The Timing of Complementary Innovations

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Abstract

Socially-valuable technologies sometimes require complementary innovations. This paper studies the development of innovations that exhibit such complementarity. At each point in time, a unit of attention is allocated across different innovation projects. The projects are completed stochastically in the form of breakthroughs. The social value of the technology depends on the set of completed projects by the time the agent decides to stop the development stage. In some cases it is optimal to develop the innovations in sequence. In others, it is optimal to develop multiple innovations simultaneously. I provide conditions that determine the efficient timing of development: sequential development is efficient when costs are high and there is more uncertainty about the innovations' rate of success. I compare the efficient allocation to the equilibrium outcome with a decentralized industry in which many firms compete for the development of the innovations.

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1 Introduction

Patent laws and other innovation policies aim to orient scarce resources toward the most socially valuable R&D projects. The value of an innovation, however, might be tied to the outcome of other developments. A case of particular interest is complementary innovations, which have been increasingly relevant in industries such as telecommunications and biotechnology. For complementary innovations, the timing is relevant. Consider the case of hardware and software: The first classical computer algorithm was written in 1843,[1] while the first computer capable of running said algorithm was developed in the 1930s. A similar process is taking place for quantum computing: Shor's quantum algorithm, a method to solve integer factorization problems in polynomial time, was written in 1994, four years before the first quantum computer prototype was developed. Today, start-ups and established companies invest hundreds of millions of dollars to develop quantum software that can only be implemented with hardware that does not yet exist,[2] and it is not clear that it ever will. What determines the timing in which complementary innovations are developed?

For some complementary innovations the timing is dictated by an exogenous order in which the developments should succeed each other.[3] For other innovations there is no exogenously imposed order, so the timing is determined endogenously by the allocation of resources to the different projects. The main research question of this paper is how are resources endogenously assigned to complementary R&D projects. In particular, how the environment (level of competition, patent rights, etc.) affects the allocation of resources to the development of complementary innovations and whether the

[1] By first classical computer algorithm I mean an algorithm written for a classical computer, that has no value given human computing power.

[2] The highest integer that has been factorized using Shor's algorithm is $21 = 7 \times 3$.

[3] For instance, no one was working on the can opener before the invention of the can—the can opener was invented decades after the can became popular. This order is natural since the problems are very much related, and a can opener cannot be invented without the specifications of the can.

allocation is efficient.

An object that is central in determining the allocation is prospects of each development. R&D projects carry high levels of uncertainty, both in terms of outcomes—the project may turn out to be successful or not—and in terms of costs—how much time and resources will be needed to complete the development. This paper combines a dynamic of beliefs with the endogenous timing of development by introducing a tractable model that features the main aspects of the R&D process: A unit of attention is allocated over a set of projects at each point in time. A success for project i arrives discretely in the form of a breakthrough. In particular a success arrives when the total amount of attention paid to a project reaches a certain level τ_i. Successes are observable, but τ_i is unknown.

The first part of this paper studies the *efficient* way to sequentially allocate attention to complementary R&D projects given the social value of innovations, which is realized when the development stage ends and is supermodular on the subset of the innovations that is successfully developed by that time. The cost of development takes the form of a constant flow throughout the development stage.

Consider two complementary projects, A and B. How should society assign the resources? Should they all be concentrated on A and then be switched to B if and only A is successfully completed? Or should both A and B be developed in parallel? Moreover, when should a project be abandoned or put on hold? A simplifying feature is that for complements having a success makes it more attractive to keep paying attention to the remaining projects. Proposition 1 shows that this implies for two complements that all that matters for efficiency is how much is invested on each remaining project before abandoning. The intuition is that given the complementarities in payoff, the amount you are willing to work on a project if there is no new success is the minimum you are going to work on that project. Since you are going to do it independently of the outcome of other projects, when you do it is not payoff-relevant.

Section 4 considers the case where the rate of success for each project λ_i is constant over time but unknown. The beliefs about λ_i evolve with the outcomes of the process of development. In particular, the lack of success is evidence in favor of λ_i being relatively low, or in other words 'the project i being relatively more challenging'. The λs are independent across projects, so working on a project does not affect the beliefs about the success rate of the others.

When the rate of success for each project is constant and known, the timing of development is irrelevant. Any project that is worth pursuing is worth completing, therefore the order of completion is not going to affect the final expected payoff.[4] In contrast, when the rate of success is uncertain, the order of development is relevant since it affects the arrival of information about the unknown parameters. The failure to develop A not only reduces the prospects of ever completing A, but also decreases the expected returns from completing B. The problem can therefore be thought as a *restless multi-armed bandit*, for which there is no general Gittins-like index rule that governs the optimal dynamic allocation.

Take as an example the case where project A is of uncertain feasibility, i.e. the success rate is either zero or λ_A, and project B has a known success rate λ_B. In this case, it is efficient to first work on project A: there is no learning by working on B, so there is no efficiency loss in back-loading all development of B. Front-loading the development of A increases the speed of learning, which is valuable because of the option given by the stopping decision. In the more general setting, the intuition from this example also applies: the efficient allocation of resources reflects the optimal learning process about the potential of the joint project. Proposition 2 claims that for two perfect complements, the nature of the efficient allocation of resources depends on the uncertainty about the projects' difficulty and the flow cost of development. For projects with high uncertainty and high costs, it is efficient to concentrate all resources in one of the projects and thus develope them in sequence. The

[4]A similar logic holds when the completion time of each project is deterministic.

more uncertain the project, the earlier in the sequence it should be developed. For projects with low uncertainty and low costs, it is efficient to spread the resources following a simple greedy strategy that maximizes at each point in time the . In the case of symmetric projects, the greedy strategy splits resources equally at all times until either one the projects succeeds or the projects are abandoned.

The second part of the paper studies private allocation of the resources. The allocation of private R&D resources depends on several factors: who assigns these resources, the appropriability of the innovations—which is in turn determined by the legal and patent system—and on how informed the agents are about successes. Section 5 analyzes two cases of private allocation: *centralized* and *decentralized* industry.

In a *centralized industry* all resources are controlled by a forward-looking innovator. The centralized industry will develop inefficiently because of the discrepancies between the social value and the value that can be appropriated by the innovator.[5] On top of the classical under/over investment, there are allocation inefficiencies: it might be optimal for society to develop the projects in sequence while the centralized industry develops them in parallel or vice versa. Section 5.1 analyzes the characteristics of the joint distributions of project successes that can be implemented by choosing the proportion of social value appropriated by the innovator.

A *decentralized industry* consists of n agents, each of who controls an equal portion of the total unit of resource available at each moment of time. The agents don't consider the informational externalities that their actions generate and in the limit, resources are always allocated to the project with higher myopic expected returns.

With substitute projects, competition biases the allocation of resources towards easy and fast projects in detriment of harder but cost-efficient ones.

[5]The two main effect governing this difference are called the appropriability effect (firms cannot capture all value generated by the innovation) and the business stealing effect (even marginal innovations might increase the market share of the innovator substantially).

This *race effect* might be a concern also with complements: if a product requires two components, and it is efficient to start developing the hard one, competition might make tempting to work on the easy component just to capture a higher share of the value generated. Section 5.2 shows that this is not the case: even if the first agent to succeed appropriates all the surplus from the joint development, the allocation of resources is not biased towards projects just because they are thought to be easier. Competition might introduce new inefficiencies by biasing the allocation towards projects where learning is slower. This inefficiencies disappear, however if the stakes are sufficiently high.[6]

1.1 Related Literature

Complementary Innovations: Main contribution is to the literature that studies research and development, and in particular complementary innovations. Scotchmer and Green [1990] and Ménière [2008] asks what is the optimal inventive requirements for a patent in the context of complementary innovations. Biagi and Denicolò [2014] study the optimal division of profits with complementary innovations. Fershtman and Kamien [1992] study the effects of cross licensing in the incentives to innovate. In these papers there is no learning in the developing stage, since successes arrives at an exponential time (arrival distribution is memoryless). A particular type of complementary innovations is the Sequential Developments or cumulative innovation e.g. in Gilbert and Katz [2011] and Green and Scotchmer [1995]. The closes paper is Bryan and Lemus [2017]. Main difference is that there is no learning in the development process.

Dynamic information acquisition from multiple sources: Recent papers that study optimal exploration with substitutes. With Poisson information structure, Nikandrova and Pancs [2018] studies the case of independent

[6]This contrasts again with the case of substitutes where higher stakes magnify the race effects.

objects while Che and Mierendorff [2017] studies substitutes that are negatively correlated. Mayskaya [2019] also studies Poisson processes with more general dependencies and payoff function. Ke and Villas-boas [2019] study a similar situation to Nikandrova and Pancs [2018] but information comes through a Brownian Process. Although this paper is not about information acquisition, we could reformulate it in that way. Liang et al. [2018] asks the question of when is it optimal for a decision maker to acquire information in a myopic way. Liang and Mu [2020] compare efficient information acquisition versus what results from the choices of short-lived agents who do not internalize the externalities of their actions.

The reminder of the paper is as follows. Section 2 introduces the general model. Section 3 presents the main results with respect to the efficient allocation. Section 5 the inefficiencies generated by the private allocation of resources. Section 6 concludes.

2 Set up

An agent is supposed to work on a finite set of projects $K := \{1, 2, ..., k\}$. The agent must decide when to stop the research activity, and before that in which way to allocate her *attention* across the various projects. Each instant before stopping, the agent allocates a unit of attention across the projects $\sum_{i \in K} \alpha_i(t) \leqslant 1$ for all $t \geqslant 0$.

Each project is completed if the cumulative attention paid to it reaches a certain amount τ_i. Project completion is observable but τ_i is unknown. The times of completion are independent across projects, with cdfs F_i.

When the agent stops, she gets a payoff $q(S)$ where $S \subseteq K$ is the set of projects that were completed.

Assumption 1. *$q(\emptyset) = 0$, $q(K) = 1$ and q is increasing in the inclusion order, i.e. $q(T) \leqslant q(S)$ for all $T \subseteq S$.*

The more interesting case concerns complementary projects.

Definition 1. *The projects of a set K are complements if the function q is supermodular, that is,*

$$q(S \cup T) - q(S) \geqslant q(T) - q(S \cap T) \qquad \forall S, T \subseteq K.$$

The projects are perfect complements if $q(S) = 0$ for all $S \subsetneq K$.

One can also define substitutes by requiring the function q to be submodular, and perfect substitutes by the property that $q(S) = 1$ for all $S \neq \emptyset$.

Working on the projects is costly. We assume a constant flow cost of c during the development stage, that is independent on which project the agent works on.[7] There is no discounting. The payoff of an agent that stops at time T and completed projects S by that time is $q(S) - c \cdot T$. The agent is an expected-payoff maximizer.

Strategies

The agent is allowed to work on several projects at the same time. More specifically, denote by $x_k(t) := \int_0^t \alpha_i(t)\ dt$ the amount of attention that the agent spent on project k up to time t. At time t the agent chooses $x'_k(t)$, the fraction of attention that the agent will pay to project k on that time. The agent must satisfy the constraint that $x'_1(t) + ... + x'_n(t) \leqslant 1$ for all t before the stopping decision.

A *strategy* is a map from the set of histories to the attention vector. Given the independence assumption, the timing at which a project is completed does not say anything about the difficulty of the remaining projects. A *stationary strategy* only looks at the cumulative attention and the set of completed projects (it does not depend on the oder in which attention was allocated so far, nor the timing of the projects' successes). A stationary strategy thus consists on a vector field for each subset of developments. Formally,

[7] This assumption is inocuous since we can normalize the time unit for different projects, by changing the distribution of τ, so that the cost is the same for all projects.

Definition 2. *A* stationary strategy *is a function* $x' : 2^K \times R_+^k \to [0,1]^k$ *such that*

$$\sum_{i \in K} x_i'(S,x) \leqslant 1 \qquad \forall x \in R_+^k \quad \forall S \subseteq K.$$

Let $\mathcal{X}'$ *be the set of stationary strategies.*

3 Optimal Allocation

The problem of the decision maker is to choose a strategy to maximize their expected payoff. Given the assumptions, we can focus on stationary strategies. Start with a initial state (S,x), a strategy $x' \in \mathcal{X}'$ and a vector $\tau \in \mathbb{R}_+^k$ of realized project completion requirements. There is a deterministic stopping time $T(x',\tau)$ and set of completed tasks $S(x',\tau)$. The allocation problem is:

$$V(S,x) = \max_{x' \in \mathcal{X}'} \int q(S(x',\tau)) - c \cdot T(x',\tau)\ dF(\tau|(S,x))$$

A related problem is when the agent has to commit at each point how many extra resources to allocate to each of the remaining projects and to abandon if non of the remaining projects turns out successful. This problem can be recursively defined as

$$\hat{V}(S,x) = \max_{\hat{x} \geqslant x} \left\{ P(\emptyset,\hat{x},x) \cdot q(S) + \sum_{\hat{S} \in 2^{K \setminus S} \setminus \{\emptyset\}} P(\hat{S},\hat{x},x) \cdot \hat{V}(S \cup \hat{S},\hat{x}) \right.$$
$$\left. - c \sum_{i \in K} \int_{x_i}^{\hat{x}_i} \frac{1-F_i(\tau)}{1-F_i(x_i)}\ d\tau \right\}$$

Where $P(\hat{S},\hat{x},x) := \prod_{i \in \hat{S}} \frac{F_i(\hat{x}_i)-F_i(x_i)}{1-F_i(x_i)} \prod_{i \in K \setminus \hat{S}} \frac{1-F_i(\hat{x}_i)}{1-F_i(x_i)}$ is the probability of completing the projects $\hat{S}$ with the extra attention $(\hat{x}-x)$.

We are interested in conditions on q and F such that these two problems equivalent, i.e. $V = \hat{V}$. In the next subsection, we analyze the case where F has constant rates of success. For this family of environments, the problems are equivalent for any q. In Section 3.2 we show that for the case of $k = 2$ the problems are equivalent for all F if an only if q is supermodular. For $k > 2$, q supermodular is necessary for the equivalence but not sufficient. An example where q is supermodular and $V \neq \hat{V}$ is presented in Appendix C.3.

3.1 Benchmark: constant rate of success

Consider the case where the rate of completion is constant, i.e. where $F_i(x) = 1 - e^{-\lambda_i x}$ for all $i \in K$. In this case, if a project is worth paying any attention, then it must be worth completing. Moreover, this implies that the order of allocation is payoff irrelevant.

$$V(S, x) = \hat{V}(S, x) = \max_{\hat{S} \in 2^{K \setminus S}} \left\{ q(S \cup \hat{S}) - c \sum_{i \in \hat{S}} \lambda_i^{-1} \right\}$$

3.2 Variable rate of success

When completing a project induces the agent to work more on all the remaining ones, the order in which the agent works on the projects is irrelevant modulo the cumulative work at the stopping points.

For any strategy x' and initial state (S, x_0), there is a *trajectory* $y_S : \mathbb{R}_+ \times \mathbb{R}^k \to \bar{\mathbb{R}}^k$ that is the (unique) solution to the differential equations $\nabla y_S(t, x_0) = x'(S, y_S(t, x_0))$ and $y_S(0, x_0) = x_0$. We will refer to $Y(S, x_0) = \lim_{t \to \infty} y_S(t, x_0)$ as the abandonment point of the strategy given an initial state (S, x_0).

Definition 3. *A strategy has* increasing abandonment points *if*

$$Y(S, x_0) \leqslant Y(\hat{S}, x_0) \text{ for all } S \subseteq \hat{S}.$$

Definition 4. *Two strategies $x', \tilde{x}'$* have the same abandonment points *if for each initial state, the abandonment point is the same for both strategies.*

Lemma 1. *If two strategies $x', \tilde{x}'$ have the same abandonment points, and these abandonment points are increasing then the two strategies have the same expected payoff.*

Proof. in the Appendix A.1 □

The intuition for Lemma 1 is the following: if the abandonment is increasing, then the current abandonment point is the least attention you are willing to put on the remaining projects by the end of the day. Since the attention it is going to be paid eventually, the order in which the agent does it is not gonna determine the outcome.

Proposition 1. *Consider $k = 2$. Any strategy that has the same abandonment points than an optimal strategy x' is also optimal if and only if the projects in K are complements.*

Proof. in the Appendix A.2. K complements is not sufficient for the claim with $k > 2$. A counterexample can be found in Appendix C.3. □

Proposition 1 implies that for complements, and only for complements, it is possible to focus on finding the optimal abandonment points. To emphasise the generality of the result, notice that we made no assumption on F_i. Thus the result holds for discrete time with arbitrary costs.[8]

There are two class of stationary strategies that we are going to be interested in, as the candidates for the solution will sometimes belong to one of these classes. The first one is the family of strategies that always maximizes the expected value increment in each period.

Let $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the completion rate of task i, $h_i = F_i'/(1 - F_i)$.

Definition 5. *A stationary strategy x' is* greedy *if for every state (S, x_0),*

$$x_i'(S, x_0) > 0 \quad \Rightarrow \quad \begin{cases} i \in \arg\max_j h_j(x_0) \cdot [V(S \cup \{j\}, x_0) - V(S, x_0)] \\ h_i(x_0) \cdot [V(S \cup \{i\}, x_0) - V(S, x_0)] \geqslant c \end{cases}$$

The second family of stationary strategies concentrates all the resources in one of the projects, and only switches projects after a success.

[8] To see this just consider an F_i with mass probabilities at differents times. The difference between the mass points can be interpreted as the cost of working on the project for an extra period.

Definition 6. *A stationary strategy x' works on the projects* in sequence *if for every state (S, x_0) there exists a project i such that*

$$Y_j(S, x_0) - x_{0,j} = 0 \qquad \forall\ j \in K \setminus i$$

4 Two perfect complements

In this section, we capture the main features of the model in the simplest possible case. Consider two perfect complements. Moreover, $F_i = 1 - p_i e^{-\lambda_i^H} - (1 - p_i)e^{-\lambda_i^L}$. The interpretation is that each project has a constant completion rate per unit of attention λ_i, unknown to the agent. The agent knows that the rate takes one of two possible values, $\lambda_i \in \{\lambda_L^i, \lambda_H^i\}$ and the rates are independent, with $\Pr(\lambda_i = \lambda_H^i) = p_i$. As attention is allocated to each project and these are not completed, the agent becomes more pessimistic about its difficulty.

The next proposition tell us that the nature of the optimal strategy depends on measure that is increasing in the normalized cost and the uncertainty about the underlying success rate.

Proposition 2. *Let $g_i := 2\frac{c}{\bar{\lambda}_i} + \frac{1}{4}\left(\frac{\delta_i}{\bar{\lambda}_i}\right)^2$, where $\delta_i := \lambda_i^H - \lambda_i^L$ and $\bar{\lambda}_i := 0.5(\lambda_i^H + \lambda_i^L)$.*

- *If $g_i > 1$ for both projects, then it is optimal to work on them in sequence.*

- *If $g_i < 1$ for both projects, then the greedy strategy is optimal. For symmetric projects $F_A = F_B$, the greedy strategy consists on splitting the resources equally.*

Proof. in the Appendix A.3. Extra conditions for the case where $g_A < 1 < g_B$ can be found in Appendix B.1. □

The result says that it is optimal to concentrate the resources (and therefore work in sequence) when the cost of development is sufficiently high, or the difference between the high and low rates is sufficiently large for both projects. The intuition is that in this case, having a single project that is difficult is sufficiently bad to abandon the project, so by concentrating the resources the agent gets to learn fast if this is the case. In the cost of development is sufficiently low, or the difference between the high and low rates is low for both projects, then it is optimal to work on the project simultaneously. In this case, having a single project that is difficult is not sufficient to stop.

How can formalize this intuition? We can interpret the result in terms of optimal information acquisition. Consider symmetric λs. There are four possible states. For the decision problem to be interesting it must be that the agent wants to work on the projects if he knew that both are easy, and he does not want to work on the projects when both of them are hard.

Suppose that the agent would be willing to work on the projects if he knew that one was difficult and the other one easy. Then the partition of the state space that is relevant for decision making is weather there is an easy tasks (continue) or not (abandon).

The probability of the event 'at least one of the tasks is easy' is $p^{\wedge} = p_A + p_B - p_A \cdot p_B$. By working on project i for a period dt and not succeeding, the change in p is

$$dp^{\wedge} = -p_i(1-p_i)(1-p_j)\delta \; dt$$

The fastest way to learn about the relevant state is to work on the task with highest probability of success, and therefore to work on the projects simultaneously.

If the agent does not want to work when one of the tasks is hard and the other one is easy, the relevant state is whether there is a hard tasks or not. There is no hard task with probability $p^{\vee} = p_A \cdot p_B$. By working on project i for a period dt and not succeeding the change in $p^{\vee}$ is

$$dp^{\vee} = -p_i p_j (1-p_i)\delta \; dt$$

The fastest way to learn about the relevant state is to work on the task with lowest probability of success, and therefore to work on the projects in sequence.

When does the agent want to continue working when one of the tasks is difficult and the other one is easy? When the expected cost of completing both projects is less than one, i.e. $\frac{c}{\lambda^L} + \frac{c}{\lambda^H} < 1$. Rearranging we can see that this is equivalently to $g < 1$.

5 Private allocation

Research and development is sometimes carried away by the private sector. The level of competition and the appropriability of the innovations will affect the incentives and thus the allocation of resources to different projects.

5.1 Centralized industry

An extreme case is a centralized industry where one player decides how to allocate resources. The problem of the agent is similar to the society problem studies before, with the caveat that she appropriates a fixed proportion γ of the social value of the innovation. We study the effects of changing the level of appropriability γ in the distribution of successes.

Lemma 2. *The probability of success of the joint project is increasing in γ, with a discrete jump at $c\frac{\lambda^L+\lambda^H}{\lambda^L\lambda^H}$.*

5.2 Decentralized industry

In this section we consider the case of multiple agents, each of who decides how to allocate an equal portion of the total resource. Let n be the number of agents in the economy, and N be the set of agents.

Definition 7. *A* stationary strategy *for agent i is a function $x_i' : N^K \times R_+^k \to [0, 1/n]^k$ such that*

$$\sum_{k\in K} x_{i,k}'(\mathbf{S}, x) \leqslant 1/n \qquad \forall x \in R_+^k \quad \forall\ \mathbf{S} \in N^K.$$

What is the payoff for each agent? We assume that there is a function $q : N^K \to [0,1]$

Definition 8. *An* equilibrium *is a strategy profile such that the expected payoff for each agent is maximized given the strategy of rest.*

Lemma 3. *There is an $\bar{n}$ such that for $n > \bar{n}$ the greedy strategy is the unique equilibrium.*

Proof. in the Appendix A.4. □

Patent races might introduce distortions. Given Lemma 3, we are going to say that there is a *race effect* when the greedy strategy is not optimal.

Two perfect complements

We can apply the results from Proposition 2 to derive sufficient conditions under which there is no race effect, i.e. the industry achieves an efficient allocation of resources. We are going to consider the most extreme version of the

Corollary 1. *If $g_i > 1$ for $i = A, B$, there is no race effect.*

Proof. This is an immediate application of Proposition 2 where we showed that for this case the greedy strategy was optimal. □

When

Corollary 2. *If $\lambda_A^L = \lambda_B^L$ and $\lambda_A^H = \lambda_B^H$ there is no race effect.*

Proof. The only remaining case is where $g_A = g_B < 1$. In this case it is efficient to work on the projects. □

6 Conclusion

Innovation is one of the main determinants of long-term economic growth and an important part of innovation is carried away by the private sector. The timing of innovation is partly determined by the investment decisions of agents whose objectives are typically misaligned from the social welfare.

Complementary innovations, which are of central and growing importance in industries such as telecommunications and bioengineering, generate investment dynamics that are different than for substitutes. In particular, the allocation of resources is not biased towards the easy and fast component in detriment of the hard but cost-effective ones.

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A Omitted proofs

A.1 Proof of Lemma 1

Lemma 1. If two strategies $x', \tilde{x}'$ have the same abandonment points, and these abandonment points are increasing then the two strategies have the same expected payoff.

Proof. The proof works by induction. The Lemma trivially holds for $k = 1$. Assume that it holds for $k = 1, 2, ..., m - 1$, we want to show that it holds for $k = m$.

Consider a strategies $x', \tilde{x}'$ and a initial state $(\emptyset, x_0)$. Let $Y(\emptyset, x_0)$ be the associated abandonment point. For each set $S \neq \emptyset$, the continuation problem is analogous to one with less than n projects, so the lemma holds. Let $v(S, x)$ be the value of the two strategies at the state (S, x) for $S \neq \emptyset$.

Consider a strategy z' with the same abandonment points than x' and such that for any $S \neq \emptyset$, $z'(S, x) = x'(\emptyset, x)$ for all x with $x'(\emptyset, x) \neq 0$. We can do this since $Y(S, x_0) \geqslant Y(\emptyset, x_0)$. For any τ, the new strategy has the same payoff than the original: either no project is successful and both abandon at the same point or the same project is succesful at the same point, and the continuation value is the same.

Similarly, we can construct a strategy $\tilde{z}'$ with the same abandonment points but such that for any $S \neq \emptyset$, $\tilde{z}'(S, x) = \tilde{x}'(\emptyset, x)$ for all x with $\tilde{x}'(\emptyset, x) \neq 0$. $\tilde{z}'$ and $\tilde{x}'$ shield the same payoff. We end the proof by showing that z' and $\tilde{z}'$ must also have the same payoff for any realization of the success times $\tau \in \mathbb{R}_+^k$.

Let $\bar{S} = \{i \in K : \tau_i < Y(\emptyset, x_0)\}$, that is the set of projects which completion time is below the abandonment point. Both z' and $\tilde{z}'$ reach $Y(\emptyset, x_0)$ with probability one. The payoff is therefore

$$v(\bar{S}, Y(\emptyset, x_0)) - c \sum_{i \in \bar{S}} \tau_i - c \sum_{i \notin \bar{S}} Y_i(\emptyset, x_0)$$

Taking expectation over the realization of τ completes the proof. $\square$

A.2 Proof of Proposition 1

Proposition 1. Consider $k = 2$. Any strategy that has the same abandonment points than an optimal strategy x' is also optimal if and only if the projects in K are complements.

($\Leftarrow$) We want to show that q supermodular implies that any strategy that has the same abandonment points than an optimal strategy is also optimal.

Lemma 4. *For two complements, any optimal strategy has increasing abandonment points.*

Proof. We want to prove that for any optimal strategy $Y_i(j, x_0) \geqslant Y_i(\emptyset, x_0)$. By supermodularity of q, the marginal value of i is larger when j was completed than when it is not. If it is optimal to work on project i when it is not clear if j is going to be completed or not, it must be optimal to work on i when j was already completed.

Formally, by contradiction assume $Y_i(j, x_0) < Y_i(\emptyset, x_0)$. Then there is a time t such that $y_{\emptyset,i}(t, x_0) = Y_i(j, x_0)$. Let $x := y_{\emptyset}(t, x_0)$. Consider the strategy that continues after x. If this strategy was copied starting on the state $(j, Y_i(j, x_0))$ with a dummy project j' that starts at x_j the result must be negative (otherwise it is worth continuing at $Y_i(j, x_0)$). But this strategy shields more than the continuation at x thus the project should stop x, so $x = Y(\emptyset, x_0)$ leading to a contraction. □

Using Lemma 1, any strategy that has the same abandonment points than x' is gonna get the same expected payoff and therefore be optimal.

($\Rightarrow$) : We prove by contrapositive. If q is *not* supermodular, there are cdfs $\{F_i, F_j\}$ such that $Y_i(j, x_0) < Y_i(\emptyset, x_0)$.

Proof. Since q is not supermodular, $q(\{i\}) > 1 - q(\{j\})$. Let $F_i = 1 - e^{-\lambda_i x}$ with λ_i such that

$$q(\{i\}) > \frac{c}{\lambda_i} > 1 - q(\{j\})$$

and let j never succeed, i.e. $F_j = 0$. Rearranging we have that

$$\lambda q(\{i\}) - c > 0 > \lambda(1 - q(\{j\})) - c$$

What implies that for any x_0, $Y_i(\emptyset, x_0) = \infty$ and $Y_i(\{j\}, x_0) = x_0$. $\square$

A.3 Proof of Proposition 2

Some preliminaries

Let δ_i be $\lambda_i^H - \lambda_i^L$. Using Bayes' rule, the beliefs $p_i(t)$ evolve

$$p_i(t) = \frac{p_i e^{-\delta_i t}}{(1 - p_i) + p_i e^{-\delta_i t}}$$

As the agent becomes more pessimistic, the subjective hazard rate $h_i(t)$ becomes lower.

$$h_i(t) = \lambda_L^i + p_i(t)\delta_i$$

Notice that

$$g_i > 1 \quad \Leftrightarrow \quad \frac{\lambda_i^L \cdot \lambda_i^H}{\lambda_i^L + \lambda_i^H} > c$$

Proposition 2:

- If $g_i > 1$ for both projects, then it is optimal to work on them in sequence.
- If $g_i < 1$ for both projects, then the greedy strategy is optimal. For symmetric projects $F_A = F_B$, the greedy strategy consists on splitting the resources equally.

The proof of the proposition is split in three lemmatas. Lemma 5 proves that g_i controls the monotonicity of project i's hazard-to-value ratio. Lemma 6 shows that when both projects have an increasing hazard-to-value ratio, it is efficient to work on them in sequence. Lemma 7 shows that when both projects have a decreasing hazard-to-value ratio the greedy strategy is efficient.

Lemma 5. *h_i/v_i is monotone. Moreover,* $\operatorname{sgn}((h_i/v_i)') = \operatorname{sgn}(g_i - 1)$.

Proof. First we show that the monotonicity of h/v depends on weather the value v is higher or lower that an expression R.

$$
\begin{aligned}
\operatorname{sgn}((h_i/v_i)') &= \operatorname{sgn}(h_i' v_i - h_i v_i') \\
&= \operatorname{sgn}(h_i' v_i - h_i(c - h_i(1 - v_i))) \\
&= \operatorname{sgn}\left(\underbrace{\frac{h_i(h_i - c)}{h_i^2 + h_i'}}_{R(t)} - v_i\right)
\end{aligned}
$$

Change of variables. In belief space, the concavexity of R is determined by weather g_i is larger or lower than one.

$$
\begin{aligned}
\hat{R}'(p) &= \frac{2\delta^2 \lambda_L \lambda_H (\lambda_L \lambda_H - c(\lambda_L + \lambda_H))}{(\lambda_L^2 + p\delta(\lambda_L + \lambda_H))^3} \\
&= M(g - 1)
\end{aligned}
$$

Now we consider two cases: $\lambda_L < c$ and $\lambda_L \geqslant c$.

Case I: $\lambda_L \geqslant c$ In this case, the agent would never stop. The value is linear in the beliefs.

$$v(p) = 1 - p\frac{c}{\lambda_H} - (1 - p)\frac{c}{\lambda_L}$$

Since $v(0) = R(0)$ and $v(1) = R(1)$,

$$g > 1 \qquad \Leftrightarrow \qquad v(p) < R(p) \qquad \forall p \in (0, 1)$$

Case II: $\lambda_L < c$ In this case, the agent abandons if sufficient time passes with no success. v is convex (information is valuable) and R is concave:

$$\lambda_L < c \qquad \Rightarrow \qquad \frac{\lambda_H}{\lambda_L + \lambda_H}\lambda_L < c \qquad \Leftrightarrow \qquad g_i > 1$$

Since $v(1) = R(1)$ and $v(\hat{p}) = R(\hat{p})$ where $\hat{p}$ is the stopping belief.

$$v(p) < R(p) \qquad \text{for any } p \in (\hat{p}, 1)$$

$\square$

Lemma 6. *If h_i/v_i is strictly increasing for $i = A, B$, it is optimal to work on the projects in sequence.*

Proof. By contradiction. Assume that $x := Y(\emptyset, x_0) > 0$. Let $r_i(t) := \frac{h_i(t)}{v_i(t)}$ and $g_i(t) := \frac{h_i'(t)}{v_i'(t) \cdot r_i(t)}$. Since x is an interior stopping point, it must be that $h_A(x_A)v_B(x_B) = h_B(x_B)v_A(x_A) = c$.

Abusing notation I write f_i instead of $f_i(x_i)$. First we show that $r_A' + r_B' > 0$ implies $\frac{h_A' v_A}{h_A v_A'} \cdot \frac{h_B' v_B}{h_B v_B'} < 1$.

$$\begin{aligned} r_A' + r_B' > 0 \qquad &\Leftrightarrow \qquad \frac{h_A' v_A - h_A v_A'}{v_A^2} + \frac{h_B' v_B - h_B v_B'}{v_B^2} > 0 \\ &\Leftrightarrow \qquad \frac{h_A v_A'}{v_A^2}\left(\frac{h_A' v_A}{h_A v_A'} - 1\right) + \frac{h_B v_B'}{v_B^2}\left(\frac{h_B' v_B}{h_B v_B'} - 1\right) > 0 \end{aligned}$$

For all (x_A, x_B) such that $h_A v_B = h_B v_A = c$,

$$\frac{h_A v_A'}{v_A^2} = \frac{h_B v_A'}{v_B v_A} = \frac{h_A v_B'}{v_A v_B} = \frac{h_B v_B'}{v_B^2}$$

Where the first and last equality use $h_A/v_A = h_B/v_B$ and the intermediate one uses that $h_B v_A' = h_B(c - h_A(1 - v_A)) = -h_B h_A(1 - v_A - v_B)$ (since $c = h_A v_B$) and equal to $h_A v_B'$ by symmetry. So,

$$\begin{aligned} r_A' + r_B' > 0 \qquad &\Leftrightarrow \qquad \frac{h_A v_A'}{v_A^2}\left[\left(\frac{h_A' v_A}{h_A v_A'} - 1\right) + \left(\frac{h_B' v_B}{h_B v_B'} - 1\right)\right] > 0 \\ &\Leftrightarrow \qquad \left[\frac{h_A' v_A}{h_A v_A'} + \frac{h_B' v_B}{h_B v_B'}\right] < 2 \\ &\Leftrightarrow \qquad \frac{h_A' v_A}{h_A v_A'} \cdot \frac{h_B' v_B}{h_B v_B'} < 1 \end{aligned}$$

Where the second implication uses that v_A is decreasing. And the last one is that the sum of two positive numbers being less than two implies that the product is less than one.

The determinant of the Hessian for the value function $V(\emptyset, x)$ is

$$\det(H) = (1 - F_A)(1 - F_B)[h'_A h'_B v_A v_B - h_A h_B v'_A v'_B]$$

So

$$\det(H) < 0 \qquad \text{iff} \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} < 1$$

And $\det(H) < 0$ rules out the candidate as an optimum (saddle point). $\square$

Lemma 7. *If h_i/v_i is strictly decreasing for $i = A, B$, the greedy strategy is optimal.*

Proof.

$$\begin{aligned} r_i \searrow \qquad &\Leftrightarrow \qquad h'_i v_i - h_i v'_i < 0 \\ &\Leftrightarrow \qquad \frac{h'_i v_i}{h_i v'_i} > 1 \end{aligned}$$

So,

$$r_A \searrow \text{ and } r_B \searrow \qquad \Rightarrow \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} > 1$$

This implies that there is at most one interior candidate for solution ($h_A v_B = h_B v_A = c$), and that if it exist it is the actual solution. We consider the two cases.

Case I: there is an interior solution candidate. Then this is the actual solution. Since at the solution $r_A = r_B$ and the r_i are decreasing, by working always on the project with highest r_i, the point is eventually reached.

Case II: there in no interior solution candidate. Then it must be that $h_i v_j = c \Rightarrow h_j v_i > c$. Thus, the solution is to work in sequence starting with project j. Moreover, $h_j/v_j > h_i/v_i$ for all x such that $h_i v_j \geqslant c$, so the greedy strategy also works in sequence starting with j. $\square$

A.4 Proof of Lemma 3

Proof. $\square$

B non-comonotonic hazard-to-value ratio

B.1 One h/v increasing and one decreasing

Lemma 8. *If the horizontal sum of the two h/v is increasing, then it is optimal to develop in sequence.*

Proof. Consider $q(y) := (h_A/v_A)^{-1}(y) + (h_B/v_B)^{-1}(y)$ decreasing for all $y \in R := (h_A/v_A)([0,\bar{t})) \cap (h_B/v_B)([0,\bar{t}))$. Taking the derivative this implies that

$$\frac{1}{(h_A/v_A)'((h_A/v_A)^{-1}(y))} + \frac{1}{(h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$\frac{(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y))}{(h_A/v_A)'((h_A/v_A)^{-1}(y)) \cdot (h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y)) > 0 \qquad \forall y \in \mathbb{R}$$

Or, in other words: $r'_A(x_A) + r'_B(x_B) > 0$ for all points (x_A, x_B) with $h_A(x_A)/v_A(x_B) = h_B(x_B)/v_B(x_B)$. We can use the same logic used in the proof of Lemma 6 to rule out interior points. $\square$

B.2 Bounds on the sum of values at the stopping point

Lemma 9. *if h_i is decreasing, at any efficient stopping point the sum of values left has to be less than one.*

Proof.

$$\begin{aligned} v'_A < 0 \quad &\Rightarrow \quad c - h_A(1 - v_A) < 0 \\ &\Rightarrow \quad h_A v_B - h_A(1 - v_A) < 0 \\ &\Rightarrow \quad v_A + v_B < 1 \end{aligned}$$

Where the second implication uses the fact that at an stopping point $c \geqslant h_A \cdot v_B$. □

Proof. If there is a fixed appropriability of value α

$$\begin{aligned} v'_A < 0 \quad &\Rightarrow \quad c - h_A(1 - v_A) < 0 \\ &\Rightarrow \quad h_A \alpha v_B - h_A(1 - v_A) < 0 \\ &\Rightarrow \quad v_A + \alpha v_B < 1 \end{aligned}$$

Symmetrically, $\alpha v_A + v_B < 1$. Hence,

$$v_A + v_B < \frac{2}{1 - \alpha}$$

□

C Extensions

C.1 Discrete time

In this appendix we will consider the discrete time case $T = \{1, 2, 3, ..., \infty\}$. At any time before stopping the agent decides which project to work on $\alpha_t \in \{A, B, \emptyset\}$. Let F_i be the distribution of successes for project i, and $h_i : T \to [0, 1]$ the respective hazard rate. Finally, let $v_i : T \to [0, 1]$ the value of the joint project when only project i is incomplete as a function of the time spent working on project i.

$$v_i(x_i) := q(j) + \max_{T \geqslant x_i} \left\{ \sum_{x=x_i+1}^{T} \frac{1-F(x)}{1-F(x_i)} \left[h(x)(1-q(j)) - c\right] \right\}$$

Proposition 3. *h_i/v_i decreasing for both projects implies that the greedy strategy is efficient.*

Proof. Grab an optimal abandonment point $x^* := Y(\emptyset, 0)$ and a trajectory to it. The trajectory has to be greedy at the time before the abandonment point. Otherwise, the optimality of x^* is violated.

Consider now a greedy trajectory and the point (x_L, x_B^*) in that trajectory where crosses $x_B = x_B^*$ (the rightmost one). If the optimum is to the right of the path ($x_A^* > x_L$) then by optimality,

$$\frac{h_A}{v_A}(x_L) \geqslant \frac{h_i}{v_i}(x_i^* - 1) \geqslant \frac{h_j}{v_j}(x_j^*)$$

If $x_L = x_i^* - 1$ then the first inequality holds with equality and there is a greedy path to the optimum: the one we considered changing at the indifferent point (x_L, x_A^*). If $x_L < x_i^* - 1$ then by strict monotonicity of h/v the inequality holds strictly, what would violate greediness of the strategy at (x_L, x_A^*) $\square$

Proposition 4. *h/v increasing for both tasks implies that the efficient allocation is in sequence.*

Proof. Suppose that the optimal stopping point $x^* = Y(\emptyset, 0)$ is interior, i.e. $x^* > 0$. Since last period is myopically optimal for each trajectory,

$$\frac{h_A}{v_A}(x_A^* - 1) \geqslant \frac{h_B}{v_B}(x_B^*) > \frac{h_B}{v_B}(x_B^* - 1)$$
$$\frac{h_B}{v_B}(x_B^* - 1) \geqslant \frac{h_A}{v_A}(x_A^*) > \frac{h_A}{v_A}(x_A^* - 1)$$

Where the strict inequalities come from the h/v being increasing for both projects. Thus, a contradiction. $\square$

C.2 Imperfect complements

$\lambda_L > c/(1-q)$ then the agent would never stop. The value is independent of q and linear. The monotonicity of h/v is equivalent to the case where $q = 0$.

Consider now $\lambda_L \in (c, c/(1-q))$. There is a belief at which the agent stops.

$$\hat{p} = \frac{c/(1-q) - \lambda_L}{\delta}$$

If $R(\hat{p}) > v(\hat{p}) = q$ and R is concave, h/v is increasing.

$R(\hat{p}) > q$

$$\frac{c^2 q}{(1-q)[c(\lambda_L + \lambda_H) - (1-q)\lambda_L\lambda_H]} > q$$

Interesting case: $[c(\lambda_L + \lambda_H) - (1-q)\lambda_L\lambda_H] > 0$.

$$\left(\frac{c}{(1-q)}\right)^2 \geqslant \frac{c}{(1-q)}(\lambda_L + \lambda_H) - \lambda_L\lambda_H$$

$$\frac{c}{(1-q)}\left(\frac{c}{(1-q)} - \lambda_L\right) \geqslant \lambda_H\left(\frac{c}{(1-q)} - \lambda_L\right)$$

$$\frac{c}{(1-q)} \geqslant \lambda_H$$

But if this is the case, then the agent does not wish to work on the development even when sure that it is relatively easy.

C.3 Supermodularity not sufficient for Lemma 1 with $k > 2$

That q supermodular implies increasing abandonment points does not hold in general for $k > 2$. Here is a counterexample:

Let $K = \{A, B, C\}$. Suppose $q(\{A, B\}) = \gamma < q(\{A, B, C\}) = 1$. $q(S) = 0$ for any subset. And suppose C is either feasible or infeasible, and that you can learn instantly about it. $\lambda_L^C = 0, \lambda_H^C = \infty$. The optimal strategy is to learn about C, and then doing the optimal thing for A and B (that might be different depending on whether C is completed or not).

In the case where

$$c < \frac{\lambda_L \lambda_H}{\lambda_L + \lambda_H} < \frac{c}{\gamma}$$

then by results when C is completed it is optimal to work simultaneously, $Y_i(\{C\}, 0) > 0$ for $i = A, B$. But when C fails, it is optimal to work in sequence, so $Y_i(\emptyset, 0) = 0$ for $i \in \{A, B\}$.