Data Mining & Machine Learning

Regularization Methods for Regression

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Ridge Regression

$$\hat{\beta}_{ridge} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} \beta_j x_{ij})^2 + \lambda \sum_{j=1}^{p} \beta_j^2 \tag{1}$$

$$\hat{\boldsymbol{\beta}}_{ridge} = \underset{\beta}{\operatorname{argmin}} (\boldsymbol{y} - \boldsymbol{x}^T \boldsymbol{\beta})^2 + \boldsymbol{\lambda} \boldsymbol{\beta}^2 \tag{2}$$

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- Method to deal with non-invertible **(X'X)** matrices

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$$\hat{\boldsymbol{\beta}}_{ridge} = (\boldsymbol{X'X} + \lambda \boldsymbol{I})^{-1}\boldsymbol{X'y} \tag{4}$$

- Equation (3) is a special case of an orthonormal design matrix

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- Back to Ordinary Least-Squares solution!

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- What happens when we change the norm? What about the L1 norm?

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- $||\beta||_1 = \sum_{j=1}^{p} |\beta_j|$, $||\beta||_2 = \sum_{j=1}^{p} \beta_j^2$, and $p :=$ the number of features

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- Difference is in the constraint of the minimization problem

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- Sparsity!
- Using LASSO results in models with less parameters (features)

LASSO Regression

$$\hat{\boldsymbol{\beta}}_{lasso}(\lambda) = S(\hat{\beta}, \lambda) \equiv sign(\hat{\beta})(|\hat{\beta}| - \lambda)_+ \tag{11}$$

$$S(\hat{\beta}, \lambda) = \begin{cases} \hat{\beta} - \lambda, & if\, \hat{\beta} > 0 \text{ and } \lambda < |\hat{\beta}| \\ \hat{\beta} + \lambda, & if\, \hat{\beta} < 0 \text{ and } \lambda < |\hat{\beta}| \\ 0, & if\, \lambda > |\hat{\beta}|. \end{cases}$$

$$\hat{\beta}^t_{j,lasso} \leftarrow S(\hat{\beta}^{t-1}_{j,lasso} + \sum_{i=1}^{n} x_{ij}(y_i - \hat{y}_i{}^{t-1}), \lambda) \tag{12}$$

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- LASSO does variable selection by setting select variables to 0!

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- Both biased in the statistical sense but asymptotically unbiased and can be shown that they select the true features with probability $\rightarrow$ 1

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- Useful in banking, tech, insurance, marketing, and other fields

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- Binary representation of all 10 features leads to a very sparse (32,769 x 15,618) matrix

LASSO Model Performance

Ridge Model Performance

Comparison of Ridge, LASSO, and OLS

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- Sweet paper on asymptotic properties of Ridge and LASSO