The Timing of Complementary Innovations

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Abstract

Socially-valuable technologies sometimes require complementary innovations. This paper studies the development of innovations that exhibit such complementarity. At each point in time, resources are allocated across different innovation projects. The projects are completed stochastically in the form of breakthroughs and the social value of the technology depends on the set of completed projects by the time development ends. In some cases it is optimal to develop the innovations in sequence. In others, it is optimal to develop multiple innovations simultaneously. I provide conditions that determine the efficient timing of development: sequential development is efficient when development costs are high and there is high uncertainty about the innovations' rate of success. I compare the efficient timing of development to the equilibrium outcome with a decentralized industry in which many firms compete on the development of the innovations.

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1 Introduction

Patent laws and innovation policies aim to orient scarce resources toward the most socially valuable R&D projects. The value of an innovation, however, might be tied to the outcome of other developments. A particularly interesting case is complementary innovations, which have been increasingly relevant in industries such as telecommunications and biotechnology. For complementary innovations, the timing is relevant. Consider the case of hardware and software: The first classical computer algorithm was written in 1843,[1] while the first computer capable of running said algorithm wasn't developed until the 1930s. A similar process is taking place for quantum computing: Shor's quantum algorithm, a method for solving integer factorization problems in polynomial time, was written in 1994, four years before the first quantum computer prototype was developed. Today, startups and established companies alike invest resources to develop quantum software that can only be implemented with hardware that does not yet exist,[2] and it is not clear that it ever will. So what determines the timing in which complementary innovations are developed?

For some complementary innovations, the timing is dictated by an exogenous order in which the developments should succeed each other.[3] For other innovations, there is no exogenously imposed order, so the timing is determined endogenously by the allocation of resources to the different projects. My main research question in this paper is, how are resources endogenously assigned to complementary R&D projects? In particular, how does the environment (for example the level of competition or patent rights) affect the allocation of resources to the development of complementary innovations,

[1] By first classical computer algorithm, I mean an algorithm written for a classical computer that has no value given human computing power.

[2] The highest integer that has been factorized using Shor's algorithm is $21 = 7 \times 3$.

[3] For instance, no one was working on the can opener before the invention of the can—the can opener was invented decades after the can became popular. This order is natural, since the problems are very much related: a can opener cannot be invented without the specifications of the can.

and is the allocation efficient?

The key to determining the allocation of resources lies in assessing the prospects of each development. R&D projects carry high levels of uncertainty, both in terms of outcomes—the project may or may not prove successful—and in terms of costs— it is not clear how much time and resources will be needed to complete the development. This paper combines a dynamic of beliefs with the endogenous timing of development by introducing a tractable model that features the main aspects of the R&D process: A unit of attention is allocated over a set of projects at each point in time. A success for project i arrives discretely in the form of a breakthrough. In particular, a success arrives when the total amount of attention paid to a project reaches a certain level τ_i. Successes are observable, but τ_i is unknown.

In the first part of this paper, I study the *efficient* way to sequentially allocate attention to complementary R&D projects given the social value of innovations, which is realized when the development stage ends and is supermodular on the subset of the innovations that have been successfully developed by that time. The cost of development takes the form of a constant flow throughout the development stage.

Consider two complementary projects, A and B. How should society assign resources to them? Should they all be concentrated on A and then be switched to B if and only when A is successfully completed? Or should both A and B be developed in parallel? Moreover, when should a project be abandoned or put on hold? For complements, the success of one project makes it more attractive to keep paying attention to the remaining projects. Proposition 1 shows that this implies for two complements that all that matters for efficiency is how much is invested on each remaining project before abandoning. The intuition is that given the complementarities in payoff, the amount you are willing to work on a project if there is no new success is the minimum you are going to work on that project. Since you are going to do it independently of the outcome of other projects, *when* you do it is not payoff-relevant.

Section 4 considers the case where the rate of success for each project λ_i is constant over time but unknown. The beliefs about λ_i evolve with the outcomes of the development process. In particular, lack of success is evidence in favor of λ_i being relatively low, or in other words "the project i being relatively more challenging." The λs are independent across projects, so working on a project does not affect beliefs about the success rate of the others.

When the rate of success for each project is constant and known, the timing of development is irrelevant. Any project that is worth pursuing is worth completing; therefore, the order of completion is not going to affect the final expected payoff.[4] In contrast, when the rate of success is uncertain, the order of development is relevant, since it affects the arrival of information about the unknown parameters. The failure to develop A not only reduces the prospects of ever completing A, but also decreases the expected returns from completing B. The problem can therefore be thought as a *restless multi-armed bandit*, for which there is no general Gittins-like index rule that governs the optimal dynamic allocation.

Take as an example the case where project A is of uncertain feasibility, that is, the success rate is either zero or λ_A, and project B has a known success rate λ_B. In this case, it is efficient to first work on project A: there is no learning by working on B, so there is no efficiency loss in back-loading all development of B. Front-loading the development of A increases the speed of learning, which is valuable because of the option given by the stopping decision. In the more general setting, the intuition from this example also applies: the efficient allocation of resources reflects the optimal learning process about the potential of the joint project. Proposition 3 claims that for two perfect complements, the nature of the efficient allocation of resources depends on the uncertainty about the projects' difficulty and the flow cost of development. For projects with high uncertainty and high costs, it is efficient to concentrate all resources in one of the projects and thus develope them

[4]A similar logic holds when the completion time for each project is deterministic.

in sequence. The more uncertain the project, the earlier in the sequence it should be developed. For projects with low uncertainty and low costs, it is efficient to spread the resources following a simple "greedy" strategy that maximizes at each point in time the expected increment in value. In the case of symmetric projects, the greedy strategy splits resources equally at all times until either one the projects succeeds or the projects are abandoned.

In the second part of this paper, I study private allocation of the resources. The allocation of private R&D resources depends on several factors: who assigns these resources, the appropriability of the innovations—which is determined by the legal and patent systems—and how informed the agents are about a given project's successes. Section 4.2 analyzes the extreme case of *decentralized* industry.

A *decentralized industry* consists of a continuum of agents, each of whom controls an equal portion of the total unit of resource available at each moment in time. The agents don't consider the informational externalities that their actions generate. With substitute projects, decentralization and competition biases the allocation of resources toward fast, easy projects to the detriment of harder but cost-efficient ones. This *race effect* might be a concern also with complements: if a product requires two components, and it is efficient to start developing the hard one, competition might make it tempting to work on the easy component just to capture a higher share of the value generated. Section 4.2 shows that this is not the case: even if the first agent to succeed appropriates all the surplus from the joint development, the allocation of resources is not biased toward projects just because they are thought to be easier. Competition might introduce new inefficiencies by biasing the allocation toward projects where learning is slower. These inefficiencies disappear, however, if the stakes are sufficiently high.[5]

[5] This contrasts again with the case of substitutes, where higher stakes magnify the race effects. See Bryan et al. [2020].

1.1 Related Literature

Complementary Innovations: The main contribution is to the literature that studies research and development, and in particular complementary innovations. Scotchmer and Green [1990] and Ménière [2008] asks what is the optimal inventive requirements for a patent in the context of complementary innovations. Biagi and Denicolò [2014] study the optimal division of profits with complementary innovations. Fershtman and Kamien [1992] study the effects of cross licensing in the incentives to innovate. Bryan and Lemus [2017] don't study complements in particular, but the direction of innovation in a general setting that accounts for complementary innovations. In all these papers there is no learning in the development stage since the process is memoryless: successes arrives at an exponential time.

A particular type of complementary innovation is that sequential developments or cumulative innovation. Papers that study sequential developments include Gilbert and Katz [2011] and Green and Scotchmer [1995]. Moroni [2019] studies a contracting environment with sequential innovations. In these papers the timing of innovation is exogenously given. To the best of my knowledge, my paper is the first one to combine an endogenous timing of development with a learning process in the development stage in the context of innovation.

Dynamic information acquisition from multiple sources: There is a recent literature that study optimal information acquisition from multiple sources. With Poisson information structure, Nikandrova and Pancs [2018] studies the case of independent objects while Che and Mierendorff [2017] studies substitutes objects that are negatively correlated. Mayskaya [2019] also studies Poisson processes with more general dependencies and payoff function. Ke and Villas-boas [2019] study a similar situation to Nikandrova and Pancs [2018] but information comes through a Brownian Process. Although this paper is not about information acquisition, we could reformulate it in that way. Liang et al. [2018] asks the question of when is it optimal for a decision maker to acquire information in a myopic way. Liang and Mu [2020] compare efficient information acquisition versus what results from the choices of short-lived agents who do not internalize the externalities of their actions.

The reminder of the paper is as follows. Section 2 introduces the model. Section 3 shows when is it possible to solve the allocation problem by looking at a auxiliary problem. In Section 4, we focus on a canonical case of perfect complements with uncertain rates of success. In Section 4.1 the efficient allocation is characterized. In Section 4.2 we identify the inefficiencies generated by the private allocation of resources by a decentralized industry. Section 6 concludes.

2 Model

A decision maker (DM) can work on a set if two projects $K := \{A, B\}$. The DM must decide when to stop the research activity, and before that in which way to allocate resources across the projects. Time is continuous and each instant before stopping, the DM allocates a unit of resource across the projects that were not completed so far. Let $\alpha_i(t)$ be the amount of resource allocated to project i at time t and $C \subseteq K$ the set of projects completed so far, the restriction is $\sum_{i \in K \setminus C} \alpha_i(t) \leqslant 1$ for all t.

Each project is completed when the cumulative resources allocated to it reaches a certain amount τ_i. Project completion is observable but τ_i is unknown. The completion times are independent across projects with F_i the cdf of project i.

When the DM stops, the payoff derived from the innovations is $q(S)$ where $S \subseteq K$ is the set of projects that were completed. If no project was completed, the value is $q(\emptyset) = 0$.

Assumption 1. *q is increasing in the inclusion order, i.e. $q(T) \leqslant q(S)$ for all $T \subseteq S$.*

Wer are interested in complementary innovations.

Definition 1. *The projects of a set K are* complements *if the function q is supermodular, that is,*

$$q(A) + q(B) \leqslant q(\{A, B\})$$

The projects are perfect complements *if $q(A) = q(B) = 0$.*[6]

Using the resources is costly: there is a constant flow cost of c during the development stage, that is independent on which project the DM works on.[7] There is no discounting. The payoff of an DM that stops at time T and completed projects S by that time is $q(S) - c \cdot T$. The DM is an expected-payoff maximizer.

Strategies

Denote by $X_k(t) := \int_0^t \alpha_i(\tau)\, d\tau$ the amount of resources that the DM allocated to project k up to time t.

A *strategy* is a map from the set of histories to the attention vector. Given that no resources can be spent on completed projects, if we start from a set of completed tasks S and the vector of cumulative resources X, we can recover the completion times $\tau_i = X_i$ for $i \in S$. A *stationary strategy* only looks at the cumulative resources and the set of completed projects (it does not depend on the order in which resources were allocated so far). A stationary strategy thus consists on a vector field for each subset of developments. Formally,

Definition 2. *A* stationary strategy *is a function $x : 2^K \times R_+^2 \to [0, 1]^2$ such that $x_i(S, X) = 0$ for all $i \in S$ and*

$$\sum_{i \in K \setminus S} x_i(S, X) \leqslant 1 \qquad \forall X \in R_+^2 \quad \forall S \subseteq K.$$

[6]Substitutes are defined by q being submodular, and perfect substitutes by the property that $q(A) = q(B) = q(\{A, B\})$.

[7]This assumption is innocuous since we can normalize the time unit for different projects, by changing the distribution of τ, so that the cost is the same for all projects.

Let $\mathcal{X}$ be the set of stationary strategies.

3 Efficient Allocation

The problem of the decision maker is to choose a strategy to maximize their expected payoff. Given the assumptions, we can focus on stationary strategies. Start with a initial state (S, X), and take a strategy $x \in \mathcal{X}$ and a vector $\tau \in \mathbb{R}_+^k$ of realized project completion requirements. There is a deterministic extra time that stopping time $\tilde{T}(x, \tau)$ and set of completed tasks $\tilde{S}(x, \tau)$. The *allocation problem* is:

$$V(S, X) = \max_{x \in \mathcal{X}} \quad \mathbb{E}_\tau \left[q(\tilde{S}(x, \tau)) - c \cdot \tilde{T}(x, \tau) \,\middle|\, (S, X) \right]$$

This problem is complicated because the set of strategies is difficult to work with. A related problem is when the agent has to commit at each point how many extra resources to allocate to each of the remaining projects and to abandon if non of the remaining projects turns out successful. We are going to call this alternative problem the *order-independent problem.*

Formally, the order independent problem can be recursively defined as follows. Let $\hat{S}_X(\tau)$ be the set of completed projects at X, that is $\hat{S}_X(\tau) := \{i \in K : \tau_i \leqslant X_i\}$ and $D_{X,\hat{X}}(\tau)$ an indicator function that takes value 1 if there are no new completed projects from X to $\hat{X}$, i.e. $D_{X,\hat{X}}(\tau) := \mathbf{1}_{\{\hat{S}_{\hat{X}}(\tau) = \hat{S}_X(\tau)\}}$.

$$\hat{V}(S, X) = \max_{\hat{X} \geqslant X} \quad \mathbb{E}_\tau \left[W(X, \hat{X}, \tau) \; - \; c \; \sum_{i \in K} (\min\{\hat{X}_i, \tau_i\} - x_i) \,\middle|\, (S, X) \right]$$

where $W(X, \hat{X}, \tau) := D_{X,\hat{X}}(\tau) \cdot q(S) + [1 - D_{X,\hat{X}}(\tau)] \cdot \hat{V}(\hat{S}_{\hat{X}}(\tau), \hat{X} \wedge \tau)$.

Let $X^* : 2^K \times \mathbb{R}_+^k \to \mathbb{R}_+^k$ be the solution to the order-independent problem. $V \geqslant \hat{V}$ since the DM can always choose a strategy that mimics the solution to the order-independent problem. We are interested in conditions on q and F such that these two problems equivalent, i.e. $V = \hat{V}$.

When completing a project induces the agent to work more on all the remaining ones, the order in which the agent works on the projects is irrelevant modulo the cumulative work at the stopping points.

Proposition 1. *If the projects are complements, then $V = \hat{V}$. If the projects are not complements, there exists a family of distribution $\{F_i\}$ such that $V \neq \hat{V}$.*

Proof. in the Appendix A.1. □

Proposition 1 implies that for complements, and only for complements, it is possible to focus on finding the *optimal abandonment points*: how much the DM is willing to work on each of the remaining projects given the current state. This problem is much simpler because we don't have to worry about the specific order in which the agent works on the projects.s To emphasize the generality of the result, notice that we made no assumption on the distributions of completion times F_i. Thus the result holds for discrete time with arbitrary costs.[8]

Let $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the completion rate of task i, $h_i = F_i'/(1 - F_i)$. When failures depress the prospects with respect to a project, we can bound the value of the projects at the stopping time.

Lemma 1. *For $K = \{A, B\}$, if h_i is decreasing for both projects and the projects are complements, then $V(\cdot, X)$ is supermodular for all $X \in X^*(\emptyset, \mathbb{R}_+^k)$.*

Sketch of the proof: If at $(\emptyset, X)$ the DM wants to stop, it must be that

$$h_B(x_B)(V(B, X) - V(\emptyset)) \leqslant c$$

. But by complementary you would be willing to work on project B if project A was completed. Thus,

$$h_B(x_B)(V(B, X) - \underbrace{V(\emptyset, X)}_{=0}) \leqslant c \leqslant h_B(x_B)(V(\{A, B\}, X) - V(\{A\}, X))$$

[8]To see this just consider an F_i with mass probabilities at different times. The difference between the mass points can be interpreted as the cost of working on the project for an extra period.

$\square$

Lemma 1 says that supermodularity of $V(\cdot, X)$ holds for all X where the agent wants to stop before the first success. $V(\cdot, X)$ is not supermodular for all X. For instance, for q modular we have

$$V(\{A, B\}, X) = q(\{A, B\}) = q(A) + q(B) < V(A, X) + V(B, X).$$

Where the last strict inequality holds for all X where the DM would like to continue working in any of the projects. However, since for the modular case the stopping problem can be thought as

There are two class of stationary strategies that we are going to be interested in, as the candidates for the solution will sometimes belong to one of these classes. The first one is the family of strategies that always maximizes the expected value increment in each period.

The second family of stationary strategies concentrates all the resources in one of the projects, and only switches projects after a success.

Definition 3. *It is efficient to work on the projects* in sequence *if for every $X \in \mathbb{R}^2_+$ there exists a project i such that $X^*_i(\emptyset, X) - X_i = 0$.*

4 Uncertain rate of success

Consider perfect complements. Moreover, each project has a constant completion rate per unit of attention λ_i, unknown to the agent. The agent knows that the rate takes one of two possible values, $\lambda_i \in \{\lambda_i^L, \lambda_i^H\}$ and the rates are independent across projects, with $\Pr(\lambda_i = \lambda_i^H) = p_i$.[9] As attention is allocated to each project and these are not completed, the agent becomes more pessimistic about its difficulty. Let $p_i(X)$ be the probability of the rate for project i being λ^H when no project was completed after resources X where allocated. We are going to use $\delta_i := \frac{1}{2}(\lambda_i^H - \lambda_i^L)$ and $\bar{\lambda}_i := \frac{1}{2}(\lambda_i^H + \lambda_i^L)$, and normalize time so that $\frac{1}{2}(\bar{\lambda}_A + \bar{\lambda}_B) = 1$.

[9] Thus, $F_i = 1 - p_i e^{-\lambda_i^H} - (1 - p_i)e^{-\lambda_i^L}$.

Definition 4. *The projects are* symmetric *if* $\lambda_A^H = \lambda_B^H$ *and* $\lambda_L^H = \lambda_B^L$. *If projects are symmetric, we say that project* i *is the* most promising project *at the state* $(\emptyset, X)$ *if* $p_i(X) > p_j(X)$.

Notice that symmetry is not requiring the initial beliefs p_A and p_B to be the same.

4.1 Efficient allocation

Claim 1. *If one of the projects has a known rate of success* ($\delta_A = 0$) *it is efficient to work on the projects in sequence, starting with the project with uncertain rate (project* B*).*

The intuition is as follows: there is no learning by working on the project with known rate. So, for any strategy with $X_i^*(\emptyset, X) > 0$, the expected return of the same strategy with $X_i^*(\emptyset, X) = 0$ is weakly larger.

The next proposition tell us that the nature of the optimal strategy depends on measure that is increasing in the normalized cost and the uncertainty about the underlying success rate.

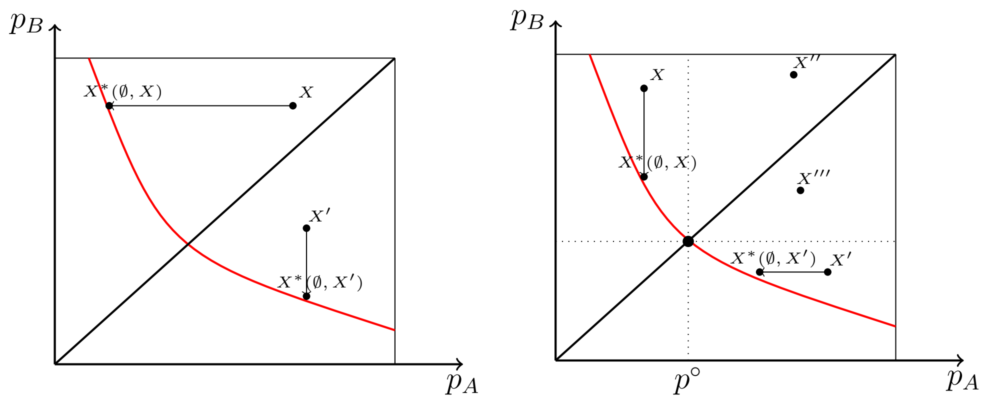
Proposition 2. *For symmetric projects, let* $g := 2\frac{c}{\gamma} + \delta^2$.

- *If* $g > 1$, *it is efficient to work on the projects in sequence, starting from the least promising project.*
- *If* $g < 1$, *it is efficient to work more on the more promising project:*

$$X_i^*(\emptyset, X) - X_i > X_j^*(\emptyset, X) - X_j \qquad \Leftrightarrow \qquad p_i(X) > p_j(X)$$

  Moreover, there exists a $p^\circ \in (0, 1)$ *such that if* $p_i(X) > p^\circ$ *for both projects then* $p_i(X^*(\emptyset, X)) = p^\circ$ *for both projects. If* $p_i(X) < p^\circ$ *for one of the projects then it is efficient to work on the projects in sequence, starting with the most promising one.*

Figure 1 shows the optimal allocation of resources before the first success for different priors, in belief space. The red curve represents the boundary of



(a) When $g > 1$ it is optimal to work on the projects in sequence, starting from the least promising one.

(b) When $g < 1$ it is optimal to work more on the more promising project.

Figure 1: Optimal allocation for symmetric projects in the first stage.

the set of points at which the agent is willing to stop before the first success $p(X^*(\emptyset, \mathbb{R}_+^2))$. In Figure 1a we see that it is optimal to work only on the project with lower prior. In Figure 1b, we see that if the initial prior $p_i > p^\circ$ for $i = A, B$, then it is optimal to work on both projects before stopping in the first stage. Thus there are multiple strategies whose trajectories arrive to that point. All these are payoff-equivalent by Proposition 1.

The result says that it is efficient to concentrate the resources (and therefore work in sequence) when the cost of development is sufficiently high, or the difference between the high and low rates is sufficiently large for both projects. The intuition is that in this case, having a single project that is difficult is sufficiently bad to abandon the joint project, so by concentrating the resources the DM gets to learn fast if this is the case. In the cost of development is sufficiently low, or the difference between the high and low rates is low for both projects, then it is optimal to work on the project simultaneously. In this case, having a single project that is difficult is not sufficient to stop.

We can formalize this intuition by interpreting the result in terms of optimal information acquisition. There are three possible states: both projects are easy ($\lambda_A = \lambda_B = \lambda^H$), both are hard ($\lambda_A = \lambda_B = \lambda^L$) or one is easy and the other one hard ($\lambda_A \neq \lambda_B$). For the decision problem to be interesting it must be that the DM would be willing to work on the projects if he knew that both are easy, and he does not want to work on the projects when both of them are hard.

Suppose that the DM would be willing to work on the projects if he knew that one was difficult and the other one was easy. Then the partition of the state space that is relevant for decision making is whether there is at least one easy project (continue) or both projects are hard (abandon).

The probability of the event 'at least one of the projects is easy' is $p^{\mathrm{OR}} = p_A + p_B - p_A \cdot p_B$. By assigning extra resources dX_i to project i and not succeeding, the change in p^{OR} is

$$\frac{dp^{\mathrm{OR}}}{dX_i} = (1 - p_j)\frac{dp_i}{dX_i} = -p_i(1 - p_i)(1 - p_j)2\delta$$

The fastest way to learn about the relevant state is to work on the project with highest p, and therefore to work on the projects simultaneously.

If the DM does not want to work when one of the tasks is hard and the other one is easy, the relevant state is whether there is a hard tasks or not. There is no hard project with probability $p^{\mathrm{AND}} = p_A \cdot p_B$. By working on project i for a period dt and not succeeding the change in p^{AND} is

$$\frac{dp^{\mathrm{AND}}}{dX_i} = p_j\frac{dp_i}{dX_i} = -p_i p_j(1 - p_i)2\delta$$

The fastest way to learn about the relevant state is to work on the project with lowest probability of success, and therefore to work on the projects in sequence.

When does the DM want to continue working when one of the tasks is difficult and the other one is easy? When the expected cost of completing both projects is less than the payoff from doing so, i.e.

$$\underbrace{\frac{c}{1+\delta} + \frac{c}{1-\delta}}_{\text{Expected cost}} < \underbrace{\gamma}_{\text{benefit}}$$

Rearranging we can see that this is equivalently to $g < 1$. Proposition 2 can be extended to asymmetric projects as follows.

Proposition 3. *Let* $g_i := 2\frac{c/\gamma}{\lambda_i} + \frac{1}{4}\left(\frac{\delta_i}{\lambda_i}\right)^2$,

1. *If* $g_i > 1$ *for both projects, then it is efficient to work on them in sequence.*

2. *If* $g_i < 1$ *for both projects, and* $\lambda_i^L > \lambda_j^H$ *then it is efficient to work on the projects in sequence, starting with the most promissing one.*

3. *If* $g_i < 1$ *for both projects and* $\lambda_i^H > \lambda_j^L$ *for* $i \neq j$*, then there exists a* $p \in (0,1)$ *such that if* $p_i(X) > p$ *for both projects, then for* $X^* = X^*(\emptyset, X)$*:*
$$\frac{h_A(X^*)}{V(K \setminus \{A\}, X^*)} = \frac{h_B(X^*)}{V(K \setminus \{B\}, X^*)}$$

Proof. in the Appendix B.1. Extra conditions for the case where $g_A < 1 < g_B$ can be found in Appendix C.2. □

4.2 Decentralized allocation

Research and development is sometimes carried away by the private sector. The level of competition and the appropriability of the innovations will affect the incentives and thus the allocation of resources to different projects.

Three important effects that arise with decentralized innovation in general and complementary innovation in particular. (1) *Effort duplication*: if agents don't know what other agents have tried and failed, it could be the case that something that was proven to not work is still tested. (2) *Hold-up problems*: agents might not be able to appropriate part of the value that innovations generate by increasing the value of complementary projects. (3) *Race effects*: the discrepancy in allocation caused by patent competition that necessarily rewards faster developments.

In this section we shut down the first two effects to focus on the race effect. We shut down the first effect by assuming that the timing of success depends on the total cumulative resources invested on a project and not on the identities of the people that invested in those projects. We shut down the second effect by assuming that the *winner-takes-it-all*: the first agent to complete one of the projects, say project i, receives the full-surplus from the remaining of the game $V(\{i\}, X)$ where $X \in \mathbb{R}^2_+$ is the current total resources allocated to each of the projects by all agents in the economy.

A concern is that with complements, the competition in development stage will bias the allocation of resources towards the projects that can be developed faster, and that these will be developed first even if it is efficient to leave these projects for later.

Decentralized industry model

There is a continuum of agents, each of who decides at each instant how to allocate an equal portion of the total resource among the different projects. We assume that the problem is non-trivial: if both projects are easy then it is efficient to work on them, and it is inefficient to stop if both projects were known to be difficult.[10]

Definition 5. *A* first-stage strategy *for agent $m \in [0, 1]$ is a function s_m : $\mathbb{R}^2_+ \to \{A, B, \emptyset\}$.*

The payoff for an agent m that was working on a project i at the moment in which this project was completed is proportional to the value of

[10]In terms of the parameters the condition is

$$\frac{\lambda_A^L \cdot \lambda_B^L}{\lambda_A^L + \lambda_B^L} < \frac{c}{\gamma} < \frac{\lambda_A^H \cdot \lambda_B^H}{\lambda_A^H + \lambda_B^H}.$$

that project $V(\{i\}, X)$ and inversely proportional to the mass of individuals working on that project at that point. Let $\pi(s, Y, x)$ be the expected payoff of following first-stage strategy s when the evolution of the process is Y and the aggregate strategy is x.

Definition 6. *A* first-stage stationary equilibrium *consists of a first-stage strategy for each agent, an industry allocation $x : \mathbb{R}_+^2 \to [0,1]^2$ and an evolution of cumulative resources $Y : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ such that*

1. *Each agent maximizes his expected payoff given the evolution of resources.*
$$s(X) \in \arg\max_{s' \in \mathcal{S}} \quad \pi(s, Y(\cdot, X), x(X))$$
2. *The industry allocation aggregates all individual strategies.*
$$x(X) = \int_0^1 s_m(X)\ dm$$
3. *The evolution of resources is consistent with the allocation of the industry.*
$$\begin{cases} \nabla Y(t, X) = x(Y(t, X)) \\ \quad Y(0, X) = X \end{cases}$$

Lemma 2. *Any stationary equilibrium satisfies*

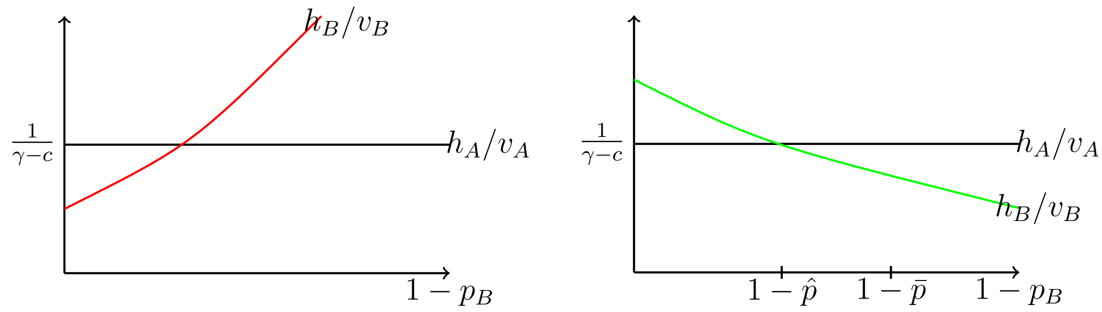
$$x_i(\emptyset, X) > 0 \qquad \Rightarrow \qquad h_i(X)V(\{i\}, X) \geqslant \max\{c, h_j(X)V(\{j\}, X)\}$$

Proof. in the Appendix B.1. □

Patent races might introduce distortions.

Definition 7. *There is a* race effect *when there is no equilibrium whose allocation is efficient.*

Proposition 4. *If the projects are symmetric or $g_i < 1$ for both projects, there is no race effect.*



(a) **Case I**: $\lambda_B^H < \lambda_A$.

(b) **Case II:** $\lambda_B^H > \lambda_A$ and $\bar{p} < \hat{p}$.

Figure 2: There is a race effect if the h_B/v_B is lower than h_A/v_A for a belief at which it is efficient to work.

If the projects are symmetric and $g > 1$, it is efficient to work on the projects in sequence, always starting with the least promising one. $g > 1$ implies that the hazard rate decreases faster than the value (in the amount of resources invested) so

$$h_i(X_i)v_j \geqslant h_j v_i(X_i) \qquad \Rightarrow \qquad h_i(X_i + \epsilon)v_j > h_j v_i(X_i + \epsilon)$$

and this implies that the equilibrium also works on the projects in sequence starting from the least promising one.

When $g_i < 1$ the value decreases faster than the hazard rate. It turns out that then it is efficient to work always on the project with highest hazard-to-value ratio. But this is exactly what the individuals do in equilibrium.

Individuals don't consider the informational externalities generated by their actions. Decentralization will bias the allocation of resources towards projects were learning is less relevant. For instance, consider the case where project A has a known rate of success λ_A. For project B the rate that is unknown. When is the decentralized allocation inefficient?

Proposition 5. *If A has a known success rate, there is a race effect if and only if $\lambda_B^H < \lambda_A$ and p_B is large enough.*

Proof. By Lemma 2, the equilibrium allocates the resources to the project with highest hazard-to-value ratio, where $v_i(X_i) = V(K \setminus \{i\}, X)$. Claim 1 says that it is efficient to work in sequence starting from project B.

Lets call $\hat{p}$ the posterior at which is efficient to stop working (interior by non-triviality of the problem). We can compute h_A/v_A, which is constant $\lambda_A/(\gamma - c/\lambda_A)$.

If h_B/v_B when the belief is close to 1 is below h_A/v_A, then there is a p such that it is optimal to work on B but the decentralized industry works in A. This is ilustrated in Figure 2a. h_B/v_B when p is close to 1 is $\lambda^H/(1 - c/\lambda^H)$, so the condition is

$$\frac{\lambda^H}{1 - c/\lambda^H} \leqslant \frac{\lambda_A}{\gamma - c/\lambda_A} \qquad \Leftrightarrow \qquad \lambda^H \leqslant \lambda_A$$

If h_B/v_B is higher than h_A/v_A at $p = 1$, the only way there could be an inefficiency is if h_B/v_B is decreasing and the stopping belief $\hat{p}$ is lower than the belief at which h_B/v_B and h_A/v_A intersect. This situation is ilustrated in Figure 2b. I now show that this is not the case:

By Lemma 1, at the stopping point X^*, $V(\{A\}, X^*) + V(\{B\}, X^*) \leqslant \gamma$. But $V(\{B\}, X^*) = \gamma - \frac{c}{\lambda_A}$, so $V(\{A\}, X^*) < c/\lambda_A$. Moreover at the stopping point, $h_B \cdot V(\{B\}, X^*) = c$, so

$$h_B(X^*)V(B, X^*) = c \geqslant \underbrace{h_A(X^*)}_{\lambda_A} v(A, X^*)$$

Rearranging we get

$$\frac{h_B(X)}{V(K \setminus \{B\}, X)} > \frac{h_B(X^*)}{V(K \setminus \{B\}, X^*)} \geqslant \frac{h_A(X^*)}{V(K \setminus \{A\}, X^*)} \qquad \forall X : X_B < X_B^*$$

Where the first inequality holds since $g_B < 1$. $\square$

The intuition that if a project is thought to be easier this would attract more attention to it is partially flawed. As the previous result shows, inefficiencies will show up if one of the projects has a higher rate of success than the other in every state, but they also require that the efficient to start project is thought to be as relatively easy.

5 Extensions

The DM can work on a finite set of projects $K := \{1, 2, ..., k\}$. As before, the DM allocates a unit of resource across the projects that were not completed so far $K \setminus C$. The vector of completion times τ is drawn from a distribution $\mu \in \Delta(\mathbb{R}^k_+)$. We will say that the projects are independent if the times of completion are independent across projects. Let F_i be the marginal distribution of project i.

Consider the case where the rate of completion are independent and constant, that is where $F_i(x) = 1 - e^{-\lambda_i x}$ for all $i \in K$. In this case, if a project is worth paying any attention, then it must be worth completing. Moreover, this implies that the order of allocation is payoff irrelevant. The follow proposition formalizes this result.

Proposition 6. *If the completion times are independent with $F_i(X) = 1 - e^{-\lambda_i X}$ then $V = \hat{V}$ for all q.*

In general, however, we cannot claim that the problems are equivalent even with complementary projects. The reason is that complementarity of the projects does not guarantee monotonicity of the solution to the problems (what was used to prove equivalence). A counterexample can be found in Appendix C.4.

Claim 2. *For $k > 2$, complementary projects is not sufficient for $\hat{V} = V$.*

6 Conclusion

Innovation is one of the main determinants of long-term economic growth. Thus, understanding the trade-offs in different approaches to innovation as well as the inefficiencies associated with economic environments is of central importance.

This paper makes substantive contributions to the understanting of the problem of development of complementary innovations:

First, the problem of efficient development of complementary innovations features different challenges than that of substitute innovations: failures in one development affect the expected returns from complementary innovations. With complementary innovations, however, successes make it more attractive to continue working on the remaining developments, what simplifies the problem substantially.

Second, allocating resources to innovation projects in an efficient way involves developing complementary innovations with a specific timing: sometimes it is efficient to develop in sequence and sometimes it is efficient to develop multiple innovations simultaneously. Sequential development is more likely to be efficient for high cost of development and higher uncertainty about the completion rates of the projects.

Third, an important part of innovation is carried away by the private sector. The timing of innovation is partly determined by the investment decisions of agents whose objectives are typically misaligned from the social welfare. Complementary innovations, generate investment dynamics that are different than for substitutes. In particular, the allocation of resources is not simply biased towards the easy and fast component to the detriment of the hard but cost-effective ones.

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A Omitted proofs from Section 3

A.1 Proof Proposition 1

For any strategy $x \in \mathcal{X}$ and initial state (S, X_0), there is a *trajectory* $y_S : \mathbb{R}_+ \times \mathbb{R}^k \to \bar{\mathbb{R}}^k$ that is the (unique) solution to the differential equations

$$\begin{cases} \nabla y_S(t, X_0) = x(S, y_S(t, X_0)) \\ \quad y_S(0, X_0) = X_0 \end{cases}$$

We will refer to $Y(S, X_0) = \lim_{t \to \infty} y_S(t, X_0)$ as the abandonment point of the strategy given an initial state (S, X_0).

Definition 8. *A strategy has* increasing abandonment points *if*

$$Y(S, X_0) \leqslant Y(\hat{S}, X_0) \textit{ for all } S \subseteq \hat{S}.$$

Definition 9. *Two strategies $x, \tilde{x}$* have the same abandonment points *if for each initial state, the abandonment point is the same for both strategies.*

Lemma 3. *If two strategies $x, \tilde{x}$ have the same abandonment points, and these abandonment points are increasing then the two strategies have the same expected payoff.*

The intuition for Lemma 3 is the following: if the abandonment is increasing, then the current abandonment point is the least attention you are willing to put on the remaining projects by the end of the day. Since the attention it is going to be paid eventually, the order in which the agent does it is not gonna determine the outcome.

Proof. The proof works by induction. The Lemma trivially holds for $k = 1$. Assume that it holds for $k = 1, 2, ..., m - 1$, we want to show that it holds for $k = m$.

Consider strategies $x, \tilde{x}$ and a initial state $(\emptyset, X_0)$. Let $Y(\emptyset, X_0)$ be the associated abandonment point. For each set $S \neq \emptyset$, the continuation problem is analogous to one with less than n projects, so the lemma holds. Let $V(S, X)$ be the value of the two strategies at the state (S, x) for $S \neq \emptyset$.

Consider a strategy z with the same abandonment points than x and such that for any $S \neq \emptyset$, $z(S, X) = x(\emptyset, X)$ for all X with $x(\emptyset, X) \neq 0$. We can do this since $Y(S, X_0) \geqslant Y(\emptyset, X_0)$. For any τ, the new strategy has the same payoff than the original: either no project is successful and both abandon at the same point or the same project is succesful at the same point, and the continuation value is the same.

Similarly, we can construct a strategy $\tilde{z}$ with the same abandonment points but such that for any $S \neq \emptyset$, $\tilde{z}(S, X) = \tilde{x}(\emptyset, X)$ for all X with $\tilde{x}\emptyset, X) \neq 0$. $\tilde{z}$ and $\tilde{x}$ shield the same payoff. We end the proof by showing that z and $\tilde{z}$ must also have the same payoff for any realization of the success times $\tau \in \mathbb{R}_+^k$.

Let $\bar{S} = \{i \in K : \tau_i < Y(\emptyset, X_0)\}$, that is the set of projects which completion time is below the abandonment point. Both z and $\tilde{z}$ reach $Y(\emptyset, X_0)$ with probability one. The payoff is therefore

$$V(\bar{S}, Y(\emptyset, X_0)) - c \sum_{i \in \bar{S}} \tau_i - c \sum_{i \notin \bar{S}} Y_i(\emptyset, X_0)$$

Taking expectation over the realization of τ completes the proof. $\square$

Proposition 1. Consider $k = 2$. If the projects are complements, then $V = \hat{V}$. If the projects are not complements, there exists a family of distribution $\{F_i\}$ such that $V \neq \hat{V}$.

($\Leftarrow$) We want to show that q supermodular implies that any strategy that has the same abandonment points than an optimal strategy is also optimal.

Lemma 4. *For two complements, any optimal strategy has increasing abandonment points.*

Proof. We want to prove that for any optimal strategy $Y_i(j, X_0) \geqslant Y_i(\emptyset, X_0)$. By supermodularity of q, the marginal value of i is larger when j was completed than when it is not. If it is optimal to work on project i when it is not clear if j is going to be completed or not, it must be optimal to work on i when j was already completed.

Formally, by contradiction assume $Y_i(j, X_0) < Y_i(\emptyset, X_0)$. Then there is a time t such that $y_{\emptyset,i}(t, X_0) = Y_i(j, X_0)$. Let $X := y_\emptyset(t, X_0)$. Consider the strategy that continues after X. If this strategy was copied starting on the state $(j, Y_i(j, X_0))$ with a dummy project j' that starts at x_j the result must be negative (otherwise it is worth continuing at $Y_i(j, X_0)$). But this strategy shields more than the continuation at X thus the project should stop X, so $X = Y(\emptyset, x_0)$ leading to a contraction. $\square$

Using Lemma 3, any strategy that has the same abandonment points than x is gonna get the same expected payoff and therefore be optimal.

($\Rightarrow$) : We prove by contrapositive. If q is *not* supermodular, there are cdfs $\{F_i, F_j\}$ such that $Y_i(j, X_0) < Y_i(\emptyset, X_0)$.

Proof. Since q is not supermodular, $q(\{i\}) > q(\{i, j\}) - q(\{j\})$. Let $F_i = 1 - e^{-\lambda_i x}$ with λ_i such that

$$q(\{i\}) > \frac{c}{\lambda_i} > q(\{i, j\}) - q(\{j\})$$

and let j never succeed, i.e. $F_j = 0$. Rearranging we have that

$$\lambda q(\{i\}) - c > 0 > \lambda(q(\{i, j\}) - q(\{j\})) - c$$

What implies that for any X_0, $Y_i(\emptyset, X_0) = \infty$ and $Y_i(\{j\}, X_0) = X_0$. $\square$

A.2 Proof of Lemma 1

Proof. h_i decreasing implies that $V(S, \cdot)$ is decreasing for all S.

If $X = X^*(\{A\}, X) = X^*(\{B\}, X)$, then $V(S, X) = q(S)$ and by complementarity of K, $V(\cdot, X)$ is supermodular. If $X \neq X^*(\{i\}, X)$ for some i,

$$\frac{\partial V(K \setminus \{i\}, X)}{\partial X_i} = c - h_i(X_i)\,[V(K, X) - V(K \setminus \{i\}, X)] \leqslant 0$$

At any point $X \in X^*(\emptyset, \mathbb{R}^k_+)$ it must be that $c \geqslant h_i(x_i) \cdot V(\{i\}, X)$. So,

$$h_i(X_i)V(\{i\}, X) - h_i(X_i)\left[V(\{A, B\}, X) - V(K \setminus \{i\}, X)\right] \leqslant 0 \qquad \forall X \in X^*(\emptyset, \mathbb{R}^k_+)$$

Rearranging,

$$V(\{B\}, X) + V(\{A\}, X) \leqslant V(\{A, B\}, X) \qquad \forall X \in X^*(\emptyset, \mathbb{R}^k_+)$$

□

B Ommitted proofs from Section 4

B.1 Proof of Propositions 2 and 3

Some preliminaries

Let δ_i be $\lambda_i^H - \lambda_i^L$. Using Bayes' rule, the beliefs $p_i(t)$ evolve

$$p_i(t) = \frac{p_i e^{-\delta_i t}}{(1 - p_i) + p_i e^{-\delta_i t}}$$

As the agent becomes more pessimistic, the subjective hazard rate $h_i(t)$ becomes lower.

$$h_i(t) = \lambda_L^i + p_i(t)\delta_i$$

Notice that

$$g_i > 1 \quad \Leftrightarrow \quad \frac{\lambda_i^L \cdot \lambda_i^H}{\lambda_i^L + \lambda_i^H} > c$$

We prove the alternative:

Proposition 3':

1. If $g_i > 1$ for both projects, then it is optimal to work on them in sequence.

2 If $g_i < 1$ for both projects, then the greedy strategy is optimal.

The proof of the proposition is split in three lemmatas. Lemma 5 proves that g_i controls the monotonicity of project i's hazard-to-value ratio. Lemma 6 shows that when both projects have an increasing hazard-to-value ratio, it is efficient to work on them in sequence. Lemma 7 shows that when both projects have a decreasing hazard-to-value ratio the greedy strategy is efficient.

Lemma 5. *h_i/v_i is monotone. Moreover,* $\operatorname{sgn}((h_i/v_i)') = \operatorname{sgn}(g_i - 1)$.

Proof. First we show that the monotonicity of h/v depends on weather the value v is higher or lower that an expression R.

$$\begin{aligned}\operatorname{sgn}((h_i/v_i)') &= \operatorname{sgn}(h_i'v_i - h_iv_i') \\ &= \operatorname{sgn}(h_i'v_i - h_i(c - h_i(1 - v_i))) \\ &= \operatorname{sgn}\left(\underbrace{\frac{h_i(h_i - c)}{h_i^2 + h_i'}}_{R(t)} - v_i\right)\end{aligned}$$

Change of variables. In belief space, the concavexity of R is determined by weather g_i is larger or lower than one.

$$\begin{aligned}\hat{R}'(p) &= \frac{2\delta^2\lambda_L\lambda_H(\lambda_L\lambda_H - c(\lambda_L + \lambda_H))}{(\lambda_L^2 + p\delta(\lambda_L + \lambda_H))^3} \\ &= M(g - 1)\end{aligned}$$

Now we consider two cases: $\lambda_L < c$ and $\lambda_L \geqslant c$.

Case I: $\lambda_L \geqslant c$ In this case, the agent would never stop. The value is linear in the beliefs.

$$v(p) = 1 - p\frac{c}{\lambda_H} - (1 - p)\frac{c}{\lambda_L}$$

Since $v(0) = R(0)$ and $v(1) = R(1)$,

$$g > 1 \qquad \Leftrightarrow \qquad v(p) < R(p) \qquad \forall p \in (0, 1)$$

Case II: $\lambda_L < c$ In this case, the agent abandons if sufficient time passes with no success. v is convex (information is valuable) and R is concave:

$$\lambda_L < c \qquad \Rightarrow \qquad \frac{\lambda_H}{\lambda_L + \lambda_H}\lambda_L < c \qquad \Leftrightarrow \qquad g_i > 1$$

Since $v(1) = R(1)$ and $v(\hat{p}) = R(\hat{p})$ where $\hat{p}$ is the stopping belief.

$$v(p) < R(p) \qquad \text{for any } p \in (\hat{p}, 1)$$

$\square$

Lemma 6. *If h_i/v_i is strictly increasing for $i = A, B$, it is optimal to work on the projects in sequence.*

Proof. By contradiction. Assume that $x := Y(\emptyset, x_0) > 0$. Let $r_i(t) := \frac{h_i(t)}{v_i(t)}$ and $g_i(t) := \frac{h_i'(t)}{v_i'(t)\cdot r_i(t)}$. Since x is an interior stopping point, it must be that $h_A(x_A)v_B(x_B) = h_B(x_B)v_A(x_A) = c$.

Abusing notation I write f_i instead of $f_i(x_i)$. First we show that $r_A' + r_B' > 0$ implies $\frac{h_A' v_A}{h_A v_A'} \cdot \frac{h_B' v_B}{h_B v_B'} < 1$.

$$\begin{aligned} r_A' + r_B' > 0 \qquad &\Leftrightarrow \qquad \frac{h_A' v_A - h_A v_A'}{v_A^2} + \frac{h_B' v_B - h_B v_B'}{v_B^2} > 0 \\ &\Leftrightarrow \qquad \frac{h_A v_A'}{v_A^2}\left(\frac{h_A' v_A}{h_A v_A'} - 1\right) + \frac{h_B v_B'}{v_B^2}\left(\frac{h_B' v_B}{h_B v_B'} - 1\right) > 0 \end{aligned}$$

For all (x_A, x_B) such that $h_A v_B = h_B v_A = c$,

$$\frac{h_A v_A'}{v_A^2} = \frac{h_B v_A'}{v_B v_A} = \frac{h_A v_B'}{v_A v_B} = \frac{h_B v_B'}{v_B^2}$$

Where the first and last equality use $h_A/v_A = h_B/v_B$ and the intermediate one uses that $h_B v_A' = h_B(c - h_A(1 - v_A)) = -h_B h_A(1 - v_A - v_B)$ (since $c = h_A v_B$) and equal to $h_A v_B'$ by symmetry. So,

$$\begin{aligned} r'_A + r'_B > 0 \quad &\Leftrightarrow \quad \frac{h_A v'_A}{v_A^2}\left[\left(\frac{h'_A v_A}{h_A v'_A} - 1\right) + \left(\frac{h'_B v_B}{h_B v'_B} - 1\right)\right] > 0 \\ &\Leftrightarrow \quad \left[\frac{h'_A v_A}{h_A v'_A} + \frac{h'_B v_B}{h_B v'_B}\right] < 2 \\ &\Leftrightarrow \quad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} < 1 \end{aligned}$$

Where the second implication uses that v_A is decreasing. And the last one is that the sum of two positive numbers being less than two implies that the product is less than one.

The determinant of the Hessian for the value function $V(\emptyset, x)$ is

$$\det(H) = (1 - F_A)(1 - F_B)[h'_A h'_B v_A v_B - h_A h_B v'_A v'_B]$$

So

$$\det(H) < 0 \qquad \text{iff} \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} < 1$$

And $\det(H) < 0$ rules out the candidate as an optimum (saddle point). □

Lemma 7. *If h_i/v_i is strictly decreasing for $i = A, B$, the greedy strategy is optimal.*

Proof.

$$\begin{aligned} r_i \searrow \quad &\Leftrightarrow \quad h'_i v_i - h_i v'_i < 0 \\ &\Leftrightarrow \quad \frac{h'_i v_i}{h_i v'_i} > 1 \end{aligned}$$

So,

$$r_A \searrow \text{ and } r_B \searrow \quad \Rightarrow \quad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} > 1$$

This implies that there is at most one interior candidate for solution ($h_A v_B = h_B v_A = c$), and that if it exist it is the actual solution. We consider the two cases.

Case I: there is an interior solution candidate. Then this is the actual solution. Since at the solution $r_A = r_B$ and the r_i are decreasing, by working always on the project with highest r_i, the point is eventually reached.

Case II: there in no interior solution candidate. Then it must be that $h_i v_j = c \Rightarrow h_j v_i > c$. Thus, the solution is to work in sequence starting with project j. Moreover, $h_j/v_j > h_i/v_i$ for all x such that $h_i v_j \geqslant c$, so the greedy strategy also works in sequence starting with j. $\square$

Proof of Lemma 2

Proof. Individual profits are

$$\int_0^t \underbrace{\frac{1 - F_A(Y_X(\tau))}{1 - F_A(X_A)} \cdot \frac{1 - F_B(Y_X(\tau))}{1 - F_B(X_B)}}_{\Pr(\text{reach } \tau)} \left[\underbrace{x_{s(Y_X(\tau))}(X) \cdot h(Y_X(\tau))}_{\text{rate success at } \tau} \cdot \underbrace{\frac{V(s(Y_X(\tau)), Y_X(\tau)}{x_{s(Y_X(\tau))}(X)}}_{\text{expected payoff if successful}} - c \right] d\tau$$

Since the individuals do not have a marginal effect on the trajectory and take it as given, the way to optimize the individual profits is every instant.

$$s \in \arg \max_{s' \in \{A,B,\emptyset\}} \quad \{h_{s'}(Y_X(\tau))V(s', Y_X(\tau)) - c\}$$

$\square$

C Extensions: proofs

Proof of Proposition 6

Proof.

$$V(S,x) = \hat{V}(S,x) = \max_{\hat{S} \in 2^{K \setminus S}} \left\{ q(S \cup \hat{S}) - c \sum_{i \in \hat{S}} \lambda_i^{-1} \right\}$$

$\square$

C.1 Discrete time

In this appendix we will consider the discrete time case $T = \{1, 2, 3, ..., \infty\}$. At any time before stopping the agent decides which project to work on $\alpha_t \in \{A, B, \emptyset\}$. Let F_i be the distribution of successes for project i, and $h_i : T \to [0, 1]$ the respective hazard rate. Finally, let $v_i : T \to [0, 1]$ the value of the joint project when only project i is incomplete as a function of the time spent working on project i.

$$v_i(x_i) := q(j) + \max_{T \geqslant x_i} \left\{ \sum_{x=x_i+1}^{T} \frac{1 - F(x)}{1 - F(x_i)} \left[h(x)(1 - q(j)) - c\right] \right\}$$

Proposition 7. *h_i/v_i decreasing for both projects implies that the greedy strategy is efficient.*

Proof. Grab an optimal abandonment point $x^* := Y(\emptyset, 0)$ and a trajectory to it. The trajectory has to be greedy at the time before the abandonment point. Otherwise, the optimality of x^* is violated.

Consider now a greedy trajectory and the point (x_L, x_B^*) in that trajectory where crosses $x_B = x_B^*$ (the rightmost one). If the optimum is to the right of the path ($x_A^* > x_L$) then by optimality,

$$\frac{h_A}{v_A}(x_L) \geqslant \frac{h_i}{v_i}(x_i^* - 1) \geqslant \frac{h_j}{v_j}(x_j^*)$$

If $x_L = x_i^* - 1$ then the first inequality holds with equality and there is a greedy path to the optimum: the one we considered changing at the indifferent point (x_L, x_A^*). If $x_L < x_i^* - 1$ then by strict monotonicity of h/v the inequality holds strictly, what would violate greediness of the strategy at (x_L, x_A^*) □

Proposition 8. *h/v increasing for both tasks implies that the efficient allocation is in sequence.*

Proof. Suppose that the optimal stopping point $x^* = Y(\emptyset, 0)$ is interior, i.e. $x^* > 0$. Since last period is myopically optimal for each trajectory,

$$\frac{h_A}{v_A}(x_A^* - 1) \geqslant \frac{h_B}{v_B}(x_B^*) > \frac{h_B}{v_B}(x_B^* - 1)$$
$$\frac{h_B}{v_B}(x_B^* - 1) \geqslant \frac{h_A}{v_A}(x_A^*) > \frac{h_A}{v_A}(x_A^* - 1)$$

Where the strict inequalities come from the h/v being increasing for both projects. Thus, a contradiction. □

C.2 One h/v increasing and one decreasing

Lemma 8. *If the horizontal sum of the two h/v is increasing, then it is optimal to develop the projects in sequence.*

Proof. Consider $q(y) := (h_A/v_A)^{-1}(y) + (h_B/v_B)^{-1}(y)$ decreasing for all $y \in R := (h_A/v_A)([0, \bar{t})) \cap (h_B/v_B)([0, \bar{t}))$. Taking the derivative this implies that

$$\frac{1}{(h_A/v_A)'((h_A/v_A)^{-1}(y))} + \frac{1}{(h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$\frac{(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y))}{(h_A/v_A)'((h_A/v_A)^{-1}(y)) \cdot (h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y)) > 0 \qquad \forall y \in \mathbb{R}$$

Or, in other words: $r_A'(x_A) + r_B'(x_B) > 0$ for all points (x_A, x_B) with $h_A(x_A)/v_A(x_B) = h_B(x_B)/v_B(x_B)$. We can use the same logic used in the proof of Lemma 6 to rule out interior points. □

C.3 Imperfect complements

$\lambda_L > c/(1-q)$ then the agent would never stop. The value is independent of q and linear. The monotonicity of h/v is equivalent to the case where $q = 0$.

Consider now $\lambda_L \in (c, c/(1-q))$. There is a belief at which the agent stops.

$$\hat{p} = \frac{c/(1-q) - \lambda_L}{\delta}$$

If $R(\hat{p}) > v(\hat{p}) = q$ and R is concave, h/v is increasing.

$R(\hat{p}) > q$

$$\frac{c^2 q}{(1-q)[c(\lambda_L + \lambda_H) - (1-q)\lambda_L \lambda_H]} > q$$

Interesting case: $[c(\lambda_L + \lambda_H) - (1-q)\lambda_L\lambda_H] > 0$.

$$\left(\frac{c}{(1-q)}\right)^2 \geqslant \frac{c}{(1-q)}(\lambda_L + \lambda_H) - \lambda_L\lambda_H$$

$$\frac{c}{(1-q)}\left(\frac{c}{(1-q)} - \lambda_L\right) \geqslant \lambda_H\left(\frac{c}{(1-q)} - \lambda_L\right)$$

$$\frac{c}{(1-q)} \geqslant \lambda_H$$

But if this is the case, then the agent does not wish to work on the development even when sure that it is relatively easy.

C.4 Supermodularity not sufficient for Lemma 3 with $k > 2$

That q supermodular implies increasing abandonment points does not hold in general for $k > 2$. Here is a counterexample:

Let $K = \{A, B, C\}$. Suppose $q(\{A, B\}) = \gamma < q(\{A, B, C\}) = 1$. $q(S) = 0$ for any subset. And suppose C is either feasible or infeasible, and that you can learn instantly about it. $\lambda_L^C = 0, \lambda_H^C = \infty$. The optimal strategy is to learn about C, and then doing the optimal thing for A and B (that might be different depending on whether C is completed or not).

In the case where

$$c < \frac{\lambda_L \lambda_H}{\lambda_L + \lambda_H} < \frac{c}{\gamma}$$

then by results when C is completed it is optimal to work simultaneously, $Y_i(\{C\}, 0) > 0$ for $i = A, B$. But when C fails, it is optimal to work in sequence, so $Y_i(\emptyset, 0) = 0$ for $i \in \{A, B\}$.