Advanced Microeconomics III

Envelope Theorem, MCS, and selling an object

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Introduction

- Often in economics, we want to know how endogenous variables depend on exogenous parameters.
  - **Example**: How does an exogenous tax t affect
    - the profits U of a firm.
    - the firm's level of production x.
- Formally, one considers a parametrized optimization problem:

$$U(t) := \max_{x \in X} \; u(x, t)$$

$$x^*(t) := \arg \max_{x \in X} \; u(x, t)$$

$$\underbrace{\frac{\partial U(t)}{\partial t} = ?}_{\text{Envelope Theorem}} \qquad\qquad \underbrace{\frac{\partial x^*(t)}{\partial t} = ?}_{\text{Comparative Statics}}$$

Overview

Envelope theorem: classical formula

- **Classical Envelope Formula**:

$$U'(t) = u_2(x^*(t), t)$$

Idea behind proof.

$$U(t) = u(x^*(t), t)$$

- Applying the chain rule:

$$U'(t) = u_1(x^*(t), t) \cdot \frac{\partial x^*(t)}{\partial t} + u_2(x^*(t), t)$$

- Because $x^*(t)$ is a maximizer, FOC

$$u_1(x^*(t), t) = 0$$

Envelope theorem: limitations

- The previous argument assumes that $x^*(\cdot)$ is differentiable.
  - This cannot be assumed directly because x^* is an endogenous object.

- Moreover, we are sometimes interested in problems for which the set X is such that we cannot use calculus.

Modern envelope theorem

- Modern version of the Envelope Theorem developed by Milgrom and Segal (2002).
- **Primitives**:
  - X choice set.
  - $T = [\underline{t}, \bar{t}]$ parameter set.
  - $u : X \times T \to \mathbb{R}$ objective function.

Assumption

The partial derivative u_2 exists and it is bounded, i.e.

$$\exists L > 0 : \text{ for all } x \in X \text{ and } t \in T, \quad |u_2(x,t)| \leq L$$

Modern envelope theorem

Modern Envelope Formula

$$U(t) = U(\underline{t}) + \int_{\underline{t}}^{t} u_2(x^*(s), s)\, ds \qquad \forall t \in T \qquad \text{(Envelope)}$$

- No assumption on X other than measurability.
- No assumptions on x^* other than existance.
- (The paper has a version with weaker assumptions.)

Modern envelope theorem: proof

Lemma

U is Lipschitz continuous, i.e. $\exists L$:

$$|U(t) - U(t')| \leq L \cdot |t' - t| \qquad \textit{for all } t, t' \in T$$

$$\begin{aligned} U(t) - U(t') &= u(x^*(t), t) - u(x^*(t'), t') \\ &\leq u(x^*(t), t) - u(x^*(t), t') \\ &= \int_{t'}^{t} u_2(x^*(t), s)\ ds \\ &\leq L \cdot |t' - t| \end{aligned}$$

- Exchanging t and t' in the previous argument, we get the desired result.

Modern envelope theorem: proof

Lemma

Any Lipschitz continuous function $f : [\underline{t}, \overline{t}] \to \mathbb{R}$ is differentiable a.e., and equals the integral over its derivative, i.e.

$$f(t) - f(\underline{t}) = \int_{\underline{t}}^{t} f'(s)\, ds$$

- For proof, see math textbook, e.g. Rudin, Real and Complex Analysis, 1987.

Modern envelope theorem: proof

- Consider t, t' such that $U'(t)$ exists.
- Notice that:

$$U(t) = u(x^*(t), t) \qquad \text{and } U(t') = u(x^*(t'), t') \geq u(x^*(t), t')$$

- Hence:

$$\frac{U(t') - U(t)}{t' - t} \geq \frac{u(x^*(t), t') - u(x^*(t), t)}{t' - t} \qquad \text{if } t' > t$$

$$\frac{U(t') - U(t)}{t' - t} \leq \frac{u(x^*(t), t') - u(x^*(t), t)}{t' - t} \qquad \text{if } t' < t$$

Modern envelope theorem: proof

$$
\begin{aligned}
u_2(x^*(t), t) &= \lim_{t' \to t} \frac{u(x^*(t), t') - u(x^*(t), t)}{t' - t} \\
&= \lim_{t' \to t} \frac{U(t') - U(t)}{t' - t} \\
&= U'(t)
\end{aligned}
$$

- Using the previous Lemma, we get the formula:

$$
U(t) = U(\underline{t}) + \int_{\underline{t}}^{t} u_2(x^*(s), s)\, ds
$$

Overview

Monotone comparative statics (MCS)

- **Comparative static question**: how do choices change with exogenous parameters.
- Models are often qualitative approximations, in many cases we are mainly interested in
  - Qualitative predictions: In what direction do endogenous variables change?
  - Predictions that are *robust* to the specifications of our models.
- These predictions are obtained using MCS techniques.
  - In this section, we will present some motivation and basic results.

Monotone comparative statics (MCS)

- Back to our problem:

$$U(t) = \max_{x \in X} \quad u(x,t)$$

$$X^*(t) = \arg\max_{x \in X} \quad u(x,t)$$

- **MCS question**: Under what conditions on u can we conclude that $X^*(t)$ is nondecreasing in t?
- (Note: when $X^*(t)$ contains more than one element, we should be more precise about what we mean by "nondecreasing.")

MCS issues

- **Immediate technical issues**:
  - **Existence**: In order to ensure that $X^*(t)$ is nonempty we need to impose some conditions (e.g. u continuous and X compact).
  - **Uniqueness**: In general $X^*(t)$ can contain several elements.
  - **Strict or weak monotonicity**: We focus on weak monotonicity here.

Traditional first-order approach

- Traditional comparative statics arguments make the following assumptions:
  - $X \subset \mathbb{R}$
  - u twice continuously differentiable.
  - $u(\cdot, t)$ concave.
  - $x^*(t)$ interior.

- Differentiating FOC with respect to t we get:

$$u_{xx}(x^*(t), t) \cdot x^{*'}(t) + u_{xt}(x^*(t), t) = 0$$

Traditional first-order approach

- Thus,

$$x^{*'}(t) = \frac{-u_{xt}(x^*(t), t)}{u_{xx}(x^*(t), t)}$$

- Under strict concavity ($u_{xx} < 0$), x is weakly increasing at t if and only if $u_{xt}(x^*(t), t) \geq 0$.

Supermodularity

A function u is supermodular if for all $x' > x$ and $t' > t$

$$u(x', t') - u(x, t') \geq u(x', t) - u(x, t)$$

Let A and B be two subsets of $\mathbb{R}$. We say that B is greater than A according to the **strong set order** iff for any $a \in A$ and $b \in B$ if $a \geq b$ then $a \in B$ and $b \in A$.

Topkis' monotonicity theorem

Topkis' Univariate Monotonicity Theorem

Suppose that u is supermodular. If $t' > t$, then $X^*(t') \geq X^*(t)$ in the strong set order.

Proof.

Consider a violation of the strong set order, i.e. assume that $x \in X^*(t)$ and $x' \in X^*(t')$. $t' > t$ and $x > x'$ with either $x' \notin X^*(t)$ or $x \notin X^*(t')$.

- Hence,

$$\begin{aligned} u(x,t) &\geq u(x',t) \\ u(x',t') &\geq u(x,t') \end{aligned}$$

- With one of the two holding with strict inequality.

Topkis' monotonicity theorem

Proof (Cont.)

- Adding the two inequalities and rearranging yields:

$$u(x, t') - u(x', t') < u(x, t) - u(x', t)$$

- This is a contradiction to $x > x'$ and supermodularity of u,

$\square$

Single crossing

A function u satisfies *single crossing* iff for all $x' > x$ and $t' > t$ we have

$$u(x', t) > u(x, t) \Rightarrow u(x', t') > u(x, t')$$

and

$$u(x', t) \geq u(x, t) \Rightarrow u(x', t') \geq u(x, t')$$

- Notice that this definition is robust to monotone transformations of u.
- This is of course related to the SC condition that we assumed in the previous models. (Exercise.)

Milgrom-Shannon

Theorem (Milgrom-Shannon)

Suppose that u satisfies single crossing. If $t' > t$, then $X^*(t') \geq X^*(t)$ in the strong set order.

Proof.

- We prove that a violation of the strong set order together with single crossing leads to a contradiction.
- Violation of SSO: $\exists t, t'$ with $t' > t$, $x \in X^*(t)$,$x' \in X^*(t')$, $x > x'$ such that either $x \notin X^*(t')$ or $x' \notin X^*(t)$.

Milgrom-Shannon

Proof (cont.)

- Thus,

$$u(x,t) \geq u(x',t)$$
$$u(x',t') \geq u(x,t')$$

- One of them is strict.
- Suppose the first one is strict:
  - Since $x > x'$, by SC,

$$u(x,t') > u(x',t') \qquad \text{Abs!}$$

- Suppose the second one is strict:
  - By the first (weak) one and SC we have

$$u(x,t') \geq u(x',t') \qquad \text{Abs!}$$

Overview

Mechanism design

- For the rest of the course, we will study **mechanism design**.
  - **game** $=$ **environment** (agents, outcome space, information)
    $+$ **rules or mechanism** (actions, map from actions to outcomes).
- Instead of taking the game as given, we fix the environment but we ask
  - What outcomes are consistent with some set of rules/mechanism?

- As a first approach and example, we will consider the problem of selling an object to a single buyer from the mechanism design perspective.

Overview

Setup

- There is a single agent (*buyer*).
- One indivisible unit of a good.
- Agent's valuation $\theta \in [0, 1]$ for the good is private information.
- Preferences are quasi-linear: her payoff from getting the good with probability q and paying p is simply

$$\theta q - p$$

- A *principal* can design any mechanism she likes to sell the good.
  - Sequence of actions that the agent can take.
  - As a function of the actions, probability q with which the agent receives the good and payment p.
  - The agent chooses optimally among actions.

Selling an object to a single buyer

- Fixing a mechanism and an optimal action of type θ, there is a probability $q(\theta)$ that she receives the object and an (expected) payment $p(\theta)$ that she makes.

$$q : [0,1] \to [0,1] \qquad \text{Induced allocation}$$

$$p : [0,1] \to \mathbb{R} \qquad \text{Induced payment rule}$$

- Which allocations and payment rules can be induced with a mechanism?

Revelation principle

- We will focus on a particular type of simple mechanisms.

A *direct revelation mechanism* (q, p) is one in which the agent is asked to make a report $\hat{\theta} \in [0, 1]$ of her type. Then, is given the good with probability $q(\hat{\theta})$ and pays $p(\hat{\theta})$.

- Note:
  - (q, p) can denote both DRM and allocation and payment rules.
  - A DRM (q, p) does not necessarily induce allocation q and payment rule p.

Revelation principle

Definition

A Direct Revelation Mechanism is Incentive Compatible (or truthful) iff every type weakly prefers to report her own type.

- Notice that if (q, p) is an IC DRM, then it induces allocation q and payment rule p.

Revelation Principle

If a mechanism induces an allocation q and payment p, then the DRM (q, p) is IC (and thus induces the allocation q and payment rule p).

Revelation principle: proof

Proof.

- Consider a mechanism that induces q and p, and a type θ.
- Since type θ behaves optimally, the payoff $q(\theta) \cdot \theta - p(\theta)$ is weakly greater than the payoff that she could get from **any** deviation.
- One particular deviation is mimicking whatever actions some other type θ' takes, in which case she would get the good with probability $q(\theta')$ and pay $p(\theta')$. So

$$q(\theta) \cdot \theta - p(\theta) \geq q(\theta') \cdot \theta - p(\theta')$$

- Now consider the DRM (q, p). The **only** deviations available are mimicking other types. We just show that all such deviations are unprofitable.

$\square$

Revelation principle

- The revelation principle is deep, trivial, and powerful.
- It allow us to restrict attention **without loss of generality** to IC DRM.
  - This is very useful for analytical proposes.
  - In practice we may be interested in *indirect* mechanisms.
  - Usually after answering what **can** be implemented (using the revelation principle) one can ask **how** can it be implemented, i.e. if it exists a natural indirect way to implement the same outcomes.

Envelope theorem revisited

- Fix a DRM (q, p).
- The problem of type θ is

$$V(\theta) \quad := \quad \max_{\hat{\theta} \in [0,1]} \quad \underbrace{q(\hat{\theta}) \cdot \theta - p(\hat{\theta})}_{\pi(\hat{\theta},\theta)}$$

- We can think of this as a parametrized optimization problem where the 'parameter' is the true type θ and the agent chooses the report.
- We can apply the Envelope Theorem.

Envelope theorem revisited

Mirrlees Envelope Theorem

Any IC DRM (q, p) satisfies the envelope formula:

$$V(\theta) = V(0) + \int_0^{\theta} \pi_2(\tilde{\theta}, \tilde{\theta}) \; d\tilde{\theta}$$

- We can rewrite as:

$$\theta \cdot q(\theta) - p(\theta) = -p(0) + \int_0^{\theta} q(\tilde{\theta}) \; d\tilde{\theta}$$

- It follows that any two indirect mechanisms that induce the same allocation q and such that $p(0) = 0$ must induce the <u>same</u> payment rule.

Characterizing incentive compatibility

- Checking whether a DRM is IC is tedious.
  - We must check that each type θ does not want to mimic any other type.
  - The Envelope Theorem gives us a necessary condition for IC.
- We are interested in a characterization.
- Say that a DRM *satisfies monotonicity* if q is weakly increasing.

Spence-Mirrlees Characterization

A DRM (q, p) is IC if and only if it satisfies the envelope formula and monotonicity.

IC characterization: proof

IC implies Monotonicity:

- Consider two types $\theta, \theta' \in [0,1]$.
- By IC:

$$\begin{aligned} \theta' \cdot q(\theta') - p(\theta') &\geq \theta' \cdot q(\theta) - p(\theta) \\ \theta \cdot q(\theta) - p(\theta) &\geq \theta \cdot q(\theta') - p(\theta') \end{aligned}$$

- Rearranging, we get:

$$\theta[q(\theta') - q(\theta)] \leq p(\theta') - p(\theta) \leq \theta'[q(\theta') - q(\theta)]$$

- Which implies:

$$(\theta' - \theta) \cdot [q(\theta') - q(\theta)] \geq 0$$

IC characterization: proof

Envelope and Monotonicity imply IC

- Payoff loss of type θ that mimics $\hat{\theta}$ is:

$$\begin{aligned} V(\theta) - \pi(\hat{\theta}, \theta) &= V(\theta) - V(\hat{\theta}) + V(\hat{\theta}) - \pi(\hat{\theta}, \theta) \\ &= \int_{\hat{\theta}}^{\theta} q(s)\ ds - (\theta - \hat{\theta}) \cdot q(\hat{\theta}) \\ &= \int_{\hat{\theta}}^{\theta} [q(s) - q(\hat{\theta})]\ ds \end{aligned}$$

- This is positive (both for $\theta > \hat{\theta}$ and $\theta < \hat{\theta}$) by monotonicity.
- Thus, (q, p) is IC.

Participation constraints

- Sometimes, the agent cannot be forced to participate in the mechanism (She might 'walk away').
- Assume that if the agent walks away she gets a payoff of zero (no good, no payment).

It is without loss of generality to focus on mechanisms that induce every type to participate.

- If type θ is not participating, one could invite her to participate and award outcome $q(\theta) = 0$ and $p(\theta) = 0$.
- Thus, we can focus on IC mechanisms that induce participation.
- We call this *Individually Rational* (IR) mechanisms.

Participation constraints

A DRM (q, p) is IC and IR if and only if it satisfies the envelope formula, monotonicity, and $p(0) \leq 0$.

Proof.

We already showed that

- IC $\Leftrightarrow$ Envelope formula and monotonicity.
- Remains to show that
  - IR $\Rightarrow p(0) \leq 0$.

$$IR \quad \Rightarrow \quad U(0) \geq 0 \quad \Rightarrow \quad -p(0) \geq 0.$$

  - $p(0) \leq 0$ and Envelope formula $\Rightarrow$ IR.

$$U(\theta) \; = \; U(0) + \int_0^\theta q(\tilde{\theta}) \, d\tilde{\theta} \; \geq \; U(0) \; = \; -p(0) \; \geq \; 0$$

Optimality of posted prices

- Suppose that the principal is a monopolist who wishes to sell the object to the agent to maximize expected profits.
- The principal can choose any mechanism that she likes, for example, *post a price*:
  - The principal sets a price P and gives the agent two options.
  - The agent can purchase the good at price P.
  - The agent can walk away.
- This is an **indirect** mechanism that induces:

$$\begin{array}{lll} q(\theta)=1 & p(\theta)=P & \text{if } \theta \geq P. \\ q(\theta)=0 & p(\theta)=0 & \text{if } \theta < P. \end{array}$$

- This mechanism does not make use of the monopolist power to allocate the good randomly.

Optimality of posted prices

Theorem (Myerson 1981)

There is a posted-price mechanism that maximizes the principal's expected revenue.

- Here we prove the result with the extra assumption that the distribution of types is absolutely continuous with a weakly increasing hazard rate.
- The result, however, holds for any distribution.

Optimality of posted prices: proof

$$
\begin{aligned}
\text{Expected revenue} &= E[p(\theta)] \\
&= E[q(\theta)\cdot\theta - V(\theta)] \\
&= \int_0^1 \left[q(\theta)\cdot\theta - V(0) - \int_0^\theta q(s)\ ds \right] f(\theta) d\theta \\
&= \int_0^1 q(\theta)\theta f(\theta) d\theta - \int_0^1 \int_0^\theta q(s)\ ds \cdot f(\theta)\ d\theta - V(0)
\end{aligned}
$$

Optimality of posted prices: proof

We will use integration by parts in the second term:

$$\begin{aligned}\int_0^1 \int_0^\theta q(s)\ ds \cdot f(\theta)\ d\theta &= F(\theta) \int_0^\theta q(s)\ ds\Big|_0^1 - \int_0^1 q(\theta)F(\theta)\ d\theta \\ &= \int_0^1 q(s)\ ds - \int_0^1 q(\theta)F(\theta)\ d\theta \\ &= \int_0^1 q(s) \cdot [1 - F(s)]\ ds\end{aligned}$$

Optimality of posted prices: proof

- Back to the expected revenue,

$$
\begin{aligned}
E[p(\theta)] &= \int_0^1 [q(s) \cdot s \cdot f(s) - q(s)[1 - F(s)]] \; ds - V(0) \\
&= \int_0^1 q(s) \left[s - \frac{1 - F(s)}{f(s)} \right] \cdot f(s) \; ds - V(0)
\end{aligned}
$$

- Thus, the problem of the seller is to chose q monotone to maximize the previous expression.

Optimality of posted prices: proof

$$VS(\theta) \quad := \quad \theta - \frac{1 - F(\theta)}{f(\theta)} \qquad \text{`Virtual surplus'}$$

- Ignoring monotonicity, we would like to choose:

$$q(\theta) = \begin{cases} 1 & \text{if } VS(\theta) \geq 0 \\ 0 & \text{if } VS(\theta) < 0 \end{cases}$$

- Under the assumption that the hazard rate is nondecreasing (and thus so is the VS), the solution is monotonic and thus solves the original problem (with the monotonicity constraint).

Optimality of posted prices

- Notice that the optimal price P^* is such that the $VS(P(\theta)) = 0$, i.e.

$$P^* = 1/h(P^*)$$

  where h is the hazard rate function.

- This corresponds to the FOC of the problem:

$$\max_P P[1 - F(p)]$$

Role of commitment

- Ability to commit is important for this result.
  - Notice that an ‘unlucky’ monopolist that offered a posted price might want to choose to offer a lower price once the agent refuses to purchase.
  - But if this is anticipated by the agent, then she has more incentives to refuse a posted price.

- Without commitment the revelation principle fails.
  - In general we also need to impose IC constraints on the principal.