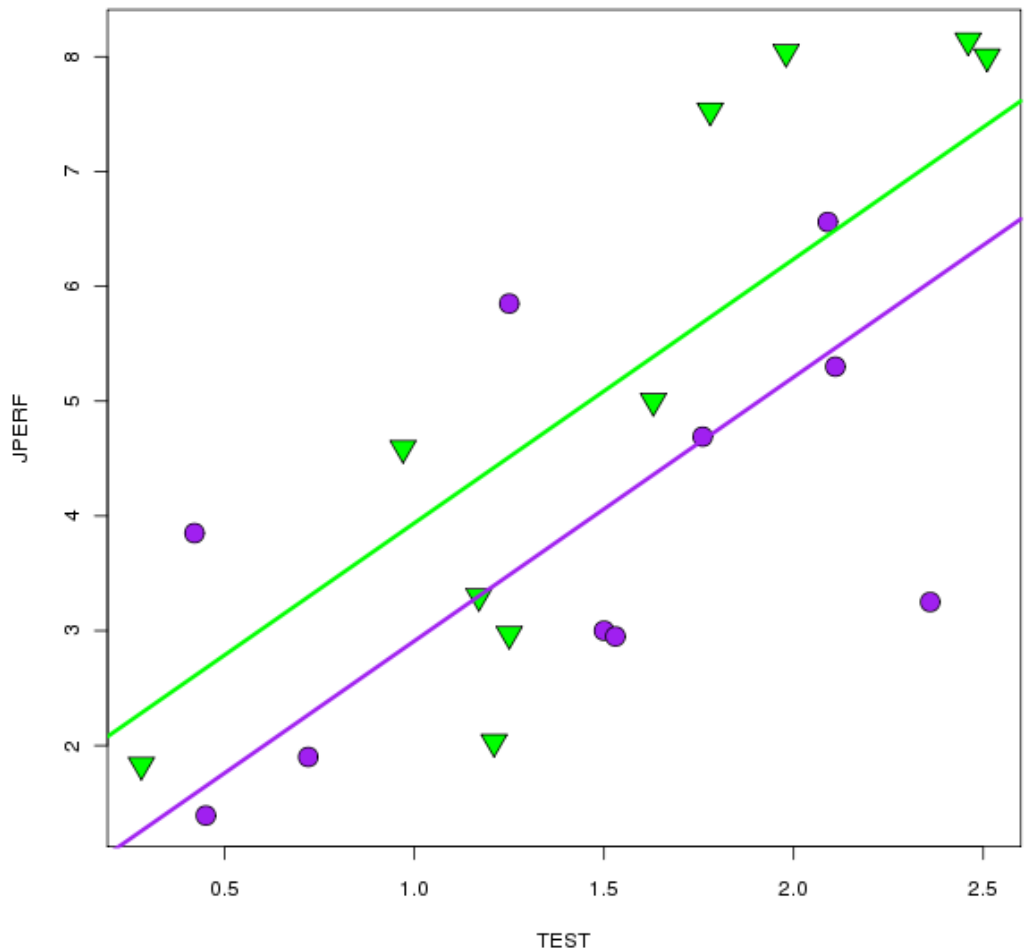

Different intercepts, different slopes

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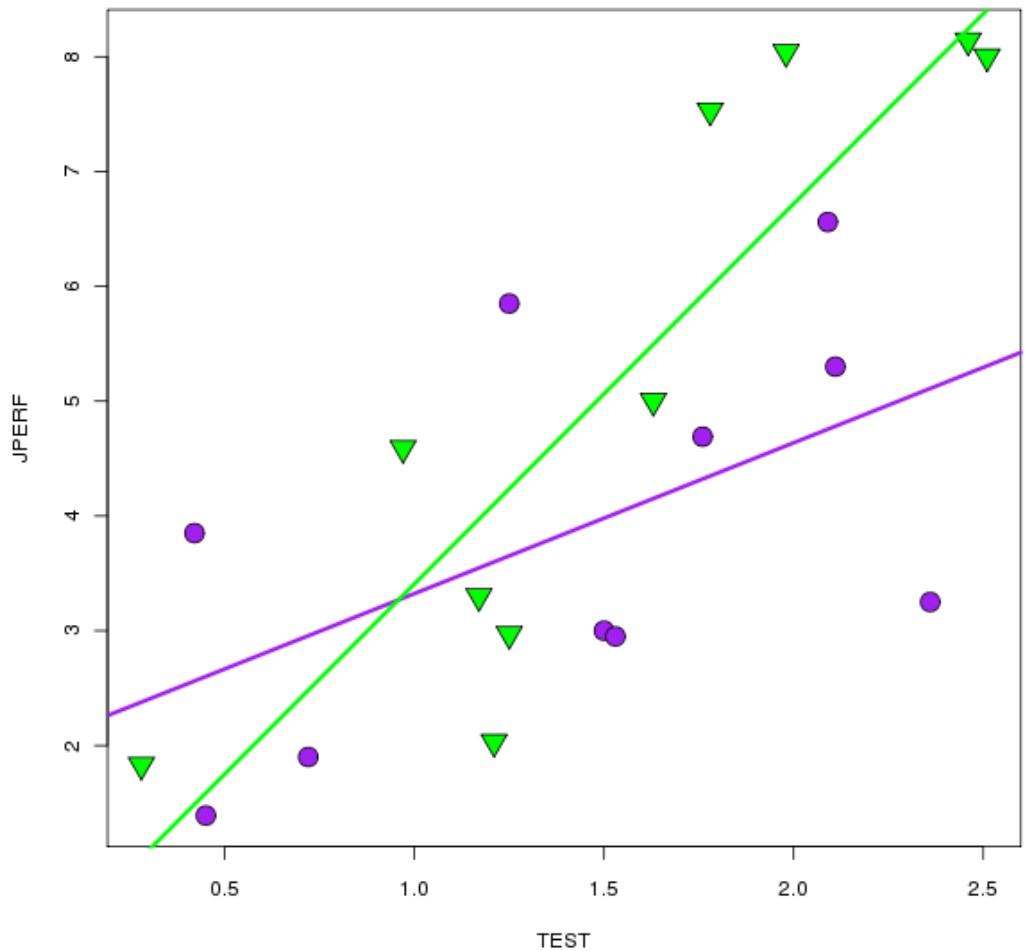

Interactions

Interpreting different models

- Both $\beta_2, \beta_3 \neq 0$ – main effect for *RACE* and interaction effect between *TEST* and *RACE*.
- $\beta_2 \neq 0, \beta_3 = 0$ – main effect for *RACE*, no interaction between *TEST* and *RACE*.
- $\beta_2 = 0, \beta_3 \neq 0$ – no main effect for *RACE*, interaction between *TEST* and *RACE*.
- R code

ANOVA models

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General definition of ANOVA model

- Models with only qualitative variables.
- Can be thought of as extensions of "two-sample" t-test to more than two groups at once, and more than one grouping variable.
- Example: in a simple experiment studying blood pressure we might start by considering only the overall health (Poor, Moderate, Good).
- Data would then have one categorical variable with three levels.

ANOVA models

Example: rehab surgery

- How does prior fitness affect recovery from surgery? Observations: 24 subjects' recovery time.
- Three fitness levels: below average, average, above average.
- If you are in better shape before surgery, does it take less time to recover?

ANOVA models

One-way ANOVA

- First generalization of two sample t-test: more than one level.
- One-way ANOVA model: observations: $Y_{ij}, 1 \leq i \leq r, 1 \leq j \leq n_i$: r groups and n_i samples in i-th group.

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \qquad \varepsilon_{ij} \sim N(0, \sigma^2).$$

- Constraint: $\sum_{i=1}^{r} \alpha_i = 0$. This constraint is needed for "identifiability". This is "equivalent" to only adding $r - 1$ columns to the design matrix for this qualitative variable.

ANOVA models

One-way ANOVA

- Model is easy to fit:

$$\widehat{Y}_{ij} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} = \overline{Y}_{i\cdot}$$

  If observation is in i-th group: predicted mean is just the sample mean of observations in i-th group.

- Simplest question: is there any group (main) effect?

$$H_0 : \alpha_1 = \cdots = \alpha_r = 0?$$

- Test is based on F-test with full model vs. reduced model. Reduced model just has an intercept.
- Other questions: is the effect the same in groups 1 and 2?

ANOVA models

ANOVA table: One-way

Source	SS	df	$E(MS)$
Treatments	$SSTR = \sum_{i=1}^r n_i \left(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot}\right)^2$	$r-1$	$\sigma^2 + \frac{\sum_{i=1}^r n_i \alpha_i^2}{r-1}$
Error	$SSE = \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_{i\cdot})^2$	$\sum_{i=1}^r n_i - r$	σ^2

- Note that $MSTR$ measures "variability" of the "cell" means. If there is a group effect we expect this to be large relative to MSE.
- We see that under $H_0 : \alpha_1 = \cdots = \alpha_r = 0$, the expected value of $MSTR$ and MSE is σ^2. This tells us how to test H_0 using ratio of mean squares, i.e. an F test.

ANOVA models

Testing for any main effect

- Rows in the ANOVA table are, in general, independent.
- Therefore, under H_0

$$F = \frac{MSTR}{MSE} = \frac{\frac{SSTR}{df_{TR}}}{\frac{SSE}{df_E}} \sim F_{df_{TR}, df_E}$$

  the degrees of freedom come from the *df* column in previous table.
- Reject H_0 at level α if $F > F_{1-\alpha, df_{TR}, df_E}$.

ANOVA models

Inference for linear combinations

- Suppose we want to "infer" something about

$$\sum_{i=1}^{r} a_i \mu_i$$

where $\mu_i = \mu + \alpha_i$ is the mean in the i-th group. For example:

$$H_0 : \mu_1 - \mu_2 = 0 \qquad \text{(same as } H_0 : \alpha_1 - \alpha_2 = 0\text{)?}$$

Is there a difference between below average and average groups in terms of rehab time?

- $$\text{Var}\left(\sum_{i=1}^{r} a_i \overline{Y}_{i\cdot}\right) = \sigma^2 \sum_{i=1}^{r} \frac{a_i^2}{n_i}.$$

ANOVA models

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Two categorical variables: kidney failure

- Time of stay in hospital depends on weight gain between treatments and duration of treatment.
- Two levels of duration, three levels of weight gain.
- Is there an interaction? Main effects?

ANOVA models

Two-way ANOVA

- Second generalization: more than one grouping variable.
- Two-way ANOVA model: observations: $(Y_{ijk}), 1 \leq i \leq r, 1 \leq j \leq m, 1 \leq k \leq n_{ij}$: r groups in first grouping variable, m groups in second and n_{ij} samples in (i, j)-"cell":

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \qquad \varepsilon_{ijk} \sim N(0, \sigma^2).$$

- In kidney example, $r = 3$ (weight gain), $m = 2$ (duration of treatment), $n_{ij} = 10$ for all (i, j).

ANOVA model

Two-way ANOVA: main questions of interest

- Are there main effects for the grouping variables?

$$H_0 : \alpha_1 = \cdots = \alpha_r = 0, \qquad H_0 : \beta_1 = \cdots = \beta_m = 0.$$

- Are there interaction effects:

$$H_0 : (\alpha\beta)_{ij} = 0, 1 \leq i \leq r, 1 \leq j \leq m.$$

ANOVA models

Constraints on the parameters

- Many constraints are needed, again for identifiability. Let's not worry about the details . . .
- Constraints:
  - $\sum_{i=1}^{r} \alpha_i = 0$
  - $\sum_{j=1}^{m} \beta_j = 0$
  - $\sum_{j=1}^{m} (\alpha\beta)_{ij} = 0, 1 \leq i \leq r$
  - $\sum_{i=1}^{r} (\alpha\beta)_{ij} = 0, 1 \leq j \leq m.$

ANOVA models

Fitting model

- Easy to fit:

$$\widehat{Y}_{ijk} = \overline{Y}_{ij\cdot} = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} Y_{ijk}.$$

- Inference for combinations

$$\text{Var}\left(\sum_{i=1}^{r} \sum_{j=1}^{m} a_{ij} \overline{Y}_{ij\cdot} \right) = \sigma^2 \cdot \sum_{i=1}^{r} \sum_{j=1}^{m} \frac{a_{ij}^2}{n_{ij}}.$$

- Usual t-tests, confidence intervals.

ANOVA models

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ANOVA table: Two-way (assuming $n_{ij} = n$)

Term	SS
A	$SSA = nm\sum_{i=1}^{r}\left(\overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdots}\right)^2$
B	$SSB = nr\sum_{j=1}^{m}\left(\overline{Y}_{\cdot j\cdot} - \overline{Y}_{\cdots}\right)^2$
AB	$SSAB = n\sum_{i=1}^{r}\sum_{j=1}^{m}\left(\overline{Y}_{ij\cdot} - \overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdot j\cdot} + \overline{Y}_{\cdots}\right)^2$
Error	$SSE = \sum_{i=1}^{r}\sum_{j=1}^{m}\sum_{k=1}^{n}(Y_{ijk} - \overline{Y}_{ij\cdot})^2$

ANOVA models

ANOVA table: Two-way (assuming $n_{ij} = n$)

SS	df	$E(MS)$
SSA	$r-1$	$\sigma^2 + nm\frac{\sum_{i=1}^r \alpha_i^2}{r-1}$
SSB	$m-1$	$\sigma^2 + nr\frac{\sum_{j=1}^m \beta_j^2}{m-1}$
$SSAB$	$(m-1)(r-1)$	$\sigma^2 + n\frac{\sum_{i=1}^r \sum_{j=1}^m (\alpha\beta)_{ij}^2}{(r-1)(m-1)}$
SSE	$(n-1)mr$	σ^2

- For instance, we see that under $H_0 : (\alpha\beta)_{ij} = 0, \forall i, j$ the expected value of $SSAB$ and SSE is σ^2 – use these for an F-test testing for an interaction.

Fixed and random effects

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Random effects

- In kidney & rehab examples, the categorical variables are well-defined categories: below average fitness, long duration, etc.
- In some designs, the categorical variable is "subject".
- Simplest example: repeated measures, where more than one (identical) measurement is taken on the same individual.
- In this case, the "group" effect α_i is best thought of as random because we only sample a subset of the entire population.

Fixed and random effects

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When to use random effects?

- A "group" effect is random if we can think of the levels we observe in that group to be samples from a larger population.
- Example: if collecting data from different medical centers, "center" might be thought of as random.
- Example: if surveying students on different campuses, "campus" may be a random effect.

Fixed and random effects

Example: sodium content in beer

- How much sodium is there in North American beer? How much does this vary by brand?
- Observations: for 6 brands of beer, we recorded the sodium content of 8 12 ounce bottles.
- Questions of interest: what is the "grand mean" sodium content? How much variability is there from brand to brand?
- "Individuals" in this case are brands, repeated measures are the 8 bottles.

One-way ANOVA (random)

One-way random effects model

- Assuming that cell-sizes are the same, i.e. equal observations for each "subject" (brand of beer).
- Observations

$$Y_{ij} \sim \mu_\cdot + \alpha_i + \varepsilon_{ij}, 1 \leq i \leq r, 1 \leq j \leq n$$

- $\varepsilon_{ij} \sim N(0, \sigma^2), 1 \leq i \leq r, 1 \leq j \leq n$
- $\alpha_i \sim N(0, \sigma^2_\mu), 1 \leq i \leq r.$
- Parameters:
  - μ is the population mean;
  - σ^2 is the measurement variance (i.e. how variable are the readings from the machine that reads the sodium content?);
  - σ^2_μ is the population variance (i.e. how variable is the sodium content of beer across brands).

One-way ANOVA (random)

Implications for model

- In random effects model, the observations are no longer independent (even if ε's are independent

$$\mathrm{Cov}(Y_{ij}, Y_{i'j'}) = \sigma^2_\mu \delta_{i,i'} + \sigma^2 \delta_{j,j'}.$$

- In more complicated models, this makes "maximum likelihood estimation" more complicated: least squares is no longer the best solution.
- Also changes the degrees of freedom for some t-statistics.

One-way ANOVA (random)

Fitting the model

- Only one parameter in the mean function $\mu_{\cdot\cdot}$.
- When cell sizes are the same (balanced),

$$\widehat{\mu}_{\cdot} = \overline{Y}_{\cdot\cdot} = \frac{1}{nr} \sum_{i,j} Y_{ij}.$$

- Unbalanced models: slightly more tricky.
- This also changes estimates of σ^2 – see ANOVA table below. We might guess that $df = nr - 1$ and

$$\widehat{\sigma}^2 = \frac{1}{nr-1} \sum_{i,j} (Y_{ij} - \overline{Y}_{\cdot\cdot})^2.$$

This is *not* the case.

One-way ANOVA (random)

ANOVA table

Source	SS	df	$E(MS)$
Treatments	$SSTR = \sum_{i=1}^r n\left(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot}\right)^2$	$r-1$	$\sigma^2 + n\sigma^2_\mu$
Error	$SSE = \sum_{i=1}^r \sum_{j=1}^n (Y_{ij} - \overline{Y}_{i\cdot})^2$	$(n-1)r$	σ^2

- Only change here is the expectation of $SSTR$ which reflects randomness of α_i's.
- ANOVA table is still useful to setup tests: the same F statistics for fixed or random will work here.
- Test for random effect: $H_0 : \sigma^2_\mu = 0$ based on

$$F = \frac{MSTR}{MSE} \sim F_{r-1,(n-1)r} \qquad \text{under } H_0.$$

One-way ANOVA (random)

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Inference for population mean: $\mu_.$

- Easy to check that

$$E(\overline{Y}_{..}) = \mu_.$$

$$\text{Var}(\overline{Y}_{..}) = \frac{n\sigma^2_\mu + \sigma^2}{rn}.$$

- To come up with a t statistic that we can use for test, CIs, we need to find an estimate of $\text{Var}(\overline{Y}_{..})$. ANOVA table says $E(MSTR) = n\sigma^2_\mu + \sigma^2$.
- Therefore,

$$\frac{\overline{Y}_{..} - \mu_.}{\sqrt{\frac{SSTR}{(r-1)rn}}} \sim t_{r-1}$$

One-way ANOVA (random)

Inference for population mean: $\mu_\cdot$

- Why $r-1$ degrees of freedom? Imagine we could record an infinite number of observations for each individual, so that $\overline{Y}_{i\cdot} \to \mu_i$, or that $\sigma^2_\mu = 0$.
- To learn anything about $\mu_\cdot$ we still only have r observations $(\mu_1, \ldots, \mu_r)$.
- Sampling more within an individual cannot narrow the CI for $\mu_\cdot$.

One-way ANOVA (random)

Estimating σ^2_μ

- From the ANOVA table

$$\sigma^2_\mu = \frac{E(SSTR/(r-1)) - E(SSE/((n-1)r))}{n}.$$

- Natural estimate:

$$S^2_\mu = \frac{SSTR/(r-1) - SSE/((n-1)r)}{n}$$

- Problem: this estimate can be negative! One of the difficulties in random effects model.