Market-Based Mechanisms*

Quitzé Valenzuela-Stookey[†] Francisco Poggi[‡]

December 7, 2022

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Abstract

We study the problem of a principal who conditions their actions on the outcomes of a competitive market as a proxy for an unobserved payoff-relevant state. Agents in the market have private information about the state, and their choices reflect both their beliefs about the state and their expectations of the principal's actions. This introduces two-way feedback between policy and the market. In a general setting, we characterize the set of joint distributions of market outcomes, principal actions, and states that can be implemented in equilibrium by a principal with commitment power. We focus in particular on implementation under constraints imposed by concerns about manipulation and equilibrium multiplicity. Our characterization of the implementable set admits a tractable representation, and significantly simplifies the principal's design problem. We apply our results to study bailout policies.

Keywords: Feedback effect, Implementation, Market manipulation, Equilibrium multiplicity, Price informativeness, Rational expectations equilibrium

*We are grateful to Marciano Siniscalchi, Jeff Ely, Eddie Dekel, Piotr Dworczak, Alessandro Pavan, Ludvig Sinander, Nicolas Schutz, and Gabriel Ziegler for discussion and feedback. We also thank seminar participants at Northwestern University, EWMES 2020, Bocconi University, California Institute of Technology, Cornell University, Carnegie Mellon, LSE, Microsoft Research, NYU Stern, Penn State, Toulouse School of Economics, UC Berkeley, UC Davis, University of Manchester, University of Mannheim, University of Pennsylvania, Vanderbilt University, 2021 MaCCI Summer Institute in Competition Policy, and University of Warwick for helpful comments.

[†]Department of Economics, UC Berkeley, Berkeley, CA 94720, U.S.A., quitze@berkeley.edu

[‡] Department of Economics, University of Mannheim, Mannheim 68161, Germany. Support by the German Research Foundation through CRC TR 224 (Project B02) is gratefully acknowledged.

Consider a mechanism design problem in which a principal wishes to learn about an unknown state before taking an action, but information about the state is dispersed among a large number of agents. In theory, the principal could contract individually with each of the agents to elicit their private information. I practice, this may be difficult if each agent has only limited information about the state, and the principal must therefore communicate with a large number of agents in order learn meaningfully.

Suppose however that the agents in question play some game, which produces an observable outcome. Call this game a market, and the outcome a price. One of the fundamental insights of information economics, going back at least to Hayek (1945), is that market outcomes can aggregate dispersed information. Thus the principal may be able to learn about the state by observing the price, without the need to contract separately with each agent. Indeed, policy makers facing uncertainty often use, or are encouraged to use, market outcomes, such as prices in financial markets, to inform their decisions.

This paper studies the general implementation/mechanism design problem of using market outcomes to inform decision making in settings where there is a *feedback* from policy to markets. To fix ideas, consider a government agency (the principal) deciding how much support to offer to a distressed company. If the company's shares are publicly traded, the principal may use the share price to learn about the company's unknown fundamentals and inform its bailout decision. The difficulty is that the information revealed by the share price depends on the joint distribution of prices and fundamentals, which is an equilibrium object. In particular, investors respond to the anticipated level of intervention by the principal, as well as their own private information. The principal must account for this feedback effect when drawing inferences from the price.

We study a model in which a principal commits ex-ante to a *decision rule* which specifies the principal's action as a function of the price in a competitive market.[1]

[1] In general the "price" could be any one-dimensional market outcome, such as the unemployment rate or order volume.

We adopt a general approach to modeling the market, which nests a wide range of market micro-structures. The equilibrium market price depends on both an unknown state and the anticipated action of the principal. The principal's payoff can be any function of the joint distribution of the state, price, and principal action.

Such market-based policies face a number of practical challenges. For one, they may be vulnerable to manipulation by market participants. For example, investors can distort their demands to shift the share price, and by extension influence the principal's action.[2] Additionally, the feedback loop between the principal's action and the endogenously determined share price can exacerbate issues of equilibrium multiplicity and induce non-fundamental market volatility (Woodford, 1994). These concerns should be accounted for when designing policy.

Feedback effects in market-based policy, and the accompanying issues or manipulation and equilibrium multiplicity, have been previously studied in various applications (see the literature review below). Our conceptual innovation is to bring a mechanism-design/implementation perspective to the general problem of market-based policy making. From a design perspective, it is natural to first ask what exactly the principal can achieve by using a market-based decision rule (a map from prices to actions). In other words, what is the feasible set of outcomes? In particular, what joint distributions of states (e.g. company fundamentals), prices, and principal actions (level of support) can the principal implement, i.e. induce in equilibrium?

Contribution

The current paper makes four major contributions relative to the existing literature. First, we provide a general framework for studying market-based interventions in environments with feedback effects. We do this by deriving a convenient way to succinctly summarize equilibrium outcomes in a market via an *invariant representation* (Section 1.1 and Appendix A). This approach is applicable to a wide range

[2] If the market is competitive and agents are small, their ability to manipulate the price will be limited. However the extent to which they can manipulate the principal's action will depend endogenously on how sensitive the principal's decision rule is to the price.

of competitive market structures, including the canonical noisy rational expectations equilibrium model of Grossman and Stiglitz (1980) which has been widely used to study market-based policy.

Second, we use this framework to fully characterize the feasible set in outcome space. An equilibrium induces a joint distribution of states, prices, and principal actions. Equivalently, the equilibrium outcomes can be described by the marginal functions mapping states to principal actions and prices, which we refer to as the *action function* and *price functions* respectively. We fully characterize the set of such joint distributions of states, prices, and principal actions that can be implemented by a principal with commitment power. More importantly, given the practical concerns of market manipulation and equilibrium multiplicity, our main characterization results (Section 2.5) characterize the set of joint distributions that can be induced as the *unique* equilibrium using a decision rule that satisfies a large-market notion of *robustness to manipulation*. The results provide novel insights into the interaction between feedback effects, equilibrium multiplicity, and market manipulation.

This characterization can be viewed as an answer to an implementation question: we characterize the set of fully implementable social choice functions, i.e. maps states to actions and prices, under a large-market incentive constraint. This characterization is also useful as the first step in solving a mechanism design problem. Rather than optimize over decision rules mapping prices to principal actions, our characterization results allow us to reformulate the principal's problem as the much simpler one of choosing an action function (mapping the state to the principal's action) subject to a simple implementability constraint. We show that in many applications, it is sufficient for the principal to choose the action function subject only to the constraint that the induced equilibrium map from states to prices is monotone.[3]

Third, we show that the constraints of unique implementation and robustness to

[3] Existing analyses of market-based policy design optimize over the space of decision rules (see for example Hauk et al. (2020)). Generally, this approach requires one to impose restrictions on the environment and parameterize the space admissible decision rules in to solve for equilibrium outcomes in closed form as a function of decision-rule parameters.

manipulation imply a natural notion of robustness to model misspecification (Section 3). This means the principal's payoff is not highly sensitive to their potentially limited understanding of market fundamentals. Finally, the results also allow us to analyze optimal policy when the requirement of unique implementation is relaxed (Section 3). In particular, we use our characterization of the implementable set to show that if the principal takes a worst-case view of equilibrium multiplicity then the restriction to unique implementation is generally without loss of optimality.

In Section 4, we apply our results to study the design of market-based bailout policy. We show how the principal must moderate intervention to account for manipulation and multiplicity concerns, and clarify the trade-off between the optimal actions and the efficiency of information aggregation.[4]

Related literature

Formally, this paper is one of mechanism design and implementation theory (see Jackson (2001) for an overview). In the language of this literature, we are interested in full implementation of a social choice function under a large-market notion of incentive compatibility. The no-manipulation criterion is similar in spirit to large-market notions of IC, such as those of Budish (2011) and Azevedo and Budish (2019), in that agents are assumed to be price takers and we require only a limiting notion of robustness to small manipulations. The connection between robustness to manipulation, equilibrium uniqueness (i.e. full implementation), and structural uncertainty, discussed in Appendix D.1, parallels Oury and Tercieux (2012), albeit in a very different setting.

The distinguishing feature of the current paper relative to the design/implementation literature is that the interaction between the principal and the agent is mediated by a market: the principal cannot observe the actions of the agents directly, but instead can only condition their action on the aggregate market outcome (price). In this sense, the

[4]Further applications to carbon cap-and-trade and monetary policies are presented in Appendix G.

paper is also related to the literature on mechanism design with limited communication, such as Mookherjee and Tsumagari (2014), with the distinguishing feature that agent's "individual messages" to the principal are aggregated into a single "aggregate message", the price, via a market.

Separately, in terms of applications, the paper is also related to a large literature in macroeconomics and finance on the two-way feedback between financial markets and the real economy, beginning with Baumol (1965). We are by no means the first to recognize the presence of two-way feedback effects when policy is conditioned on markets. These forces are important in many contexts, such as monetary policy. Important contributions include Bernanke and Woodford (1997), Ozdenoren and Yuan (2008), Bond and Goldstein (2015), Glasserman and Nouri (2016), Boleslavsky et al. (2017), and Hauk et al. (2020). For a survey of this literature see Bond et al. (2012).[5]

Broadly, our contribution relative to this literature is to bring a design and implementation perspective to policy-making in these settings. We formalize the problem of policy design under commitment in a general setting and provide a full characterization of feasible policy outcomes, while accounting for manipulation and equilibrium multiplicity concerns. This implementation question has not previously been studied in the literature. Thus while we study similar environments, our analysis is formally quite different from the literature on feedback effects in macroeconomics and finance.

Others have noted that policy based on market outcomes may be vulnerable to manipulation. Goldstein and Guembel (2008) study manipulation by strategic traders when firms use share prices in secondary financial markets to guide investment decisions. In Lee (2019) a regulator uses stock-price movements of affected firms to determine whether or not to move forward with new regulation. The discontinuous nature of the policy considered in their model opens the door to manipulation. Motivated by these concerns, we focus on policies that are robust to small manipulations.

The literature has also documented the fact that feedback effects may induce

[5]Closely related is the literature on prediction markets and conditional decision markets, e.g. Teschner et al. (2017), in which a principal conditions their actions on a market outcome.

equilibrium multiplicity (see among others Dow and Gorton (1997), Bernanke and Woodford (1997), Angeletos and Werning (2006), Glasserman and Nouri (2016)). However necessary and sufficient conditions for a policy to induce a unique equilibrium have not been established in a general setting, nor under the additional constraint of robustness to manipulation. To our knowledge, we are the first to study market-based policy under commitment subject to equilibrium uniqueness as a constraint.[6]

On a methodological level, our results characterizing the implementable set in *outcome space* (the set of action and price functions) greatly simplify the principal's problem. Hauk et al. (2020) develop variational techniques optimizing directly over the space of decision rules in settings with feedback effects. In contrast, the problem of optimizing over action and price functions is often much more tractable, allowing for flexible policy design and weaker assumptions on the market.

1 Model

The baseline model consists of the following primitive objects.

i. A *state space* Θ, which is a convex subset of a topological vector space, endowed with the Borel σ-algebra and a probability measure ν.

ii. A convex and compact set $\mathcal{A}$ of *principal actions*, a subset of a Banach space.

iii. A set $\mathcal{P} = \mathbb{R}$ of *prices*.

iv. A set $\mathcal{W} \subseteq \mathcal{A}^{\mathcal{P}}$ of *admissible decision rules*.[7]

The set $\mathcal{W}$ embodies restrictions on the principal's policy. For example, the principal might be constrained to use only continuous mappings from $\mathcal{P}$ to $\mathcal{A}$.[8] We assume that all constant policies are admissible. Formally, letting M_a be the constant decision rule for action $a \in \mathcal{A}$, we have $M_a \in \mathcal{W}$ for all $a \in \mathcal{A}$. The timing of interaction is as

[6] A related concern is equilibrium non-existence, which is a particularly salient issue when the principal lacks commitment power, see for example Bond et al. (2010) and Siemroth (2019). These papers focus on the fact that (in the language of the current paper) if the first-best action and price functions violate measurability (i.e. there are distinct states with the same price but different principal actions) then the first-best is not an equilibrium outcome, and an equilibrium may fail to exist. Siemroth (2019) identifies conditions for existence in a model without commitment.

[7] $\mathcal{A}^{\mathcal{P}}$ is the set of functions from $\mathcal{P}$ to $\mathcal{A}$.

[8] We discuss the relationship between robustness to manipulation and a relaxed notion of continuity in Section 2.1.

follows.[9]

1. The principal publicly commits to a *decision rule* $M \in \mathcal{W}$.[10]
2. A "market game" is played and a price is determined in equilibrium.
3. If the price is p, the principal takes the action $M(p)$.

It remains to specify what precisely is meant by the "market game". This is the final primitive feature of the model.

1.1 The market

Defining the market in suitable generality takes some work, but the upside is that we are able to treat many different types of markets within a unified framework. To preview, the following are examples of markets covered by our analysis.

1. *Private values.* A continuum of price-taking agents have private values for an asset. Let $x_i(s_i, p, a)$ be the demand of agent i with value s_i when the price is p and the agent anticipates the principal will take action a. Assume $p \mapsto x_i(s_i, p, a)$ is strictly decreasing for all s_i, a. Let $\theta = \{s_i\}_{i \in I}$ be the profile of types. An equilibrium is defined by a price function $P : \Theta \mapsto \mathcal{P}$ satisfying market clearing

$$0 = \int_{i \in I} x_i(s_i, P(\theta), M \circ P(\theta)) di.$$

   For example, the agents may be firms trading carbon credits (as in Appendix G.1). The firm's type is it's abatement cost (cost of reducing emissions). The principal's action is some policy that affects the abatement cost.

2. *Labor market.* There are a continuum of firms and workers. The principal is a government agency which commits to a map M from the unemployment rate p to a level of unemployment benefits a. Each worker's search effort is determined by the level of benefits, and their probability of being unemployed depends on their search effort and labor demand conditions θ. Given benefit level a,

[9] Alternatively, one can interpret the model as representing the steady state of a repeated interaction, in which steps 2 and 3 alternate indefinitely.

[10] In Appendix G.1 we briefly discuss the implications of our model for the no-commitment case.

worker i's probability of being unemployed can therefore be summarized by a function $u_i(a, \theta)$. An equilibrium is defined by a function $P : \Theta \mapsto \mathcal{P}$ such that $\int_i u_i(M \circ P(\theta), \theta) = P(\theta)$.[11]

3. *Common values, imperfect information.* Consider a standard rational expectations equilibrium (REE) model of an asset market. The asset's ex-post value given state $\theta \in \mathbb{R}$ and principal action a is $\pi(\theta, a)$. Each agent observes a private signal $s_i = \theta + \varepsilon_i$. Agents then submit a demand schedule $X_i(p, s_i)$ to a market maker, and have utility $u_i : \mathbb{R} \to \mathbb{R}$. Given a decision rule M, P is called a REE price function iff there exists a function $X_i(p, s_i)$ s.t.
   (a) $X_i(p, s_i) = \arg\max_x \mathbb{E}\left[u_i\big(x \cdot (\pi(M(p), \theta) - p)\big) \mid s_i, P(\theta) = p\right]$, and
   (b) $\int X_i\left(P_M(\theta), s_i\right) di = 0 \quad \forall \quad \theta \in \Theta$.
4. *Noisy REE (Grossman and Stiglitz, 1980).* The model is as in the previous example, except that there is aggregate uncertainty in the form of a supply shock z. This is a workhorse model of asymmetric information in asset markets. The asset value is $\pi(a, \omega)$ and signals are $s_i = \omega + \varepsilon_i$. The state includes the payoff-relevant state and supply shock: $\theta = (z, \omega)$. Market clearing means

$$\int X_i\left(P(\theta), s_i\right) di = z \quad \forall \quad \theta \in \Theta.$$

We now define formally the conditions characterizing the set of markets to which our analysis applies. After the principal chooses the decision rule M, a set of market participants play some continuation game. The actions of the market participants in this continuation game, together with the state, determine a price in $\mathcal{P}$. Let this relationship be represented by $\tilde{P}(\theta, \psi)$ where ψ is a profile of actions taken by the market participants.

The game is paired with a solution concept, which for each M defines a set of equilibria in the continuation game. We call the game/solution-concept pair a *market* if for any decision rule M and equilibrium strategy profile ψ^*_M, the map $\theta \mapsto \tilde{P}(\theta, \psi^*_M)$

[11] We can interpret this fixed-point condition as a steady-state condition in a dynamic model.

is a deterministic function.[12] In other words, given a decision rule M, for any equilibrium in the market there exists a *price function* $P : \Theta \to \mathcal{P}$ describing the price which realizes in each state. We maintain the assumption that there exists at least one equilibrium for every constant decision rule, i.e. every fixed principal action.

There may be multiple equilibria given M, with distinct price functions. We say that P is an *equilibrium price function* given M if it is the price function in some equilibrium given decision rule M. The fact that the set of equilibrium price functions depends on M is the source of the feedback effect.

We restrict attention to the subset of markets that are "competitive". Loosely, a market is competitive if agents behave as price takers. We show that many competitive markets possess the following convenient property.

Definition 1. A market *admits an invariant representation* given $(\mathcal{W}, \hat{\mathcal{P}})$, where $\hat{\mathcal{P}} \subset \mathcal{P}^{\Theta}$, if there exists a function $R : \mathcal{A} \times \Theta \to \mathcal{P}$ such that

1. If P is an eq. price function given $M \in \mathcal{W}$, then $P(\theta) = R(M \circ P(\theta), \theta)$ $\forall \theta$.
2. If $P \in \hat{\mathcal{P}}$ and $P(\theta) = R(M \circ P(\theta), \theta)$ for all θ then P is an eq. price function given M.

The key general observation for our analysis is that, for the purposes of choosing M, a market with an invariant representation can be fully summarized by R: that is, $P \in \hat{\mathcal{P}}$ is an equilibrium price function given M iff $P(\theta) = R(M \circ P(\theta), \theta)$.[13]

It may be that a given market only admits an invariant representation given a $\hat{\mathcal{P}}$ that is a strict subset of $\mathcal{P}^{\Theta}$, and $\mathcal{W}$ that is a strict subset of $\mathcal{A}^{\mathcal{P}}$. In this case, the extent to which the property is useful depends on whether $\hat{\mathcal{P}}$ and $\mathcal{W}$ contain

[12] The assumption that this function is deterministic just means that θ captures all the relevant uncertainty in the market. This definition of a market does not imply that the agents' strategies must be measurable with respect to the state. The state may contain dimensions that are not directly payoff relevant for the principal. For example, in a noisy REE model of an asset market, as in Grossman and Stiglitz (1980), the state will include the supply shock and the "payoff relevant state", but not the agents' private signals. In other settings, the state can represent the entire profile of agents' private signals, as in Jordan (1982).

[13] Moreover, since R does not depend on the decision rule M, it can be estimated using data from a market in which the principal's action is not conditioned on the price, or in which some other decision rule was used. Thus a principal contemplating the introduction of a market-based decision rule can use historical aggregate data to estimate the function R and design the decision rule, without being subject to the Lucas critique that a change in the policy regime will change the relationship between the fundamentals (state and principal action) and the price (Lucas et al., 1976).

the relevant price and action functions (see Appendix A for further discussion). The smaller the sets $(\mathcal{W}, \hat{\mathcal{P}})$, the easier it is to satisfy the conditions' definition. Indeed, if R is an invariant representation given $(\mathcal{W}'', \mathcal{P}'')$ and $\mathcal{W}' \subset \mathcal{W}''$, $\mathcal{P}' \subset \mathcal{P}''$, then R is also an invariant representation given $(\mathcal{W}', \mathcal{P}')$. On the other hand, the smaller is $(\mathcal{W}, \mathcal{P})$, the less useful the property.

Determining whether or not the market admits an invariant representation with respect to the desired $(\mathcal{W}, \hat{\mathcal{P}})$ is the first step in our analysis. Based on Definition 1, it is not immediately obvious which markets admit an invariant representation. In part, this is because of the existential qualifier in the definition. In Appendix A we provide an axiomatic characterization of the markets that admit an invariant representation. This makes it relatively easy to check whether a given market satisfies this property.

In most applications it is enough, for practical purposes, that there exists a function R which describes the equilibrium for almost all states. The market admits an *a.e. invariant representation* given $(\mathcal{W}, \hat{\mathcal{P}})$ if Condition 1 in Definition 1 holds for almost all θ (with Condition 2 unchanged). To facilitate applications, we show the following.

Proposition 1. In the above examples[14]

- Markets 1-2 admit an invariant representation in $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$.
- Market 3 admits an a.e invariant representation in $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$. The a.e. qualifier can be dropped if $\theta \mapsto \pi(a, \theta)$ is strictly increasing for all a.
- Market 4 admits an a.e invariant representation, where $\mathcal{W}$ is the set of decision rules that induce a unique equilibrium, and $\hat{\mathcal{P}}$ is the set of price functions with "non-intersecting level sets".[15] (Under the assumptions of CARA utility, π affine in ω, additive normal signal structure and normally distributed supply shocks, which are standard in the literature.)

[14]Many of the parametric assumptions imposed on these markets can be relaxed. See Appendix F.2.

[15]These sets $\mathcal{W}$ and $\mathcal{P}$ are discussed in detail in Section 4.2. We are interested in unique implementation, so the restriction to such decision rules is without loss. Moreover, we show that the "non-intersecting level sets" property is a necessary condition for unique implementation, and so the restriction to this set of price functions is also without loss.

Proof. In Appendix C.1. □

These results make use of the axiomatic characterization in Appendix A. Again, the important takeaway from Proposition 1 is that in all the markets in question, we know that P is an equilibrium price function given $M \in \mathcal{W}$ if $P \in \hat{\mathcal{P}}$ and $P(\theta) = R(M \circ P(\theta), \theta)$; and that in any equilibrium this condition holds almost everywhere (so for design purposes, we can focus on equilibria such that it holds for all θ).

2 Main results

Given a decision rule M and an associated equilibrium price function P, let $Q := M \circ P$ be the equilibrium *action function*. That is, Q is the induced equilibrium map from states to actions, and, together with the prior distribution on Θ, the pair (Q, P) fully describes the equilibrium joint distribution of states, prices, and principal actions. Since this joint distribution is ultimately what the principal cares about, it is convenient to formulate the design problem directly over price and action functions, i.e. the *equilibrium outcomes*, rather than the decision rule M. In order to do this, we must characterize the set of (Q, P) that are equilibrium outcomes for some $M \in \mathcal{W}$. This is the focus of this section.

Definition 2. (Q, P) is *implementable in* $\mathcal{W}$ if there exists $M \in \mathcal{W}$ such that P is an equilibrium price function given M and $Q = M \circ P$. Say simply that (Q, P) is implementable if it is implementable in $\mathcal{A}^{\mathcal{P}}$.

Observation 1. If the market admits an invariant representation R given $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$ then following are equivalent[16]

i. (Q, P) is implementable.

ii. $Q(\theta) \neq Q(\theta') \Rightarrow P(\theta) \neq P(\theta')$, and $P(\theta) = R(Q(\theta), \theta)$ for all θ.

We refer to $Q(\theta) \neq Q(\theta') \Rightarrow P(\theta) \neq P(\theta')$ as the *measurability condition*, and $P(\theta) = R(Q(\theta), \theta)$ for all θ as the *market-clearing* condition. The "only if" direction

[16]In fact, the converse holds as well: if (i) and (ii) are equivalent then R is an invariant representation of the market.

of Observation 1 characterizes the set of implementable (Q, P) for markets that admit an invariant representation.

Unfortunately, implementability in Observation 1 relies on the ability to use any decision rule. However, some of these may have undesirable properties, such as vulnerability to manipulation and equilibrium multiplicity. We therefore refine the implementability criterion to address these concerns.

2.1 Manipulation

Market manipulation is a salient concern in many market-based policy-making environments. Agents in the market may attempt to manipulate the price in order to influence the principal's action by buying/selling an asset, releasing false information, or other means.[17] While, in the models we consider, agents are generally assumed to behave as price takers, we view the price-taking assumption as an idealization of a world in which agents are small, but may have some non-zero market power. The ability of a small (but not infinitesimal) agent to manipulate the principal depends on the sensitivity of the principal's decision rule mapping prices to actions. If, for example, the decision rule is discontinuous, then an agent will be able to induce a significant change in the principal's action by manipulating the price, even if their individual price impact is small.

In order to maintain consistency between the idealized model in which agents are price takers and one in which agents are small, but may have a non-zero price impact, it seems natural to restrict the principal to use a continuous decision rule. However the restriction to everywhere-continuous decision rules is stronger than is needed to address these concerns, and may come at a cost.[18] If a discontinuity in the decision rule of the principal occurs at a price that is far from any which could arise in equilibrium then manipulation via a small price impact will not be possible.

Definition 3. A decision rule $M : \mathcal{P} \to \mathcal{A}$ is **essentially continuous** if, for any

[17]Goldstein and Guembel (2008) discusses manipulation of this sort.

[18]Appendix G.2 studies an example in which discontinuities *away* from equilibrium prices are necessary for unique implementation.

equilibrium price P given M, M is continuous on an open set containing the closure of $P(\Theta)$.

In other words, an essentially continuous decision rule can have discontinuities only where there are no nearby equilibrium prices. Let $\mathcal{M}$ be the set of essentially continuous decision rules. We also refer to such decision rules as "robust to manipulation". Importantly, essential continuity (or even continuity) of the decision rule does not imply that equilibrium price functions must be continuous.

2.2 Multiplicity

A second concern is that the dependence of the principal's action on the endogenously determined price can lead to multiple equilibria, since there may be multiple self-fulfilling beliefs that agents in the market can hold about what action the principal will take (Bernanke and Woodford, 1997). This type of multiplicity is pervasive in market-based policy problems. In reality, the principal is often unable to select which equilibrium will be played. Moreover, the fact that there are multiple equilibria could lead to non-fundamental volatility in the market, as agents coordinate on one or another belief about what action the principal will take. This type of volatility is a first-order concern in many settings in which market-based policies are used, such as monetary policy (Woodford, 1994). We are therefore interested primarily in unique implementation.[19]

Definition 4. M is **robust to multiplicity** if there is at most one equilibrium price function P given M.

Unique implementation is desirable in many settings, especially those in which non-fundamental volatility is a first-order concern. Moreover, we show in Section 3 how this restriction is without loss of optimality when the principal takes a worst-case approach to equilibrium multiplicity.

[19]In other words, fully implementable of a social choice functions.

2.3 Robust implementation

With the definitions of robustness to manipulation and multiplicity, we are ready to define the refined implementation criterion, which is our primary focus.

Definition 5. Let $\mathcal{W}^*$ be the set of essentially continuous decision rules that are robust to multiplicity. We say that (Q, P) is *continuously uniquely implementable* (CUI) iff (Q, P) is implementable in $\mathcal{W}^*$.

In some cases, it will also be useful to consider a slightly weaker notion of robustness to multiplicity. The idea is that the principal might be willing to have multiple equilibria as long as different equilibria lead to the same outcomes almost surely.

Definition 6. A decision rule M is *weakly robust to multiplicity* if for any two equilibrium price functions P, P' given M it holds that $P(\theta) - P'(\theta) = 0$ for almost every state θ.

Definition 7. Let $\mathcal{W}^-$ be the set of essentially continuous decision rules that are weakly robust to multiplicity. We say that (Q, P) is *continuously weakly uniquely implementable* (CWUI) iff (Q, P) is implementable in $\mathcal{W}^-$.

Before proceeding to the characterization of CUI and CWUI pairs (Q, P) we illustrate, with a simple example, the issues that arise because of the interaction between feedback effects, equilibrium multiplicity, and robustness to manipulation.

2.4 A brief illustration: Bailouts

To fix ideas consider the following application, which we return to in greater detail in Section 4. The principal must choose a level of support $a \in \mathcal{A} = [0, 1]$ to provide to a public company. The company's prospects $\theta \in \Theta \subseteq \mathbb{R}$ (representing the demand environment, competition, future costs, etc.) are unknown. Higher states represent better prospects: the ex-post cash flow from a share of the company is $\pi(a, \theta) = \beta_0(a) + \beta_1(a)\theta$, where β_0 is increasing and β_1 is strictly positive and decreasing.[20]

[20] Linearity of the cash-flow is not important here, but is useful in the noisy REE model of Section 4.2, and therefore maintained for clarity.

In other words, greater intervention reduces the sensitivity of the cash flow to the state. As a result, greater intervention has a larger positive impact on cash flow when the state is low, and may decrease this value when the state is high.[21] For simplicity, assume also that $\beta_0'(a)/\beta_1'(a)$ is constant, so there exists a state θ^* such that $a \mapsto \pi(a, \theta^*)$ is increasing below θ^* and decreasing above.

Shares are traded in a competitive stock market, as in Example 3 in Section 1.1. In Proposition 1 we show that there exists an invariant representation R for this type of market. In fact, for this example the equilibrium price must be fully revealing, so $R = \pi$ and thus $\theta \mapsto R(a, \theta)$ is continuous and strictly increasing for every a, and $a \mapsto R(a, \theta)$ is increasing for states below θ^*, and decreasing above.

The principal's ex-post payoff is $u(a, \theta) = (1 - a)b\theta - ca$. Thus, they would like to choose the maximal intervention $a = 1$ if $\theta \leq -\frac{c}{b}$, and $a = 0$ otherwise. We refer to this as the first-best action function. We say that the principal is *hawkish* if $-\frac{c}{b} > \theta^*$, and *dovish* if $-\frac{c}{b} < \theta^*$.[22] The price functions corresponding to the first best action functions are illustrated by the dotted blue lines in Figure 1.[23]

In the hawkish case, the first best is implementable. In fact, we show that it is CUI: it can be implemented uniquely with a decision rule that is robust to manipulation. However the dovish first-best is not implementable: the first-best action function is not measurable with respect to the price for prices between p' and p''.[24]

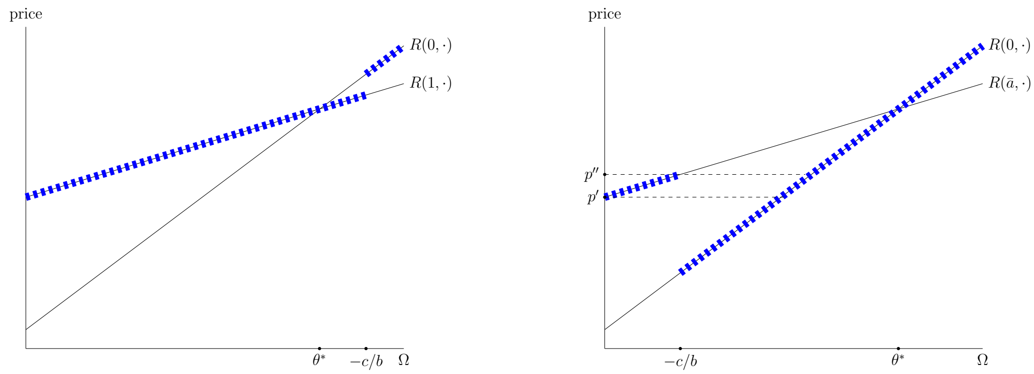
[21] This reflects the fact that investors view government involvement in the firm as reducing upside when business prospects are good; for example because the bailout involves the government taking an active role in management, or carries negative stigma (Che et al., 2018). $a \mapsto \pi_2(a, \theta)$ will also be decreasing if the bailout takes the form of forgivable loans, where the ex-post liability is increasing in the state (which will be revealed *ex-post*).

The German government's 2020 bailout of Lufthansa illustrates this pattern. In this case, one large shareholder threatened to veto the proposed bailout. This shareholder was reportedly concerned that the government stake would make it harder to restructure and cut jobs. On the other hand, the supervisory board chairman emphasized Lufthansa's dire prospects: "We don't have any cash left. Without support, we are threatened with insolvency in the coming days." (Wissenbach and Taylor, 2020).

[22] If the principal is hawkish there are states in which the principal would like to intervene even if investors perceive a negative impact on cash flows. This is the case if the social benefit of intervention is large relative to the cost, for example, because the company is considered strategically important, employs a large number of workers, or engages in production which has large technological spillovers. Conversely, the principal is dovish when the social benefit or intervention is judged to be low relative to the cost. In this case there are states in which the principal would like to forgo intervention even though intervening would increase the cash flow.

[23] Given an action function Q, the corresponding price function is $P(\theta) = R(Q(\theta), \theta)$.

[24] In a similar example, Bond et al. (2010) observe that if the principal observes a sufficiently precise exogenous signal then they can overcome the measurability issue, and restore implementability.



(a) Hawkish, implementable

(b) Dovish, not implementable

Figure 1: First-best

What is a dovish principal to do? Notice that the "problematic" range of prices (p', p'') is relatively small. One natural solution to the measurability problem is to move away from the first best slightly so as to fix the measurability issue. Indeed, when (p', p'') is small, this can be done with a relatively small perturbation to the action function. Figure 2 illustrates the price function corresponding to the action function which agrees with the first-best action function for all states except (θ', θ''). On (θ', θ''), the action function specifies intermediate actions so as to induce the depicted price function. If the interval (θ', θ'') has low probability, this modification provides a good approximation to the first-best payoff, and is implementable by Observation 1.

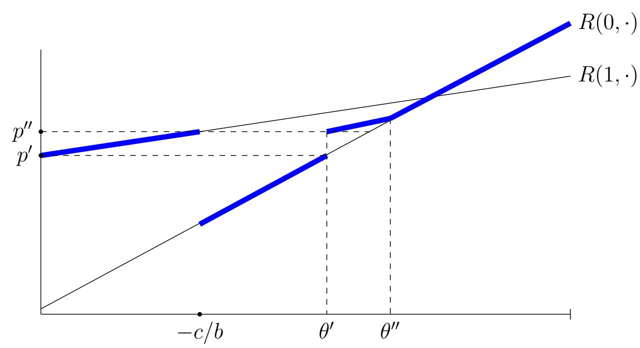


Figure 2: Restoring implementability

Without any concern for manipulation or a desire for unique implementation, the optimal policy for a dovish principal may well take the form depicted in Figure 2. In fact, if the principal is concerned about either manipulation or equilibrium multiplicity, but not both, it is easy to show that an action function nearly identical to that in Figure 2 can be implemented. However, surprisingly, no nearby action function can be implemented if *both* concerns are present, i.e. if the principal wants to implement something close to first-best as the unique equilibrium using an essentially continuous decision rule (by Theorem 1). We turn now to establishing this result.

2.5 Characterizing CUI outcomes

In this section, we establish that, when the market admits an invariant representation, the defining feature of CUI and CWUI outcomes is a monotone price. Specifically, we show that a monotone price function is necessary when the invariant representation R is weakly increasing in the state (Theorem 1), and essentially sufficient under additional mild conditions (Theorem 2).

For simplicity, we assume in this section that the market admits and invariant representation in$(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$. This assumption can be easily relaxed, and we do so in Section 2.7. To simplify the statement of the results, we also assume that the state space is an open bounded interval $\Theta = (\underline{\theta}, \bar{\theta})$.[25] We then extend the results to multidimensional Θ. We also assume that the invariant representation R is continuous in (a, θ) and (weakly) increasing in the state θ for all actions. [26] Finally, we add two technical conditions on the invariant representation at the extreme states. Let $R(a, \underline{\theta}) := \inf_{\theta \in \Theta} R(a, \theta)$ and $R(a, \bar{\theta}) := \sup_{\theta \in \Theta} R(a, \theta)$. First, we assume that R converges uniformly to the extremes. In other words, $R(\cdot, \theta_n)$ converges uniformly as $\theta_n \to \underline{\theta}$ and $\bar{\theta}$. This guarantees that continuity is preserved for the limit functions $R(a, \underline{\theta})$ and $R(a, \bar{\theta})$. Second, we assume that for all $p \in \mathcal{P}$ and $\theta \in \{\underline{\theta}, \bar{\theta}\}$, the set of actions for which $R(a, \theta) = p$ is the union of finitely many connected subsets of $\mathcal{A}$.[27] These technical assumptions are satisfied in all applications we consider.

[25] The results for closed θ are the same, except that it is necessary to modify the boundary conditions in Theorem 2. We omit this result in the interest of brevity.

[26] Both monotonicity and continuity of R can be justified by natural assumptions on primitives in many micro-foundations, as discussed in Appendix F.

[27] Given continuity, this assumption means that the market-clearing price at the extremes does not oscillate too frequently (as a function of the action).

Theorem 1 (Necessity)**.** If (Q, P) is CWUI, then P is monotone.

Proof. In Appendix B. □

In other words, if M is essentially continuous and induces a price function P that is non-monotone then M induces at least two equilibria with different equilibrium prices for a positive mass of states. It is worth emphasizing that a CWUI P need not be increasing; it may be monotonically decreasing, even when R is strictly increasing.

Theorem 1 reveals a surprising interaction between unique implementation and robustness to manipulation. Singly, neither constraint imposes a substantive restriction on the implementable set.[28] Jointly, however, they have important implications for what the principal can achieve (the price must be monotone).

The monotonicity condition of Theorem 1 bears no direct relation to the types of monotonicity conditions that are common in the implementation literature (such as Maskin (1999) and Myerson (1981)). These latter concern monotonicity (in various forms) of the social choice function. This distinction should not be surprising, given that the aggregate "price" has no immediate analog in the typical setting.

A monotone price function, together with the market-clearing condition $P(\theta) = R(M \circ P(\theta), \theta)$, is nearly, but not exactly, sufficient for CUI. We require an additional technical condition.

Definition 8. The action function Q satisfies boundary condition 1 (BC1) if there are $\bar{Q}, \underline{Q} \in \mathcal{A}$ such that $\bar{Q} = \lim_{\theta \to \bar{\theta}} Q(\theta)$ and $\underline{Q} = \lim_{\theta \to \underline{\theta}} Q(\theta)$. Moreover, Q satisfies boundary condition 2 (BC2) if it satisfies BC1, $R(\underline{Q}, \underline{\theta}) \neq \inf \mathcal{P}$ implies that $R(\cdot, \underline{\theta})$ doesn't have a local maximum at $\underline{Q}$, and $R(\bar{Q}, \bar{\theta}) \neq \sup \mathcal{P}$ implies that $R(\cdot, \bar{\theta})$ doesn't have a local minimum at $\bar{Q}$.

Theorem 2. Assume R is strictly increasing in θ.[29] Then (Q, P) is CUI iff

[28] Suppose (Q, P) is implementable. If we do not restrict to essentially continuous M, then we can define M outside of $P(\Theta)$ to guarantee that no additional equilibria exist. If we are not concerned about multiplicity, we can always define an essentially continuous M that approximates well the decision rule needed to implement (Q, P).

[29] In Proposition 11 we extend the result to weakly increasing R.

1. $P(\theta) = R(Q(\theta), \theta)$ for all θ,
2. P is strictly monotone.
3. Q is continuous and satisfies BC1. Moreover, if P is decreasing, Q satisfies BC2.

Proof. In Appendix B.2. □

2.5.1 Theorem 2: technical discussion

The first point in Theorem 2 is simply the market clearing condition that was already necessary for implementation (Observation 1). It is worth noting that continuity of Q is not implied by the continuity of M, but is instead a consequence of requiring unique implementation. In Section 2.6 we show that by slightly relaxing to weak robustness to multiplicity we get a characterization that allows for discontinuous Q, so we do not view condition 3 as a critical characteristic of implementable pairs. The monotonicity of P is the essential point.

Notice that for any (Q, P) that is CUI, the continuity of Q implies continuity of P on Θ, and thus $P(\Theta)$ must be convex. Given (Q, P) satisfying condition 1 of Theorem 2, and with P increasing, it is straightforward to construct an M that continuously uniquely implements it: for prices in $P(\Theta)$ simply choose the action that is consistent $M(p) = Q \circ P^{-1}(p)$, and then use $\bar{Q}$ for prices above $\sup P(\Theta)$ and $\underline{Q}$ for prices below $\inf P(\Theta)$. Moreover, this implies that if (Q, P) is CUI and P is increasing then it can be implemented by a continuous M.

When P is decreasing, the construction of M leaving actions constant for prices outside of $P(\Theta)$ does not work. The last part of condition 3 of Theorem 2 guarantees that there is a way to define a continuous M for prices slightly above $\sup P(\Theta)$ and slightly below $\inf P(\Theta)$ such that these prices can never occur in equilibrium.

Since Theorem 2 contains a strict monotonicity condition, the CUI set may not be closed. Thus it is useful to introduce a notion of virtual optimality. Say that (Q, P) is *virtually CUI* (virtually CWUI) if for any $\varepsilon > 0$ there exist CUI (CWUI) (Q', P') such that (Q, P) is within ε of (Q', P') in the sup-norm. Say that (Q, P) is *virtually* *optimal* if it is optimal among either the virtually CUI or CWUI set.

2.5.2 Theorem 2: practical implications

In words, Theorem 2 characterizes the set of fully (i.e. uniquely) implementable action-price function pairs, under the robustness to manipulation constraint. The result is also useful as the first step in solving a mechanism design problem: we can optimize directly over the space of continuous action functions Q, subject only to the constraint that the induced price function is monotone (moreover, it is generally without loss of optimality to ignore the continuity condition, as we show in Proposition 2). This is a standard control problem analogous to the first step in Myerson (1981), where we appeal to the revelation principle and the characterization of incentive compatibility to transform the auction design problem into a control problem with a monotonicity constraint. The alternative is to optimize directly over the decision rule (see for example Hauk et al. (2020) and Lee (2019) for fruitful applications of this approach). The downside of this approach is that one generally needs to make strong assumptions on the environment, and restrict attention exogenously to a parametric set of decision rules, in order to solve for equilibrium outcomes in closed form as a function of decision-rule parameters.

2.6 Characterizing CWUI outcomes

To characterize the set of CWUI outcomes, i.e. those (Q, P) that are implemented by a decision rule in $\mathcal{W}^-$, we make the assumption that the market is *fully bridgeable*.

Definition 9 (Full bridgeability)**.** For any state $\theta \in \Theta$ and actions a, a' such that $R(a, \theta) \neq R(a', \theta)$, there exists a continuous function $\gamma : [0, 1] \to \mathcal{A}$ such that

- $\gamma(0) = a$, $\gamma(1) = a'$.
- $x \mapsto R(\gamma(x), \theta)$ is strictly monotone.

If $\mathcal{A} = [0, 1]$ then full bridgeability is satisfied iff $a \mapsto R(a, \theta)$ is monotone for every θ. For more general action spaces, weaker notions of monotonicity suffice.[30]

[30] For example, suppose the principal's action consists of mixtures over a set of consequences, i.e.

Proposition 2. Assume R is strictly increasing in θ and the market is fully bridgeable. Then (Q, P) is CWUI iff

1. $P(\theta) = R(Q(\theta), \theta)$ for all θ.
2. P is strictly monotone.
3. If Q is discontinuous at θ^* then P is also discontinuous at θ^*.
4. Q is BC1. Moreover, if P is decreasing, Q is BC2.

Proof. In Appendix C.2 □

The main substantive difference between CUI and CWUI outcomes is that the action function need not be continuous.

2.7 Multidimensional state space

Suppose that Θ is an open subset of $\mathbb{R}^N$, endowed with the usual product partial order. When working with a one-dimensional state space, we were able to prove the existence of an invariant representation given $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$ by imposing monotonicity of some primitive objects, such as asset dividends, in θ (see Appendix F.2). The difficulty with moving to a multi-dimensional state space is that we cannot in general identify a complete order on Θ for which such monotonicity conditions hold. It is therefore useful, for example for the application of Section 4.2, to relax the assumption that the market admits an invariant representation given $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$.

We first introduce an additional condition: say the market has *level sets represented by* a (possibly empty valued) set function $L : \mathcal{A} \times \mathcal{P} \to 2^{\Theta}$ if $M(p) = a$ implies that there is an equilibrium with $P(\theta) = p$ for all $\theta \in L(a, p)$. We can think of the states in $L(a, p)$ as "payoff equivalent in equilibrium", given action a.

Notice that if the market admits an invariant representation given $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$ then there is an L that represents the level sets; define $L(a, p) = \{\theta : R(a, \theta) = p\}$. The

$\mathcal{A} = \Delta(Z)$ for some finite set Z, where each consequence is associated with a value $\pi(z, \theta)$ (for example, Z may be a set of conditions that the principal can attach to a bailout, and π the cash-flow of the company). Fixing the state θ, any action $a \in \Delta(Z)$ induces a distribution over the set of values $\pi(Z, \theta)$. If $R(a'', \theta) > R(a', \theta)$ whenever the distribution induced by a'' first-order stochastically dominates that induced by a', then the environment is fully bridgeable. In other words, a weak monotonicity notion suffices for full bridgeability. This result, along with more general sufficient conditions for bridgeability, is discussed in Appendix E, where we also relax the assumption.

point of identifying L is precisely to relax the assumption that there is an invariant representation given $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$.

Suppose M is a decision rule for which there exists a unique equilibrium. Let (P_M, Q_M) be the price and action functions in this equilibrium. Then it must be that if $P_M(\theta) = p$ then $P_M(\theta') = p$ for all $\theta' \in L(Q_M(\theta), p)$; if not then by definition of L there is equilibrium multiplicity in the states $L(Q_M(\theta), p) \setminus \{\theta' : P_M(\theta') = p\}$. Thus we can define an invariant representation R for the market by $R(a, \theta) = \{p : \theta \in L(a, p)$. This is a well-defined function iff there is a unique p such that $\theta \in L(a, p)\}$; in which case we say that the market has *unique level sets represented by* L. In Appendix F.3 we show that it is satisfied in the noisy REE model.

Given a market with unique level sets represented by L, say that a price function P has *complete level sets* if for all $p \in P(\Theta)$ there exists $a \in \mathcal{A}$ such that $\{\theta \in \Theta : P(\theta) = p\} = L(a, p)$. Let $\tilde{P}$ be the set of price functions with complete level sets. Let $\mathcal{W}^U$ be the set of decision rules for which there exists a unique market equilibrium. Then the preceding discussion is summarized by the following lemma.

Lemma 1. If the market has unique level sets represented by L, then it admits an invariant representation given $(\mathcal{W}^U, \tilde{\mathcal{P}})$, defined by $R(a, \theta) := \{p : \theta \in L(a, p)\}$.

Notice that in deriving the representation in Lemma 1, we are using the restrictions implied by the unique implementation requirement. Such a market need not admit an invariant representation given $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$.

Define $\bar{T}(a, \theta) = \{\theta' : R(a, \theta') = R(a, \theta)\}$. The following result is analogous to Theorem 2 in the uni-dimensional case.

Proposition 3. Assume the market level sets uniquely represented by L, and so admits an invariant representation in $(\mathcal{W}^U, \tilde{\mathcal{P}})$. Assume moreover that the representation R is strictly increasing.[31] If (Q, P) is CUI, then

i. $P(\theta) = R(Q(\theta), \theta)$.

[31] By this we mean strictly increasing in the usual product partial order on $\mathbb{R}^N$.

ii. P is strictly monotone (in the product partial order on Θ).

iii. Q is continuous.

iv. For all θ, $Q(\theta') = Q(\theta)$ for all $\theta' \in \bar{T}(Q(\theta), \theta)$.

v. $Q(\theta) \neq Q(\theta') \Rightarrow \bar{T}(Q(\theta), \theta) \cap \bar{T}(Q(\theta'), \theta') = \varnothing$.

Proof. In Appendix C.3. □

The conditions of Proposition 3 are also sufficient, except we require analogs to BC1 and BC2 for the multi-dimensional space. We omit the details for brevity. Since any market that admits an invariant representation in $(\mathcal{A}^{\mathcal{P}}, \mathcal{P}^{\Theta})$ has unique level sets represented by some L, Proposition 3 applies to such markets.

3 Properties and extensions

In Appendix D we discuss properties of CUI policies and study optimal policy when the unique implementation requirement is relaxed. Briefly, these results can be summarized as follows.

Structural uncertainty. In many settings there may be uncertainty regarding the relationship between the principal action, price, and the state. For example, there may be noise traders in the market who induce variability into the price.[32] Additionally, the principal may have limited data with which to estimate the representation R. It is therefore desirable to use a decision rule such that outcomes are suitably continuous with respect to small perturbations of R. We show that in fact any CUI action and price functions are implementable in a way that is *robust to structural uncertainty*, i.e. such that outcomes are suitably continuous in R (Theorem 4). (CWUI outcomes can be implemented so as to satisfy a weaker notion of robustness).

Beyond uniqueness In some cases, the principal may be willing to tolerate the existence of multiple equilibria. The restriction to unique implementation would be especially unappealing if the principal could choose some decision rule which induces multiple equilibria, but such that all of these equilibria dominate, from the principal's

[32]As shown in appendix F.3, an alternative way to deal with such noise is to fold it into the state.

perspective, the best equilibrium that can be implemented uniquely. We show that this is never the case. The key insight is that even if a decision rule induces multiple equilibria, at least one of these will be weakly uniquely implementable (Theorem 5). Thus for a principal who evaluates a set of possible equilibria according to the worst case, the restriction to (weak) unique implementation is without loss of optimality (Corollary 2). In other words, unique (i.e. full) implementation is without loss of optimality in a "robust mechanism design" sense (e.g. Carroll (2015)). We also study the case where the principal takes a less extreme approach to multiplicity (Proposition 9).

4 Application: bailouts

In this section we consider in greater detail the bailout setting introduced in Section 2.4. The same analysis applies to an international lender such as the IMF using a country's bond price to inform a bailout decision.[33]

4.1 Markets without noise

The principal seeks to learn about the company's degree of distress by observing its stock price.[34] We first study a market in which information aggregates perfectly, and illustrate qualitative changes in the principal's policy as a function of the relative social benefit the principal attaches to strengthening the company. In Section 4.2 we study the same problem when the market is described by the canonical noisy REE model of Grossman and Stiglitz (1980), in which there a much wider range of ways in which information can be aggregated.

Shares are traded in a competitive stock market, as in Section 2.4. We relax for now the assumption that the ex-post asset value $\pi(a, \theta)$ is linear in the state: we assume only that $\theta \mapsto \pi(a, \theta)$ is increasing for all a, and $a \mapsto \frac{\partial}{\partial \theta}\pi(a, \theta)$ is decreasing (as discussed in Section 2.4). By Proposition 1, the market admits an invariant rep-

[33]Similarly, it is argued that state-contingent debt instruments, in which payments are conditioned on variables such as GDP or commodities prices, should be used to reduce the need for protracted and costly sovereign debt restructurings (Cohen et al., 2020).

[34]Versions of this problem have been studied by Bond and Goldstein (2015) and Lee (2019), among others. The analysis here is also applies to the problem of an international lender using sovereign debt prices to inform emergency lending decisions.

resentation R. Moreover, as in Section 2.4, $R = \pi$, so $\theta \mapsto R(a, \theta)$ is continuous and strictly increasing for every a and $a \mapsto \frac{\partial}{\partial\theta}R(a, \theta)$ is decreasing.

As before, the principal wishes to choose the maximal intervention $a = 1$ if $\theta \leq -\frac{c}{b}$, and make no intervention otherwise. We refer to this as the *first-best* action function. We generalize the definitions of a hawkish and dovish principal.

Definition 10. The principal is **hawkish** if $a \mapsto \pi(a, \theta)$ is decreasing in a neighborhood of $\theta = \frac{c}{b}$, **dovish** if $a \mapsto \pi(a, \theta)$ is increasing in a neighborhood of $\theta = \frac{c}{b}$.

As discussed, the principal is hawkish (dovish) if the social benefit of intervention is large (small) relative to the cost. The following result uses an extension of Proposition 2 to environments that are not necessarily fully bridgeable (Proposition 12).[35]

Proposition 4. If the principal is hawkish, then the first-best action function is CWUI. In this case, under the optimal decision rule, letting $p' = R(1, -c/b) < p'' = R(0, -c/b)$, the principal chooses full intervention below p', no intervention above p'', and a continuously decreasing level of intervention on (p', p'').

Say that π has the *pivot property* if there exists a state θ^* such that $a \mapsto \pi(a, \theta)$ is increasing (decreasing) for $\theta < (>)\theta^*$ (as is the case in Section 2.4).

Lemma 2. If π has the pivot property then the converse to Proposition 4 holds as well: the first-best is CWUI if an only if the principal is hawkish.

If the principal is hawkish, the first best action function, which is a step function that goes from 1 to 0 at $\theta = -\frac{c}{b}$, is implemented robustly by a decision rule for which the action decreases gradually as a function of the price over a range of intermediate prices. In contrast, when the principal is dovish intermediate actions are taken in equilibrium for some states, and the (virtually) optimal decision rule features sharp jumps in the action as a function of the price. This is most clearly illustrated under the assumption that π has the pivot property.

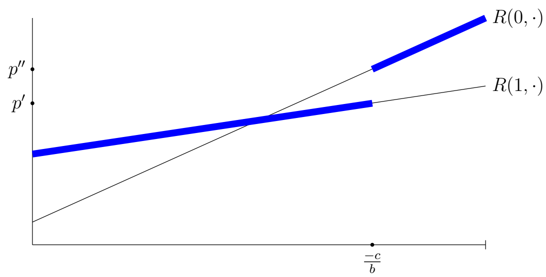
[35]The market here may not be fully bridgeable, but it is bridgeable in a neighborhood of $\frac{c}{b}$ since by assumption $a \mapsto \pi(a, \frac{c}{b})$ is monotone.

Proposition 5. Assume the principal is dovish and π has the pivot property. Then there exist states $\theta' < -\frac{c}{b} < \theta''$ such that

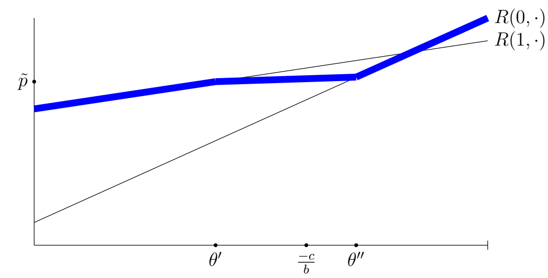
- The virtually optimal action function Q^* features full intervention below θ', no intervention above θ'', and a level of intervention that is continuously decreasing as a function of the state over (θ', θ'').
- The virtually optimal decision rule is a continuous approximation of a step function, which for some $\tilde{p}$ switches from full intervention below $\tilde{p}$ to no intervention above $\tilde{p}$.

Proof. In Appendix C.5. □

Figure 3a illustrates the case of a hawkish principal (under the assumption that π has the pivot property). The blue lines correspond to the price function induced by the first-best action function.



(a) Hawkish principal, first-best is CWUI



(b) Dovish principal, virtually optimal

Figure 3: First best

Figure 3b illustrates the case of a dovish principal. In this case, the monotonicity constraint on the equilibrium price function (Theorem 1) binds. The blue lines correspond to the price function induced by the action function that is optimal under this constraint. The fact that $\theta' < -\frac{c}{b} < \theta''$ implies that the virtually optimal policy entails two types of errors, resulting from the desire for robust implementation: too little intervention for states in $(\theta', -\frac{c}{b})$, and too much for states in $(-\frac{c}{b}, \theta'')$.

4.2 Bailouts: imperfect information aggregation

Consider now the bailout problem in a framework with aggregate uncertainty. The market is modeled as in Section 1, example 4. The state $\theta = (\omega, z)$ consists of the payoff-relevant state $\omega \in \Omega$, interpreted as the strength of the company, and the shock to aggregate supply $z \in \mathcal{Z}$.[36] Agents' signals are equal to ω plus a normally distributed noise term with variance σ_i^2. Agent i has CARA utility with risk-aversion coefficient τ_i. The supply shock has a truncated-normal distribution on $(\underline{z}, \bar{z})$, where $\underline{z} = -\bar{z}$ (we allow $\bar{z} = \infty$). Additionally, we make the simplifying assumption that ω is uniformly distributed. The cash flow (the ex-post value of the asset) is assumed to be linear, given by $\pi(a, \omega) = \beta_0(a) + \beta_1(a)\omega$. Assume that $\beta_1(a)$ is strictly positive, strictly decreasing, weakly convex, and twice differentiable.

We normalize the social benefit of intervention to 1, so the principal's ex-post payoff is $u(a, \theta) := (1 - a)\omega - ac$. The first-best action function is to make the maximal intervention if $\omega \leq -c$, and make zero intervention otherwise.

The invariant representation for this market is discussed in detail in Appendix F.3. In brief, we show that any CWUI price function P must belong to the set $\tilde{\mathcal{P}}$ with the "no intersecting level sets" property. In particular, for any $p \in P(\Theta)$ there exists $\ell \in \mathbb{R}$ and $a \in \mathcal{A}$ such that $\{(\omega, z) : P(\omega, z) = p\} = \{(\omega, z) : L^*(\omega, z|a) = \ell\}$, where

$$L^*(\omega, z|a) := \frac{\kappa}{\beta_1(a)}\omega - z = \ell \tag{1}$$

and $\kappa := \int_i \frac{\tau_i}{\sigma_i^2} di$.[37] Given that $P \in \tilde{\mathcal{P}}$ is a necessary condition for CWUI, it suffices to show that the market admits an invariant representation in $(\mathcal{W}^*, \tilde{\mathcal{P}})$.

Lemma 3. The above market admits an a.e. invariant representation given $(\mathcal{W}^*, \tilde{\mathcal{P}})$.

[36] We denote the strength of the company as ω, rather than θ as in Section 4.1, to maintain consistency with the rest of the paper: θ must contain all variables which determine the price.

[37] Note that the higher is β_1, i.e. the more responsive is the cash flow to ω, the less informative is the price about ω. This is because when β_1 is high, investors' private signals of ω are less informative about the cash flow π. As a result, investors put less weight on their private signals relative to the public information contained in the price, and so the price is less informative. Since β_1 is decreasing in the level of intervention, this implies that higher levels of intervention make the price more informative about the payoff-relevant state. Bond and Goldstein (2015) also study how market-based interventions affect the efficiency of information aggregation by prices.

Moreover, $(\omega, z) \mapsto R(a, \omega, z)$ is strictly increasing for all a.

Proof. In Appendix F.3. □

4.3 Reformulating the problem

To characterize optimal policy in this setting, we use Proposition 3 to reduce the problem of choosing $Q : \Omega \times \mathcal{Z} \to [0, 1]$ to the simpler one-dimensional problem of choosing the marginal action function $\omega \mapsto Q(\omega, \bar{z})$.[38] For this purpose, it is convenient to define action functions on $\mathbb{R} \times \mathcal{Z}$, rather than just $\Omega \times \mathcal{Z}$.[39]

By Proposition 3, if Q is a CUI action function and $Q(\omega, \bar{z}) = a$, then the state $(\omega, \bar{z})$ belongs to a level set of the equilibrium price function which is defined by $\{(\omega', x') : R(a, \omega', z') = R(a, \omega, \bar{z})\}$. Since the principal's action must be measurable with respect to the price, this implies that $Q(\omega', z') = Q(\omega, z)$ for all (ω', z') in this level set. Let $w(a, z, x) : \mathcal{A} \times \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$ be the unique $\omega \in \mathbb{R}$ such that $R(a, w(a, z, x), z) = R(a, x, \bar{z})$.[40] From eq. (1), $w(a, z, x) := x - \frac{1}{\kappa}\beta_1(a)\left(\bar{z} - z\right)$.

Thus, any CUI $Q : \mathbb{R} \times \mathcal{Z} \to [0, 1]$ is uniquely identified among the set of CUI action functions by its marginal $\omega \mapsto Q(\omega, \bar{z})$ (to visualize this approach, see Figure 4). Conversely, Proposition 3 implies that $\alpha : \mathbb{R} \to [0, 1]$ is the marginal of a CUI Q if and only if it satisfies the following properties: *i)* α is continuous, *ii)* $x \mapsto R(\alpha(x), x, \underline{z})$ is monotone, and *iii)* $x \mapsto w(\alpha(x), \underline{z}, x)$ is strictly increasing. Condition *(iii)* says that the linear statistics for the price function do not intersect. Thus we optimize over $\alpha : \mathbb{R} \to [0, 1]$ that satisfy conditions (i)-(iii).

4.4 Characterization of optimal policy

Recall that in the model with supply shocks, we said that the principal was hawkish if $a \mapsto \pi(a, \omega)$ was decreasing for all $\omega \geq -c$, where $\omega = -c$ is the highest state at which the principal wants to intervene. In the current model, we introduce a slightly different notion of hawkish, which accounts for the imperfection in the aggregation

[38] For simplicity we focus on CUI; the analysis for CWUI is nearly identical. In fact, the optimal CUI and CWUI policies coincide.

[39] We still consider $\Omega \times \mathcal{Z}$ to be the state space, for the purposes of defining CUI.

[40] It is convenient to define w for x that are not in Ω.

of information by the price. Here we say that the principal is *hawkish* if $a \mapsto \pi(a, \omega)$ is decreasing for all $\omega \geq -\frac{1}{\kappa}\bar{z}\beta_1(1) - c$. The additional term $-\frac{1}{\kappa}\bar{z}\beta_1(1)$ reflects the fact that information aggregation is necessarily imperfect, and so the principal may choose less than full intervention for some ω below $-c$.

The characterization of optimal policy is most easily stated under the following assumption, which says that the range of fundamental uncertainty, i.e. $[\underline{\omega}, \bar{\omega}]$, is large relative to the range of supply shocks $[\underline{z}, \bar{z}]$ and the marginal cost of intervention c. This implies that $a = 0$ is optimal for low ω and $a = 1$ for high ω.[41]

Assumption 1. $\bar{\omega} \geq \frac{1}{\kappa}\bar{z}\left(\beta_1(0) - \beta_1'(0)\right) - c$ and $\underline{\omega} \leq \frac{1}{\kappa}\bar{z}\beta_1(1) - c$.[42]

In Section 4, where information was perfectly aggregated by the price, the first-best action function was CUI when the principal was hawkish. This is not the case with aggregate uncertainty.

Theorem 3. When the principal is hawkish, the optimal CUI action function Q^* is characterized by $\alpha^*(\omega) := Q^*(\omega, \bar{z})$ given by

- $\alpha^*(\omega) = 1$ for $\omega \leq \omega^* := \frac{1}{\kappa}\bar{z}\beta_1(1) - c$
- $\alpha^*(\omega) = 0$ for $\omega \geq \omega^{**} := \frac{1}{\kappa}\bar{z}\left(\beta_1(0) - \beta_1'(0)\right) - c$
- α^* is continuous and strictly decreasing on (ω^*, ω^{**}), defined as the unique solution to $\frac{1}{\kappa}\bar{z}\big(\beta_1(a) - (1-a)\beta_1'(a)\big) - c = \omega$.

Thus Q^* features maximal intervention for low-ω/high-z states and no intervention for high-ω/low-z states. Moreover, intermediate actions are taken with positive probability if and only if $a \mapsto \beta_1(a)$ is non-constant.

Proof. In Appendix C.6. □

The level sets of the equilibrium price function are depicted in Figure 4. Northwest of the blue dotted line full intervention occurs. Southeast of the solid red line there is no intervention. In between, Q^* is a continuous function that takes intermediate values. The equilibrium price is increasing in the southeast direction.

[41] This assumption is made purely for the purpose of exposition; it simplifies the proof of Theorem 3. Dropping this assumption does not change the qualitative results.

[42] Since $\beta_1(a)$ is decreasing, $\beta_1(0) - \beta_1'(0) \geq \beta_1(1)$, so the conditions are not redundant.

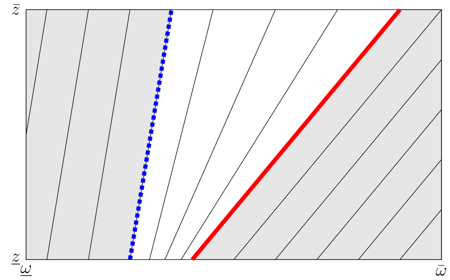


Figure 4: Level sets of optimal action function.

The first-best cannot be implemented when ω is not perfectly revealed. Aside from providing an analytical characterization of the optimal policy, the interesting feature of Theorem 3 is that intermediate actions play a significant role. Recall that this is not the case for a hawkish principal when the price is fully revealing, as in Section 4.1. Intermediate actions are optimal here—even though the principal's ex-post payoff is linear—because the principal recognizes the impact their action has on the degree to which information is aggregated by the price. This effect operates via changes in the slope of the linear statistic in eq. (1). For higher states, the principal balances the desire to reduce the level of intervention against the reduction in the informativeness of the price. Only when $a \mapsto \beta_1(a)$ is constant does the principal take only extreme actions in equilibrium. The following can be observed from Theorem 3 and eq. (1).

Corollary 1. Under the optimal policy

i. The set of states for which intermediate actions are taken is increasing (in the set inclusion order) in $\beta_1(0) - \beta_1'(0)$, and decreasing in $\beta_1(1)$.

ii. The probability of taking an intermediate action decreases if private signals become more precise, i.e. σ_i decreases for almost all i.

Appendix

A Axiomatic approach to invariant representation

In this section, we characterize markets that admit an invariant representation in terms of primitive properties.

Definition 11. Let $\hat{\mathcal{P}} \subset \mathcal{P}^{\Theta}$. The market is *competitively identified in* $(\mathcal{W}, \hat{\mathcal{P}})$ if

1. For any $M, M' \in \mathcal{W}$, any equilibrium price functions P given M and P' given M', and any state $\theta \in \Theta$, $M(P(\theta)) = M'(P'(\theta)) \quad \Rightarrow \quad P(\theta) = P'(\theta)$.
2. Let $\{M_a\}_{a \in K}$ be a family of constant decision rules and $\{P_a\}_{a \in K}$ a family of respective equilibrium price functions, where $K \subset \mathcal{A}$. Let $\{I_a\}_{a \in K}$ be a partition of Θ such that $P_a(I_a) \cap P_{a'}(I_{a'}) = \varnothing$ for all $a, a' \in K$ s.t. $a \neq a'$.[43] Let $P : \Theta \to \mathcal{P}$ be defined by $P := P_a$ on I_a, for each $a \in K$. Then $P \in \hat{\mathcal{P}}$ implies that P is an equilibrium price function for some $M \in \mathcal{W}$.

Both conditions in Definition 11 represent a sense in which equilibrium outcomes are separable across states, which is true in general only if agents are price takers. The first condition says that the equilibrium principal action in state θ uniquely identifies the equilibrium price in state θ, across all $M \in \mathcal{W}$ and all associated equilibria. The second part of Definition 11 says that we can generate new equilibria by stitching together equilibrium price functions, provided the resulting function is in some predetermined set $\hat{\mathcal{P}}$.

Proposition 6. A market is competitively identified in $(\mathcal{W}, \hat{\mathcal{P}})$ iff it admits an invariant representation given $(\mathcal{W}, \hat{\mathcal{P}})$.

Proof. In Appendix C.4 □

Proposition 6 and Part 1 in Definition 1 tells us that $P(\theta) = R(M \circ P(\theta), \theta)$ for all θ is a necessary condition for P to be an equilibrium price function. This is immediately implied by part 1 in Definition 11. Part 2 of Definition 1, which is implied by condition

[43]For a function $f : X \mapsto Y$ we use the notation $f(I) := \{f(x) : x \in I\}$ for $I \subset X$.

2 in Definition 11, says the converse. In some settings, before identifying an invariant representation, the requirements of robustness (to manipulation and multiplicity) imply restrictions on the set of possible equilibrium price functions. Thus for the purposes of identifying the invariant representation, we can restrict attention to a set of price functions $\hat{\mathcal{P}} \subsetneq \mathcal{P}$. Condition 2 in the definition of competitively identified may not hold for all $P \in \mathcal{P}$, but it is enough to know that it holds on $\hat{\mathcal{P}}$. The utility of this approach is illustrated in Appendix F.3.

A.1 Importance of price taking

Since agents in competitive markets behave as price takers, they do not internalize the effect that their actions have on the principal's action. This is the key property that allows for an invariant representation of the market. Markets in which agents are not price takers generally fail to admit an invariant representation.

Example 1. Consider the standard auction environment. The market participants are the bidders and the seller. An equilibrium consists of a mechanism for the seller and strategies for the bidders, specifying their bids as a function of their type. This environment is a market, where we let the price be that paid by the winning bidder. The principal could be a regulator who commits to taking some action as a function of the price. This action may affect the value of the object.

If $\mathcal{W}$ contains non-constant decision rules, it is easy to construct examples showing that the market does not admit an a.e. invariant representation. This is because bidders internalize the effect that their actions have on the price (winning bid) and thus respond to global properties of the principal's decision rule, which implies a violation of Definition 11, part 1.

B Appendix: Proofs of the main results

We begin with some preliminary results. Use $\theta_M(p)$ to indicate the states that are consistent with a price p, given a policy function M, i.e., the states for which there exists a P implementing $\theta_M := \{\theta \in \Theta : M \text{ implements } P \text{ and } P(\theta) = p\}$.

Observe that for markets that admit a invariant representation, $\theta_M(p) := \{\theta \in \Theta : R(M(p), \theta) = p\}$

Lemma 4. If R is weakly increasing in θ then $\theta_M(p)$ is convex valued.

Proof. $\theta_M(p) = \{\theta \in \Theta : R(M(p), \theta) = p\}$. If $R(M(p), \cdot)$ is monotone, $R(M(p), \theta') = R(M(p), \theta'') = p$ implies $R(M(p), \theta) = p$ for all $\theta \in (\theta', \theta'')$. □

Lemma 5. For any M, each p such that $\theta_M(p) = \varnothing$ is of one and only one of the following two types:

- *Type L:* $\quad R(M(p), \theta') > p \qquad \forall \theta' \in \Theta$.
- *Type H:* $\quad R(M(p), \theta') < p. \qquad \forall \theta' \in \Theta$.

Proof. If p is of neither type, there exists a pair of states θ', θ'' such that $R(M(p), \theta') - p > 0 > R(M(p), \theta'') - p$. Then by continuity, there is a state $\theta \in (\theta', \theta'')$ such that $R(M(p), \theta) - p = 0$. But then $\theta(p)$ is not empty. □

Lemma 6. (Generalized intermediate value theorem). Let $F : [0, 1] \to [0, 1]$ be a non-empty, compact, and convex valued, upper hemicontinuous correspondence. Let $p_1 < p_2$. Let $y_1 \in F(p_1)$ and $y_2 \in F(p_2)$. Then for any $\tilde{y} \in (\min\{y_1, y_2\}, \max\{y_1, y_2\})$ there exists $p \in [p_1, p_2]$ such that $\tilde{y} \in F(p)$.

Proof. Assume that $y_2 > y_1$ (the case with $y_2 = y_1$ is trivial and $y_2 < y_1$ is symmetric). We prove by contrapositive: assume that there exists a $\tilde{y} \in (y_1, y_2)$ such that $\tilde{y} \notin F(p)$ for all $p \in [p_1, p_2]$. Since $F(p)$ is convex, for every p either $\max F(p) < \tilde{y}$ or $\min F(p) > \tilde{y}$. Let $p^* = \sup\{p \in [p_1, p_2) : \max F(p) < \tilde{y}\}$.

Suppose that $\max F(p^*) < \tilde{y}$. Notice that this is only compatible with $p^* < p_2$. Consider the open set $V := (\min F(p^*) - \epsilon, \max F(p^*) + \epsilon)$ with $\epsilon < \tilde{y} - \max F(p^*)$. By upper hemicontinuity, there exists a neighborhood of p^* such that $F(p) \subset V$ for all p in such neighborhood. Thus, in a neighbourhood of p^*, $F(p) < \tilde{y}$, which violates the definition of p^*.

Suppose that $\min F(p^*) > \tilde{y}$. Notice that this is only compatible with $p^* > p_1$. Using upper hemicontinuity as before, we get that there is a neighbor of p^* such that $F(p) > \tilde{y}$ for all p in that neighborhood, what violates the definition of p^*. □

Lemma 7. For any M, $p \mapsto \theta_M(p)$ is compact-valued. If M is continuous at p' and $\theta_M(p')$ is not empty for every p in a neighborhood of p' then $p \mapsto \theta_M(p)$ is upper hemicontinuous at p'.

Proof. Compact valuedness is easy: if $R(M(p), \theta) - p \neq 0$ then by continuity of R this holds for all θ' in a neighborhood of θ. Now upper hemicontinuity. Let $\bar{\Theta}$ be the closure of Θ, and let $\bar{\theta}_M(p) := \{\theta \in \bar{\Theta} : R(M(p), \theta) = p\}$ (where for $\theta \in \bar{\Theta} \setminus \Theta$ we define $R(a, \theta)$ as the limit as $\theta' \to \theta$ for $\theta' \in \Theta$). Let V be an open set containing $\bar{\theta}_M(p)$. Since $\bar{\theta}_M(p)$ is compact Then $\Theta \setminus V$ is compact, so there exists $\kappa > 0$ such that $|R(M(p), \theta) - p| > \kappa$ for all $\theta \in \bar{\Theta} \setminus V$. Then by continuity of R, M there exists an open neighborhood U of p such that $|R(M(p'), \theta) - p'| \geq \kappa$ for all $p' \in U, \theta \in \bar{\Theta} \setminus V$. Thus $\theta_M(p') \subseteq V$. Thus $p \mapsto \bar{\theta}_M(p)$ is upper hemicontinuous at p'.

Now since $\theta_M(p')$ is non empty, if $\bar{\theta} \in \bar{\theta}_M(p')$ then there is some $\theta_1 < \bar{\theta}$ such that $[\theta_1, \bar{\theta}) \subset \theta_M(p')$. Similarly if $\underline{\theta} \in \bar{\theta}_M(p')$. Then any open set containing $\theta_M(p')$ also contains $\bar{\theta}_M(p')$. Thus θ_M is upper hemicontinuous at p' as well. □

B.1 Proof of Theorem 1

Let (P, Q) be CWUI and $M \in \mathcal{W}^*$ a policy that implements (P, Q). Let $P_M(\theta) := \{p \in \mathbb{R} : R(M(p), \theta) = p\}$. For any $\theta \in \Theta$, Let $a_\theta := \sup\{p \leq P(\theta) : \theta_M(p) = \emptyset\}$ and $b_\theta := \inf\{p \geq P(\theta) : \theta_M(p) = \emptyset\}$.

Lemma 8. For any θ^*, if $a_{\theta^*} = b_{\theta^*} = P(\theta^*)$ then either $R(M(P(\theta^*)), \theta) = P(\theta^*)$ for all $\theta \leq \theta^*$, or $R(M(P(\theta^*)), \theta) = P(\theta^*)$ for all $\theta \geq \theta^*$ (or both).

Proof. Since M is essentially continuous, it is continuous in a neighborhood of $P(\theta^*)$. Let $(\underline{p}, \bar{p})$ be such a neighborhood.

Claim 1. Either (1.1) For every $p \in (P(\theta^*), \bar{p})$ there exists $p \in (P(\theta^*), p)$ such that $R(M(p'), \theta) > p'$ for all $\theta \in \Theta$, and/or (1.2) For every $p \in (P(\theta^*), \bar{p})$ there exists $p' \in (P(\theta^*), p)$ such that $R(M(p'), \theta) < p'$ for all $\theta \in \Theta$. Similarly, either (1.3) For every $p \in (\underline{p}, P(\theta^*))$ there exists $p' \in (p, P(\theta^*))$ such that $R(M(p'), \theta) > p'$ for all $\theta \in \Theta$, and/or (1.4) For every $p \in (\underline{p}, P(\theta^*))$ there exists $p' \in (p, P(\theta^*))$ such that $R(M(p'), \theta) < p'$ for all $\theta \in \Theta$.

Claims (1.1)-(1.4) follow from Lemma 5 and the assumption that $a_{\theta^*} := \sup\{p \leq P(\theta) : \theta_M(p) = \emptyset\} = P(\theta^*) = b_{\theta^*} := \inf\{p \geq P(\theta) : \theta_M(p) = \emptyset\}$.

Claim 2. $R(M(P(\theta^*)), \theta) = P(\theta^*)$ for all $\theta \leq \theta^*$ if either (1.1) or (1.3) hold. $R(M(P(\theta^*)), \theta) = P(\theta^*)$ for all $\theta \geq \theta^*$ if either (1.2) or (1.4) hold.

Suppose (1.1) holds. Then we can find a sequence $\{p_n\}$ with $p_n < P(\theta^*)$ and $p_n \to P(\theta^*)$ such that $R(M(p_n), \theta) > p_n$ for all θ. By continuity, $R(M(P(\theta^*)), \theta) \geq P(\theta^*)$, and since $\theta \mapsto R(M(P(\theta^*)), \theta)$ is non-decreasing, we have $R(M(P(\theta^*)), \theta) = P(\theta^*)$ for all $\theta \leq \theta^*$.

By a symmetric argument, if (1.3) holds, then $R(M(P(\theta^*)), \theta) \geq P(\theta^*)$, and since $\theta \mapsto R(M(P(\theta^*)), \theta)$ is non-decreasing and $R(M(P(\theta^*)), \theta^*) = P(\theta^*)$, we have $R(M(P(\theta^*)), \theta) = P(\theta^*)$ for all $\theta \leq \theta^*$. A symmetric argument applies to (1.2) and (1.4). Combined, Claims 1 and 2 complete the proof. □

Lemma 9. For any θ^*, if $a_{\theta^*} < b_{\theta^*}$ then $P_M(\theta) \cap (a_{\theta^*}, b_{\theta^*}) \neq \emptyset$ for all $\theta \in \Theta$.

Proof. Note that, by definition of a_θ and b_{θ^*}, $(a_{\theta^*}, b_{\theta^*}) \subseteq P_M(\Theta)$. By definition of essential continuity, M is continuous in $(a_{\theta^*}, b_{\theta^*})$. First, we prove that either (i) $R(M(a_{\theta^*}), \underline{\theta}) = a_{\theta^*}$ and $R(M(b_{\theta^*}), \bar{\theta}) = b_{\theta^*}$ or (ii) $R(M(a_{\theta^*}), \bar{\theta}) = a_{\theta^*}$ and $R(M(b_{\theta^*}), \underline{\theta}) = b_{\theta^*}$.

Consider a sequence of prices p_n such that $p_n \in (a_{\theta^*}, b_{\theta^*})$ and p_n converges to a_{θ^*}. For every n , since $p_n \in (a_{\theta^*}, b_{\theta^*}) \subset P_M(\Theta)$, there exists $\theta_n \in \Theta$ such that $R(M(p_n), \theta_n) = p_n$. Thus, for each n, $R(M(p_n), \underline{\theta}) - p_n \leq R(M(p_n), \theta_n) - p_n = 0$ where the first inequality holds by monotonicity of R.

Since M is continuous on a neighbourhood of a_{θ^*}, taking limits side-by-side, we get that $R(M(a_{\theta^*}), \underline{\theta}) \leq a_{\theta^*}$. Likewise, we can prove that $a_{\theta^*} \leq R(M(\bar{a}_{\theta^*}), \theta)$.

It remains to show that a_{θ^*} is not in the interior. Suppose by contradiction that it is. Then $0 \in (R(M(a_{\theta^*}), \underline{\theta}) - a_{\theta^*}, R(M(a_{\theta^*}), \bar{\theta}) - a_{\theta^*}))$. For an $\epsilon > 0$ small enough, for every $p \in (a_{\theta^*} - \epsilon, a_{\theta^*})$, $0 \in (R(M(p), \underline{\theta}) - p, R(M(p), \bar{\theta}) - p))$. Thus, applying the intermediate value theorem, $(a_{\theta^*} - \epsilon, a_{\theta^*}] \subset P_M(\Theta)$, what violates the definition of a_{θ^*}. The same argument can be applied to prove that $b_{\theta^*} \in \{R(M(b_{\theta^*}), \underline{\theta}), R(M(b_{\theta^*}), \bar{\theta})\}$.

Next we show that it cannot be the case that $R(M(a_{\theta^*}), \underline{\theta}) = a_{\theta^*}$ and $R(M(b_{\theta^*}), \underline{\theta}) = b_{\theta^*}$. If that is the case, since continuity of M in $(a_{\theta^*}, b_{\theta^*})$ implies that $\theta_M(p)$ is upper hemicontinuous in $(a_{\theta^*}, b_{\theta^*})$, $P_M(\tilde{\theta})$ is not a singleton for a small $\delta > 0$, $\tilde{\theta} \in [\underline{\theta}, \underline{\theta} + \delta)$. A symmetric argument rules out the case in which $R(M(a_{\theta^*}), \bar{\theta}) = a_{\theta^*}$ and $R(M(b_{\theta^*}), \bar{\theta}) = b_{\theta^*}$.

We finish the proof by showing that $P_M(\theta') \cap (a_{\theta^*}, b_{\theta^*}) \neq \emptyset$ for all $\theta' \in \Theta$. Suppose case (i) holds, i.e. that $\underline{\theta} \in \theta_M(a_{\theta^*})$ and $\bar{\theta} \in \theta_M(b_{\theta^*})$. By the intermediate value theorem in Lemma 6, for every $\theta' \in (\underline{\theta}, \bar{\theta})$ there exists a $p \in (a_{\theta^*}, b_{\theta^*})$ such that $\theta \in \theta_M(p)$. Thus, $p \in P_M(\theta') \cap (a_{\theta^*}, b_{\theta^*})$. If case (ii) holds, a symmetric argument proves the claim. $\square$

Proof of Theorem 1. Suppose, towards a contradiction, that (Q, P) is implemented by $M \in \mathcal{W}^*$ and P is not monotone. Assume in particular that there is $\theta_1 < \theta_2 < \theta_3$ such that $P(\theta_1) > P(\theta_3) > P(\theta_2)$. (The other cases of non-monotonicity are symmetric.)

Claim 1. Either (1.1) $a_{\theta_1} = a_{\theta_2} = a_{\theta_3}$ and $b_{\theta_1} = b_{\theta_2} = b_{\theta_3}$, or (1.2) $a_{\theta_1} = b_{\theta_1}$, $a_{\theta_2} = b_{\theta_2}$, and $a_{\theta_3} = b_{\theta_3}$.

To shop Claim 1, either there exists $i \in \{1, 2, 3\}$ such that $a_{\theta_i} < b_{\theta_i}$, or not. Suppose such an i does exist and there exists $j \neq i$ such that $a_{\theta_i} \neq a_{\theta_j}$ (which also implies $b_{\theta_i} \neq b_{\theta_j}$). Then by Lemma 9, for any $\theta \in \Theta$ we have $P_M(\theta) \cap (a_{\theta_i}, b_{\theta_i}) \neq \varnothing$. If $a_{\theta_j} < b_{\theta_j}$ then by Lemma 9 for any $\theta \in \Theta$ we have $P_M(\theta) \cap (a_{\theta_j}, b_{\theta_j}) \neq \varnothing$, so there is multiplicity in all states. If $a_{\theta_j} = b_{\theta_j}$ then by Lemma 8 either there is multiplicity above θ_j or below θ_j.

Claim 2. If (1.1) holds, then we have a contradiction.

M is continuous in $(P(\theta_1), P(\theta_2))$. Thus we can apply Lemma 6: for every $\theta \in (\theta_1, \theta_3)$ there exists a price $p_\theta \in (P(\theta_3), P(\theta_1))$ such that $p_\theta \in P_M(\theta)$. Likewise, for every $\theta \in (\theta_2, \theta_3)$ there exists a price in $p'_\theta \in (P(\theta_2), P(\theta_3))$ such that $p'_\theta \in P_M(\theta)$. Thus, for all states in $\theta \in (\theta_2, \theta_3)$, $p'_\theta \neq p_\theta$ and therefore the set $P_M(\theta)$ has more than one element, what implies a violation of robustness to multiplicity.

Claim 3. If (1.2) holds, then we have a contradiction.

In this case, Lemma 8 implies that for each $i \in \{1, 2, 3\}$, either $P(\theta_i) \in P_M(\theta)$ for all $\theta \leq \theta_i$, or for all $\theta \geq \theta_i$. Any combination of these conditions for $i \in \{1, 2, 3\}$ implies multiplicity on a positive measure set. $\square$

B.2 Proof of Theorem 2

Lemma 10. Assume R is weakly increasing in θ. For any $M \in \mathcal{M}$ that is robust to multiplicity, Let $p_1 < p_2$ such that there are states $\underline{\theta}$ and $\bar{\theta}$ with $\underline{\theta} < \theta < \bar{\theta}$ for each $\theta \in \theta(p_1) \cup \theta(p_2)$. Then $[p_1, p_2] \in P(\Theta)$.

Proof. By Theorem 1, the price function P is monotone, so without loss of generality assume that it is increasing, and let $p_1, p_2 \in P(\Theta)$ with $p_2 > p_1$. Assume towards a contradiction that there exists $p \in (p_1, p_2)$ such that $p \notin P(\Theta)$. By Lemma 5 p is either type H or type L. Suppose it is type L, i.e. $R(M(p), \theta) - p > 0$ for all θ. Since $\theta_M(p_1) \neq \varnothing$, it must be that $R(M(p_1), \underline{\theta}) - p_1 \leq 0$. Moreover, since $\underline{\theta} \notin \theta_M(p_1)$ by assumption, the inequality is strict: $R(M(p_1), \underline{\theta}) - p_1 < 0$. Then by continuity there exists $p' \in (p_1, p)$ such that $R(M(p'), \underline{\theta}) - p' = 0$. Let $\theta_1 = \min \theta_M(p_1)$, which exists by Lemma 7 (by assumption $\theta_1 > \underline{\theta}$). Since P is increasing, $p' > p_1 > P(\theta)$ for all $\theta \in [\underline{\theta}, \theta_1)$. Then by Lemma 6 there is multiplicity for all states in $\theta \in [\underline{\theta}, \theta_1)$, which is a contradiction. If p is type H then the proof is symmetric, using p_2 for p_1. $\square$

Proof. ($\Rightarrow$) Part 1 follows trivially from the market clearing condition necessary for implementation (see Observation 1).

Theorem 1 states that P must be *weakly* monotone. To prove strict monotonicity (part 2) consider $P(\theta) = P(\theta')$. Then, $R(Q(\theta), \theta) = R(Q(\theta'), \theta')$. By measurability, $Q(\theta) = Q(\theta')$, so $R(Q(\theta), \theta) = R(Q(\theta), \theta')$ which, since R is strictly increasing in θ implies that $\theta = \theta'$. Thus, P is strictly monotone.

Now we prove that Q is continuous for any interior state. Since $R(a, \theta)$ is strictly monotone in θ, we have $|\theta_M(p)| \leq 1$ for all p. To see this, consider $\theta, \theta' \in \theta_M(p)$. This means that $R(M(p), \theta) = p = R(M(p), \theta')$ which, by strict monotonicity of R, means that $\theta = \theta'$.

For some interior state θ', let $p^- := \lim_{\theta \searrow \theta'} P(\theta)$ and $p^+ := \lim_{\theta \nearrow \theta'} P(\theta)$. Since M is essentially continuous, M is continuous in an open neighborhood N of $P(\theta')$. This, together with continuity of R, implies that $\theta_M(p)$ is continuous on N. Thus, there is a neighborhood of θ' such that $P(\theta) \cap N$ is not empty for all θ in the neighborhood. Therefore, p^- and p^+ must be equal to $P(\theta)$ or multiplicity would be violated.

Given that P is continuous for interior states, a discontinuity of Q in an interior state will necessarily imply a discontinuity of M for a price in $\bar{P}$, which would violate essential continuity. Thus, Q must be continuous for all interior states.

P is monotone and bounded (below by $\min_{a\in\mathcal{A}} R(a, \underline{\theta})$ and above by $\max_{a\in\mathcal{A}} R(a, \bar{\theta})$), so $\underline{P} := \lim_{\theta\searrow\underline{\theta}} P(\theta)$ and $\bar{P} := \lim_{\theta\nearrow\bar{\theta}} P(\theta)$ exist. Let M be the policy function that continuously uniquely implements (Q, P). By essential continuity, M is continuous at $\underline{P}$, so $\lim_{p\searrow\underline{P}} M(p) = M(\underline{P})$. But then, since $Q(\theta) = M(P(\theta))$ for all θ, $\lim_{\theta\searrow\underline{\theta}} Q(\theta) = \lim_{\theta\searrow\underline{\theta}} Q(\theta) = Q(M(\underline{P}))$. The same arguments hold for the other extreme state $\bar{\theta}$.

Finally, for the case in which P is strictly decreasing, we need to show that $\underline{Q}$ is not maximal at the bottom, and $\bar{Q}$ is not minimal at the top. Since P is decreasing, for prices right above $\underline{P}$, $\theta_M(p)$ should be empty. $\bar{Q}$ is maximal at the bottom so $R(\cdot, \underline{\theta})$ has a local maximum at $\underline{Q}$. This means that there is a neighborhood around $\underline{Q}$ such that $R(q', \underline{\theta}) < p$ for all q' in the neighborhood. By essential continuity, for prices slightly above p the action is in such neighborhood. So for any $\varepsilon > 0$ there exists a $p' \in (p, p+\varepsilon)$ such that $R(M(p'), \underline{\theta}) \leq p$. Since $\theta \mapsto R(a, \theta)$ is strictly increasing and R is continuous, for ε small enough we will also have $R(M(p'), \bar{\theta}) > p'$. But then by continuity of R there exists θ such that $R(M(p'), \theta) = p'$, so $\theta_M(p')$ is not empty. A symmetric argument rules out Q being minimal at the top.

($\Leftarrow$) M can be easily defined on $P(\Theta)$ as follows. Since P is injective, define M on $P(\Theta)$ as $M(p) = Q(P^{-1}(p))$. Notice that M is continuous (by 1 and 3).

The challenge is to define the function M for prices outside $P(\Theta)$. The constructions differs for increasing and decreasing P. If P is increasing then define $M(p) = \bar{Q}$ for all prices above $\bar{P}$ and $M(p) = \underline{Q}$ for all prices below $\underline{P}$. We want to check that for all these prices $\theta_M(p) = \emptyset$. For prices above $\bar{P}$, $p \geq \bar{P} = R(\bar{Q}, \bar{\theta}) = R(M(\bar{P}), \bar{\theta}) >$

$R(M(p), \theta)$ where the last inequality holds for all $\theta \in \Theta$. A symmetric argument proves that $p < R(M(p), \theta)$ for prices below $\underline{P}$. Thus, (Q, P) is CUI.[44]

Now for decreasing P, we need to show that there exists a continuous decision rule for prices right above $\underline{P}$ so that $\theta_M(p)$ is empty. Let $\underline{R}(a) = R(a, \underline{\theta})$ and consider a finite partition $\{A_i\}_{i=1}^k$ of $\mathcal{A}$ such that the sets $\{A_i \cap \underline{R}^{-1}(\underline{P})\}_{i=1}^k$ are connected. Moreover, by continuity, we can pick the partition $\{A_i\}_{i=1}^k$ such that the distance between two of the subsets is greater than zero: For A, A' two elements of the partition, if the distance between $A \cap \underline{R}^{-1}(\underline{P})$ and $A' \cap \underline{R}^{-1}(\underline{P})$ is zero, then there is a sequence of actions $\{a_i\}_{i=1}^{\infty}$ such that $a_i \in A \cap \underline{R}^{-1}(\underline{P})$ and $a = \lim_{i \to \infty} a_i \in A' \cap \underline{R}^{-1}(\underline{P})$. By continuity, the sets are connected. Thus, in a neighbourhood of $\underline{Q}$, $\underline{R}^{-1}(\underline{P})$ is connected. By continuity, this splits the neighbourhood of $\underline{Q}$ in sets for which $\underline{R}(a) > \underline{P}$ and sets for which $\underline{R}(a) < \underline{P}$. Since $\underline{Q}$ is not a local maximum, there exists at least one set for which $\underline{R}(a) > \underline{P}$ that is at a distance 0 of $\underline{Q}$.

Pick a continuous path $\hat{a} : [0, 1] \to \mathcal{A}$ such that $\hat{a}(0) = \underline{Q}$ and $\hat{a}(t) \in \mathcal{A}^-$ for all $t > 0$. There exists an increasing function $h : [0, 1] \to \mathcal{P}$ such that $h(t) < \underline{R}(\hat{a}(t))$. Thus, we can make $M(\underline{P} + t\epsilon) = h(t)$. Then for all $\tilde{P} \in (\underline{P}, \underline{P} + \epsilon)$, $R(M(\tilde{P}), \theta) > Q(M(\tilde{P}), \underline{\theta}) > \tilde{P}$.

Use a symmetric construction for M below $\bar{P}$. Beyond these prices, at the neighborhood of $P(\Theta)$, essential continuity is not binding, so any actions that do not generate equilibria work for the construction. By 1, if for a price all actions generate an equilibrium, then that price must be in $P(\Theta)$. □

C Other omitted proofs

C.1 Proof of Proposition 1

1. Private values: Take two action functions M, M' and respective equilibrium price functions P, P'. Suppose that at θ, $M(P(\theta)) = M'(P'(\theta))$. The optimal demand schedule for agent i is $x^*(p|s_i) := x_i(s_i, p, M(p))$. If $P(\theta) > (<) P'(\theta)$ then for all i, $x_i(P(\theta)) < (>) x_i(P'(\theta))$. But then markets cannot clear in state

[44] Moreover, any (Q, P) that is CUI and such that P is increasing, can be implemented by an M that is continuous.

θ for both P and P'.

2. Labor market: This is immediate, let $R(a, \theta) = \int_i u_i(a, \theta)$.
3. Common values, imperfect information: This follows from Proposition 14.
4. Noisy REE: See Corollary 4.

C.2 Proof of Proposition 2

Proof. Necessity for condition 1 is immediate, and condition 2 follows from Theorem 1. The argument for condition 4 is the same as in Theorem 2.

It remains to show that Q can have a discontinuity at θ^* iff P has a discontinuity at θ^*. To see this, notice that Q can be discontinuous at θ^* iff P is continuous at θ^* (otherwise M would need to be discontinuous at $P(\theta^*)$). As shown in the proof of Theorem 1, P can be discontinuous at θ^* only if $\theta_M(p) = \{\theta^*\}$ on $(\min\{\lim_{\theta \nearrow \theta^*} P(\theta), \lim_{\theta \searrow \theta^*} P(\theta)\}, \max\{\lim_{\theta \nearrow \theta^*} P(\theta), \lim_{\theta \searrow \theta^*} P(\theta)\})$ (otherwise there is multiplicity for all states in a neighborhood of θ^*). Such a P can be implemented by an essentially continuous M iff there exists γ satisfying the definition of bridgeability (in which case we take $M = \gamma$ on this interval). Once M is defined over these intervals corresponding to the discontinuities in P, the argument for sufficiency of conditions (1)-(4) is the same as in Theorem 2. □

C.3 Proof of Proposition 3

Proof. Note that $\bar{T}(a, \theta) = L(a, R(a, \theta))$. Then, as outlined in the discussion preceding Proposition 3, conditions *iv* and *v* are necessary given the definition of L, since otherwise there would be multiplicity. *i* is obviously necessary. To show necessity of *ii* and *ii*, restrict attention to a one-dimensional strictly ordered chain in Θ (e.g. the diagonal). For the restriction of Q to this chain, necessity of monotonicity and continuity for interior states follow from the same arguments as in the uni-dimensional case. The key step is that under *iii* and *iv*, this implies that *ii* holds; if there is a non-monotonicity on some chain then there will be a non-monotonicity on every chain. Similarly, Q must be continuous on the interior. □

C.4 Proof of Proposition 6

The "if" is immediate. For the "only if", note that there is at most one equilibrium for any constant policy function M_a: take two equilibrium price functions P and P' given M_a, since $M_a(P(\theta)) = M_a(P'(\theta)) = a$, by Part 1 of the definition of a competitively identified, $P(\theta) = P'(\theta)$. Let $R(a, \theta)$ be the equilibrium price function given M_a.

1. Let P be an equilibrium price function given M. Fix θ, Let $a = M(P(\theta))$. Then consider M_a and respective equilibrium price function $R(a, \theta)$. By part 1, since $M(P(\theta)) = a = M_a(R(a, \theta))$ it must be that $P(\theta) = R(a, \theta)$.
2. Let $I_a = \{\theta : M \circ P(\theta) = a\}$. Then $P(I_a) \cap P(I_{a'}) = \varnothing$ for all $a, a' \in M \circ P(\Theta)$. (If $p \in P(I_a) \cap P(I_{a'})$, the definition of I_a says that $M(p) = a' \neq a = M(p)$). The families $\{M_a\}_{a \in M \circ P(\Theta)}$ and $\{P_a\}_{a \in M \circ P(\Theta)}$, where $P_a = R(a, \cdot)$, satisfy the conditions in part 2 of the definition of competitively identified.

C.5 Proof of Proposition 5

Proof. Since π has the pivot property, the environment is fully bridgeable. Since the principal is dovish, the price-monotonicity constraint from Proposition 2 binds. For any $\theta' < \frac{-c}{b}$ the principal can choose M to generate a payoff arbitrarily close to

$$\int_{\underline{\theta}}^{\theta'} u(\bar{a}, \theta) dH(\theta) + \int_{\theta'}^{t(\theta')} u(\alpha(\theta, \theta'), \theta) dH(\theta) + \int_{t(\theta')}^{\bar{\theta}} u(0, \theta) dH(\theta), \tag{2}$$

where $\alpha(\theta, \theta')$ is defined by $R(\alpha(\theta, \theta'), \theta) = R(\bar{a}, \theta')$, and $t(\theta')$ by $R(0, t(\theta') = R(\bar{a}, \theta')$. Here $\alpha(\theta, \theta')$ is continuous and decreasing in its first argument and increasing in the second, and $t(\theta')$ is decreasing. The principal chooses θ' to maximize eq. (2). We want to show under the optimal policy $\theta' < \frac{-c}{b} < t(\theta')$. To see this, suppose $\theta' = \frac{-c}{b}$. There is a first-order gain from lowering θ' by a small ε, since this means that a lower action is take on the entire interval $(\frac{-c}{b}, t(\frac{-c}{b}))$. The loss from a lower action on $(\frac{-c}{b} - \varepsilon, \frac{-c}{b})$ is second order. An analogous argument applies to raising θ' when $t(\theta') = \frac{-c}{b}$. $\square$

C.6 Proof of Theorem 3

Proof. Let $\mathcal{C}^+$ be the set of continuous and decreasing functions from $\mathbb{R}$ to $[0,1]$. Then the principal's program can be written as follows.

$$\max_{\alpha\in\mathcal{C}^+}\int_{-\infty}^{\infty}\int_{\underline{z}}^{\bar{z}} u\big(\alpha(x), w(z,\alpha(x),x)\big) f_\omega(w(z,\alpha(x),x)) f_z(z)\ dz\ dx \tag{3}$$
$$s.t.\quad x\mapsto R(\alpha(x),x,\bar{z}) \text{ increasing}\quad \text{and}\quad x\mapsto w(\alpha(x),\underline{z},x) \text{ increasing.}$$

We solve this program ignoring the constraints that $x \mapsto R(\alpha(x), x, \bar{z}$ and $x \mapsto w(\alpha(x), \underline{z}, x)$ are increasing, and then verify that they are satisfied.

Let $\omega^{**} := \frac{1}{\kappa}\bar{z}\left(\beta_1(0)-\beta_1'(0)\right)-c$, and $\omega^* := \frac{1}{\kappa}\bar{z}\beta_1(1)-c$. First, consider $x\in(\omega^*,\omega^{**})$. Using the assumption that ω is uniformly distributed, the derivative of the objective at x with respect to a is proportional to

$$\begin{aligned}
&\int_{\underline{z}}^{\bar{z}}\left[u_1\big(a,w(a,z,x)\big)+u_2(a,w(a,z,x))w_1(a,z,x)\right] f_z(z)dz\\
&\quad=\int_{\underline{z}}^{\bar{z}}\left[-\left(x+c-\frac{1}{\kappa}\beta_1(a)(\bar{z}-z)\right)-(1-a)\frac{1}{\kappa}\beta_1'(a)\left(\bar{z}-z\right)\right] f_z(z)dz\\
&\quad=\frac{1}{\kappa}\bar{z}\left(\beta_1(a)-(1-a)\beta_1'(a)\right)-(x+c)
\end{aligned}\tag{4}$$

Under the maintained assumption that β_1 is convex, $a\mapsto\beta_1(a)-(1-a)\beta_1'(a)$ strictly positive and strictly decreasing; and the expression in eq. (4) is strictly decreasing in a. Thus setting eq. (4) equal to 0 defines a decreasing and continuous (in fact, differentiable) $\alpha_*:\Omega\mapsto[0,1]$.

For $x\in\mathbb{R}$ such that either $x>\bar{\omega}$ or $w(0,\underline{z},x)<\underline{\omega}$ the derivative with respect to a does not take the form in eq. (4). Consider $x>\bar{\omega}$. For any $a\in[0,1]$ and a policy α such that $\alpha(x)=a$, the posterior over $\mathbb{R}$ induced by the public information $\{(x',z):L(x',z|a)=L(x,\bar{z}|a)\}$ first-order stochastically dominates that induced by $\{(x',z):L(x',z|0)=L(\omega^{**},\bar{z}|0)\}$. Since no intervention is optimal at ω^{**}, no intervention remains optimal at $x>\omega^{**}$. Full intervention is optimal for $x<\omega^*$.

We now argue that α_* also satisfies the constraints in the principal's program, and so constitutes an optimal policy. First, we show that $x \mapsto w(\alpha_*(x), \underline{z}, x)$ is increasing.

$$\frac{d}{dx}w(\alpha_*(x), \underline{z}, x) = 1 - \frac{1}{\kappa}\beta_1'(\alpha_*(x))(\bar{z} - \underline{z})\alpha_*'(x)$$

Implicit differentiation of the condition in eq. (4) over the range of x for which α_* is non-constant yields $\alpha_*'(x) = \left(\bar{z}\frac{1}{\kappa}\left(2\beta_1'(\alpha_*(x)) - (1-\alpha_*(x))\beta_1''(\alpha_*(x))\right)\right)^{-1}$. Substituting this into the previous expression and using the fact that $\underline{z} = -\bar{z}$ yields

$$\frac{d}{dx}w(\alpha_*(x), \underline{z}, x) = 1 - \frac{2\beta_1'(\alpha_*(x))}{(2\beta_1'(\alpha_*(x)) - (1-\alpha_*(x))\beta_1''(\alpha_*(x)))}$$

Since $\beta_1' < 0$ and $\beta_1'' \geq 0$, we have $\frac{d}{dx}w(\alpha_*(x), \underline{z}, x) \geq 0$ as desired.

We now show that $\omega \mapsto R(\alpha_*(\omega), \omega, z)$ is increasing. From eq. (4) we have $\alpha_*(\omega) = 1$ for all $\omega \leq \frac{1}{\kappa}\bar{z}\beta_1(z) - c$. Under the assumption that the principal is hawkish, $a \mapsto \pi(a, \omega)$ is decreasing for all $\omega > -w(1, \underline{z}, \frac{1}{\kappa}\bar{z}\beta_1(z) - c) = -\frac{1}{\kappa}\bar{z}\beta_1(z) - c$. Since α_* is decreasing above $\frac{1}{\kappa}\bar{z}\beta_1(z) - c$, this implies that $\omega \mapsto R(\alpha_*(\omega), \omega, z)$ is increasing. $\square$

D Properties and extensions

D.1 Structural uncertainty

Assume throughout this section that Θ is closed and bounded. Endow the space of invariant representations $R : \mathcal{A} \times \Theta \to \mathbb{R}$ with the sup-norm. For a given decision rule M and invariant representation R, let $\tilde{Q}_R(\theta|M) := \{a \in \mathcal{A} : M(R(a, \theta)) = a\}$. In words, $\tilde{Q}_R(\theta|M)$ is the set of actions that are consistent with market clearing in state θ.

An open neighborhood of $\tilde{Q}_R(\cdot|M)$ is a set-valued and open-valued correspondence $U : \Theta \rightrightarrows 2^{\mathcal{A}}$ such that $\tilde{Q}_R(\theta|M) \subset U(\theta)$ for all θ. The map $R \rightrightarrows \tilde{Q}_R(\theta|M)$ is *uniformly continuous* at R if it is uniformly upper and lower hemicontinuous. That is, for any open neighborhood U of $\tilde{Q}_R(\cdot|M)$ and any open-valued correspondence $V : \Theta \rightrightarrows 2^{\mathcal{A}}$ such that $\tilde{Q}_R(\theta|M) \cap V(\theta) \neq \varnothing$ for all θ, there exists a neighborhood N of R such that $\hat{R} \in N$ implies, for all $\theta \in \Theta$, *i)* $\tilde{Q}_{\hat{R}}(\theta|M) \subset U(\theta)$, and *ii)* $\tilde{Q}_{\hat{R}}(\theta|M) \cap V(\theta) \neq \varnothing$.

Definition 12. A decision rule M is **robust to structural uncertainty at** R if $R \rightrightarrows \tilde{Q}_R$ is uniformly continuous at R

For any $S \subseteq \Theta$ let $\tilde{Q}_{R|S}$ be the restriction of $\tilde{Q}_R$ to S. Say that $R \rightrightarrows \tilde{Q}_R$ is *almost uniformly continuous* at R if $\forall\ \varepsilon > 0\ \exists\ S \subseteq \Theta$ with $\lambda(S) > 1 - \varepsilon$ such that $R \rightrightarrows \tilde{Q}_{R|S}(\theta|M)$ is uniformly continuous at R (where S replaces Θ in the definition of uniform continuity).

Definition 13. A decision rule M is **weakly robust to structural uncertainty at** R if $R \rightrightarrows \tilde{Q}_R$ is almost uniformly continuous at R

The interpretation of robustness to structural uncertainty is that the decision rule should induce almost the same joint distribution of states and actions for small perturbations to the invariant representation. This in turn implies that the principal's expected payoff will be continuous in the function R. It turns out CUI (CWUI) implies implementability via a decision rule that is (weakly) robust to structural uncertainty.

Theorem 4. If (Q, P) are CUI then they are implementable given invariant representation R with an essentially continuous decision rule that is robust to multiplicity and structural uncertainty at R. If (Q, P) are CWUI then they are implementable with an essentially continuous decision rule that is weakly robust to multiplicity and weakly robust to structural uncertainty at R.

To prove Theorem 4, we make use of the following intermediate results.

Lemma 11. Given a function $F : X \times [0, 1] \to X$ on a compact subset X of an Euclidean space, define the function $G(t) = \{x \in X : F(x, t) = x\}$. Assume $t \mapsto F(x, t)$ is continuous. If $G(t)$ is single-valued and $x \mapsto F(x, t)$ is continuous on an open neighborhood of $G(t)$ then G is upper and lower hemicontinuous at t.

Proof. Since $G(t)$ is single-valued upper hemicontinuity implies lower hemicontinuity. We want to show that for any open neighborhood V of $G(t)$ there exists a neighborhood U of t such that $G(t') \subseteq V$ for all $t' \in U$.

Claim 1. For any open neighborhood V of $G(t)$ there exists a $\kappa > 0$ such that $|F(x,t) - x| > \kappa \quad \forall \ x \in X \setminus V$. The proof of claim 1 is as follows. $X \setminus V$ is a closed subset of a compact set, and thus compact. The function $x \mapsto |F(x,t) - x|$ is continuous, so it attains its minimum on $X \setminus V$. Since $G(t)$ is unique and $G(t) \notin X \setminus V$, this minimum is strictly greater than zero, so the desired κ exists.

To complete the proof of Lemma 11, we need to show that there exists an open neighborhood U of t such $|F(x,t') - x| > \kappa \quad \forall \ x \in X \setminus V, \quad t' \in U$. By continuity of $t' \mapsto F(x,t') - x$, for each x there exists a ε_x such that $|t' - t| < \varepsilon_x$ implies $|F(x,t') - x| > \kappa$. For each x, define $\ell(x,\varepsilon) = \min\{|F(x,t') - x| : |t' - t| \leq \varepsilon/2\}$, which exists by continuity of F and compactness of $|t' - t| \leq \varepsilon/2$.

Define $B(x) = \{x' \in X : \ell(x', \varepsilon_x) > \kappa\}$. By continuity of $x \mapsto F(x,t') - x$, $B(x)$ contains an open neighborhood of x (Berge's maximum theorem). Let $\tilde{B}(x)$ be this open neighborhood. The set $\cup_{x \in X \setminus V} \tilde{B}(x)$ covers $X \setminus V$. Then by compactness of $X \setminus V$ there exists a finite sub-cover. Let u be the smallest ε_x corresponding to an x such that $\tilde{B}(x)$ is in the finite sub-cover. Then $U = \{t' \in (0,1) : |t' - t| < u\}$. □

Proposition 7. Given a continuous function $F : X \times \Theta \times (0,1) \to X$ on a compact subset X of a Euclidean space, define the function $G(t,\theta) = \{x \in X : F(x,\theta,t) = x\}$. Let S be any compact subset of Θ such that $G(t,\theta)$ is single-valued for all $\theta \in S$. Then $t \rightrightarrows G(t,\theta)$ is upper and lower hemicontinuous at t, uniformly over S.

Proof. Since $G(t,\theta)$ is single-valued on S it suffices to show upper hemicontinuity. Let $V(\theta)$ be an open neighborhood of $\theta \mapsto G(t,\theta)$ on S. Without loss of generality (since Θ is compact and $G(t,\theta)$ single-valued on S), let $V(\theta) = \{x \in X : |G(t,\theta) - x| < \delta\}$ for some $\delta > 0$, or equivalently, $V(\theta) = \cup_{x \in G(t,\theta)} N_\delta(x)$. We want to show that there exists a neighborhood U of t such that $t' \in U$ implies $G(t',\theta) \subseteq V(\theta)$ for all $\theta \in S$.

Claim 1. $X \setminus V(\theta)$ is upper and lower hemicontinuous on S: Since $G(t,\theta)$ is single-valued, $X \setminus V(\theta) = X \setminus N_\delta(G(t,\theta))$ where $N_\delta(x)$ is the open ball around x with radius δ. We first show upper hemicontinuity. Let W be an open set containing $X \setminus V(\theta)$. Without loss of generality, let $W = X \setminus \bar{N}_{\delta-\rho}(G(t,\theta))$ for some $\rho \in (0,\delta)$ where $\bar{N}_{\delta-\rho}(x)$ is the closed ball around x with radius $\delta - \rho$.[45] By Lemma 11, we know that $\theta \mapsto G(t,\theta)$ is upper and lower hemicontinuous at all $\theta \in S$. By upper hemicontinuity of $\theta \mapsto G(t,\theta)$ at θ, there exists an open neighborhood B of θ such that $\theta' \in B$ implies $|x - G(\theta,t)| < (\delta-\rho)/2$ for all $x \in G(\theta',t)$. Then $\bar{N}_{\delta-\rho}(G(t,\theta)) \subset \cup_{x \in G(t,\theta')} N_\delta(x) = V(\theta')$ for all $\theta' \in B$. Thus $V(\theta') \subset W$ for all $\theta' \in B$, which shows upper hemicontinuity.

For lower hemicontinuity, let $W \subset X$ be an open set intersecting $X \setminus V(\theta)$. This holds if and only if there exists $x' \in W$ such that $|x' - G(t,\theta)| > \delta$. By upper hemicontinuity of $\theta \mapsto G(t,\theta)$ at θ, there exists an open neighborhood B of θ such that $\theta' \in B$ implies $|x - G(\theta,t)| < (|x' - G(t,\theta)| - \delta)/2$ for all $x \in G(\theta',t)$. Then $\theta' \in B$ implies $|x' - x| > \delta$ for all $x \in G(t,\theta')$. Thus $x' \notin \cup_{x \in G(t,\theta')} N_\delta(x) = V(\theta')$, so $W \cap X \setminus V(\theta') \neq \varnothing$ for all $\theta' \in B$, which shows lower hemicontinuity. This completes the proof of Claim 1.

We know from Lemma 11 that for each $\theta \in S$ there exists $\varepsilon_\theta, \kappa_\theta > 0$ such that

$$|t' - t| < \varepsilon_\theta \Longrightarrow |F(x,\theta,t') - x| > \kappa_\theta \;\; \forall\, x \in X \setminus V(\theta). \tag{5}$$

Claim 2. For each $\theta \in S$ there exists an open neighborhood $B(\theta)$ of θ such that

$$\theta' \in B(\theta) \text{ and } |t' - t| < \varepsilon_\theta \Longrightarrow |F(x,\theta,t') - x| > \kappa_\theta \;\; \forall\, x \in X \setminus V(\theta'),$$

where $\varepsilon_\theta, \kappa_\theta$ satisfy (5). The proof of this claim is as follows. Define

$$z(\theta,\varepsilon) := \min\{|F(x,\theta,t') - x| : |t' - t| \le \varepsilon/2,\; x \in X \setminus V(\theta)\},$$

[45] W so defined is open in X, but not in the space of which X is a subset.

which is well defined by compactness of $X \setminus V(\theta)$. By Berge's maximum theorem and Claim 1, $\theta \mapsto z(\theta, \varepsilon)$ is continuous at any $\theta \in S$. By (5) we know that $z(\theta, \varepsilon_\theta) > \kappa_\theta$ for all $\theta \in S$. Then for any $\theta \in S$ there exists an open neighborhood $B(\theta)$ of θ such that $\theta' \in B(\theta)$ implies $z(\theta', \varepsilon_\theta) > \kappa_\theta$. This proves Claim 2.

To complete the proof of Proposition 7, note that $\cup_{\theta \in S} B(\theta)$ is an open cover of S. By compactness of S there exists a finite sub-cover. Let I be the set of $\theta \in S$ that index this sub-cover. Let $\varepsilon = \min\{\varepsilon_\theta : \theta \in I\}/2$. Then

$$|t' - t| < \varepsilon \Longrightarrow |F(x, \theta, t') - x| > 0 \quad \forall\, x \in X \setminus V(\theta) \text{ and } \theta \in S.$$

Since $G(t', \theta)$ is non-empty for all t', θ we have that $|t' - t| < \varepsilon$ implies that for all θ, $G(t', \theta) \subseteq V(\theta)$, which shows upper hemicontinuity as desired. $\square$

Proof. There are two cases to consider: $P(\underline{\theta}) \leq P(\bar{\theta})$ or $P(\underline{\theta}) > P(\bar{\theta})$.

If $P(\underline{\theta}) \leq P(\bar{\theta})$ P is weakly increasing by Theorem 1. Then as noted in Section 2.5, (Q, P) can be implemented by a decision rule M that is continuous. Let $F(a, \theta, t) = M(R(a, \theta, t))$, where t continuously parameterizes the function R. Then F is continuous since M is continuous. Moreover, $G(t, \theta) = \tilde{Q}(\theta, t)$ will be single-valued on all but a zero-measure set of states when M is weakly robust to multiplicity, and single-valued everywhere when M is robust to multiplicity. Therefore for any $\varepsilon > 0$ we can find a compact set S such that $G(t, \theta)$ is single-valued for all $\theta \in S$. When M is robust to multiplicity let $S = \Theta$. Then Proposition 7 applies, which gives the result.

If $P(\underline{\theta}) > P(\bar{\theta})$ then Theorem 1 implies that P is weakly decreasing. As shown in the proof of Theorem 1, there exists a closed set $C \supset [P(\bar{\theta}), P(\underline{\theta})]$ such that M is continuous on C, but may have discontinuities outside of C. We are free to define M outside of C, so long as there is no $p \notin C$ such that $R(M(p), \theta) = p$. Let $M(p) = Q(\bar{\theta})$ if $p \notin C$ and $p > P(\underline{\theta})$, and let $M(p) = Q(\underline{\theta})$ if $p \notin C$ and $p < P(\bar{\theta})$. Since $P(\underline{\theta}) > P(\bar{\theta})$ by assumption, and $\theta \mapsto R(a, \theta)$ is weakly increasing for all a, there exists $\varepsilon > 0$ such that *(i)* $p - R(M(p), \theta) > \varepsilon$ for all θ and all $p \notin C$, $p > P(\underline{\theta})$, and *(ii)* $R(M(p), \theta) - p < \varepsilon$ for all θ and all $p \notin C$, $p < P(\bar{\theta})$. Therefore, conditions *(i)* and *ii* will continue to hold for some $\varepsilon' > 0$ and any R' that is sufficiently close to R in the sup-norm. This implies that it is sufficient to establish upper and lower hemicontinuity of $R \rightrightarrows \tilde{Q}_R$ for the restriction of M to C. Since M is continuous on C the argument applied to above the $P(\underline{\theta}) \leq P(\bar{\theta})$ P case holds here as well. □

The important implication of Theorem 4 is that small changes in R lead to small changes in the principal's expected payoff. Even though under the perturbed invariant representation R' there may be multiple equilibria, the joint distribution of states, prices and actions associated with each one will be close to that of the original equilibrium under R.

If M is robust to multiplicity but has discontinuities on $\bar{P}_M$ then it is not robust to structural uncertainty, at least when the discontinuity is not essential, i.e. when the left and right limits of M exist.[46] As discussed in Section 1, this further motivates the restriction to essentially continuous decision rules. Let $\theta_M(p|R) = \{\theta \in \Theta : R(M(p), \theta) = p\}$ be the set of states at which p could be an equilibrium price under M and R, and let $\bar{P}_M(R) := \{p \in \mathcal{P} : \theta_M(p) \neq \varnothing\}$ be the set of prices that could arise in equilibrium.

Lemma 12. Assume that M is robust to multiplicity. If M has a non-essential discontinuity on $\bar{P}_M(R)$ then it is not robust to structural uncertainty at R.

Proof. Suppose M is discontinuous at p', and let $\theta' \in \theta_M(p'|R)$. First, suppose that $p \mapsto R(M(p), \theta')$ is continuous at p'. Since M is discontinuous, there exists an open neighborhood U or $M(p')$ such that for any $\varepsilon > 0$ there exists $p'' \in N_\varepsilon(p')$ with $M(p) \notin U$. Since $p \mapsto R(M(p), \theta')$ is continuous at p', for any $\delta > 0$ we can choose ε small to guarantee $|R(M(p''), \theta') - R(M(p'), \theta')| < \delta$. But then let $\hat{R}$ be a continuous

[46] Given that $\mathcal{A}$ is compact, an essential discontinuity can be pictured as a point at which M oscillates with vanishing wavelength. The only potential benefit to the principal of using a discontinuous M is to avoid multiplicity, but an essential discontinuity is not useful in this regard.

function in a δ-neighborhood of R such that $\hat{R}(M(p''),\theta') = p'$, so $M(p'') \in \tilde{Q}_{\hat{R}}(\theta'|M)$. Therefore we cannot have upper hemicontinuity of $R \mapsto \tilde{Q}_R(\theta'|M)$ at R.

Now, suppose $p \mapsto R(M(p),\theta')$ is discontinuous at p'. Assume M is left-continuous at p' (symmetric argument for right-continuous, and similar for removable discontinuity). Then there exists $\varepsilon > 0$ such that either $R(M(p),\theta') < p$ for all $p \in [p'-\varepsilon, p')$ or $R(M(p),\theta') > p$ for all $p \in [p'-\varepsilon, p')$. Assume without loss of generality that the former holds. Then let $\hat{R}$ be a continuous function such that $\hat{R}(M(p),\theta') > R(M(p),\theta')$ for all $p \in [p'-\varepsilon, p')$. For $\hat{R}$ close to R there will be a neighborhood U or p' such that $\hat{R}(M(p),\theta') \neq p$ for all $p \in U$. This is because M is discontinuous at p'. Then $R \mapsto \tilde{Q}_R(\theta'|M)$ cannot be lower hemicontinuous at R. $\square$

Lemma 12 shows that essential continuity is, to an extent, necessary for robustness to structural uncertainty.

D.2 Beyond uniqueness

The key insight is that even if a decision rule induces multiple equilibria, at least one of these will be weakly uniquely implementable. This is established via the following intermediate result.

Proposition 8. Assume R is weakly increasing in θ. If $M \in \mathcal{M}$ induces multiple equilibria then at least one has a monotone price function (strictly monotone if R is strictly increasing in θ).

Proof. Claim 0. For any $\theta' \in (\underline{\theta}, \bar{\theta})$ and p' such that $\theta' \in \theta_M(p')$, there exist p'' such that $\theta_M(p'') \cap \{\underline{\theta}, \bar{\theta}\} \neq \varnothing$, $\theta_M(p) \neq \varnothing$ for all $p \in (\min\{p',p''\}, \max\{p',p''\})$ and M is continuous on $(\min\{p',p''\}, \max\{p',p''\})$ (when this interval is non-empty).

Let $\theta' \in (\underline{\theta}, \bar{\theta})$ be arbitrary, and let p' be such that $\theta' \in \theta_M(p')$. If $\{p \leq p' : \theta_M(p) = \varnothing\}$ is empty then $p'' = \arg\min_{a \in \mathcal{A}} R(a, \underline{\theta})$ satisfies the conditions of the claim. Similarly, if $\{p \geq p' : \theta_M(p) = \varnothing\}$ is empty then $p'' = \arg\max_{a \in \mathcal{A}} R(a, \bar{\theta})$ satisfies the conditions of the claim. Assume that $\{p \leq p' : \theta_M(p) = \varnothing\} \neq \varnothing$ and $\{p \geq p' : \theta_M(p) = \varnothing\} \neq \varnothing$. Let $\underline{p} = \sup\{p \leq p' : \theta_M(p) = \varnothing\}$ and $\bar{p} = \inf\{p \geq p' :$

$\theta_M(p) = \varnothing\}$. Since $M \in \mathcal{M}$, we have $\underline{p} < p' < \bar{p}$. Since M must be continuous on $(\underline{p}, \bar{p})$, we have $\theta_M(\underline{p}) \cap \{\underline{\theta}, \bar{\theta}\} \neq \varnothing$ and $\theta_M(\bar{p}) \cap \{\underline{\theta}, \bar{\theta}\} \neq \varnothing$. This proves Claim 0.

Claim 1. Let $\theta' \in (\underline{\theta}, \bar{\theta})$ and p' be such that $\theta' \in \theta_M(p')$. Let p'' be such that $\theta_M(p) \neq \varnothing$ for all $p \in (\min\{p', p''\}, \max\{p', p''\})$ and M is continuous on $(\min\{p', p''\}, \max\{p', p''\})$ (when this interval is non-empty). Then if $\underline{\theta} \in \theta_M(p'')$ and $p'' \leq p'$ ($p'' \geq p'$) there exists an equilibrium with a price function that is increasing (decreasing) on $[\underline{\theta}, \theta']$. Similarly, if $\bar{\theta} \in \theta_M(p'')$ and $p'' \geq p'$ ($p'' \leq p'$) there exists an equilibrium with a price function that is increasing (decreasing) on $[\theta', \bar{\theta}]$.

We will show the claim for $\bar{\theta} \in \theta_M(p'')$ and $p'' \geq p'$; all others cases are symmetric. For any θ, the set $\theta_M^{-1}(\theta)$ is compact: if $R(M(p), \theta) \neq p$ then this holds for all $\tilde{p}$ in a neighborhood p, since $M \in \mathcal{M}$ is continuous around equilibrium prices. If $p' = p''$ then we are done: convexity of $\theta_M(p)$ (Lemma 4) implies that there is a constant, and thus monotone, equilibrium price function on $[\theta', \bar{\theta}]$. Assume instead that $p'' > p'$. If there exists $\theta^* \in (\theta', \bar{\theta})$ such that $p^* > p''$ for any $p^* \in \theta_M^{-1}(\theta'')$ then there exists $\tilde{\theta} \in (\theta', \bar{\theta})$ such that $p'' \in \theta_M^{-1}(\tilde{\theta})$, by continuity of M on (p', p'') and Lemma 6. Then convexity of $\theta_M(p'')$ implies that we can construct a flat price function above $\tilde{\theta}$. Therefore assume no such θ^* exists. By a symmetric argument, we can assume that $\theta_M^{-1}(p) \cap [p', p''] \neq \varnothing$ for all $\theta \in [\theta', \bar{\theta}]$.

We want to construct an increasing equilibrium price function on $[\theta', \bar{\theta}]$. Consider an arbitrary price function $\tilde{P}$ such that $\tilde{P}(\theta) \in \theta_M^{-1}(\theta) \cap [p', p'']$ for all $\theta \in [\theta', \bar{\theta}]$, $\tilde{P}(\underline{\theta}) = p'$, and $\tilde{P}(\bar{\theta}) = p''$. We will show that any violations of monotonicity can be ironed without leading to further violations.

Claim 1.2. Suppose $\tilde{P}(\theta_2) < \tilde{P}(\theta_1) < \tilde{P}(\theta_3)$ for $\bar{\theta} > \theta_3 > \theta_2 > \theta_1 >$. Then there exists $p \in \theta_M^{-1}(\theta_2) \cap [\tilde{P}(\theta_1), \tilde{P}(\theta_3)]$.

Claim 1.2 follows immediately from Lemma 6. This in turn shows that Claim 1 holds for $\bar{\theta} \in \theta_M(p'')$ and $p'' \geq p'$, which is what we wished to show.

Claim 0 and Claim 1 together imply the existence of a monotone price function. If R is strictly increasing in θ then measurability of the action with respect to the price implies that P must be strictly monotone. $\square$

Theorem 1 says that monotonicity of the price function is a necessary condition for CWUI. Monotonicity is not in general sufficient. However, if we know that P is monotone *and* is induced by some $M \in \mathcal{M}$ then monotonicity of P suffices for CWUI in many settings. This is the case when the environment is fully bridgeable, as defined in Section 2.5. Under this assumption, any increasing selection from the price functions induced by M is CWUI.

Theorem 5. Assume R is strictly increasing in θ and the environment is fully bridgeable. If $M \in \mathcal{M}$ induces multiple equilibria then at least one is characterized by (Q, P) that are CWUI.

Proof. Let (Q, P) be an equilibrium induced by M, such that P is strictly monotone, which exists by Proposition 8. Since $M \in \mathcal{M}$ induces (Q, P), P can have no degenerate discontinuities. Let $\hat{M} = M$ on $P(\Theta)$ and $\mathcal{P} \setminus [\inf P(\Theta), \sup P(\Theta)]$. We show how to define $\hat{M}$ for the remaining prices such that it is essentially continuous and weakly uniquely implements (Q, P).

Suppose P has a non-degenerate discontinuity at θ^*, and let $\underline{p} = \lim_{\theta \nearrow \theta^*} P(\theta)$ and $\bar{p} = \lim_{\theta \searrow \theta'} P(\theta)$. If the discontinuity at θ^* is bridgeable then we can define $\hat{M}$ on $[\min\{\underline{p}, \bar{p}\}, \max\{\underline{p}, \bar{p}\}]$ such that *(i)* $\hat{M}(\underline{p}) = \lim_{\theta \nearrow \theta^*} Q(\theta)$, *(ii)* $\hat{M}(\bar{p}) = \lim_{\theta \searrow \theta^*} Q(\theta)$, and *(iii)* $p = R(\hat{M}(p), \theta^*)$ for all $p \in [\min\{\underline{p}, \bar{p}\}, \max\{\underline{p}, \bar{p}\}]$. Since the environment is fully bridgeable, this can be done for all discontinuities. Thus $\hat{M}$ so defined is continuous on $[\inf P(\Theta), \sup P(\Theta)]$ and coincides with M on $\hat{M} = M$ on $P(\Theta)$ and $\mathcal{P} \setminus [\inf P(\Theta), \sup P(\Theta)]$. Since M was essentially continuous, so is $\hat{M}$. Moreover, there are multiple market-clearing prices only in states at which P had a discontinuity. Since P is monotone, this set has measure zero. $\square$

Say that the principal is pessimistic if they evaluate a set of possible equilibria according to the worst case.

Corollary 2. Assume R is strictly increasing in θ and the environment is fully bridgeable. If the principal is pessimistic then the restriction to (Q, P) that are CWUI is without loss of optimality.

The conclusion of Theorem 5 can be extended in two ways. First, the result extends to weakly increasing R, when the environment satisfies a slightly stronger notion of bridgeability. Second, since CWUI price and action functions can generally be very well approximated by CUI price and action functions, we can replace CWUI with virtually CUI in the conclusion of Theorem 5. This requires some mild additional conditions, which guarantee that any CWUI (Q, P) can be approximated arbitrarily well by some CUI $(\hat{Q}, \hat{P})$.

Theorem 5 also simplifies the problem of a principal who takes a less extreme approach to multiplicity than the strict worst-case preferences described above. Consider a principal who lexicographically evaluates policies that induce multiple equilibria: the principal first evaluates a decision rule according to the worst equilibrium that it induces. Among those decision rules with the same worst-case equilibrium payoff, the principal chooses based on the best equilibrium that each induces (or indeed some other function of the remaining equilibria).[47] By Theorem 5 we know that the highest worst-case guarantee is exactly the maximum payoff over the subset of decision rules in $\mathcal{M}$ that are weakly robust to multiplicity. Once this value has been determined, the goal of the principal is to choose the decision rule with the best equilibrium outcome, subject to not inducing any equilibrium with a payoff below this worst-case bound.

Assume first that the principal's payoffs do not depend directly on the price; the principal cares only about the joint distribution of states and actions (a similar discussion will apply to other preferences). Assume that there is a unique optimal CWUI action function Q^*, implemented uniquely by decision rule M^* (if there are multiple optimal CWUI action functions then Condition 1 in Proposition 9 below must hold for one of them). If this is the case then, by Theorem 5, the principal with

[47] Such preferences are similar in spirit to these studied in the context of robust mechanism design (Börgers, 2017) and information design (Dworczak and Pavan, 2020).

lexicographic preferences wants to choose a decision rule that implements Q^* as one of its equilibrium outcomes; if Q^* is not one of the equilibrium outcomes then there will be some other CWUI action function induced by the decision rule, which will be worse than Q^* by definition. This pins down the decision rule for all prices in the range $\{R(Q^*(\theta), \theta) : \theta \in \Theta\}$; any optimal decision rule must coincide with M^* for such prices. Moreover, Q^* will be an equilibrium outcome of any such decision rule. This discussion implies the following.

Proposition 9. Let Q^* be the set of optimal CWUI action functions. Then the optimization constraints of the principal with lexicographic multiplicity preferences can be stated as follows: choose $\hat{M}$ subject to

1. $\exists Q \in Q^*$ such that $\hat{M}(R(Q(\theta), \theta)) = Q(\theta)$ for all $\theta \in \Theta$,
2. $\hat{M} \in \mathcal{M}$.

These constraints can greatly simplify the problem of finding optimal policies for a principal with lexicographic preferences over multiple equilibria. Consider the example of Section 4.1. The question is if, for a dovish principal who cannot implement the first best, it is possible to improve the best-case equilibrium while still guaranteeing that no equilibrium gives a payoff less than that of the virtually optimal CWUI policy. In this problem, the answer is no.

Proposition 10. Assume that π has the pivot property and the principal is dovish. If the principal takes the lexicographic approach to equilibrium multiplicity, there is no gain from considering action functions that are not CWUI.

Proof. Consider a virtually optimal action function, defined by the interval (θ', θ'') over which ironing occurs. Then, as shown above, the virtually optimal decision rule is pinned down on $P^*(\Theta)$. The only potential changes that could be made to the decision rule when allowing for multiplicity are on $(R(0, \underline{\theta}), R(1, \underline{\theta}))$. Changing the decision rule on this range will can only induce equilibria in which lower actions are taken for states below $\frac{-c}{b}$, which is worse for the principal. $\square$

D.3 CUI with weakly increasing R.

Relaxing the assumption of strictly increasing $\theta \mapsto R(a,\theta)$ to weakly increasing, we obtain a similar characterization to Theorem 2. It is necessary, however, to add a condition to account for actions for which the induced price is constant over an interval of states.

Consider (P,Q) implementable. (P,Q) satisfies market-clearing and measurability by Observation 1. Moreover suppose that $R(Q(\theta),\theta) = R(Q(\theta),\theta')$ with $\theta \neq \theta'$. If $Q(\theta') \neq Q(\theta)$ then there will be multiplicity, since by measurability $P(\theta') \neq P(\theta)$ but $P(\theta) = R(Q(\theta),\theta')$ is a market clearing price in state θ'. The only modifications needed to extend Theorem 2 are those that rule out such instances of multiplicity.

Proposition 11. Assume R is weakly increasing in θ. Then (Q,P) is CUI iff

1. $P(\theta) = R(Q(\theta),\theta)$ for all θ,
2. P is weakly monotone.
3. Q is continuous and BC1. Moreover, if P is decreasing, then Q is BC2.
4. For all θ,θ', $P(\theta) = P(\theta')$ or $P(\theta) = R(Q(\theta),\theta')$ implies $Q(\theta') = Q(\theta)$.

Proof. ($\Rightarrow$) The main difference with Theorem 2 lies in part 4. The first part is the measurability condition that was already necessary for implementation. For the second part, first notice that if $P(\theta) = R(Q(\theta),\theta')$ then $P(\theta') = P(\theta)$: otherwise there are multiple equilibria at the state θ', one with price $P(\theta)$ and one with price $P(\theta')$. Measurability implies that $Q(\theta) = Q(\theta')$.

($\Leftarrow$) We cannot construct the function M that continuously uniquely implements (P,Q) in the same way as the one in Theorem 2 since P is not necessarily injective: $P^{-1}(p)$ is not necessarily a singleton anymore. However, the measurability condition in 4 guarantees that $\theta,\theta' \in P^{-1}(p)$ then $Q(\theta) = Q(\theta')$ so $Q(P^{-1}(p))$ is a singleton for all $p \in P(\Theta)$. The construction of M outside of $P(\Theta)$ is the same as in Theorem 2. $\square$

E Bridgeability and CWUI

To characterize the set of outcomes (Q, P) that are CWUI, first, note that if (Q, P) are CWUI, then, since P must be monotone by Theorem 1, any discontinuity in P must be a jump discontinuity, and P can have at most countably many discontinuities. Moreover, Q can be discontinuous at θ only if P is as well: otherwise it would not be possible for Q to be implemented by an M that is continuous at $P(\theta)$. Thus Q can also have at most countably many discontinuities. Finally, essential continuity of the implementing M gives the following result.

Lemma 13. The one-sided limits of any CWUI Q, denoted by $\lim_{\theta \nearrow \theta'} Q(\theta)$ and $\lim_{\theta \searrow \theta'} Q(\theta)$, must exist for all θ'.

Proof. Let $\{\theta_n\}$ be an increasing sequence converging to θ'. Suppose P is increasing (the argument is symmetric if P is decreasing). Then $\{P(\theta_n)\}$ is an increasing and bounded sequence, and so converges. Denote this limit by $\bar{p}$. Since M is essentially continuous, it is continuous in a neighborhood of $\bar{p}$. Hence $\lim_{n\to\infty} Q(\theta_n) = \lim_{n\to\infty} M(P(\theta_n)) = M(\bar{p})$. $\square$

Suppose P has a discontinuity at θ^*, and let $\underline{p} = \lim_{\theta \nearrow \theta^*} P(\theta)$ and $\bar{p} = \lim_{\theta \searrow \theta'} P(\theta)$.

Definition 14. Say that a discontinuity in P at θ^* is *bridgeable* given Q if there exists a continuous function $\gamma : [\min\{\underline{p}, \bar{p}\}, \max\{\underline{p}, \bar{p}\}] \to \mathcal{A}$ such that *i)* $\gamma(\underline{p}) = \lim_{\theta \nearrow \theta^*} Q(\theta)$, *ii)* $\gamma(\bar{p}) = \lim_{\theta \searrow \theta^*} Q(\theta)$, and *iii)* $p = R(\gamma(p), \theta^*)$ for all $p \in [\min\{\underline{p}, \bar{p}\}, \max\{\underline{p}, \bar{p}\}]$. We say that the environment is *fully bridgeable* if for any (Q, P), all discontinuities in P are bridgeable.

Observation 2. A discontinuity in P at θ^* is bridgeable iff there exists a continuous function $\gamma : [0, 1] \to \mathcal{A}$ such that *i)* $\gamma(0) = \lim_{\theta \nearrow \theta^*} Q(\theta)$, *ii)* $\gamma(1) = \lim_{\theta \searrow \theta^*} Q(\theta)$, and *iii)* $x \mapsto R(\gamma(x), \theta)$ is strictly monotone.

Observation 2 is useful because the condition that $x \mapsto R(\gamma(x), \theta)$ is strictly monotone is easier to check than the fixed-point condition in the definition of bridgeability.

Proposition 12. Assume R is strictly increasing in θ. Then (Q, P) is CWUI iff

1. $P(\theta) = R(Q(\theta), \theta)$ for all θ.
2. P is strictly monotone.
3. If Q is discontinuous at θ^* then P has a bridgeable discontinuity at θ^*.
4. $\bar{Q} := \lim_{\theta \to \bar{\theta}} Q(\theta)$ and $\underline{Q} := \lim_{\theta \to \underline{\theta}} Q(\theta)$ exist. Moreover, if P is decreasing, then $\underline{Q}$ is not maximal at the bottom and $\bar{Q}$ is not minimal at the top.

Proof. Given Theorem 2, we need only show that Q can have a discontinuity at θ^* iff P has a bridgeable discontinuity at θ^*. Clearly Q can be discontinuous at θ^* iff P is continuous at θ^* (otherwise M would need to be discontinuous at $P(\theta^*)$). As shown in the proof of Theorem 1, P can be discontinuous at θ^* only if $\theta_M(p) = \theta^*$ on $(\min\{\lim_{\theta \nearrow \theta^*} P(\theta), \lim_{\theta \searrow \theta^*} P(\theta)\}, \max\{\lim_{\theta \nearrow \theta^*} P(\theta), \lim_{\theta \searrow \theta^*} P(\theta)\})$. This is possible iff there exists γ satisfying the definition of bridgeability (in which case we take $M = \gamma$ on this interval). □

Of the conditions in Proposition 12, the bridgeability condition is in theory the most difficult to verify. Fortunately, we show that most relevant environments are fully bridgeable and so this condition can be ignored. If the environment if fully bridgeable then we can replace condition 3 in Proposition 12 with the following: Q can be discontinuous at θ^* iff P is as well. (Or equivalently: $\lim_{\theta \nearrow \theta^*} R(Q(\theta), \theta) \neq \lim_{\theta \searrow \theta^*} R(Q(\theta), \theta)$.) Thus, from a practical perspective, many applied problems can be solved simply by optimizing over the action function Q subject to the constraint that $R(Q(\theta), \theta)$ be strictly monotone.

E.1 Bridgeability: Sufficient Conditions

This section discusses bridgeability further. We provide sufficient conditions for the various notions of bridgeability, and show that they are satisfied in common settings.

Let $(\mathcal{A}, \succ)$ be a partially ordered set. Say $(\mathcal{A}, \succ)$ is *upward directed* if for any two $a'', a' \in \mathcal{A}$ there exists $c \in \mathcal{A}$ such that $c \succ a''$ and $c \succ a'$. Downward directed is defined analogously.[48] We use the notation $a''_\alpha a' \equiv \alpha a'' + (1-\alpha)a'$. Say that $\succ$ is preserved by mixtures if for any $a'' \succ a'$ and $\alpha \in (0,1)$, $a'' \succ a''_\alpha a' \succ a'$. Finally, say that $a \mapsto R(a,\theta)$ is *strongly monotone with respect to* $\succ$ if $a'' \succ a'$ and $a'' \neq a'$ implies $R(a'',\theta) > R(a',\theta)$. We use the notation $a''_\alpha a' \equiv \alpha a'' + (1-\alpha)a'$. The following proposition gives sufficient conditions for full bridgeability, but it is also useful because the proof of the existence of a monotone path is constructive. This construction could potentially be useful in applications.

Proposition 13. Let $(\mathcal{A}, \succ)$ be a partially ordered set that is both upward and downward directed, and such that $\succ$ is preserved by mixtures. If $R(\cdot,\theta)$ is strongly monotone with respect to $\succ$ then there is a monotone path between a' and a'' at θ iff $R(a'',\theta) \neq R(a',\theta)$

Proof. The condition $R(a',\theta) \neq R(a'',\theta)$ is obviously necessary. It remains to show that it is sufficient. That is, we want to show that there exists a monotone path between any $a'', a' \in \mathcal{A}$ such that $R(a',\theta) \neq R(a'',\theta)$. Assume without loss that $R(a'',\theta) > R(a',\theta)$. If $a'' \succ a'$ then the ray from a'' to a' is a monotone path. This follows since $\succ$ is preserved by mixtures and $R(\cdot,\theta)$ is strongly monotone.

Suppose a' and a'' are not ordered. Let $\bar{a}$ be an upper bound for a'', a', i.e. $\bar{a} \succ a''$ and $\bar{a} \succ a'$, and let $\underline{a}$ be a lower bound. Both exist since $(\mathcal{A}, \succ)$ is upward and downward directed. By continuity of R, there exists $\bar{\lambda} \in (0,1)$ such that $R(\bar{a}_{\bar{\lambda}} a', \theta) = R(a'',\theta)$. Similarly there exists $\underline{\lambda} \in (0,1)$ such that $R((a''_{\underline{\lambda}}\underline{a}),\theta) = R(a',\theta)$.

We will now construct one-half of the monotone path from a' to a''. Let $t : [0,1] \to [\bar{\lambda}, 1] \times [0,1]$ be a continuous and strictly monotone function, and let $t_i(x)$ be the i^{th} coordinate of $t(x)$. For each $x \in (0,1)$, we have $R(\bar{a}_{t_1(x)}a',\theta) > R(a'',\theta)$, $R(\underline{a}_{t_2(x)}a',\theta) < R(a',\theta)$, and $\bar{a}_{t_1(x)}a' \succ \bar{a}_{t_1(x)}a'$. These properties follow from strong monotonicity of R and the fact that $\succ$ is preserved under mixtures.

For each $x \in (0,1)$, define $f(x)$ by $R((\bar{a}_{t_1(x)}a')_{f(x)}(\underline{a}_{t_2(x)}a'),\theta) = xR(a'',\theta) + (1-x)R(a',\theta)$. We claim that $x \mapsto (\bar{a}_{t_1(x)}a')_{f(x)}(\underline{a}_{t_2(x)}a')$ is a continuous function. It is

[48] A lattice is an upward and downward directed set, but the converse is not true.

a well-defined function by strong monotonicity of R. It is continuous since R and t are continuous. Moreover, by construction $x \mapsto R((\bar{a}_{t_1(x)}a')_{f(x)}(\underline{a}_{t_2(x)}a'), \theta)$ is strictly increasing, and $(\bar{a}_{t_1(0)}a')_{f(0)}(\underline{a}_{t_2(0)}a') = a'$. Therefore $x \mapsto (\bar{a}_{t_1(x)}a')_{f(x)}(\underline{a}_{t_2(x)}a')$ forms one half of a monotone path from a' to a''. The other half of the monotone path is defined analogously, using a'' and $\underline{\lambda}$ in place of a' and $\bar{\lambda}$. $\square$

Proposition 13 makes it easy to identify when a discontinuity will be bridgeable. For example, it implies that when $\mathcal{A}$ is a chain a gap between a' and a'' will be bridgeable at θ iff $R(\cdot, \theta)$ is strictly monotone on (a', a'').

More importantly, Proposition 13 implies that every discontinuity will be bridgeable when $\mathcal{A} = \Delta(Z)$, i.e. the set of distributions on some set Z, under mild assumptions on R. Let $\pi(z, \theta)$ be a real-valued function, with $\theta \mapsto \pi(z, \theta)$ continuous for all z. For example, $\pi(a, \theta)$ could represent a company's cash flow as a function of the state and government intervention $z \in Z$. In state θ, any $a \in \mathcal{A}$ in induces a distribution $F(a, \theta)$ on $\mathbb{R}$ via $\pi(\cdot, \theta)$. Let $\succ_{FOSD}$ be the first-order stochastic dominance order. This partial order on $\Delta(\mathbb{R})$ induces a preorder $\succeq$ on $\mathcal{A}$. Define $a'' \succ a'$ by $a'' \succeq a'$ and $\neg(a' \succeq a'')$ if $a'' \neq a'$, and $a' \succ a'$ for all a'. If $\pi(z', \theta) \neq \pi(z'', \theta)$ for all $z'' \neq z'$ then $\succeq = \succ$. Then $a \mapsto R(a, \theta)$ is strongly monotone if $F(a'', \theta) \succ F(a', \theta)$ implies $R(a'', \theta) \succ R(a', \theta)$. The partially ordered set $(\mathcal{A}, \succ)$ satisfies the conditions of Proposition 13 (when $\pi(z', \theta) \neq \pi(z'', \theta)$ for all $z' \neq z''$ it is in fact a lattice).

Corollary 3. If $\mathcal{A} = \Delta(Z)$ and for all θ $a \mapsto R(a, \theta)$ is strongly monotone with respect to the order induced by first-order stochastic dominance, then the environment is fully bridgeable.

F Representing the market

One reason for the market to fail to admit an invariant representation is if for some decision rule M, there exist multiple equilibria with the same action function, but different price functions. However, it is relatively straightforward to extend our analysis to this type of market, using a representation via a market-clearing correspondence

$R : \mathcal{A} \times \Theta \mapsto 2^{\mathcal{P}}$. The more interesting and challenging scenario, in terms of representing the market, is when there are *global effects*: it is not sufficient to know the equilibrium action in state θ in order to determine the equilibrium price in that state (or even the set of equilibrium prices in state θ). Nonetheless, we obtain representations.

F.1 Equilibrium inferences and global effects

To illustrate the challenges involved in modeling markets in which global effects are present, and to understand the reasons such effects might arise, consider the REE asset market of Section 1.1, Example 3. Recall that as stated in Proposition 1, this markets admits an invariant representation.

The key feature of this environment is that in addition to their private signals, investors learn about the state from the price. In contrast to the private-values setting of Section 1.1 Example 1, other investors' signals are informative about a payoff-relevant state, and thus investors draw inferences from the price. Formally, this can be seen in the demand optimality condition

$$X_i(p, s_i) = \arg\max_x \mathbb{E}\left[u_i\big(x \cdot (\pi(M(p), \theta) - p)\big) \mid s_i, P_M(\theta) = p\right] \tag{6}$$

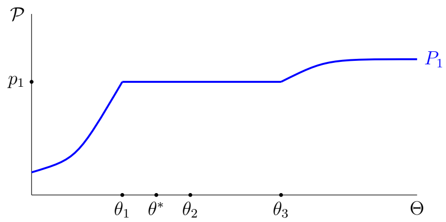
The difficulty with analyzing market-based policy in this environment can be seen by examining (6). The principal's decision rule affects investors in two ways. The first is a direct *forward guidance* effect: the decision rule determines what action investors anticipate, conditional on the price, and thus affects the anticipated dividend $\pi(M(p), \theta)$. However there's also an indirect *informational effect*, arising from the fact that when formulating their demand for the price of p, investors condition on the event $\{\theta \in \Theta : P_M(\theta) = p\}$. The decision rule shapes the entire equilibrium price function P_M, and thus determines what information investors infer about the state from the price. The subtlety of this informational effect is that investor beliefs in a give state will depend on the equilibrium price and principal actions in other states. Thus global properties of the decision rule and the equilibrium price and action functions will matter in determining the price in a given state. Such global dependence makes it more difficult to analyze the principal's problem in outcome space (the space of price and action functions); modifying the action and price function for some states may necessitate modifications elsewhere. This introduces global constraints into the principal's problem.

To understand this difficulty, consider the REE asset market model described above, and let Q_1, P_1 an implementable action and price function. The price function, depicted in Figure 5a, is constant over the interval $[\theta_1, \theta_3]$. Let Q_2 be another action function, such that $Q_2(\theta) = Q_1(\theta)$ for $\theta \leq \theta_2$ and $Q_2 \neq Q_1$ elsewhere. We want to know if Q_2 is implementable, and if so, what the corresponding price function will look like. It is natural to expect that if Q_2 is implementable, the corresponding price function P_2 will differ from P_1 for states above θ_2. Suppose that $P_2 > P_1$ above θ_2. However, can it be the case that the price functions also differ below θ_2, where the action functions are the same? Suppose that this is not the case; $P_2 = P_1$ below θ_2. Let $\theta^* \in (\theta_1, \theta_2)$ be a state in which Q_1 and Q_2 coincide, so $Q_1(\theta^*) = Q_2(\theta^*) = a^*$. In the Q_1 equilibrium, the information revealed by a price of $P_1(\theta^*)$ is $\{\theta : P_1(\theta) = P_1(\theta^*)\} = [\theta_1, \theta_3]$. Therefore, in state θ^* investor i's demand is given by

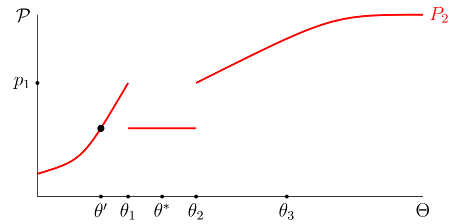
$$X_i(P_1(\theta^*), s_i) = \arg\max_x \mathbb{E}\left[u_i\big(x \cdot (\pi(a^*, \theta) - p)\big) \mid s_i, \theta \in [\theta_1, \theta_3]\right]$$

Similarly, in the Q_2 equilibrium, a price of $P_2(\theta)$ reveals that $\theta \in [\theta_1, \theta_2]$, so i's demand in state θ is $X_i(P_2(\theta^*), s_i) = \arg\max_x \mathbb{E}\left[u_i\big(x \cdot (\pi(a^*, \theta) - p)\big) \mid s_i, \theta \in [\theta_1, \theta_2]\right]$

Notice that the beliefs of investor i, in this case, are first-order stochastically dominated by those in the Q_1 equilibrium. If $\theta \mapsto \pi(a^*, \theta)$ is strictly increasing then the quantity demanded by every investor will be higher under the FOSD dominant beliefs. This means that in order for markets to clear at state θ^* in the Q_2 equilibrium, the price must be lower than in the Q_1 equilibrium. Thus it cannot be that $P_1 = P_2$ for all states below θ_2. However, if in the Q_2 equilibrium the price must be lower



(a) Q_1 equilibrium



(b) Q_2 not implementable

Figure 5: Information effects and global dependence

for states in $[\theta_1, \theta_2]$, as depicted in Figure 5b, then it may be that (Q_2, P_2) is not even implementable. This will be the case if there is some state $\theta' < \theta_1$ such that $P_2(\theta') = P_2(\theta^*)$, but $Q_2(\theta') \neq Q_2(\theta^*)$, as the principal's action must be measurable with respect to the price.

F.2 Invariant representation in REE

Despite the presence of informational effects, an invariant representation can be derived under quite mild conditions.

Consider a more general version of the asset market model described above. There are a unit mass of investors. Investors receive conditionally independent signals s_i about the state, with conditional distribution $h(\cdot|\theta)$ on $[\underline{s}, \bar{s}]$. The ex-post payoff to investor i who purchases a quantity x of the asset when the principal takes action a, the state is θ, and the asset price is p is given by $V_i(a, \theta, x, p)$, which is assumed to be strictly decreasing in p, strictly concave in x (to guarantee a unique solution), and continuous in x, θ.[49] For a fixed action a the demand of investor i who observes signal s and knows that the state is in $\mathcal{I} \subseteq \Theta$ is given by $x_i(p|a, s_i, \mathcal{I}) = \max_x E[V_i(a, \theta, x, p)|s, \mathcal{I}]$.

Assume $p \mapsto x_i$ is strictly decreasing for all i (which holds if, for example, that $(x, p) \mapsto V_i(a, \theta, x, p)$ satisfies strict single crossing). Investor heterogeneity, both of utilities and beliefs, is allowed for, but for simplicity assume that there are finitely many investor types, meaning finitely many distinct demand functions in the pop-

[49]For example, each investor has a strictly increasing Bernoulli utility function u_i and wealth w_i, and $V_i(a, \theta, x, p) \equiv u_i(x(\pi(a, \theta) - p) + w_i)$.

ulation. Normalizing the aggregate supply of the asset to zero, the market clearing condition is $\int_0^1 x_i(p|a, s_i, \mathcal{I})di = 0$. Since there is a continuum of investors and a finite number of investor types aggregate demand is deterministic, conditional on the state and the principal action a. Thus we can write market clearing in state θ as $X(p|a, \mathcal{I}, \theta) = 0$. Let $P^*(a, \mathcal{I}, \theta)$ be the unique price that clears the market.

Given any price function $\tilde{P} : \Theta \to \mathbb{R}$, let $\mathcal{I}_{\tilde{P}} : \Theta \to 2^\Theta$ be the coarsest partition with respect to which $\tilde{P}$ is measurable. We say that $\tilde{P}$ *induces* partition $\mathcal{I}_{\tilde{P}}$. A *rational expectations equilibrium* (REE) given decision rule M consists of a price function $\tilde{P}$ such that $X(\tilde{P}(\theta)|M(\tilde{P}(\theta)), \mathcal{I}_{\tilde{P}}(\theta), \theta) = 0$ for all θ.

In many settings, there is a monotone relationship between investors' private signals and their actions. It turns out that this is sufficient to guarantee the existence of an a.e. invariant representation. Let $\geq$ be a complete order on the state space. Define the level set of $\geq$ as $L_\theta \equiv \{\theta' \in \Theta : \theta' \geq \theta\} \cap \{\theta' \in \Theta : \theta \geq \theta'\}$, and let the upper-set be $U_\theta = \{\theta' \in \Theta : \theta' \geq \theta\}$.

Single Crossing. $V_i(a, \theta, x, p)$ satisfies single crossing in x, θ. Moreover, if $\theta' \in L_\theta$ then $V_i(a, \theta', x, p) - V_i(a, \theta', x', p) = V_i(a, \theta, x, p) - V_i(a, \theta, x', p)$.

Belief Monotonicity. $h(\cdot|\theta'')$ strictly MLRP dominates $h(\cdot|\theta')$ for $\theta'' > \theta'$.

The second piece of the Single Crossing assumption says that i has the same preferences over quantities in states $\theta, \theta' \in L_\theta$, conditional on a, p.

To see how these two assumptions imply that the market admits an invariant representation, consider the example illustrated in Figure 5. The issue encountered there is that since state θ^* belonged to different public information sets in the Q_1 and Q_2 equilibria, i.e. different level sets of the equilibrium price function, the demands in state θ^* could also differ. In particular, we posited that if higher states are associated with higher aggregate beliefs in the population (Belief Monotonicity) then demand would be higher in state θ^* when this state belongs to the public information set $[\theta_1, \theta_3]$ then when it belongs to the public information set $[\theta_1, \theta_2]$. This conclusion holds when higher beliefs are associated with higher demands (an implication of Single Crossing and Athey (2002)). The flaw with the above of reasoning is that if demands are strictly increasing in private signals conditional on the public information set $[\theta_1, \theta_3]$ then we cannot have a constant price over this interval to begin with: demand would be higher at higher states within this interval. Thus it must be that demand is constant as a function of private signals, which in turn implies that aggregate demand will be the same whether the public information is $[\theta_1, \theta_3]$ or $[\theta_1, \theta_2]$.

The key observations that we make use of to prove that the market admits an invariant representation are 1) that the principal's action is measurable with respect to the price, and 2) that public information sets revealed to investors are exactly the level sets the price function. The following proposition formalizes the above argument.

Proposition 14. Assume there is a complete order on Θ such that Single Crossing and Belief Monotonicity are satisfied. Then the market admits an a.e invariant representation given $(\mathcal{P}^{\mathcal{A}}, \Theta^{\mathcal{P}})$. In particular, $R(a, \theta) = P^*(a, L_\theta, \theta)$.

Proof. First, suppose (Q, P) are equilibrium outcomes given M. We want to show that $P(\theta) = P^*(Q(\theta), L_\theta, \theta)$ for almost all θ. Fix a state θ, and let $\mathcal{I}_P(\theta)$ be the public information set to which θ belongs. If $\mathcal{I}_P(\theta) \subseteq L_\theta$ then we are done, so suppose $\mathcal{I}_P(\theta) \setminus L_\theta$ is non-empty. Let $x_i^*(s) = x_i(P(\theta)|Q(\theta), s, \mathcal{I}_P(\theta))$. Under Single Crossing and Belief Monotonicity, $x_i^*(s)$ is weakly increasing in s. Suppose $x_i^*(s)$ is strictly increasing in s. Then (using the so-called "continuum law of large numbers" convention) Belief Monotonicity implies that for any $\theta' \in \mathcal{I}_P(\theta) \setminus L_\theta$, we have $X(P(\theta)|Q(\theta), \mathcal{I}_P(\theta), \theta') > (<) X(P(\theta)|Q(\theta), \mathcal{I}_P(\theta), \theta)$ if $\theta' > (<) \theta$. In either case, $P^*(Q(\theta), \mathcal{I}_P(\theta), \theta') \neq P^*(Q(\theta), \mathcal{I}_P(\theta), \theta)$. But contradicts the assumption that $\theta' \in \mathcal{I}_P(\theta)$. Thus it must be that $s \mapsto x_i^*(s)$ is constant. We now show that this implies the result.

Assume $s \mapsto x_i^*(s)$ is constant, and let $x^* = x_i^*(s)$. Suppose there exists a measurable set $A \subset \mathcal{I}_P(\theta)$, and x' such that

i. $V_i(Q(\theta), \theta', x', P(\theta)) > V_i(Q(\theta), \theta', x^*, P(\theta))$ for all $\theta' \in A$.

ii. $\mu(A|\mathcal{I}_P(\theta)) > 0$.

Then Single Crossing and Belief Monotonicity imply that $s \mapsto x_i^*(s)$ is not constant, which violates our previous conclusion. Therefore, it must be that no such A, x' exist. If no A, x' satisfy condition i. then there exists x^* such that $V_i(Q(\theta), \theta', x', P(\theta)) \leq V_i(Q(\theta), \theta', x^*, P(\theta))$ for all x and $\theta \in \mathcal{I}_P$. Then it must be that for all $\theta' \in \mathcal{I}_P(\theta)$, we have $P(\theta') = P^*(Q(\theta), L_\theta, \theta) = P^*(Q(\theta'), L_{\theta'}, \theta')$ as desired. The only other possibility is that any A, x' that satisfy condition i., do not satisfy condition ii., so $\mu(A|\mathcal{I}_P(\theta)) = 0$. Let $\{(A_n, x_n')\}_{n\geq 0}$ be the set of all such pairs. These can be divided into two groups: $x_n' > x^*$ and $x_n' < x^*$. Assume that all are of the $x_n' > x^*$ group (a symmetric argument applies to the $x_n' < x^*$ group). Notice that there must exist $\theta^* \in \mathcal{I}_P(\theta)$ such that $V_i(Q(\theta), \theta^*, x^*, P(\theta)) > V_i(Q(\theta), \theta^*, x', P(\theta))$ for all $x' > x^*$ (otherwise x^* could not be optimal under any signal). Moreover, for any n, we have $(\cup_{\theta' \in A_n}(U_{\theta'} \cap \mathcal{I}_P(\theta)), x_n') \in \{(A_n, x_n')\}$ by Single Crossing, so without loss of generality, assume that $A_n = \cup_{\theta' \in A_n}(U_{\theta'} \cap \mathcal{I}_P(\theta))$ for all n, and assume $A_n \subset A_{n'}$ for $n' > n$. Then we can define a decreasing countable sequence θ_n such that $U_{\theta_n} \subseteq A_n \subseteq U_{\theta_{n+1}}$ for all n and $\cup_n A_n \subseteq \cup_n U_{\theta_n}$. Since $U_{\theta_n} \subseteq A_n = \cup_{\theta' \in A_n}(U_{\theta'} \cap \mathcal{I}_P(\theta))$, Single Crossing implies that x', U_{θ_n} satisfy condition i., so $\mu(U_{\theta_n}|\mathcal{I}_P(\theta)) = 0$. Then countable additivity of μ implies $\mu(\cup_n U_{\theta_n}|\mathcal{I}_P(\theta)) = 0$, so $\mu(\cup_n A_n|\mathcal{I}_P(\theta)) = 0$.

But then $V_i(Q(\theta), \theta', x', P(\theta)) \leq V_i(Q(\theta), \theta', x^*, P(\theta))$ for all x and all but a conditionally-zero-measure subset of $\mathcal{I}_P(\theta)$. Thus for all $\theta' \in \mathcal{I}_P(\theta) \setminus A$, we have $P(\theta') = P^*(Q(\theta), L_\theta, \theta) = P^*(Q(\theta'), L_{\theta'}, \theta')$ as desired. Thus far, we have reasoned for a fixed public information set $\mathcal{I}_P(\theta)$. However since for any information set. However since $P(\theta) = P^*(Q(\theta), L_\theta, \theta)$ can fail for at most a conditionally-zero-measure subset of any information set, the set of all such θ has zero measure in Θ.

For the converse direction, we want to show that if $R(a, \theta) = P^*(a, L_\theta, \theta)$ and Q, P, M satisfy commitment and market clearing then (Q, P) are equilibrium outcomes given M. Thus we need to check that $X(P(\theta)|M(P(\theta)), \mathcal{I}_P(\theta), \theta) = 0$ for all θ. Fix a public information set $\mathcal{I}_P(\theta)$. The first part of the above proof for the other direction continues to hold: it must be that $s \mapsto x_i^*(s) \equiv x_i(P(\theta)|Q(\theta), s, \mathcal{I}_P(\theta))$ is constant, otherwise P could not be constant on $\mathcal{I}_P(\theta)$. But the second part of the above proof tells us that for all but a conditionally-zero-measure subset of $\mathcal{I}_P(\theta)$, we have $V_i(Q(\theta), \theta', x^*, P(\theta)) > V_i(Q(\theta), \theta', x', P(\theta))$ for all $x' \neq x^*$. Let θ'' be a state such that this inequality holds. Then $x_i(P(\theta)|Q(\theta), s, \mathcal{I}_P(\theta)) = x_i(P(\theta)|Q(\theta), s, L_{\theta''})$ for all s, so markets clear in state θ if and only if $P = P^*(a, L_\theta, \theta)$. □

Continuity of the invariant representation $R(a, \theta) = P^*(a, L_\theta, \theta)$ is guaranteed by continuity of $\theta \mapsto h(\cdot|\theta)$ and continuity of V_i.

F.3 Noisy REE in asset markets

This section illustrates a market in which restrictions on P derived without reference to an invariant representation are then used to identify the invariant representation. We do this in the canonical noisy REE model of Grossman and Stiglitz (1980) and Hellwig (1980) by extending results from Breon-Drish (2015) to a setting with feedback effects.

The setting is as follows. There is a single asset that pays an ex-post dividend of $\pi(a, \omega)$, where $\omega \in \Omega$ is referred to as the payoff-relevant state. We assume that π is continuous and is affine in ω for all a; $\pi(a, \omega) = \beta_0^a + \beta_1^a \omega$. Each investor observes an additive signal $s_i = \omega + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma_i^2)$, where σ_i^2 lies in a bounded set. The supply shock is a random variable z taking values in $\mathcal{Z}$. We assume that z has a truncated normal distribution. That is, z is the restriction of a normal random variable $\hat{z} \sim N(0, \sigma_Z^2)$ to the interval $[b_1, b_2]$, with $-\infty \leq b_1 \leq 0 \leq b_2 \leq \infty$ (note that this assumption accommodates un-truncated supply shocks as well). For simplicity, let $b_1 = -b_2$; this does not affect the results. The state θ consists of both the payoff-relevant state ω and the supply shock z.

There are are a continuum of investors $i \in [0, 1]$, each with CARA utility $u(w) = -\exp\left\{-\frac{1}{\tau_i} w\right\}$. The ex-post payoff to an investor who purchases x units of the asset at price p when the principal takes action a is given by $-\exp\left\{-\frac{1}{\tau_i} x(\pi(a, \omega) - p)\right\}$, where τ_i lies in some bounded set. We assume that the distribution of private signals in the population is uniquely determined by the state ω (this is the usual "continuum law of large numbers" convention). Let $x_i(p|a, \mathcal{I}, s_i)$ be the demand of investor i when the price is p, the anticipated principal action is a, and the public information is that $(\omega, z) \in \mathcal{I}$, and i's private signal is s_i. Aggregate demand is $X(p|a, \mathcal{I}, \omega)$.

We generalize the standard noisy REE definition. Define a *public information function* as $\lambda : \mathcal{P} \to 2^{(\Omega, \mathcal{Z})}$. Given a pair (P, λ) of price and public information functions, say that markets clear if $X(P(\omega, z)|M(P(\omega, z)), \lambda(P(\omega, z)), \omega) = z$ for all (ω, z). An equilibrium given M consists of a pair (P, λ) such that

1. Markets clear given (P, λ).
2. Information is consistent: there exists a price function P' such that markets clear given (P, λ), and such that $\lambda(p) = \{(\omega, z) : P'(\omega, z) = p\}$

The standard definition of REE would require that $P' = P$, meaning public information is consistent with the equilibrium price function. If this holds then we say that P characterizes a *fully consistent* equilibrium. We will show in the end that under the constraint of equilibrium uniqueness, it is without loss of generality to restrict attention to fully consistent equilibria. However it is precisely because of the concern with equilibrium multiplicity that we introduce the more general notion. To see why, fix the decision rule M and suppose that (P, λ) is a fully consistent equilibrium. Recall that we interpret this market as one in which investors submit limit orders to a market maker. Suppose there exists another price function $\hat{P} \neq P$ such that markets clear given $(\hat{P}, \lambda)$. Then one may well be concerned that for a realized state (ω, z) the market maker sets the price $\hat{P}(\omega, z)$, rather than $P(\omega, z)$. If there are multiple equilibria in this sense, then moving between them requires no change in the behavior of market participants, simply a change in the selection of the market clearing price by the market maker. We want a notion of equilibrium uniqueness which rules out this type of multiplicity, hence our more general equilibrium definition.

On the other hand, the search for truly unique implementation is hopeless in the noisy REE model studied here, even restricting to fully consistent equilibria, since there are multiple (meaningfully different) fully consistent equilibria even when there is no policy feedback, that is, fixing the principal's action (Pálvölgyi and Venter, 2015). This is because even for a fixed principal action, there may be multiple fully consistent price functions (which of course correspond to different public information functions). We therefore need to consider a weaker uniqueness notion.

What we really want to rule out is multiplicity arising from the endogeneity of the principal's action. We therefore in this context say that M is robust to multiplicity if for any equilibrium (P, λ) given, there is no other price function $\hat{P}$ such that $(\hat{P}, \lambda)$ is also an equilibrium. In other words, we fix the public information function, and require uniqueness of equilibrium price functions. We show (Proposition 16) that any P which is uniquely implementable according to this notion must characterize a fully consistent equilibrium.

We turn now to establishing existence of a suitable invariant representation. The complication in this setting is that there is no easy way to narrow down the space of possible public information sets that can be revealed by the price. We therefore analyse directly the problem of characterizing what equilibria can be induced with a decision rule $M \in \mathcal{M}$ that is robust to multiplicity. To do this, we first need some preliminary results.

Lemma 14. For and $\mathcal{I} \subseteq \Omega \times \mathcal{Z}$, $p \in \mathcal{P}$, and $a \in \mathcal{A}$, the function $\omega \mapsto X(p|a, \mathcal{I}, \omega)$ is Lipschitz continuous.

Proof. First note that $s_i \mapsto x_i(p|a, \mathcal{I}, s_i)$ is Lipschitz continuous since Ω is bounded and $s_i = \omega + \varepsilon_i$ for a normally distributed ε_i. Increasing ω by δ has the same effect on aggregate demand as increasing s_i by δ for all i. Then $\omega \mapsto X(p|a, \mathcal{I}, \omega)$ is Lipschitz continuous since σ_i and τ_i are bounded in the population. $\square$

Note that since the distribution of signals in the population is uniquely determined by ω (following the usual "continuum law of large numbers" convention) it cannot be that any public information set $\mathcal{I}$ contains states (ω', z') and (ω', z'') with $z'' \neq z'$, since the aggregate demand would not be the same in both cases. Therefore, the distribution of ω conditional on $\mathcal{I}$ cannot have atoms.

Lemma 15. For any $p \in \mathcal{P}$ and $a \in \mathcal{A}$, let $\mathcal{I}$ be a set satisfying $\mathcal{I} \subseteq \{(\omega, z) : X(p|a, \mathcal{I}, \omega) = z\}$. The distribution of ω conditional on $\mathcal{I}$ is absolutely continuous.

Proof. First, note if (ω'', z'') and (ω', z') are elements of $\mathcal{I}$, with $\omega'' > \omega'$ then it must be that $z'' > z'$. This follows from the fact that aggregate demand is strictly increasing in ω and strictly decreasing in p.

The function $\omega \mapsto X(p|a, \mathcal{I}, \omega)$ is Lipschitz continuous by Lemma 14. So for any $\kappa > 0$ there exists $\delta > 0$ such that for any $(\omega'', z''), (\omega', z') \in \mathcal{I}$, we have $|\omega'' - \omega'| < \delta$ implies $|z'' - z'| < \kappa$. In other words, there is a uniform bound on the "slope" of $\mathcal{I}$ in $\Omega \times \mathcal{Z}$ space. Since the prior distribution on $\Omega \times \mathcal{Z}$ is absolutely continuous, this implies the desired result. $\square$

Lemma 16. Let $\mathcal{I}$ satisfy $\mathcal{I} \subseteq \{(\omega, z) : X(p|a, \mathcal{I}, \omega) = z\}$ for some p, a. Then there exists $k > 0$ and ℓ such that $\mathcal{I} \subseteq \{(\omega, z) : k \cdot \omega - z = \ell\}$

Proof. Define the random variable $\tilde{V}^a := \pi(a, \theta) = \beta_0^a + \beta_1^a \theta$. Then define $\tilde{S}_i^a := \beta_1^a s_i + \beta_0^a = \tilde{V}^a + \beta_1^a \varepsilon_i$. Thus conditional on knowing the principal's action, investor i's observation of s_i is equivalent to observing a signal $\tilde{S}_i^a$ which is equal to the true dividend $\tilde{V}^a$ plus normal random noise, where the variance of the noise term depends on a; it is given by $\sigma_{ai}^2 = (\beta_1^a)^2 \sigma_i^2$. The results then follow from the proof of Proposition 2.2 in Breon-Drish (2015) (Online Appendix). The proposition in Breon-Drish (2015) pertains to the information sets revealed by equilibrium price functions which are continuous and satisfy a differentiability assumption. However, for the relevant direction of the proof, these conditions are only needed to guarantee that the distribution of $\tilde{V}^a$ conditional on $\mathcal{I}$ has a density, which is implied here by Lemma 15. $\square$

In words, Lemma 16 says that any public information set, either one revealed on-path by the price or by the off-path inference function, must lie in a linear subset of $\Omega \times \mathcal{Z}$. In other words, and such $\mathcal{I}$ must be a subset of some set of the form

$\{(\omega, z) : k \cdot \omega - z = \ell\}$ for some $k > 0$ and ℓ. The following proposition identifies exactly which hyperplanes the public information sets can lie in.

Proposition 15. Assume CARA utility, π affine in θ and continuous, additive normal signal structure and truncated-normally distributed supply shocks. Then there exists a unique (up to positive transformations) function $L^* : \Omega \times \mathcal{Z} \times \mathcal{A} \to \mathbb{R}$ defined by

$$L^*(\omega, z|a) = \left(\frac{1}{\beta_1^a} \int_i \frac{\tau_i}{\sigma_i^2} di\right) \cdot \omega - z \tag{7}$$

such that for any M, if $\mathcal{I}$ is the public information revealed at price p (in which case $\mathcal{I} \subseteq \{(\omega, z) : X(p|a, \mathcal{I}, \omega) = z\}$) then $L^*(\omega'', z''|M(p)) = L^*(\omega', z'|M(p))$ for almost all $(\omega'', z''), (\omega', z') \in \mathcal{I}$

Proof. Given Lemma 16, we just need to identify what the coefficients on the linear statistic are. Fix M, and let $L_M : \Omega \times \mathcal{Z} \times \mathcal{A} \to \mathbb{R}$ be the equilibrium statistic in a generalized linear equilibrium in which the price reveals exactly a hyperplane. Define the random variable $\tilde{V}^a := \pi(a, \omega) = \beta_0^a + \beta_1^a \omega$. Then define $\tilde{S}_i^a := \beta_1^a s_i + \beta_0^a = \tilde{V}^a + \beta_1^a \varepsilon_i$. Thus conditional on knowing the principal's action, investor i's observation of s_i is equivalent to observing a signal $\tilde{S}_i^a$ which is equal to the true dividend $\tilde{V}^a$ plus normal random noise, where the variance of the noise term depends on a; it is given by $\sigma_{ai}^2 = (\beta_1^a)^2 \sigma_i^2$. Let $\tilde{L}^a$ be the random variable $L_M(\omega, z, a)$.

We first fix the principal's action at a, and generalize Breon-Drish (2015) Proposition 2.1 to allow for supply shocks with a truncated normal distribution. We will therefore suppress dependence of $\tilde{S}_i^a, \tilde{V}^a, \tilde{L}^a$ on the action a for the time being. Abusing notation, write the statistic L in terns of v, rather than ω; that is, $L(v, z|a) = \alpha v - z$, suppressing the dependence on M.[50] For fixed a, the truncation is the only difference between the current setting and that of Breon-Drish (2015) Proposition 2.1. By the same steps as the proof for Proposition 2.1 in Breon-Drish (2015) Online Appendix,

[50] This abuse of notation is done to match the notation of Breon-Drish (2015). Note that in that paper "a" is used in place of α to denote the slope of the equilibrium statistic. The reader examining Breon-Drish (2015) should not confuse this with the notation for the principal action used in the current paper.

we can show that the conditional distribution of $\tilde{V}^a$ conditional on $\tilde{S}_i^a = s_i$ and $\tilde{L}^a = \ell$ is given by

$$dF_{\tilde{V}|\tilde{S},\tilde{L}}(v|s_i,\ell) = \frac{\mathbb{1}[\ell - \alpha v \in (-b,b)] \exp\left\{ \left(\frac{1}{\sigma_{ai}^2} s_i + \frac{\alpha}{\sigma_Z^2}\ell \right) v - \frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2} \right) v^2 \right\} dF_{\tilde{V}}(v)}{\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{ \left(\frac{1}{\sigma_{ai}^2} s_i + \frac{\alpha}{\sigma_Z^2}\ell \right) x - \frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2} \right) x^2 \right\} dF_{\tilde{V}}(x)}, \tag{8}$$

where $\mathbb{1}[\cdot]$ is the indicator function. This is not in the *exponential family* of distributions, as defined in Breon-Drish (2015) Assumption 10. Nonetheless, it will have similar properties. We can write the conditional distribution in (8) as

$$\mathbb{1}[\ell - \alpha v \in (-b,b)] \exp\left\{ \hat{L}(s_i,\ell)v - g\left(\hat{L}(s_i,\ell);\alpha,\ell\right) \right\} dH(v;\alpha),$$

where $\hat{L}(s,\ell) = \left(\frac{1}{\sigma_{ai}^2} s_i + \frac{\alpha}{\sigma_Z^2}\ell \right)$ and

$$\begin{aligned} g_i(\hat{L};\alpha,\ell) &= \log\left(\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{ \left(\frac{1}{\sigma_{ai}^2} s_i + \frac{\alpha}{\sigma_Z^2}\ell \right) x - \frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2} \right) x^2 \right\} dF_{\tilde{V}}(x) \right) \\ dH_i(v;\alpha) &= \exp\left\{ -\frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2} \right) v^2 \right\} dF_{\tilde{V}}(v) \end{aligned}$$

This has the following important implication (essentially the same as Lemma A6 in Breon-Drish (2015)). The conditional distribution integrates to 1, i.e. $\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{ \hat{L}(s_i,\ell)v - g\left(\hat{L}(s_i,\ell);\alpha,\ell\right) \right\} dH(v;\alpha) = 1$, so $\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{ \hat{L}(s_i,\ell)v \right\} dH(v;\alpha) = \exp\left\{ g\left(\hat{L}(s_i,\ell);\alpha,\ell\right) \right\}$. As a result, for any $t \in \mathbb{R}$ we have

$$\mathbb{E}\left[\exp\{t\tilde{V}\} | s,\ell \right] = \exp\left\{ g\left(t + \hat{L}(s_i,\ell);\alpha,\ell\right) - g\left(\hat{L}(s_i,\ell);\alpha,\ell\right) \right\}.$$

The remainder of the proof for the fixed-action case proceeds as in Breon-Drish (2015) Proposition 2.1. In particular, this shows that in any generalized linear equilibrium with fixed action a, we have $\alpha = \int_i \frac{\tau_i}{\sigma_{ai}^2} di$. Since $v = \beta_0^a + \beta_1^a \omega$ and $\sigma_{ai}^2 = (\beta_1^a)^2 \sigma_i^2$ we have $L^*(\omega, z|a) = \beta_0^a \int_i \frac{\tau_i}{\sigma_{ai}^2} di + \left(\frac{1}{\beta_1^a} \int_i \frac{\tau_i}{\sigma_{ai}^2} di \right) \cdot \omega - z$ Since information revelation is characterized by the level sets of L^*, we can ignore the first term.

We now show that the result holds under feedback as well. Given M, the investor knows which action the principal will take conditional on the price. In a generalized linear equilibrium, the investor's demand is therefore determined by maximizing utility given that the price is p, the action is $M(p)$, the observed signal is $\tilde{S}_i^a$, and the extended state is in $\{(\omega, z) : L_M(\omega, a|a) = \ell\}$ for the value of ℓ corresponding to price level p. The remaining question is which $L_M(\cdot|a)$ could constitute equilibrium statistics given action a and decision rule M. The first part of the proof shows that if the principal's action is fixed at a then there is a unique equilibrium statistic $L^*(\omega, z|a)$. Since all investors know the principal's action once they observe the price, this L^* must be the equilibrium statistic, regardless of M. $\square$

We now wish to use these properties, in particular Lemma 16, to identify features of equilibrium. Proposition 15 identifies the hyperplane to which each information set belongs. Following Breon-Drish (2015), we refer to these hyperplanes as *linear statistics*. So in other words, the public information will always reveal *at least* the associated linear statistic. In fact, under robustness to multiplicity and $M \in \mathcal{M}$, the equilibrium price function will reveal *exactly* the linear statistic, and no more.

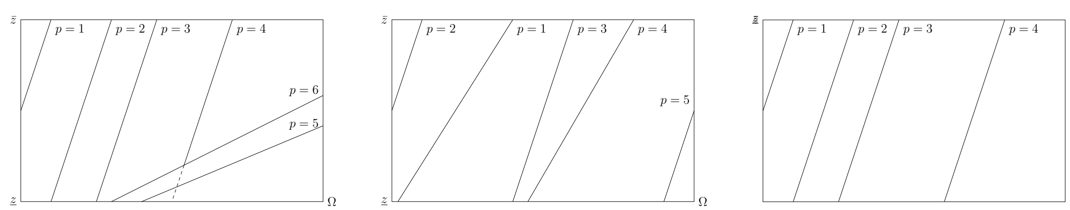
Proposition 16. Maintain the assumptions of Proposition 15. If $M \in \mathcal{M}$ is robust to multiplicity then (up to zero-measure violations) the level sets of the equilibrium price function $\tilde{P}$ are given by $\{(\omega, z) : L^*(\omega, z|M(p)) = \ell\}$ for some ℓ, where L^* is given by (7). Moreover, $\tilde{P}$ characterizes a fully consistent equilibrium.

Proof. Proposition 15 says that the equilibrium price must reveal at least the linear statistic. We want to show that the price can reveal no more than this. For $p \in \tilde{P}(\Omega, \mathcal{Z})$ let $l^*(p)$ be the linear statistic revealed by p. Suppose that $\mathcal{I}(p) := \{(\omega, z) : \tilde{P}(\omega, z) = p\} \neq l^*(p)$, so that the price reveals more than the linear statistic. We show that in this case there will be multiplicity. This follows from the fact that the set of states $\{(\omega, z) : X(p|M(p), \mathcal{I}(p), \omega) = z\}$ is the entire linear statistic $l^*(p)$. This follows from the proof of Lemma 15 and Proposition 2.2 in Breon-Drish (2015)

(Online appendix), which shows that individual demands will be linear in signals for any price (so aggregate demand is linear in the state). This implies also that $\tilde{P}$ is a fully consistent equilibrium. $\square$

Let $\tilde{\mathcal{P}}$ be the set of price functions P such that every level set of P given by $\{(\omega, z) : L^*(\omega, z|M(p)) = \ell\}$ for some ℓ. We refer to $\tilde{\mathcal{P}}$ as the price functions with *non-intersecting level sets.*

The idea behind Proposition 16 is illustrated in Figure 6. Figure 6a illustrates a situation in which the level set of the price function at $p = 4$ is a strict subset of the linear statistic $L^*(\omega, z|M(4))$. The dotted line is the segment of the linear statistic which is omitted from the level set. Since conditioning on the truncated level set induces higher posterior beliefs about ω than conditioning on the entire linear statistic, the price in these states would be lower than in an equilibrium in which the action was fixed at $M(4)$ for all prices. This would imply that there does not exist an invariant representation. The representation is saved, however, by the uniqueness requirement. In the situation depicted in Figure 6a, we show that there are additional equilibria in which the action $M(4)$ is taken for states on the dotted line segment, violating the uniqueness requirement.



(a) Multiplicity (b) No multiplicity (c) Fixed action $a \in \mathcal{A}$

Figure 6: Linear statistics

From (7) we can see how the principal's action affects information aggregation; the higher is β_1^a, i.e. the more sensitive the asset value is to the state, the smaller the coefficient on θ in the equilibrium statistic. As a result, the price is less informative about the state. This is because when β_1^a is high, each trader's private signal is less informative about the asset value. As a result, traders place less weight on their private information relative to the information revealed by the price. The linear statistics for a fixed action $a \in \mathcal{A}$ are pictured in Figure 6c. The slope of the linear statistics is $-\frac{1}{\beta_1^a}\int_i \frac{\tau_i}{\sigma_i^2} di$, which again illustrates that the price reveals more precise information about ω the lower is β_1^a. The proof of Proposition 15 also yields an expression for $R(a, \omega, z)$, although for the current purposes it is sufficient to note simply that such a function exists and is strictly increasing (with the product partial order on $\Omega \times \mathcal{Z}$).

Define $L(a,p) := \{(\omega, z) : X(p|a, L^*(\omega, z|a), \omega) = z\}$. Note that by definition, $L(a,p) = \{(\omega, a) : L^*(\omega, z|a) = \ell\}$ for some ℓ.

Corollary 4. Assume CARA utility, π affine in ω, additive normal signal structure and truncated-normally distributed supply shocks. Then the market has unique level sets represented by L. As a result, the market admits an a.e. invariant representation given $(\mathcal{W}^*, \tilde{\mathcal{P}})$. Moreover, $(\omega, z) \mapsto R(a, \omega, z)$ is strictly increasing for all a.

There are several differences between the environment of Proposition 15 and that of Breon-Drish (2015) Proposition 2.1. Most importantly, the current setting features a feedback effect, whereas asset returns follow a fixed distribution in Breon-Drish (2015).[51] The approach here is similar to Siemroth (2019). Aside from the fact that in Siemroth (2019) the principal cannot commit, that paper also assumes that the asset value is additively separable in the state and the principal's action. This is more than a technical assumption; it implies, as the author demonstrates, that the information revealed by the price is the same in all equilibria, regardless of the principal's actions. Siemroth (2019) also restricts attention to equilibria in which the price function is continuous (not to be confused with continuity of the principal's decision rule). This is a substantive assumption, as it implies that the equilibrium, when it exists, is unique. We are interested precisely in characterizing uniquely implementable outcomes, so

[51]Additionally, the signal σ_i observed by each investor is given by the state plus noise, as opposed to the asset return plus noise as in Breon-Drish (2015). This is immediately handled by a suitable change of variables, given the assumption that $\theta \mapsto \pi(a, \theta)$ is affine for all a. We allow here for the supply shock to follow a truncated normal distribution, where Breon-Drish (2015) considers only the un-truncated distribution.

such an assumption on the (endogenous) price function would be inappropriate.

G Further applications

G.1 Variable-volume carbon credits

Consider the problem of the emissions regulator discussed in the introduction. Such a policy is referred to by Karp and Traeger (2021) as a "smart cap".[52] The socially optimal level of emissions is determined by the marginal cost to firms of reducing their emissions, known as the abatement cost, and the marginal social benefit of reducing emissions. Assume that the regulator knows the social benefit of reducing emissions, but does not know firms' abatement costs.[53] Firms have private information about these costs.

Let q be the quantity of "clean air" produced by society. The societal benefit of clean air is given by $B(q)$. The social cost of producing q units of clean air is unknown to the regulator. This cost depends on the cost to emissions-producing firms of reducing their emissions. We parameterize the cost by $C(q, \theta)$, where θ is unknown to the regulator.

Under a variable-volume credits policy, the regulator issues a unit mass of credits. The regulator's action space $\mathcal{A} = [0, 1]$ is the per-credit emissions volume allowance. If the per-credit volume allowance is a, the quantity of clean air is given by $1 - a$. The regulator's decision rule specifies the per-credit volume as a function of the price for credits.

There are a continuum of firms $i \in [0, 1]$. Each firm observes its own cost type s_i. The distribution of costs in population is F_θ, where $\theta \in [0, 1]$ and $\theta \mapsto F_\theta$ is increasing in the FOSD order. A firm's payoff is given by $u(a \cdot x, s_i) - p \cdot x$, where a is per-credit volume and x is the number of credits purchased. Assume u is continuous; strictly increasing and strictly concave in its first argument; and strictly decreasing

[52]Karp and Traeger (2021) show that a smart cap can implement the regulator's first best, but do not consider uniqueness and manipulation constraints.

[53]In reality, there may also be uncertainty about the social benefit of reducing emissions.

and convex in its second argument. Notice that this is a private-values setting; a firm's payoff does not depend directly on the abatement costs of others. Credits are traded in a competitive market. Denote the firm's demand by

$$X(p, a, s_i) = \arg\max_x \quad u(a \cdot x, s_i) - p \cdot x.$$

Demands are unique under the maintained assumptions on u, and strictly decreasing in p. Given an action function $Q : \Theta \mapsto \mathcal{A}$, it must be that the equilibrium in state θ is the unique value satisfying

$$\int_\theta X(p, Q(\theta), s) dF_\theta(s) = Q(\theta) \tag{9}$$

Thus condition (9) implicitly defines an invariant representation for the credits market, where the market-clearing function $R(a, \theta)$ is continuous, strictly decreasing in its first argument, and strictly increasing in its second.

The regulator's first-best action function is given by

$$Q^*(\theta) = \arg\max_a B(1 - a) - C(1 - a, \theta).$$

Assume that $\theta \mapsto C_1(q, \theta)$ is continuous and strictly increasing. Then the first-best cannot be implemented by setting prices or quantities alone, since Q^* is strictly increasing. However, since Q^* is continuous and the associated price function P^* is strictly increasing, the first-best can be implemented uniquely by a decision rule that is robust to manipulation, by Theorem 2. The implementing decision rule in this case is continuous and strictly increasing.

In fact, since the first best price function P^* fully reveals the state and the first-best is CUI, the first-best here can also be implemented even if the principal lacks commitment power. Formally, $M(p) = Q^*(P^{*-1}(p))$. In other words, in equilibrium the principal learns the state perfectly and does not need to commit to taking an ex-post sub-optimal action in order to induce this equilibrium.

G.2 Moving against the market

In this section, we explore the distinctive features of a set of applications in which the principal would like to induce a *decreasing* price. As before, $\theta \mapsto R(a, \theta)$ is increasing. These are therefore situations in which the principal is working to move prices against the market. The following are two such instances.

Monetary policy in a crisis

During the financial crisis of 2008 and the Covid-19 recession of 2020, central banks moved aggressively to lower interest rates. In this application, the unknown state is the severity of the liquidity crisis faced by firms and the market price is the interest rate. The action is the size of asset purchases made by the central bank through open market operations. The central bank's objective is to implement an interest rate that is decreasing in the state via its open market operations.

Grain reserves

Many developing countries manage grain reserves as a tool for stabilizing grain prices and responding to food shortages. The state here is the size of a demand or supply shock, the price is the grain price, and the action is the size of grain purchases/sales. Depending on the nature of the crisis and the structure of the grain market, the government may wish to implement a decreasing price. If the government has limited capacity to make direct transfers to households it may wish to implement transfers by lowering the grain price when there is a severe crisis. For example, suppose that grain is a Giffen good. If there is an employment crisis outside of agriculture the price of grain may rise, absent government intervention.[54] In this case, the government may wish to subsidize non-agricultural households by lowering the grain price.

Throughout this section, we maintain the assumptions that $\mathcal{A} = [\underline{a}, \bar{a}] \in \mathbb{R}$ and that $\theta \mapsto R(a, \theta)$ is strictly increasing for all a, and that $a \mapsto R(a, \theta)$ is strictly

[54] There is empirical evidence that food staples are Giffen goods for extremely poor households (Jensen and Miller, 2008).

decreasing for all θ (that this function is decreasing as opposed to increasing is simply a normalization). A decreasing price function is possible if and only if $R(\underline{a}, \underline{\theta}) > R(\bar{a}, \bar{\theta})$. Figure 7 depicts such an environment.

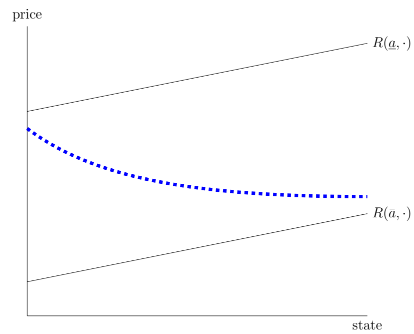


Figure 7: Decreasing price function

The following observation shows that implementing an increasing price function in this setting is easy.

Lemma 17. If $a \mapsto R(a, \theta)$ is strictly decreasing for all θ then any strictly increasing $M \in \mathcal{M}$ induces an increasing and continuous price function as the unique equilibrium.

Proof. An equilibrium exists for any increasing M by Tarski's fixed point theorem. That the price function will be increasing follows from the fact that $a \mapsto R(a, \theta)$ is decreasing and $\theta \mapsto R(a, \theta)$ is increasing. If P is increasing and M is increasing, there will be no equilibrium involving prices above $P(\bar{\theta})$ or below $P(\underline{\theta})$.

We show that M can have no discontinuities on $[P(\underline{\theta}), P(\bar{\theta})]$, which implies that P is continuous. Suppose, towards a contradiction that there is a non-empty set D of discontinuities in this region, and let $p' = \inf D$. By definition of $\mathcal{M}$, $p' \in (P(\underline{\theta}), P(\bar{\theta}))$. Let $a' = \lim_{p \nearrow p'} M(p)$. For any $p \in (P(\underline{\theta}, p')$ and any $a \in (M(P(\underline{\theta}), a')$ there exists $\theta \in (\underline{\theta}, \bar{\theta})$ such that $R(a, \theta) = p$. This follows from the fact that $a \mapsto R(a, \theta)$ is decreasing. Then for any $p \in (P(\underline{\theta}), p')$ there exists θ such that $R(M(p), \theta) = p$, since M is increasing and continuous on $(P(\underline{\theta}, p')$. This contradicts the definition of p'. $\square$

An equilibrium exists for any increasing M by Tarski's fixed point theorem. That the price function is increasing follows from the fact that $a \mapsto R(a, \theta)$ is decreasing and $\theta \mapsto R(a, \theta)$ is increasing. If P is increasing and M is increasing, there will be no equilibrium involving prices above $P(\bar{\theta})$ or below $P(\underline{\theta})$. Moreover, we show that M cannot have a discontinuity on $[P(\underline{\theta}), P(\bar{\theta})]$, which implies that P is continuous.

Decreasing price functions are more interesting in this setting. Non-monotonicity of M will be necessary to robustly implement a decreasing price.

Lemma 18. Assume $a \mapsto R(a, \theta)$ is strictly decreasing for all θ, and let P be a decreasing price function. If $M \in \mathcal{M}$ uniquely implements P then

i. $M(p)$ is decreasing and continuous on an open interval containing $(P(\bar{\theta}), P(\underline{\theta}))$,

ii. M has discontinuities in $(P(\underline{\theta}), R(\underline{a}, \underline{\theta})]$ and $(R(\bar{a}, \bar{\theta}), P(\bar{\theta})]$.

iii. There exist $p'' > p' > P(\underline{\theta})$ such that $M(p'') > M(p')$.

iv. There exist $p' < p'' < P(\bar{\theta})$ such that $M(p'') > M(p')$.

Proof. Condition *i* is immediate. For *ii*, first note that for $p \in (R(\underline{a}, \underline{\theta}), R(\underline{a}, \bar{\theta}))$ it must be that $M(p) > \underline{a}$; if not then $R(M(p), \theta) = p$ for some $\theta \in (\underline{\theta}, \bar{\theta})$. Suppose there is no discontinuity on $(P(\underline{\theta}), R(\underline{a}, \underline{\theta})]$. Then M must be decreasing over this domain to prevent multiplicity, and $\lim_{p \searrow R(\underline{a}, \underline{\theta}} M(p) = \underline{a}$. But for $p \in (R(\underline{a}, \underline{\theta}), R(\underline{a}, \bar{\theta}))$ it must be that $M(p) > \underline{a}$, so there must be a discontinuity. A symmetric argument applies to $(R(\bar{a}, \bar{\theta}), P(\bar{a})]$

Conditions *iii* and *iv* follow from a similar argument. Define $\bar{p}$ by $\bar{p} = \sup\{p : M \text{ is decreasing on } (P(\bar{\theta}), \bar{p})\}$. The argument above implies that $\bar{p} \leq R(\underline{a}, \underline{\theta})$. This implies *iii.* A symmetric argument implies *iv.* □

Lemma 18 shows that discontinuous and non-monotone M is necessary to implement a decreasing price. The intuition comes from the fact that the government is attempting to move against the market. Suppose the principal uses a strictly decreasing decision rule. If the lower bound $\underline{a}$ on the action is reached at some price p, then the principal will no longer have the capacity to move against the market for prices below p. Thus for such prices, the market forces generating an increasing price will dominate, and there will be multiple equilibria.

More formally, there are only two ways to guarantee that $\theta_M(p) = \varnothing$, i.e. that there are no equilibria with price p. Either M must specify an action that is too high, meaning $R(M(p), \bar{\theta}) < p$, or too low, so that $R(M(p), \underline{\theta}) > p$. If neither of these holds, there will be some θ such that $R(M(p), \theta) = p$, by continuity of R. The only way to ensure that there are no equilibria with prices in $[R(\underline{a}, \underline{\theta}), R(\underline{a}, \bar{\theta})]$ is to take a high enough action for such prices; it must be that $R(M(p), \bar{\theta}) < p$ for all such prices. At the same time, M must be decreasing on $(P(\bar{\theta}), P(\underline{\theta}))$ in order to implement a decreasing P. This tension is what necessitates discontinuities and non-monotonicities in M.

Lemma 18 is important in applications because it highlights the danger of artificially restricting the class of permissible decision rules. If, for example, one restricts attention to monotone decision rules, it is not possible to uniquely implement a decreasing price. It is nonetheless common practice in the literature to focus on monotone, or even linear, decision rules (see for example Bernanke and Woodford (1997)). Most papers that make this type of linearity assumption do so in models where the action space is unbounded. The fact that the action space is bounded here is an important driver of the non-monotonicity result in Lemma 18. However in reality there are often bounds on the available set of actions.[55] In the grain reserves example, the government cannot sell more grain than it has in reserve. Similarly, central banks in developing countries cannot make unlimited asset purchases without creating significant balance sheet risks (Crowley, 2015). Lemma 18 shows that such restrictions on the feasible actions can interact in surprising ways with conditions on the decision rules used to gain tractability. Our general framework allows us to avoid the need to impose such conditions.

[55]Notice that the conclusion of Lemma 18 does not depend on how "tight" the bounds are; non-monotone decision rules are necessary even if the range of admissible actions is arbitrarily large.

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