Journal notes on:

Introduction to topological superconductivity

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I. INTRODUCTION TO SUPERCONDUCTIVITY

Mostly from Tinkham's book

A. London theory

B. BCS theory

C. Ginzburg-Landau theory

1. *Classic type II superconductors*

2. *Josephson effect*

D. Time-dependent Ginzburg-Landau theory

II. INTRODUCTION TO TOPOLOGICAL SUPERCONDUCTIVITY

Mostly from Akhmerov's course

A. SSH model

B. Integer quantum Hall effect

C. Quantum spin Hall effect

D. Quantum anomalous Hall effect

E. Fraction Hall effect

F. Kane-Mele model

G. Kitaev model

In condensed matter physics, the *Kitaev chain* or *Kitaev–Majorana chain,* first proposed by Alexei Kitaev in 2000 **??**, is a toy model for a topological superconductor using a 1D hybrid (semiconductor+superconductor) nanowires featuring Majorana bound states.

The Kitaev chain model consists of a 1D linear lattice of N site and spinless fermions at zero temperature, subjected to nearest neighbor hoping interactions. The real-space tight-binding Hamiltonian describing such model reads

$$H = \mu \sum_{i=1}^{N} \left(c_i^\dagger c_i - \frac{1}{2} \right) - t \sum_{i=1}^{N-1} \left(c_{i+1}^\dagger c_i + h.c \right) + \Delta \sum_{i=1}^{N-1} \left(c_{i+1}^\dagger c_i^\dagger + h.c \right) \tag{1}$$

with $c_i^\dagger$ (c_i) fermionic creation (annihilation) operators, μ the chemical potential, t the hopping energy and Δ an proximity induced superconducting p-wave pairing.

The objective of this model definition is to be able to have a Majorana bound states on the edges mode. For this, let us engineering the Hamiltonian in such a special way that it is actually possible to separate two Majoranas. Foremost, we define each site n as if it has two sublattices, $s = A$ and $s = B$. We then define Majorana operators relating to the fermionic operators as

$$\gamma_i^A = c_i^\dagger + c_i \qquad \text{and} \qquad \gamma_i^B = i\left(c_i^\dagger - c_i\right) \tag{2}$$

or rather, in the opposite way, as

$$c_i^\dagger = \frac{1}{2}\left(\gamma_i^A - i\gamma_i^B\right) \qquad \text{and} \qquad c_i = \frac{1}{2}\left(\gamma_i^A + i\gamma_i^B\right) \tag{3}$$

Indeed, each site can host a fermion or, equivalently, each site hosting two Majorana modes. These Majorana operators are Hermitian $\gamma_i^s = (\gamma_i^s)^\dagger$, unitary $(\gamma_i^s)^2 = 1$ and anticommute as $\left\{\gamma_i^s, \gamma_j^{s'}\right\} = 2\delta_{ij}\delta_{ss'}$.

Substituting directly into the Hamiltonian of Eq.(1) the fermionic operators as given by Eqs.(3) we obtain

$$H = -i\mu\frac{1}{2}\sum_{i=1}^{N}\gamma_i^B\gamma_i^A + i\frac{1}{2}\sum_{i=1}^{N-1}\left(\omega_+\gamma_i^B\gamma_{i+1}^A + \omega_-\gamma_{i+1}^B\gamma_i^A\right), \quad \text{with } \omega_\pm = \Delta \pm t \tag{4}$$

From it we can distinguish between two phases—trivial and topological—, corresponding, respectively, to two different ways of pairing these Majoranas states— no unpaired modes or one isolated mode on both edges, both depicted in Fig.1 in blue and red respectively. This phases can be identified, respectively, in their limiting regimes where one sets $\Delta = t = 0$ and $\mu = 0$ with $\Delta = t \neq 0$.

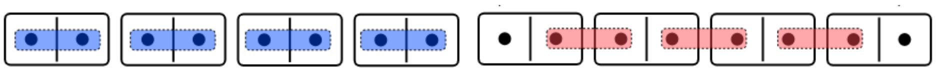

Figure 1. Kitaev chain Majorana modes pairing possibilities

Indeed, see that by setting $\Delta = t = 0$ within the Hamiltonian of Eq.(4) we obtain

$$H_{\text{trivial}} = -i\mu\frac{1}{2}\sum_{i=1}^{N}\gamma_i^B\gamma_i^A, \tag{5}$$

which corresponds to the "no unpaired Majorana modes" configuration. The energy cost for each fermion to be occupied is μ, with all excitations having an energy of either $\pm\mu/2$. The band structure will then have a gapped bulk and no zero energy edge states.

On the other hand, see that by setting $\mu = 0$ with $\Delta = t \neq 0$ we obtain

$$H_{\text{topological}} = it\sum_{n=1}^{N-1}\gamma_i^B\gamma_{i+1}^A \tag{6}$$

which corresponds to the "unpaired edge Majorana mode" configuration where every Majorana operator is coupled to a Majorana operator of a different kind in the next site. Note that the summation only goes up to $n = N - 1$. Moreover, see that by assigning a new fermion operator $\tilde{c}_i = 1/2\left(\gamma_i^B + i\gamma_{i+1}^A\right)$, the Hamiltonian can be otherwise expressed as

$$H_{\text{topological}} = 2t\sum_{n=1}^{N-1}\left(\tilde{c}_i^\dagger\tilde{c}_i + \frac{1}{2}\right) \tag{7}$$

which describes a new set of $N-1$ Bogoliubov quasiparticles with energy t. For every Majorana pair we assign an energy difference $2t$ between the empty and filled state. All states which are not at the ends of the chain have an energy of $\pm t$ and thus the bands structure has a gapped bulk. However, see that the missing mode $\tilde{c}_N = 1/2\left(\gamma_N^B + i\gamma_1^A\right)$, which couples the Majorana operators from the two endpoints of the chain, does not appear in the Hamiltonian and does most have zero energy. This mode is called a Majorana zero mode and is highly delocalized. As the presence of this mode does not change the total energy, the ground state is two-fold degenerate. This condition is a topological superconducting non-trivial phase.

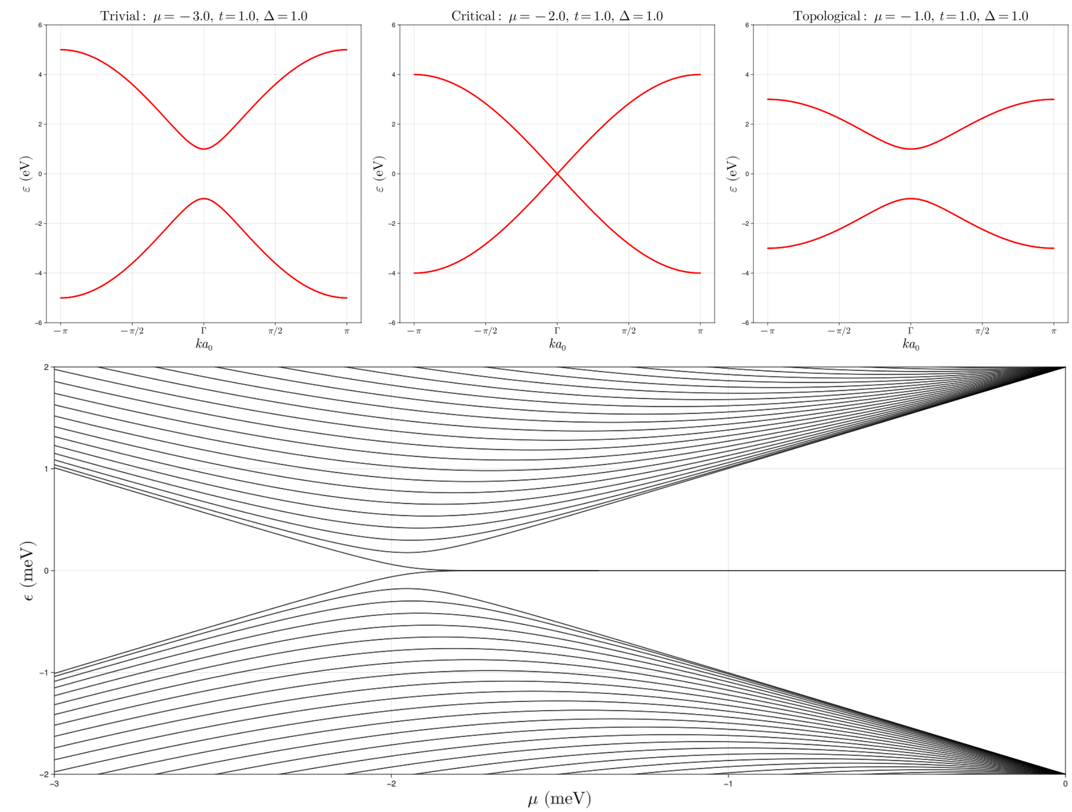

Figure 2. Kitaev chain **(top)** band structure and **(bottom)** band spectrum for a chain length of $L = 50$ with lattice spacing $a_0 = 1$ fixing $\Delta = t = 1.0$. The critical μ shifts forward to infinity as $L \to 0$.

Numerical implementation in Quantica.jl

Shown below are the broad strokes of a numerical implementation of the Hamiltonian in Julia using the Quantica.jl. However, prior to this implementation, we will be needing the Bogoliubov-de Gennes formalism. For this, we define a Nambu spinor

$$\check{c}_i^\dagger = \begin{pmatrix} c_i^\dagger & c_i \end{pmatrix} \quad \text{and} \quad \check{c}_i = \begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix} \tag{8}$$

such that that Hamiltonian in Eq.(1) reads

$$H = \mu \frac{1}{2} \sum_i \check{c}_i^\dagger \tau_z \check{c}_i - t \sum_{i=1}^{N-1} \check{c}_{i+1}^\dagger \tau_z \check{c}_i + \Delta \sum_{i=1}^{N-1} \check{c}_{i+1}^\dagger i\tau_y \check{c}_i \tag{9}$$

with τ_x, τ_y, τ_z Pauli matrices in the particle-hole subspace.

To understand why this is the case check:

$$\mu : \quad \check{c}_i^\dagger \tau_z \check{c}_i = \begin{pmatrix} c_i^\dagger & c_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix} = c_i^\dagger c_i - c_i c_i^\dagger = 2c_i^\dagger c_i - 1 \tag{10}$$

$$t : \quad \check{c}_j^\dagger \tau_z \check{c}_i = \begin{pmatrix} c_j^\dagger & c_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix} = c_j^\dagger c_i - c_j c_i^\dagger = c_j^\dagger c_i + h.c \tag{11}$$

$$\Delta : \quad \check{c}_j^\dagger i\tau_y \check{c}_i = \begin{pmatrix} c_j^\dagger & c_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix} = c_j^\dagger c_i^\dagger - c_j c_i = c_j^\dagger c_i^\dagger + h.c \tag{12}$$

where we the fermionic anti-commutation properties $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$, such that $c_i c_i^\dagger = 1 - c_i^\dagger c_i$, $c_j c_i^\dagger = -c_i^\dagger c_j$ and $c_j c_i = -c_i c_j$.

Also, see that these Hamiltonian has particle-hole symmetry, i.e $\mathcal{P}H\mathcal{P}^{-1} = -\tau_x H^* \tau_x = -H$ with $\mathcal{P} = \tau_x \mathcal{K}$ and $\mathcal{K}$ complex conjugation, as well as time reversal symmetry, i.e $\mathcal{T}H\mathcal{T}^{-1} = H^* = H$ with $\mathcal{T} = \mathcal{K}$ for this spinless case (for reference, $\mathcal{T} = i\sigma_y \mathcal{K}$ for a 1/2-spin system). Once again, to understand why this is the case check.

Therefore, within this Nambu orbital space in mind, we can define the Kitaev hamiltonian as:

```julia

build_Kitaev_chain(; kw...) = build_Kitaev_chain(LatticeParams(; kw...))
function build_Kitaev_chain(p::LatticeParams)
    @unpack L, a0 = p

    lat = LP.linear(; a0)

    model_normal = -@onsite((; μ=0.0) -> μ*τz) - @hopping((; t=0.0) -> t*τz)
    model_anomalous = @hopping((; Δ=0.0) -> Δ*1im*τy, region = (r, dr) -> dr[1] > 0)
    model_anomalous_dagger = @hopping((; Δ=0.0) -> -Δ*1im*τy, region = (r, dr) -> dr[1] < 0)
    model = model_normal + model_anomalous + model_anomalous_dagger

    h = lat |> hamiltonian(model, orbitals=2)

    if isfinite(L) h=supercell(h, region = r -> 0 <= r[1] <= L) end

    return h
end

```

The key factor of this implementation is that it is necessary to define the p-wave pairing (Nambu) hopping with a sign difference as it hops front and backwards, otherwise it will not be an hermitian term.

H. Oreg-Lutchyn model

The Oreg-Lutchyn Majorana minimal model consists of a finite 1D semiconductor (SM) nanowire with strong spin-orbit coupling (SOC) α and a tunable chemical potential μ, in proximity of a superconductor (SC) of homogeneous pairing Δ, having a magnetic field B_z applied along it's length, defined as the $\hat{z}$ direction. The Rashba effect describes the coupling of an electric field E_x that breaks inversion symmetry breaking in the direction perpendicular to the wire, to the electron's spin, i.e $\propto (i\vec{\nabla} \times \hat{x}) \cdot \vec{\sigma} = i\sigma_y \partial_z$ with $\vec{\sigma} = (\sigma_x,\, \sigma_y,\, \sigma_z)$. The Zeeman effect described the spin splitting due to the in-plane magnetic field B_z. The pairing term describes the Cooper pairs from BCS theory than could tunnel from the SM to the SC.

The tight-binding Hamiltonian describing such system can then be decomposed as

$$H = H_{\mathrm{K}} + H_{\mathrm{SOC}} + H_{\mathrm{Z}} + H_{\mathrm{SC}} \tag{13}$$

$$H_{\mathrm{K}} = (2t - \mu) \sum_{i\sigma} c^{\dagger}_{i\sigma} c_{i\sigma} - t \sum_{\langle i,j \rangle \sigma} c^{\dagger}_{i\sigma} c_{j\sigma} \tag{14}$$

$$H_{\mathrm{SOC}} = \frac{\alpha}{2a_0} \sum_{i\sigma} \left(c^{\dagger}_{i+1\bar{\sigma}} c_{i\sigma} + h.c \right) \tag{15}$$

$$H_{\mathrm{Z}} = V_{\mathrm{Z}} \sum_{i} \left(c^{\dagger}_{i\uparrow} c_{i\uparrow} - c^{\dagger}_{i\downarrow} c_{i\downarrow} \right) \tag{16}$$

$$H_{\mathrm{SC}} = \Delta \left(c^{\dagger}_{i\downarrow} c^{\dagger}_{i\uparrow} + \mathrm{h.c.} \right) \tag{17}$$

with $c^{\dagger}_i$ (c_i) fermionic creation (annihilation) operators, μ the chemical potential, $t = \eta/a_0^2$ the hopping energy into $\langle i, j \rangle$ nearest-neighbouring sites with a_0 the lattice constant and $\eta = \hbar^2/2m^*$ with m^* the effective mass of the electrons, $V_{\mathrm{Z}} = g_J \mu_{\mathrm{B}} B_z/2$ the Zeeman potential with g_J the Landé gyromagnetic moment and μ_{B} Bohr's magneton, α the Rashba SOC strength and Δ proximity induced superconducting s-wave pairing.

A paragraph explaining the bands.

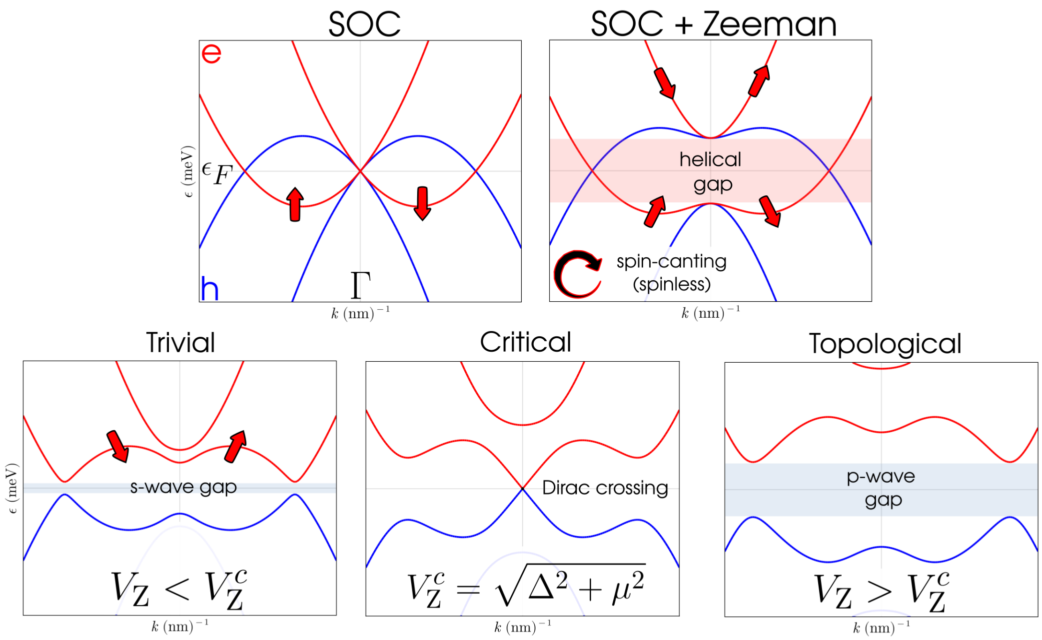

Figure 3.

A paragraph explaining the phase-diagram, pfaffian and band spectrum.

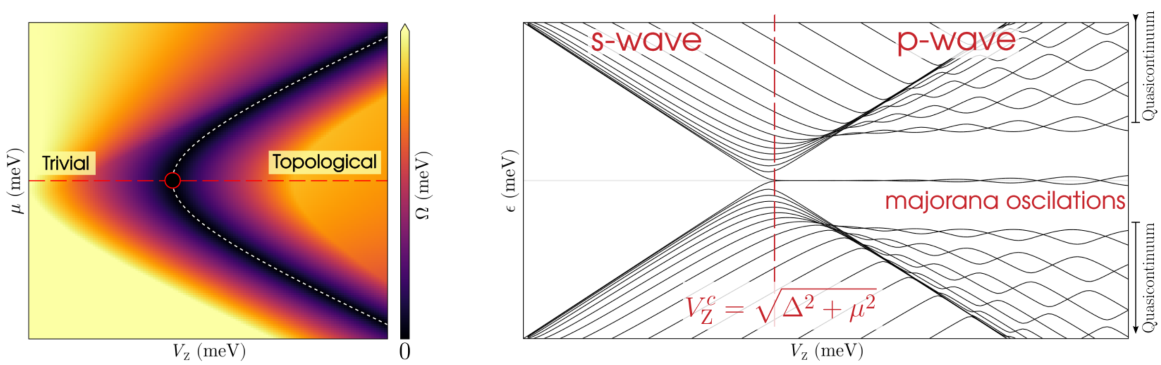

Figure 4.

Numerical implementation in Quantica.jl

Shown below are the broad strokes of a numerical implementation of the Hamiltonian in Julia using the Quantica.jl. However, prior to this implementation, we will be needing the Bogoliubov-de Gennes formalism. For this, need to double the degrees of freedom through the Nambu-spinor. In the so called unrotated-spin basis we define a Nambu spinor as

$$\check{c}_i^\dagger = \begin{pmatrix} c_i^\dagger & c_i \end{pmatrix} = \begin{pmatrix} c_{i\uparrow}^\dagger & c_{i\downarrow}^\dagger & c_{i\uparrow} & c_{i\downarrow} \end{pmatrix} \tag{18}$$

In this Nambu⊗spin orbital space the Hamiltonian in Eq.(13) reads

$$
\begin{aligned}
H =& H_{\mathrm{K}} + H_{\mathrm{SOC}} + H_{\mathrm{Z}} + H_{\mathrm{SC}} && (19)\\
H_{\mathrm{K}} =& (2t-\mu)\sum_i \check{c}_i^\dagger[\tau_z\otimes\sigma_0]\check{c}_i - \frac{1}{2}t\sum_{\langle i,j\rangle}\check{c}_i^\dagger[\tau_z\otimes\sigma_0]\check{c}_j && (20)\\
H_{\mathrm{SOC}} =& \frac{\alpha}{2a_0}\sum_i \check{c}_i^\dagger[\tau_z\otimes i\sigma_y]\check{c}_{i+1} && (21)\\
H_{\mathrm{Z}} =& V_Z\sum_i \check{c}_i^\dagger[\tau_z\otimes\sigma_z]\check{c}_i && (22)\\
H_{\mathrm{SC}} =& \frac{1}{2}\Delta\sum_i \check{c}_i^\dagger[\tau_y\otimes\sigma_y]\check{c}_i && (23)
\end{aligned}
$$

with τ Pauli matrices in the particle-hole subspace and σ in the spin subspace.

To understand why this is the case check we show explicitly the derivation for the pairing term as an example. It reads:

$$
\begin{aligned}
\check{c}^\dagger[\tau_y\otimes\sigma_y]\check{c} &= \begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger & c_\uparrow & c_\downarrow \end{pmatrix}\left(\begin{array}{cc|cc} 0 & 0 & 0 & -1\\ 0 & 0 & +1 & 0\\ \hline 0 & +1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{array}\right)\begin{pmatrix} c_\uparrow\\ c_\downarrow\\ c_\uparrow^\dagger\\ c_\downarrow^\dagger \end{pmatrix} \qquad (24)\\
&= -c_\uparrow^\dagger c_\downarrow^\dagger + c_\downarrow^\dagger c_\uparrow^\dagger + c_\uparrow c_\downarrow - c_\downarrow c_\uparrow = 2\left(c_\downarrow^\dagger c_\uparrow^\dagger + \text{h.c.}\right)
\end{aligned}
$$

where we the fermionic anti-commutation properties $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$.
The remaining terms derivation is analogous but even simpler because there is will be no mixing of particle with particle-hole components; the holeonic terms will correspond to the negative of the electronic terms, meaning that one just needs to expand the space according to $\tau_z\otimes$ the respective spin matrix. For the kinetic term there is no mixing of spin so it must trivially have the spin Pauli matrix σ_0. Similarly, for the Zeeman term there is only the same-spin mixing of the type $\uparrow\uparrow - \downarrow\downarrow$ so it must have σ_z . As for the SOC term there is spin-mixing of opposing spins, so the options are either σ_x or $i\sigma_y$ (with a i for it to be hermitian). One can check with the fermionic anti-commutation properties that it is indeed $i\sigma_y$.

Therefore, within this spin⊗Nambu orbital space in mind, we can define the Oreg-Lutchyn Hamiltonian as:

```julia

build_OregLutchyn_wire(; kw...) = build_OregLutchyn_wire(Params(; kw...))
function build_OregLutchyn_wire(p::Params)
    @unpack L, a0, m0 = p
    t = ħ2ome/(2m0*a0^2)

    lat = LP.linear(; a0)

    model_K = @onsite((; μ=0.0) -> (2*t-μ)*σ0τz) - hopping(t*σ0τz)
    model_Z = @onsite((; Vz=0.0) -> Vz*σzτz)
    model_α = @hopping((r, dr; α=0.0) -> α*(im*dr[1]/(2a0^2))*σyτz)
    model_Δ = @onsite((; Δ0=0.0) -> Δ0*σyτy)
    model = model_K + model_Z + model_α + model_Δ

    h = lat |> hamiltonian(model, orbitals=4)
    if isfinite(L) h=supercell(h, region = r -> 0 <= r[1] <= L) end

    return h
end

```

Alternative Nambu basis

It is common for people to define instead the Nambu spinor in a rotated basis as such

$$\bar{\boldsymbol{c}}_i^\dagger = \begin{pmatrix} \boldsymbol{c}_i^\dagger & [i\sigma_y \boldsymbol{c}_i] \end{pmatrix} = \left(\begin{array}{cc|cc} c_{i\uparrow}^\dagger & c_{i\downarrow}^\dagger & c_{i\downarrow} & -c_{i\uparrow} \end{array} \right) \tag{25}$$

As also explained in section II.C.1 of the previous part, these basis' operators relate to each other as

$$\bar{\boldsymbol{c}}_i = \bar{\boldsymbol{\mathcal{U}}} \check{\boldsymbol{c}}_i \Leftrightarrow \check{\boldsymbol{c}}_i = \bar{\boldsymbol{\mathcal{U}}}^\dagger \bar{\boldsymbol{c}}_i \tag{26}$$

$$\bar{\boldsymbol{c}}_i^\dagger = \check{\boldsymbol{c}}_i^\dagger \bar{\boldsymbol{\mathcal{U}}}^\dagger \Leftrightarrow \check{\boldsymbol{c}}_i^\dagger = \bar{\boldsymbol{c}}_i^\dagger \bar{\boldsymbol{\mathcal{U}}} \tag{27}$$

and, consequently, for a generic $\check{\boldsymbol{M}}$ matrix,

$$\bar{\boldsymbol{M}} = \bar{\boldsymbol{\mathcal{U}}} \check{\boldsymbol{M}} \bar{\boldsymbol{\mathcal{U}}}^\dagger \tag{28}$$

with $\bar{\boldsymbol{\mathcal{U}}}$ is a unitary matrix (i.e $\bar{\boldsymbol{\mathcal{U}}}^\dagger \bar{\boldsymbol{\mathcal{U}}} = \bar{\boldsymbol{\mathcal{U}}} \bar{\boldsymbol{\mathcal{U}}}^\dagger = \mathbb{1}$)

$$\bar{\boldsymbol{\mathcal{U}}} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \imath\sigma_y \end{pmatrix} \tag{29}$$

Making use of Pauli matrices' property

$$\sigma_\alpha \sigma_\beta = \sigma = \sigma_0 \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma \tag{30}$$

one can check that

$$H_{\mathrm{K}} : \bar{\boldsymbol{\mathcal{U}}} [\tau_z \otimes \sigma_0] \bar{\boldsymbol{\mathcal{U}}}^\dagger = [\tau_z \otimes \sigma_0] \tag{31}$$

$$H_{\mathrm{SOC}} : \bar{\boldsymbol{\mathcal{U}}} [\tau_z \otimes \imath\sigma_y] \bar{\boldsymbol{\mathcal{U}}}^\dagger = [\tau_z \otimes \imath\sigma_y] \tag{32}$$

$$H_{\mathrm{Z}} : \bar{\boldsymbol{\mathcal{U}}} [\tau_z \otimes \sigma_z] \bar{\boldsymbol{\mathcal{U}}}^\dagger = [\tau_z \otimes \sigma_z] \tag{33}$$

$$H_{\mathrm{SC}} : \bar{\boldsymbol{\mathcal{U}}} [\tau_y \otimes \sigma_y] \bar{\boldsymbol{\mathcal{U}}}^\dagger = [\tau_x \otimes \sigma_0] \tag{34}$$

meaning that, in this the rotated basis, only the pairing Hamiltonian has it's Pauli matrices changed. Concretely,

$$H_{\mathrm{SC}} = \frac{1}{2}\Delta \sum_i \bar{\boldsymbol{c}}_i^\dagger [\tau_x \otimes \sigma_0] \bar{\boldsymbol{c}}_i \tag{35}$$

I. Topological insulators