Data Mining & Machine Learning

Regularization Methods for Regression

Francisco Javier Arceo
Senior Data Scientist

NYU Polytechnic School of Engineering
Commonwealth Bank of Australia

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Ridge Regression

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- Method to deal with non-invertible **(X'X)** matrices

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$$\hat{\beta}_{j,ridge} = \frac{\sum_{i=1}^{n}(x_{ij} - \bar{x_j})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_{ij} - \bar{x_j})^2} \frac{1}{1+\lambda} = \frac{\hat{\beta}_{j,ols}}{1+\lambda} \tag{3}$$

$$\hat{\beta}_{ridge} = (\boldsymbol{X'X} + \lambda \boldsymbol{I})^{-1}\boldsymbol{X'y} \tag{4}$$

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- Back to Ordinary Least-Squares solution!

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- What happens when we change the norm? What about the L1 norm?

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- Difference is in the constraint of the minimization problem

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- Sparsity!
- Using LASSO results in models with less parameters (features)

LASSO Regression

$$\hat{\boldsymbol{\beta}}_{lasso}(\lambda) = S(\hat{\beta}, \lambda) \equiv sign(\hat{\beta})(|\hat{\beta}| - \lambda)_+ \tag{11}$$

$$S(\hat{\beta}, \lambda) = \begin{cases} \hat{\beta} - \lambda, & if\, \hat{\beta} > 0\ and\ \lambda < |\hat{\beta}| \\ \hat{\beta} + \lambda, & if\, \hat{\beta} < 0\ and\ \lambda < |\hat{\beta}| \\ 0, & if\, \lambda > |\hat{\beta}|. \end{cases}$$

$$\hat{\beta}^t_{j,lasso} \leftarrow S(\hat{\beta}^{t-1}_{j,lasso} + \sum_{i=1}^{n} x_{ij}(y_i - \hat{y_i}^{t-1}), \lambda) \tag{12}$$

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- LASSO does variable selection by setting select variables to 0!

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- Both methods reduce the size of the coefficients but Ridge shrinks to 0 continuously; LASSO shrinks discretely
- Both biased in the statistical sense but asymptotically unbiased and can be shown that they select the true features with probability $\rightarrow 1$

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- Useful in banking, tech, insurance, marketing, and other fields

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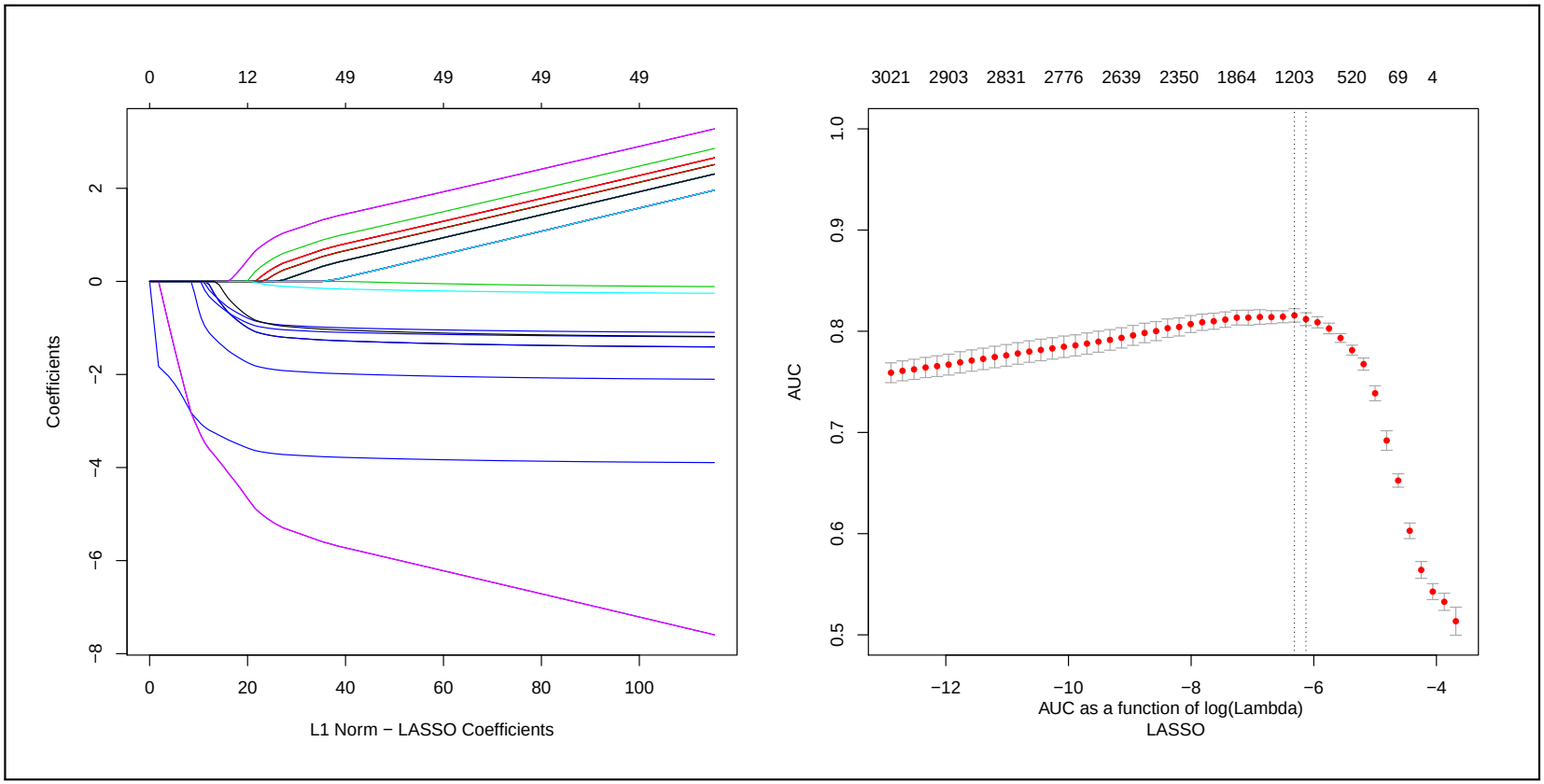
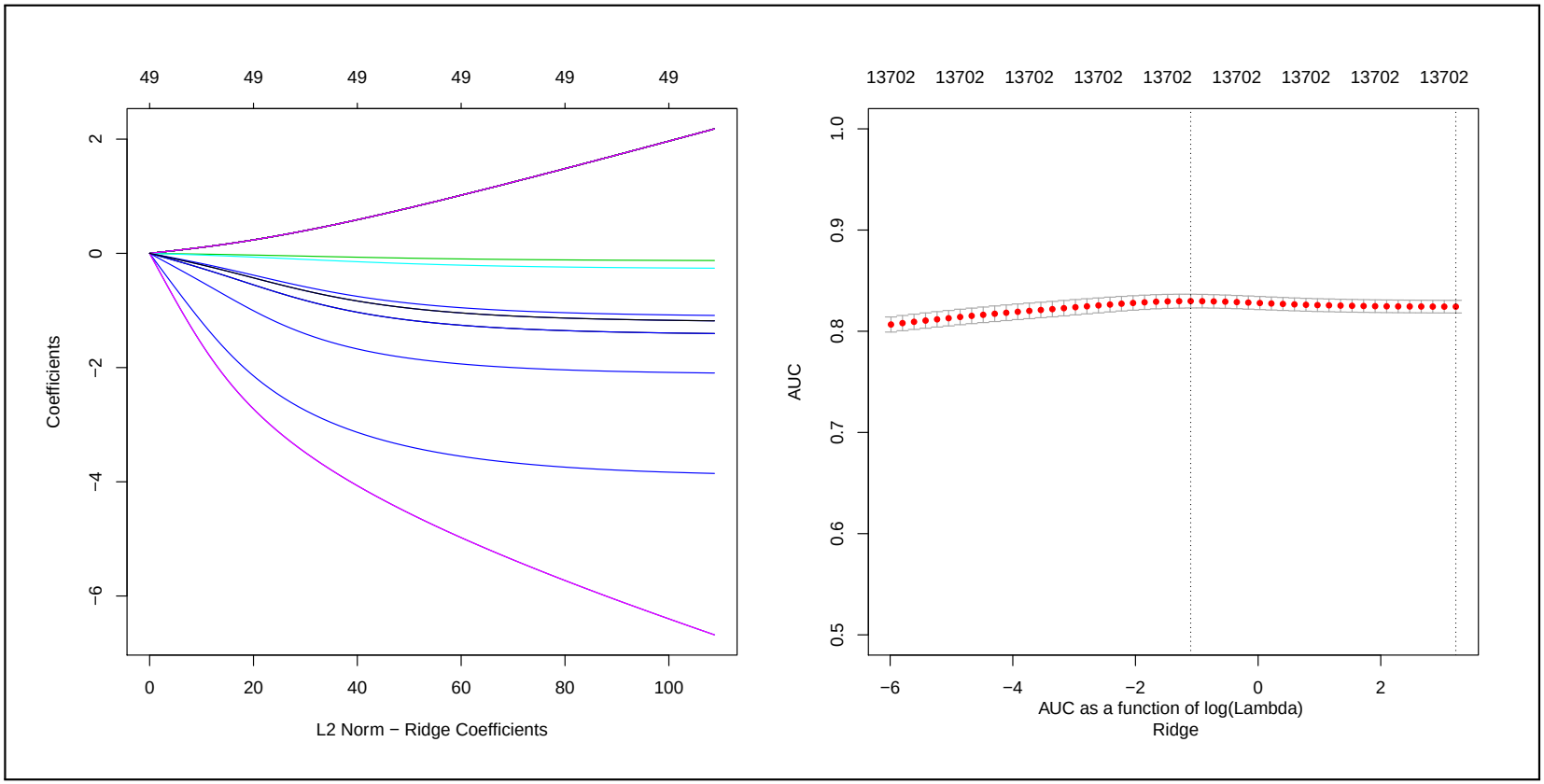
- Binary representation of all 10 features leads to a very sparse (32,769 x 15,618) matrix

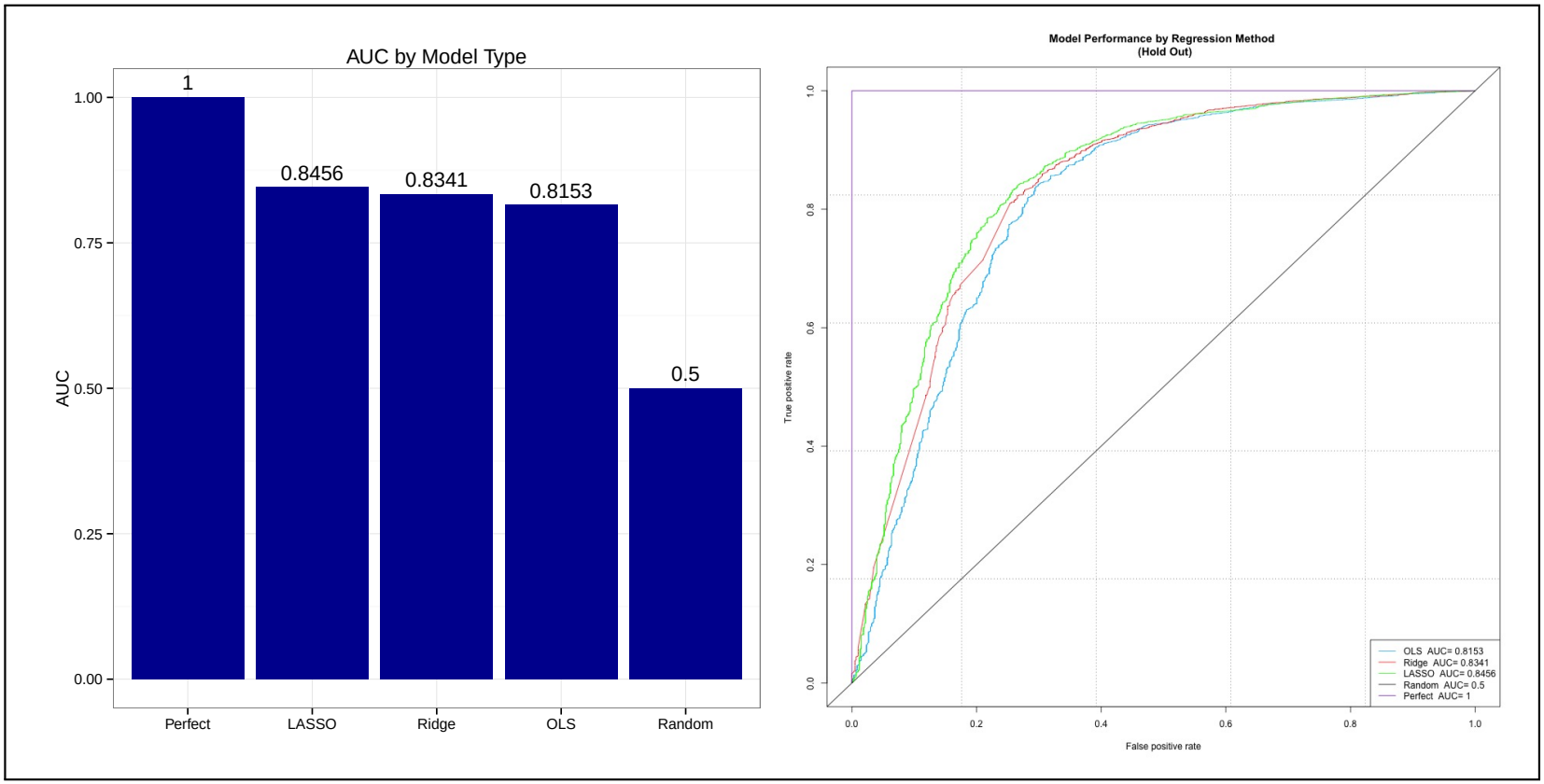
LASSO Model Performance



Ridge Model Performance



Comparison of Ridge, LASSO, and OLS



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- Sweet paper on asymptotic properties of Ridge and LASSO