The Timing of Complementary Innovations

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November 30, 2020

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Abstract

This paper studies the development of socially-valuable technologies that require complementary innovations. At each point in time, resources are allocated to innovation projects that are completed stochastically in the form of breakthroughs. The social value of the technology depends on the set of projects that was completed by an endogenous stopping time. I solve the problem of efficient dynamic allocation of resources by showing that, for complements, this problem is equivalent to an auxiliary static problem. In some cases, the solution involves developing the innovations in sequence. In others, it is optimal to develop multiple innovations simultaneously. I provide a simple condition that determines the efficient timing of development. Then, I compare the solution to a decentralized allocation that is the equilibrium outcome when a continuum of firms that race to innovate. The decentralized allocation is efficient when the projects are symmetric or the stakes are sufficiently high, provided that the innovators are rewarded for the full *potential* of their innovations.

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1 Introduction

Firms, legislators, venture capitalists, and other decision makers aim to orient scarce resources toward the most profitable or socially valuable R&D projects. The usefulness of an innovation, however, is sometimes tied to the uncertain outcome of other developments. Consider, as an example, the development of hardware and software. The first computer algorithm was written in 1843,[1] while the first computer capable of running said algorithm wasn't developed until the 1930s. A similar process is taking place for quantum computing: Shor's quantum algorithm, a method for solving integer factorization problems in polynomial time, was written in 1994, four years before the first quantum computer prototype was developed.

This paper studies the allocation of resources toward the completion of complementary innovations. Complementarities are important—and increasingly relevant—in many industries such as telecommunications and biotechnology. A new medical treatment for early stages of a disease is more valuable if there are better diagnosis methods, and vice versa. Vaccines for different strains of a virus are more valuable when the other ones are developed. To fix ideas, consider two abstract complementary innovations, A and B. Should all the resources be concentrated on the development of A and then be switched to B if and only when A is successfully completed? Or should both A and B be developed in parallel? Moreover, when should a project be abandoned or put on hold?

In many industries, R&D is carried away by agents with private interests. Going back to the example of hardware and software we see that today, startups and established companies alike invest resources to develop quantum software that can only be implemented with hardware that does not yet exist,[2] and it is not clear that it ever will. Is the allocation carried away by private agents efficient? Moreover, how does the environment (for example, the level of concentration of the industry or the requirements for patentability) affect the allocation of resources to the development of complementary innovations?

One of the keys to determining the efficient allocation of resources lies in assessing, at each point in time, the prospects of each development. R&D projects carry high levels of uncertainty, both in terms of outcomes—the project may or may not prove successful—and in

[1]By *the first computer algorithm*, I mean the first algorithm specifically tailored to be implemented on a classical computer. This algorithm was developed by Ada Lovelace, and it is a method to generate Bernoulli numbers.

[2]The current quantum computers seem to be insufficiently powerful to be useful. The highest integer that has been factorized using Shor's algorithm is $21 = 7 \times 3$. Moreover, it is not clear that it will be possible to develop a sufficiently powerful quantum computer, at least in the near future.

terms of costs—it is not clear how much time and how many resources will be required to complete the development. This paper attempts to give an answer to the previous questions by introducing a tractable model that features the main aspects of the R&D process.

A unit of a resource (attention) is allocated over two complementary projects at each point in time. For each of the projects, successes arrive discretely in the form of a breakthrough. More precisely, a success arrives when the total amount of attention paid to a project reaches a certain level. Successes are observable, but the amount of attention required to complete a project is unknown. Moreover, success times are independent across projects, so that working on one of the developments does not provide any information about how demanding will be the other one. The only interaction between the projects comes through the complementarity in payoffs.[3] Finally, the cost of development takes the form of a constant flow throughout the development stage.

In Section 3, I study the *efficient* way to sequentially allocate attention to complementary R&D projects. The value of the innovations is realized when the development stage ends and the complementarities of the innovations are captured by the value function being supermodular on the subset of the innovations that have been successfully developed by that time.

For complements, the success of one project makes it more attractive to keep working on the remaining projects. Proposition 1 shows that this implies that all that matters for efficiency is how much is invested on each remaining project before abandoning. The intuition is that given the complementarities in payoff, the amount that the planner is willing to work on a project if there is no new success is the minimum she is going to work on that project. Since the planner would work this amount independently of the outcome of other projects, *when* does she do it is not payoff-relevant. Thus, for complements, the dynamic allocation problem can be solved by solving an auxiliary simple static problem. Moreover, I provide bounds on the boundary of the efficient stopping region. These bounds are given by the supermodularity of the value function at any stopping point.

Section 4 specializes in the canonical case where the rate of success of each project is constant over time but unknown. The beliefs about λ_i evolve with the outcomes of the development process. In particular, lack of success is evidence in favor of λ_i being relatively low, or in other words "the project i being relatively more challenging."

If the rate of success for each project were known, the timing of development is irrelevant. Any

[3] In the extended model of Section 5.1, the assumption of independence is weakened.

project that is worth pursuing is worth completing; therefore, the order of completion is not going to affect the final expected payoff.[4] In contrast, when the rate of success is uncertain, the order of development is relevant, since it affects the arrival of information about the unknown parameters. An initial failure to develop A not only reduces the prospects of ever completing A, but also decreases the expected returns from completing B. The problem can therefore be thought as a *restless multi-armed bandit*, for which there is no general Gittins-like index rule that governs the optimal dynamic allocation.

Take as an example the case where project A is of uncertain feasibility, that is, the success rate is either zero or λ_A, and project B has a known success rate λ_B. In this case, it is efficient to first work on project A: there is no learning by working on B, so there is no efficiency loss in back-loading all development of B. Front-loading the development of A increases the speed of learning, which is valuable because of the option given by the stopping decision. The intuition from the previous case applies more generally: the efficient allocation of resources reflects the optimal learning process about the potential of the joint project. For symmetric projects, Proposition 2 partitions the parameter space in two. For projects with high uncertainty about the rate of success and high costs, it is efficient to work in sequence starting with the less promising project. For other parameters, it is optimal to work on the projects in simultaneous. This results is generalized to asymmetric projects in Proposition 3.

Using the results on the efficient allocation, we can switch to study whether the private allocation is efficient. The allocation of private R&D resources depends on several factors: who assigns these resources, the appropriability of the innovations—which is determined by the legal and patent systems—and how informed the agents are about a given project's successes. In Section 4.2, I study private allocation of the resources in the extreme case of a *decentralized* allocation.

A *decentralized allocation* myopically allocates the resources to the projects maximizing a flow payoff. In particular, the decentralized allocation does not consider how the allocation changes the dynamics of beliefs about the feasibility and difficulty of the innovative projects. The decentralized allocation is equivalent to the equilibrium outcome of an industry that consists of a continuum of firms that race to develop the innovations, and each of whom controls an equal portion of the total unit of resource available at each moment in time. The firms don't consider the informational externalities that their actions generate. Before the first success, the firm take actions considering the value of the continuation game that they would be able to appropriate in case they succeed. With substitute projects, Bryan and

[4] A similar logic holds when the completion time for each project is deterministic.

Lemus [2017] show how decentralization and competition biases the allocation of resources toward fast, easy projects to the detriment of harder but cost-efficient ones. Although one might think that this *race effect* is also a concern with complements, Proposition 4 shows that this is not the case: the decentralized allocation of resources is not biased toward projects just because they are thought to be easier. In particular, if the only difference between the projects is the beliefs about the success rate being high or low, the decentralized allocation of resources is efficient.

For asymmetric projects, decentralization might introduce novel inefficiencies by biasing the allocation toward projects where learning is slower. These inefficiencies disappear, however,m if the stakes are sufficiently high.[5] Proposition 6 considers the case where only one of the projects is of uncertain rate and shows that there is inefficiencies in the decentralized allocation if an only if the high rate for the uncertain project is lower than the rate of the certain project and the uncertain project is thought to be of high rate with high probability.

Next, I discuss the relevant literature. The reminder of the paper is as follows. Section 2 introduces the model. Section 3 shows when is it possible to solve the allocation problem by looking at a auxiliary problem. In Section 4, we focus on a canonical case of perfect complements with uncertain rates of success. In Section 4.1 the efficient allocation is characterized. In Section 4.2 we identify the inefficiencies generated by the private allocation of resources by a decentralized industry. Section 6 concludes.

1.1 Related Literature

The main contribution is to the literature that studies complementary innovations. Scotchmer and Green [1990] and Ménière [2008] ask what is the optimal inventive requirements for a patent in the context of complementary innovations. Biagi and Denicolò [2014] study the optimal division of profits with complementary innovations. Fershtman and Kamien [1992] study the effects of cross licensing in the incentives to innovate. Bryan and Lemus [2017] study the direction of innovation in a general setting that accounts for complementary innovations. Bryan et al. [2020] focus on the effect of a *crises*—a proportional increase in the payoff from innovations—in the direction of innovation with partial substitutes. In these papers there is no learning in the development stage since the process of information arrival is memoryless.

A particular type of complementary innovation is that sequential developments or cumulative

[5]This contrasts again with the case of substitutes, where higher stakes magnify the race effects. See Bryan, Lemus, and Marshall [2020].

innovation. Papers that study sequential developments include Gilbert and Katz [2011] and Green and Scotchmer [1995]. Moroni [2019] studies a contracting environment with sequential innovations. In these papers the timing of innovation is exogenously given. I focus on complementary innovations in which the timing is determined endogenously by the allocation of resources. To the best of my knowledge, my paper is the first one to combine an endogenous timing of development with a learning process in the development stage in the context of innovation.

Another important related literature studies the problem of dynamic information acquisition from multiple sources. With Poisson information structure, both Nikandrova and Pancs [2018] and Che and Mierendorff [2017] study an agent that acquires information before an irreversible decision. Nikandrova and Pancs [2018] studies the case of independent processes while Che and Mierendorff [2017] studies processes that are negatively correlated. Mayskaya [2019] also studies irreversible decision with Poisson structure in a general setting. Ke and Villas-boas [2019] study problem of independent information sources where the agent learns about the state by observing a Brownian process. Klabjan, Olszewski, and Wolinsky [2014] study the problem of sequential acquisition of information about the attributes of an object.

Liang, Mu, and Syrgkanis [2018] and Liang and Mu [2020] compare the performance of optimal strategies and a different strategy. Liang et al. [2018] asks the question of how well does a strategy that neglects all dynamic considerations and acquire information in a myopic way performs with respect to the optimal information acquisition strategy. Liang and Mu [2020] compare efficient information acquisition to what results from the choices of short-lived agents who do not internalize the externalities of their actions.

2 Model

A decision maker (DM) can work on two research projects, A and B. The DM decides when to stop the research activity and, before that, in which way to allocate resources across the projects. Time is continuous and each instant before stopping, the DM allocates a unit of resource across the projects that were not completed so far. $\mathcal{A} := [0, 1]^2 \cup a_0$ is set of available actions at each time, where a_0 is the stopping action.

Let $\alpha_i(t)$ be the amount of resource allocated to project i at time t. Each project is completed when the cumulative resources allocated to it $X(t) := \int_0^t \alpha_i(\tilde{t})\ d\tilde{t}$ reaches a certain amount τ_i. Project completion is observable but τ_i is unknown. Formally, the agent observes the stochastic process $S(t) := \{i : \tau_i \leqslant X(t)\}$ that represents the set of projects that was completed so far. The completion times of the projects are independent with F_i the cdf of project i.

At each point in time, the agent decides either to stop or how to allocate a unit of the resource across the incomplete projects. The resource is scarce: $\sum_{i \in S(t)^c} \alpha_i(t) \leqslant 1$ for all t.

If the DM stops at time t, the payoff derived from the innovations is $q(S(t))$. As a normalization, if no project was completed the payoff is zero.

Assumption 1 (valuable innovations)**.** *q is increasing in the inclusion order, i.e.*

$$q(S) \leqslant q(S') \quad \textit{for all } S \subseteq S'.$$

Wer are interested in complementary innovations.

Definition 1. *The projects are* complements *if the function q is supermodular, that is,*

$$q(A) + q(B) \leqslant q(\{A, B\})$$

The projects are perfect complements *if $q(A) = q(B) = 0$.*[6]

Finally, using the resources is costly: there is a constant flow cost of c during the development stage, that is independent on which project the DM works on.[7] There is no discounting.[8] Thus, the payoff of an DM that stops at time t and completed projects S by that time is $q(S(t)) - c \cdot t$. The DM is an expected-payoff maximizer.

Strategies

A *strategy* is a map from the set of histories $\mathcal{H}$ to the set of actions $\mathcal{A}$. A *stationary strategy* only uses part of the information contained in the history, namely the cumulative resources and the set of completed projects. In particular, in a stationary strategy the actions of the agent do not depend on the order in which resources were allocated so far. Notice that since no resources can be spent on completed projects, by knowing the set of completed tasks S and

[6]Substitutes are defined by q being submodular, and perfect substitutes by the property that $q(A) = q(B) = q(\{A, B\})$.

[7]This assumption is innocuous since we can normalize the time unit for different projects, by changing the distribution of τ, so that the cost is the same for all projects.

[8]Alternatively, we could have had a discount factor instead of the linear cost in time. The qualitative features of the solution remain unchanged for this alternative version of the model.

the vector of cumulative resources X, the agent can recover the completion times: $\tau_i = X_i$ for $i \in S$. Let $\mathcal{H}^\circ := 2^{\{A,B\}} \times \mathbb{R}^2_+$ be the set of stationary histories. Formally, a stationary strategy consists on a vector field for each subset of developments.

Definition 2. *A* stationary strategy *is a function $x : \mathcal{H}^\circ \to \mathcal{A}$ such that $x_i(S, X) = 0$ for all $i \in S$ and*

$$\sum_{i \in K \setminus S} x_i(S, X) \leqslant 1 \qquad \forall\ (S, X) \in \mathcal{H}$$

Let $\mathcal{X}$ be the set of stationary strategies.

3 Efficient Allocation

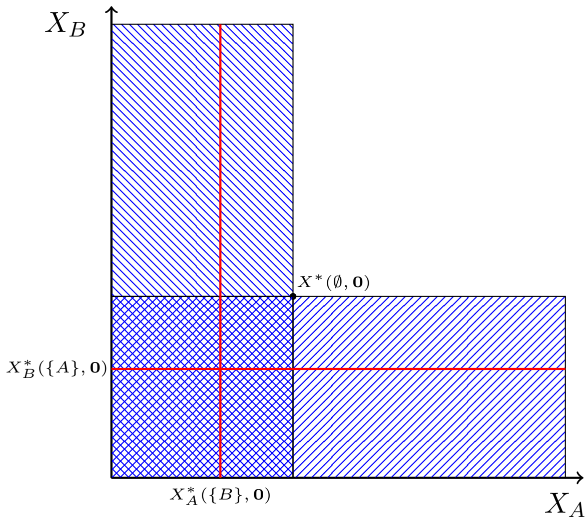
The problem of the decision maker is to choose a strategy to maximize their expected payoff. Since all the information about the underlying uncertainty is embedded in the stationary history, it is without loss of optimality to focus on stationary strategies. For an initial state (S, X), strategy $x \in \mathcal{X}$, and vector τ of completion times, there is a deterministic extra time that the agent that follows strategy x will work $\tilde{T}(x, \tau)$ and deterministic set of projects that will be completed by that time $\tilde{S}(x, \tau)$. The *allocation problem* can be expressed as:

$$V(S, X) \quad = \quad \max_{x \in \mathcal{X}} \quad \mathbb{E}_\tau \left[q(\tilde{S}(x, \tau)) - c \cdot \tilde{T}(x, \tau) \,\middle|\, (S, X) \right]$$

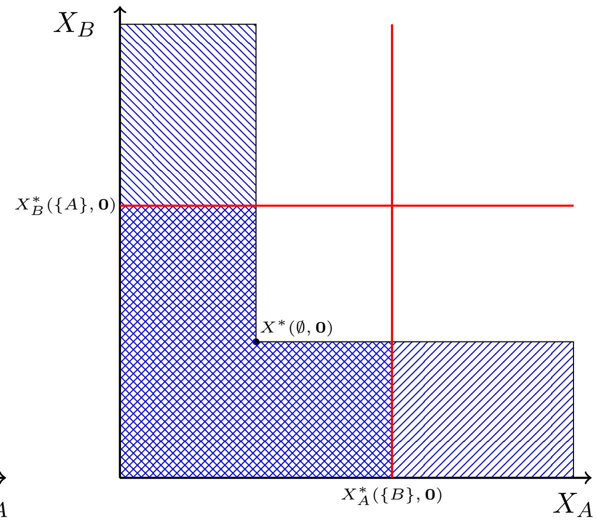
A different problem is as follows. The agent decides how many resources to *pledge* to each of the remaining projects, and then allocates them in some order. If none of the projects is completed after all the pledged resources are allocated, the agent has to stop. If at least one of the projects is completed, more resources can be allocated to the remaining ones. We are going to refer to this alternative problem as the *order-independent problem.* The name stems from the fact that for a given pledge, the order in which the resources are allocated will not affect the outcome.

Formally, the order-independent problem can be recursively defined as follows. Let $\hat{S}_X(\tau)$ be the set of completed projects at X, that is $\hat{S}_X(\tau) := \{i \in K : \tau_i \leqslant X_i\}$ and $D_{X,\hat{X}}(\tau)$ an indicator function that takes value 1 if there are no new completed projects from X to $\hat{X}$, i.e. $D_{X,\hat{X}}(\tau) := \mathbf{1}_{\left\{\hat{S}_{\hat{X}}(\tau) = \hat{S}_X(\tau)\right\}}$.

$$\hat{V}(S, X) \quad = \quad \max_{\hat{X} \geqslant X} \quad \mathbb{E}_\tau \left[W(X, \hat{X}, \tau) \ - \ c \sum_{i \in K} (\min\{\hat{X}_i, \tau_i\} - x_i) \,\middle|\, (S, X) \right]$$



(a) Substitute projects.



(b) Complementary projects.

Figure 1: Solution to the order-independent problem. The agent completes project A (project B) when the vector of completion times falls in the area with north east (west) pattern.

where $W(X, \hat{X}, \tau) := D_{X,\hat{X}}(\tau) \cdot q(S) + [1 - D_{X,\hat{X}}(\tau)] \cdot \hat{V}(\hat{S}_{\hat{X}}(\tau), \hat{X} \wedge \tau)$.

Let $X^*(S, X)$ be the solution to the order-independent problem. It must be that $V \geqslant \hat{V}$ since the DM can always choose a strategy that implements the solution to the order-independent problem. We are interested in conditions on q and F such that these two problems equivalent, i.e. $V = \hat{V}$.

When completing a project induces the agent to work more on the remaining one, the order in which the agent works on the projects before the first success irrelevant modulo the cumulative work at the stopping decision. This is formalized in the following proposition.

Proposition 1. *If the projects are complements, then $V = \hat{V}$. If the projects are not complements, there exists a family of distributions $\{F_i\}$ such that $V \neq \hat{V}$.*

Proof. in the Appendix A.1. Here is a sketch that provides the intuition of the result.

For any strategy, there is a (potentially infinite) amount of resources $\hat{X}$ that the agent allocates to each project before stopping, conditional on that no project is ever completed. For complementary projects, the amount the agent allocates on a project for an *optimal* strategy must be less than the amount that the DM would be willing to allocate to the project if the other one was completed: $\hat{X}_i \leqslant X_i^*(\{j\}, X)$. This means that a pledge is not binding: if the agent pledges $\hat{X}$, independently of the outcomes of the process, he was going to put more than $\hat{X}_i$ resources on project i.

Proposition 1 implies that for complements, and only for complements, it is possible to solve the dynamic optimal allocation by finding the *optimal pledge*. The solution to this static problem is a simple maximization. To emphasize the generality of the result, notice that no assumption was made on the distributions of completion times F_i. Thus, the result holds even for discrete time with arbitrary costs.[9]

Let $h_i := F_i'/(1 - F_i)$ be the *completion rate* of project i. When the completion rate is decreasing, failures depress the prospects of each project, we can bound the stopping frontier—the set of states at which the agent stops—using the following lemma.

Lemma 1. *If h_i is decreasing for one of the projects and the projects complements, then $V(\cdot, X)$ is supermodular for all $X^*(\emptyset, X) \neq X$.*

Sketch of the proof: Since the projects are complements, completion of one of the projects weakly increases the willingness to work on the remaining one. If at $(\emptyset, X)$ the DM wants to stop, it must be that

$$h_j(\emptyset, X)(V(\{j\}, X) - V(\emptyset)) \leqslant c$$

But the agent would be weakly willing to work on project B if project A was completed. Thus, since

$$h_j(\emptyset, X)(V(\{i\}, X) - \underbrace{V(\emptyset, X)}_{=0}) \leqslant c \leqslant h_j(\{i\}, X)(V(\{i, j\}, X) - V(\{i\}, X))$$

By independence, $(V(\{i\}, X) \leqslant (V(\{i, j\}, X) - V(\{i\}, X))$.

Lemma 1 says that supermodularity of $V(\cdot, X)$ holds for all X where the agent wants to stop before the first success. $V(\cdot, X)$ is not supermodular for all X. For instance, for q modular we have

$$V(\{A, B\}, X) = q(\{A, B\}) = q(A) + q(B) \leqslant V(A, X) + V(B, X).$$

Where the last inequality holds strictly for all X where the DM would like to continue working on any of the projects. However, since for the modular case the stopping problem can be thought as independent across projects, at a stopping point $v(\{i\}, X) = q(i)$ for both projects.

[9] To see this just consider an F_i with mass probabilities at different times. The difference between the mass points can be interpreted as the cost of working on the project for an extra period.

Change of variables To solve the order-independent problem, it is convenient to change variables and work with probabilities of success. Thus, instead of choosing a pledge, the DM chooses the probability with witch she wants to succeed.

$$\max_{\bar{p}_A, p_A, \bar{p}_B, p_B} (p_A \bar{p}_B + p_B \bar{p}_A - p_A p_B) q(\{A,B\}) + \sum_{i=A,B} p_i (1-\bar{p}_j) q(i) - \sum_{i=A,B} [p_i C_j(\bar{p}_j) + (1-p_i) C_j(p_j)]$$

Where $C_i(p)$ is the expected cost associated with completing a project with probability p.

$$C_i(p) := \int_0^{F^{-1}(p)} (1 - F(\tau)) c \, d\tau$$

When the hazard rate of project i is decreasing, the cost function is convex, and the solution to the problem can be characterized by the first order conditions: $C_i'(\bar{p}_i) = q(\{A,B\}) - q(j)$ and $C_i'(p_i) = q(i) + \frac{\bar{p}_j - p_j}{1-p-j} q(\{A,B\})$.

A particular family of solutions to the order-independent problem is one where pledges are concentrated on one of the projects.

Definition 3. *It is efficient to work on the projects* in sequence *if for every X there exists a project i such that $X_i^*(\emptyset, X) - X_i = 0$.*

In the next section, we focus on a family of canonical problems and use the simplification result from Proposition 1 to answer qualitative aspects of the solution—when is it optimal to work on the projects in sequence and when it is not—and to analyze the effects of industry concentration on the allocation of resources.

4 Uncertain Rate of Success

The focus of this section is on a set of canonical problems: perfect complements with constant but unknown completion rate. The agent knows that the rate takes one of two possible values, $\lambda_i \in \{\lambda_i^L, \lambda_i^H\}$ and, as before, the rates are independent across projects, with $p_i := \Pr(\lambda_i = \lambda_i^H)$.[10] As resources are allocated to a project and this is not completed, the agent becomes more pessimistic about its difficulty. Let $p_i(X)$ be the probability of the rate for project i being λ^H when no project was completed after resources X where allocated. We are going to use $\delta_i := \frac{1}{2}(\lambda_i^H - \lambda_i^L)$ and $\bar{\lambda}_i := \frac{1}{2}(\lambda_i^H + \lambda_i^L)$, and normalize time so that $\frac{1}{2}(\bar{\lambda}_A + \bar{\lambda}_B) = 1$.

[10]Thus, $F_i = 1 - p_i e^{-\lambda_i^H} - (1-p_i) e^{-\lambda_i^L}$.

Definition 4. *The projects are* symmetric *if* $\lambda_A^H = \lambda_B^H$ *and* $\lambda_L^H = \lambda_B^L$*. If projects are symmetric, we say that project* i *is the* most promissing project *at the state* $(\emptyset, X)$ *if* $p_i(X) > p_j(X)$*.*

Notice that symmetry is not requiring the initial beliefs p_A and p_B to be the same.

4.1 Efficient Allocation

Observation: If project A has a known rate of success, it is efficient to work on the projects in sequence. The efficient sequence starts with the project of uncertain rate (project B).

The reason is that there is no learning by working on the project with known rate. So, for any strategy with $X_i^*(\emptyset, X) > 0$, the expected return of the same strategy with $X_i^*(\emptyset, X) = 0$ is weakly larger.

The next proposition tell us that the nature of the optimal strategy depends on measure that is increasing in the normalized cost and the uncertainty about the underlying success rate.

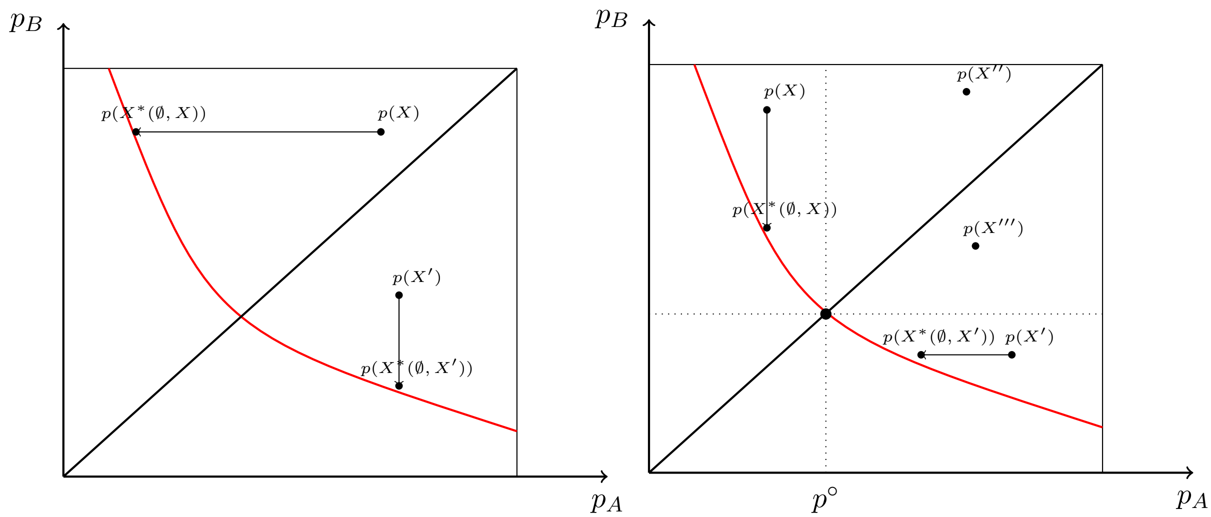
Proposition 2. *For symmetric projects, let* $g := 2\frac{c}{\gamma} + \delta^2$*.*

- *If* $g > 1$*, it is efficient to work on the projects in sequence, starting from the least promising project.*
- *If* $g < 1$*, it is efficient to work more on the more promising project:*

$$X_i^*(\emptyset, X) - X_i > X_j^*(\emptyset, X) - X_j \qquad \Leftrightarrow \qquad p_i(X) > p_j(X)$$

Moreover, there exists a $p^\circ \in (0,1)$ *such that if* $p_i(X) > p^\circ$ *for both projects then* $p_i(X^*(\emptyset, X)) = p^\circ$ *for both projects. If* $p_i(X) < p^\circ$ *for one of the projects then it is efficient to work on the projects in sequence, starting with the most promising one.*

Figure 2 shows the optimal allocation of resources before the first success for different priors, in belief space. The red curve represents the boundary of the set of points at which the agent is willing to stop before the first success $p(X^*(\emptyset, \mathbb{R}_+^2))$. In Figure 2a, when $g > 1$, it is optimal to work only on the project with lower prior. In Figure 2b, when $g < 1$, to the left of the 45° line the initial prior $p_i > p^\circ$ for $i = A, B$, then it is optimal to work on both projects before stopping in the first stage. More precisely, for an initial state $(\emptyset, X)$, $X_i^* = p_i^{-1}(p^\circ) - X$. Thus, there are multiple stationary strategies that are optimal. All these are payoff-equivalent by Proposition 1.



(a) When $g > 1$ it is optimal to work on the projects in sequence, starting from the least promising one.

(b) When $g < 1$ it is optimal to work more on the more promising project.

Figure 2: Optimal allocation for symmetric projects in the first stage.

The result says that it is efficient to concentrate the resources (and therefore work in sequence) when the cost of development is sufficiently high, or the difference between the high and low rates is sufficiently large for both projects. The intuition is that in this case, having a single project that is difficult is sufficiently bad to abandon the joint project, so by concentrating the resources the DM gets to learn fast if this is the case. In the cost of development is sufficiently low, or the difference between the high and low rates is low for both projects, then it is optimal to work on the project simultaneously. In this case, having a single project that is difficult is not sufficient to stop.

We can formalize this intuition by interpreting the result in terms of optimal information acquisition. There are three possible states: both projects are easy ($\lambda_A = \lambda_B = \lambda^H$), both are hard ($\lambda_A = \lambda_B = \lambda^L$) or one is easy and the other one hard ($\lambda_A \neq \lambda_B$). For the decision problem to be interesting it must be that the DM would be willing to work on the projects if he knew that both are easy, and he does not want to work on the projects when both of them are hard.

Suppose that the DM would be willing to work on the projects if he knew that one was difficult and the other one was easy. Then the partition of the state space that is relevant for decision making is whether there is at least one easy project (continue) or both projects are hard (abandon).

The probability of the event 'at least one of the projects is easy' is $p^{\mathrm{OR}} = p_A + p_B - p_A \cdot p_B$. By assigning extra resources dX_i to project i and not succeeding, the change in p^{OR} is

$$\frac{dp^{\mathrm{OR}}}{dX_i} = (1-p_j)\frac{dp_i}{dX_i} = -p_i(1-p_i)(1-p_j)2\delta$$

The fastest way to learn about the relevant state is to work on the project with highest p, and therefore to work on the projects simultaneously.

If the DM does not want to work when one of the project is hard and the other one is easy, the relevant state is whether there is a hard project or not. There is no hard project with probability $p^{\mathrm{AND}} = p_A \cdot p_B$. By working on project i for a period dt and not succeeding the change in p^{AND} is

$$\frac{dp^{\mathrm{AND}}}{dX_i} = p_j\frac{dp_i}{dX_i} = -p_i p_j(1-p_i)2\delta$$

The fastest way to learn about the relevant state is to work on the project with lowest probability of success, and therefore to work on the projects in sequence.

When does the DM want to continue working when one of the projects is difficult and the other one is easy? When the expected cost of completing both projects is less than the payoff from doing so, i.e.

$$\underbrace{\frac{c}{1+\delta} + \frac{c}{1-\delta}}_{\text{Expected cost}} < \underbrace{\gamma}_{\text{benefit}}$$

Rearranging we can see that this is equivalently to $g < 1$. Proposition 2 can be extended to asymmetric projects as follows.

Proposition 3. *Let* $g_i := 2\frac{c/\gamma}{\lambda_i} + \frac{1}{4}\left(\frac{\delta_i}{\lambda_i}\right)^2$,

1. *If* $g_i > 1$ *for both projects, then it is efficient to work on them in sequence.*
2. *If* $g_i < 1$ *for both projects, and* $\lambda_i^L > \lambda_j^H$ *then it is efficient to work on the projects in sequence, starting with the most promissing one.*
3. *If* $g_i < 1$ *for both projects and* $\lambda_i^H > \lambda_j^L$ *for* $i \neq j$*, then there exists a* $p \in (0,1)$ *such that if* $p_i(X) > p$ *for both projects, then for* $X^* = X^*(\emptyset, X)$*:*

$$\frac{h_A(X^*)}{V(K\setminus\{A\}, X^*)} = \frac{h_B(X^*)}{V(K\setminus\{B\}, X^*)}$$

Proof. in the Appendix B.1. Extra conditions for the case where $g_A < 1 < g_B$ can be found in Appendix D.3. ☐

4.2 Decentralized Allocation

So far, we focus on the efficient allocation: how would resources be allocated by a single forward-looking decision maker that internalizes the social value of the innovations. Research and development is rarely carried away by such decision maker. The resources in the economy might be controlled by different agents, with private incentives. The level of concentration and the appropriability of the innovations affect the incentives of the agents and, ultimately, the allocation of resources to different projects.

In this section, we consider a strategy that, at each state, maximizes a flow payoff. The flow payoff depends on the reward that is expected from innovation $R : \mathcal{H}^\circ \to \mathbb{R}_+$. We consider a family of rewards that consider the current use of the innovation as well as the potential uses, if complementary technologies are developed.

Definition 5. *A strategy x is the* decentralized allocation *for appropriation α if for each history (S, X) maximizes*

$$\sum_{i=A,B} x_i[h_i(X) \cdot R_\alpha(S, X) - c]$$

Where

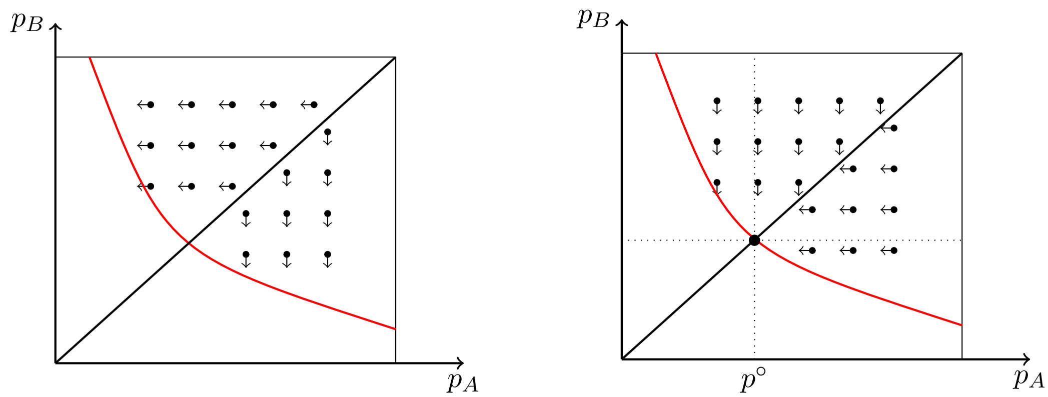
$$R_\alpha(S, X) := [\underbrace{(q(S \cup \{i\}) - q(S))}_{immediate} + \alpha \underbrace{(V(S \cup \{i\}, X) - q(S \cup i))}_{potential}$$

An innovation's worth is composed of its immediate how useful it is given the current technology. It also generates *potential reward* in the form of future prospects. The parameter α captures the level of appropriability of the innovation rents. When $\alpha = 0$, the decentralized allocation assigns resources to the project that creates higher expected immediate reward net of the cost of development. When $\alpha > 0$ the reward internalizes, at least partly, the potential uses of the innovation in the future.

The decentralized allocation turns out to be the equilibrium allocation in a game with a continuum of firms that compete in the development stage when the first firm to succeed gets the whole surplus of the development process. This microfoundation is explored in Appendix C.

Proposition 4. *If the projects are symmetric, the decentralized allocation is efficient if and only if $\alpha = 1$.*

When $\alpha = 1$, the decentralized allocation chooses to work always on the project with highest hazard-to-value ratio $\frac{h_i(X)}{V(\{j\},X)}$, which is constant on X_j. We know, from Proposition 2, that



(a) When $g > 1$ the decentralized allocation works always on the least promising project.

(b) When $g < 1$ the decentralized allocation works always on the most promising project.

Figure 3: Decentralized allocation for $\alpha = 1$ with symmetric projects in the first stage.

when projects are symmetric and $g > 1$, it is efficient to work on them in sequence, always starting with the least promising one. $g > 1$ implies that the hazard rate $h_i(X)$ decreases faster than the value $V(\{j\}, X)$ so for $X'_j = X_j$ and $X'_i > X_i$

$$h_i(X)V(\{i\}, X) \geqslant h_j(X)V(\{j\}, X) \qquad \Rightarrow \qquad h_i(X')V(\{i\}, X') \geqslant h_j(X')V(\{j\}, X')$$

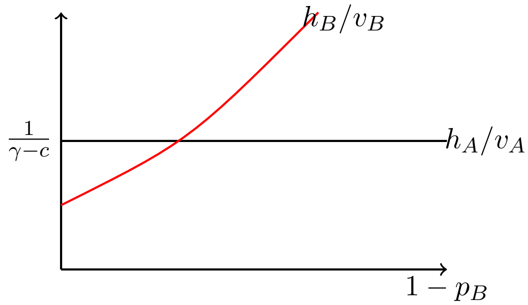
and this implies that the decentralized allocation for $\alpha = 1$ also works on the projects in sequence starting from the least promising one.

Proposition 5. *If $g_i < 1$ for both projects, the decentralized allocation is efficient for $\alpha = 1$.*

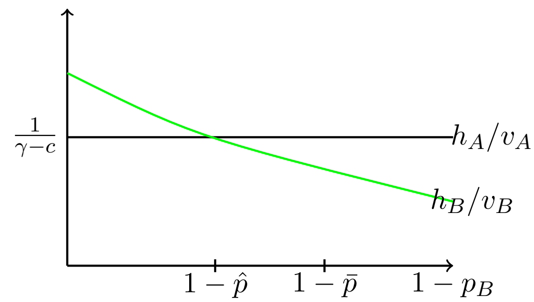
When $g_i < 1$ the value decreases faster than the hazard rate. It turns out that when this holds for both projects, it is efficient to work always on the project with highest hazard-to-value ratio. This is exactly what the decentralized strategy does.

The decentralized allocation does not consider the information generated by the allocation of resources. Decentralization will thus bias the allocation of resources towards projects where there is less learning. The next proposition considers the case where project A has a known rate of success. For project B the rate that is unknown.

Proposition 6. *If project A has a known success rate, the decentralized allocation for $\alpha = 1$ is inefficient if and only if $\lambda_B^H < \lambda_A$ and p_B is large enough.*



(a) **Case I**: $\lambda_B^H < \lambda_A$.



(b) **Case II:** $\lambda_B^H > \lambda_A$ and $\bar{p} < \hat{p}$.

Figure 4: The decentralized allocation is inefficient if h_B/v_B is lower than h_A/v_A for a belief for which it is efficient to work.

Proof. The decentralized allocation allocates resources to the project with highest hazard-to-value ratio, where $v_i(X_i) = V(K \setminus \{i\}, X)$. The observation from below says that it is efficient to work in sequence starting from project B.

Lets call $\hat{p}$ the posterior at which is efficient to stop working (interior if the proble is not trivial). h_A/v_A is constant and equal to $\lambda_A/(\gamma - c/\lambda_A)$.

If h_B/v_B is below h_A/v_A when the belief is close to 1, then there is an initial belief p such that it is optimal to work on B but the decentralized allocation works instead on A. This is ilustrated in Figure 4a. h_B/v_B when p is close to 1 is $\lambda^H/(1 - c/\lambda^H)$, so the condition is

$$\frac{\lambda^H}{1 - c/\lambda^H} \leqslant \frac{\lambda_A}{\gamma - c/\lambda_A} \qquad \Leftrightarrow \qquad \lambda^H \leqslant \lambda_A$$

If h_B/v_B is higher than h_A/v_A at $p = 1$, the only way there could be an inefficiency is if h_B/v_B is decreasing and the stopping belief $\bar{p}$ is higher than the belief at which h_B/v_B and h_A/v_A intersect, $\hat{p}$. This situation is ilustrated in Figure 4b. This, however, is never the case:

By Lemma 1, at the efficient stopping point X^*, $V(\{A\}, X^*) + V(\{B\}, X^*) \leqslant \gamma$. But $V(\{B\}, X^*) = \gamma - \frac{c}{\lambda_A}$, so $V(\{A\}, X^*) < c/\lambda_A$. Moreover at the stopping point, $h_B \cdot V(\{B\}, X^*) = c$, so

$$h_B(X^*) \cdot V(\{B\}, X^*) = c \geqslant \underbrace{h_A(X^*)}_{\lambda_A} \cdot V(\{A\}, X^*)$$

Rearranging we get

$$\frac{h_B(X)}{V(K \setminus \{B\}, X)} > \frac{h_B(X^*)}{V(K \setminus \{B\}, X^*)} \geqslant \frac{h_A(X^*)}{V(K \setminus \{A\}, X^*)} \qquad \forall X : X_B < X_B^*$$

Where the first inequality holds since $g_B < 1$. $\square$

The intuition that if a project is thought to be easier this would attract more attention to it is partially flawed. As the previous result shows, inefficiencies will show up if one of the projects has a higher rate of success than the other in every state, but they also require that the efficient to start project is thought to be as relatively easy.

5 Extensions

5.1 Relaxing Independence

So far we assumed that the projects are independent. Proposition 1 can be generalized more generally to affiliated projects. Let the vector of completion times τ be drawn from a distribution with density function f.

Definition 6. *The projects are* affiliated *if for every $\tau, \hat{\tau} \in \mathbb{R}^2_+$*

$$f(\tau \wedge \hat{\tau}) \cdot f(\tau \vee \hat{\tau}) > f(\tau) \cdot f(\hat{\tau})$$

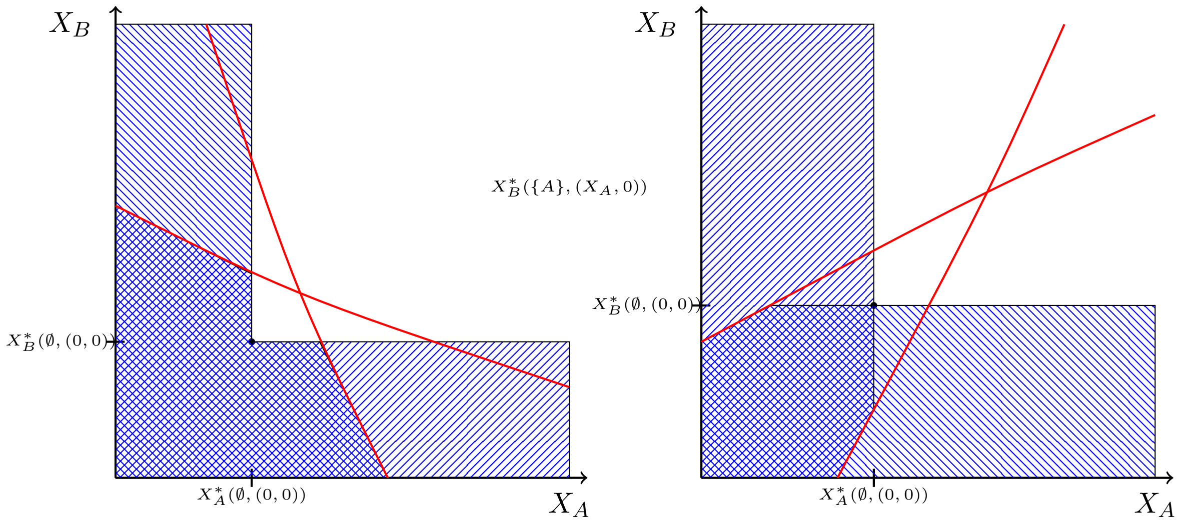
Proposition 7. *If the projects are complements and affiliated, then $V = \hat{V}$.*

The result of the proposition hinges on the success in one project incentivizing the agent to work more on the remaining one. For projects that are affiliated, a success in one of them is good news about the distribution of the completion times of the other, what makes it more attractive to continue working on it.

If the completion times are not affiliated, like in the Figure 5b, it can be that a very early success in one of the projects leads to optimally lowering the amount that the DM wants to work on the remaining project. Thus, pledging could be costly and the two problems are not equivalent.

5.2 More than two projects

The DM can work on a finite set of projects $K := \{1, 2, ..., k\}$. As before, the DM allocates a unit of resource across the projects that were not completed so far $S(t)^c$. Again, we define complementarity by the supermodularity of the value of innovations.



(a) Positively affiliated projects.

(b) Negatively affiliated projects.

Figure 5: Dependent completion times. The agent completes project A (project B) when the vector of completion times falls in the area with north east (west) pattern.

Definition 7. *Projects in the set K are* complements *if q is supermodular*

$$q(A \cup B) + q(A \cap B) \geqslant q(A) + q(B) \qquad \forall A, B \subseteq K$$

If the hazard rate is weakly increasing for all projects, the allocation problem and the order-independent problem.

Proposition 8. *If the projects are independent and h_i is weakly increasing for all $i \in K$ then $V = \hat{V}$ for all q.*

The reason is that when the hazard rate is increasing, if a project is worth allocating any resource, then it must be worth completing. Thus, for the optimal strategy, the set of completed projects is the same for all realizations of τ.

For more than two projects, however, we cannot claim that the problems are equivalent even with complementary, affiliated projects. The reason is that complementarity of the projects does not guarantee monotonicity of the solution to the problems that was used to prove the equivalence with two projects.

Claim 1. *For $k > 2$, independent and complementary projects is* not *sufficient for $\hat{V} = V$.*

We show this claim by means of the following counterexample:

Example 1. *Let* $K = \{A, B, C\}$*. Suppose* $q(\{A, B\}) = \gamma < q(\{A, B, C\}) = 1$*.* $q(S) = 0$ *for any subset. And suppose* C *is either feasible or infeasible, and that you can learn instantly about it.* $\lambda_L^C = 0, \lambda_H^C = \infty$*. The optimal strategy is to learn about* C*, and then doing the optimal thing for* A *and* B *(that might be different depending on whether* C *is completed or not).*

In the case where

$$c < \frac{\lambda_L \lambda_H}{\lambda_L + \lambda_H} < \frac{c}{\gamma}$$

then by results when C *is completed it is optimal to work simultaneously,* $Y_i(\{C\}, 0) > 0$ *for* $i = A, B$*. But when* C *fails, it is optimal to work in sequence, so* $Y_i(\emptyset, 0) = 0$ *for* $i \in \{A, B\}$*.*

6 Conclusion

Innovation is one of the main determinants of long-term economic growth. Thus, understanding the trade-offs in different approaches to innovation as well as the inefficiencies associated with economic environments is of central importance.

This paper makes substantive contributions to the understanting of the problem of development of complementary innovations:

First, the problem of efficient development of complementary innovations features different challenges than that of substitute innovations: failures in one development affect the expected returns from complementary innovations. With complementary innovations, however, successes make it more attractive to continue working on the remaining developments, what simplifies the problem substantially.

Second, allocating resources to innovation projects in an efficient way involves developing complementary innovations with a specific timing: sometimes it is efficient to develop in sequence and sometimes it is efficient to develop multiple innovations simultaneously. Sequential development is more likely to be efficient for high cost of development and higher uncertainty about the completion rates of the projects.

Third, an important part of innovation is carried away by the private sector. The timing of innovation is partly determined by the investment decisions of agents whose objectives are typically misaligned from the social welfare. Complementary innovations, generate investment dynamics that are different than for substitutes. In particular, the allocation of resources is not simply biased towards the easy and fast component to the detriment of the hard but cost-effective ones.

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A Omitted proofs from Section 3

A.1 Proof Proposition 1

The proof is based on Lemma 2 that is stated for k projects. First, some preliminaries.

For any stationary strategy $x \in \mathcal{X}$ and initial state (S, X_0), there is a *trajectory* $y_S : \mathbb{R}_+ \times \mathbb{R}^k \to \bar{\mathbb{R}}^k$ that is the (unique) solution to the differential equations

$$\begin{cases} \nabla y_S(t, X_0) = x(S, y_S(t, X_0)) \\ y_S(0, X_0) = X_0 \end{cases}$$

We will refer to $Y(S, X_0) = \lim_{t\to\infty} y_S(t, X_0)$ as the abandonment point of the strategy given an initial state (S, X_0).

Definition 8. *A strategy has* increasing abandonment points *if*

$$Y(S, X_0) \leqslant Y(\hat{S}, X_0) \text{ for all } S \subseteq \hat{S}.$$

Definition 9. *Two strategies $x, \tilde{x}$* have the same abandonment points *if for each initial state, the abandonment point is the same for both strategies.*

Lemma 2. *If two strategies $x, \tilde{x}$ have the same abandonment points, and these abandonment points are increasing then the two strategies have the same expected payoff.*

Proof. The proof works by induction. The Lemma trivially holds for $k = 1$. Assume that it holds for $k = 1, 2, ..., m - 1$, we want to show that it holds for $k = m$.

Consider strategies $x, \tilde{x}$ and a initial state $(\emptyset, X_0)$. Let $Y(\emptyset, X_0)$ be the associated abandonment point. For each set $S \neq \emptyset$, the continuation problem is analogous to one with less than n projects, so the lemma holds. Let $V(S, X)$ be the value of the two strategies at the state (S, x) for $S \neq \emptyset$.

Consider a strategy z with the same abandonment points than x and such that for any $S \neq \emptyset$, $z(S, X) = x(\emptyset, X)$ for all X with $x(\emptyset, X) \neq 0$. We can do this since $Y(S, X_0) \geqslant Y(\emptyset, X_0)$. For any τ, the new strategy has the same payoff than the original: either no project is successful and both abandon at the same point or the same project is succesful at the same point, and the continuation value is the same.

Similarly, we can construct a strategy $\tilde{z}$ with the same abandonment points but such that for any $S \neq \emptyset$, $\tilde{z}(S, X) = \tilde{x}(\emptyset, X)$ for all X with $\tilde{x}\emptyset, X) \neq 0$. $\tilde{z}$ and $\tilde{x}$ shield the same payoff. We end the proof by showing that z and $\tilde{z}$ must also have the same payoff for any realization of the success times $\tau \in \mathbb{R}_+^k$.

Let $\bar{S} = \{i \in K : \tau_i < Y(\emptyset, X_0)\}$, that is the set of projects which completion time is below the abandonment point. Both z and $\tilde{z}$ reach $Y(\emptyset, X_0)$ with probability one. The payoff is therefore

$$V(\bar{S}, Y(\emptyset, X_0)) - c\sum_{i\in\bar{S}} \tau_i - c\sum_{i\notin\bar{S}} Y_i(\emptyset, X_0)$$

Taking expectation over the realization of τ completes the proof. □

The intuition for Lemma 2 is the following: if the abandonment is increasing, then the current abandonment point is the least attention you are willing to put on the remaining projects by the end of the day. Since the attention it is going to be paid eventually, the order in which the agent does it is not gonna determine the outcome.

Proposition 1. Consider $k = 2$. If the projects are complements, then $V = \hat{V}$. If the projects are not complements, there exists a family of distribution $\{F_i\}$ such that $V \neq \hat{V}$.

(⇐) We want to show that q supermodular implies that any strategy that has the same abandonment points than an optimal strategy is also optimal.

Lemma 3. *For two affiliated and complementary projects, any optimal strategy has increasing abandonment points.*

Proof. We want to prove that for any optimal strategy $Y_i(j, X_0) \geqslant Y_i(\emptyset, X_0)$. By supermodularity of q, the marginal value of i is larger when j was completed than when it is not. If it is optimal to work on project i when it is not clear if j is going to be completed or not, it must be optimal to work on i when j was already completed.

Formally, by contradiction assume $Y_i(j, X_0) < Y_i(\emptyset, X_0)$. Then there is a time t such that $y_{\emptyset,i}(t, X_0) = Y_i(j, X_0)$. Let $X := y_\emptyset(t, X_0)$. Consider the strategy trajectory starting at X. If this strategy was copied starting on the state $(j, Y_i(j, X_0))$ with a dummy project j'

that starts at x_j the expected continuation payoff must be negative (otherwise it is worth continuing at $Y_i(j, X_0)$).

$$
\begin{aligned}
q(j) + \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1-F_i(\tau_i|X_0)}{1-F_i(Y_i(j,X_0))} \left[h_i(\tau_i)(q(ij)-q(j)) - c\right] \, d\tau_i &\geqslant \\
q(j) + \int_0^T \frac{1-F(Y(j,X_0)+y_\emptyset(\tau,X)-X)}{1-F(Y(j,X_0))} \alpha_i(\tau_i)h_i(\tau_i)(q(ij)-q(j)) \, d\tau & \\
- \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1-F_i(\tau_i|X_0)}{1-F_i(Y_i(j,X_0))} c \, d\tau &\geqslant \\
\int_0^T \frac{1-F(Y(j,X_0)+y_\emptyset(\tau,X)-X)}{1-F(Y(j,X_0))} \left[\alpha_i(\tau_i)h_i(\tau_i)(q(ij)-q(j)) + \alpha_j(\tau)h_j(\tau)q(j)\right] \, d\tau & \\
- \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1-F_i(\tau_i|X_0)}{1-F_i(Y_i(j,X_0))} c \, d\tau &\geqslant \\
\int_0^T \frac{1-F(Y(j,X_0)+y_\emptyset(\tau,X)-X)}{1-F(Y(j,X_0))} \left[\alpha_i(\tau_i)h_i(\tau_i)q(i) + \alpha_j(\tau)h_j(\tau)q(j)\right] \, d\tau & \\
- \int_{Y_i(j,X_0)}^{Y_i(\emptyset,X_0)} \frac{1-F_i(\tau_i|X_0)}{1-F_i(Y_i(j,X_0))} c \, d\tau &\geqslant \\
V(\emptyset, Y(\emptyset, X)) &\geqslant 0
\end{aligned}
$$

But this strategy shields more than the continuation at X thus the project should stop X, so $X = Y(\emptyset, x_0)$ leading to a contraction. □

Using Lemma 2, any strategy that has the same abandonment points than x is gonna get the same expected payoff and therefore be optimal.

(⇒) : We prove by contrapositive. If q is *not* supermodular, there are cdfs $\{F_i, F_j\}$ such that $Y_i(j, X_0) < Y_i(\emptyset, X_0)$.

Proof. Since q is not supermodular, $q(\{i\}) > q(\{i,j\}) - q(\{j\})$. Let $F_i = 1 - e^{-\lambda_i x}$ with λ_i such that

$$q(\{i\}) > \frac{c}{\lambda_i} > q(\{i,j\}) - q(\{j\})$$

and let j never succeed, i.e. $F_j = 0$. Rearranging we have that

$$\lambda q(\{i\}) - c > 0 > \lambda(q(\{i,j\}) - q(\{j\})) - c$$

What implies that for any X_0, $Y_i(\emptyset, X_0) = \infty$ and $Y_i(\{j\}, X_0) = X_0$. □

A.2 Proof of Lemma 1

Proof. h_i decreasing implies that $V(S,\cdot)$ is decreasing for all S.

If $X = X^*(\{A\},X) = X^*(\{B\},X)$, then $V(S,X) = q(S)$ and by complementarity of K, $V(\cdot,X)$ is supermodular. If $X \neq X^*(\{i\},X)$ for some i,

$$\frac{\partial V(K\setminus\{i\},X)}{\partial X_i} = c - h_i(X_i)\left[V(K,X) - V(K\setminus\{i\},X)\right] \leqslant 0$$

At any point $X \in X^*(\emptyset,\mathbb{R}^k_+)$ it must be that $c \geqslant h_i(x_i)\cdot V(\{i\},X)$. So,

$$h_i(X_i)V(\{i\},X) - h_i(X_i)\left[V(\{A,B\},X) - V(K\setminus\{i\},X)\right] \leqslant 0 \qquad \forall X \in X^*(\emptyset,\mathbb{R}^k_+)$$

Rearranging,

$$V(\{B\},X) + V(\{A\},X) \leqslant V(\{A,B\},X) \qquad \forall X \in X^*(\emptyset,\mathbb{R}^k_+)$$

□

B Ommitted proofs from Section 4

B.1 Proof of Propositions 2 and 3

Some preliminaries

Let δ_i be $\lambda_i^H - \lambda_i^L$. Using Bayes' rule, the beliefs $p_i(X)$ evolve

$$p_i(X) = \frac{p_i e^{-\delta_i x_i}}{(1-p_i) + p_i e^{-\delta_i x}}$$

As the agent becomes more pessimistic, the subjective hazard rate $h_i(t)$ becomes lower.

$$h_i(X) = \lambda_L^i + p_i(X)\delta_i$$

Notice that

$$g_i > 1 \quad \Leftrightarrow \quad \frac{\lambda_i^L \cdot \lambda_i^H}{\lambda_i^L + \lambda_i^H} > c$$

We prove the alternative:

Proposition 3':

1. If $g_i > 1$ for both projects, then it is optimal to work on them in sequence.

2 If $g_i < 1$ for both projects, then the greedy strategy is optimal.

The proof of the proposition is split in three lemmatas. Lemma 4 proves that g_i controls the monotonicity of project i's hazard-to-value ratio. Lemma 5 shows that when both projects have an increasing hazard-to-value ratio, it is efficient to work on them in sequence. Lemma 6 shows that when both projects have a decreasing hazard-to-value ratio the greedy strategy is efficient.

Lemma 4. *h_i/v_i is monotone. Moreover,* $\operatorname{sgn}((h_i/v_i)') = \operatorname{sgn}(g_i - 1)$.

Proof. First we show that the monotonicity of h/v depends on weather the value v is higher or lower that an expression R.

$$\begin{aligned}
\operatorname{sgn}((h_i/v_i)') &= \operatorname{sgn}(h_i'v_i - h_i v_i') \\
&= \operatorname{sgn}(h_i'v_i - h_i(c - h_i(1 - v_i))) \\
&= \operatorname{sgn}\left(\underbrace{\frac{h_i(h_i - c)}{h_i^2 + h_i'}}_{R(t)} - v_i\right)
\end{aligned}$$

Change of variables. In belief space, the concavexity of R is determined by weather g_i is larger or lower than one.

$$\begin{aligned}
\hat{R}'(p) &= \frac{2\delta^2\lambda_L\lambda_H(\lambda_L\lambda_H - c(\lambda_L + \lambda_H))}{(\lambda_L^2 + p\delta(\lambda_L + \lambda_H))^3} \\
&= M(g - 1)
\end{aligned}$$

Now we consider two cases: $\lambda_L < c$ and $\lambda_L \geqslant c$.

Case I: $\lambda_L \geqslant c$ In this case, the agent would never stop. The value is linear in the beliefs.

$$v(p) = 1 - p\frac{c}{\lambda_H} - (1 - p)\frac{c}{\lambda_L}$$

Since $v(0) = R(0)$ and $v(1) = R(1)$,

$$g > 1 \qquad \Leftrightarrow \qquad v(p) < R(p) \qquad \forall p \in (0, 1)$$

Case II: $\lambda_L < c$ In this case, the agent abandons if sufficient time passes with no success. v is convex (information is valuable) and R is concave:

$$\lambda_L < c \qquad \Rightarrow \qquad \frac{\lambda_H}{\lambda_L + \lambda_H}\lambda_L < c \qquad \Leftrightarrow \qquad g_i > 1$$

Since $v(1) = R(1)$ and $v(\hat{p}) = R(\hat{p})$ where $\hat{p}$ is the stopping belief.

$$v(p) < R(p) \qquad \text{for any } p \in (\hat{p}, 1)$$

$\square$

Lemma 5. *If h_i/v_i is strictly increasing for $i = A, B$, it is optimal to work on the projects in sequence.*

Proof. By contradiction. Assume that $x := Y(\emptyset, x_0) > 0$. Let $r_i(t) := \frac{h_i(t)}{v_i(t)}$ and $g_i(t) := \frac{h_i'(t)}{v_i'(t)\cdot r_i(t)}$. Since x is an interior stopping point, it must be that $h_A(x_A)v_B(x_B) = h_B(x_B)v_A(x_A) = c$.

Abusing notation I write f_i instead of $f_i(x_i)$. First we show that $r_A' + r_B' > 0$ implies $\frac{h_A' v_A}{h_A v_A'} \cdot \frac{h_B' v_B}{h_B v_B'} < 1$.

$$\begin{aligned} r_A' + r_B' > 0 \qquad &\Leftrightarrow \qquad \frac{h_A' v_A - h_A v_A'}{v_A^2} + \frac{h_B' v_B - h_B v_B'}{v_B^2} > 0 \\ &\Leftrightarrow \qquad \frac{h_A v_A'}{v_A^2}\left(\frac{h_A' v_A}{h_A v_A'} - 1\right) + \frac{h_B v_B'}{v_B^2}\left(\frac{h_B' v_B}{h_B v_B'} - 1\right) > 0 \end{aligned}$$

For all (x_A, x_B) such that $h_A v_B = h_B v_A = c$,

$$\frac{h_A v_A'}{v_A^2} = \frac{h_B v_A'}{v_B v_A} = \frac{h_A v_B'}{v_A v_B} = \frac{h_B v_B'}{v_B^2}$$

Where the first and last equality use $h_A/v_A = h_B/v_B$ and the intermediate one uses that $h_B v_A' = h_B(c - h_A(1 - v_A)) = -h_B h_A(1 - v_A - v_B)$ (since $c = h_A v_B$) and equal to $h_A v_B'$ by symmetry. So,

$$
\begin{aligned}
r'_A + r'_B > 0 \qquad &\Leftrightarrow \qquad \frac{h_A v'_A}{v_A^2}\left[\left(\frac{h'_A v_A}{h_A v'_A} - 1\right) + \left(\frac{h'_B v_B}{h_B v'_B} - 1\right)\right] > 0 \\
&\Leftrightarrow \qquad \left[\frac{h'_A v_A}{h_A v'_A} + \frac{h'_B v_B}{h_B v'_B}\right] < 2 \\
&\Leftrightarrow \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} < 1
\end{aligned}
$$

Where the second implication uses that v_A is decreasing. And the last one is that the sum of two positive numbers being less than two implies that the product is less than one.

The determinant of the Hessian for the value function $V(\emptyset, x)$ is

$$\det(H) = (1 - F_A)(1 - F_B)[h'_A h'_B v_A v_B - h_A h_B v'_A v'_B]$$

So

$$\det(H) < 0 \qquad \text{iff} \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} < 1$$

And $\det(H) < 0$ rules out the candidate as an optimum (saddle point). $\square$

Lemma 6. *If h_i/v_i is strictly decreasing for $i = A, B$, the greedy strategy is optimal.*

Proof.

$$
\begin{aligned}
r_i \searrow \qquad &\Leftrightarrow \qquad h'_i v_i - h_i v'_i < 0 \\
&\Leftrightarrow \qquad \frac{h'_i v_i}{h_i v'_i} > 1
\end{aligned}
$$

So,

$$r_A \searrow \text{ and } r_B \searrow \qquad \Rightarrow \qquad \frac{h'_A v_A}{h_A v'_A} \cdot \frac{h'_B v_B}{h_B v'_B} > 1$$

This implies that there is at most one interior candidate for solution ($h_A v_B = h_B v_A = c$), and that if it exist it is the actual solution. We consider the two cases.

Case I: there is an interior solution candidate. Then this is the actual solution. Since at the solution $r_A = r_B$ and the r_i are decreasing, by working always on the project with highest r_i, the point is eventually reached.

Case II: there in no interior solution candidate. Then it must be that $h_i v_j = c \Rightarrow h_j v_i > c$. Thus, the solution is to work in sequence starting with project j. Moreover, $h_j/v_j > h_i/v_i$ for all x such that $h_i v_j \geqslant c$, so the greedy strategy also works in sequence starting with j. □

C Equilibrium

There is a continuum of agents, $m \in [0,1]$. Each agen decides, at each instant, what project to work on $\alpha_m(t) \in \{A, B\}$. Once all agents stop developing, the value of the joint development is split across the agents. The payoff of an agent will depend on what innovations are successfully developed, who developed the innovations, and the timing of development.

We take a reduced-form approach to the problem, with focus on the first stage: let $W_i(X)$ be the expected payoff that is captured by the first agent to innovate when the innovation is i and the state was $(\emptyset, X)$.

Definition 10. *A* first-stage strategy *for agent* $m \in [0,1]$ *is a function* $s_m : \mathbb{R}^2_+ \to \{A, B, \emptyset\}$.

We assume that the problem is non-trivial: if both projects are easy then it is efficient to work on them, and it is efficient to abandon immediately if both projects were known to be difficult.[11]

A concern is that with complements, the competition in development stage will bias the allocation of resources toward the projects that can be developed faster, and that these will be developed first even if it is efficient to leave these projects for later.

The payoff for an agent m that was working on a project i at the moment in which this project was completed is proportional to the value of that project $V(\{i\}, X)$ and inversely proportional to the mass of individuals working on that project at that point. Let $\pi(s, Y, x)$ be the expected payoff of following first-stage strategy s when the evolution of the process is Y and the aggregate strategy is x.

Definition 11. *A* first-stage stationary equilibrium *consists of a first-stage strategy for each agent, an industry allocation* $x : \mathbb{R}^2_+ \to [0,1]^2$ *and an evolution of cumulative resources* $Y : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2_+$ *such that*

[11] In terms of the parameters the condition is

$$\frac{\lambda_A^L \cdot \lambda_B^L}{\lambda_A^L + \lambda_B^L} < \frac{c}{\gamma} < \frac{\lambda_A^H \cdot \lambda_B^H}{\lambda_A^H + \lambda_B^H}.$$

1. *Each agent maximizes his expected payoff given the evolution of resources.*

$$s(X) \in \arg\max_{s' \in \mathcal{S}} \quad \pi(s, Y(\cdot, X), x(X))$$

2. *The industry allocation aggregates all individual strategies.*

$$x(X) = \int_0^1 s_m(X)\ dm$$

3. *The evolution of resources is consistent with the allocation of the industry.*

$$\begin{cases} \nabla Y(t,X) = x(Y(t,X)) \\ \quad Y(0,X) = X \end{cases}$$

Lemma 7. *Any stationary equilibrium satisfies*

$$x_i(\emptyset, X) > 0 \qquad \Rightarrow \qquad h_i(X)V(\{i\}, X) \geqslant \max\{c, h_j(X)V(\{j\}, X)\}$$

Proof. in the Appendix C.1. □

Patent races might introduce distortions.

Definition 12. *There is a* race effect *when there is no equilibrium whose allocation is efficient.*

C.1 Proof of Lemma 7

Proof. Individual profits are

$$\int_0^t \underbrace{\frac{1-F_A(Y_X(\tau))}{1-F_A(X_A)} \cdot \frac{1-F_B(Y_X(\tau))}{1-F_B(X_B)}}_{\Pr(\text{reach } \tau)} \left[\underbrace{x_{s(Y_X(\tau))}(X) \cdot h(Y_X(\tau))}_{\text{rate success at } \tau} \cdot \underbrace{\frac{V(s(Y_X(\tau)), Y_X(\tau)}{x_{s(Y_X(\tau))}(X)}}_{\text{expected payoff if successful}} - c \right] d\tau$$

Since the individuals do not have a marginal effect on the trajectory and take it as given, the way to optimize the individual profits is every instant.

$$s \in \arg \max_{s' \in \{A,B,\emptyset\}} \quad \left\{ h_{s'}(Y_X(\tau))V(s', Y_X(\tau)) - c \right\}$$

□

D Extensions: proofs

D.1 Proof of Proposition 8

Proof. Suppose that

$$V(S,x) = \hat{V}(S,x) = \max_{\hat{S}\in 2^{K\setminus S}} \left\{ q(S\cup\hat{S}) - c\sum_{i\in\hat{S}} \lambda_i^{-1} \right\}$$

□

D.2 Discrete time

In this appendix we will consider the discrete time case $T = \{1, 2, 3, ..., \infty\}$. At any time before stopping the agent decides which project to work on $\alpha_t \in \{A, B, \emptyset\}$. Let F_i be the distribution of successes for project i, and $h_i : T \to [0,1]$ the respective hazard rate. Finally, let $v_i : T \to [0,1]$ the value of the joint project when only project i is incomplete as a function of the time spent working on project i.

$$v_i(x_i) := q(j) + \max_{T\geqslant x_i} \left\{ \sum_{x=x_i+1}^{T} \frac{1-F(x)}{1-F(x_i)} \left[h(x)(1-q(j)) - c\right] \right\}$$

Proposition 9. *h_i/v_i decreasing for both projects implies that the greedy strategy is efficient.*

Proof. Grab an optimal abandonment point $x^* := Y(\emptyset, 0)$ and a trajectory to it. The trajectory has to be greedy at the time before the abandonment point. Otherwise, the optimality of x^* is violated.

Consider now a greedy trajectory and the point (x_L, x_B^*) in that trajectory where crosses $x_B = x_B^*$ (the rightmost one). If the optimum is to the right of the path ($x_A^* > x_L$) then by optimality,

$$\frac{h_A}{v_A}(x_L) \geqslant \frac{h_i}{v_i}(x_i^* - 1) \geqslant \frac{h_j}{v_j}(x_j^*)$$

If $x_L = x_i^* - 1$ then the first inequality holds with equality and there is a greedy path to the optimum: the one we considered changing at the indifferent point (x_L, x_A^*). If $x_L < x_i^* - 1$ then by strict monotonicity of h/v the inequality holds strictly, what would violate greediness of the strategy at (x_L, x_A^*) □

Proposition 10. *h/v increasing for both tasks implies that the efficient allocation is in sequence.*

Proof. Suppose that the optimal stopping point $x^* = Y(\emptyset, 0)$ is interior, i.e. $x^* > 0$. Since last period is myopically optimal for each trajectory,

$$\frac{h_A}{v_A}(x_A^* - 1) \geqslant \frac{h_B}{v_B}(x_B^*) > \frac{h_B}{v_B}(x_B^* - 1)$$
$$\frac{h_B}{v_B}(x_B^* - 1) \geqslant \frac{h_A}{v_A}(x_A^*) > \frac{h_A}{v_A}(x_A^* - 1)$$

Where the strict inequalities come from the h/v being increasing for both projects. Thus, a contradiction. □

D.3 One h/v increasing and one decreasing

Lemma 8. *If the horizontal sum of the two h/v is increasing, then it is optimal to develop the projects in sequence.*

Proof. Consider $q(y) := (h_A/v_A)^{-1}(y) + (h_B/v_B)^{-1}(y)$ decreasing for all $y \in R := (h_A/v_A)([0,\bar{t})) \cap (h_B/v_B)([0,\bar{t}))$. Taking the derivative this implies that

$$\frac{1}{(h_A/v_A)'((h_A/v_A)^{-1}(y))} + \frac{1}{(h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$\frac{(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y))}{(h_A/v_A)'((h_A/v_A)^{-1}(y)) \cdot (h_B/v_B)'((h_B/v_B)^{-1}(y))} < 0 \qquad \forall y \in \mathbb{R}$$

$$(h_A/v_A)'((h_A/v_A)^{-1}(y)) + (h_B/v_B)'((h_B/v_B)^{-1}(y)) > 0 \qquad \forall y \in \mathbb{R}$$

Or, in other words: $r_A'(x_A) + r_B'(x_B) > 0$ for all points (x_A, x_B) with $h_A(x_A)/v_A(x_B) = h_B(x_B)/v_B(x_B)$. We can use the same logic used in the proof of Lemma 5 to rule out interior points. □

D.4 Imperfect complements

$\lambda_L > c/(1-q)$ then the agent would never stop. The value is independent of q and linear. The monotonicity of h/v is equivalent to the case where $q = 0$.

Consider now $\lambda_L \in (c, c/(1-q))$. There is a belief at which the agent stops.

$$\hat{p} = \frac{c/(1-q) - \lambda_L}{\delta}$$

If $R(\hat{p}) > v(\hat{p}) = q$ and R is concave, h/v is increasing.

$R(\hat{p}) > q$

$$\frac{c^2 q}{(1-q)[c(\lambda_L + \lambda_H) - (1-q)\lambda_L\lambda_H]} > q$$

Interesting case: $[c(\lambda_L + \lambda_H) - (1-q)\lambda_L\lambda_H] > 0$.

$$\left(\frac{c}{(1-q)}\right)^2 \geqslant \frac{c}{(1-q)}(\lambda_L + \lambda_H) - \lambda_L\lambda_H$$

$$\frac{c}{(1-q)}\left(\frac{c}{(1-q)} - \lambda_L\right) \geqslant \lambda_H\left(\frac{c}{(1-q)} - \lambda_L\right)$$

$$\frac{c}{(1-q)} \geqslant \lambda_H$$

But if this is the case, then the agent does not wish to work on the development even when sure that it is relatively easy.