# Advanced Microeconomics III

## Spence's Signaling Model

Francisco Poggi

# Introduction

- **Akerlof**: markets with privately informed participants are often inefficient.
- Agents whose information is *favorable* may have an incentive to find means to convey this information.

- **Signaling**: information can be conveyed, but only indirectly.

# Introduction

- Examples:
  - A *warranty* may signal good quality of a used car.
  - *education* may signal workers' ability.
- Questions:
  - How can signaling occur in equilibrium?
  - Is signaling always welfare-improving?

# Spence's model

- A single worker and many (at least 2) firms.
- Worker can be of two types: $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L$.
- Only the worker knows $\theta$.
- If employed by a firm, worker produces output $\theta$.
- Firm's payoff:
  - $\theta - w$ if employs the worker at wage $w$.
  - zero otherwise.

# Spence's model

- Worker moves first: chooses an observable education level $e \in [0, \infty)$
- Firms observe $e$ (again: not $\theta$).
- Cost of education $c(e|\theta)$.
- Worker payoff when education $e$ and employed at wage $w$:

$$u(w, e|\theta) = w - c(e|\theta)$$

- Notice that education in this model is unproductive.

# Spence's model

- Extra assumptions:
  - Cost of no education is zero.

$$c(0|\theta) = 0 \qquad \text{for all } \theta$$

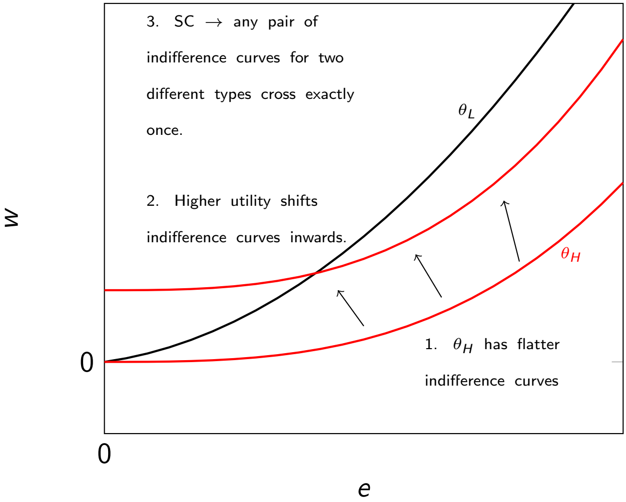
  - Cost of education increasing and convex in education.

$$c'(e|\theta) > 0 \qquad \text{and} \qquad c''(e|\theta) > 0$$

  - High type worker has a smaller education cost.
    Moreover: High type has a smaller <u>marginal</u> cost of education.

$$c'(e|\theta_H) < c'(e|\theta_L) \qquad \forall e > 0 \qquad\qquad \text{(Single-crossing)}$$

# Indifference Curves



# PBE Analysis

- **Solution concept**: (Pure-strategy) Perfect Bayesian Equilibrium.
- Described by:
  - A choice of education level for each worker type $e_L, e_H$.
  - $\mu(e)$ firms' posterior beliefs that worker is of type $H$.
  - Wage offers of the firms $w(e)$.
- Satisfying:
  - Optimality of education choices given wage offers.
  - Beliefs $\mu(e)$ consistent with Bayes' Rule where possible.
  - Wage offers constitute a Nash equilibrium at each subgame.
  - **Symmetry**: All firms hold the same beliefs after observing $e$.
    - (Not implied by weak PBE.)
  - Firms believe other firms conform to equilibrium wage offer $w(e)$ both on and off path.

# PBE Analysis

- Competition among firms leads to the following wage offers (why?):

$$w(e) = E_{\mu(e)}[\theta] = \mu(e) \cdot \theta_H + (1 - \mu(e)) \cdot \theta_L$$

- Two types of pure-strategy equilibria:
  - **Separating equilibria**: each type chooses a different education level ($e_H \neq e_L$).
  - **Pooling equilibria**: types choose the same education level ($e_H = e_L$).

## Separating Equilibria

- $e_H \neq e_L$.
- Bayes' rule where possible: $\mu(e_L) = 0$ and $\mu(e_H) = 1$.
- By competition:

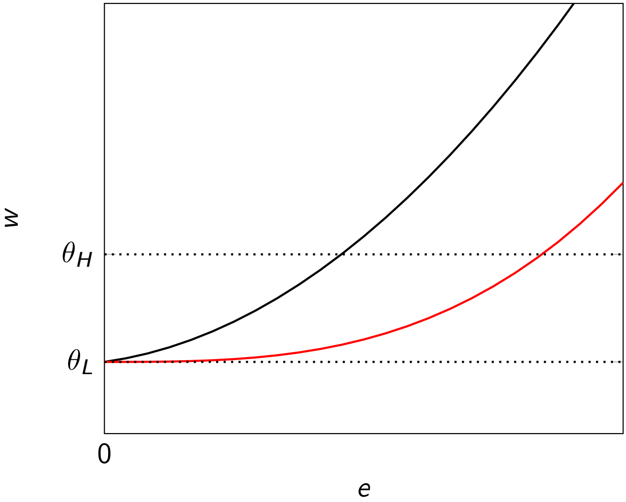
$$w(e_L) = \theta_L \qquad w(e_H) = \theta_H$$

**Lemma**

In any separating equilibrium, $e_L = 0$.

- PBE implies that $w(e) \in [\theta_L, \theta_H]$.
- So, if $e_L > 0$, the deviation to $e = 0$ is profitable for type $\theta_L$.

# Separating Equilibria



## Separating Equilibria: Incentive Compatibility

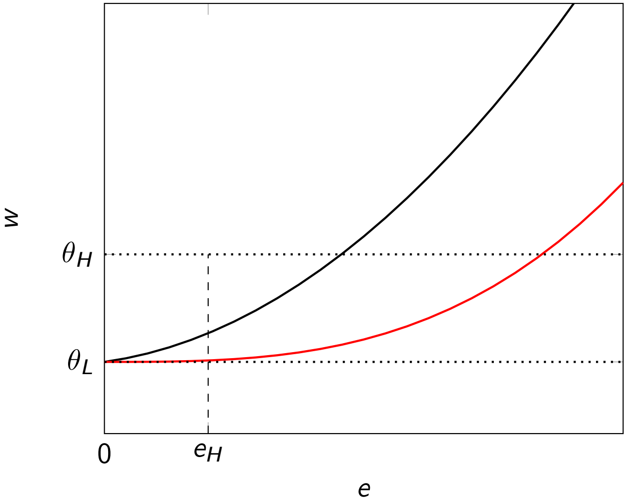
**Lemma**

*In a separating equilibrium, type $H$ chooses $e_H > 0$ such that*

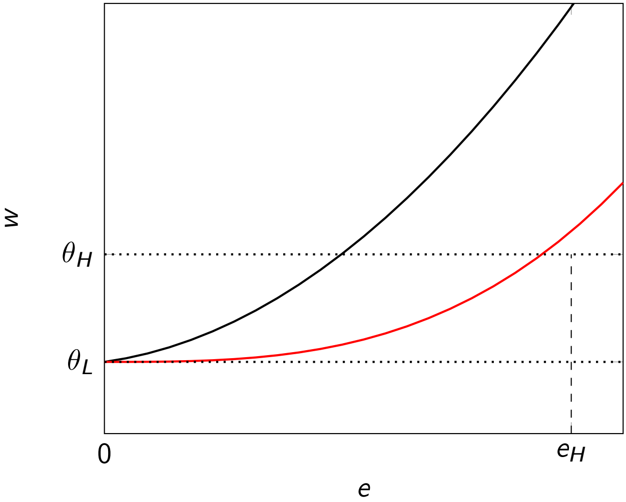
$$\theta_H - c(e_H|\theta_H) \ \geq \ \theta_L \ \geq \ \theta_H - c(e_H|\theta_L) \tag{IC}$$

- First inequality: type $H$ prefers his education $e_H$ rather than zero.
- Second inequality: type $L$ prefers zero rather than $e_H$.

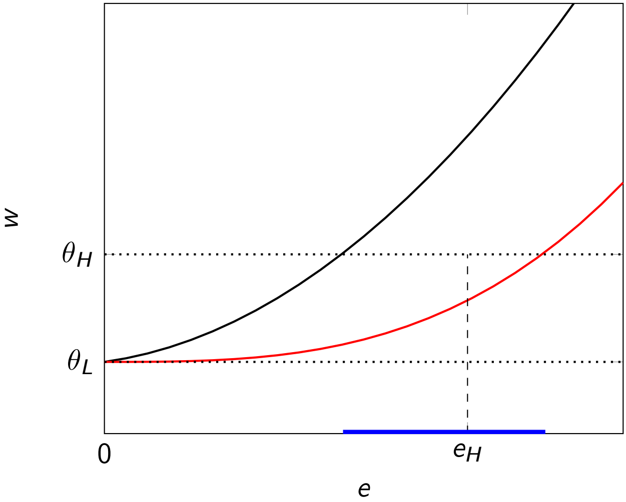
# Separating Equilibrium: IC



# Separating Equilibrium: IC



# Separating Equilibrium: IC



# Separating Equilibria

- Previous lemmata describe *necessary* conditions for separating equilibrium.

- These are also *sufficient*: remains to specify out-of-equilibrium beliefs.
  - Suppose any deviation is considered to be by a low type.
  - Then wage would be $\theta_L$ for any worker with an education level different than $e_H$.
  - <u>Any</u> deviation would be unprofitable.

# Equilibrium Multiplicity

- We have **multiple** separating equilibria.
    - These equilibria can be ranked in Pareto sense.
    - Best separating equilibrium: the one with lowest education $e_H$.

$$c(e_H|\theta_L) = \theta_H - \theta_L$$

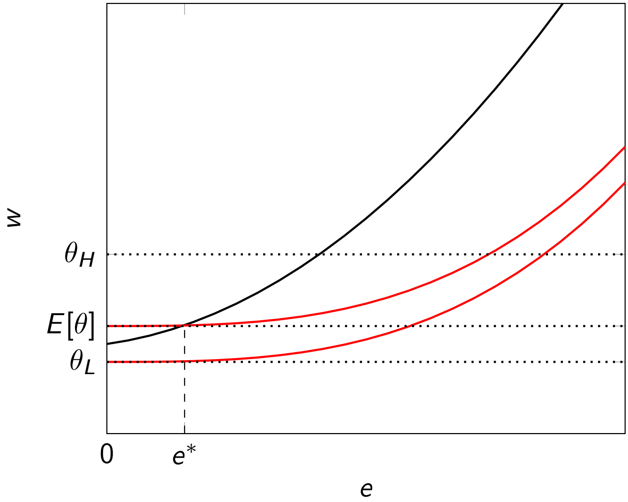
# Pooling Equilibria

- Pooling equilibrium: $e_L = e_H = e^*$.

- Bayes' rule where possible: $\mu(e^*) = \Pr(\theta = \theta_H)$
- Competition implies that $w(e^*) = E[\theta]$.

- Out-of-equilibrium beliefs: $\mu(e) = 0$ for $e \neq e^*$.
  - Then $w(e) = \theta_L$ for $e \neq e^*$.

# Pooling Equilibria



# Multiple Pooling Equilibria

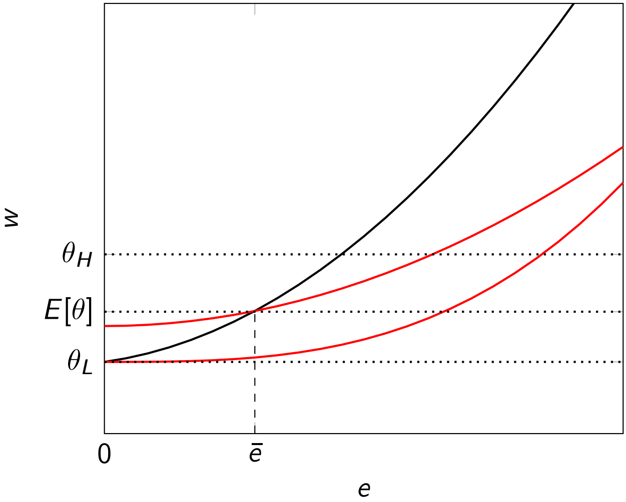
- **Again**: Best pooling equilibrium is the one with the lowest level of education ($e^* = 0$).

- What about the *worst* one?

$$E[\theta] - c(\bar{e}|\theta_L) = \theta_L$$

$$c(\bar{e}|\theta_L) = E[\theta] - \theta_L$$

# Worst Pooling Equilibria



# Comparing Pooling and Separating Equilibra

- The best pooling equilibrium may or may not Pareto dominate the best separating equilibrium.
  - High types not always benefit from the availability of a signaling device. Only if their fraction is small enough.

- The best separating equilibrium *never* Pareto dominates the best pooling equilibrium.
  - The low type is always worse-off.

# Reasonable Beliefs (Equilibrium Refinements)

- There are multiple separating and multiple pooling equilibria.
- Which equilibrium is more likely to emerge?
  - One approach: Pareto dominance. Not a game-theoretical argument.
- Forward induction arguments can be used to refine the equilibrium.
- PBE allows for any beliefs off the equilibrium path.
- Refinements put conditions on these off equilibrium beliefs.
- Most refinements in this game uniquely select the least costly separating equilibrium.

# Intuitive Criterion

- Cho and Kreps (1987) 'Intuitive criterion':
  - Key question: Who might benefit from the deviation?

**Definition**

A deviation $e'$ is dominated in equilibrium for type $\theta$ if, for any sequentially rational response by the receivers $w' = E_{\mu'}[\theta]$ for some beliefs $\mu'$, the resulting payoff $u(e', w', \theta)$ is less than the equilibrium payoff $u(e(\theta), w(e(\theta)), \theta)$.

**Definition**

A PBE *passes the Intuitive Criterion Test (ICT)* if no type $\theta$ would be better off deviating to an action $e' \neq e(\theta)$ should the receivers' beliefs following $e'$ assign zero probability to types $\theta'$ for whom the deviation is *dominated in equilibrium*.

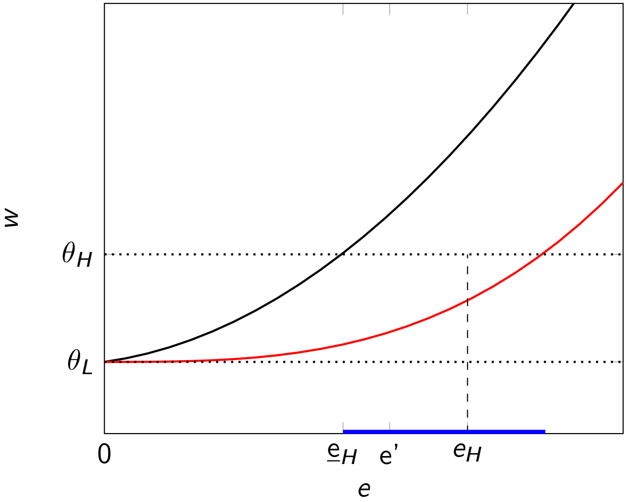
# Intuitive Criterion: Separating Equilibrium

- Let $\underline{e}_H$ be the minimal education for the high type that can be sustained in a separating equilibrium.
- Start from a separating equilibrium with $e_H > \underline{e}_H$. We will show that this equilibrium does not satisfy the ICT.
  - Consider a deviation to $e' \in (\underline{e}_H, e_H)$ (This is off the equilibrium path).
  - A type $\theta_L$ can guarantee a payoff of $\theta_L$ by following equilibrium strategies. The deviation can bring type $\theta_L$ at most:

$$\theta_H - c(e'|\theta_L) < \theta_L$$

  - Thus, a type $\theta_L$ would never deviating to $e'$. Formally $e'$ is dominated in equilibrium for type $\theta_L$.
  - The PBE does not pass the ICT: If $\mu(e') = 1$, type $\theta_H$ would benefit from deviating to $e'$.

# Intuitive Criterion: Separating Equilibrium



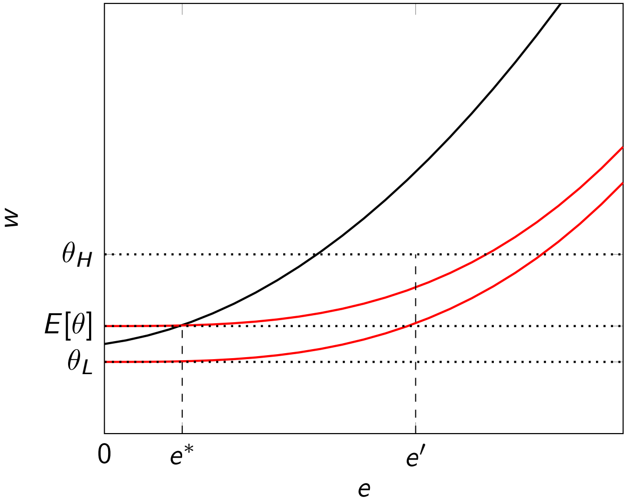
# Intuitive Criterion: Pooling Equilibrium

- Let start instead from a pooling equilibrium at $e^*$.
- **Claim**: there exists $e'$ such that

$$E[\theta] - c(e^*|\theta_H) < \theta_H - c(e'|\theta_L) < E[\theta] - c(e^*|\theta_L)$$

- Deviating to $e'$ is dominated in equilibrium for type $\theta_L$.
- So, the pooling PBE does not pass theå ICT. If $\mu(e') = 1$, type $\theta_H$ would benefit from deviating to $e'$.

# Intuitive Criterion: Pooling Equilibrium



# Intuitive Criterion

- Only the best separating PBE passes the ICT.
- Notice that sometimes *forced pooling* generates a Pareto improvement.
  - In particular, when the share of high types is sufficiently large.
- Another Pareto improvement can arise with *cross-subsidization*.

# Model with Continuum of Types

- Consider a model with a continuous of types.
  - Support in $[\underline{\theta}, \bar{\theta}]$.
  - Density function $f$ strictly positive everywhere in the support.

- **Question**: Is there a separating equilibrium? Is it unique?
  - Parametric assumption: $c(e|\theta) = \alpha \cdot e^2/\theta$.

# Empirical Evidence

- Education is sometimes used as a signal of ability.
- Bedard, Kelly. Human Capital Versus Signaling Models.
  - Effect of constraining access to university in high school graduation levels.
  - **Finding**: In regions with a university, high school drop-out rates are higher.
  - This would be difficult to explain in model of human capital.
  - Signaling theory provides an explanation:
    - No university is close, more high-ability kids will stop education after high-school.
    - This gives low-ability kids incentives to pool with the high-ability kids.
  - Policy implications:
    - Improving access to university might increase drop-out rates and depress wages for some kids.

# Other Models of Signaling

- Evidence and voluntary disclosure of verifiable information. Grossman (1981) Milgrom (1981) Dye (1985)

- Costless signaling (cheap talk): might work if preferences between sender and receiver are partially aligned. (Crawford Sobel (1982))

# Classical Evidence Models

Seminal model developed by Grossman (1981) and Milgrom (1981)

- Similar to the previous model.
  - One worker, more than 2 firms.
  - Worker has private type $\theta$ with cdf $F$.
  - Firms compete offering wages.

- Instead of choosing a level of education, worker can take a (free) test that perfectly reveals his type.
  - Formally, worker can send a message in $\{\emptyset, \theta\}$.
  - Market observes the message before making wage offers.

# Unraveling

- Again, our solution concept is PBE.
- let $w(m)$ be the wage that firms offer to an agent that sends message $m$.
- Let $\Theta_\circ$ be the subset of types that chooses the empty message in equilibrium.
- For simplicity, assume that indifferent types take the test.
- Suppose that $w(\emptyset) > \underline{\theta}$.
- It must be that $\Theta_\circ = [\underline{\theta}, w(\emptyset))$
- $w(\emptyset) = E[\theta|\Theta_\circ] < w(\emptyset)$. Abs!
- So $w(\emptyset) = \underline{\theta}$ and $\Theta_\circ$ is empty.

# Partial Unraveling

- Dye (1985) and Jung and Kwon (1988): Worker has evidence with some probability $\lambda$, and no evidence otherwise (independent of type).

- **Partial unraveling**:
  - Let $w$ be the wage for a worker in the absence of evidence.
  - Any type with $\theta < w$ will not present evidence.
  - Equilibrium $w$ is the unique solution to:

$$\begin{aligned} w \quad = \quad E[\theta|m=\emptyset] \quad &= \quad E[\theta| \text{ no evidence or } \theta < w]. \\ &= \quad q(w)\cdot E[\theta] + (1-q(w))\cdot E[\theta|\theta < w] \end{aligned}$$

where $q(w) = Pr($ no evidence $|$ no evidence or $\theta < w)$.

## Partial Unraveling

- Example: $\theta \sim U[0, b]$.

$$q(w) = \frac{p}{p + (1-p)F(w)} = \frac{p \cdot b}{p \cdot b + (1-p) \cdot w}$$

- So,

$$E[\theta | m = \emptyset] = \frac{p \cdot b}{p \cdot b + (1-p) \cdot w} \cdot \frac{b}{2} + \frac{(1-p)w}{p \cdot b + (1-p) \cdot w} \cdot \frac{w}{2}$$

- Solving $E[\theta | m = \emptyset] = w$ we get

$$w = \frac{\sqrt{p} \cdot b}{1 + \sqrt{p}}$$

## Cheap Talk

Classical setup by Crawford and Sobel:

- Two agents, sender and receiver $i \in \{S, R\}$.
- Nature picks $\theta$ from prior $F(\cdot)$.
- Sender observes private information $\theta \in [0, 1]$.
- Sender sends a message $m \in M$ to receiver.
- Receiver takes action $a \in \mathcal{A}$.

# Cheap Talk

- Preferences $U^i(\theta, a)$.
- $a^i(\theta)$: preferred action of $i$ when state is $i$. Solves max $U^i(\theta, a)$.
- Preferences are partially aligned: $a^i(\theta)$ increases in $\theta$.
- But not perfectly: $a^S(\theta) > a^R(\theta)$.

- **Example**: quadratic preferences.

$$U^R(a, \theta) = -(a - \theta)^2 \qquad U^S(a, \theta) = -(a - \theta - b)^2$$

- We assume these preference from now on.

# Cheap Talk

- **Key assumption**: No commitment.

- **Equilibrium**:

  - Message for each type: $\mu : [0,1] \to M$. For each $\theta$,
  $$\mu(\theta) \in \arg\max_{m \in M} U^S(\alpha(m), \theta).$$

  - Action for each message: $\alpha : M \to \mathcal{A}$. For each $m \in M$,
  $$\alpha(m) \in \max_{a \in \mathcal{A}} \int_0^1 U^R(a, \theta)\beta(\theta|m)\ d\theta$$

  - Interpretation of message: $\beta(\theta|m)$. Consistent with $\mu$ and Bayes' Rule.

# Cheap Talk

**Proposition**

An uninformative equilibrium always exists.

- This is the *babbling* equilibrium:
  - Strategy of sender is state independent.
  - Posterior equals prior.
  - Receiver takes optimal action given prior independently of the message.

- Notice that this is true even for the case in which incentives are perfectly aligned ($b = 0$).

# Cheap Talk

- Any informative equilibria?
  - With deterministic messages, the receiver's best response is:

$$\alpha(m) = E[\theta|\mu(\theta) = m]$$

- Suppose that on the path all messages of the sender are interpreted to be in one of two categories: $l$ and $h$.
- Suppose that $\alpha(l) < \alpha(h)$. By single-crossing we have cutoff equilibria.
- There is at most one agent that is indifferent given two actions.

# Cheap Talk

Generalizing properties of the equilibrium:

- The equilibrium message strategy is *partitional*: In any PBE there exist $k$ cutoff types $\theta_1, \theta_2, ...\theta_k$ such that for each k,

$$\mu(\theta) = \mu(\theta') \qquad \forall \theta, \theta' \in (\theta_k, \theta_{k+1})$$

- Suppose $N$ is the maximum number of partitions that can be sustained in equilibrium. It is possible to show that:
  - There exist a PBE with $n$ partitions for all $n < N$.
  - The ex-ante expected payoff of both sender and receiver increases in the number of partitions.