Advanced Microeconomics III

Mechanism Design - 2

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Introduction

- Consider an auction but where the revenue-collector is included as an "agent zero".
- The revenue-maximizing scf is a constrained Pareto efficient allocation!
  - It's not possible to increase the payoff of agents without decreasing the payoff of agent zero.
- In general, the set of constrained Pareto efficient scf is difficult to characterize.
- We ask instead: is there a constrained efficient scf that is Pareto-efficient?

Efficient Mechanisms

- Quasilinear environment with private values:
    - Set of agents I.
    - Types Θ_i.
    - Preferences given by $u_i(x, \theta_i) - t_i$.

- We say that an allocation rule $\alpha : \Theta \to X$ is *efficient* iff

$$\alpha(\theta) \in \arg\max_{x \in X} \sum_{i \in I} u_i(x, \theta_i) \qquad \forall \theta \in \Theta.$$

Efficient Mechanisms

- We analyze mechanisms that implement an efficient allocation.
- As before, by the Revelation Principle, we restrict attention to DRM.
- We fix an efficient allocation α^* and consider DRMs that differ only in the transfer rule.

- Other properties that are interesting beyond efficiency:
  - Incentive Compatibility (BIC and DSIC)
  - Voluntary participation ("individual rationality")
  - That no money is required to run the mechanism ("Budget-balanced").
- Usually, there is tension between the last two properties.

Example

- Consider the DRM (α^*, t), such that
  - Each agent **receives** a transfer equivalent to the sum of other agents' allocation payoffs.

$$t_i(\theta) = -\sum_{j \neq i} u_j(\alpha^*(\theta), \theta_j)$$

- This DRM is DSIC.

- Given others' reports θ_{-i}, agent's i problem is:

$$\max_{\hat{\theta}_i \in \Theta_i} \quad u_i(\alpha^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + \underbrace{\sum_{j \neq i} u_j(\alpha^*(\hat{\theta}_i, \theta_{-i}), \theta_j)}_{t_i(\hat{\theta}_i, \theta_{-i})}$$

- or

$$\max_{\hat{\theta}_i \in \Theta_i} \quad \sum_{k \in I} u_k(\alpha^*(\hat{\theta}_i, \theta_{-i}), \theta_k)$$

- Since α^* is efficient,

$$\sum_{k \in I} u_k(\alpha^*(\theta), \theta_k) \geq \sum_{k \in I} u_k(\alpha^*(\hat{\theta}_i, \theta_{-i}), \theta_k) \qquad \forall\ \hat{\theta}_i \in \Theta_i$$

- Thus, it is a dominant strategy to report truthfully.
- **Problem**: this requires large positive transfers to participants. i.e. to 'run a deficit'.

VCG Mechanisms

- **VCG mechanisms** (due to Vickrey, Clark and Groves) is a family of DRM in which the allocation α^* is efficient and the transfer is given by

$$t_i(\theta) = \sum_{j \neq i} u_j(\alpha^*(\bar{\theta}_i, \theta_{-i}), \theta_j) - \sum_{j \neq i} u_j(\alpha^*(\theta), \theta_j)$$

  where $\bar{\theta}_i$ is a **default type** for player i.

- **Intuition**: each agent pays *what other agents can achieve without i* minus *what the other agents get if i is present*. In other words, the externality imposed on others.

Application: VCG in auctions

- Let $\Theta_i = [0,1]$ for all i.
- let $\bar{\theta} = (0, 0, ..., 0)$.

- **Efficient allocation**: assign the object to the agent that values the item the most.

- VCG payments are:

$$
\begin{aligned}
t_i(\theta) &= \sum_{j \neq i} u_j(\alpha^*(\bar{\theta}_i, \theta_{-i}), \theta_j) - \sum_{j \neq i} u_j(\alpha^*(\theta), \theta_j) \\
&= \begin{cases} \max_{j \neq i} \theta_j & \theta_i > \theta_j \quad \forall j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

- The VCG mechanism is equivalent to the second price auction.

Application II: Binary public good provision

- Let $X = \{0, 1\}$ and $I = \{1, 2\}$.
- Let $\Theta_i = [0, 1]$.
- Let $u_i(x, \theta_i) = x \cdot (\theta_i - \frac{c}{2})$
- c: cost of providing the public good.

- The efficient allocation is:

$$\alpha^*(\theta) = \begin{cases} 1 & \text{if } \theta_1 + \theta_2 \geq c \\ 0 & \text{otherwise} \end{cases}$$

Application II: Binary public good provision

- VCG transfers with $\bar{\theta}_i = 0$.

$$\begin{aligned} t_i(\theta) &= u_j(\alpha^*(0, \theta_{-i}), \theta_j) - u_j(\alpha^*(\theta), \theta_j) \\ &= 0 - \theta_j \cdot 1_{\{\theta_i + \theta_j > 1\}} \end{aligned}$$

- VCG transfers with $\bar{\theta}_i = 1$.

$$\begin{aligned} t_i(\theta) &= u_j(\alpha^*(1, \theta_{-i}), \theta_j) - u_j(\alpha^*(\theta), \theta_j) \\ &= \theta_j - \theta_j \cdot 1_{\{\theta_i + \theta_j > 1\}} = \theta_j \cdot 1_{\{\theta_i + \theta_j < 1\}} \end{aligned}$$

VCG is DSIC

For any profile of default types $\bar{\theta}$, the VCG mechanism is DSIC.

Proof.

$$\begin{aligned} U_i(\theta) &= u_i(\alpha(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta_i, \theta_{-i}) \\ &= u_i(\alpha(\theta_i, \theta_{-i}), \theta_i) - C(\bar{\theta}_i, \theta_{-i}) + \sum_{j \neq i} u_j(\alpha(\theta_i, \theta_{-i}), \theta_j) \\ &\geq u_i(\alpha(\hat{\theta}_i, \theta_{-i}), \theta_i) - C(\bar{\theta}_i, \theta_{-i}) + \sum_{j \neq i} u_j(\alpha(\hat{\theta}_i, \theta_{-i}), \theta_j) \end{aligned}$$

Where

$$C(\bar{\theta}_i, \theta_{-i}) := \sum_{j \neq i} u_j(\alpha^*(\bar{\theta}_i, \theta_{-i}), \theta_j)$$

Application III: Bilateral Trade

- Two agents (*buyer* and *seller*): $I = \{B, S\}$.
- Single object. $X = \{\text{trade}, \text{no trade}\}$
- Buyer's valuation is θ_b. Seller's cost is θ_s.

$$u_s(x, \theta_s) = -1_{\{x=\text{trade}\}} \cdot \theta_s \qquad u_b(x, \theta_b) = 1_{\{x=\text{trade}\}} \cdot \theta_b$$

- **Efficient allocation rule**:

$$\alpha^*(\theta) = \begin{cases} \text{trade} & \text{if } \theta_b > \theta_s \\ \text{no trade} & \text{if } \theta_b < \theta_s \end{cases}$$

Application III: Bilateral Trade

- We construct a VCG mechanism (α^*, t) with default types $\bar{\theta}_b = 0$ and $\bar{\theta}_s = 1$.

- In this mechanism,
  - When there is no trade $(\theta_b < \theta_s)$,

$$t_b(\theta) = t_s(\theta) = 0$$

  - When there is trade $(\theta_b \geq \theta_s)$,

$$t_b(\theta) = \theta_s \qquad \text{and} \qquad t_s(\theta) = -\theta_b$$

- This VCG mechanism incurs a **deficit** whenever there is trade.

Participation and Budget-balanced condition

- **Voluntary participation** (IR):

$$U_i(\theta) \geq 0 \qquad \forall i \in I.$$

  - This is an **ex-post** notion.
  - Each agent-type is happy to participate for **all types of others**.

- **Budget-balanced condition (BB)**:

$$S(\theta) := \sum_{k \in I} t_k(\theta) = 0 \qquad \forall \theta \in \Theta$$

  - This is an **ex-post** notion, in the belief-free spirit of DSIC.
  - If $S(\theta) \geq 0$ for all θ we say that the mechanism *never runs a deficit*.
    - **Note**: this equivalent to voluntary participation of an "agent zero".

Myerson-Satterthwaite

In the bilateral trade environment there is no mechanism that is efficient, DSIC, satisfies voluntary participation, and that is budget-balanced.

Proof.

- Deferred for later.

□

- Comments:
  - The result generalizes to environments with asymmetric supports, as long as the supports intersect.
  - Inefficiencies persist but gets smaller quickly when the number of traders on each side of the market increases.

Beyond Dominant Strategies

- Budget-balanced is a strong condition. Sometimes, we are interested in the mechanism not running a deficit in expectation.
- Likewise, IR requires that all agents are happy to participate in the mechanism ex-post. Sometimes, we are interested in interim incentives to participate (IIR).
- Finally, in Bayesian environments, we can relax our solution concept to Bayesian implementation.

Expected Externality Mechanism (AGV)

- **Idea**: instead of making each player pay the realized externality imposed on others, make them pay the **expected** externality.
- Consider a Bayesian environment with independent types.
- **AGV mechanism** (Arrow and d'Aspremont and Gerard-Varet): DRM (α^*, t^{AGV}) in which α^* is efficient and the payment rule is given by

$$t_i^{AGV}(\theta) = \frac{1}{N-1} \sum_{j \neq i} \tilde{t}_j(\theta_j) - \tilde{t}_i(\theta_i)$$

where

$$\tilde{t}_i(\theta_i) = E_{\theta_{-i}} \left[\sum_{j \neq i} u_j(\alpha^*(\theta), \theta_j) \right]$$

AGV is budget-balanced

Observe that, for all θ,

$$\sum_{i \in I} t_i^{AGV}(\theta) = \sum_{i \in I} \left[\frac{1}{N-1} \sum_{j \neq i} \tilde{t}_j(\theta_j) - \tilde{t}_i(\theta_i) \right]$$
$$= \frac{1}{N-1} \underbrace{\sum_{i \in I} \sum_{j \neq i} \tilde{t}_j(\theta_j)}_{(N-1) \sum_{i \in I} \tilde{t}_i(\theta_i)} - \sum_{i \in I} \tilde{t}_i(\theta_i)$$
$$= 0$$

AGV is BIC

- If i tells the truth, she gets an interim utility

$$
\begin{aligned}
U_i(\theta_i) &= E_{\theta_{-i}}\left[u_i(\alpha^*(\theta),\theta_i) - t_i^{AGV}(\theta)\right] \\
&= E_{\theta_{-i}}\left[u_i(\alpha^*(\theta),\theta_i) + E_{\theta_{-i}}\left[\sum_{j\neq i} u_j(\alpha^*(\theta),\theta_j)\right]\right] \\
&\quad \underbrace{- \frac{1}{N-1} E_{\theta_{-i}}\left[\sum_{j\neq i} u_j(\alpha^*(\theta),\theta_j)\right]}_{\text{constant}}
\end{aligned}
$$

- By the efficiency of α^*, $\sum_{k\in I} u_k(\alpha^*(\theta),\theta_k) \geq \sum_{k\in I} u_k(x,\theta_k) \quad \forall x$.
- Hence, truthful reporting is a BNE.

AGV may not be IIR

- There is nothing that guarantees that IIR is satisfied.

- There might be an agent i and type θ_i such that AGV involves

$$U_i(\theta_i) < 0$$

Generalized VCG

- We allow for type dependent outside options $\underline{U}_i(\theta_i)$.
- Given the efficient allocation rule α^*, let

$$\bar{\theta}_i \in \arg\min_{\theta_i \in \Theta_i} E_{\theta_{-i}} \left[\sum_{j=1}^{N} u_j(\alpha^*(\theta_i, \theta_{-i}), \theta_j) - \underline{U}_i(\theta_i) \right]$$

- We refer to $\bar{\theta}_i$ as the *least charitable type* of agent i.

Generalized VCG

- **GVCG**: DRM in which the allocation rule α^* is efficient and the payments are given by:

$$t_i^{GVCG}(\theta) = \sum_{j \neq i} u_j(\alpha^*(\bar{\theta}_i, \theta_{-i}), \theta_j) + u_i(\alpha^*(\bar{\theta}_i, \theta_{-i}), \bar{\theta}_i) - \sum_{j \neq i}^{N} u_j(\alpha^*(\theta_i, \theta_{-i}), \theta_j) - \underline{U}_i(\bar{\theta}_i)$$

Theorem (Krishna-Perry)

The GVCG mechanism is IIR and BIC, and maximizes the expected surplus among all mechanisms that are IIR, BIC and implement the efficient allocation rule.

Proof - BIC

Exercise.

Proof - IIR

Exercise.

Proof - Revenue Maximizing

- Consider a mechanism Γ that is efficient, BIC, and IIR.
- Let $\hat{u}_i : \Theta_i \to \mathbb{R}$ be the payoff function for agent i in Γ. By payoff equivalence:

$$\hat{U}_i(\theta_i) - \hat{U}_i(\bar{\theta}_i) = U_i(\theta_i) - U_i(\bar{\theta}_i)$$

- Where u_i is the utility of agent i in the GVCG mechanism.
- Notice that

$$\hat{U}_i(\bar{\theta}) \geq \underline{U}_i(\bar{\theta}_i) = U_i(\bar{\theta}_i)$$

  Where the inequality holds by IIR of Γ.
- Then,

$$\hat{U}_i(\theta_i) - \hat{U}_i(\bar{\theta}_i) \geq U_i(\theta_i) - U_i(\bar{\theta}_i)$$

Proof - Revenue Maximizing

- Thus,

$$\hat{U}_i(\theta_i) \geq U_i(\theta_i)$$

- Since the allocation for Γ and GVCG is the same, it must be that

$$E_{\theta_{-i}}\left[t_i^{\Gamma}(\theta)|\theta_i\right] \leq E_{\theta_{-i}}\left[t_i^{GVCG}(\theta)|\theta_i\right] \qquad \forall\theta_i \in \Theta_i$$

- Taking expectation over θ_i and adding up all types we get the desired result.

Proof - Myerson-Satterthwaite

- In the bilateral trade environment, $\bar{\theta}_b = 0$ and $\bar{\theta}_s = 1$ are the least charitable types.
- Thus, the VCG mechanism we considered coincides with the GVCG mechanism.
- But this mechanism runs a deficit, thus:

There does not exist an efficient, BIC, IIR, ex-ante BB mechanism in the bilateral trade environment.