The Timing of Complementary Innovations

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Abstract

This paper studies the dynamic completion of complementary tasks or projects. At each point in time, resources are allocated to projects that are completed as breakthroughs. I solve the problem of efficient dynamic allocation of resources by showing that, for complements, the solution must satisfy a regret-free property. I apply these results to study the problem of innovation when there is uncertainty about the difficulty of innovations. In some cases, the solution involves completing the projects in sequence. In others, it is optimal to work on multiple projects simultaneously. I provide simple conditions that determine the efficient timing of development of innovations and analyze when is it possible to implement this efficient timing with decentralized incentives.

1 Introduction

One of the principal aims of innovation policy is to orient scarce resources toward the most socially valuable R&D projects. The value of an innovation, however, is sometimes tied to the uncertain outcome of other developments: A new treatment for a medical condition is more valuable if there is a novel diagnosis method that helps identify the condition at an early stage, when the treatment is more effective; likewise, the novel diagnosis method is more valuable if there are new treatments for said condition; quantum software can be used to solve problems that classical computers cannot solve, but can only be implemented with quantum hardware.[1] These complementarities in innovation are important—and increasingly relevant—in many industries. Moreover, R&D projects carry high levels of uncertainty, both in terms of outcomes—projects may or may not prove to be successful—and in terms of costs—it is not clear how much time and how many resources will be required to complete the development.

This paper studies the development of complementary innovations when innovation requires non-specific resources such as time, money, or attention. To fix ideas, consider two abstract complementary innovations, A and B. The first question is what is the *efficient* way to develop these innovations. Should resources be first concentrated on the development of A and then on B if and only when A was successfully completed? Or vice versa? Or should both A and B be developed in parallel? Moreover, when should a project be abandoned or put on hold? The question of efficient development of innovations is relevant for both innovation policy and for firms or venture capitalists that maximize profits.

[1] Shor's quantum algorithm, a method for solving integer factorization problems in polynomial time, was written in 1994, four years before the first quantum computer prototype was developed.

The first contribution of this paper is to introduce a tractable model that features key aspects of the R&D process when the development of innovations requires non-specific resources. A unit of a resource (attention) is allocated over two complementary projects at each point in time. For each of the projects, a success arrives discretely in the form of a breakthrough. More precisely, a success arrives when the cumulative amount of attention paid to a project reaches a certain level. Successes are observable, but the amount of attention required to succeed is unknown. The joint value of the innovations is realized when the development stage is endogenously terminated and depends on the set of projects that was completed by that time. This paper is particularly concerned with complementary innovations, where the marginal value of completing each project is increasing in the set of projects that was successfully completed.

The second contribution is to introduce techniques to solve the dynamic problem of attention allocation. More precisely, I use properties of the optimal allocation to find the solution via an alternative problem. When projects are complements and positively affiliated, the dynamic problem of attention allocation is equivalent to a recursive problem in which the decision maker pledges a certain amount of attention to each of the remaining projects, and commits to abandon the projects if no success is obtained after the attention is allocated. Thus, all that matters for efficiency is how much total attention the DM is willing to pledge to each project, and not the specific order in which the agent allocates the attention. This is useful because it reduces the strategy space to a few parameters. The intuition is that given the complementarities in payoff, the amount that the planner is willing to work on a project if there is no new success is the minimum she is going to work on that project on any optimal strategy. Since the planner would work this amount independently of the outcome of other projects, *when* she does it is not payoff-relevant.

I apply these results to a canonical problem where projects are perfect complements and the rate of success of each project is constant over time but unknown. The beliefs about these rates evolve with the outcomes of the development process. In particular, lack of success is evidence in favor of the rates being relatively low, or in other words "the project i being relatively more challenging."

If the rate of success for each project were known, the timing of development is irrelevant. Any project that is worth pursuing is worth completing; therefore, the order of completion is not going to affect the final expected payoff.[2] In contrast, when the rate of success is uncertain, the order of development is relevant, since it affects the arrival of information about the unknown parameters. An initial failure to develop A not only reduces the prospects of ever completing A, but also decreases the expected returns from completing B. The problem therefore does not fit in the classical experimentation framework and thus there is no general result (such as the Gittins index) that governs the optimal dynamic allocation.[3]

Consider the case where project A is of uncertain feasibility, that is, the success rate is either zero or λ_A, and project B has a known success rate λ_B. In this case, it is efficient to first work on project A: there is no learning by working on B, so there is no efficiency loss in back-loading all development of B. Front-loading the development of A increases the

[2] A similar logic holds when the completion time for each project is deterministic.

[3] The problem fits the framework of *restless multi-armed bandits*, for which there is no general index solution.

speed of learning, which is valuable because of the option given by the stopping decision. The intuition from the previous case applies more generally: the efficient allocation of resources reflects the optimal learning process about the potential of the joint project. For symmetric projects, Proposition 3 partitions the parameter space in two. For projects with high uncertainty about the rate of success and high costs, it is efficient to work in a sequence starting with the less promising project. For other parameters, it is optimal to work on the projects simultaneously.

The final contribution of the paper is the study of inefficiencies that arise due to decentralization and whether the efficient allocation can be implemented with private incentives. The allocation of private R&D resources depends on several factors: who assigns these resources, the appropriability of the innovations—which is determined by the legal and patent systems—and how informed the agents are about a given project's successes.

The reminder of the paper is as follows. In the next section, I discuss the relevant related literature. Section 2 introduces the model and preliminary results. The main results are presented in Section 3. In Section 4 we apply the results to study the incentives for efficient production of innovations when the rate of innovation is constant but unknown.

1.1 Related Literature

The problem of incentives in the context of complementary complementary innovations was study by Scotchmer and Green [1990] and Ménière [2008], who ask what is the optimal inventive requirements for a patent in the context of complementary innovations. Biagi and Denicolò [2014] study the optimal division of profits with complementary innovations. Fershtman and Kamien [1992] study the effects of cross licensing in the incentives to innovate. Bryan and Lemus [2017] study the direction of innovation in a general setting that accounts for complementary innovations. Bryan et al. [2020] focus on the effect of a *crises*—a proportional increase in the payoff from innovations—in the direction of innovation with partial substitutes. In these papers there is no learning in the development stage since the process of information arrival is memoryless.

Some complementary innovations are sequential or cumulative. Papers that study sequential developments include Gilbert and Katz [2011] and Green and Scotchmer [1995]. Moroni [2019] studies a contracting environment with sequential innovations. In these papers the timing of innovation is exogenously given. I focus on complementary innovations in which the timing is determined endogenously by the allocation of resources to the projects. To the best of my knowledge, this paper is the first one to combine an endogenous timing of development with a learning process in the development stage.

Another relevant branch of the literature studies the problem of dynamic information acquisition from multiple sources. With Poisson information structure, both Nikandrova and Pancs [2018] and Che and Mierendorff [2019] study an agent that acquires information before an irreversible decision. Nikandrova and Pancs [2018] studies the case of independent processes while Che and Mierendorff [2019] studies processes that are negatively correlated. Mayskaya [2019] also studies irreversible decision with Poisson structure in a general setting. Ke and Villas-boas [2019] study problem of independent information sources where the agent learns about the state by observing a Brownian process. Klabjan et al. [2014] study the problem of sequential acquisition of information about the attributes of an object.

Liang et al. [2018] and Liang and Mu [2020] compare the performance of optimal strategies and a different strategy. Liang et al. [2018] asks the question of how well does a strategy that neglects all dynamic considerations and acquire information in a myopic way performs with respect to the optimal information acquisition strategy. Liang and Mu [2020] compare efficient information acquisition to what results from the choices of short-lived agents who do not internalize the externalities of their actions.

The paper shares many key elements with the theory of scheduling in operations research. This literature is concerned with the problem of specifying an order in which jobs or tasks should be completed. Although there are papers in this literature that incorporate uncertainty in the amount of resources that each task demands, the objective functions is typically different. A classical question in this literature is how to complete a certain set of tasks in the least possible expected time. In this paper, the set of tasks that end up being completed is endogenous.

2 Setup

A decision maker (DM) works on two projects, A and B. Time is continuous and the DM decides when to stop for working on the projects for good and, before that, how to allocate resources across the projects. In particular, each instant before stopping, the DM allocates a unit of a *attention* across the projects that were not completed so far.

Let $\alpha_i(t)$ be the amount of attention allocated to project i at time t. Attention is scarce: $\alpha_A(t) + \alpha_B(t) \leqslant 1$ for all $t \geqslant 0$. Each project is completed when the cumulative attention that is allocated to it $x_i(t) := \int_0^t \alpha_i(\tilde{t})\, d\tilde{t}$ reaches a certain amount τ_i. Project completion is observable but τ_i is unknown. Formally, the DM observes the stochastic process $S_t := \{i : \tau_i \leqslant x_i(t)\}$ that represents the set of projects that were completed so far. The completion times $\tau = (\tau_A, \tau_B)$ are jointly continuous with density function $f : [0,\infty]^2 \to \mathbb{R}_+$ and marginal density function f_A and f_B.

Definition 1. *The projects are* independent *if* $f(x_A, x_B) = f_A(x_A) \cdot f_B(x_B)$. *The projects are* positively (negatively) affiliated *if f is log-supermodular (log-submodular), i.e.* $f(\tau \vee \tau') \cdot f(\tau \wedge \tau') \geqslant (\leqslant) f(\tau) \cdot f(\tau')$

When the DM stops the research activity, she receives a payoff that depends on the set of projects that were completed and the total attention that he allocated to each of the projects. In particular, the payoff of a DM that stops at time T and completed projects S_T by that time is $q(S_T) - c_A \cdot x_A(T) - c_B \cdot x_B(T)$. The DM is an expected-payoff maximizer.[4] We also assume free disposal: q is non-decreasing in the inclusion order. Moreover, we normalize $q(\emptyset) = 0$.

Definition 2. *The projects are* complements *if the function q is supermodular, that is,*

$$q(A) + q(B) \leqslant q(\{A, B\})$$

[4] In this model there is no discounting. Alternatively, we could have had a discount factor instead of the linear cost in time. The qualitative features of the solution remain unchanged for this alternative version of the model.

The projects are perfect complements *if* $q(A) = q(B) = 0$.[5]

2.1 Preliminary Analysis

The problem that the DM faces after one of the projects is completed, say project j, is straightforward: he will continue working on the remaining project i as long as it is profitable doing so.

Assumption 1. *The conditional hazard rate* $h_i(x_i|\tau_j) := f_i(x_i|\tau_j)/[1 - F_i(x_i|\tau_j)]$ *is non-increasing for* $i = A, B$.

We maintain Assumption 1 for the main sections of the paper.[6] Under this assumption, the DM works on the remaining project as long as

$$h_i(x_i|\tau_j) \cdot [q(\{i,j\}) - q(\{j\})] - c_i \geqslant 0$$

The reason is that, once the expected flow payoff of working on the remaining project becomes negative, it will remain negative forever. So it makes no sense for the DM to continue paying attention to that remaining project.

Let $\bar{x}_i(\tau_j)$ be the first time that the DM would like to stop working on project i when j was completed at τ_j, that is:

$$\bar{x}_i(\tau_j) = \min\{\ x :\ h_i(x|\tau_j) \cdot [q(\{i,j\}) - q(\{j\})] - c_i \geqslant 0\ \}$$

Lemma 1. *For positive (negative) affiliated projects,* $\bar{x}_i(\tau_j)$ *is decreasing (increasing) in* τ_j *for* $i = A, B$.

Sketch of the proof. Affiliation implies that the conditional hazard rates are ordered (increasing for positive affiliation and decreasing for negative affiliation), as it is shown in Lemma 2. This immediate implies monotonicity of $\bar{x}_i(\cdot)$ when Assumption 1 holds. A proof of the more general result when Assumption 1 does not necessarily hold can be found in Appendix A.1. □

It will be useful to define the continuation value when j was completed at τ_j and x_i attention was allocated to project i. This continuation value is:

$$v_i(x_i|\tau_j) \ := \ \max_{\hat{x}} \int_{x_i}^{\hat{x}} \underbrace{\frac{1 - F(x|\tau_j)}{1 - F(x_i|\tau_j)}}_{\substack{\text{Conditional probability} \\ \text{of reaching } x.}} \cdot \underbrace{[h_i(x|\tau_j) \cdot [q(\{i,j\}) - q(\{j\})] - c]}_{\substack{\text{Net flow expected payoff} \\ \text{of working on } i \text{ at } x.}} \, dx$$

What remains to be determined is what is the optimal allocation of attention in the first-stage, i.e. before the first project completion.

Definition 3. *A* first-stage strategy α *is a pair of functions* (α_A, α_B) *with* $\alpha_i : \mathbb{R}_+ \to [0,1]$ *such that* $\alpha_A(t) + \alpha_B(t) \leqslant 1$ *for all* $t \geqslant 0$*. Let* $\mathcal{A}$ *be the set of first-stage strategies.*

[5] Substitutes are defined by q being submodular, and perfect substitutes by the property that $q(A) = q(B) = q(\{A,B\})$.

[6] Assumption 1 is relaxed in **??**.

For any $\alpha \in \mathcal{A}$ and time t, there is a amount of attention allocated to each of the projects in the case of no project completion. Formally,

$$x_i^\alpha(t) := \int_0^t \alpha_i(\tilde{t})\ d\tilde{t}$$

Without loss of generality we can focus on first-stage strategies such that $\alpha_A(t)+\alpha_B(t) = 1$ for $t \leqslant T^\alpha$ and 0 for $t > T^\alpha$ for some $T^\alpha \in [0,\infty]$.[7] Let $X^\alpha := x^\alpha(T^\alpha)$ be the amount of attention that the agent allocates to each project before stopping when there is no success.

Let $V(\alpha)$ be the value of using first-stage strategy α in combination with optimal continuation after the first success. we can write:

$$V(\alpha) = \int_0^{T^\alpha} \underbrace{[1 - F(x(t))]}_{\substack{\text{probability of reaching} \\ \text{time } t \text{ in the first stage.}}} \cdot \sum_{i=A,B} \alpha_i(t) \underbrace{[f(x^\alpha(t)) \cdot v_j(x_j(t)) - c_i]}_{\substack{\text{flow net expected payoff of} \\ \text{working on project } i \text{ in the first-stage.}}}\ dt$$

The DM chooses the first-stage strategy that maximizes V.

3 Main Results

The main results of the paper establish relationships between properties of q and f and properties of the optimal first-stage strategy.

Definition 4. *A first-stage strategy α is* regret-free *if for every $t \leqslant T^\alpha$,*

$$x_A^\alpha(t) \leqslant \bar{x}_A(x_B^\alpha(t)) \quad \text{and} \quad x_B^\alpha(t) \leqslant \bar{x}_B(x_A^\alpha(t)).$$

When the DM uses a first-stage strategy that is regret-free, then after completing one of the projects he doesn't regret the amount of attention that he already allocated to the remaining project. Regret free strategies dominate other first-stage strategies where the DM stops at the same point, as is formalized in the following proposition.

Proposition 1. *If $\alpha, \hat{\alpha}$ are such that $X^\alpha = X^{\hat{\alpha}}$ and α is regret-free, then $V(\alpha) \geqslant V(\hat{\alpha})$.*

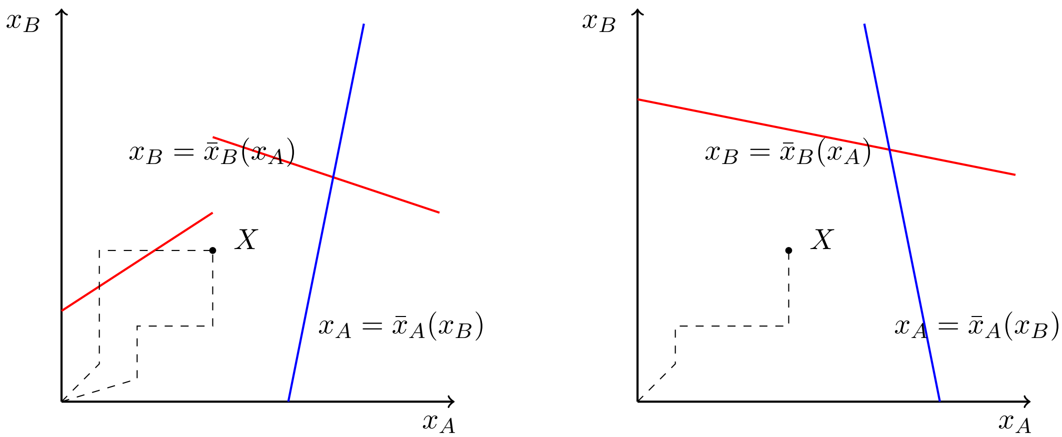
Proof. in Appendix B.1. □

The intuition for Proposition 1 is that, compare to a first-stage strategy with regret, one without regret provides the individual with the option to stop earlier, and this is valuable.

Notice that in Lemma 1 we showed that for positively affiliated projects, $\bar{x}_i$ is weakly decreasing. Thus, in that case, a first-stage strategy is regret-free if and only if $X_A^\alpha \leqslant \bar{x}_A(X_B^\alpha)$ and $X_B^\alpha \leqslant \bar{x}_B(X_A^\alpha)$.

Corollary 1. *When projects are positively affiliated, if α and $\hat{\alpha}$ are such that $X^\alpha = X^{\hat{\alpha}} = X$ satisfying $X_A \leqslant \bar{x}_A(X_B)$ and $X_B \leqslant \bar{x}_B(X_A)$ then $V(\alpha) = V(\hat{\alpha})$.*

[7] The reason is that, in this model, there is no benefit for the DM in delaying the allocation of attention. This fact remains true if we incorporate a discount factor to the objective function.



(a) Substitute projects. (b) Positive affiliated projects.

Figure 1: Path. $\bar{x}$. $\bar{x}$ decreasing for positive affiliated projects. Intuition for Proposition 1.

Proof. For $t \leqslant T^\alpha$,

$$x_i^\alpha(t) \leqslant X_i^\alpha \leqslant \bar{x}_i(X_j^\alpha) \leqslant \bar{x}_i(x_j^\alpha(t))$$

Where the first inequality holds because x_i^α is increasing by definition. The second inequality holds by hypothesis, and the third inequality holds since, by Lemma 1, $\bar{x}_i(\cdot)$ is decreasing. □

Intuitively, for positively affiliated projects, when the first-stage strategy is regret-free, the amount of attention that the DM is willing to allocate to one of the projects if there is no news is less than the amount of attention that he would allocate if the other project is completed. Thus, he is willing to allocate said attention independently of the outcome of the other project. The order in which the agent allocates this attention does not affect then the total payoff.

Corollary 1 means that for positively affiliated projects the value V of regret-free first-stage strategies is pinned-down by X^α. Moreover, to compute the value of any regret-free first-stage strategy α, we can simply assume that attention is first allocated to any of the projects.

Let $\hat{V}_i(X)$ be the expected value that the DM gets when, in the first-stage, he puts first all his attention to project i until X_i, and only then he works on project j until X_j.

$$\begin{aligned} \hat{V}_i(X) := \int_0^{X_i} & [1 - F_i(\tilde{x})] \cdot [f_i(\tilde{x}) \cdot v_j(0|\tilde{x}) - c_i] \, d\tilde{x} \\ & + [1 - F_i(X_i)] \cdot \int_0^{X_j} [1 - F(\tilde{x})] \cdot [f_j(\tilde{x}|x_i) \cdot v_i(X_i|\tilde{x}) - c_j] \, d\tilde{x} \end{aligned} \tag{1}$$

Corollary 2. *If projects are positively affiliated, then for any strategy α that is regret-free $V(\alpha) = \hat{V}_A(X^\alpha) = \hat{V}_B(X^\alpha)$.*

Solving the problem $\max_{x\in\mathbb{R}^2} \hat{V}_i(x)$ is allegedly easier than solving the problem $\max_{\alpha\in\mathcal{A}} V(\alpha)$, since it is possible to apply standard optimization techniques. Thus, we are interested to know when is that the optimal first-stage strategy is regret-free.

Although it might seem intuitive that, with complementary projects, completing a project should induce the agent to allocate more attention to the remaining one, this intuition is, in principle, flawed. The next example illustrates the difficulty.

Example 1. *Suppose that projects are independent. Moreover, suppose that project A has a constant hazard rate λ and project B has a continuous and strictly decreasing hazard rate $h_B(x_B)$ that converges to zero as x_B goes to infinity.*

When the DM completes project A, he is willing to work on project B until $\bar{x}_B(x_A)$ which is a constant defined by $h_B(\bar{x}_B)\cdot[q(\{A,B\})-q(\{A\})] = c_B$. When project A is not completed, the DM would be willing to work on project B until $h_B(x_B)\cdot[q(\{B\})+v_A(x_A)] = c_B$, where

$$v_A(x_A) = \max\{q(\{A,B\}) - q(\{B\}) - c_A/\lambda\ ,\ 0\}.$$

*Notice that q supermodular does **not** guarantee that the incentives to complete B when A was completed ($q(\{A,B\} - q(\{A\}))$ are larger than the incentives to complete B when A was not completed ($q(\{B\}) + v_A(x_A)$). In fact,*

$$q(B) + v_A(x_A)\ >\ q(\{A,B\}) - q(\{A\}) \quad \Leftrightarrow \quad c_A/\lambda\ <\ \min\{q(A)\ ,\ q(\{A,B\}) - q(\{B\})\}$$

For q supermodular $q(\{A\}) > q(\{A,B\}) - q(\{B\})$. Thus, in that case, the incentives to work on B are higher when A was not completed iff $c_A/\lambda < q(A)$.

If $c_A/\lambda < q(A)$, however, completing project A is profitable independently of whether project B is completed or not. Thus, completing project A first is optimal.

Theorem 1 gives us a sufficient condition for optimal first-stage strategies to be regret-free.

Theorem 1. *If projects are complements and positively affiliated then there is an optimal first-stage strategy α^* that is regret-free.*

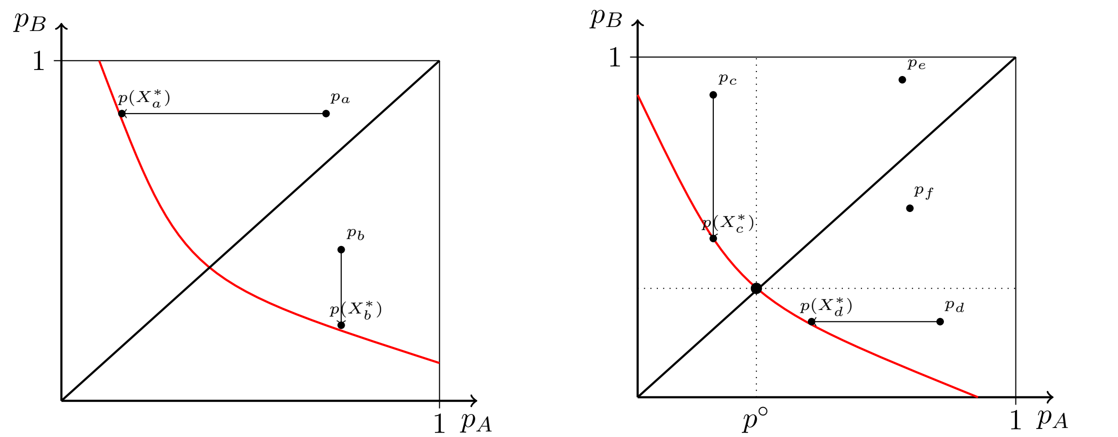
Proof. in Appendix B.2. □

As in the example, the key to prove this result is to show that if the DM finds it optimal to work on a project in the first-stage beyond what he would have liked to work on the project if the other one was completed, then it must be that working more on the other project is also optimal.

Supermodularity of q is not only sufficient for regret-free optimal strategies with positive affiliated project, but it is also necessary, in the sense that if it is violated one can construct independent distributions of completion amounts F_A and F_B such that no optimal first-stage strategy is regret-free, and moreover the order in which the attention is allocated in the first-stage affects the expected payoff. This claim is formalized in the following proposition.

Proposition 2. *If the projects are not complements, there exists a distribution f such that the projects are independent and $\max_{\alpha\in\mathcal{A}} V(\alpha) > \max_{X\in\mathbb{R}^2} \hat{V}_i(X)$ for $i = A, B$.*

Proof. in Appendix B.3. □



(a) Projects are cost-effective when assorted. (b) Projects are not cost-effective when assorted.

Figure 2: Projects in belief space.

4 Constant but unknown rates of success

There are two projects, each of which is of one of two types: *difficult* (with a low hazard rate λ_L) or *easy* (with a high hazard rate λ_H). The difficulty of the projects is independent. Project i has an ex ante probability of being easy equal to p_i. If $p_i < p_j$ we say that i is the *least promising* project and j the *most promising* one.

The projects are perfect complements: if both are completed a value $q > 0$ is generated. The cost of working on the projects is proportional to the total attention allocated, i.e. $c_A = c_B = c$.

Definition 5. *The projects are* cost-effective when assorted *if it is worth to pursue the projects when it is known that one is difficult and the other one is easy. I.e., when*

$$q > \frac{c}{\lambda_H} + \frac{c}{\lambda_L}$$

Proposition 3.

- *If the projects are not cost-effective when assorted, then it is efficient to work on them in sequence starting with the least promising one.*
- *If the projects are cost-effective when assorted, then it is efficient to work more on the most promising project.*

Proof. in Appendix C. □

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A Omitted Proofs

A.1 Proof of Lemma 1

Given a continuous distribution with density f, define the marginals, f_X and f_Y, and the conditional distributions $f_{Y|X}$ and $f_{X|Y}$.

Lemma 2. *If f is log-supermodular (log-submodular) then the conditional hazard rate*

$$h_{Y|X}(y|x) := \frac{f_{Y|X}(y|x)}{1 - F_{Y|X}(y|x)}$$

is decreasing (increasing) in x for all y.

Proof. Let f be log-supermodular. For $x' \geqslant x$ and $y' \geqslant y$,

$$f(x', y) \cdot f(x, y') \leqslant f(x, y) \cdot f(x', y')$$

$$f_X(x') \cdot f_{Y|X}(y|x') \cdot f_X(x) \cdot f_{Y|X}(y'|x) \leqslant f_X(x) \cdot f_{Y|X}(y|x) \cdot f_X(x') \cdot f_{Y|X}(y'|x')$$

$$f_{Y|X}(y|x') \cdot f_{Y|X}(y'|x) \leqslant f_{Y|X}(y|x) \cdot f_{Y|X}(y'|x')$$

$$\int_y^\infty f_{Y|X}(y|x') \cdot f_{Y|X}(y'|x) \, dy' \leqslant \int_y^\infty f_{Y|X}(y|x) \cdot f_{Y|X}(y'|x') \, dy'$$

$$f_{Y|X}(y|x') \cdot [1 - F_{Y|X}(y|x)] \leqslant f_{Y|X}(y|x) \cdot [1 - F_{Y|X}(y|x')]$$

$$\frac{f_{Y|X}(y|x')}{1 - F_{Y|X}(y|x')} \leqslant \frac{f_{Y|X}(y|x)}{1 - F_{Y|X}(y|x)}$$

(The proof for the case of f sub-modular is similar but with all inequalities reversed.) □

Lemma 3. *If f is log-supermodular, For $x' > x$,*

$$A(y; x, x') := \frac{1 - F_{Y|X}(y|x)}{1 - F_{Y|X}(y|x')}$$

is increasing in y.

Proof. Taking derivatives we have,

$$\begin{aligned}
\frac{\partial A(y; x, x')}{\partial y} &= \frac{f_{Y|X}(y|x)[1 - F_{Y|X}(y|x')] - f_{Y|X}(y|x')[1 - F_{Y|X}(y|x)]}{[1 - F_{Y|X}(y|x')]^2} \\
&= \frac{1 - F_{Y|X}(y|x)}{1 - F_{Y|X}(y|x')} \cdot \underbrace{\left[\frac{f_{Y|X}(y|x)}{1 - F_{Y|X}(y|x)} - \frac{f_{Y|X}(y|x')}{1 - F_{Y|X}(y|x')}\right]}_{>0 \text{ by Lemma 2}} \geqslant 0
\end{aligned}$$

□

Lemma 4. *If projects are positively (negatively) affiliated, then $\bar{x}_j(x_i)$ is decreasing (increasing) in x_i.*

Proof. Consider the problem of an agent that completed project j at x and already spent y in the remaining project i. The marginal value of continuing until $\bar{y} \geqslant y$ is:

$$\pi_i(\bar{y}; x) := \underbrace{\frac{F_{Y|X}(\bar{y}|x) - F_{Y|X}(y|x)}{1 - F_{Y|X}(y|x)}}_{\text{Prob. of completing } i} \cdot [q(\{i,j\}) - q(\{j\})] - c_i \cdot \underbrace{\int_y^{\bar{y}} \frac{1 - F_{Y|X}(\tilde{y}|x)}{1 - F_{Y|X}(y|x)} \, d\tilde{y}}_{\text{Expected remaining time.}}$$

Then,

$$\begin{aligned}
\frac{\partial \pi(\bar{y}; x)}{\partial y} &= \frac{f_{Y|X}(\bar{y}|x)}{1 - F_{Y|X}(y|x)} \cdot [q(\{i,j\}) - q(\{j\})] - c_i \cdot \frac{1 - F_{Y|X}(\bar{y}|x)}{1 - F_{Y|X}(y|x)} \\
&= \frac{1 - F_{Y|X}(\bar{y}|x)}{1 - F_{Y|X}(y|x)} \cdot \left[\frac{f_{Y|X}(\bar{y}|x)}{1 - F_{Y|X}(\bar{y}|x)} \cdot [q(\{i,j\}) - q(\{j\})] - c_i \right] \\
&\geqslant \frac{1 - F_{Y|X}(\bar{y}|x)}{1 - F_{Y|X}(y|x)} \cdot \left[\frac{f_{Y|X}(\bar{y}|x')}{1 - F_{Y|X}(\bar{y}|x')} \cdot [q(\{i,j\}) - q(\{j\})] - c_i \right] \\
&= \underbrace{\frac{1 - F_{Y|X}(\bar{y}|x)}{1 - F_{Y|X}(y|x)} \cdot \frac{1 - F_{Y|X}(y|x')}{1 - F_{Y|X}(\bar{y}|x')}}_{A(\bar{y};x,x')} \cdot \underbrace{\left[\frac{f_{Y|X}(\bar{y}|x')}{1 - F_{Y|X}(y|x')} \cdot [q(\{i,j\}) - q(\{j\})] - c_i \cdot \frac{1 - F_{Y|X}(\bar{y}|x')}{1 - F_{Y|X}(y|x')} \right]}_{\pi'(\bar{y};x')}
\end{aligned}$$

$A(\bar{y}; x, x')$ is positive and increasing by Lemma 3. Thus, by Proposition 2 and Theorem 1 in Quah and Strulovici [2009], $\pi(\bar{y}; x)$ interval dominates $\pi(\bar{y}; x')$ and

$$\arg \max_{\bar{y} \geqslant y} \pi(\bar{y}; x) \;\geqslant_{SSO}\; \arg \max_{\bar{y} \geqslant y} \pi(\bar{y}; x')$$

Where S dominates S' in the strong set order ($S \geqslant_{SSO} S'$) if for any $s \in S$ and $s' \in S'$, $\max\{s, s'\} \in S$ and $\min\{s, s'\} \in S'$. This implies that

$$\bar{x}(y; x) = \min \arg \max_{\bar{y} \geqslant y} \pi(\bar{y}; x) \;\geqslant\; \min \arg \max_{\bar{y} \geqslant y} \pi(\bar{y}; x') = \bar{x}(y; x')$$

(For log-submodular f the proof works the same choosing $x' < x$.)

□

B Proofs of Section 3

B.1 Proof of Proposition 1

Proposition 1. If $\alpha, \hat{\alpha}$ are such that $X^{\alpha} = X^{\hat{\alpha}}$ and α is regret-free, then $V(\alpha) \geqslant V(\hat{\alpha})$.

Proof. For any strategy $\tilde{\alpha}$, project i and completion ammount τ_i, let $\mathbf{x}_j^{\tilde{\alpha}}(\tau_i)$ be the amount of attention allocated to project j when project i is completed first at τ_i. Formally,

$$\mathbf{x}_j^{\tilde{\alpha}}(\tau_i) = \max\{x_j : \exists\ t \geqslant 0 \text{ with } x_i^{\alpha}(t) = \tau_i \text{ and } x_j^{\alpha}(t) = x_j\}$$

Notice that if $\tilde{\alpha}$ is regret-free, $\mathbf{x}_j^{\tilde{\alpha}}(\tau_i) \leqslant \bar{x}_j(\tau_i)$.

Consider a strategy that is equal to α in the first stage but if any project i is completed at τ_i, the DM works on the remaining project j until

$$x_j^*(\tau_i; \hat{\alpha}) \ := \ \max \ \ \{ \ \bar{x}_j(\tau_i) \ , \ \mathbf{x}_j^{\hat{\alpha}}(\tau_i) \ \}$$

Notice that this is always possible since α is regret-free, and thus at $x_j^{\alpha}(\tau_i) \leqslant \bar{x}_j(\tau_i)$.

This strategy shields a value lower than $V(\alpha)$, since they have the same first stage but the continuation is not necessarily optimal. We will show that this strategy gives the DM the same ex-post payoff as $\hat{\alpha}$ with optimal continuation.

We can split the problem in three cases, depending on τ:

- **$\hat{\alpha}$ completes project i first and then project j.**

  This means that $\tau_i < X_i^{\hat{\alpha}}$ and $x_j^{\hat{\alpha}}(\tau_i) < \tau_j < \bar{x}_j(\tau_i)$.

  Since τ is not larger than X^{α}, the strategy that uses α also completes at least one project.

  Suppose that the strategy that starts with α completes project i first. Then since $\tau < \bar{x}_j(\tau_i) \leqslant x_j^*(\tau_i; \hat{\alpha})$, the DM that follows this strategy also completes project j. The payoff for both strategies is $q(\{A, B\}) - c_A \cdot \tau_A - c_B \cdot \tau_B$.

  Suppose instead that the strategy that starts with α completes project j first. Since $\hat{\alpha}$ completes i first, $\tau_i < \mathbf{x}_i^{\hat{\alpha}}(\tau_j) \leqslant x_i^*(\tau_j, \hat{\alpha})$. Thus, the DM also completes project i. The payoff for both strategies is the same: $q(\{A, B\}) - c_A \cdot \tau_A - c_B \cdot \tau_B$.

- **$\hat{\alpha}$ completes project i first but doesn't complete project j.**

  Since $\hat{\alpha}$ completes i first, it must be that $\tau_i < X_i^{\hat{\alpha}}$ and $\tau_j > \mathbf{x}_j^{\hat{\alpha}}(\tau_i)$. Moreover, since $\hat{\alpha}$ does not complete project j in the second stage, it must be that $\tau_j > \bar{x}_j(\tau_i)$. Thus $\tau_j > x_j^*(\tau_i; \hat{\alpha})$.

  Also, since α is regret-free $\bar{x}_j(\tau_i) \geqslant \mathbf{x}_j^{\alpha}(\tau_i)$. Thus $\tau_j > \bar{x}_j(\tau_i) \geqslant \mathbf{x}_j^{\alpha}(\tau_i)$ what implies that the DM that starts with α will also complete project i first. The DM does not complete project j since $\tau_j > x_j^*(\tau_i, \hat{\alpha})$.

  The payoff for both strategies is thus, $q(\{i\}) - c_i \cdot \tau_i - c_j \cdot x_j^*(\tau_i, \hat{\alpha})$.

- **$\hat{\alpha}$ does not complete any of the projects.**

  This means that $\tau > X^{\hat{\alpha}} = X^{\alpha}$. Thus, the DM that starts with α in the first stage does not complete any of the projects either. The payoff for both strategies is $-c_A X_A^{\alpha} - c_B X_B^{\alpha}$.

$\square$

B.2 Proof of Theorem 1

Theorem 1 If projects are complements and positively affiliated then there is an optimal first-stage strategy α that is regret-free.

Lemma 5. *Any strategy α such that $X_A^\alpha \leqslant \bar{x}_A(X_B^\alpha)$ and $X_B^\alpha > \bar{x}_B(X_A^\alpha)$ is dominated by a strategy $\hat{\alpha}$ with $X^\alpha = X^{\hat{\alpha}}$ and such that $\hat{\alpha}_A(t) = 0$ for all t such that $x_B^{\hat{\alpha}}(t) > \bar{x}_B(x_A^\alpha)$.*

Proof. Suppose α is such that $x_A^\alpha \leqslant \bar{x}_A(x_B^\alpha)$ and $x_B^\alpha > \bar{x}_B(x_A^\alpha)$.

Let $\hat{t}$ be the time at which the X_B^α reaches the point at which it would have been optimal [By Lemma 4, $\bar{x}_B(x_A(t))$ is non-increasing, so $x_B^\alpha(t)$ intersects $\bar{x}_B(x_A(t))$ once]:

$$\hat{t} := \max \ \{ \ \tilde{t} \geqslant 0 \ : \ x_B^\alpha(\tilde{t}) \leqslant \bar{x}_B(X_A^\alpha) \ \}$$

Let's define a new strategy $\hat{\alpha}$ as follows:

$$\hat{\alpha}_A(t) = \begin{cases} \alpha_A(t) & \text{for } t \leqslant \tilde{t} \\ 1 & \text{for } t \in [\tilde{t}, \tilde{t} + X_A^\alpha - x_A^\alpha(\tilde{t})] \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\hat{\alpha}_B(t) = \begin{cases} \alpha_B(t) & \text{for } t \leqslant \tilde{t} \\ 1 & \text{for } t \in \left[\tilde{t} + X_A^\alpha - x_A^\alpha(\tilde{t}) \ , \ \tilde{t} + X_A^\alpha - x_A^\alpha(\tilde{t}) + X_B^\alpha - \bar{x}_B(X_A^\alpha)\right] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Suppose that the DM uses the first-stage strategy $\hat{\alpha}$ but in case of a success in project j, he continues working on the remaining project i until $\max\{x_i, \bar{x}_i(x_i^\alpha(x_j))\}$. This strategy would give the DM the same ex-post payoff that using α in combination with optimal second stage:

If for one of the projects $\tau_i \leqslant x_i^\alpha(\tilde{t})$, then both strategies have the same ex-post payoff, since they are equivalent before $\tilde{t}$ in the first stage and have the same continuation. If $\tau \geqslant X^\alpha$, then none of the strategies completes any of the projects in the first stage and the ex-post payoff is therefore $-\sum_{i=A,B} c_i \cdot X_i^\alpha$ for both strategies.

If τ is not lower than $x(\tilde{t})$ and not larger than X^α, then strategy α will complete at least one of the projects.

If τ is such that the α completes A first, then $\hat{\alpha}$ also completes A first (since front-loads working on project A). Project B is thus completed in the second stage if and only if $\tau_B \leqslant \bar{x}_B(\tau_A)$ for both strategies.

If τ is such that α completes B first, then there are two cases: either (i) $\tau_B \leqslant X_B^\alpha$ [in which case $\hat{\alpha}$ completes project A first but both strategies end up completing both projects [since $\bar{x}_A(x_B) > X_A^\alpha$.] or (ii) $\tau_B > X_B^\alpha$, in which case $\hat{\alpha}$ also completes project B first and both strategies work on A until $\bar{x}_A(\tau_B)$. $\square$

Lemma 6. *If h is decreasing, projects are complements and positively affiliated, then for any optimal strategy α and project i it must be that $X_i^\alpha \leqslant \bar{x}_i(X_j^\alpha)$.*

Proof. Suppose α is such that $X_A^\alpha \leqslant \bar{x}_A(X_B^\alpha)$ and $X_B^\alpha > \bar{x}_B(X_A^\alpha)$. Then by Lemma 5, there is an optimal strategy α^* with the same $X^{\alpha^*} = X^\alpha$ such that the strategy ends up working on project B, i.e. $\alpha_B^*(t) = 1$ in $[T^{\alpha^*} - \epsilon, T^{\alpha^*}]$ for some small $\epsilon > 0$.

$$h_B(X^\alpha) \cdot [q(\{B\}) + v_A(X^\alpha)] \geqslant c_B$$

Since h_B is decreasing, it must be that

$$h_B(X^\alpha) \cdot [q(\{A, B\}) - q(\{A\})] < c_B$$

Thus,

$$q(\{B\}) + v_A(X^\alpha) > q(\{A, B\}) - q(\{A\}).$$

$$v_A(X^\alpha) = Q^\alpha \cdot [q(\{A, B\}) - q(\{B\})] - K$$

Where

$$Q^\alpha = \int_{X_A^\alpha}^{\bar{x}_A(X_B^\alpha)} \frac{f_A(x|X_B^\alpha)}{1 - F_A(X_A^\alpha|X_B^\alpha)}\ dx = \frac{F_A(\bar{x}_A(X_B^\alpha)|X_B^\alpha) - F_A(X_A^\alpha|X_B^\alpha)}{1 - F_A(X_A^\alpha|X_B^\alpha)}$$

is the probability of completing project A if B is completed at the last instant of the first stage and

$$K^\alpha = c_B \cdot \int_{X_A^\alpha}^{\bar{x}_A(X_B^\alpha)} \frac{1 - F_A(x|X_B^\alpha)}{1 - F_A(X_A^\alpha|X_B^\alpha)}\ dx$$

is the expected cost of working optimally on A after B is completed at the last instant of the first stage.

So,

$$q(\{B\}) + Q^\alpha \cdot [q(\{A, B\}) - q(\{B\})] - K^\alpha > q(\{A, B\}) - q(\{A\}).$$

Rearranging:

$$Q^\alpha \cdot q(\{A\}) - K^\alpha > \underbrace{(1 - Q^\alpha)}_{\geqslant 0} \cdot \underbrace{[q(\{A, B\}) - q(\{B\}) - q(\{A\})]}_{\geqslant 0 \text{ for complements.}} \geqslant 0$$

Thus, α^* is not optimal: the DM would benefit from not stopping at X^α and work on project A instead.

It remains to show that it cannot be that α is such that $X_A^\alpha > \bar{x}_A(X_B^\alpha)$ and $X_B^\alpha > \bar{x}_B(X_A^\alpha)$.

If that is the case, at the very last instant the worker would stop after any success. Let i be a project in which the DM puts positive attention on the last instant before stopping. Then the payoff if succeeds is $q(\{i\})$ which is less than $q(\{A, B\}) - q(\{A\})$ by complementarity of the projects and this was not sufficient for him to put effort. □

B.3 Proof of Proposition 2

Lemma 7. *If projects are perfect substitutes, projects are independent with h_i decreasing then it is efficient to always work on the project with highest flow net payoff $h_i(x_i) \cdot q(\{i\}) - c_i$.*

Proposition 2 If the projects are not complements, there exists a distribution f such that the projects are independent and $\max_\alpha V(\alpha) \neq \max_x \hat{V}(x)$.

Proof. Since q is not supermodular, $q(\{A\}) + q(\{B\}) > q(\{A, B\})$. Thus

$$\frac{c_A}{q(\{A\})} < \frac{c_A}{q(\{A, B\}) - q(\{B\})}$$

Let i be the project with highest $c_i(q(A, B) - q(\{i\}))$.

We construct the hazard rate of each project be such that:

- h_i is strictly decreasing.
- $h_i(0) = \frac{c_i}{q(\{A,B\}) - q(j)}$.
- $\lim_{x\to\infty} h_i(x) = \frac{c_i}{q(\{i\})}$.

Given that $h_i < \frac{c_i}{q(\{A,B\}) - q(j)}$, the agent will stop after the first success: at most one project will be developed.

Given that $h_i > \frac{c_i}{q(\{i\})}$, it will never be optimal to stop before the first success. Thus, exactly one project will be developed.

The projects are thus in some sense as perfect substitutes. It is optimal for the agent to always work on the project with highest flow payoff, i.e. in project i iff

$$h_i q(\{i\}) - c_i > h_j q(\{j\}) - c_j$$

Given 2, the agent will start with project i. However after some time with no success, the agent would like to switch to project j since

$$\lim_{x\to\infty} h_i \cdot q(\{i\}) - c_i = 0 < h_j(0) \cdot q(\{j\}) - c_j$$

$\square$

C Proofs of Section 4

C.1 Preliminaries

Let $\delta := \lambda^H - \lambda^L$. Using Bayes' rule, the beliefs $p_i(X)$ evolve

$$p_i(X) = \frac{p_i e^{-\delta x_i}}{(1-p_i) + p_i e^{-\delta x}}$$

As the agent becomes more pessimistic, the subjective hazard rate $h_i(x)$ becomes lower.

$$h_i(x) = \lambda_L + p_i(x) \cdot \delta$$

For $x < \bar{x}$,

$$v_i(x) = \frac{1}{1-F(x)} \cdot \int_x^{\bar{x}} [1-F(\tilde{x})] \cdot (h(\tilde{x}) \cdot q - c) \; d\tilde{x} \tag{4}$$

Next two important lemmata: Lemma 8 proves that whether the projects are is sufficient to identity the monotonicity of project i's hazard-to-value ratio. Lemma 9 shows that when whether hazard-to-value ratios are increasing or decreasing changes the sign of the determinant of the Hessian of the optimization problem $\max_X \hat{V}(X)$.

Lemma 8. *$h_i(x)/v_i(x)$ is monotone. Moreover, $h_i(x)/v_i(x)$ is increasing if and only if projects are cost effective when assorted.*

Proof. For this proof, we only focus on one of the projects, so we drop the subscript. Deriving Equation (4),

$$\begin{aligned} v'(x) &= \frac{f(x)}{[1-F(x)]^2} \int_x^{\bar{x}} [1-F(\tilde{x})] \cdot (h(\tilde{x}) \cdot q - c) \; d\tilde{x} \; - \; (h(x) \cdot q - c) \\ &= h(x) \cdot v(x) + c - h(x) \cdot q \\ &= c - h(x) \cdot [q - v(x)]. \end{aligned}$$

Now we show that the monotonicity of $h(x)/v(x)$ depends on weather $v(x)$ is higher or lower that an expression $R(x)$.

$$\begin{aligned} \operatorname{sgn}\left(\frac{\partial(h(x)/v(x))}{\partial x}\right) &= \operatorname{sgn}\left[h'(x) \cdot v(x) - h(x) \cdot v'(x)\right] \\ &= \operatorname{sgn}\left[h'(x) \cdot v(x) - h(x) \cdot [c - h(x) \cdot (q - v(x))]\right] \\ &= \operatorname{sgn}\left(\underbrace{\frac{h(x) \cdot [q \cdot h(x) - c\,]}{h(x)^2 + h'(x)}}_{R(x)} \; - \; v(x)\right) \end{aligned}$$

$$\frac{h(x)}{v(x)} \quad \text{decreasing} \qquad \Leftrightarrow \qquad R(x) < v(x) \tag{5}$$

It will be useful to switch to belief space.
Let $\hat{h} := \lambda_L + p \cdot \delta$. Notice that $\hat{h}(p(x)) = h(x)$. Then

$$h'(x) = \underbrace{\hat{h}'(p)}_{\delta} \cdot \underbrace{\frac{\partial p(x)}{\partial x}}_{-\delta \cdot p \cdot (1-p)}$$

$$\hat{R}(p) := \frac{\hat{h}(p) \cdot (q \cdot \hat{h}(p) - c)}{\hat{h}(p)^2 + \delta^2 \cdot p \cdot (1-p)} \tag{6}$$

Deriving Equation (6) twice,

$$\hat{R}''(p) = \frac{2\ \delta^2 \cdot \lambda_L \cdot \lambda_H \cdot (q \cdot \lambda_L \cdot \lambda_H - c \cdot (\lambda_L + \lambda_H))}{(\lambda_L^2 + p\ \delta\ (\lambda_L + \lambda_H))^3}$$

$$\begin{aligned} \hat{R}''(p) > 0 \qquad &\Leftrightarrow \qquad q \cdot \lambda_L \cdot \lambda_H - c \cdot (\lambda_L + \lambda_H) > 0 \\ &\Leftrightarrow \qquad q > \frac{c}{\lambda_L} + \frac{c}{\lambda_H} \\ &\Leftrightarrow \qquad \begin{array}{c}\text{project are not} \\ \text{cost-effective when assorted.}\end{array} \end{aligned} \tag{7}$$

There are two cases to be considered separately: $\lambda_L < c$ and $\lambda_L \geqslant c$.

Case I: $\lambda_L \cdot q \geqslant c$ Since $h(x) \cdot q > \lambda_L \cdot q \geqslant c$, the agent that completes a project does never stop in the second stage. The value $\hat{v}$ is linear in the beliefs:

$$\hat{v}(p) = q - p \cdot \frac{c}{\lambda_H} - (1-p) \cdot \frac{c}{\lambda_L}$$

Since

$$\hat{v}(0) = q - \frac{c}{\lambda_L} = \hat{R}(0)$$

and

$$\hat{v}(1) = q - \frac{c}{\lambda_H} = \hat{R}(1)$$

If projects are effective when assorted, R is concave by Equation (7) and thus

$$v(p) < R(p) \qquad \forall p \in (0,1)$$

And then $h(x)/v(x)$ is increasing by Equation (6).

If, on the other hand, projects are not effective when assorted, R is convex by Equation (7) and thus

$$v(p) > R(p) \qquad \forall p \in (0,1)$$

And then $h(x)/v(x)$ is decreasing by Equation (6).

Case II: $\lambda_L \cdot q < c$ In this case, the agent stops putting attention to the remaining project if sufficient attention is allocated without success. More specifically, the agent will stop allocating attention to the remaining project when p reaches $\hat{p} := \frac{c/q-l}{h-l}$.

v is strictly convex (information is valuable). Moreover, we can show that R is concave:

$$\lambda_L \cdot q < c \qquad \Rightarrow \qquad \frac{\lambda_H}{\lambda_L + \lambda_H} \cdot \lambda_L \cdot q < c \qquad \Leftrightarrow \qquad q < \frac{c}{\lambda_L} + \frac{c}{\lambda_H}$$
$$\Leftrightarrow \qquad \text{project are not cost-effective when assorted.}$$

Since $\hat{v}(1) = q - \frac{c}{\lambda_H} = \hat{R}(1)$ and $v(\hat{p}) = 0 = R(\hat{p})$,

$$\hat{v}(p) < \hat{R}(p) \qquad \text{for any } p \in (\hat{p}, 1)$$

$$h(x)/v(x) \text{ decreasing}$$

$\square$

Lemma 9. *If $h_i(x)/v_i(x)$ is strictly decreasing for $i = A, B$, it is optimal to work on the projects in sequence.*

Proof. Since projects are complements and independent (thus positive affiliated), Theorem 1 indicates that there is an optimal first-stage strategy α that is regret-free. Let $X = X^\alpha$. By Corollary 1, $\hat{V}_A(X) = \hat{V}_B(X) = V(X)$. Moreover, by optimality,

$$X \in \arg\max_{\tilde{X}} \hat{V}_i(\tilde{X}) \qquad \text{for } i = A, B$$

By contradiction assume that X is interior. Then the first-order conditions from deriving Equation (1) give us that

$$h_A(X_A) \cdot v_B(X_B) = h_B(X_B) \cdot v_A(X_A) = c \tag{8}$$

Claim: For an interior optimal point X,

$$\sum_{i=A,B} \frac{\partial h_i(X)/v_i(X)}{\partial X_i} > 0 \qquad \Rightarrow \qquad \prod_{i=A,B} \frac{h_i'(X_i) \cdot v_i(X_i)}{h_i(X_i) \cdot v_i'(X_i)} < 1$$

Using Equation (8),

$$\frac{h_A(X_A) \cdot v_A'(X_A)}{v_A^2(X_A)} = \frac{h_B(X_B) \cdot v_A'(X_A)}{v_B(X_B) \cdot v_A(X_A)} = \frac{h_A(X_A) \cdot v_B'(X_B)}{v_A(X_A) \cdot v_B(X_B)} = \frac{h_B(X_B) \cdot v_B'(X_B)}{v_B^2(X_B)}$$

Where the first and last equality use $h_A(X_A)/v_A(X_A) = h_B(X_B)/v_B(X_B)$ and the intermediate one uses that

$$\begin{aligned} h_B(X_B) \cdot v_A'(X_A) &= h_B(X_B) \cdot [c - h_A(X_A) \cdot (q - v_A(X_A)] \\ &= -h_B(X_B) \cdot h_A(X_A) \cdot [q - v_A(X_A) - v_B(X_B)] \end{aligned}$$

Since $c = h_A(X_A) \cdot v_B(X_B))$ and equal to $h_A(X_A) \cdot v'_B(X_B)$ by symmetry. So,

$$\begin{aligned}
\sum_{i=A,B} \frac{\partial h_i(X)/v_i(X)}{\partial X_i} > 0 \qquad &\Leftrightarrow \qquad \sum_{i=A,B} \frac{h'_i(X_i) \cdot v_i(X_i) - h_i(X_i) \cdot v'_i(X_i)}{v_i^2(X_i)} > 0 \\
&\Leftrightarrow \qquad \sum_{i=A,B} \frac{h_i(X_i) \cdot v'_i(X_i)}{v_i^2(X_i)} \left(\frac{h'_i(X_i) \cdot v_i(X_i)}{h_i(X_i) \cdot v'_i(X_i)} - 1 \right) > 0 \\
&\Leftrightarrow \qquad \left[\frac{h'_A(X_A) \cdot v_A(X_A)}{h_A(X_A) \cdot v'_A(X_A)} + \frac{h'_B(X_B) \cdot v_B(X_B)}{h_B(X_B) \cdot v'_B(X_B)} \right] < 2 \\
&\Leftrightarrow \qquad \frac{h'_A(X_A) \cdot v_A(X_A)}{h_A(X_A) \cdot v'_A(X_A)} \cdot \frac{h'_B(X_B) \cdot v_B(X_B)}{h_B(X_B) \cdot v'_B(X_B)} < 1
\end{aligned}$$

Where the third implication uses that v_A is decreasing and the last one uses that the sum of two positive numbers being less than two implies that the product is less than one.

The determinant of the Hessian H for $V_A(X)$ is

$$\begin{aligned}
\det(H) \quad = \quad & [1 - F_A(X_A)] \cdot [1 - F_B(X_B)] \cdot \\
& \cdot [h'_A(X_A) \cdot h'_B(X_B) \cdot v_A(X_A) \cdot v_B(X_B) \; - \; h_A(X_A) \cdot h_B(X_B) \cdot v'_A(X_A) \cdot v'_B(X_B)]
\end{aligned}$$

So

$$\det(H) < 0 \qquad \Leftrightarrow \qquad \frac{h'_A(X_A) \cdot v_A(X_A)}{h_A(X_A) \cdot v'_A(X_A)} \cdot \frac{h'_B(X_B) \cdot v_B(X_B)}{h_B(X_B) \cdot v'_B(X_B)} < 1$$

And $\det(H) < 0$ rules implies that X is a saddle point, and thus not optimal. Thus it must be that the solution is not interior. □

Proposition 3

- If the projects are not cost-effective when assorted, then it is efficient to work on them in sequence starting with the least promising one.
- If the projects are cost-effective when assorted, then it is efficient to work more on the most promising project.

Proof. We prove the two parts of the proposition separately.

Part I : If projects are not cost-effective when assorted, then by Lemma 8 we know that $h_i(x)/v_i(x)$ is decreasing for $i = A, B$. By Lemma 9 we know that it must be optimal to work on the projects in sequence. It remains to show that it is efficient to start with the less promising project.

Assume WLOG that $p_A > p_B$ and consider the alternative problem in which the initial beliefs are symmetric p_A for both projects. By symmetry, there must be two solutions $(X^*, 0)$ and $(0, X^*)$. Consider the second solution, the one that works on project B. The original problem is the same as the continuation problem of that problem after x attention was allocated to project B without success, where x is such that $p_A(x) = p_B$. Since the continuation strategy cannot be suboptimal, it must be that $(0, X^* - x)$ is a solution to the original problem. Thus, the agent works on the projects in sequence starting from the least promising one.

Part II :

$$\frac{h_i(X_i)}{v_i(X_i)} \searrow \quad \Leftrightarrow \quad h_i'(X_i) \cdot v_i(X_i) - h_i(X_i) \cdot v_i'(X_i) < 0$$

$$\Leftrightarrow \quad \frac{h_i'(X_i) \cdot v_i(X_i)}{h_i(X_i) \cdot v_i'(X_i)} > 1$$

So,

$$\frac{h_i(X_i)}{v_i(X_i)} \searrow \text{ for } \quad i = A, B \quad \Rightarrow \quad \det(H) = \frac{h_A'(X_A) \cdot v_A(X_A)}{h_A(X_A) \cdot v_A'(X_A)} \cdot \frac{h_B'(X_B) \cdot v_B(X_B)}{h_B(X_B) \cdot v_B'(X_B)} > 1$$

This implies that there is at most one interior candidate for solution that satisfies the first order conditions $h_A(X_A) \cdot v_B(X_B) = h_B(X_B) \cdot v_A(X_A) = c$, and that if this candidate exist, it is the actual solution.

As before, assume WLOG that $p_A > p_B$ and consider the alternative problem in which the initial beliefs are symmetric p_A for both projects. By symmetry, the solution candidate is symmetrical (X^*, X^*). If $p_B > p_A(X^*)$, the original problem is the same as the continuation problem of that problem after x attention was allocated to project B without success, where x is such that $p_A(x) = p_B$. Since the continuation strategy cannot be suboptimal, it must be that $(X^*, X^* - x)$ is a solution to the original problem. If $p_B < p_A(X^*)$, the solution will not be interior. In that case, the agent works only on project i in the first stage with $h_i(X_i^*) \cdot v_j(0) = c > h_j(0) \cdot v_i(X_i^*)$. Since $h_i(x)/v_i(x)$ is decreasing, it must be that $h_i(0)/v_i(0) > h_j(0)/v_j(0)$, what proves that i is the project with highest prior (since p is decreasing in x, $\hat{h}(p)/\hat{v}(p)$ is increasing).

□