Implied Volatility Surfaces

Relatório Projecto de Investigação em Matemática e Aplicações

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The term "market" makes reference to the financial market, in which securities and derivatives are traded at low transaction costs relative to the assets themselves. The securities include, but are not limited to, public traded companies' stocks and bonds by any company whose willing to issue them. There are two types of popular derivative products, forward contracts and options. In this work we will be focused on the latter and methods involving it's contract pricing.

1 Options

An option is a contract between two parties. There are two main types of option contracts: calls and puts. The call contract gives the buyer the right to buy the underlying asset, the put gives the buyer the right to sell. The other party, the seller, has the obligation to sell (call) or buy (put) if the specifications of the contract are met, and the buyer chooses to exercise his right. Each contract has usually the following specific properties [10]:

- The type of contract, whether it is a call option or a put option;
- The specifications of the underlying asset, e.g. quantity and type of asset;
- The strike price, K;
- The duration of the contract or time to maturity, T.
- The contract price.

The strike price is the threshold price for the buyer to exercise the option, e.g. for a call option, the buyer may exercise for all asset prices above the strike price, i.e. the seller must sell the asset at the strike price even though the market price of this asset is higher, leading to the buyer selling this asset at a profit, disregarding the contract price. On the contrary, for a put option, the seller must sell the asset for market asset prices below the strike price. The contract price is also referred to as the premium of the contract and is usually designated by c in case of a call option contract or p in case of a put option contract.

When the buyer of the contract may exercise his right subdivides the types of options further into American options or European options. In the latter the buyer can only exercise at the end of the duration of the contract, in contrast, when it comes to American options the buyer may exercise the option at any time before the terminus of the contract, including at the expiration time. In this work we will only deal with European call options or European put options or strategies involving both, so we will refer to these as simply call options or put options.

Usually the underlying assets in these contracts are stocks and their behaviour on the stock market, i.e. their market price at time t, S_t, dictates the contract pricing, that's why modeling their properties may be useful to accurately and fairly price option contracts.

We will consider following functions $c \equiv c(K, T, S_0, \sigma)$ and $p \equiv p(K, T, S_0, \sigma)$, where S_0 is the present market value of the underlying of the contract and σ is the volatility of the underlying, and try to model them throughout this work.

1.1 Put-call parity

We can get a glimpse of the put-call parity by imagining the following portfolio [1]: a short European put plus a long European call with the same maturity, strike price, underlying and with no dividends replicates a forward with the same maturity and strike. Intuitively, as the seller of the put and buyer of the call, you will buy the underlying at strike price either way the underlying price is at maturity, above or bellow the strike, which is what is mandated by a forward contract, F. This hints us to a certain duality between these two types of options.

To define the put-call parity, consider the following portfolios:

- V_1: a put and a share of the stock;
- V_2: a call and Ke^{-rT} of cash;

with r the risk-free rate. Let's analyse the outcome of both positions at maturity. If $S(T) < K$:

- V_1: the put is exercised and so you will sell your a share of the stock for K;
- V_2: the call is not exercised and so you have K.

If $S(T) > K$:

- V_1: the put is not exercised and so you will sell your a share of the stock for $S(T)$;

- V_2: the call is exercised and so you buy the stock with the funds you have and sell it for $S(T)$.

This implies that the both portfolios have the same result and so should be valued the same at the present time:

$$p + S_0 = c + Ke^{-rT}, \tag{1}$$

this equality is known as the put-call parity.

1.2 Strategies using options

When engaging in a option contract the parts express some implicit belief as to the behaviour of the underlying. For example, if you partake in a long position of a call you would expect the price of the underlying to go up, specifically, above the strike price at the time of maturity with no upper bound for the profit, and vice versa with a put. With this, comes the risk of your expectations not being met, and the risk may be very high leading to a very large loss. There are ways to limit risk employing strategies that involve more than one option, in exchange for the limitless profit when your expectations about the market are met if you opted for vanilla option.

1.2.1 Spreads

The spreads are the family of strategies that require more than option of the same type (i.e. call or put), but different positions (i.e. long or short).

- **Bull Call Spread**: take the long on a call of a certain underlying with a certain strike price, and take the short on a call of the same underlying but with higher strike price;
- **Bear Put Spread**: similar to the call spread, take the long on a put of a certain underlying with a certain strike price , and take the short on a call of the same underlying but, now instead, with lower strike price.

	Payoff bull spread	Payoff bear spread
$S_T \leq K_1$	0	$K_2 - K_1$
$K_1 < S_T < K_2$	$S_T - K_1$	$K_2 - S_T$
$S_T \geq K_2$	$K_2 - K_1$	0

Table 1: The payoff of these spreads at different scenarios has always the lower bound of 0 and upper bound of $K_2 - K_1$. Please note that this does not guarantee profit greater or equal than 0 since payoff and profit are not the same and you have to take into account the premium of the options agent in the spread.

The denomination "bull" or "bear" is referent to the payoff profile, i.e. the bull spread is the two option spread with the payoff profile of the first column of Table 1, and a bear spread has the payoff of the second column. We can construct bull put spreads and bear call spreads as well, but the former aren't as common as the ones presented earlier.

- **Calendar Spread**: also refereed to as time spread, is created by selling one call with a certain strike price K and time to maturity T_1, and buying one call with the same strike price but longer time to maturity T_2.

Usually, for the same strike price, the call options with lesser time to maturity have a lower premium than ones with a longer time to maturity, therefore this spread, requires initial investment.

This spread is profitable near the strike price K and the loss occurs considerably far from K.

This spread can also be constructed using put options, but in that case, the short put has longer time to maturity. The behaviour is identical, just factor in the put call parity.

- **Ratio Spread**: hold the short in two options with time to maturity T and strike K_1, and hold a long in another one with the same time to maturity but strike K_2. Consider K_1 to be at-the-money, it is requires that K_2 be in-the-money in relation to K_1, so that in the case of call options $K_2 < K_1$ and in the case of put options $K_2 > K_1$.

For simplicity, assume that K_1 and K_2 are related by a factor of two, i.e. if the option type requires $K_2 > K_1$ then, we have $K_2 = 2K_1$. Vice-versa, if $K_2 < K_1$ then $K_2 = \frac{1}{2}K_1$. We have the following payoff for a call ratio spread when $S_T > K_1$ (all options are exercised):

$$(S_T - K_2) - 2(S_T - K_1) = (S_T - 2K_1) - 2S_T + 2K_1 = -S_T, \tag{2}$$

which leads to unlimited loss in case the stock price goes high. Of course if $S_T < K_1$ no option is exercised and the payoff is 0. This is a good strategy to employ when the investor believes that the stock price won't move to far away from K_1, and it has limited risk in case the stock price goes low (Table 2). The same computations can be made in case of puts with the payoff profile being a mirror image of the call ratio spread (Table 3).

	Payoff call ratio spread
$S_T \leq K_2$	0
$K_2 < S_T < K_1$	S_T
$S_T \geq K_1$	$-S_T$

Table 2: Payoff of a call ratio spread when $K_2 = \frac{1}{2}K_1$.

	Payoff put ratio spread
$S_T \leq K_1$	S_T
$K_1 < S_T < K_2$	$-S_T$
$S_T \geq K_2$	0

Table 3: Payoff of a put ratio spread when $K_2 = 2K_1$.

1.2.2 Straddle

A straddle consists in a strategy that uses both types of options with same time to maturity and strike price. Buy both a call and a put option with strike K and time to maturity T. So you'll have positive payoff with it being 0 at $S_T = 0$ (Table 4).

	Payoff straddle
$S_T < K$	$K - S_T$
$S_T = K$	0
$S_T > K$	$S_T - K$

Table 4: When $S_T < K$ only the put option is exercised, similarly when $S_T > K$ only the call is exercised, and so this strategy exhibits the respective option type payoff. When $S_T = 0$ no option is exercised.

2 Probability Theory

In regards to stock price behaviour the best we can do is trying to predict what will be the most frequent prices and how often and by how much will the price deviate from them. For that, the tool we use is probability theory. We shall define the basis for probability theory in general terms, it will become clear how this will be applicable to stock prices, or more generally, the financial market. All considerations in probability theory are in respect to events, ω. The set of all events is called the sample space Ω and subsets of it are called compound events and include Ω itself and $\phi := \{\}$, the empty set. There is a bijection between events and subsets of the sample space, so the terms event and subset will be used equivalently.

There are remarkable relations between events that must be defined, for instance $A \cap B$ reads "intersection of A with B" and represents the points common to both the A subset and the B subset.

A probability measure $\mathbb{P}$ is a real-valued function that associates to each event A a value. This family of functions follows the axioms bellow[11]:

- $\mathbb{P}(A) \geq 0, \forall A \in \Omega$;
- $\mathbb{P}(\Omega) = 1$;

- If $A \cap B = \phi$, then $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$. That is, if two events don't share a subset of Ω, their joint probability is the sum of their individual probabilities.

- For any sequence $\{A_i\}$ of pairwise disjoint events, i.e. $A_i \cap A_j = \phi$ for $i \neq j$, we have that $\mathbb{P}(A_1 \cup A_2 \cup ...) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + ...$ This axiom is only needed when considering infinite sample spaces, otherwise it can be derived from the first three.

The probability of an event A occurring given that event B has occurred is called the conditional probability, and is defined as:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \tag{3}$$

with $\mathbb{P}(B) \neq 0$ and it is said that events A and B are independent if:

$$\mathbb{P}(A|B) = \mathbb{P}(A). \tag{4}$$

2.1 Random Variables

A random variable $\mathbf{X}$ is a real or complex valued function on a sample space, assigns values to points in the sample place like so, $\mathbf{X}(\omega) = x$, for some $\omega \in \Omega$. There are similarities between probabilities and random variables, the difference resides in that a probability is a set function and a random variable is a point function, both of the sample space.

Some structure regarding a random variable may be detectable and may provide deeper information as to it's behaviour, so let us define the distribution function of the random variable $\mathbf{X}$:

$$F_{\mathbf{X}}(x) = \mathbb{P}(\mathbf{X} \leq x) \tag{5}$$

which reads "the probability that the variable $\mathbf{X}$ takes a value lesser or equal to x".

$F_{\mathbf{X}}(x)$ may be written as an integral of some integrable function $f(x)$ as follows:

$$F_{\mathbf{X}}(x) = \int_{-\infty}^{x} f_{\mathbf{X}}(u)du, \tag{6}$$

and if this function is continuous, we can define:

$$f_{\mathbf{X}}(x) \equiv ddxF_{\mathbf{X}}(x), \tag{7}$$

this function is called the probability density distribution function of $\mathbf{X}$, and $f_{\mathbf{X}}(x)dx$ can be thought as the probability of $\mathbf{X}$ taking a value between x and $x + dx$.

2.2 Summary statistics

One interesting aspect of a random variable **X** distribution would be the center value to which all the values that **X** may take are around. This value is designated as the expected value of **X**, or mean, and is as follows:

$$E[\mathbf{X}] = \int_{-\infty}^{+\infty} x f_{\mathbf{X}}(x)dx. \tag{8}$$

Further along this work we will use μ to refer to the mean of the random variable we are considering, and so

$$E[\mathbf{X}] := \mu, \tag{9}$$

although this description, μ and $E[\mathbf{X}]$ may be associated, both, with the mean of the distribution **X**. The $E[\mathbf{X}]$ representation will become useful when referencing the expected values of different distributions of the same ransom variable **X**, e.g. $E[\mathbf{X}^n]$.

We can use the mean to compute some important statistics, a class described as the central moments of order n and defined as:

$$E[(\mathbf{X} - \mu)^n]. \tag{10}$$

The moment of order $n = 1$ is trivially null, the more interesting moment would be the one of order $n = 2$, as it provides us with information about how much dispersion we have around the mean value. We call this moment, the variance, σ^2

$$\sigma^2 = \int_{-\infty}^{+\infty} x - \mu f_{\mathbf{X}}(x)dx = E[\mathbf{X}^2] - \mu^2. \tag{11}$$

The variance grows with the dispersion of **X** around the mean, and therefore will never be negative, as it can be perceived by consequence of the definition. We will be rather interested in it's square root, $\sigma(\mathbf{X})$, the standard deviation of **X**.

2.2.1 Normal Distribution

A random variable that follows the normal distribution, or Gaussian distribution, has the following probability density distribution function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}. \tag{12}$$

The two determining parameters in this distribution μ and σ represent, respectively, the mean and the standard deviation. Two variables with follow a normal distribution with same mean and standard deviation are said to be identically distributed and for many practical interests would be regarded as the same.

3 Stock Prices

Public companies trade equity in the form of stock shares. The price of each stock depends greatly on the company's product demand, and overall earnings. It's behaviour is practically erratic, though some properties may be extracted from historical data and used to forecast future stock price's trend and volatility.

3.1 Volatility

Roughly speaking, volatility measures the uncertainty associated with the stock price fluctuations. A commonly used statistical measure for volatility based on historic data is the standard deviation, from which it borrows it's symbol σ, but they are not the same. Volatility is an intrinsic and implicit property to the stock price, it may be modeled and approximated, but there's no way for it to be known.

One of the ways to estimate volatility is to consider the logarithmic return,

$$u_i = log\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, ..., n \ , \tag{13}$$

and it's standard deviation,

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \overline{u})^2}. \tag{14}$$

This value does not correspond directly with the volatility, but is an estimate of $\sigma\sqrt{\tau}$, where τ is constant the time interval between stock prices $S_i - 1$ and S_i, for $i = 1, 2, ..., n$, and so the estimate of the volatility is given by

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}. \tag{15}$$

If the volatility was constant in time this would be a good enough approximation, but despite some models considering it to be constant, it appears this is not the case. If it were constant the more historical data considered the better the results, but given it is not, how much data to consider, n, must be regarded, and regularly used rule of thumb is to account for as much data before as you are trying to forecast. Also, this computation must be done regularly in order to adjust the models that subside on the volatility value to be correct as time passes.

Later, we will present other models of estimating volatility such as SABR models, in which volatility changes over time, much in the same way as stock prices do.

3.2 Interest Rate and Time Value of Money

Interest rates are fruit of the relation between lenders and borrowers, more specifically with the risk associated with lending money. The higher the risk, the higher the interest rate. Naturally different credit have different risks and different lenders measure these risks differently, but the market stabilizes the interest rate transversely between all lenders by the way of offer and demand, and risks associated as stated before.

The time value of money is direct product of interest. From an investors point of view, a given amount of money today is more valuable and one year from now, for example, since the investor can take the money to a bank and retrieve it a year later with added interest.

Dictating the rate of return of investments is the risk-free rate, r, such that future value[3], f, of an initial investment p continuously compounded over time period t is:

$$f = pe^{rt} \ . \tag{16}$$

4 Stochastic Processes

A random variable is a function that assigns a number to an event, a stochastic process $\mathbf{X}$ is a function that assigns a number to an event and real variable, $\mathbf{X}(w, t)$, the latter parameter is often called time, even when it does not correspond to time in the real physical application of the underlying process. Stochastic processes, also referred to as random processes, can be seen as a family of random variables indexed by the parameter t,

$$\mathbf{X} \equiv \{\mathbf{X}_t(\omega), t \in \mathbb{R}\}, \tag{17}$$

for some interval of t, which can be either continuous or discrete. The opposite point of view is useful as well, that is, a stochastic process can be viewed as a collection of functions of t for a given event w,

$$\mathbf{X} \equiv \{\mathbf{X}_\omega(t), w \in \Omega\}, \tag{18}$$

for the sake of simplicity the argument ω is usually omitted, and $\mathbf{X}(t)$ for a fixed ω is called a realization of the process. Both representations are correct and equivalent, and may be used in accordance with their convenience.

4.1 Random Walk

Let's construct a process by means of independent binomial random variables $\mathbf{X}_1, \mathbf{X}_2, ...$, where $\mathbf{X}_i, \forall i \geq 1$ has the following distribution

$$Pr(\mathbf{X} = 1) = Pr(\mathbf{X} = -1) = \frac{1}{2}. \tag{19}$$

We can define the iterative process $W_n(t)$ as

$$\{ \ W_n(0) = 0, \mathbf{W}_n\left(\frac{i}{n}\right) = \mathbf{W}_n\left(\frac{i-1}{n}\right) + \frac{\mathbf{X}_i}{\sqrt{n}} \forall i \geq 1, \tag{20}$$

where the index n is linked with two important properties of the process. Firstly, the time spacing is $1/n$ and secondly, the process may only move up or down by $1/\sqrt{n}$. This process is known as the random walk[1].

4.2 Brownian Motion

Dr. Robert Brown observed that dust particles suspended in the air described an erratic behaviour and their path was random [5]. This became known as the Brownian motion, the family of stochastic processes used to describe these phenomena have the same name but are also referred to as Wiener processes. This kind of behaviour was since discovered in many more instances, for example, an electron under black body radiation or the financial market, more specifically, it models the random behaviour stock prices seem to have.

Einstein was one of the people who made an effort to formalize this phenomenon[7], and one of the assumptions made is that the future behaviour of the particle is only dependent on it's position at present time and probability of Δ displacement in time instant δt. This is called the Markov property, and are called Markov processes the stochastic processes that exhibit it. For instance, it is also assumed that the future stock prices depend only on the present price and the density probability distribution for the movement of said stock price in the next clock cycle Δt, or in continuous terms, the infinitesimal time change dt, and therefore follow a Markov process.

The Brownian motion follows a $\mathbb{P}$-Wiener process $B = \{B(t) : t \geq 0\}$, for a probability measure $\mathbb{P}$, and has the properties bellow[12]:

- $B(0) = 0$;
- $B(t)$ is continuous;
- Under $\mathbb{P}$, $B(t)$ is normally distributed $N(0, t)$;
- $B(t) - B(s) \sim N(0, t - s)$, and is independent of the history of the process up to time s.

The generalized Wiener process $W = \{W(t) : t \geq 0\}$ is defined as:

$$dW(t) = adt + bdB(t), \tag{21}$$

where a, b are constants and $B(t)$ is a Brownian motion as defined above. The first term is responsible for the drift and the second can be seen as to introduce noise to the path of $dW(t)$. We can easily see that Eq. 21 implies that $W(t)$ drifts a per unit of time. With some further analysis, and taking into account the properties of Brownian motions described above, see that the variance of the Wiener process $dW(t)$ is b^2dt. So, usually the notation is

$$dW(t) = \mu dt + \sigma dB(t), \tag{22}$$

as to indicate the roles of these constants as drift and standard deviation.

4.3 Stock Price as Geometric Brownian Motion

It could be the case that the stock prices follow a generalized Wiener process but the rate of return must be a percentage of the stock price, as well as the volatility, and so these take the decimal form leading the stock price process $S = \{S(t) : t \geq 0\}$ being modeled by:

$$dS(t) = \mu S dt + \sigma S dB(t), \tag{23}$$

where μ is the rate of return and σ is the volatility. This is the most widely used model for stock prices and is the geometric Brownian motion. This process is the continuous limit of the random walk process, as individual realizations of both processes are identically distributed when the number of steps of the random walk n tends to infinity and the time step becomes infinitesimal[1].

4.4 Black-Scholes Pricing Model

4.4.1 Martingale Processes

Before deriving the Black-Scholes pricing formula we should emphasize some useful definitions. In first place we should have the notion of a martingale. A discrete process $X = \{X_n : n \geq 0\}$ is a martingale if and only if:

- $\mathbb{E}(|X_n|) < \infty$;
- $\mathbb{E}(X_{n+1}|X_1, ..., X_n) = X_n$.

For our purpose we can consider that if X is a stochastic process such that $\mathbb{E}\left[\int_0^T \sigma^2(s)ds\right] < \infty$, then X is a martingale if and only if X is driftless ,i.e. $\mu = 0$ [1].

4.4.2 Girasnov Theorem

Along the way of deriving the Black-Scholes pricing formula it will be useful to turn a geometric Brownian motion to a Martingle process, the way to do this is by chage of probability measure and the meathod is Girasnov's theorem[8]. So, let's start with a $\mathbb{P}-$Brownian motion and γ_t is a process satisfying $\mathbb{E}_{\mathbb{P}}\left[exp\left(\int_0^T \gamma^2(t)dt\right)\right] < \infty$, then, there exists a measure $\mathbb{Q}$ that satisfies:

- $\mathbb{Q}$ is equivalent to $\mathbb{P}$;
- $\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left(-\int_0^T \gamma^2(t)dW(t) - \frac{1}{2}\int_0^T \gamma^2(t)dt\right)$;
- $Z(t) = W(t) + \int_0^t \gamma^2(s)ds$ is a $\mathbb{Q}-$ Brownian motion.

$\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon–Nikodym derivative. Analogous with the first fundamental theorem of calculus, given two continuous positive measures $\mathbb{P}$ and $\mathbb{Q}$, there is a function f, such that for any measurable set E:

$$\mathbb{Q}(E) = \int_E f d\mathbb{P}, \tag{24}$$

then, f is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$[2].

4.4.3 Black-Scholes pricing formula derivation

Black, Scholes and Merton [4] predicate some assumptions about the market and stock behaviour as follows:

- The stock price behaviour is that of a geometric Brownian motion as described before;
- Doesn't apply to derivatives dependent on dividend-paying stocks;
- No arbitrage opportunities exist;
- Continuous time frame;
- The risk-free interest rate, r, is constant.

Therefore, intuitively, the valuation at time t of an arbitrary claim $X(S)$ would be, expiring by the time horizon T would be:

$$f(t,T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[X(S)|\mathcal{F}_t] \tag{25}$$

which is futures price of the claim $X(S)$ under the martingale measure $\mathbb{Q}$, instead of $\mathbb{P}$, the real world probability measure. This is the solution of the Black-Scholes-Merton differential equation,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf, \tag{26}$$

which the particular solutions are dependent on the specific derivative's boundary conditions. For example, for European call and put options, we solve for the following boundary conditions, respectively,

$$f_c(t=T,T) = X_c = max(S_t - K, 0), \tag{27}$$

$$f_p(t=T,T) = X_p = max(K - S_t, 0). \tag{28}$$

We can derive then the pricing formula for the first case, the European call option. We'll begin by finding the stock price's dynamics under $\mathbb{Q}$. For that we'll use Girasnov's theorem with $\gamma(t) = (\mu - r)/\sigma$, then there exists an equivalent measure $\mathbb{Q}$ under which

$$dB(t) = dW(t) + \frac{\mu - r}{\sigma}dt \tag{29}$$

is a $\mathbb{Q}$-Brownian motion. Then,

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(dB(t) - \frac{\mu - r}{\sigma}\sigma dt) = rdt + \sigma dB(t) \tag{30}$$

is now a geometric Brownian motion with drift r, and so the dynamics of $S(t)$ under the new measure are as follows:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma B(t)}. \tag{31}$$

Now we're in position to compute the valuation of a call, $X = max(S(T)-K, 0)$ under this new measure. For simplicity sake we'll take $t = 0$ without loss of generality,

$$f(0,T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S(T)-K)1_{S(T)>K}], \tag{32}$$

where 1 is the indicator function. Since K is a constant, we cant separate the equation above into two terms

$$f(0,T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S(T)1_{S(T)>K}] - e^{-rT}K\mathbb{Q}(S(T) > K), \tag{33}$$

where $\mathbb{Q}(S_(T) > K)$ is the probability of $S(T)$ being greater than K under measure $\mathbb{Q}$:

$$\mathbb{Q}(S(T) > K) = \mathbb{Q}(B(T) > log\left(\frac{K}{S_0}\right) - \frac{r-\sigma^2}{2}T = N(d_2), \tag{34}$$

since $B(T)$, under measure $\mathbb{Q}$ is normally distributed with variance T. As for the first term in the valuation equation we again use Girasnov's theorem with $\gamma(t) = \sigma$, then we have,

$$dZ(t) = dB(t) - \sigma dt \tag{35}$$

which is a Brownian motion under new measure $\mathbb{S}$. We then have that the Radon–Nikodym derivative is a martingale,

$$\frac{d\mathbb{S}}{d\mathbb{Q}} = e^{\frac{\sigma^2}{2}t+\sigma B(t)}. \tag{36}$$

Finally,

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}[S(T)1_{S(T)>K}] = S(0)\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{S}}{d\mathbb{Q}}1_{S(T)>K}\right] = S(0)\mathbb{S}(S(T) > K), \tag{37}$$

this last equality is by definition of Radon-Nikodym derivative. Similarly with the second term of the equation we get

$$\mathbb{S}(S(t) > K) = \mathbb{S}(Z(T) > log\left(\frac{K}{S_0}\right) - \frac{r+\sigma^2}{2}T = N(d_1), \tag{38}$$

since $Z(T)$ is normally distributed with variance T under measure $\mathbb{S}$. So we have that the value of a European call option under the Black-Scholes assumptions is:

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2), \tag{39}$$

with

$$d_1 = \frac{log(S_0/K) + (r + \sigma^2/2)T)}{\sigma\sqrt{T}}, \tag{40}$$

$$d_2 = \frac{log(S_0/K) + (r - \sigma^2/2)T)}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \tag{41}$$

The European put option valuation can be computed using the same method or by the put-call parity, and it's value is

$$p = Ke^{-rT}N(-d_2) - S_0 N(-d_1). \tag{42}$$

5 Volatility Surfaces

5.1 Implied Volatility

How the volatility of a stock price may be estimated was discussed earlier, but in fact, volatility can not be known as it is an intrinsic property of the stock price. We may not need to know the volatility of the stock price to trade fairly, as it is the perceived volatility by the market that matters, and that's what traders usually work with and it is known as the implied volatility. Since volatility is the only parameter of the Black-Scholes equation that we don't have before hand, it is possible, once we have the option price, to derive it. Unfortunately, the Black-Scholes equation is not invertable. To calculate the implied volatilities we must employ numerical methods that we will present further in this work. The implied volatilities are more robust than option prices, and that's why traders prefer it. To find the implied volatility is to find the solution, σ_{imp}, to the following equation:

$$C_{BS}(\cdot, \sigma_{imp}) - \hat{C} = 0, \tag{43}$$

where $\hat{C}$ is the observed value of the price of the contract and $C_{BS}(\cdot, \sigma_{imp})$ is the Black-Scholes price for all of the known parameters of the contract and depends on the volatility.

In this work we will employ methods that use implied volatilites of actively traded options to predict implied volatilities for other options. There are many algorithms designed to find roots of functions $f(x)$, in our case we chose Newton-Raphson's method which consists on guessing the initial value of x_0 and then employing the following recursive formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{44}$$

where $f'(x_n)$ is the derivative of $f(x_n)$. This algorithm should converge to the root in $O(n^2)$ and the stoppage condition is:

$$\frac{|x_{n+1} - x_n|}{x_{n+1}} < \varepsilon, \tag{45}$$

for some predefined ε, in which case it is determined that the root of $f(x)$ is approximated by x_{n+1}.

In our case we chose $\sigma_0 = 0.5$ and computed:

$$\sigma_{n+1} = \sigma_n - \frac{C_{BS}(\cdot, \sigma_{n+1}) - \hat{C}}{\nu} \tag{46}$$

where ν is often referred to as "vega" and is defined as follows:

$$\nu = \partial C_{BS}(\cdot, \sigma)\partial\sigma = S_0\phi(d_1)\sqrt{T} \tag{47}$$

with $\phi(x)$ the normal density function($\phi(x) \cong N'(x)$), and it is valued the same for Black-Scholes puts and calls [6].

5.2 SABR model

Although Black-Scholes model is a good model and, historically, provided a good insight to the mechanisms of the stock prices and basis to option pricing, it is not enough by today's standards given it's accuracy. The main improvement that SABR have on the Black-Scholes model is now the volatility a function of time as well, as follows:

$$dS(t) = rS(t)dt + e^{-r(T-t)(1-\beta)}\sigma(t)S(t)^{\beta}dW_S(t) \tag{48}$$

$$d\sigma(t) = \nu\sigma(t)dW_{\sigma}(t) \tag{49}$$

where β is the constant elasticity of variance valued between 0 and 1, and ν is the sensitivity of the volatility, or, in other words, the volatility of the volatility process. We define $\alpha := \sigma(0)$, and it deserves it's own designation since it is a crucial parameter in the model. Finally, both processes must be correlated, intuitively this is true because volatility behaviour must be somehow related to the stock price, and so these processes are correlated as

$$\mathbb{E}\left[dW_S(t)dW_{\sigma}(t)\right] = \rho dt. \tag{50}$$

Hence the name SABR: stochastic-α-β-ρ, these are the parameters that comprise the calibration of the model volatility surface [9].

We are interested in using SABR for volatility surface interpolation, i.e. given a set of prices of a type of derivative, the SABR model is calibrated to fit existing implied volatilities and use it as an interpolation tool for other strikes. All of this resides in that different strike prices K result in diferent volatilities for a given derivative price, and therefore the Black-Scholes model is not enough since it considers the volatility to be constant.

One very useful property of this model is that it allows for a closed solution for the implied volatility [13], for that, it is more convenient to use the foward price $f = e^{rT}S$, yielding:

$$\sigma_{imp} = \frac{\alpha\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\alpha\beta v}{(fK)^{(1-\beta)/2}}+\frac{(2-3\rho^2)v^2}{24}\right)T\right)}{(fK)^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}ln^4\left(\frac{f}{K}\right)\right)}\cdot\frac{z}{\chi(z)}, \tag{51}$$

where,

$$z = \frac{v}{\alpha}(fK)^{(1-\beta)/2}ln\frac{f}{K}, \tag{52}$$

and,

$$\chi(z) = ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right). \tag{53}$$

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