Problem Set 1

Advanced Microeconomics III

Spring 2022

Problem 1 Based on MWG 13.B.3.

Consider a positive selection version of the adverse selection model in which $r(\cdot)$ is continuous and strictly decreasing. Also assume that F has a strictly positive density on $[\underline{\theta}, \bar{\theta}]$

a. Show that the *more capable* workers are the ones choosing to work for any given wage.

***Solution*:** *A worker accepts the job if*

$$w > r(\theta)$$

If θ accepts the job, then $\theta' > \theta$ also accepts the job.

$$w > r(\theta) > r(\theta')$$

The first inequality holds since θ accepts the job and the second because r is decreasing. ■

b. Show that if $r(\theta) > \theta$ for all θ, then the resulting competitive equilibrium is Pareto efficient.

***Solution*:**

$$r(\theta) \geq r(\bar{\theta}) > \bar{\theta}$$

So, there cannot be an equilibrium in which workers do work. If a worker works the wage has to be larger than $\bar{\theta}$ but then firms are loosing money.

Thus, an equilibrium wage $w^* \in (\bar{\theta}, r(\bar{\theta}))$ with $\Theta^* = \emptyset$.

This is Pareto efficient since each worker does the most productive activity. ■

c. Show that in any competitive equilibrium the trading activity is inefficiently high if the following assumption is satisfied:

- there exists $\hat{\theta} \in (\underline{\theta}, \bar{\theta})$ such that $r(\theta) < \theta$ for all $\theta > \hat{\theta}$ and $r(\theta) > \theta$ for all $\theta < \hat{\theta}$.

***Solution*:** *Let's assume that $w \leq \hat{\theta}$. Then*

$$E[\theta | r(\theta) \leq w] = E[\theta | \theta \geq r^{-1}(w)] > r^{-1}(w) \geq r^{-1}(\hat{\theta}) = \hat{\theta} \geq w$$

$$E[\theta | r(\theta) > w] > w$$

Thus, in every competitive equilibrium $w > \hat{\theta}$. This implies that, in every competitive equilibrium, $[\hat{\theta}, \bar{\theta}] \subset \Theta^*$. ■

Problem 2 Based on MWG 13.B.6 and Wilson (1980).

Consider the following extension of the adverse selection model. There is a mass N' of buyers, each of which wants to buy at most one car. The buyers differ in their willingness to pay for the car: a buyer of type γ has the willingness to pay $\gamma\theta$ for a car bought from a seller of type θ. Be aware that now each seller has some private information θ and each buyer has some private information γ. Assume that γ is distributed with a strictly positive density function g on $[0, \infty)$; let G denote the corresponding cumulative distribution function.

a. Let $z(p, \mu)$ denote the aggregate demand for cars when the price is p and the average quality of cars offered at price p is μ. Derive an expression for the function z in terms of G.

***Solution*:** *A buyer would be willing to buy a car iff $E[\gamma\theta - p | \text{car offered}] \geq 0$, i.e. if $\gamma\mu - p \geq 0$, or $\gamma \geq p/\mu$.*

$$z(p, \mu) = N \cdot [1 - G(p/\mu)]$$

■

b. Let $\mu(p) = E[\theta | r(\theta) \leq p]$ and define the aggregate demand for cars by $z^*(p) = z(p, \mu(p))$. Assuming that μ is differentiable, show that z^* is strictly increasing around a point $\bar{p}$ if, at $p = \bar{p}$, the elasticity of μ with respect to p exceeds 1, and is strictly decreasing if the elasticity is below 1. Interpret!

***Solution*:**

$$z^*(p) = N \cdot \left[1 - G\left(\frac{p}{\mu(p)}\right)\right]$$

Then

$$\begin{aligned}\frac{\partial z^*(p)}{\partial p} &= -N \cdot g\left(\frac{p}{\mu(p)}\right) \cdot \left[\frac{1}{\mu(p)} - \frac{p}{\mu(p)^2} \cdot \frac{\partial \mu(p)}{\partial p}\right] \\ &= -N \cdot g\left(\frac{p}{\mu(p)}\right) \cdot \frac{1}{\mu(p)} \cdot [1 - \varepsilon_\mu(p)]\end{aligned}$$

Where $\varepsilon_\mu(p)$ is the elasticity of μ with respect to p. Interpretation: when the elasticity is greater than 1, an increase in the price is more than compensated by an increase in quality μ, and thus the demand increases with price. ■

c. Assume that r is strictly increasing and continuous. Let $s(p) = N \cdot F(r^{-1}(p))$ denote the aggregate supply of cars, and define a competitive equilibrium price p^* by the equation $z^*(p^*) = s(p^*)$. Show that if there are multiple competitive equilibria, then the one with the highest price Pareto dominates all others.

***Solution*:**

- *For sellers, it is immediate to see that they are weakly better off with a higher price.*
- *For buyers not so clear: higher quality, but pay higher price. Let p_L and p_H be two competitive equilibrium prices with $p_H > p_L$. Since $r()$ is strictly increasing, $s(p)$ is increasing in p. Thus, $z^*(p_H) > z^*(p_L)$. There must be some buyers that were not willing to buy at the low price that are willing to but at the high price. I.e. there exists a γ such that*

  $$p_L/\mu(p_L) \geq \gamma \geq p_H/\mu(p_H)$$

  This implies that $p_L/\mu(p_L) \geq p_H/\mu(p_H)$.
- *Every type γ that was not buying at the low price either is not buying at the high price or is buying (in both cases it must be better off).*
- *We want to prove that if a buyer was buying at the low price ($\gamma \geq p_L/\mu(p_L)$), they have a higher payoff at the high price ($\gamma\mu(p_H) - p_H \geq \gamma\mu(p_L) - p_L$).*

- *We know that* $p_L/\mu(p_L) \geq p_H/\mu(p_H)$. *Rearranging and substracting* $p_L\mu(p_L)$ *side by side,*

$$p_L\mu(p_H) - p_L\mu(p_L) \geq p_H\mu(p_L) - p_L\mu(p_L)$$

- *Or,*

$$\frac{p_L}{\mu(p_L)} \geq \frac{p_H - p_L}{\mu(p_H) - \mu(p_L)}$$

- *So,* $\gamma > \frac{p_L}{\mu(p_L)}$ *implies* $\gamma \geq \frac{p_H - p_L}{\mu(p_H)-\mu(p_L)}$. *Rearranging we get the desired result.*

■

d. Consider a game-theoretic model in which buyers make simultaneous price offers. Show that (1) only the highest competitive equilibrium price can arise as a SPNE and (2) the highest-price competitive equilibrium p^* is a SPNE if and only if $z^*(p) \leq z^*(p^*)$ for all $p > p^*$.

***Solution*:**

1. *Consider two competitive prices* p_L *and* p_H *with* $p_H > p_L$. *We want to show that* p_L *cannot arise as a SPNE.*
   - *Suppose the equilibrium is* p_L. *There are nonowners that would be willing to purchase at* p_H *but not at* p_L.
2. *Suppose there is a* $p > p^*$ *such that* $z^*(p) > z^*(p^*)$. *Then again there are nonowners willing to buy at p that were not buying at* p^*. *This cannot be an equilibrium. If there is no such p, then all buyers are happy: if they choose a lower price they will not buy and that is strictly worse, if they choose a higher price they also do worse (requires proof).*

■

Problem 3

Consider the adverse selection model and assume that the distribution of θ is exponential with parameter λ.

a. Write down an expression for $E[\theta|\theta < \hat{\theta}]$.

***Solution*:**

$$E[\theta] = Pr(\theta < \hat{\theta})E[\theta|\theta < \hat{\theta}] + Pr(\theta > \hat{\theta})E[\theta|\theta > \hat{\theta}]$$

$$\frac{1}{\lambda} = (1 - e^{-\lambda\hat{\theta}})E[\theta|\theta < \hat{\theta}] + e^{-\lambda\hat{\theta}}\left(\hat{\theta} + \frac{1}{\lambda}\right)$$

$$E[\theta|\theta < \hat{\theta}] = \frac{1}{\lambda} - \frac{e^{-\lambda\hat{\theta}}\hat{\theta}}{1 - e^{-\lambda\hat{\theta}}}$$

■

b. Assume that $r(\theta) = \alpha\theta$.

i. For which α there exists a CE involving a complete market breakdown?

ii. For which α there exists a CE without a complete market breakdown?

***Solution*:**

$$E[\theta|r(\theta) < p] = E[\theta|\alpha\theta < p] = E[\theta|\theta < p/\alpha] = \frac{1}{\lambda} - \frac{p/\alpha}{e^{\lambda p/\alpha} - 1}$$

First derivative:

$$-\frac{\alpha(e^{\lambda p/\alpha} - 1) - \lambda p e^{\lambda p/\alpha}}{\alpha^2(e^{\lambda p/\alpha} - 1)^2}$$

The limit to $p = 0$ is indeterminate. We use l'Hopital's rule:

$$\frac{-\alpha\frac{\lambda}{\alpha}(e^{\lambda p/\alpha}) + \lambda e^{\lambda p/\alpha} + \lambda p\frac{\lambda}{\alpha}e^{\lambda p/\alpha}}{\alpha^2 2(e^{\lambda p/\alpha} - 1)\frac{\lambda}{\alpha}e^{\lambda p/\alpha}} = \frac{\lambda p\frac{\lambda}{\alpha}e^{\lambda p/\alpha}}{2\alpha\lambda(e^{\lambda p/\alpha} - 1)e^{\lambda p/\alpha}} = \frac{\lambda p e^{\lambda p/\alpha}}{2\alpha^2(e^{\lambda p/\alpha} - 1)e^{\lambda p/\alpha}}$$

The limit to $p = 0$ continues to be indeterminate. We use l'Hopital's rule again:

$$\frac{\lambda[e^{\lambda p/\alpha} + p\frac{\lambda}{\alpha}e^{\lambda p/\alpha}]}{2\alpha^2(\frac{2\lambda p}{\alpha}e^{2\lambda/\alpha} - \frac{\lambda}{\alpha}e^{\lambda p/\alpha})}$$

Evaluating at zero:

$$\frac{\lambda}{2\alpha^2(\frac{\lambda}{\alpha})} = \frac{1}{2\alpha}$$

Thus, if $\alpha > 1/2$ there is complete market breakdown, whereas if $\alpha < 1/2$ there is some trade. ■