Market-Based Mechanisms[*]

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Abstract

Decision makers frequently condition their actions on economic outcomes, e.g. asset prices, that they believe convey information about an unknown state. However the decision maker's action, or expectations thereof, may also influence the economic outcome. In this paper we study the general problem of choosing decision rules mapping outcomes to actions in the presence of such feedback effects. We characterize the set of joint distributions of outcomes, actions, and states that can be implemented as the *unique* equilibrium by such a decision rule. Moreover, we show that any uniquely implementable equilibrium will be robust to model misspecification. This characterization of the feasible set greatly simplifies the problem of choosing decision rules. A simple graphical technique allows us to identify qualitative features of optimal policies. We illustrate the power of this approach with an application to corporate bailouts. The results are also useful for characterizing optimal decision rules when the requirement of unique implementation is relaxed.

1 Introduction

This paper is motivated by two observations. First, many economic agents base their decisions on aggregate outcomes such as prices, the unemployment rate, or the rate of inflation. Second, the aggregate outcomes used in decision making may in turn be affected by the actions taken,

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or expectations thereof. Together, these two conditions give rise to a feedback loop: outcomes affect actions, expectations of which in turn affect outcomes. We are interested in the use of aggregate outcomes to make decisions in the presence of such feedback effects. We refer to these as market-based decision problems.

Decision making in the presence of feedback effects appears in wide range of economic environments. Central banks often adjust the nominal interest rate in response to changes in inflation rate, or expert forecasts thereof, and a large literature exists discussing the efficacy of such policies given the feedback loop that they create.[1] Governments frequently decide on fiscal policy based on macroeconomic indicators, which are impacted by expectations of policy decisions. The debate in the U.S. surrounding the Paycheck Protection Program (PPP) serves as a recent example. Members of Congress cited rising unemployment as evidence of the need for robust interventions, while business based their personnel decisions in part on expectations of future PPP support.[2] Feedback effects are also present in corporate decision making and financial markets. Inclusion in the S&P 500 is based in part on market capitalization, which creates feedback between share prices and the inclusion decision (the decision maker here being the manager of the index). Part of the recent rise in the share price of Tesla may have been driven by speculation that the company would soon be included in the S&P 500, which would trigger a jump in demand from passive funds that track the index.[3] The fact that Tesla's shares fell 21% following its omission from the index supports this hypothesis. Alternatively, the managers of a publicly traded firm may abandon a new project if the share price drops following the project's announcement. In each of these situations, the action taken, or the expectation of which action will be taken, will in turn affect the variable on which decision making is conditioned.

In our model, a principal commits to a decision rule M mapping some outcome, which we will refer to here as the price, to actions. There is a payoff relevant state of nature which is unknown to the principal. We require that agents' beliefs about the principal's action be consistent with the principal's announced decision rule, given the realized equilibrium price. This gives rise to the feedback effect. The price is the equilibrium outcome of some underlying price-formation game played by the agents. Our model accommodates a broad class of price-formation games which share the feature that the equilibrium price can be summarized by a function of agents' expectations of the principal's action and the state. These include asset markets and settings in which experts forecast macroeconomic variable such as the unemployment rate. Of particular interest is the noisy rational expectation

[1] See for example Bernanke and Woodford (1997).

[2] See Thomas and Cutter (2020) and Hughes and Morath (2020) for contemporaneous media coverage.

[3] Stevens (2020).

equilibrium model, of the type introduced by Grossman and Stiglitz (1980) and Hellwig (1980), which is the workhorse model of asymmetric information in asset markets, and which we study in detail in Section 6.1.

The principal's payoff depends on the joint distribution of the action, state, and price. Therefore, rather than directly studying the choice of decision rules, we instead focus on the induced mappings from states to actions and prices, referred to as action and price functions respectively. We then ask which action and price functions are *implementable*, i.e. can be induced as equilibrium outcomes by some decision rule. An action and price function pair will be *virtually implementable* if there are implementable functions arbitrarily nearby. Focusing on implementability, rather than on the decision rule directly, greatly simplifies the study of optimal policies.

This problem can also be viewed as an instance of constrained mechanism design. The principal is limited in that they can respond to the agent's actions/messages only through the market outcome (or price) which aggregates them. Our problem is further distinguished from other multi-agent mechanism design problems by two key features. First, we generally (although not always) think of the market as being large. When there are many agents in the market, their incentive compatibility constraints are relaxed; each agent's actions can have at most a small effect on the price. This may enable the principal to elicit information even in environments in which the usual single-crossing conditions on agent payoffs are violated. Second, there is a feedback effect. The price is determined by the state and agents' beliefs about the principal's action. We study problems in which each agent's beliefs about the principal's action are also shaped by the price. In other words, the price serves as a public signal which agents observe, in addition to any private information they may have. The feedback effect limits what the principal can achieve.

Practical concerns often constrain which decision rules the principal can use. First, we restrict attention to (essentially) continuous decision rules (we do in fact allow for some discontinuities, provided these occur away from prices that could occur in equilibrium, as discussed in Section 2.4). This is motivated by manipulation concerns; if the decision rule is discontinuous then it would be possible to induce a discrete change in the principal's action by making arbitrarily small perturbations to the price function. We view continuity as a minimal necessary condition for preventing manipulation. Continuity can also be viewed of as a limiting criterion for preventing manipulation as the cost of manipulation grows large.

Second, the principal may be concerned about indeterminacy of outcomes when there are multiple equilibria under the announced decision rule. Non-fundamental volatility is well documented in environments such as asset markets where expectations play an important role in determining outcomes. There is therefore great interest in designing policies for which a unique equilibrium outcome exists, as discussed in Woodford (1994). This is especially true in problems, such as managing inflation, in which stability is a paramount concern. Moreover, conditioning policy decisions on prices often exacerbates equilibrium multiplicity Bernanke and Woodford (1997). Multiplicity of equilibria is a central focus of this paper. We say that a decision rule is *robust to multiplicity* if it induces a unique outcome in all states, and *weakly robust to multiplicity* if it does so only in almost all states.

Finally, the principal in general has limited information about the fundamentals of the economy. In particular, the precise map from states and expected actions to prices may be unknown. It is therefore desirable for the principal to use a decision rule that is robust to such uncertainty; small perturbations to the fundamentals should not lead to drastic changes in the joint distribution of states, prices, and actions. When a decision rule induces a map from fundamentals to outcomes that is suitably continuous, we say that the decision rule is *robust to structural uncertainty.*

We say that an action and price function pair is *continuously uniquely implementable (CUI)* if it is implementable by a continuous decision rule that is robust to multiplicity, and *continuously weakly uniquely implementable (CWUI)* if the decision rule is only weakly robust to multiplicity.[4] Our first major results concern the characterization of the set of CUI and CWUI price and action functions. We show first that all continuous decision rules that are robust to multiplicity induce a price that is monotone in the state.[5] The proof of this result is involved, but the basic reasoning can be summarized as follows. First, for a price function to be implementable and non-monotone it must be discontinuous; it cannot be that different actions are taken in different states yet induce the same price, since the decision rule is measurable with respect to the price. However, when there are discontinuities and non-monotonicities in the price functions there will be multiple equilibria. This is due to the fact that while the price function may be discontinuous, the decision rule must be continuous. Continuously "bridging the gaps" where the price function is discontinuous creates multiplicity.

The monotonicity of the price function, moreover, essentially characterizes CWUI action

[4]Again, we do not in fact require continuity everywhere. The precise continuity requirement is discussed in Section 2.4.

[5]Readers familiar with the mechanism design literature may suppose that monotonicity of the price in this setting is related to the monotonicity of feasible allocations that arises in many mechanism design problems, and which is generally the consequence of single-crosssing payoffs. This is not the case; our environment does not share the important features of classical mechanism design problems. Price monotonicity reflects an entirely different set of factors.

functions. Under general conditions, satisfied in most applications, an action function is CWUI if and only if the associated price function is monotone. Additionally, we show that any continuous decision rule that is robust to multiplicity is robust to structural uncertainty. In other words, given CUI, the principal gets robustness to structural uncertainty for free (CWUI implies a weaker notion of robustness to structural uncertainty).

The characterizations of CUI and CWUI price functions greatly simplify the problem of finding optimal decision rules. The set of action functions which induce increasing price functions is much smaller than the set of all action functions. Moreover, when the first-best is not CUI (or CWUI), the optimal policy can often be solved for via a simple ironing procedure, involving only a few scalar parameters. This is illustrated in Section 5.1.

When non-fundamental volatility of market outcomes is not a primary concern, the principal may be willing to tolerate equilibrium indeterminacy, provided all equilibria give the principal a high payoff. We therefore consider decision making when the requirement of unique implementation is relaxed. We show that *any* continuous decision rule will induce at least one equilibrium that could in fact be implemented uniquely by an appropriate modification of the decision rule. This result has a number of important implications. First, suppose the principal takes a strict worst-case view of multiplicity, i.e. evaluates decision rules based only on the worst equilibrium that they could induce. When CWUI is characterized by monotonicity of the price function, the above result implies that it is without loss of optimality to restrict attention to CWUI outcomes.

This result also helps to analyze the problem of a principal who takes a less extreme approach to multiplicity than the strict worst-case preferences described above. Consider a principal who takes a lexicographic approach to multiple equilibria: the principal first evaluates a decision rule according to the worst equilibrium that it induces. Among those decision rules with the same worst-case equilibrium payoff, the principal chooses based on the best equilibrium that each induces (or some other function of the remaining equilibria).[6] The above result implies that the best "worst-case guarantee" can be found by optimizing over the set of CWUI mechanisms. Suppose there is a unique decision rule M that achieves this worst case guarantee. Then the need to satisfy the worst-case guarantee implies that the principal must use a decision rule that coincides with M for all prices that can arise in the equilibrium under M. This observation can greatly simplify the problem of solving for optimal decision rules when the principal takes a lexicographic approach to multiplicity.

We illustrate the power of our results with an application to government bailouts of a

[6]Such preferences are similar in spirit to these studied in the context of robust mechanism design (Börgers, 2017) and information design (Dworczak and Pavan, 2020).

company or industry. The government's first best policy will be CUI if and only if the positive social externalities from bailing out the company are high. We characterize the optimal CUI policy when first-best is not feasible, and show which policies will be optimal when the uniqueness requirement is relaxed. In a separate set of applications, we discuss the distinctive features that arise when the principal attempts to "move against the market" by inducing a decreasing price function. For example, central banks often aim to reduce the interest rate during severe crises, while absent intervention, interest rates would be increasing in the severity of the crisis.

This paper is part of a large literature related to the two-way feedback between financial markets and the real economy, beginning with Baumol (1965). For a survey of this literature see Bond et al. (2012). Among other contributions, this literature identified multiplicity of equilibria as a fundamental feature of feedback environments. Multiplicity is discussed in Dow and Gorton (1997), Bernanke and Woodford (1997), and Angeletos and Werning (2006). The current paper contributes to this literature by characterizing the set of policies under which multiplicity arises.

More specifically, this paper relates to decision making under commitment in the presence of two-way feedback. Important contributions include Bernanke and Woodford (1997), Ozdenoren and Yuan (2008), Bond et al. (2010), Bond and Goldstein (2015), and Boleslavsky et al. (2017). Bernanke and Woodford (1997) shows how the use of inflation forecasts to inform monetary policy can reduce the informativeness of forecasts. In the language of our paper, this occurs when induced market-outcome function (in this case the inflation forecast) violates the necessary monotonicity condition. Bernanke and Woodford (1997) restrict attention to linear decision rules, and show that equilibrium multiplicity can arise. Our analysis show that non-monotone decision rules may in fact be *necessary* to prevent multiplicity (Section 5.2). Ozdenoren and Yuan (2008) study an asset market in which aggregate investor demand, which is only partially revealed by the asset price, directly determines cash flows through some exogenous function. Multiplicity also arises in this setting. Bond and Goldstein (2015) focuses on the how market-based interventions affect the efficiency of information aggregation by prices. In contrast to the current paper, traders in Bond and Goldstein (2015) care about the state only insofar as it allows them to predict the government's action. Since traders have no incentive to trade on their private information if they know the state, information aggregation is highly dependent on the decision rule.

Other papers have noted that decisions based on market outcomes may be vulnerable to manipulation. Goldstein and Guembel (2008) study manipulation by strategic traders when firms use share prices in secondary financial markets to guide investment decisions. In

Lee (2019) a regulator uses stock-price movements of affected firms to determine whether or not to move forward with new regulation. In Lee (2019), the discontinuous nature of the policy considered opens the door to manipulation. While we do not explicitly model manipulation here, the restriction to continuous decision rules is motivated in part by concerns of manipulation. Our characterization results provide an important starting point for more explicit application-specific models of manipulation.

In general, the current paper makes four major contributions to the existing literature. First, we provide a general framework for studying market-based decision making with feedback effects. By focusing on implementable price and action functions, rather than directly on the decision rule, we are able to shed new light on the general structure of the problem. Moreover, we show how the information content of prices can be separated from their market-clearing role in a broad class of problems, including the canonical noisy REE model (Section 6.1). This separation significantly reduces the complexity of analyzing market-based decision rules. Second, and most importantly, we characterize the feasible set of uniquely implementable outcomes. This characterization greatly simplifies the analysis of optimal policy in applications. Third, we show that continuity and robustness to multiplicity imply robustness to structural uncertainty. This means the decision maker's payoff will not be overly sensitive to their limited understanding of market fundamentals. Finally, our results also allow us to analyze optimal policy when the principal is willing to tolerate multiplicity. As the issue of equilibrium indeterminacy is central to the literature, as noted by Woodford (1994) among others, we see this as an important contribution.

1.1 A brief illustration: emergency lending

Before proceeding to the formal model and results, we will illustrate the key ideas with a discussion of emergency lending. A similar application to corporate bailouts is discussed in greater detail in Section 5.1. Consider an international lender such as the IMF or World Bank deciding on the size of an emergency loan to be extended to a country experiencing a crisis. The severity of the crisis is determined by a number of factors, such as anticipated changes in the balance of payments, prospects of domestic manufacturers, and the government's capacity for reform. The lender is unaware of the precise severity of the crisis, which is represented by an unknown state $\theta \in [\underline{\theta}, \bar{\theta}]$, with lower states representing greater severity. Dispersed information regarding the state may be at least partially reflected in the price of government bonds. For the purposes of this example, imagine that all traders in the bond market know the true state (this assumption is purely for illustrative purposes; it does not affect the results discussed here and will not be a part of the general model). The lender would therefore like to use bond prices to inform its lending decision.

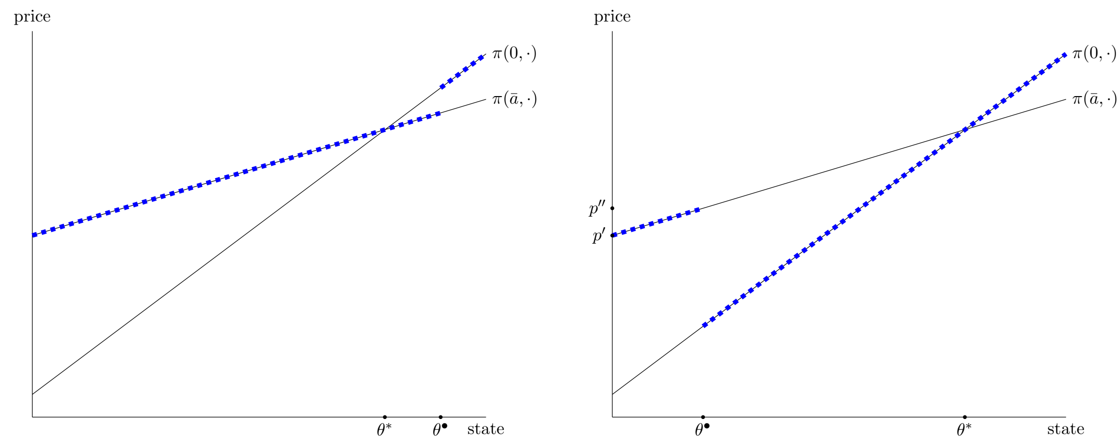
Let $\pi(a,\theta)$ be the value of government bonds if the lender extends a loan of size $a \in [0,\bar{a}]$ and the state is θ. For any loan amount a, bond values are increasing in θ. A large emergency loan leads to higher bond prices when the crisis is severe, as it reduces the probability of default in the short term. However bondholders may also worry that the increase in the country's debt burden could have adverse long-term affects. For example, the increase in the dept burden may push the country down the back side of the dept Laffer curve, as investors worry that long-term growth will be negatively affected by the higher taxes needed to pay for the increased debt burden. If the current crisis is very mild this effect may dominate, and bond prices may react negatively to the lender's intervention. Indeed, Cordella et al. (2010) find that the strongest empirical evidence of a negative relationship between debt and growth seems to be for countries with relatively good policies and institutions. These considerations are captured by the following two assumptions on bond values

1. There exists $\theta^* \in [\underline{\theta},\bar{\theta}]$ such that $\pi(\cdot,\theta)$ is increasing for $\theta \leq \theta^*$ and decreasing for $\theta > \theta^*$.

2. $\pi_2(a,\theta)$ is decreasing in a

The lender would like to extend emergency relief only when the crisis is indeed severe. For simplicity, assume there exists a state $\theta^\bullet$ such that the the lender's payoff is increasing in a when $\theta \leq \theta^\bullet$, and decreasing in a when $\theta > \theta^\bullet$. As a result, the lender would ideally like to extend the maximal loan amount $\bar{a}$ if and only if $\theta \leq \theta^\bullet$, and otherwise make no loan. We refer to this policy as the first-best action function. Higher $\theta^\bullet$ corresponds to a more interventionist policy on the part of the lender. Figure 1a illustrates an interventionist first-best action function in which $\theta^\bullet > \theta^*$. The solid lines denote the bond values as a function of the state under the two extreme actions 0 and $\bar{a}$. The dashed blue line is the price function P^* induced by the first-best action function. Note that for each price p there is at most a single state θ such that $P^*(\theta) = p$. It will therefore be possible to choose a decision rule mapping prices to actions that implements this action function. In fact, Corollary 3 implies that this first best will be continuously weakly uniquely implementable (there will only be multiple equilibrium actions in state $\theta^\bullet$).

Figure 1b illustrates a conservative first-best policy. In this case the lender is unwilling to intervene in some states in which bondholders would like the government to receive a large emergency loan. This is likely the most realistic scenario. For each price in (p',p'') there are two states such that $P^*(\theta) = p$: a low state in which the action $\bar{a}$ is taken and a high state in which the action 0 is taken. As a result, this action function can not be implemented by a market-based decision rule. In other words, the action is not measurable with respect to the price.



(a) First-best is CWUI

(b) First-best not implementable

Consider the modification of the conservative first-best action function illustrated in Figure 2, which eliminates the measurability problem discussed above. This requires making an intermediate level of intervention for states in (θ', θ''), where the lender would prefer not to intervene. Given this modification, for any price p there is a unique state θ such that $P^*(\theta) = p$, and so this action function is implementable. In fact, it will be implementable with a continuous decision rule. If the probability associated with such states is low, and/or the loss from making larger loans then desired is small, then the payoff from this type of policy will be close to that of the first-best. The lender is still able to take it's preferred action in all other states

Unfortunately, it will not be possible to continuously and uniquely implement a policy resembling that of Figure 2. In fact, any continuous decision rule M that implements this action function will induce at least one equilibrium in which large loans are made for all states in (θ', θ''). This is precisely because of the non-monotonicity in the induced price function (Theorem 1). The best CWUI action function will be monotone, as illustrated in Figure 3; the lender trades off lower than desired lending for some states below $\theta^\bullet$ with higher than desired lending for states above $\theta^\bullet$. Lending as a function of θ decreases gradually in order to maintain price monotonicity. In fact, this policy will be CUI (Theorem 2). Moreover, Proposition 4 implies that if decision rule M^* is optimal within the set of continuous and decision rules that are robust to multiplicity, i.e. those that induce an equilibrium of the form in Figure 3, then M^* will be in the set of optimal decision rules even if the uniqueness requirement is relaxed and the principal takes a worst-case approach to multiplicity. However,

M^* will no longer be optimal if the principal takes a lexicographic approach to multiplicity, as detailed in Section 4. In this case, Proposition 6 implies that the optimal decision rule will in fact be similar to the one which induces the price function illustrated in Figure 2.

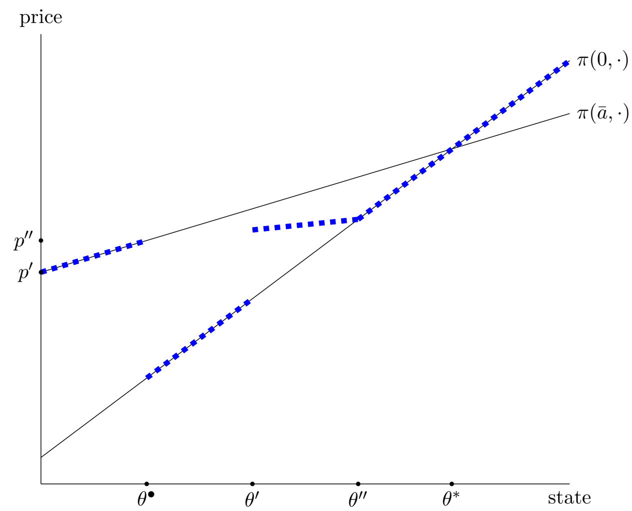


Figure 2: Implementable, not CWUI

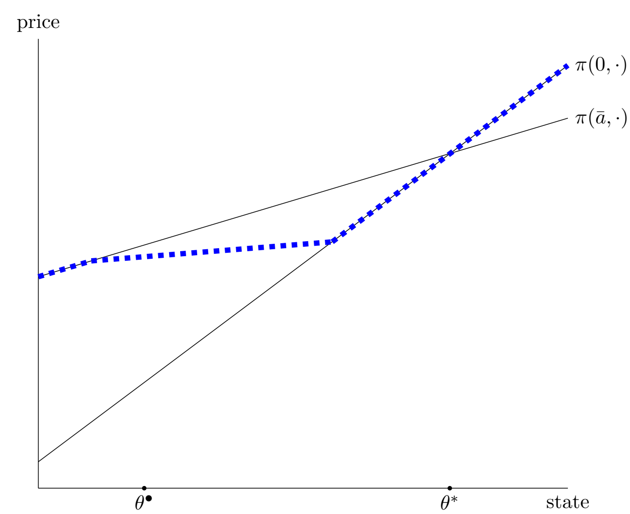


Figure 3: (virtually) optimal CWUI policy

The remainder of the paper is organized as follows. Section 2 introduces the model, and discusses the various robustness notions considered. Section 3 presents the main characterization results. Section 4 discusses optimal policy when the unique implementation restriction is relaxed. Section 5.1 explores the application to bailouts. Section 6 extends the model to multi-dimensional state spaces, and discusses the spacial case of noisy REE.

2 The model

The model consists of the following primitive objects.

i. The state space, denoted by Θ, which is a compact interval in $\mathbb{R}$.[7]

ii. A convex set $\mathcal{A}$ of principal actions, which is a subset of a Banach space.

iii. A convex set $\mathcal{P} \subseteq \mathbb{R}$ of aggregate outcomes.

For clarity, we will refer to the aggregate outcome as the price, although the model applies to many situations in which the aggregate outcome is not a price, as is discussed below. There are three periods; $0, 1, 2$. The timing of interaction is as follows.

0. The principal publicly commits to a decision rule $M : \mathcal{P} \mapsto \mathcal{A}$ specifying an action for each price.

1. The price is determined.

2. If the price is p, the principal takes the action $M(p)$.

It only remains to describe how the price is determined in period 1. This is covered in the following section.

2.1 Price formation

In order to facilitate a wide range of applications, we take a modular approach to modeling price formation. This approach is based on the observation that the equilibrium outcome in a broad class of market models can be summarized as a function of the state and anticipated principal action. We first present a general reduced form representation of price formation. This will form the basis for most of the analysis of the paper. The reduced form summarizes the equilibrium outcomes of the game through which the price is determined. This approach encompasses many of the leading applications of decision making under feedback. We discuss various price formation games which are consistent with this reduced-form approach.

[7]It is immediate to extend the results to arbitrary dimensions when there is a complete order on the state space with respect to which we can make certain monotonicity assumptions discussed below. Multi-dimensional state spaces are discussed in more detail in Section 6. Most results also continue to hold if Θ is unbounded.

2.1.1 Price formation: reduced form

For the reduced form, we assume that the price is a function of the state and the anticipated action of the principal. Thus the price is given by a function $R : \mathcal{A} \times \Theta \mapsto \mathcal{P}$. The precise interpretation of R is that if the state is θ and all agents believe that the principal will take action a then the price will be $R(a, \theta)$. Throughout, we maintain the assumption that R is continuous.

The defining feature of environments with feedback is that if the principal has announced decision rule M and the price is p then all agents anticipate that the principal will take action $M(p)$. This situation arises naturally in many applications, and will be discussed further in the micro-foundations. The meaning of the function R will be further clarified when we discuss equilibrium. Before turning to equilibrium analysis we will discuss how equilibrium outcomes in various games can be summarised by a function $R : \mathcal{A} \times \Theta \mapsto \mathcal{P}$.

2.1.2 Price formation: micro-foundations

A formal treatment of various micro-foundations is presented in Appendix C. Here we will simply discuss informally two micro-foundations.

Asset market. This is one of the leading examples of decision making under feedback. Consider an environment in which there is fixed supply of a single asset and a continuum of traders.[8] The asset pays a dividend that is a function of the state and the principal's action. Each trader receives a private signal that is partially informative about the state. Traders base their demand on *a*) the market price, *b*) the anticipated action of the principal, and *c*) their belief about the state. The latter is a function of both their private signal and information conveyed by the asset price. Since the principal's action is a function of the price, there is no ambiguity about the action given the observed price. A rational expectations equilibrium (REE) in this environment consists of a price function $\tilde{P} : \Theta \mapsto \mathcal{P}$ such that markets clear in each state θ given

- The anticipated action $M(\tilde{P}(\theta))$,
- The inferences made from the price given the function $\tilde{P}$.

In Appendix C we show that in such an environment, under some assumptions on information and payoffs, there exists a function $R : \mathcal{A} \times \Theta \mapsto \mathcal{P}$ that gives the REE clearing price given any decision rule M.

[8]In Section 6.1 we show how the results can be extended to markets with aggregate supply shocks.

Expert forecasts.

In many situations agents may not observe the aggregate outcome when making the decisions that will, taken together, determine the aggregate outcome. For example, the unemployment rate in a given month is the result of the decisions of firms and workers who act without observing the realized unemployment rate. If, in such a situation, the principal makes a decision that is relevant for agents, based on the aggregate outcome then agents will need to predict the action that the principal will take. In many such settings, expert forecasts play an important role in agent decision making.

Suppose an economist receives a signal $\theta \in \Theta$ about the underlying state of the economy ω, and reports publicly their expectation $\hat{p}$ of the unemployment rate p. At the end of the month, the government observes p and chooses $a \in \mathcal{A}$ according to $M(p)$. The action here could be, for example, the amount of money to put into an employment subsidy program. The realized unemployment rate will depend on firm's expectations about a and the underlying state θ. Assume that firms trust the economist's forecast; they take it as an accurate prediction of the unemployment rate. Firm's them make their personnel decisions. The realized unemployment rate will given by $J(M(\hat{p}), \theta)$.

The economist recognizes the effect that their forecast has on firm behavior, and thus on the realized unemployment rate. The economist will take this into account when making their prediction. Thus their expectation of the unemployment rate will be given by

$$\hat{p} = \mathbb{E}[J(M(\hat{p}), \omega)|\theta] \equiv R(M(\hat{p}), \theta).$$

Such a fixed point exists when $\mathcal{A}$ is compact and M continuous. Note that R here is a function of the economist's signal θ, rather than the underlying state ω.

2.2 Implementation

We have not yet discussed the preferences of the principal. We will not make assumptions on these, other than that they do not depend directly on the announced decision rule M. Rather, the principal cares only about the joint distribution of states, actions, and prices. In other words, the principal cares about the equilibrium maps from states to actions and prices induced by their announced decision rule.

A rational expectations equilibrium (REE) under M consists of a price function $P : \Theta \mapsto \mathcal{P}$ such that $P(\theta) = R(M \circ P(\theta), \theta)$ for all $\theta \in \Theta$. Let $Q : \Theta \mapsto \mathcal{A}$ be an *action function* and $P : \Theta \mapsto \mathcal{P}$ a *price function.* The price and action functions are the objects of interest from the principal's perspective. It is therefore important to understand which price and action functions can be implemented using market-based decision rules.

Definition. (Q, P) *is* ***implementable*** *if there exists* $M : \mathcal{P} \mapsto \mathcal{A}$ *such that*

1. $P(\theta) = R(M \circ P(\theta), \theta) \quad \forall\, \theta \in \Theta$ *(RE)*

2. $Q = M \circ P$. *(commitment)*

The RE (rational expectations) condition requires that the realized price be consistent with the anticipated action given decision rule M. The commitment condition simply says that the principal is in fact using decision rule M.

Implementability can be equivalently defined without making explicit reference to the implementing decision rule M.

Observation 1. (Q, P) *is implementable iff*

1. $P(\theta) = R(Q(\theta), \theta) \quad \forall\, \theta \in \Theta$ *(RE)*

2. $Q(\theta) \neq Q(\theta') \qquad \Rightarrow \qquad P(\theta) \neq P(\theta')$. *(measurability)*

Here the measurability condition guarantees that there exists a P measurable function M that induces action function Q. Clearly if this condition is violated there can exist no such M.

For any continuous decision rule M, a rational expectations equilibrium exists (for any $\theta \in \Theta$, the function $a \mapsto M(R(a, \theta))$ has a fixed point by the Schauder fixed point theorem.) However it is possible to define discontinuous decision rules for which no REE exists. Non-existence of REE is a manifestation of the incompleteness of the description of the model. The true meaning of equilibrium non-existence will depend on the nature of the fundamental game played by agents in the market, i.e. on the micro-foundation for the function R. For example, Bond et al. (2010) show that non-existence in a setting with feedback can be translated to a breakdown in trade: the market-maker abstains from making markets because they would lose money by doing so. We will not explicitly model market outcomes when a REE fails to exist. Rather, our focus will be on the more limited set of outcomes that can be implemented as REE.

Observation 1 gives a characterization of the set of implementable (Q, P). However it is not, on its own, a very useful characterization for two reasons. First, it does not point to any general qualitative features of implementable mechanisms. Second, it ignores many important practical concerns that the principal may consider when choosing a decision rule. It turns out that when these are taken into account a more meaningful characterization of the set of implementable mechanisms can be given.

2.3 Uniqueness

A number of practical concerns naturally arise when contemplating the use of market based decision rules. We focus on two restrictions of the set of decision rules that the principal may use, which reflect such practical considerations.

Our primary focus is on issues related to multiplicity of equilibria. The approach to multiple equilibria depends on the type of analysis being conducted. From an implementation perspective, the question is how to induce a given (Q, P) as equilibrium outcomes. In the implementation literature, this means that (Q, P) should be the unique equilibrium outcomes induced by some decision rule. The mechanism design perspective, on the other hand, is that the principal can choose from any of the equilibria induced by a given decision rule M. From this perspective, the goal of the principal is simply to induce (Q, P) as *an* equilibrium outcome.

We will consider both perspectives in this paper. To begin, we will take the implementation perspective that outcomes must be induced uniquely. We will then show how the results obtained can be related to a more permissive attitude towards multiplicity. We say that (Q, P) are *uniquely implementable* if they are the unique equilibrium outcomes given some decision rule M. In this case, we say that M is *robust to multiplicity.* Equivalently, a decision rule M is *robust to multiplicity* if $\{p \in \mathcal{P} : p = R(M(p), \theta)\}$ is singleton for all θ.

It will also be helpful to consider a slightly weaker notion of robustness to multiplicity. A decision rule M is *weakly robust to multiplicity* if $\{p \in \mathcal{P} : p = R(M(p), \theta)\}$ is singleton for almost all θ. This definition of robustness makes most sense when the principal maximizes expected utility and has an absolutely continuous prior H. If instead H has atoms then the definition should be modified so that the requirement of a unique price holds almost everywhere under H. There is no difficulty in accommodating this modification into the analysis, although it requires rewording some of the results.

2.4 Continuity

For a number of reasons, the principal may be concerned about using a discontinuous decision rule. One important reason is manipulation. An agent may manipulate the price by buying/selling the asset, releasing false information, or other means.[9] A discontinuous decision rule will be vulnerable to manipulation, as an agent can induce a significant change in the principal's action by making an arbitrarily small change to the price.[10]

[9]Goldstein and Guembel (2008) discusses manipulation of this sort.

[10]Continuity may well be insufficient to prevent manipulation in some settings; additional restrictions may be required in specific applications, and will imply refinements of the set of admissible decision rules. However

Similarly, if M is discontinuous then the set of equilibrium outcomes may be overly sensitive to the model fundamentals, in particular to the function R, about which the principal may well have imperfect knowledge. Indeed, Lemma 2 shows that if M has a discontinuity at at some price which could occur in equilibrium then the equilibrium outcomes will respond discontinuously to changes in R. Decision rules for which the equilibrium outcomes respond continuously to perturbations of R, which we refer to as *robust to structural uncertainty*, are discussed in Section 3.3.

The concerns about manipulation and model misspecification discussed above suggest that we should restrict attention to continuous decision rules. However the restriction to everywhere-continuous decision rules is stronger than is needed to address these concerns. As Theorem 4 shows, it is enough to have continuity in the neighborhood of any equilibrium price to guarantee robustness to structural uncertainty. Similarly, if a discontinuity in M occurs at a price which is far from any which could arise in equilibrium then there should be no reason to worry that the equilibrium is vulnerable to manipulation. We therefore allow for discontinuities in the decision rule, provided they do not occur near equilibrium prices. Formally, let $C \subset \mathcal{A}^{\mathcal{P}}$ be the set of continuous functions from $\mathcal{P}$ to $\mathcal{A}$. For any M, let $\bar{P}_M = \cup_{\theta \in \Theta}\{p \in \mathcal{P} : R(M(p), \theta) = p\}$ be the set of possible equilibrium prices given M, and let $cl(\bar{P}_M)$ be its closure.

Definition. *A function $M : \mathcal{P} \mapsto \mathcal{A}$ is* ***essentially continuous*** *if i) there exists $\hat{M} \in C$ and an open set $U \subseteq \mathcal{P}$ with $cl(\bar{P}_{\hat{M}}) \subset U$ such that $\hat{M}(p) = M(p)$ for all $p \in U$, and ii) M is continuous on an open set containing $cl(\bar{P}_M)$.*

Condition i in this definition says that M must be close to some continuous $\hat{M}$: they can differ only outside of the set of equilibrium prices for $\hat{M}$. Condition ii says that, moreover, M cannot have discontinuities near prices that could arise in equilibrium under M.

Let $\mathcal{M}$ be the set of essentially continuous functions. Clearly $C \subset \mathcal{M}$. In words, $\mathcal{M} \setminus C$ consists only of functions that can be found by introducing discontinuities to some continuous M at prices that are bounded away from the set of equilibrium prices. Throughout, we will restrict attention to decision rules in $\mathcal{M}$. We will at times refer to this as a continuity

continuity of the decision rule can be interpreted as a limiting notion of preventing manipulation as moving the price becomes very costly. Assume that the cost of price manipulation is proportional to the magnitude of the induced price change. For fixed costs of price manipulation and fixed private benefits, manipulation will not be profitable if and only if the decision rule is Lipschitz continuous, where the Lipshitz constant is determined by the costs and benefits of manipulation. In the limit, as the costs of manipulation grow, we simply require continuity. Requiring Lipschitz continuity of the decision rule, rather than simply continuity, would not substantively change the analysis.

requirement; although it does not imply that M must be everywhere continuous, it has the same intuitive content.

Condition i in the definition of essentially continuous could be dropped without affecting the main results of the paper. The only difference is that without this condition we cannot guarantee that a rational expectations equilibrium, defined in the next section, exists for any $M \in \mathcal{M}$. We could equivalently replace condition i with the requirement that a rational expectations equilibrium exist given M. Alternatively, all results would continue to hold hold if condition ii were dropped from the definition. This condition is included because it captures the intuition regarding which discontinuities should be allowed.

2.5 Unique implementation

We first analyse the problem of unique implementation. Unique implementation is a primary objective in many market-based decision settings.

Definition. (Q, P) *is* ***continuously uniquely implementable (CUI)*** *if it is implementable uniquely by an* $M \in \mathcal{M}$.

In other words, (Q, P) is continuously uniquely implementable if there exists $M \in \mathcal{M}$ such that:

1. $Q = M \circ P$

2. $P(\theta)$ is the unique solution to
$$p = R(M(p), \theta)$$
for all θ.

3. $$Q(\theta) \neq Q(\theta') \qquad \Rightarrow \qquad P(\theta) \neq P(\theta')$$

There are two differences between implementability and CUI; the uniqueness requirement in condition 2 and the continuity requirement that $M \in \mathcal{M}$. Continuity, as discussed above, reflects manipulation concerns. If condition 2 holds for almost all θ, rather than all θ, then we say that (Q, P) is *continuously weakly uniquely implementable (CWUI)*. There is no substantive difference between the two notions, but it is sometimes easier to state results for the weaker notion.

We will sometimes refer to an action function Q as CUI, by which we mean that there exists a P such that the pair (Q, P) is CUI, in similarly for price functions P.

At times, it will be convenient to discuss approximate, rather than exact, implementation. As is standard, we say that (P, Q) is virtually implementable if it can be approximated arbitrarily well by some implementable $(\hat{P}, \hat{Q})$. Say that Q' is an ε-approximation of Q if the set $\{\theta : Q(\theta) \neq Q'(\theta)\}$ has measure less than ε.

Definition. *(P, Q) is* ***virtually CUI*** *if for any $\varepsilon > 0$ there exists an ε-approximation of Q that is CUI.*

The characterization of CUI (and virtually CUI) outcomes will be one of the main results of this paper. It turns out that this characterization is also central to understanding optimal decision rules even when the uniqueness requirement is relaxed.

3 Main results

We turn now to the main results of the paper. Some preliminary definitions and results are first needed. The following assumption on price formation will be maintained for most results.

Definition. *R is* ***weakly increasing in*** *θ if $\theta \mapsto R(a, \theta)$ is weakly increasing for all $a \in \mathcal{A}$*

We say that R is *strictly increasing* in θ if $\theta \mapsto R(a, \theta)$ is strictly increasing for all $a \in \mathcal{A}$. Note that the order used on Θ is irrelevant. All results that assume that R is weakly increasing continue to hold under the weaker assumption that R is comonotone in θ; $R(a, \theta'') \geq R(a, \theta')$ implies $R(a', \theta'') \geq R(a', \theta'')$ for all a', a''. Similarly results that assume that R is strictly increasing continue to hold as long as there exists some order on Θ such that $\theta \mapsto R(a, \theta)$ is strictly increasing for all a. Both strictly and weakly increasing R can be justified by natural assumptions on primitives in many micro-foundations, and is satisfied in all applications that we have come across.

For any decision rule M, define $\theta_M(p) = \{\theta \in \Theta : R(M(p), \theta) = p\}$. When R is strictly increasing in θ, $\theta_M(p)$ will be a function from $\mathcal{P} \mapsto \Theta \cup \varnothing$. When R is only weakly increasing $\theta_M(p)$ may be set valued.

The defining feature of CUI outcomes is a monotone price. This condition is necessary under weakly increasing R.

Theorem 1. *Assume R is weakly increasing in θ. If $M \in \mathcal{M}$ is weakly robust multiplicity then it induces a monotone price function.*

Proof. In Appendix A.2 □

In other words, Proposition 1 says that if M is continuous and it induces a price function P that is non-monotone then there will be multiple equilibria. Monotonicity essentially characterizes CUI outcomes, as the following two sections will show. Two features of Theorem 1 are worth emphasising. First, the induced equilibrium price function P need not be increasing; it may be monotonically decreasing. Second, monotonicity of the induced P is not simply a consequence of measurability. This would be the case if we required Q to be continuous. However Q need not be continuous. The assumption that M is continuous does imply continuity of Q. Nonetheless, we will see that continuity of Q may be an implication of robustness to multiplicity.

3.1 Implementable action functions

In most situations the principal cares about the actions that they take. They may also care about the price, but since the price is determined in equilibrium by the action function it is sufficient to look only at actions.[11] In such cases the relevant question is what action functions Q are CUI.

To understand the sufficient conditions for CUI, assume first that $\theta \mapsto R(a, \theta)$ is strictly monotone for all a. If Q is a continuous action function and $\theta \mapsto R(Q(\theta), \theta)$ is strictly monotone then there will be no multiplicity for prices in $P(\Theta)$: define $M(p)$ on $P(\Theta)$ as the unique function satisfying $M(R(Q(\theta), \theta)) = Q(\theta)$. This is well defined when $\theta \mapsto R(Q(\theta), \theta)$ is strictly monotone. M is continuous since Q is continuous, and there can be no multiplicity involving prices in $P(\theta)$ since $|\{\theta \in \Theta : R(a, \theta) = p\}| \leq 1$ for all p under strict monotonicity of $\theta \mapsto R(a, \theta)$. While it remains to define M on $\mathcal{P} \setminus P(\Theta)$, this argument suggests that strict monotonicity of the induced price function and continuity of Q are sufficient for CUI. With a minor caveat, this is indeed the case. The difficulty is that continuity of Q is not implied by continuity of M. It turns out however that Theorem 1 implies that continuity of Q is necessary a condition for CUI. This gives the characterization of CUI when R is strictly increasing, which requires only minor modifications when R is weakly increasing. Before stating the theorem, we require the following definitions.

Definition. ***Local upper monotonicity*** *is satisfied at* (a, θ) *if there exists* $\varepsilon > 0$ *and a continuous function* $m : [R(a, \theta), R(a, \theta)+\varepsilon] \mapsto \mathcal{A}$ *with* $m(R(a, \theta)) = a$ *such that* $R(m(p), \theta) > p$ *for all* $p \in (R(a, \theta), R(a, \theta) + \varepsilon]$.

Definition. ***Local lower monotonicity*** *is satisfied at* (a, θ) *if there exists* $\varepsilon > 0$ *and a continuous function* $m : [R(a, \theta)-\varepsilon, R(a, \theta)] \mapsto \mathcal{A}$ *with* $m(R(a, \theta)) = a$ *such that* $R(m(p), \theta) <$

[11] In Appendix A.9 we explore the case in which the principal only cares about the price function.

p for all $p \in (R(a,\theta) - \varepsilon, R(a,\theta)]$.

While these conditions may seem dense, they essentially require that $a' \mapsto R(a',\theta)$ does not have weak local extremum at a (maximum for upper, minimum for lower). This is always necessary, it is sufficient when R is smooth; if $a' \mapsto R(a',\theta)$ is continuously differentiable at a then local upper (lower) monotonicity is satisfied at (a,θ) if and only if a is not a weak local maximum (minimum) of $a' \mapsto R(a',\theta)$. These conditions will be relevent only when the price function is decreasing.

Theorem 2. *Assume R is strictly increasing in θ. Then*

i. If Q is CUI then it is continuous on the interior of Θ and induces a strictly monotone price function. Moreover if P is decreasing then local upper monotonicity is satisfied at $(Q(\underline{\theta}), \underline{\theta})$ and local lower monotonicity is satisfied at $(Q(\bar{\theta}), \bar{\theta})$.

ii. If Q is continuous and induces a strictly increasing price function then it is CUI. If it induces a strictly decreasing price function, satisfies local upper monotonicity at $(Q(\underline{\theta}), \underline{\theta})$, and satisfies local lower monotonicity at $(Q(\bar{\theta}), \bar{\theta})$, then Q is CUI.

Proof. Proof in Appendix A.3. □

The only difference between the necessary conditions of Theorem 2, *i* and the sufficient conditions of *ii* concerns discontinuities at the extreme states $\underline{\theta}$ and $\bar{\theta}$. In most applications this gap can be closed: the necessary and sufficient conditions for CUI will be strict monotonicity of P and continuity of Q on the interior of Θ. This will hold precisely when any discontinuity at $\underline{\theta}$ is lower-bridgeable, and any discontinuity at $\bar{\theta}$ is upper-bridgeable, as defined in Section 3.2. Since discontinuities at the extreme states are of no substantive interest we deffer discussion of these conditions.

It is worth pointing out that any CUI Q that induces an increasing price can be implemented with a continuous M. Discontinuities in M (which must occur outside of the set of equilibrium prices) are only useful for implementing decreasing price functions. This is shown in the construction of M given in the proof of Theorem 2.

Lemma 1. *A continuous and CUI Q can be uniquely implemented by a continuous M if and only if it induces an increasing price.*

Relaxing strict monotonicity to weak monotonicity we can obtain a similar characterization to Theorem 2. It is necessary however to add an additional condition to account for actions for which the induced price is constant over an interval of states. Let $r(a,p) =$

$\{\theta \in \Theta : R(a,\theta) = p\}$. Under strict monotonicity $r(a,p)$ contains at most one state for all $a \in \mathcal{A}, p \in \mathcal{P}$. Under weak monotonicity however $r(a,p)$ may be a non-degenerate interval.

Let $P(\theta) = R(Q(\theta),\theta)$, and suppose $r(Q(\theta'),P(\theta')$ is non-degenerate. If $Q(\theta'') \neq Q(\theta')$ for some $\theta'' \in r(Q(\theta'),P(\theta')$ then clearly there will be multiplicity, since $R(Q(\theta'),\theta')$ is an REE price in state θ''. The only modifications needed to extend Theorem 2 are those that rule out such instances of multiplicity.

Corollary 1. *Assume R is weakly increasing in θ. Then*

- *If Q is CUI then it is continuous on the interior of Θ, induces a weakly monotone price function, and for all θ, $\theta' \in r(Q(\theta),P(\theta))$ we have $Q(\theta') = Q(\theta)$. Moreover if P is decreasing then local upper monotonicity is satisfied at $(Q(\underline{\theta}),\underline{\theta})$ and local lower monotonicity is satisfied at $(Q(\bar{\theta}),\bar{\theta})$.*

- *If Q is continuous, induces a weakly increasing price function, and for all θ, $\theta' \in r(Q(\theta),P(\theta))$ we have $Q(\theta') = Q(\theta)$, then Q is CUI. If it induces a weakly decreasing price function and additionally satisfies local upper monotonicity at $(Q(\underline{\theta}),\underline{\theta})$ and local lower monotonicity at $(Q(\bar{\theta}),\bar{\theta})$, then Q is CUI.*

Under the condition that for all θ, $\theta' \in r(Q(\theta),P(\theta))$ we have $Q(\theta') = Q(\theta)$, Corollary 1 follows from the same argument as Theorem 2. Again, the gap between the necessary and sufficient conditions is closed under mild assumptions, discussed in Section 3.2.

3.2 Continuous weakly unique implementation

The necessity of continuity of Q in Theorem 2 and Corollary 1 is an implication of Theorem 1 and the requirement of uniqueness for all θ. The substantive characteristic of CUI outcomes however is monotonicity of P. This section formalizes this assertion. We show that in many common settings continuous *weakly* unique implementability will be characterized fully by monotonicity of P.

First, since we only require uniqueness *almost everywhere*, Q can have countably many discontinuities, provided the discontinuities satisfy a certain condition. Roughly, this condition says that Q can be well approximated by a continuous Q'. For any two actions $a', a'' \in \mathcal{A}$, a *path* from a' to a'' is a continuous function $\gamma : [0,1] \mapsto \mathcal{A}$ such that $\gamma(0) = a', \gamma(1) = a''$. Say that there exists a *monotone path* from a' to a'' at θ if there exists a path γ from a' to a'' such that $x \mapsto R(\gamma(x),\theta)$ is strictly monotone.

Definition. *A discontinuity in Q at θ' is* ***bridgeable*** *if there exists a monotone path from $\lim_{\theta \nearrow \theta'} Q(\theta)$ to $\lim_{\theta \searrow \theta'} Q(\theta)$ at θ'.*

The following are weaker notions of bridgeability, which it will only be necessary to define on the extreme states $\underline{\theta}, \bar{\theta}$.

Definition. *A discontinuity in Q at $\bar{\theta}$ is **upper-bridgeable** if there exists a path γ from $\lim_{\theta \nearrow \bar{\theta}} Q(\theta)$ to $Q(\bar{\theta})$ such that $R(\gamma(x), \bar{\theta}) \leq \max\{\lim_{\theta \nearrow \bar{\theta}} R(Q(\theta), \theta), R(Q(\bar{\theta}), \bar{\theta})\}$ for all $x \in [0,1]$, with equality iff*

$$\gamma(x) = \underset{a \in \{\lim_{\theta \nearrow \bar{\theta}} Q(\theta), Q(\bar{\theta})\}}{\arg\max} R(a, \bar{\theta}).$$

Definition. *A discontinuity in Q at $\underline{\theta}$ is **lower-bridgeable** if there exists a path γ from $\lim_{\theta \searrow \underline{\theta}} Q(\theta)$ to $Q(\underline{\theta})$ such that $R(\gamma(x), \underline{\theta}) \geq \min\{\lim_{\theta \searrow \underline{\theta}} R(Q(\theta), \theta), R(Q(\underline{\theta}), \underline{\theta})\}$ for all $x \in [0,1]$, with equality iff*

$$\gamma(x) = \underset{a \in \{\lim_{\theta \searrow \underline{\theta}} Q(\theta), Q(\bar{\theta})\}}{\arg\min} R(a, \underline{\theta}).$$

Notice that a necessary condition for a discontinuity at θ to be bridgeable (or upper/lower bridgeable) is $\lim_{\theta \nearrow \theta'} R(Q(\theta), \theta) \neq \lim_{\theta \searrow \theta'} R(Q(\theta), \theta)$. We say that the environment is *fully bridgeable* if for every θ, this condition is also sufficient for bridgeability. Finally, say that the environment is *continuously bridgeable* if for any $\theta^* \in \Theta$ there exists $\varepsilon > 0$ such that if a', a'' is bridgeable at θ^* and $R(a'', \theta) \neq R(a', \theta)$ for all $\theta \in [\theta^*, \theta^* + \varepsilon]$ then there exists a sup-norm continuous function $\sigma(\cdot | a', a'') : [\theta^*, \theta^* + \varepsilon] \mapsto \mathcal{A}^{[0,1]}$ such that $\sigma(\theta | a', a'')$ is a monotone path from a' to a'' for all $\theta \in [\theta^*, \theta^* + \varepsilon]$. Say that the environment is *continuously fully bridgeable* if it is full bridgeable and continuously bridgeable. Bridgeability, and the related notions, will be discussed further following the statement of the results.

Theorem 3. *Assume $\theta \mapsto R(a, \theta)$ is strictly increasing for all $a \in \mathcal{A}$. Then Q is CWUI iff*

i. Either $P(\theta) := R(Q(\theta), \theta)$ is strictly increasing; or it is strictly decreasing and satisfies local upper monotonicity at $(Q(\underline{\theta}), \underline{\theta})$ and local lower monotonicity at $(Q(\bar{\theta}), \bar{\theta})$.

ii. Any discontinuity in Q on the interior of Θ is bridgeable.

iii. If Q is discontinuous at $\underline{\theta}$ then local upper monotonicity is satisfied at $(\min\{\lim_{\theta \searrow \underline{\theta}} Q(\theta), Q(\underline{\theta})\}, \underline{\theta})$ and local lower monotonicity at $(\max\{\lim_{\theta \searrow \underline{\theta}} Q(\theta), Q(\underline{\theta})\}, \underline{\theta})$.

iv. If Q is discontinuous at $\bar{\theta}$ then local upper monotonicity is satisfied at $(\min\{\lim_{\theta \nearrow \bar{\theta}} Q(\theta), Q(\bar{\theta})\}, \bar{\theta})$ and local lower monotonicity at $(\max\{\lim_{\theta \nearrow \bar{\theta}} Q(\theta), Q(\bar{\theta})\}, \bar{\theta})$.

Proof. In Appendix A.4. □

If the principal's payoffs are invariant to changes on zero measure sets then conditions *iii.* and *iv.* can be ignored for the purposes of choosing optimal policies; we can restrict attention to Q that are continuous at the endpoints. When the environment is fully bridgeable the type of discontinuities in Q that are allowed can be more easily characterized.

Corollary 2. *Assume $\theta \mapsto R(a,\theta)$ is strictly monotone for all $a \in \mathcal{A}$, and the environment is fully bridgeable. Then Q is CWUI iff*

i. $P(\theta) := R(Q(\theta),\theta)$ is strictly monotone. Moreover if P is decreasing then local upper monotonicity is satisfied at $(Q(\underline{\theta}),\underline{\theta})$ and local lower monotonicity is satisfied at $(Q(\bar{\theta}),\bar{\theta})$.

ii. If Q is discontinuous at θ then so is P.

Finally, it will be useful to know when condition *ii* in Corollary 2 is redundant. This will be the case when any discontinuities that violate this condition can be well approximated. Say that Q has a *degenerate discontinuity* at θ if Q is discontinuous at θ and P is not. The environment is *correctable* if for and $\varepsilon > 0$, any strictly monotone Q, and any θ at which Q has a degenerate discontinuity, there exists a monotone Q' that has no degenerate discontinuities in $(\theta-\varepsilon,\theta+\varepsilon)$ and such $Q' = Q$ on $\Theta \setminus (\theta-\varepsilon,\theta+\varepsilon)$. Sufficient conditions for correctability are discussed in Appendix B.

Corollary 3. *Assume $\theta \mapsto R(a,\theta)$ is strictly monotone for all $a \in \mathcal{A}$, and the environment is fully bridgeable and correctable. Then Q is virtually CWUI iff*

i. $P(\theta) := R(Q(\theta),\theta)$ is strictly monotone. Moreover if P is decreasing then local upper monotonicity is satisfied at $(Q(\underline{\theta}),\underline{\theta})$ and local lower monotonicity is satisfied at $(Q(\bar{\theta}),\bar{\theta})$.

ii. The set of states at which Q is discontinuous has zero measure.

Proof. In Appendix A.5. □

Bridgeability of discontinuities and correctability of the environment are less transparent conditions than monotonicity of the price, and so it will be useful to know general conditions under which they satisfied. If $\mathcal{A}$ is a subset of $\mathbb{R}$ then clearly a discontinuity at θ with left limit $\underline{a}$ and right limit $\bar{a}$ is bridgeable iff $a \mapsto R(a,\theta)$ is strictly monotone on $[\min\{\underline{a},\bar{a}\},\max\{\underline{a},\bar{a}\}]$.[12] When the action space is multi-dimensional the condition becomes more difficult to check, but also easier to satisfy. For example, Proposition 14 shows that full bridgeability is satisfied when the action space is multi-dimensional and R satisfies a weak monotonicity condition.

[12]This does not mean that $a \mapsto (a,\theta)$ is monotone in the same direction in every state; it could be increasing in some states and decreasing in others.

A full discussion of bridgeability, corrrectability, and related notions is contained in Appendix B. This section gives general conditions under which every discontinuity is bridgeable. In most applications encountered in the literature it is easy to verify that the environment is continuously fully bridgeable and correctable. Even when it is not, the states at which these conditions fail are readily identifiable.

An alternative way to understand the conditions of Theorem 3 is in terms of approximations to Q. Say that Q' is a continuous ε-approximation of Q if Q' is continuous and $\lambda\left(\{\theta \in \Theta : Q(\theta) \neq Q'(\theta)\}\right) < \varepsilon$, where λ is Lebesgue measure.[13]

Proposition 1. *Assume $\theta \mapsto R(a,\theta)$ is strictly increasing for all $a \in \mathcal{A}$ and the environment is continuously fully bridgeable. Then if Q is CWUI there exists a continuous ε-approximation Q' that is CUI, for any $\varepsilon > 0$.*

Proof. In Appendix A.6. □

In other words, Proposition 1 says that the space $\mathcal{Q}$ of continuous Q which induce a strictly monotone price is dense in the space of CWUI Q. Proposition 1 can help simplify the problem of solving for an optimal policy. Any such $Q \in \mathcal{Q}$ will be CUI, and by Proposition 1 there is no loss of optimality, provided the principal's payoffs are continuous.

Say γ is a *proper monotone path* from a' to a'' at θ if it is a monotone path, and moreover $r(\gamma(x), R(\gamma(x),\theta)) = \theta$ for all $x \in [0,1]$. A discontinuity in Q at θ is *properly bridgeable* if there exists a proper monotone path from $\lim_{\theta' \nearrow \theta} Q(\theta)$ to $\lim_{\theta' \searrow \theta} Q(\theta)$ at θ. The environment is fully properly bridgeable if all non-degenerate discontinuities are properly bridgeable. Note that if $\theta \mapsto R(a,\theta)$ is strictly increasing for all a then proper bridgeability is equivalent to bridgeability.

Proposition 2. *Assume $\theta \mapsto R(a,\theta)$ is weakly increasing for all a. Then Q is CWUI iff*

i. $P := R(Q(\theta),\theta)$ is weakly monotone. Moreover if P is decreasing then local upper monotonicity is satisfied at $(Q(\underline{\theta}),\underline{\theta})$ and local lower monotonicity is satisfied at $(Q(\bar{\theta}),\bar{\theta})$.

ii. Any discontinuity in Q on the interior of Θ is properly bridgeable.

iii. A discontinuity in Q at $\underline{\theta}$ is lower bridgeable, and at $\bar{\theta}$ is upper bridgeable.

iv. $Q(\theta) = Q(\theta')$ for all $\theta' \in r(Q(\theta), P(\theta))$ and all θ.

Proof. In Appendix A.8. □

[13] Since Θ and $\mathcal{A}$ are compact, if there is an ε-approximation for any ε then there is a sequence that approaches Q in the L^p norm, for any $p < \infty$.

The first condition is necessary by Theorem 1. It is sufficient given the other two conditions. Under monotonicity, condition *iii.* guarantees that the measurability restriction is satisfied, so an implementing M can be found. Condition *ii.* guarantees that an $M \in \mathcal{M}$ can be found that implements Q.

3.3 Structural uncertainty

Another practical concern of the principal is that the price may be influenced by uncertain factors other than the state in which the principal is interested. For example, the presence of noise/liquidity traders in an asset market could introduce aggregate uncertainty. As a consequence, the price may not be a deterministic function of the state and anticipated action. Additionally, the principal may simply have limited information about market fundamentals, which within the model translates into uncertainty about the function R. The principal will want to choose a decision rule that is robust to these types of uncertainty.

Endow the space of market-clearing functions $R : \mathcal{A} \times \Theta \mapsto \mathbb{R}$ with the sup-norm. Let $\mathcal{C}$ be the set of continuous functions on $\mathcal{A} \times \Theta$. For a given decision rule M and market clearing function R, let $\tilde{Q}_R(\theta|M) := \{a \in \mathcal{A} : M(R(a,\theta)) = a\}$. In words, $\tilde{Q}_R(\theta|M)$ is the set of actions that are consistent with rational expectations in state θ. An open neighborhood of $\tilde{Q}_R(\Theta|M)$ is an open set $U \subset \mathcal{A} \times \Theta$ (with the product topology) such that $\tilde{Q}_R(\Theta|M) \subset U$. The map $R \rightrightarrows \tilde{Q}_R(\theta|M)$ is *uniformly continuous* at R if it is uniformly upper and lower hemicontinuous. That is, for any open neighborhood U of $\tilde{Q}_R(\Theta|M)$ and any open set V such that $\tilde{Q}_R(\theta|M) \cap V \neq \varnothing$ for all θ, there exists an open neighborhood N of R such that $\hat{R} \in N$ implies *i)* $\tilde{Q}_{\hat{R}}(\Theta|M) \subset U$, and *ii)* $\tilde{Q}_{\hat{R}}(\theta|M) \cap V \neq \varnothing$ for all θ.

For any $S \subseteq \Theta$ let $\tilde{Q}_{R|S}$ be the restriction of $\tilde{Q}_R$ to S. Say that $R \rightrightarrows \tilde{Q}_R$ is *almost uniformly continuous* at R if $\forall\, \varepsilon > 0\ \exists\, S \subseteq \Theta$ with $\lambda(S) > 1 - \varepsilon$ such that $R \rightrightarrows \tilde{Q}_{R|S}(\theta|M)$ is uniformly continuous at R (where S replaces Θ in the definition of uniform continuity).

Definition. *A decision rule M is* ***(weakly) robust to structural uncertainty*** *if $R \rightrightarrows \tilde{Q}_R$ is (almost) uniformly continuous at R.*

The interpretation of this definition is that the decision rule should induce almost the same joint distribution of states and actions for small perturbations to the market clearing function. This in turn implies that the principal's expected payoff will be continuous in the function R. It turns out that continuous decision rules that are robust to multiplicity are robust to structural uncertainty.

Theorem 4. *If $M \in \mathcal{M}$ is (weakly) robust to multiplicity then it is (weakly) robust to structural uncertainty.*

Proof. Proof in Appendix A.11.1 □

The important implication of Theorem 4 is that small changes in R lead to small changes in the principal's expected payoff. Formally, for any selection from $\theta \mapsto \tilde{Q}(\theta, R)$, i.e. any function $Q : \Theta \mapsto \mathcal{A}$, such that $Q(\theta) \in \{\tilde{Q}(\theta, R)\}$ for all θ, abuse notation and write $Q \in \tilde{Q}(\cdot, R)$. Let the principal's expected payoff for a Q be given by

$$U(Q) = \int_\Theta u(\theta, Q(\theta))dH(\theta)$$

where $u : \Theta \times \mathcal{A} \mapsto \mathbb{R}$ is continuous and H is absolutely continuous with respect to Lebesgue measure.[14] Let $\mathcal{U}(R) = \{v \in \mathbb{R} : \exists\ Q \in \tilde{Q}(\cdot, R) \text{ with } U(Q) = v\}$ be the set of payoffs consistent with equilibria induced by M, given market clearing function R.

Proposition 3. *Let $M \in \mathcal{M}$ be a decision rule that is weakly robust to multiplicity for market clearing function R. Then $\mathcal{U}(R)$ is upper and lower hemicontinuous at R on $\mathcal{C}$.*

Proof. in the Appendix A.11.2. □

The same conclusion holds *a fortiori* if M is robust to multiplicity.

If M is robust to multiplicity but has jump discontinuities on $\bar{P}_M$ then it will not be robust to structural uncertainty under robustness to multiplicity. As discussed in Section 2, this further motivates the restriction to continuous decision rules. Let $\theta_M(p|R) = \{\theta \in \Theta : R(M(p), \theta) = p\}$ is the set of states at which p could be an equilibrium price under M and R, and $\bar{P}_M(R) := \{p \in \mathcal{P} : \theta_M(p) \neq \varnothing\}$ is the set of prices that could arise in equilibrium.

Lemma 2. *Assume that M satisfies robustness to multiplicity. If M has a jump discontinuity on $\bar{P}_M(R)$ then it is not robust to structural uncertainty.*

Proof. Proof in Appendix A.11.3. □

This result can be generalized to other types of discontinuities, and to M that don't satisfy robustness to multiplicity.

4 Beyond uniqueness

The following two propositions are extremely useful when relaxing the requirement of unique implementation. They allow us to use the previous characterization to study problems in which the strict uniqueness requirement is not imposed.

[14] Alternatively, we could dispense with absolute continuity and define robustness to multiplicity in terms of H.

Proposition 4. *Assume R is weakly increasing in θ. If $M \in \mathcal{M}$ induces multiple equilibria then at least one has monotone price function.*

Proof. In Appendix A.9.4 $\square$

Proposition 5. *Assume R is weakly increasing in θ. If $M \in \mathcal{M}$ induces multiple equilibria then at least one is characterized by (Q, P) that are CWUI.*

Proof. In Appendix A.9.5 $\square$

One implication Proposition 4 is that if the principal takes a strict worst case view of multiplicity (but is willing to tolerate weak unique implementation) then it is without loss of optimality to restrict attention to CWUI outcomes. That is, if the principal evaluates a decision rule M according to the worst equilibrium that it induces, then the principal may as well restrict attention to M that are weakly robust to multiplicity.

The strict worst case view of multiplicity may be too extreme for some principals. Consider now a principal who takes a lexicographic approach to multiplicity. The principal first ranks decision rules according to their worst case equilibrium outcomes. Among those decision rules with the same worst-case payoff, the principal then chooses the one with the highest best-case payoff. By Proposition 4 we know that the highest worst-case guarantee is exactly the maximum payoff within the class of decision rules that are weakly robust to multiplicity. Once this value has been determined, the goal of the principal is to choose the decision rule with the best equilibrium outcome, subject to not inducing any equilibrium with a payoff below this worst-case bound.

Assume first that the principal's payoffs do not depend directly on the price; the principal cares only about the joint distribution of states and actions (similar discussion will apply to other preferences). Assume that there is a unique optimal CWUI action function Q^* (similar discussion applies to virtual implementation), implemented uniquely by decision rule M^* (if there are multiple optimal CWUI action functions then then condition 1 below must hold for one of them). If this is the case then, by Proposition 4, the principal needs to choose a decision rule that implements Q^* as one of its equilibrium outcomes. This pins down the decision rule for all prices in the range $\{R(Q^*(\theta), \theta) : \theta \in \Theta\}$; any optimal decision rule must coincide with M^* for such prices. Moreover, Q^* will be an equilibrium outcome of any such decision rule. This discussion implies the following.

Proposition 6. *Let Q^* be the set of virtually CWUI action functions. Then the constraints of the principal with lexicographic multiplicity preferences can be stated as follows: choose $\hat{M}$ subject to*

1. $\exists Q \in Q^*$ *such that* $\hat{M}(R(Q(\theta), \theta)) = Q(\theta)$ *for all* $\theta \in \Theta$,

2. $\hat{M} \in \mathcal{M}$.

As will be illustrated in the application of Section 5.1, these constraints can greatly simplify the problem of finding optimal policies for a principal with lexicographic preferences over multiple equilibria.

5 Applications

5.1 Bailouts

The government is considering a bailout for a publicly traded company, which it considers strategically important.[15] The company's business prospects $\theta \in \Theta$, representing the demand environment, competition, future costs, etc., are unknown. The government chooses a level of support $a \in \mathcal{A} = [0, \bar{a}]$. For each level of support the share price is a strictly increasing function of the state. We make two additional assumptions regarding the share price.[16]

1. The slope of $\theta \mapsto R(a, \theta)$ is decreasing in a.

2. There exists a state θ^* such that $a \mapsto R(a, \theta)$ is strictly increasing for $\theta < \theta^*$ and strictly decreasing for $\theta > \theta^*$.

The first assumption represents the belief on the part of investors that government involvement in the firm will reduce upside when business prospects are good. This could be because the bailout involves the government taking a role in management, for example by gaining seats on the board. An alternative interpretation is that the bailout takes the form of forgivable loans, such that the amount owed is increasing in the state (which will be revealed *ex post*). The second assumption captures the fact that when business prospects are sufficiently bad, the bailout is necessary to sustain the operations of the business. When business prospects are sufficiently good however, the adverse effects of government intervention dominate. These features are derived from the discussion around recent bailouts, for example that of Lufthansa by the German government.[17]

[15] Alternatively, the bailout could be for an entire industry, in which many of the firms are publicly traded.

[16] These assumptions can be directly related to the asset dividends, as discussed in Appendix C.

[17] In the Lufthansa case, one large shareholder, Heinz Hermann Thiele, threatened to veto the proposed bailout, which involved the government taking a 20% stake in the company and receiving seats on the board. Thiele was reportedly concerned that the government stake would make it harder to restructure and cut jobs. On the other hand, supervisory board chairman Karl-Ludwig Kley emphasised Lufthansa's dire prospects:

The government does not wish to give any support to the company if the state is below some threshold θ'. In such cases the business is not considered viable, and the government prefers to let it fail. On the other hand, if the state is above some threshold $\theta'' > \theta'$, the government would also like to offer no support. In this case the government believes that the business can survive without intervention. The government's payoff $u(a, \theta)$ is therefore decreasing in a for $\theta \notin (\theta', \theta'')$. The government would like to intervene when the state is in $[\theta', \theta'']$. In these states the government's payoff $u(a, \theta)$ is increasing in a. The principal maximizes expected utility, and has an absolutely continuous prior H.

Figure 4 illustrates the situation in which $\theta^* \in [\theta', \theta'']$. The blue lines correspond to the price function P^* induced by the first-best action function Q^*. Assumption 2 on R implies that the environment is continuously fully bridgeable and correctable. Since the price function induced by $Q*$ is strictly increasing, the first-best is CWUI by Theorem 3.

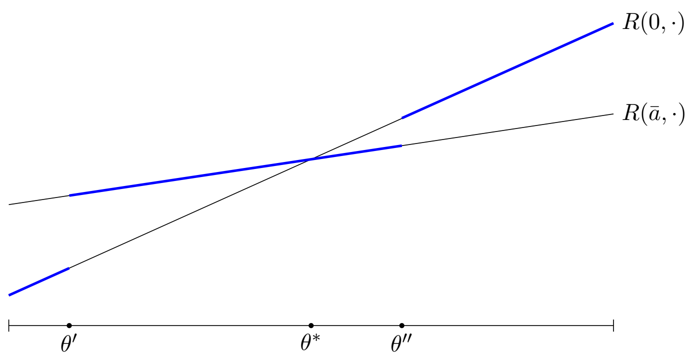


Figure 4: First-best is CWUI

A relatively interventionist government is represented by $\theta^* < \theta''$. In this case the government would like to intervene even in states in which investors would prefer no bailout. This will be the case when the strategic importance of the company is high, for example when the company is involved in national security, employs a large number of workers, or engages in production which has large technological spillovers.

Although the first-best is CWUI when $\theta^* < \theta''$, the government must take care in choosing the appropriate implementing decision rule, so as to avoid multiplicity. There are a continuum of decision rules that implement the first-best as an equilibrium outcome. The decision rule for prices in $P^*(\Theta)$ is clearly determined by the desired action function. However the action function alone does not pin down the decision rule for prices in $\tilde{P} \setminus P^*(\Theta)$. Consider the

"We don't have any cash left. Without support, we are threatened with insolvency in the coming days." Lufthansa shares rose 20% when Thiele announced that he would support the deal (Wissenbach and Taylor, 2020).

prices in the range $(R(0,\theta'), R(\bar{a},\theta'))$. For such prices, M must satisfy $p = R(M(p),\theta')$. If the government responds to much to price changes in this range, meaning that M increases faster than what this condition implies, then there will equilibria in which action $a > 0$ is taken for states below θ'. Similarly if the government under-responds then there will be equilibria in which action $a < \bar{a}$ is taken for states above θ'. Similar restrictions apply to the discontinuity in P^* at θ''.

Suppose instead that $\theta^* > \theta''$. In this case the government is lassiez faire; it does not wish to intervene in states (θ'', θ^*) in which investors would welcome a bailout. The price function associated with the first-best outcome is depicted in Figure 5. In this case the price is non-monotone, and is therefore not CWUI. In fact, in this case it is not even implementable, as it violates measurability. The optimal CWUI outcome is found by ironing the price function to eliminate non-monotonicity.

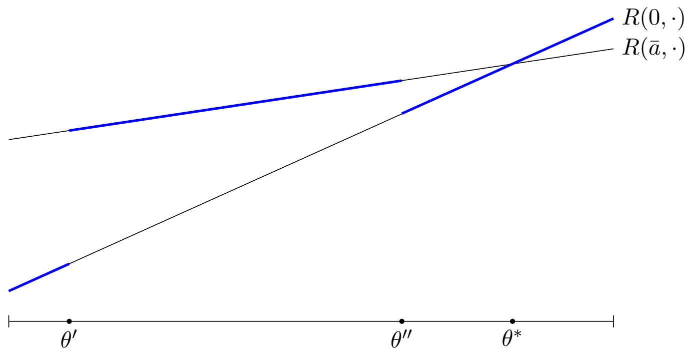


Figure 5: First-best not implementable

The price function for the virtually optimal decision rule is pictured in Figure 6. By virtually optimal, we mean that the outcome can be arbitrarily well approximated by a CWUI decision rule (the approximation is necessary when the induced price is fla). It is characterized by a state $\hat{\theta}$ at which the ironing begins. For any $\hat{\theta} \in [\theta', \theta'']$ the government's payoff is given by

$$\int_{\underline{\theta}}^{\theta'} u(0,\theta)dH(\theta) + \int_{\theta'}^{\hat{\theta}} u(\bar{a},\theta)dH(\theta) + \int_{\hat{\theta}}^{t(\hat{\theta})} u(\alpha(\theta,\hat{\theta}),\theta)dH(\theta) + \int_{t(\hat{\theta})}^{\bar{\theta}} u(0,\theta)dH(\theta),$$

where $\alpha(\theta,\hat{\theta})$ is defined by $R(\alpha(\theta,\hat{\theta}),\theta) = R(\bar{a},\hat{\theta})$, and $t(\hat{\theta})$ by $R(0,t(\hat{\theta}) = R(\bar{a},\hat{\theta})$. Here $\alpha(\theta,\hat{\theta})$ is decreasing in its first argument and increasing in the second, and $t(\hat{\theta})$ is decreasing. Assuming R and u are differentiable, the optimal $\hat{\theta}$ can be identified via the first order condition.

Suppose that in the case of $\theta^* > \theta''$ the government is willing to tolerate some multiplicity, and takes the lexicographic approach described in Section 4. The question is whether or

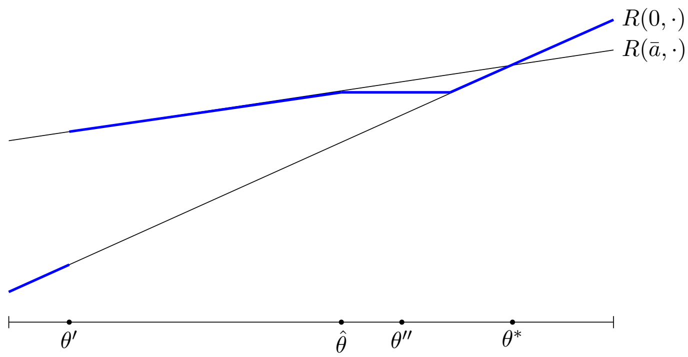


Figure 6: virtually optimal decision rule

not the government can improve their upside while still guaranteeing the payoff given by the virtually optimal decision rule. Assume for simplicity that there is a unique $\hat{\theta}$ that defines the virtually optimal price function (if there are multiple such $\hat{\theta}$ the same analysis applies to any selection). Then, as shown in Section 4, the virtually optimal decision rule is pined down on $P^*(\Theta)$. The only potential changes that could be made to the decision rule when allowing for multiplicity are on $(R(0,\theta'), R(\bar{a},\theta'))$. It is easy to see from Figure 6 however, that changing the decision rule on this range will can only induce equilibria in which lower actions are taken on (θ', θ'') or higher actions are taken on $[\underline{\theta}, \theta')$. Neither of these modifications benefits the principal. Thus relaxing the unique implementation requirement does not change the optimal decision rule.

5.2 Moving against the market

In this section I explore the distinctive features of a set of applications in which the principal would like to induce a *decreasing* price. As before, $\theta \mapsto R(a,\theta)$ is increasing. These are therefore situations in which the principal is working to move prices against the market. The following are two such applications.

Monetary policy in a crisis

During the financial crisis of 2008 and the ongoing Covid-19 recession, central banks have moved aggressively to lower interest rates. In this application the unknown state is the severity of the liquidity crisis faced by firms, and the market price is the interest rate. Central banks' objective is to implement an interest rate that is decreasing in the state via their open market operations. The action is the size of asset purchases.

Grain reserves

Many developing countries manage grain reserves as a tool for stabilizing the grain price and responding to food shortages. The state here is the size of a demand or supply shock, the price is the grain price, and the action is the size of grain purchases/sales. Depending on the nature of the crisis and the structure of the grain market, the government may wish to implement a decreasing price. If the government has limited capacity to make direct transfers to households it may wish to implement transfers by lowering the grain price when there is a severe crisis. For example, suppose that grain is a Giffen good.[18] If there is an employment crisis outside of agriculture the price of grain may rise, absent government intervention.

Throughout this section, we maintain the assumptions that $\mathcal{A} = [\underline{a}, \bar{a}] \in \mathbb{R}$ and that $\theta \mapsto R(a, \theta)$ is strictly increasing for all a, and that $a \mapsto R(a, \theta)$ is strictly decreasing for all θ (that this function is decreasing as opposed to increasing is simply a normalization). A deceasing price function is possible if and only if $R(\underline{a}, \underline{\theta}) > R(\bar{a}, \bar{\theta})$. Figure 7 depicts such an environment.

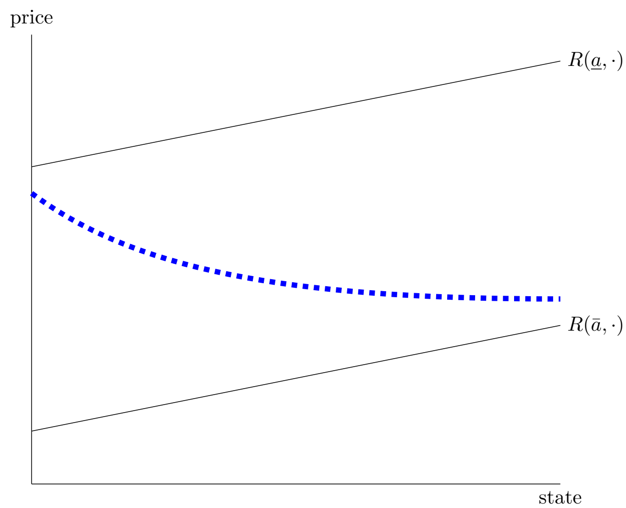


Figure 7: Decreasing price function

The following observation is immediate in this setting.

Lemma 3. *If $a \mapsto R(a, \theta)$ is strictly decreasing for all θ then any strictly decreasing $M \in \mathcal{M}$ induces an increasing price function.*

If $M \in \mathcal{M}$ is decreasing it will therefore be robust to manipulation. Decreasing price functions are more interesting in this setting. Non-monotonicity of M will be necessary to robustly implement a decreasing price.

[18]There is empirical evidence (Jensen and Miller, 2008) that food staples are Giffen goods for extremely poor households.

Lemma 4. *Assume $a \mapsto R(a,\theta)$ is strictly decreasing for all θ, and let P be a decreasing price function. If $M \in \mathcal{M}$ uniquely implements P then*

i. $M(p)$ is decreasing on an open interval containing $(P(\bar{\theta}), P(\underline{\theta}))$,

ii. M has discontinuities in $(P(\underline{\theta}), R(\underline{a}, \underline{\theta})]$ and $(R(\bar{a}, \bar{\theta}), P(\bar{\theta})]$.

iii. There exist $p'' > p' > P(\underline{\theta})$ such that $M(p'') > M(p')$.

iv. There exist $p' < p'' < P(\bar{\theta})$ such that $M(p'') > M(p')$.

Proof. Proof in Appendix A.10. □

Lemma 4 shows that discontinuous and non-monotone M will be necessary to implement a decreasing price. The intuition comes from the fact that there are only two ways to guarantee that $\theta_M(p) = \varnothing$, i.e. that there are no equilibria with price p. Either M must specify an action that is too high, meaning $R(M(p), \bar{\theta}) < p$, or too low, so that $R(M(p), \underline{\theta}) > p$. If neither of these hold then there will be some θ such that $R(M(p), \theta) = p$, by continuity of R. The only way to ensure that there are no equilibria with prices in $[R(\underline{a}, \underline{\theta}), R(\underline{a}, \bar{\theta})]$ is to take a high enough action for such prices; it must be that $R(M(p), \bar{\theta}) < p$ for all such prices. At the same time, M must be decreasing on $(P(\bar{\theta}), P(\underline{\theta}))$ in order to implement a decreasing P. This tension is what necessitates discontinuities and non-monotonicities in M.

Lemma 4 is important in applications because it highlights the danger of artificially restricting the class of permissible decision rules. If, for example, one restricts attention to monotone decision rules, it will not be possible to uniquely implement a decreasing price. It is nonetheless common practice in the literature to focus on monotone, or even linear, decision rules (see for example Bernanke and Woodford (1997)).

6 Extension: multi-dimensional state space

The assumption of a one-dimensional state space is not as restrictive as it may seem. This restriction only enters the analysis through the assumption that $\theta \mapsto R(a,\theta)$ is increasing (strictly or weakly) for all $a \in \mathcal{A}$. This is a purely order-theoretic assumption; it does not directly concern the dimensionality of the state space. We could alternatively allow the state space be an arbitrary subset of a Banach space, but assume that there exists a complete order $\succsim$ on the state space, with respect to which we can assume $\theta \mapsto R(a,\theta)$ is increasing (provided the interval structure is preserved by $\succsim$, i.e. $\theta'' \succsim \theta' \Rightarrow \theta'' \succsim \alpha\theta'' + (1-\alpha)\theta' \geq \theta'$ for all $\alpha \in [0,1]$). In this case the state space can be effectively compressed to a single dimension, and the analysis for the one-dimensional case applies.

In this section we discuss the extension of the results to settings in which it is not possible to immediately map the problem to one with a uni-dimensional state space. We will pay particular attention to the common example of an asset market with aggregate supply shocks, as this model is commonly used in the finance literature on feedback effects.

Suppose that Θ is a compact subset of $\mathbb{R}^N$, endowed with the usual product partial order. Assume that $R : \mathcal{A} \times \Theta \mapsto R$ is continuous, and is increasing with respect to the partial order on Θ, i.e.

Strictly increasing R**.** $\theta'' > \theta'$ implies $R(a, \theta'') > R(a, \theta')$ for all a.[19]

Define $\bar{R}(a, \theta) = \{\theta' : R(a, \theta') = R(a, \theta)\}$. That is, $\bar{R}(a, \theta)$ is the level set of $R(a, \cdot)$ corresponding to the price $R(a, \theta)$. Under the assumptions of continuous and strictly increasing R, $\bar{R}(a, \theta)$ is a one-dimensional curve in Θ for any a, θ. The problem can be reduced to one with a uni-dimensional state space if and only if $\bar{R}(a, \theta) = \bar{R}(a', \theta)$ for all a, a' and θ; if this condition does not hold then there is no complete order on Θ with respect to which R is monotone for any a. Nonetheless, we will be able to characterize CUI in this setting.

Clearly if there is an equilibrium under M in which the principal takes action a in state θ then there is an equilibrium in which the principal takes action a for all states in $\bar{R}(a, \theta)$. Therefore robustness to multiplicity implies that for all θ, $Q(\theta') = Q(\theta)$ for all $\theta' \in \bar{R}(Q(\theta), \theta)$. Robustness to multiplicity also implies that $Q(\theta) \neq Q(\theta') \Rightarrow \bar{R}(Q(\theta), \theta) \cap \bar{R}(Q(\theta'), \theta') = \varnothing$; otherwise there would be multiple equilibrium actions for any states in the intersection. As a result of these two observations, the previous characterizations of CUI action and price functions can be extended without much difficulty. The following result is analogous to Theorem 2 in the uni-dimensional case.

Proposition 7. *Assume strictly increasing R. Then Q is CUI if*

i. $P(\theta) := R(Q(\theta), \theta)$ is strictly monotone (in the product partial order on Θ). Moreover if P is decreasing then local upper monotonicity and local lower monotonicity are satisfied at at the actions and states corresponding to the highest and lowest prices respectively.

ii. Q is continuous.

iii. For all θ, $Q(\theta') = Q(\theta)$ for all $\theta' \in \bar{R}(Q(\theta), \theta)$.

iv. $Q(\theta) \neq Q(\theta') \Rightarrow \bar{R}(Q(\theta), \theta) \cap \bar{R}(Q(\theta'), \theta') = \varnothing$.

These conditions are also necessary, except that it may be possible for Q to have discontinuities at the states associated with the highest prices (see Section 3 for discussion).

[19] This can be easily relaxed to weakly increasing.

Proof. Proof in Appendix A.11.4 □

Proposition 7 continues to hold when θ is unbounded. This is useful to note because some applications, such as the noise REE model of the next section, will make use of an unbounded state space.

6.1 Noisy REE in asset markets

An important setting in which decision making under feedback effects occurs is in asset markets. Since Grossman and Stiglitz (1980) and Hellwig (1980), the noisy rational expectations model has been a workhorse model for studying asymmetric information in asset markets. This model adds shocks to aggregate supply, interpreted as noise or liquidity traders, to rational expectations model of the asset market. The standard approach, without feedback effects, is to assume joint normality of asset returns and aggregate demand shocks, and look for equilibria in which the price is linear in trader's private signals. Breon-Drish (2015) generalizes the noisy REE model to allow for non-normal distributions of states and supply shocks. This section extends results from Breon-Drish (2015) to a setting with feedback effects. First, we show that in a general model, nesting the joint normal model of Grossman and Stiglitz (1980), there exists a function $R : \mathcal{A} \times \Theta \times \mathcal{Z} \mapsto \mathcal{P}$ which summarizes the equilibrium price in the noisy REE model, where $\mathcal{Z}$ is the space of supply shocks. In other words, after extending the state-space to include the supply shock, the model is consistent with our reduced form representation. Second, we use this result and Proposition 7 to characterize the set of CUI action and price functions when the asset price is determined in a noisy REE model.

To show that the desired market-clearing function $R : \mathcal{A} \times \Theta \times \mathcal{Z} \mapsto \mathcal{P}$ exists, we apply the results of Breon-Drish (2015) to an environment with feedback effects. The key is to show that we can identify the information revealed by the price separately from the decision rule. There are are a continuum of investors $i \in [0, 1]$, each with CARA utility $u(w) = -\exp\left\{-\frac{1}{\tau_i} w\right\}$. The ex-post payoff to an investor who purchases x units of the asset at price p when the principal takes action a is given by $-\exp\left\{-\frac{1}{\tau_i} x(\pi(a, \theta) - p)\right\}$.

Assume that the asset value (i.e. the ex-post payoff from holding the asset) π is affine in θ for all a; $\pi(a, \theta) = \beta_0^a + \beta_1^a \theta$. Each investor observes a additive signal $\sigma_i = \theta + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma_i^2)$. Finally, assume that z has a truncated normal distribution. That is, z is the restriction of a normal random variable $\hat{z} \sim N(0, \sigma_Z^2)$ to the interval $[b_1, b_2]$, with $-\infty \le b_1 \le 0 \le b_2 \le \infty$ (note that this assumption accommodates un-truncated supply shocks as well). For simplicity, let $b_1 = -b_2$; this does not affect the results.

Given M, say that an equilibrium characterized by $P : \Theta \times \mathcal{A} \mapsto \mathcal{P}$ is a *generalized linear* *equilibrium* if there exists function $L_M : \Theta \times \mathcal{Z} \times \mathcal{A} \mapsto \mathbb{R}$ such that $(\theta, z) \mapsto L_M(\theta, z|a)$ is linear for all a, and $M(P(\theta, z)) = a$ implies $P(\theta', z') = P(\theta, z)$ for all θ', z' such that $L_M(\theta', z'|a) = L_M(\theta, z|a)$. Given the monotonicity of "truth plus noise" signals, if $(\theta'', z'') > (\theta', z')$ then it cannot be that $P(\theta'', z'') = P(\theta', z')$ in any equilibrium. Thus in any generalized linear equilibrium it will also be the case that $P(\theta, z) = P(\theta', z')$ implies $L_M(\theta, z|M(P(\theta, z))) = L_M(\theta', z'|M(P(\theta', z')))$. In other words, a generalized linear equilibrium is one in which the price reveals exactly that the extended state lies on a line segment in $\Theta \times \mathcal{Z}$. We refer to L_M as the equilibrium statistic. The following is an extension of Breon-Drish (2015) Proposition 2.1 (Online Appendix) to a setting with feedback effects.

Proposition 8. *Assume CARA utility, π affine in θ, additive normal signal structure and truncated-normally distributed supply shocks. Then there exists a unique (up to positive transformations) function $L^* : \Theta \times \mathcal{Z} \times \mathcal{A} \mapsto \mathbb{R}$ such that for any M and any generalized linear equilibrium given M, the equilibrium statistic is L^*, given by*

$$L^*(\theta, z|a) = \left(\frac{1}{\beta_1^a} \int_i \frac{\tau_i}{\sigma_i^2} di \right) \cdot \theta - z \tag{1}$$

Proof. Proof in Appendix A.11.5. $\square$

From (1) we can see how the principal's action affects information aggregation; the higher is β_1^a, i.e. the more sensitive the asset value is to the state, the smaller the coefficient on θ in the equilibrium statistic. As a result, the price is less informative about the state. This is because when β_1^a is high, each trader's private signal is less informative about the asset value. As a result, traders place less weight on their private information relative to the information revealed by the price. The proof of Proposition 8 also yields an expression for $R(a, \theta, z)$, although for the current purposes it is sufficient to note simply that such a function exists.

Corollary 4. *Assume CARA utility, π affine in θ, additive normal signal structure and truncated-normally distributed supply shocks. Restrict attention to generalized linear equilibria. Then there exists a unique function $R : \mathcal{A} \times \Theta \times \mathcal{Z} \mapsto \mathcal{P}$ such that in any equilibrium in which action a is taken in extended state (θ, z), the price is $R(a, \theta, z)$.*

There are three differences between the environment of Proposition 8 and that of Breon-Drish (2015) Proposition 2.1. First, the single σ_i observed by each investor is given by the state plus noise, as opposed to the asset return plus noise as in Breon-Drish (2015). This is immediately handled by a suitable change of variables, given the assumption that $\theta \mapsto \pi(a, \theta)$ is affine for all a. Second, we allow here for the supply shock to follow a truncated normal distribution, where Breon-Drish (2015) considers only the un-truncated distribution. This requires generalizing Breon-Drish (2015) Proposition 2.1, which is relatively straightforward. Finally, and most importantly, the current setting features a feedback effect, whereas asset returns follow a fixed distribution in Breon-Drish (2015). We show how the results for the fixed-action case imply the desired result when there is feedback.

Note that the restriction to generalized linear equilibria is not in general without loss. Breon-Drish (2015) shows that this is without loss if we assume that the equilibrium price function is continuous. Nonetheless, restricting attention to generalized linear equilibria is not incompatible with the focus on unique implementation in this paper. There is a separation between multiplicity that may arise because of the principal's decision rule and multiplicity that could arise even if the principal's action was fixed, i.e. in the absence of feedback effects. The restriction to generalized linear equilibria can be interpreted as a behavioral assumption: investors make their inferences by running generalized linear regressions. Indeed, it is standard on the literature on noisy REE to restrict attention linear equilibria. The focus of this paper is on multiplicity that arises because of the feedback effect, rather than multiplicity that is a fundamental part of the trading game.

Given Corollary 4, we can apply Proposition Proposition 7 to the noisy REE setting. The problem of finding optimal policies is generally complicated by the additional restrictions *iii* and *iv*, relative to the uni-dimensional case. In some cases however, these constraints simplify the problem. For example, if the noise distribution is unbounded, these conditions and the expression for L^* in (1) have the following implication.

Lemma 5. *In the noise REE model with normally distributed supply shocks, any CUI action function must be such that* $\beta_1^{Q(\theta,z)} = \beta_1^{Q(\theta',z')}$ *for all* $(\theta, z), (\theta', z') \in \Theta \times \mathcal{Z}$.

In other words, Lemma 5 says that any CUI action function can only use actions for which the slope of the asset payoff in the state is the same. This will not be true when the supply shocks are bounded; in this case additional action functions will be CUI.

Appendix

A Proofs of Section 3

A.1 Preliminary results

It will be useful to establish some properties of θ. Let $\bar{P}$ be the set of p for which $\theta_M(p) \neq \varnothing$. Corollary 5 below implies that for continuous M, the set of p for which $\theta_M(p) = \varnothing$ is open.

Lemma 6. *If M is continuous then $p \mapsto \theta_M(p)$ is compact-valued, and it is upper hemicontinuous on $\bar{P}$.*

Proof. Compact valued is easy: if $R(M(p), \theta) - p \neq 0$ then by continuity of R this holds for all θ' in a neighborhood of θ.

Now upper hemicontinuity. Let V be an open neighborhood of $\theta_M(p)$. Then $\Theta \setminus V$ is compact, so there exists $\kappa > 0$ such that $R(M(p), \theta) - p > \kappa$ for all $\theta \in \Theta \setminus V$. Then by continuity of R, M there exists an open neighborhood U of p such that $R(M(p'), \theta) - p' > \kappa$, and thus $\theta_M(p') \in V$, for all $p' \in U \cap \bar{P}$. Thus $p \mapsto \theta_M(p)$ is upper hemicontinuous. □

Lemma 7. *If R is weakly increasing in θ then $\theta_M(p)$ is convex valued.*

Proof. $\theta_M(p) = \{\theta \in \Theta : R(M(p), \theta) = p\}$. If $R(M(p), \cdot)$ is monotone, $R(M(p), \theta') = R(M(p), \theta'') = p$ implies $R(M(p), \theta) = p$ for all $\theta \in (\theta', \theta'')$. □

A.2 Proof of Theorem 1

Proof. We first prove the result for continuous M, and then extend it to all of $\mathcal{M}$. Assume without loss of generality that that $R(a, \cdot)$ is increasing for all a. Lemmas 10 and 11 and Corollary 5 apply.

Let $\theta_1 < \theta_2 < \theta_3$ be interior, and suppose $P(\theta_1) > P(\theta_2)$ and $P(\theta_3) > P(\theta_2)$ (the other type of non-monotonicity is dealt with symmetrically). We first want to show that $[P(\theta_2), \max\{P(\theta_1), P(\theta_3)\}] \subseteq \bar{P}$. Suppose not, so there is some $p' \in [P(\theta_2), \max\{P(\theta_1), P(\theta_3)\}]$ such that $\theta(p) = \varnothing$. Then p' is either Type L or Type H. Assume it is Type H (symmetric argument using Lemma 10 if it is Type L). First, assume $p' \in [P(\theta_2), \min\{P(\theta_1), P(\theta_3)\}]$. By Lemma 11, part (i) there is a $p > p'$ such that $\theta_2 \in \theta(p)$. Thus there is multiplicity in state θ_2. Then by continuity of M and R, and R weakly increasing in θ, there is multiplicity for all θ in $[\theta_2, \theta_2 + \varepsilon)$ and/or $[\theta_2, \theta_2 - \varepsilon)$ for some $\varepsilon > 0$, violating multiplicity. Thus $[P(\theta_2), \min\{P(\theta_1), P(\theta_3)\}] \subseteq \bar{P}$. Suppose instead that $p' \in [\min\{P(\theta_1), P(\theta_3)\}, \max\{P(\theta_1), P(\theta_3)\}]$.

The by Lemma 5, either part (i) or part (ii), there is multiplicity in one of θ_1, θ_3. Then there is multiplicity on a positive measure set, since these are interior.

Assume that $P(\theta_3) \geq P(\theta_1)$ (symmetric argument for reverse inequality). Suppose there exists $\theta' \geq \theta_2$ such that $\theta' \in \theta(P(\theta_1))$. Note that R weakly increasing in θ implies that $\{\theta \in \Theta : R(a, \theta) = p\}$ is convex for all a, so $\theta_2 \in \theta(P(\theta_1))$. Thus if such a θ' exists there will be multiplicity in state θ_2, and, by the same argument as above, there will be multiplicity for a positive measure of states.

It remains to show that the existence of such a θ' is implied by our assumptions. Suppose instead that $\theta(P(\theta_1)) \subseteq [\underline{\theta}, \theta_2)$. We will show that this implies that there exists $p > P(\theta_1)$ such that $\theta_2 \in \theta(p')$, so there is multiplicity in θ_2, and as before this will imply multiplicity for a positive measure of states. Suppose no such p' exists. Let $\tilde{p} = \sup\{p \in [P(\theta_1), P(\theta_3)] : \max \theta(p) < \theta_2\}$, which is well defined by Lemma 6. Since $\theta(\tilde{p})$ is convex, the assumption that no such p' exists implies that either $\max \theta(\tilde{p}) < \theta_2$ or $\min \theta(\tilde{p}) > \theta_2$ Then we have a violation of upper hemicontinuity at $\tilde{p}$. Thus there exists $p > P(\theta_1)$ such that $\theta_2 \in \theta(p')$, as desired.

Now, for the extreme states we want to see that the monotonicity is maintained. Let θ be interior, then $p(\bar{\theta}) = R(M(p(\bar{\theta})), \bar{\theta}) > R(M(p(\bar{\theta})), \theta)$ and $R(M(p(\bar{\theta})), \theta) > \underline{p} := \inf\{\bar{P}\}$. So there is an equilibrium price $\tilde{p} \in (p(\bar{\theta}), \underline{p})$. If $p(\bar{\theta}) > \bar{p} := \sup\{\bar{P}\}$, then the previous does not imply multiplicity, but if $p(\bar{\theta}) < \bar{p}$, it does for all θ with associated prices in $(p(\bar{\theta}), \bar{p})$. The same argument holds to prove that $p(\underline{\theta}) \leq \underline{p}$.

It remains to show that the result holds for discontinuous $M \in \mathcal{M}$. For P to be non-monotone without violating robustness to multiplicity it must be that P is discontinuous. Suppose P is discontinuous at θ, and assume without loss of generality that P is decreasing below θ and left-continuous at θ. Let $\theta' > \theta$ be such that $P(\theta') > P(\theta)$. Then it must be that M is discontinuous on $(P(\theta), P(\theta'))$, otherwise the argument above for continuous M would apply. Let $\underline{p} = \inf\{p \in (P(\theta), P(\theta')) : M \text{ is discontinuous at } p\}$. Since M is continuous on $(P(\theta), \underline{p})$, the argument for continuous M implies that $\theta_M(p) = \theta$ for all $p \in (P(\theta), \underline{p})$. But then, by the definition of $\mathcal{M}$, M must be continuous in a neighborhood of $\underline{p}$, which is a contradiction. □

A.3 Theorem 2

Proof. First, note that under the assumption that R is strictly increasing in θ, Theorem 1 implies that P must be *strictly* monotone; otherwise measurability would be violated. Given this, to show necessity of i we first show that continuity of Q is necessary for CUI.

Suppose first that Q is discontinuous at an interior state θ'. If $P(\theta) := R(Q(\theta), \theta)$ is not also discontinuous at θ' then there can be no $M \in \mathcal{M}$ that implements Q. Assume P

is discontinuous at θ. Suppose without loss of generality that P is increasing. Under strict monotonicity of $\theta \mapsto R(a, \theta)$, we have $|\theta_M(p)| \leq 1$ for all p.

Assume first that M is continuous. Thus Lemma 8 implies that $(\lim_{\theta \nearrow \theta'} P(\theta), \lim_{\theta \searrow \theta'} P(\theta)) \subseteq \bar{P}$. If $\theta_M(p) \neq \theta'$ for some $p \in (\lim_{\theta \nearrow \theta'} P(\theta), \lim_{\theta \searrow \theta'} P(\theta))$ then there will by multiplicity in some state $\theta'' \neq \theta'$ by Lemma 12. But if $\theta_M(p) = \theta'$ for some $p \in (\lim_{\theta \nearrow \theta'} P(\theta), \lim_{\theta \searrow \theta'} P(\theta))$ then there is multiplicity in state θ'.

If M is discontinuous it must still be continuous in a neighborhood N of $P(\theta')$. Then the same argument implies that $\theta_M(p) = \theta'$ for all $p \in N \cap (\lim_{\theta \nearrow \theta'} P(\theta), \lim_{\theta \searrow \theta'} P(\theta))$.

Suppose that P is decreasing; we want to show that local upper monotonicity is satisfied at $(Q(\underline{\theta}, \underline{\theta})$ and local lower monotonicity is satisfied at $(Q(\bar{\theta}), \bar{\theta})$. We prove the former here, the latter is symmetric. Suppose local upper monotonicity is not satisfied at $(Q(\underline{\theta}, \underline{\theta})$. Since M must be continuous in a neighborhood of $P(\underline{\theta})$ (by definition of $\tilde{M}$) the violation of local upper monotonicity implies that for any $\varepsilon > 0$ there exists a $p \in (P(\underline{\theta}, P(\underline{\theta}) + \varepsilon)$ such that $R(M(p), \underline{\theta}) \leq p$. Since $\theta \mapsto R(a, \theta)$ is strictly increasing and R is continuous, for ε small enough we will also have $R(M(p), \bar{\theta}) > p$. But then by continuity of R there exists θ such that $R(M(p), \theta) = p$. Since $p > P(\underline{\theta})$ this implies that there is multiplicity at θ.

Now for sufficiency of *ii*. The argument immediately preceding Theorem 2 implies that continuity of Q and strict monotonicity of P are sufficient to rule out multiplicity involving prices in $P(\Theta)$. If P is increasing then define $M(p) = M(P(\bar{\theta}))$ for all $p > P(\bar{\theta})$ and $M(p) = M(P(\underline{\theta}))$ for all $p < P(\underline{\theta}))$. Then there can be no equilibria involving prices in $\mathcal{P} \setminus P(\Theta)$. When P is decreasing let m, ε satisfy the conditions of local upper monotonicity at $Q((\underline{\theta}), \theta)$. Then defining $M = m$ on $(P(\underline{\theta}), P(\underline{\theta}) + \varepsilon)$ guarantees that there is no equilibrium at any such prices. Therefore $M \in \mathcal{M}$ can be discontinuous at $P(\underline{\theta}) + \varepsilon$. Defining $M(p) = Q(\bar{\theta})$ for all $p > P(\underline{\theta}) + \varepsilon$ guarantees that there can be no equilibria involving prices above $P(\underline{\theta})$. A symmetric construction is used to guarantee that there are no equilibrium prices below $P(\bar{\theta})$. $\square$

A.4 Theorem 3

Proof. First for necessity. Theorem 1 implies that P must be weakly monotone. If it is not strictly monotone then it will violate measurability, given that $R(a, \cdot)$ is strictly monotone. The necessity of the local upper/lower monotonicity conditions follows from the same argument as Theorem 2. This proves necessity of *i*.

To show necessity of *ii*., suppose Q has a discontinuity at an interior state θ' that is not bridgeable. Given that we have established *i*, assume without loss or generality that P is strictly increasing and left continuous. We first show that M must be continuous on

$(P(\theta'), \lim_{\theta \searrow \theta'} P(\theta))$. Suppose not, and let $\underline{p} = \inf\{p \geq P(\theta') : M \text{ is discontinuous at } p\}$. By definition of $\mathcal{M}$, it must be that $\underline{p} > P(\theta')$. Under strict monotonicity of $\theta \mapsto R(a, \theta)$, we have $|\theta_M(p)| \leq 1$. If $\theta_M(p) \neq \theta'$ for some $p \in (P(\theta'), \underline{p})$ then there will by multiplicity by Lemma 12. But then, by definition of $\mathcal{M}$, M must be continuous on a neighborhood of $\underline{p}$, which contradicts the definition of $\underline{p}$.

We have established that $(\lim_{\theta \nearrow \theta'} P(\theta), \lim_{\theta \searrow \theta'} P(\theta))$. If the discontinuity at θ' is not bridgeable then there is no continuous M such that the following three conditions hold: *a*) $\lim_{\theta \nearrow \theta'} M(P(\theta)) = \lim_{\theta \nearrow \theta'} Q(\theta)$, *b*) $\lim_{\theta \searrow \theta'} M(P(\theta)) = \lim_{\theta \searrow \theta'} Q(\theta)$ and, *c*) $\theta_M(p) = \theta'$ for all $p \in (\lim_{\theta \nearrow \theta'} P(\theta), \lim_{\theta \searrow \theta'} P(\theta))$. To see this, notice that any such M would constitute a monotone path from $\lim_{\theta \nearrow \theta'} Q(\theta)$ to $\lim_{\theta \searrow \theta'} Q(\theta)$.

Now for necessity of *iii* and *iv*. Note that this conditions will be satisfied if the discontinuities are bridgeable. If they are not bridgeable then the argument for necessity of the local upper/lower monotonicity conditions is the same as that given for Theorem 2.

Now for sufficiency. Assume without loss that P is strictly increasing. Define $M(p) = Q(P^{-1}(p))$, which is well defined on $P(\Theta)$ by *i*. Moreover M is continuous on $P(\Theta)$ under *ii*. Q is continuous at any interior state θ at which P is continuous. It remains to define M on $\mathcal{P} \setminus P(\Theta)$. This is done as in Theorem 2. $\square$

A.5 Corollary 3

Proof. First necessity. If Q is not strictly monotone then for ε small enough there will be no ε-approximation that is strictly monotone. Thus by Theorem 1 there are no CWUI ε-approximations. Suppose *ii* is violated. Since P is strictly monotone it can have at most countably many discontinuities. Thus Q must have a positive measure δ of degenerate discontinuities. If $\varepsilon < \delta$ then for any ε-approximation of Q there will be a degenerate discontinuity that is outside of the set of states for which $Q' \neq Q$. But then Q' has a degenerate discontinuity, and so is not CWUI.

Now for sufficiency. Given Corollary 2 we need only show that Q can be approximated around all degenerate discontinuities. This follows immediately from the definition of correctable. $\square$

A.6 Proposition 1

Proof. Corollary 2 implies that P is strictly monotone and that whenever Q is discontinuous so is P. Thus P (and Q) can have at most countably many discontinuities. The proposition will follow if we can show that for any $\varepsilon > 0$ and any θ^* at which Q is discontinuous, we can continuously approximate Q around θ^* without changing Q outside of $(\theta^* - \varepsilon, \theta^* + \varepsilon)$.

Since R is continuous and P is discontinuous at θ^*, there exists $\delta < \varepsilon$ and $\theta' \in (\theta^*, \theta^* + \delta)$ such that $Q(\theta^*), Q(\theta')$ and δ satisfy the conditions of continuous bridgeability. Thus there exists a continuous Q' on $[\theta^*, \theta^* + \delta]$ such that $Q'(\theta^*) = \lim_{\theta \nearrow \theta^*} Q(\theta)$, $Q'(\theta^* + \delta) = Q(\theta^* + \delta)$, and $R(Q'(\theta), \theta)$ is strictly increasing on $[\theta^*, \theta^* + \delta]$. Since ε was arbitrary, this gives the desired approximation. □

A.7 Lemma 8

Lemma 8. *Assume R is weakly increasing in θ. For any continuous M that is robust to multiplicity, let p_1, p_2 be prices such that $\theta_M(p_1)$ and $\theta_M(p_2)$ are contained in the interior of θ. Then*

$$[\min\{p_1, p_2\}, \max\{p_1, p_2\}] \in \bar{P}.$$

Proof. By Theorem 1, the price function P is monotone, so without loss of generality assume that it is increasing, and let $p_2 > p_1$. Assume towards a contradiction that there exists $p \in (p_1, p_2)$ such that $\theta_M(p) = \varnothing$. By Lemma 9 p is either type H or type L. Suppose it is type L, i.e. $R(M(p), \underline{\theta}) - p > 0$. Since $\theta_M(p_1) \neq \varnothing$, it must be that $R(M(p_1), \underline{\theta}) - p_1 \leq 0$. Moreover, since $\underline{\theta} \notin \theta_M(p_1)$ by assumption, the inequality is strict: $R(M(p_1), \underline{\theta}) - p_1 < 0$. Then by continuity there exists $p' \in (p_1, p)$ such that $R(M(p'), \underline{\theta}) - p' = 0$. Let $\theta_1 = \min \theta_M(p_1)$, which exists by Lemma 6 (by assumption $\theta_1 > \underline{\theta}$). Since P is increasing, $p' > p_1 > P(\theta)$ for all $\theta \in [\underline{\theta}, \theta_1)$. Then by Lemma 12 there is multiplicity for all states in $\theta \in [\underline{\theta}, \theta_1)$, which is a contradiction. If p is type H then the proof is symmetric, using p_2 rather than p_1. □

A.8 Proof of Proposition 2

Proof. This essentially follows from Theorem 3. The only modifications are the following. Condition *iv* is clearly necessary and sufficient to for there to be no monotonicity involving actions in $Q(\Theta)$. The modification of *ii* from bridgeable to properly bridgeable is necessary and sufficient for there to be no multiplicity involving actions not in $Q(\Theta)$. There is no need to modify condition *iii* since it guarantees existence of that are all type H (at $\bar{\theta}$) or type L (at $\underline{\theta}$), and thus involve no multiplicity.

□

A.9 Implementable price functions

In some cases the principal may not care directly about the actions they take, only about the price that they induce. In this section we ask the following question: for which price functions there exists an action function such that (Q, P) is CWUI. We call such a P CWUI.

Definition. *A price function* $P : \Theta \to \mathbb{R}$ *is in range if for each* $\theta \in \Theta$, $P(\theta) \in R(\mathcal{A}, \theta)$.

Proposition 9. *Under strict monotonicity of* R *in* θ*, a price function is implementable if and only if it is in range and an injection.*

Proof. in the Appendix A.9.1 □

We will call intersection states the ones where there is at least two different actions a_1, a_2 with $R(a_1, \theta) = R(a_2, \theta)$. Let Θ_I be the set of such states. We will make the following extra assumptions on R:

Mixture continuity. For any θ, any a'', a' such that $R(a'', \theta) > R(a', \theta)$, and any $p \in [R(a', \theta), R(a'', \theta)]$, there is a unique α such that $R(\alpha a' + (1-\alpha)a'', \theta) = p$.

Isolated intersections. For every $\theta \in \Theta_I$, there exists an $\epsilon > 0$ such that $B_\epsilon(\theta) \cap \Theta_I = \{\theta\}$.

Intersection smoothness. $R_2(a, \theta)$ exists for every intersection state θ and a that puts weight only on intersecting actions for that state.

Definition. *A price function* $P : \Theta \to \mathbb{R}$ *satisfies the* ***kink's condition*** *iff there exist* C^1 *functions* $\bar{P}$ *and* $\underline{P}$ *in range and such that* $\bar{P}(\theta) \geq P(\theta) \geq \underline{P}(\theta)$.

The kink's condition effectively means that every kink of P in the upper envelope of $R(a, \theta)$ is concave, and every kink in the lower envelope is convex. Moreover, the kink's condition implies that if there is a θ such that $R(\mathcal{A}, \theta)$ is a singleton, the price function has to be differentiable at θ.

Proposition 10. *Under strict monotonicity of* R *in* θ*, mixture continuity, isolated intersections, and intersection smoothness, a price function is CWUI if and only if it is in range, strictly monotone, and satisfies the kink's condition.*

Proof. in the Appendix A.9.2. □

There are two primary components of the proof of Proposition 10. The first is that $\bar{P}$ is convex. The second is to show that if $\theta(p)$ is non-monotone then there will be multiplicity. Identification is used to prove both parts of the proposition, but it is not necessary for either. One simple relaxation under which the result is preserved is to allow for actions with constant payoffs.

Weak identification. R is weakly increasing in θ. Moreover, if $R(a, \cdot)$ is not strictly increasing then it is constant.

Proposition 11. *Under weak identification, a pair (Q, P) is implementable if and only if $Q(P^{-1}(p))$ is a singleton for all $p \in P(\Theta)$.*

Proposition 12. *Under R weakly increasing in θ, a price function is CWUI if and only if it is in range and weakly monotone and whenever it is flat at a price p, it is so for the whole set $\theta_M(p)$.*

Proof. in the Appendix A.9.3. $\square$

A.9.1 Proof of Proposition 9

Proof. ($\Rightarrow$): suppose not an injection. There are θ and θ' with $P(\theta) = P(\theta')$. By identification, $R(Q(\theta), \theta) \neq R(Q(\theta'), \theta')$, which by rational expectations means that $P(\theta) \neq P(\theta')$, a contradiction. If not in range, then there exist a $\theta \in \Theta$ such that $P(\theta) \notin R(\mathcal{A}, \theta)$, i.e. there is no $a \in \mathcal{A}$ such that $R(a, \theta) = P(\theta)$, so $R(Q(\theta), \theta) \neq P(\theta)$, violating rational expectations.

($\Leftarrow$): Since $P(\theta)$ is in range, for each $\theta \in \Theta$ there exists a a with $R(a, \theta) = P(\theta)$. let's define $Q(\theta)$ by a selection in the rational expectations condition: $R(Q(\theta), \theta) = P(\theta)$. Measurability is satisfied trivially since $P(\theta) \neq P(\theta')$ for all $\theta \neq \theta'$. $\square$

A.9.2 Proof of Proposition 10

Proof. ($\Rightarrow$): Take M that implements P. For all $p \in R(\mathcal{A}, \Theta)$ there is at most a unique $\theta \in \Theta$ that satisfies $R(M(p), \theta) = p$. Otherwise identification would be violated. This defines a function $\theta(p)$.

Let $\bar{P}$ be the set of all prices for which there is an interior solution. We want to show that $\bar{P}$ is convex. Pick $p, p' \in \bar{P}$ and $\alpha \in (0, 1)$ we want to see that $p_\alpha := \alpha p + (1-\alpha)p' \in \bar{P}$. Let θ and θ' the associated states of p and p'. Continuity of R plus identification imply strict monotonicity of R in θ and for all a. Assume without loss that $\theta' > \theta$.

We will prove that

$$R(M(p_\alpha), \theta) \leq p_\alpha \leq R(M(p_\alpha), \theta') \tag{2}$$

Consider a violation of the second inequality. If $p_\alpha > R(M(p_\alpha), \theta')$ notice that also, $p = R(M(p), \theta) < R(M(p), \theta')$. Therefore, we have

$$p_\alpha - R(M(p_\alpha), \theta') > 0 \qquad \text{and} \qquad p - R(M(p), \theta') < 0$$

By continuity and since $\theta' \in \Theta^o$, there exists an $\bar{\varepsilon} > 0$ such that for all $\tilde{\theta} \in B_{\bar{\varepsilon}}(\theta')$

$$p_\alpha - R(M(p_\alpha), \tilde{\theta}) > 0 \qquad \text{and} \qquad p - R(M(p), \tilde{\theta}) < 0$$

By continuity there is a $p_1 \in (p_\alpha, p)$ with $p_1 - R(M(p_1), \theta') = 0$. But for $0 < \epsilon < \bar{\varepsilon}$ we have $p' = R(M(p'), \theta') < R(M(p'), \theta' - \epsilon)$. There exists a $p_2 \in (p_\alpha, p')$ such that $R(M(p_2), \theta' - \epsilon) = p_2$. $p_1 \neq p_2$, so there is multiplicity in a set of states $[\theta', \theta' + \epsilon)$. With a similar logic we can rule out $p_\alpha \leq R(M(p_\alpha), \theta)$.

Finally, by continuity of π in θ and using Equation (2), there is a $\hat{\theta}$ in (θ, θ') such that $p_\alpha - R(M(p_\alpha), \hat{\theta}) = 0$, therefore $p_\alpha \in \bar{P}$.

The function $\theta(p)$ is continuous in the set $\bar{P}$. However we could have discontinuities for the two prices that are associated with the extreme states $\bar{\theta}$ and $\underline{\theta}$.

We show now that $\theta(p)$ is monotone in $\bar{P}$. Suppose that is not, i.e. there are prices $p_l < p_m < p_h$ such that either $\theta(p_m) < \min\{\theta(p_l), \theta(p_h)\}$ or $\theta(p_m) > \max\{\theta(p_l), \theta(p_h)\}$. Suppose the first (the symmetric argument holds for the other case). Then for all $\theta \in (\theta(p_m), \min\{\theta(p_l), \theta(p_h)\})$ and by continuity there are prices $p^1_\theta \in (p_l, p_m)$ and $p^2_\theta \in (p_m, p_h)$ with $\theta(p^1_\theta) = \theta(p^2_\theta)$. This violates multiplicity.

We can invert $\theta(p)$ in $\bar{P}$. The only problem is when $\theta(p)$ is flat, but any selection would give us that the inverse is strictly monotone.

Now, for the extreme states we want to see that the monotonicity is maintained. Let θ be interior, then $p(\bar{\theta}) = R(M(p(\bar{\theta})), \bar{\theta}) > R(M(p(\bar{\theta})), \theta)$ and $R(M(p(\bar{\theta})), \theta) > \underline{p} := \inf\{\bar{P}\}$. So there is an equilibrium price $\tilde{p} \in (p(\bar{\theta}), \underline{p})$. If $p(\bar{\theta}) > \bar{p} := \sup\{\bar{P}\}$, then the previous does not imply multiplicity, but if $p(\bar{\theta}) < \bar{p}$ It does for all θ with associated prices in $(p(\bar{\theta}), \bar{p})$. The same argument holds to prove that $p(\underline{\theta}) \leq \underline{p}$.

($\Leftarrow$): P is strictly monotone and bounded so there is a countable number of discontinuities. Fill those to get a continuous and monotone $\theta(p) := \sup\{\theta : P(\theta) < p\}$.

Let $\bar{M} : P(\Theta) \rightrightarrows \mathcal{A}$ be the set of actions that give price p at the corresponding state i.e. $a \in \bar{M}(p)$ if and only if $R(a, \theta(p)) = p$.

If p is not an intersection price, then $\bar{M}$ is LHC at p. Therefore in a ball around p there is a continuous selection. If p is a interior intersection, then we can consider the set of actions that are not involved in the intersection and select a continuous M. $\square$

A.9.3 Proof of Proposition 12

Proof. Assume without loss of generality that that $R(a, \cdot)$ is increasing for all a. Lemmas 10 and 11 and Corollary 5 apply.

Let $\theta_1 < \theta_2 < \theta_3$ be interior, and suppose $P(\theta_1) > P(\theta_2)$ and $P(\theta_3) > P(\theta_2)$ (the other type of non-monotonicity is dealt with symmetrically). We first want to show that $[P(\theta_2), \max\{P(\theta_1), P(\theta_3)\}] \subseteq \bar{P}$. Suppose not, so there is some $p' \in [P(\theta_2), \max\{P(\theta_1, P(\theta_3)\}]$

such that $\theta(p) = \varnothing$. Then p' is either Type L or Type H. Assume it is Type H (symmetric argument using Lemma 10 if it is Type L). First, assume $p' \in [P(\theta_2), \min\{P(\theta_1, P(\theta_3)\}]$. By Lemma 11, part (i) there is a $p > p'$ such that $\theta_2 \in \theta(p')$. Thus there is multiplicity in state θ_2. Then by continuity of M and R, and R weakly increasing in θ, there is multiplicity for all θ in $[\theta_2, \theta_2 + \varepsilon)$ and/or $[\theta_2, \theta_2 - \varepsilon)$ for some $\varepsilon > 0$, violating multiplicity. Thus $[P(\theta_2), \min\{P(\theta_1, P(\theta_3)\}] \subseteq \bar{P}$. Suppose instead that $p' \in [\min\{P(\theta_1, P(\theta_3), \max\{P(\theta_1, P(\theta_3)\}]$. The by Lemma 5, either part (i) or part *(ii)*, there is multiplicity in one of θ_1, θ_3. Then there is multiplicity on a positive measure set, since these are interior.

Assume that $P(\theta_3) \geq P(\theta_1)$ (symmetric argument for reverse inequality). Suppose there exists $\theta' \geq \theta_2$ such that $\theta' \in \theta(P(\theta_1))$. Note that R weakly increasing in θ implies that $\{\theta \in \Theta : R(a, \theta) = p\}$ is convex for all a, so $\theta_2 \in \theta(P(\theta_1))$. Thus if such a θ' exists there will be multiplicity in state θ_2, and, by the same argument as above, there will be multiplicity for a positive measure of states.

It remains to show that the existence of such a θ' is implied by our assumptions. Suppose instead that $\theta(P(\theta_1)) \subseteq [\underline{\theta}, \theta_2)$. We will show that this implies that there exists $p > P(\theta_1)$ such that $\theta_2 \in \theta(p')$, so there is multiplicity in θ_2, and as before this will imply multiplicity for a positive measure of states. Suppose no such p' exists. Let $\tilde{p} = \sup\{p \in [P(\theta_1), P(\theta_3)] : \max \theta(p) < \theta_2\}$, which is well defined by Lemma 6. Since $\theta(\tilde{p})$ is convex, the assumption that no such p' exists implies that either $\max \theta(\tilde{p}) < \theta_2$ or $\min \theta(\tilde{p}) > \theta_2$ Then we have a violation of upper hemicontinuity at $\tilde{p}$. Thus there exists $p > P(\theta_1)$ such that $\theta_2 \in \theta(p')$, as desired.

Now, for the extreme states we want to see that the monotonicity is maintained. Let θ be interior, then $p(\bar{\theta}) = R(M(p(\bar{\theta})), \bar{\theta}) > R(M(p(\bar{\theta})), \theta)$ and $R(M(p(\bar{\theta})), \theta) > \underline{p} := \inf\{\bar{P}\}$. So there is an equilibrium price $\tilde{p} \in (p(\bar{\theta}), \underline{p})$. If $p(\bar{\theta}) > \bar{p} := \sup\{\bar{P}\}$, then the previous does not imply multiplicity, but if $p(\bar{\theta}) < \bar{p}$, it does for all θ with associated prices in $(p(\bar{\theta}), \bar{p})$. The same argument holds to prove that $p(\underline{\theta}) \leq \underline{p}$. $\square$

Strictly monotone price functions may have jump discontinuities. However such price functions can always be approximated arbitrarily well by continuous and strictly increasing functions.

The principal might be willing to accept multiplicity if all the equilibria induced by a mechanism are good. In particular, it is reasonable to assume that if all equilibria induced by a given mechanism M are better then the best CWUI equilibrium then the principal will prefer M to any CWUI. We first need some intermediate results.

Lemma 9. *Fix a continuous M. Assume $R(a, \cdot)$ is (weakly) increasing for all a (the same holds if decreasing, with $\underline{\theta}$ and $\bar{\theta}$ switched). Then each p such that $\theta(p) = \varnothing$ is of one and* *only one of the following two types:*

- *Type L:* $R(M(p), \underline{\theta}) > p$.
- *Type H:* $R(M(p), \bar{\theta}) < p$.

Proof. Since $\theta \mapsto R(M(p), \theta)$ is increasing p cannot be of both types. If p is of neither then by continuity there exists a $\theta \in [\underline{\theta}, \bar{\theta}]$ such that $R(M(p), \theta) = p$. But then $\theta(p)$ is not empty. □

Corollary 5. *The set of prices* $\{p : \theta(p) = \varnothing\}$ *is open.*

Lemma 10. *Assume* $R(a, \cdot)$ *is (weakly) increasing for all* a *(the same holds if decreasing, with* $\underline{\theta}$ *and* $\bar{\theta}$ *switched) and* M *is continuous. Let* p *be Type L and* $\theta'' > \theta'$.

i. *If there exists* $p'' > p$ *such that* $\theta'' \in \theta(p'')$ *then there exists* $p' \in (p, p'']$ *such that* $\theta' \in \theta(p')$.

ii. *If there exists* $p'' < p$ *such that* $\theta'' \in \theta(p'')$ *then there exists* $p' \in [p'', p)$ *such that* $\theta' \in \theta(p')$.

Proof. We will prove (i), the proof for (ii) is symmetric. $R(M(p), \underline{\theta}) > p$ since p is type L. Moreover, under monotonicity

$$p'' = R(M(p''), \theta'') \geq R(M(p''), \theta') \geq R(M(p''), \underline{\theta}).$$

Then by continuity of R and M, there exists $\underline{p} \in (p, p'']$ such that $R(M(\underline{p}), \underline{\theta}) = \underline{p}$. By monotonicity we have $R(M(\underline{p}), \theta') \geq R(M(\underline{p}), \underline{\theta}) = \underline{p}$ and $p'' = R(M(p''), \theta'') \geq R(M(p''), \theta')$. Then by continuity of R, M there exists $p' \in [\underline{p}, p'']$ such that $R(M(p'), \theta') = p'$, so $\theta' \in \theta(p')$ as desired. □

Lemma 11. *Assume* $R(a, \cdot)$ *is (weakly) increasing for all* a *(the same holds if decreasing, with* $\underline{\theta}$ *and* $\bar{\theta}$ *switched) and* M *is continuous. Let* p *be type H and* $\theta'' > \theta'$.

i. *If there exists* $p' > p$ *such that* $\theta' \in \theta(p')$ *then there exists* $p'' \in (p, p']$ *such that* $\theta'' \in \theta(p'')$.

ii. *If there exists* $p' < p$ *such that* $\theta' \in \theta(p')$ *then there exists* $p'' \in [p', p)$ *such that* $\theta'' \in \theta(p'')$.

Proof. Analogous to that of Lemma 10. □

Lemma 12. *(Generalized intermediate value theorem). Let $F : [0,1] \mapsto [0,1]$ be a compact and convex valued, upper hemicontinuous correspondence. Let $p_1 < p_2$. Let $y_1 \in F(p_1)$ and $y_2 \in F(p_2)$. Then for any $\tilde{y} \in (\min\{y_1, y_2\}, \max\{y_1, y_2\})$ there exists $p \in [p_1, p_2]$ such that $\tilde{y} \in F(p)$.*

Proof. Define $p^* := \sup\{p \in [p_1, p_2) : \max F(p) < \tilde{y}\}$. If $p^* = p_1$ then $\max F(p) \geq \tilde{y}$ for all $p \in (p_1, p_2)$. Assume none of these hold with equality (otherwise we are done). Then if $\min F(p) \leq \tilde{y}$ for some $p \in (p_1, p_2]$ then we are done, by convexity of F. So suppose $\min F(p) > \tilde{y}$ for all $p \in (p_1, p_2]$. Then $\tilde{y} \in F(p_1)$: otherwise, by convexity of $F(p_1)$, we have $\max F(p_1) < \tilde{y}$, which violates upper hemicontinuity. Thus we are done if $p* = p_1$.

Suppose instead that $p^* = p_2$. If $\min F(p_2) \leq \tilde{y}$ then we are done, by convexity of $F(p)$. Suppose $\min F(p_2) > \tilde{y}$. Then by the definition of p^*, it must be that for any $\varepsilon > 0$ there exists $p \in (p_2 - \varepsilon, p_2)$ such that $\max F(p) < \tilde{y}$. But this violates upper hemicontinuity of F at p_2. Thus we are done if $p^* = p_2$.

It only remains to address the case of $p^* \in (p_1, p_2)$. It must be that $\max F(p^*) \geq \tilde{y}$: if not then by upper hemicontinuity there exists $\varepsilon > 0$ such that $\max F(p) < \tilde{y}$ for all $p \in [p^*, p^* + \varepsilon)$, but this would contradict the definition of p^*. If $\min F(p^*) \leq \tilde{y}$ then we are done, by convexity. So suppose $\min F(p^*) > \tilde{y}$. Then by upper hemicontinuity there exists $\varepsilon > 0$ such that $\min F(p) > \tilde{y}$ for all $p \in (p^* - \varepsilon, p^*]$. But this contradicts the definition of p^*. □

A.9.4 Proof of Proposition 4

Proof. Claim 0. For any $\theta' \in (\underline{\theta}, \bar{\theta})$ and p' be such that $\theta' \in \theta_M(p')$, there exist p'' such that $\theta_M(p'') \cap \{\underline{\theta}, \bar{\theta}\} \neq \varnothing$, $\theta_M(p) \neq \varnothing$ for all $p \in (\min\{p', p''\}, \max\{p', p''\})$ and M is continuous on $(\min\{p', p''\}, \max\{p', p''\})$ (when this interval is non-empty).

Let $\theta' \in (\underline{\theta}, \bar{\theta})$ be arbitrary, and let p' be such that $\theta' \in \theta_M(p')$. If $\{p \leq p' : \theta_M(p) = \varnothing\}$ is empty then $p'' = \arg\min_{a \in \mathcal{A}} R(a, \underline{\theta})$ satisfies the conditions of the claim. Similarly, if $\{p \geq p' : \theta_M(p) = \varnothing\}$ is empty then $p'' = \arg\max_{a \in \mathcal{A}} R(a, \bar{\theta})$ satisfies the conditions of the claim. Assume that $\{p \leq p' : \theta_M(p) = \varnothing\} \neq \varnothing$ and $\{p \geq p' : \theta_M(p) = \varnothing\} \neq \varnothing$. Let $\underline{p} = \sup\{p \leq p' : \theta_M(p) = \varnothing\}$ and $\bar{p} = \inf\{p \geq p' : \theta_M(p) = \varnothing\}$. Since $M \in \mathcal{M}$, we have $\underline{p} < p' < \bar{p}$. Since M must be continuous on $(\underline{p}, \bar{p})$, we have $\theta_M(\underline{p}) \cap \{\underline{\theta}, \bar{\theta}\} \neq \varnothing$ and $\theta_M(\bar{p}) \cap \{\underline{\theta}, \bar{\theta}\} \neq \varnothing$. This proves Claim 0.

Claim 1. Let $\theta' \in (\underline{\theta}, \bar{\theta})$ and p' such that $\theta' \in \theta_M(p')$. Let p'' be such that $\theta_M(p) \neq \varnothing$ for all $p \in (\min\{p', p''\}, \max\{p', p''\})$ and M is continuous on $(\min\{p', p''\}, \max\{p', p''\})$ (when this interval is non-empty). Then if $\underline{\theta} \in \theta_M(p'')$ and $p'' \leq p'$ ($p'' \geq p'$) there exists an equilibrium with a price function that is increasing (decreasing) on $[\underline{\theta}, \theta']$. Similarly, if $\bar{\theta} \in \theta_M(p'')$

and $p'' \geq p'$ ($p'' \leq p'$) there exists an equilibrium with a price function that is increasing (decreasing) on $[\theta', \bar{\theta}]$.

We will show the claim for $\bar{\theta} \in \theta_M(p'')$ and $p'' \geq p'$; all others cases are symmetric. For any θ, the set $\theta_M^{-1}(\theta)$ is compact: if $R(M(p), \theta) \neq p$ then this holds for all $\tilde{p}$ in a neighborhood p, since $M \in \mathcal{M}$ is continuous around equilibrium prices. If $p' = p''$ then we are done: convexity of $\theta_M(p)$ (Lemma 7) implies that there is a constant, and thus monotone, equilibrium price function on $[\theta', \bar{\theta}]$. Assume instead that $p'' > p'$. If there exists $\theta^* \in (\theta', \bar{\theta})$ such that $p^* > p''$ for any $p^* \in \theta_M^{-1}(\theta'')$ then there exists $\tilde{\theta} \in (\theta', \bar{\theta})$ such that $p'' \in \theta_M^{-1}(\tilde{\theta})$, by continuity of M on (p', p'') and *Lemma* 12. Then convexity of $\theta_M(p'')$ implies that we can construct a flat price function above $\tilde{\theta}$. Therefore assume no such θ^* exists. By a symmetric argument, we can assume that $\theta_M^{-1}(p) \cap [p', p''] \neq \varnothing$ for all $\theta \in [\theta', \bar{\theta}]$.

We want to construct an increasing equilibrium price function on $[\theta', \bar{\theta}]$. Consider an arbitrary price function $\tilde{P}$ such that $\tilde{P}(\theta) \in \theta_M^{-1}(\theta) \cap [p', p'']$ for all $\theta \in [\theta', \bar{\theta}]$, $\tilde{P}(\underline{\theta}) = p'$, and $\tilde{P}(\bar{\theta}) = p''$. We will show that any violations of monotonicity can be ironed without leading to further violations.

Claim 1.2. Suppose $\tilde{P}(\theta_2) < \tilde{P}(\theta_1) < \tilde{P}(\theta_3)$ for $\bar{\theta} > \theta_3 > \theta_2 > \theta_1 >$. Then there exists $p \in \theta_M^{-1}(\theta_2) \cap [\tilde{P}(\theta_1), \tilde{P}(\theta_3)]$.

Claim 1.2 follows immediately from Lemma 12. This in turn shows that Claim 1 holds for $\bar{\theta} \in \theta_M(p'')$ and $p'' \geq p'$, which is what we wished to show.

Claim 0 and Claim 1 together imply the existence of a monotone price function. □

A.9.5 Proof of Proposition 5

Proof. By Proposition 4, M admits an equilibrium with a monotone price function P. Let Q be the associated action function. For any state θ such that $r(Q(\theta), P(\theta))$ is non-degenerate, let $\hat{Q}(\theta') = R(Q(\theta), \theta)$ for all $\theta' \in r(Q(\theta), P(\theta))$. Clearly $\hat{P}(\theta) := R(\hat{Q}(\theta), \theta)$ will also be monotone, and $(\hat{Q}, \hat{P})$ is also implemented by M. It remains to show that M can be modified on $\mathcal{P} \setminus \hat{P}(\Theta)$ in order to implement $(\hat{Q}, \hat{P})$ uniquely. This follows from Proposition 2. Note that $\hat{Q}$ will have no degenerate discontinuities since M is continuous on $\hat{P}(\Theta)$. □

A.10 Proof of Lemma 4

Proof. Condition i is immediate. For ii, first note that for $p \in (R(\underline{a}, \underline{\theta}), R(\underline{a}, \bar{\theta}))$ it must be that $M(p) > \underline{a}$; if not then $R(M(p), \theta) = p$ for some $\theta \in (\underline{\theta}, \bar{\theta})$. Suppose there is no discontinuity on $(P(\underline{\theta}), R(\underline{a}, \underline{\theta})]$. Then M must be decreasing over this domain to prevent multiplicity, and $\lim_{p \searrow R(\underline{a}, \underline{\theta}} M(p) = \underline{a}$. But for $p \in (R(\underline{a}, \underline{\theta}), R(\underline{a}, \bar{\theta}))$ it must be that $M(p) > \underline{a}$, so there must be a discontinuity. A symmetric argument applies to $(R(\bar{a}, \bar{\theta}), P(\bar{a})]$

Conditions *iii* and *iv* follow from a similar argument. Define $\bar{p}$ by $\bar{p} = \sup\{p : M \text{ is decreasing on } (P(\bar{\theta}), \bar{p})\}$. The argument above implies that $\bar{p} \leq R(\underline{a}, \underline{\theta})$. This implies *iii*. A symmetric argument implies *iv*. □

A.11 Proof of Section 3.3

Lemma 13. *Given a continuous function $F : X \times (0, 1) \mapsto X$ on a compact, convex subset X of a Euclidean space, define the function*

$$G(t) = \{x \in X : F(x, t) = x\}.$$

Then $G(t)$ is non-empty for all t (Brouwer's fixed point theorem). Moreover, if $G(t)$ is single valued then G is upper and lower hemicontinuous at t.

Proof. Since $G(t)$ is single valued upper hemicontinuity implies lower hemicontinuity. We want to show that for any open neighborhood V of $G(t)$ there exists a neighborhood U of t such that $G(t') \subseteq V$ for all $t' \in U$.

Claim 1. For any open neighborhood V of $G(t)$ there exists a $\kappa > 0$ such that

$$|F(x, t) - x| > \kappa \quad \forall \;\; x \in X \setminus V.$$

The proof of claim 1 is as follows. $X \setminus V$ is a closed subset of a compact set, and thus compact. The function $x \mapsto F(x, t) - x$ is continuous, so it attains its minimum on $X \setminus V$. Since $G(t)$ is unique and $G(t) \notin X \setminus V$, this minimum is strictly greater then zero, so the desired κ exists.

To complete the proof of Lemma 13, we need to show that there exists an open neighborhood U of t such

$$|F(x, t') - x| > \kappa \quad \forall \;\; x \in X \setminus V, \quad t' \in U.$$

By continuity of $t' \mapsto F(x, t') - x$, for each x there exists a ε_x such that $|t' - t| < \varepsilon_x$ implies $|F(x, t') - x| > \kappa$. For each x, define $\ell(x, \varepsilon) = \min\{|F(x, t') - x| : |t' - t| \leq \varepsilon/2\}$, which exists by continuity of F and compactness of $|t' - t| \leq \varepsilon/2$. Define

$$B(x) = \{x' \in X : \ell(x', \varepsilon_x) > \kappa\}.$$

By continuity of $x \mapsto F(x, t') - x$, $B(x)$ contains an open neighborhood of x (Berge's maximum theorem). Let $\tilde{B}(x)$ be this open neighborhood. The set $\cup_{x \in X \setminus V} \tilde{B}(x)$ covers $X \setminus V$. Then by compactness of $X \setminus V$ there exists a finite sub-cover. Let u be the smallest ε_x corresponding to an x such that $\tilde{B}(x)$ is in the finite sub-cover. Then $U = \{t' \in (0, 1) : |t' - t| < u\}$. □

Proposition 13. *Given a continuous function $F : X \times \Theta \times (0,1) \mapsto X$ on a compact, convex subset X of a Euclidean space, define the function*

$$G(t, \theta) = \{x \in X : F(x, \theta, t) = x\}.$$

Then $G(t,\theta)$ is non-empty for all t, θ (Brouwer's fixed point theorem). Moreover, let S be any compact subset of Θ such that $G(t,\theta)$ is single valued for all $\theta \in S$. Then $t \rightrightarrows G(t,\theta)$ is upper and lower hemicontinuous at t, uniformly over S.

Proof. Since $G(t,\theta)$ is single valued on S it suffices to show upper hemicontinuity. Let $V(\theta)$ be an open neighborhood of $\theta \mapsto G(t,\theta)$ on S. Without loss of generality (since Θ is compact and $G(t,\theta)$ single valued on S), let $V(\theta) = \{x \in X : |G(t,\theta) - x| < \delta\}$ for some $\delta > 0$, or equivalently, $V(\theta) = \cup_{x \in G(t,\theta)} N_\delta(x)$. We want to show that there exists an open neighborhood U of t such that $t' \in U$ implies $G(t', \theta) \subseteq V(\theta)$ for all $\theta \in S$.

Claim 1. $X \setminus V(\theta)$ is upper and lower hemicontinuous on S.

The proof of Claim 1 is as follows. Since $G(t,\theta)$ is single valued,

$$X \setminus V(\theta) = X \setminus N_\delta(G(t,\theta))$$

where $N_\delta(x)$ is the open ball around x with radius δ. We first show upper hemicontinuity. Let W be an open set containing $X \setminus V(\theta)$. Without loss of generality, let

$$W = X \setminus \bar{N}_{\delta-\rho}(G(t,\theta))$$

for some $\rho \in (0,\delta)$ where $\bar{N}_{\delta-\rho}(x)$ is the closed ball around x with radius $\delta - \rho$.[20] By Lemma 13, we know that $\theta \mapsto G(t,\theta)$ is upper and lower hemicontinuous at all $\theta \in S$. By upper hemicontinuity of $\theta \mapsto G(t,\theta)$ at θ, there exists an open neighborhood B of θ such that $\theta' \in B$ implies $|x - G(\theta,t)| < (\delta-\rho)/2$ for all $x \in G(\theta', t)$. Then $\bar{N}_{\delta-\rho}(G(t,\theta)) \subset \cup_{x \in G(t,\theta')} N_\delta(x) = V(\theta')$ for all $\theta' \in B$. Thus $V(\theta') \subset W$ for all $\theta' \in B$, which shows upper hemicontinuity.

For lower hemicontinuity, let $W \subset X$ be an open set intersecting $X \setminus V(\theta)$. This holds if and only if there exists $x' \in W$ such that $|x' - G(t,\theta)| > \delta$. By upper hemicontinuity of $\theta \mapsto G(t,\theta)$ at θ, there exists an open neighborhood B of θ such that $\theta' \in B$ implies $|x - G(\theta,t)| < (|x' - G(t,\theta)| - \delta)/2$ for all $x \in G(\theta',t)$. Then $\theta' \in B$ implies $|x' - x| > \delta$ for all $x \in G(t,\theta')$. Thus $x' \notin \cup_{x \in G(t,\theta')} N_\delta(x) = V(\theta')$, so $W \cap X \setminus V(\theta') \neq \varnothing$ for all $\theta' \in B$, which shows lower hemicontinuity. This completes the proof of Claim 1.

We know from Lemma 13 that for each $\theta \in S$ there exists $\varepsilon_\theta, \kappa_\theta > 0$ such that

$$|t' - t| < \varepsilon_\theta \Longrightarrow |F(x,\theta,t') - x| > \kappa_\theta \;\; \forall\, x \in X \setminus V(\theta). \tag{3}$$

[20] W so defined is open in X, but not in the space of which X is a subset.

Claim 2. For each $\theta \in S$ there exists an open neighborhood $B(\theta)$ of θ such that

$$\theta' \in B(\theta) \text{ and } |t' - t| < \varepsilon_\theta \Longrightarrow |F(x, \theta, t') - x| > \kappa_\theta \;\; \forall\, x \in X \setminus V(\theta'),$$

where $\varepsilon_\theta, \kappa_\theta$ satisfy (3).

The proof of this claim is as follows. Define

$$z(\theta, \varepsilon) := \min\{|F(x, \theta, t') - x| : |t' - t| \leq \varepsilon/2,\; x \in X \setminus V(\theta)\},$$

which is well defined by compactness of $X \setminus V(\theta)$. By Berge's maximum theorem and Claim 1, $\theta \mapsto z(\theta, \varepsilon)$ is continuous at any $\theta \in S$. By (3) we know that $z(\theta, \varepsilon_\theta) > \kappa_\theta$ for all $\theta \in S$. Then for any $\theta \in S$ there exists an open neighborhood $B(\theta)$ of θ such that $\theta' \in B(\theta)$ implies $z(\theta', \varepsilon_\theta) > \kappa_\theta$. This proves Claim 2.

To complete the proof of Proposition 13, note that $\cup_{\theta \in S} B(\theta)$ is an open cover of S. By compactness of S there exists a finite sub-cover. Let I be the set of $\theta \in S$ that index this sub-cover. Let $\varepsilon = \min\{\varepsilon_\theta : \theta \in I\}/2$. Then

$$|t' - t| < \varepsilon \Longrightarrow |F(x, \theta, t') - x| > 0 \;\; \forall\, x \in X \setminus V(\theta) \text{ and } \theta \in S.$$

Since $G(t', \theta)$ is non-empty for all t', θ we have that $|t' - t| < \varepsilon$ implies that for all θ, $G(t', \theta) \subseteq V(\theta)$, which shows upper hemicontinuity as desired. □

A.11.1 Proof of Theorem 4

Proof. We first show that the result holds for continuous M. Since any discontinuity in $M \in \mathcal{M}$ must be bounded away from the set of equilibrium prices, this implies that the result also holds for all $M \in \mathcal{M}$.

Let M be continuous. Let $F(a, \theta, t) = M(R(a, \theta, t))$, where t continuously parameterizes the function R. Then F is continuous since M is continuous. Moreover, $G(t, \theta) = \tilde{Q}(\theta, t)$ will be single valued on all but a zero-measure set of states when M is weakly robust to multiplicity, and single valued everywhere when M is robust to multiplicity. Therefore for any $\varepsilon > 0$ we can find a compact set S such that $G(t, \theta)$ is single valued for all $\theta \in S$. When M is robust to multiplicity let $S = \Theta$. Then Proposition 13 applies, which gives the result. □

A.11.2 Proof of Proposition 3

Proof. First, note that $|\mathcal{U}(R)| = 1$. It is non-empty since Lemma 13 and robustness to multiplicity imply that $\theta \mapsto \tilde{Q}(\theta, R)$ is a continuous function on all but a zero measure set of states, and is thus $\theta \mapsto u(\theta, Q(\theta)$ integrable for all $Q \in \tilde{Q}(\cdot, R)$. It is single valued since all $Q \in \tilde{Q}(\cdot, R)$ are the same on all but a zero measure set of states.

Since $|\mathcal{U}(R)| = 1$, upper hemicontinuity implies lower hemicontinuity, so it suffices to show the former. Thus we want to show that for any $\delta > 0$ there exists an open neighborhood $B \subseteq \mathcal{C}$ of R such that $R' \in B$ implies $\mathcal{U}(R') \subseteq (\mathcal{U}(R) - \delta, \mathcal{U}(R) + \delta)$. Since the set of states at which $\tilde{Q}(\theta, R)$ is not single valued has zero measure, for any $\varepsilon > 0$ there exists a compact set S such that $\{\theta \in \Theta : |\tilde{Q}(\theta, R)| \neq 1\} \subset \Theta \setminus S$ and $\lambda(\Theta \setminus S) < \varepsilon$ (where λ is Lebesgue measure). For any such S, there exists a neighborhood $B_S \subset C$ of R such that $R' \in B_S$ implies

$$\int_S u(\theta, Q(\theta)) dH(\theta) - \mathcal{U}(R)| < \delta/2$$

Taking ε small enough gives implies $\mathcal{U}(R') \subseteq (\mathcal{U}(R) - \delta, \mathcal{U}(R) + \delta)$ as desired. □

A.11.3 Proof of Lemma 2

Proof. Suppose M is discontinuous at p', and let $\theta' \in \theta_M(p|R)$. Assume M is left-continuous at p' (symmetric argument for right-continuous). Then there exists $\varepsilon > 0$ such that either $R(M(p), \theta') > p$ or $R(M(p), \theta') < p$ for all $p \in (p' - \varepsilon, p')$. Suppose without loss that the latter holds. Let $\hat{R}$ be a continuous function such that $\hat{R}(M(p), \theta') = R(M(p), \theta') - \delta(p' + \varepsilon - p)$ for $p \in (p' - \varepsilon, p' + \varepsilon)$ and $\hat{R}(a, \theta') = R(a, \theta)$ elsewhere. Then for any $\delta > 0$ $\theta' \in \theta_M(p|\hat{R})$ iff $p \in (p', p' + \varepsilon)$. Since M has a jump discontinuity at p', and since δ was arbitrary, taking ε small enough implies that $R \mapsto \tilde{Q}_R(\theta'|M)$ is not upper hemicontinuous. □

A.11.4 Proof of Proposition 7

Proof. Sufficiency is obvious. Conditions *iii* and *iv* are necessary, as discussed in the paragraph preceding Proposition 7. To show necessity of *i* and *ii*, restrict attention to a one-dimensional strictly ordered chain in Θ (e.g. the diagonal). For the restriction of Q to this chain, necessity of *i* and continuity for interior states then follow from the same arguments as in the uni-dimensional case. Under *iii* and *iv*, this implies that *i* holds; if there is a non-monotonicity on some chain then there will be a non-monotonicity on every chain. Similarly Q must be continuous on the interior. □

A.11.5 Proof of Proposition 8

Proof. Fix M, and let $L_M : \Theta \times \mathcal{Z} \times \mathcal{A} \mapsto \mathbb{R}$ be the equilibrium statistic in a generalized linear equilibrium. Define the random variable $\tilde{V}^a := \pi(a, \theta) = \beta_0^a + \beta_1^a \theta$. Then define $\tilde{S}_i^a := \beta_1^a \sigma + \beta_0^a = \tilde{V}^a + \beta_1^a \varepsilon_i$. Thus conditional on knowing the principal's action, investor i's observation of σ_i is equivalent to observing a signal $\tilde{S}_i^a$ which is equal to the true dividend $\tilde{V}^a$ plus normal random noise, where the variance of the noise term depends on a; it is given by $\sigma_{ai}^2(\beta_1^a)^2\sigma_i^2$. Let $\tilde{L}^a$ be the random variable $L_M(\theta, z, a)$.

We first fix the principal's action at a, and generalize Breon-Drish (2015) Proposition 2.1 to allow for supply shocks with a truncated normal distribution. We will therefore suppress dependence of $\tilde{S}_i^a, \tilde{V}^a, \tilde{L}^a$ on the action a for the time being. Abusing notation, write the statistic L in terns of v, rather than θ; that is, $L(v, z|a) = \alpha v - z$, suppressing the dependence on M.[21] For fixed a, the truncation is the only difference between the current setting and that of Breon-Drish (2015) Proposition 2.1. By the same steps as the proof for Proposition 2.1 in Breon-Drish (2015) Online Appendix, we can show that the conditional distribution of $\tilde{V}^a$ conditional on $\tilde{S}_i^a = s_i$ and $\tilde{L}^a = \ell$ is given by

$$dF_{\tilde{V}|\tilde{S},\tilde{L}}(v|s_i, \ell) = \frac{\mathbb{1}[\ell - \alpha v \in (-b, b)] \exp\left\{\left(\frac{1}{\sigma_{ai}^2}s_i + \frac{\alpha}{\sigma_Z^2}\ell\right)v - \frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2}\right)v^2\right\} dF_{\tilde{V}}(v)}{\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{\left(\frac{1}{\sigma_{ai}^2}s_i + \frac{\alpha}{\sigma_Z^2}\ell\right)x - \frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2}\right)x^2\right\} dF_{\tilde{V}}(x)}, \quad (4)$$

where $\mathbb{1}[\cdot]$ is the indicator function. This is not in the *exponential family* of distributions, as defined in Breon-Drish (2015) Assumption 10. Nonetheless, it will have similar properties. We can write the conditional distribution in (4) as

$$\mathbb{1}[\ell - \alpha v \in (-b, b)] \exp\left\{\hat{L}(s_i, \ell)v - g\left(\hat{L}(s_i, \ell); \alpha, \ell\right)\right\} dH(v; \alpha),$$

where

$$\begin{aligned}
\hat{L}(s, \ell) &= \left(\frac{1}{\sigma_{ai}^2}s_i + \frac{\alpha}{\sigma_Z^2}\ell\right) \\
g_i(\hat{L}; \alpha, \ell) &= \log\left(\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{\left(\frac{1}{\sigma_{ai}^2}s_i + \frac{\alpha}{\sigma_Z^2}\ell\right)x - \frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2}\right)x^2\right\} dF_{\tilde{V}}(x)\right) \\
dH_i(v; \alpha) &= \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_{ai}^2} + \frac{\alpha^2}{\sigma_Z^2}\right)v^2\right\} dF_{\tilde{V}}(v)
\end{aligned}$$

This has the following important implication (essentially the same as Lemma A6 in Breon-Drish (2015)). Since the conditional distribution must integrate to 1, i.e.

$$\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{\hat{L}(s_i, \ell)v - g\left(\hat{L}(s_i, \ell); \alpha, \ell\right)\right\} dH(v; \alpha) = 1$$

[21] This abuse of notation is done to match the notation of Breon-Drish (2015). Note that in that paper "a" is used in place of α to denote the slope of the equilibrium statistic. The reader examining Breon-Drish (2015) should not confuse this with the notation for the principal action used in the current paper.

we have that

$$\int_{\frac{\ell-b}{\alpha}}^{\frac{\ell+b}{\alpha}} \exp\left\{\hat{L}(s_i,\ell)v\right\} dH(v;\alpha) = \exp\left\{g\left(\hat{L}(s_i,\ell);\alpha,\ell\right)\right\}.$$

As a result, for any $t \in \mathbb{R}$ we have

$$\mathbb{E}\left[\exp\{t\tilde{V}\}|s,\ell\right] = \exp\left\{g\left(t+\hat{L}(s_i,\ell);\alpha,\ell\right) - g\left(\hat{L}(s_i,\ell);\alpha,\ell\right)\right\}.$$

The remainder of the proof for the fixed-action case proceeds as in Breon-Drish (2015) Proposition 2.1. In particular, this shows that in any generalized linear equilibrium with fixed action a,

$$\alpha = \int_i \frac{\tau_i}{\sigma_{ai}^2} di.$$

Since $v = \beta_0^a + \beta_1^a\theta$ and $\sigma_{ai}^2 = (\beta_1^a)^2\sigma_i^2$ we have

$$L^*(\theta, z|a) = \beta_0^a \int_i \frac{\tau_i}{\sigma_{ai}^2} di + \left(\frac{1}{\beta_1^a}\int_i \frac{\tau_i}{\sigma_{ai}^2} di\right)\cdot\theta - z$$

Since only the level sets of L^* matter, we can ignore the first term.

We now show that the result holds under feedback as well. Given M, the investor knows which action the principal will take conditional on the price. In a generalized linear equilibrium, the investor's demand is therefore determined by maximizing utility given that the price is p, the action is $M(p)$, the observed signal is $\tilde{S}_i^a$, and the extended state is in $\{(\theta,z) : L_M(\theta,a|a) = \ell\}$ for the value of ℓ corresponding to price level p. The remaining question is which $L_M(\cdot|a)$ could constitute equilibrium statistics given action a and decision rule M. The first part of the proof shows that if the principal's action is fixed at a then there is a unique equilibrium statistic $L^*(\theta,z|a)$. Since all investors know the principal's action once they observe the price, this L^* must be the equilibrium statistic, regardless of M. □

B Bridgeability

This section discusses bridgeability further. We provide sufficient conditions for the various notions of bridgeability, and show that they are satisfied in common settings.

Let $(\mathcal{A},\succ)$ be a partially ordered set. Say $(\mathcal{A},\succ)$ is *upward directed* if for any two $a'', a' \in \mathcal{A}$ there exists $c \in \mathcal{A}$ such that $c \succ a''$ and $c \succ a'$. Downward directed is defined analogously.[22] We use the notation $a''_\alpha a' \equiv \alpha a'' + (1-\alpha)a'$. Say that $\succ$ is preserved by mixtures if for any $a'' \succ a'$ and $\alpha \in (0,1)$, $a'' \succ a''_\alpha a' \succ a'$. Finally, say that $a \mapsto R(a,\theta)$ is *strongly*

[22] A lattice is an upward and downward directed set, but the converse is not true.

monotone with respect to $\succ$ if $a'' \succ a'$ and $a'' \neq a'$ implies $R(a'', \theta) > R(a', \theta)$. We use the notation $a''_\alpha a' \equiv \alpha a'' + (1-\alpha)a'$. The following proposition gives sufficient conditions for full bridgeability, but it is also useful because the proof of the existence of a monotone path is constructive. This construction could potentially be useful in applications.

Proposition 14. *Let $(\mathcal{A}, \succ)$ be a partially ordered set that is both upward and downward directed, and such that $\succ$ is preserved by mixtures. If $R(\cdot, \theta)$ is strongly monotone with respect to $\succ$ then there is a monotone path between a' and a'' at θ iff $R(a'', \theta) \neq R(a', \theta)$*

Proof. The condition $R(a', \theta) \neq R(a'', \theta)$ is obviously necessary. It remains to show that it is sufficient. That is, we want to show that there exists a monotone path between any $a'', a' \in \mathcal{A}$ such that $R(a', \theta) \neq R(a'', \theta)$. Assume without loss that $R(a'', \theta) > R(a', \theta)$. If $a'' \succ a'$ then the ray from a'' to a' is a monotone path. This follows since $\succ$ is preserved by mixtures and $R(\cdot, \theta)$ is strongly monotone.

Suppose a' and a'' are not ordered. Let $\bar{a}$ be an upper bound for a'', a', i.e. $\bar{a} \succ a''$ and $\bar{a} \succ a'$, and let $\underline{a}$ be a lower bound. Both exist since $(\mathcal{A}, \succ)$ is upward and downward directed. By continuity of R, there exists $\bar{\lambda} \in (0,1)$ such that $R(\bar{a}_{\bar{\lambda}} a', \theta) = R(a'', \theta)$. Similarly there exists $\underline{\lambda} \in (0,1)$ such that $R(a''_{\underline{\lambda}} \underline{a}), \theta) = R(a', \theta)$.

We will now construct one half of the monotone path from a' to a''. Let $t : [0,1] \mapsto [\bar{\lambda}, 1] \times [0,1]$ be a continuous and strictly monotone function, and let $t_i(x)$ be the i^{th} coordinate of $t(x)$. For each $x \in (0,1)$, we have $R(\bar{a}_{t_1(x)} a', \theta) > R(a'', \theta)$, $R(\underline{a}_{t_2(x)} a', \theta) < R(a', \theta)$, and $\bar{a}_{t_1(x)} a' \succ \bar{a}_{t_1(x)} a'$. These properties follow from strong monotonicity of R and the fact that $\succ$ is preserved under mixtures.

For each $x \in (0,1)$, define $f(x)$ by $R((\bar{a}_{t_1(x)} a')_{f(x)} (\underline{a}_{t_2(x)} a'), \theta) = xR(a'', \theta) + (1-x)R(a', \theta)$. We claim that $x \mapsto (\bar{a}_{t_1(x)} a')_{f(x)} (\underline{a}_{t_2(x)} a')$ is a continuous function. It is a well defined function by strong monotonicity of R. It is continuous since R and t are continuous. Moreover, by construction $x \mapsto R((\bar{a}_{t_1(x)} a')_{f(x)} (\underline{a}_{t_2(x)} a'), \theta)$ is strictly increasing, and $(\bar{a}_{t_1(0)} a')_{f(0)} (\underline{a}_{t_2(0)} a') = a'$. Therefore $x \mapsto (\bar{a}_{t_1(x)} a')_{f(x)} (\underline{a}_{t_2(x)} a')$ forms one half of a monotone path from a' to a''. The other half of the monotone path is defined analogously, using a'' and $\underline{\lambda}$ in place of a' and $\bar{\lambda}$. $\square$

Proposition 14 makes it easy to identify when a discontinuity will be bridgeable. For example, it implies that when $\mathcal{A}$ is a chain a gap between a' and a'' will be bridgeable at θ iff $R(\cdot, \theta)$ is strictly monotone on (a', a'').

More importantly, Proposition 14 implies that every discontinuity will be bridgeable when $\mathcal{A} = \Delta(Z)$, i.e. the set of distributions on some set Z, under mild assumptions on R. Let $\pi(z, \theta)$ be a real valued function, with $\theta \mapsto \pi(z, \theta)$ continuous for all z. For example, $\pi(a, \theta)$

could represent a company's cash flow as a function of the state and government intervention $z \in Z$. In state θ, any $a \in \mathcal{A}$ in induces a distribution $F(a,\theta)$ on $\mathbb{R}$ via $\pi(\cdot,\theta)$. Let $\succ_{FOSD}$ be the first-order stochastic dominance order. This partial order on $\Delta(\mathbb{R})$ induces a preorder $\succeq$ on $\mathcal{A}$. Define $a'' \succ a'$ by $a'' \succeq a'$ and $\neg(a' \succeq a'')$ if $a'' \neq a'$, and $a' \succ a'$ for all a'. If $\pi(z',\theta) \neq \pi(z'',\theta)$ for all $z'' \neq z'$ then $\succeq = \succ$. Then $a \mapsto R(a,\theta)$ is strongly monotone if $F(a'',\theta) \succ F(a',\theta)$ implies $R(a'',\theta) \succ R(a',\theta)$. The partially ordered set $(\mathcal{A},\succ)$ satisfies the conditions of Proposition 14 (when $\pi(z',\theta) \neq \pi(z'',\theta)$ for all $z' \neq z''$ it is in fact a lattice).

Corollary 6. *If $\mathcal{A} = \Delta(Z)$ and for all θ $a \mapsto R(a,\theta)$ is strongly monotone with respect to the order induced by first-order stochastic dominance, then the environment is fully bridgeable.*

It will also be useful to establish a related notion of bridgeability. Say that there exists a monotone path from (a',θ') to (a'',θ'') if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{A} \times [\theta',\theta'']$ such that $\gamma(0) = (a',\theta')$, $\gamma(1) = (a'',\theta'')$, $x \mapsto \gamma_1(x)$ is weakly increasing and $R(\gamma_1(x),\gamma_2(x))$ is strictly increasing. The path is *strongly monotone* if moreover $\gamma_1(x)$ is strictly increasing.

Recall that the environment is *continuously bridgeable* if for any $\theta^* \in \Theta$ there exists $\varepsilon > 0$ such that if a', a'' is bridgeable at θ^* and $R(a'',\theta) \neq R(a',\theta)$ for all $\theta \in [\theta^*,\theta^*+\varepsilon]$ then there exists a sup-norm continuous function $\sigma(\cdot|a',a'') : [\theta^*,\theta^*+\varepsilon] \mapsto \mathcal{A}^{[0,1]}$ such that $\sigma(\theta|a',a'')$ is a monotone path from a' to a'' for all $\theta \in [\theta^*,\theta^*+\varepsilon]$. Say that the environment is *continuously fully bridgeable* if it is full bridgeable and continuously bridgeable.

Lemma 14. *Assume $\theta \mapsto R(a,\theta)$ is strictly monotone for all $a \in \mathcal{A}$ and the environment is continuously fully bridgeable. Then the environment is correctable if for all θ such that $R(a',\theta) = R(a'',\theta)$ for all $a \in \mathcal{A}$ the following holds: there exists $a_1 \in \mathcal{A}$ and $\delta > 0$ such that*

i. $R(a_1,\theta') > R(a,\theta')$ *for all* $a \neq a_1$ *and* $\theta' \in (\theta-\delta,\theta)$.

ii. $R(a,\theta') > R(a_1,\theta')$ *for all* $a \neq a_1$ *and* $\theta' \in (\theta,\theta+\delta)$.

Proof. If P is decreasing then the existence of an approximating Q' around any degenerate discontinuity follows immediately from continuous bridgeability. Assume therefore that P is increasing.

First suppose that there is a degenerate discontinuity at some θ such that there exist a', a'' with $R(a',\theta) \neq R(a'',\theta)$. Assume there exists $\bar{a}$ such that $R(\bar{a},\theta) > R(Q(a),\theta)$ (the argument for the reverse inequality is symmetric). Since P is increasing and continuous at θ, for any $\varepsilon > 0$ there exists $\theta'' \in (\theta,\theta+\varepsilon)$ and $\theta' \in (\theta-\varepsilon,\theta)$ such that $R(\bar{a},\theta) > R(Q(\theta''),\theta)$ for all $\theta \in (\theta',\theta'')$. Then since the environment is continuously bridgeable (in particular between $Q(\theta'')$ and $\bar{a}$) there exists a continuous Q' on $[\theta', \theta'']$, with the corresponding P' strictly increasing, such that $R(Q'(\theta'), \theta') \in (R(Q(\theta'), \theta'), R(Q(\theta''), \theta''))$ and $Q'(\theta'') = Q(\theta'')$. Thus the environment is correctable.

Now suppose there is a degenerate discontinuity at some θ such that $R(a'', \theta) = R(a', \theta)$ for all $a', a'' \in \mathcal{A}$. Then under conditions i and ii the following Q' satisfies the conditions for correcting the degenerate discontinuity: for any $\varepsilon < \delta$, $Q' = a_1$ on $(\theta - \varepsilon, \theta + \varepsilon)$ and equals Q elsewhere. □

Note that Lemma 14 implies that the environment is correctable if for all θ there exist a', a'' such that $R(a', \theta) \neq R(a'', \theta)$. Except for unusual cases, the environment will be continuously fully bridgeable when it is fully bridgeable. For example, the environment of Corollary 6 is continuously fully bridgeable when Z is finite and $\theta \mapsto \pi(z, \theta)$ is differentiable for all z.

Lemma 15. *Assume $\mathcal{A} = \Delta(Z)$ for some finite Z, $\pi(z, \theta)$ is differentiable for all z, and for all θ, $a \mapsto R(a, \theta)$ is strongly monotone with respect to the first-order stochastic dominance induced order. Then the environment is continuously fully bridgeable.*

Proof. First suppose $\min_{z'', z' \in Z} |\pi(z'', \theta^*) - \pi(z', \theta^*)| > 0$. Then by continuity of $\theta \mapsto \pi(z, \theta)$, there exists $\varepsilon > 0$ such that $\pi(z'', \theta) > \pi(z', \theta) \Leftrightarrow \pi(z'', \theta^*) > \pi(z', \theta^*)$ for all $\theta \in [\theta^*, \theta^* + \varepsilon]$ and z', z''. Thus the partial order on $\mathcal{A}$ induced by first-order stochastic dominance is the same for all $\theta \in (\theta^* - \varepsilon, \theta^* + \varepsilon)$. This implies that the join and meet are the same for any a', a'', and so the construction used in the proof of Proposition 14 can make use of the same join and meet. Then the conditions of continuous bridgeability are implied by continuity of R.

Now suppose $\pi(z'', \theta^*) = \pi(z', \theta^*)$ for all $z'', z' \in B \subset Z$. Suppose that for any $\delta > 0$ there exists $\theta \in [\theta^*, \theta^* + \delta]$ and $z'', z'' \in B$ such that $\pi(z'', \theta) > \pi(z', \theta)$. Then by differentiability of π in θ, there exists a set $C \subset B$ and $\varepsilon > 0$ such that such that i) $\pi(z'', \theta) = \pi(z', \theta)$ for all $\theta \in [\theta^*, \theta^* + \delta]$ and all $z', z'' \in C$, and ii) $\pi(z'', \theta) > \pi(z', \theta) \Leftrightarrow \pi(z'', \theta') > \pi(z', \theta')$ for all $\theta, \theta' \in (\theta^*, \theta^* + \varepsilon]$ and all $z', z'' \in Z \setminus C$. Then the FOSD-induced order on $\mathcal{A}$ is the same for any $\theta', \theta'' \in [\theta^*, \theta^* + \delta]$. Moreover, this order is a superset of the FOSD-induced order at θ^*: if a'' first-order stochastically dominates a' at $\theta' \in (\theta^*, \theta^* + \delta]$ then it will also do so at θ^*. Thus for any a', a'' we can use the join and meet for the FOSD order induced by $\theta \in (\theta^*, \theta^* + \delta]$ to construct the monotone path θ^* as well. Then the conditions of continuous bridgeability are implied by continuity of R. □

C Micro-founding R

C.1 Asset market

We show here that summarizing the market through the function R is consistent with a model of information aggregation. Suppose there is a unit mass of traders. Traders receive conditionally independent signals σ_i about the state, with conditional distribution $h(\cdot|\theta)$. Assume that $h(\cdot|\theta) \neq h(\cdot|\theta')$ for all $\theta \neq \theta'$. Traders are expected utility maximizers. The payoff to trader i who purchases a quantity x of the asset when the principal takes action a, the state is θ, and the asset price is p is given by $V_i(a, \theta, x, p)$, which is assumed to be strictly decreasing in p.[23] For a fixed action a the demand of trader i who observes signal σ and knows that the state is in $\mathcal{I} \subseteq \Theta$ is given by

$$x_i(p|a, \sigma_i, \mathcal{I}) = \max_x E[V_i(a, \theta, x, p)|\sigma, \mathcal{I}].$$

Assume $p \mapsto x_i$ is strictly decreasing for all i (which is implied by assuming, for example, that $(x, p) \mapsto v_i(a, \theta, x, p)$ satisfies strict single crossing). Trader heterogeneity, both of utilities and beliefs, is allowed for, but for simplicity assume that there are are finitely many trader types, meaning finitely many distinct demand functions in the population. Normalizing the aggregate supply of the asset to zero, the market clearing condition is

$$\int_0^1 x_i(p|a, \sigma_i, \mathcal{I}) di = 0.$$

Since there is a continuum of traders and a finite number trader types aggregate demand is deterministic, conditional on the state and the principal action a. Thus we can write market clearing in state θ as

$$X(p|a, \mathcal{I}, \theta) = 0.$$

Let $P^*(a, \mathcal{I}, \theta)$ be the unique price that clears the market.

Given any price function $\tilde{P} : \Theta \mapsto \mathbb{R}$, let $\mathcal{I}_{\tilde{P}} : \Theta \mapsto 2^\Theta$ be the coarsest partition with respect to which $\tilde{P}$ is measurable. We say that $\tilde{P}$ *induces* partition $\mathcal{I}_{\tilde{P}}$.

A *rational expectations equilibrium* (REE) given decision rule M consists of a price function $\tilde{P}$ such that $X(\tilde{P}(\theta)|M(\tilde{P}(\theta)), \mathcal{I}_{\tilde{P}}(\theta), \theta) = 0$ for all θ. Let $\mathcal{M}$ be the set of decision rules for which there exists a REE. For any decision rule $M \in \mathcal{M}$, let $\tilde{P}_M$ be the associated REE price function.

[23]For example, each trader has a strictly increasing Bernoulli utility function u_i and wealth w_i, and $V_i(a, \theta, x, p) \equiv u_i(x(\pi(a, \theta) - p) + w_i)$.

The population distribution of signals is different for any distinct $\theta, \theta' \in \mathcal{I}$. It is therefore natural to assume that, unless all states in $\mathcal{I}$ are payoff equivalent, there will exist some pair of states $\theta, \theta' \in \mathcal{I}$ such that $P^*(a, \mathcal{I}, \theta) \neq P^*(a, \mathcal{I}, \theta')$.

A1. For any $a \in \mathcal{A}$ and $\mathcal{I} \subseteq \Theta$, if $P^*(a, \mathcal{I}, \theta) = P^*(a, \mathcal{I}, \theta')$ for all $\theta, \theta' \in \mathcal{I}$ then $P^*(a, \mathcal{I}, \theta) = P^*(a, \theta, \theta)$ for all $\theta \in \mathcal{I}$.

This assumption is discussed further following the statement of the proposition.

We want to show the equivalence between implementable mechanisms and rational expectations equilbiria.

Proposition 15. *Under A1, there exists a function $R : \mathcal{A} \times \Theta \mapsto \mathbb{R}$ such that for any decision rule M there exists a rational expectations equilibrium with price function $\tilde{P}$ if and only if M implements $\tilde{P}$ given market clearing function R.*

Proof. First, we want to show that there exists an R such that for any decision rule M, if there exists a REE given M, with price function $\tilde{P}$, then M implements $\tilde{P}$ given market clearing function R. Suppose that for decision rules M_1, M_2 there exist REE, with price functions $\tilde{P}_1$ and $\tilde{P}_2$ respectively. Let $\mathcal{I}_{\tilde{P}_1}$ and $\mathcal{I}_{\tilde{P}_2}$ be the partitions of Θ induced by $\tilde{P}_1$ and $\tilde{P}_2$ respectively.

Define $R(a, \theta) = \{\tilde{P}_M(\theta) : M \in \mathcal{M},\ M(\tilde{P}_M(\theta)) = a\}$. That is $R(a, \theta)$ is the set of prices that can be supported as part of a REE for which the equilibrium action in state θ is a.

We want to show that R as defined above is a function. In other words, we want to show that if for some state θ, the equilibrium mixed is a under both M_1 and M_2 (that is, $M_1(\tilde{P}_1(\theta)) = M_2(\tilde{P}_2(\theta)) = a$), then $\tilde{P}_1(\theta) = \tilde{P}_2(\theta)$. Since $\tilde{P}_j$ induces $\mathcal{I}_{\tilde{P}_j}$, it must be that $P^*(a, \mathcal{I}_{\tilde{P}_j}, \theta') = \tilde{P}_j(\theta)$ for all $\theta' \in \mathcal{I}_{\tilde{P}_j}$ for $j \in \{1, 2\}$. Then A1 implies that $P^*(a, \mathcal{I}_{\tilde{P}_j}, \theta) = P^*(a, \theta, \theta)$ for $j \in \{1, 2\}$, so $\tilde{P}_1(\theta) = \tilde{P}_2(\theta)$ as desired.

The other direction is straightforward. By the definition of implementation, if M implements $\tilde{P}$ given market clearing function R then $\tilde{P}$ is a REE price function given decision rule M. □

A1 is an assumption on the payoff structure and the information structure. It is satisfied in typical models of the asset market. For example, A1 will hold if the function $\theta \mapsto v_i(a, \theta, x, p)$ is strictly monotone for all a, p, $x > 0$, and all i; and the distribution of posteriors induced by $h(\cdot|\theta)$ is monotone (in an appropriate sense) in θ.[24]

[24] A sufficient condition for the monotonicity of $\theta \mapsto v_i(a, \theta, x, p)$ is co-monotonicity of $\theta \mapsto v_i(a, \theta, x, p)$ for all a (when $v_i(a, \theta, x, p) \equiv u_i(x \cdot (\pi(a, \theta) - p) + w_i)$ this is equivalent to co-monotonicity of $\pi(a, \cdot)$). Posterior monotonicity will hold, for example, if $\sigma = \theta + \delta$ for some continuously distributed zero mean random variable

The following are sufficient conditions for A1, along with a concrete example that satisfies these conditions. For the example, let $\mathcal{A} = [0,1]$, $v_i(a, \theta, x, p) = u(x \cdot (\pi(a, \theta) - p) + w_i)$ and assume that $\pi(a, \theta)$ is weakly increasing in θ.

1. *Ordered signals.* Assume that $h(\cdot|\theta'') >_{MLR} h(\cdot|\theta')$ for all $\theta'' > \theta'$. This implies that the posteriors induced by signals are also ordered by MLR; higher signals induce MLR higher posteriors over Θ.

   Example: $\sigma = \theta + \varepsilon$, where ε is zero-mean noise.

2. *Single-crossing between x, θ.* We want individuals to demand more of the asset when they get a high signal. Assume therefore that $V_i(a, \theta, x, p)$ satisfies single crossing between x and θ. Monotonicity of demand is implied by standard MCS results (see Athey (2001)).

   Example: $u(x(\pi(a, \theta) - p) + w_i)$ satisfies single crossing in x and θ when $\theta \mapsto \pi(a, \theta)$ is increasing.

3. *Payoff equivalence.* For any $\mathcal{I}$, we want demand to be strictly increasing in θ unless $V_i(a, \theta, x, p) = V_i(a, \theta', x, p)$ for all $\theta, \theta' \in \mathcal{I}$.

   Example: This holds given the assumptions made thus far (in particular, monotonicity of π).

C.2 Forecasts and macro aggregates

Many policy decisions are made with reference to macroeconomic outcomes. For example, the government may decide to increase the amount of unemployment benefits or fund worker-retention programs depending on initial jobless claims or the unemployment rate. Many such problems also have a dynamic component. For example, businesses deciding whether or not to fire employees may care about the future unemployment rate both as a signal of demand and as a determinant of government worker-retention policies. In such settings, forward looking agents often make use of expert forecasts of the relevant macro variables, such as the unemployment rate.

C.2.1 Policy decision up-front

The policy maker may prioritize timeliness over accuracy when making certain policy decisions. In such cases it will be necessary for the policy maker to take an action before the

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relevant aggregate outcome has been realized. The policy maker will therefore make use of expert forecasts. For example, consider the problem of the government choosing the level of unemployment benefits. The policy maker may wish to act before relevant data, such as the unemployment rate in the coming month, has been collected. It must therefore relay on forecasts of the relevant variables. For simplicity, assume that the government conditions its benefits policy exclusively on expert forecasts of the unemployment rate for the coming month (it is straightforward to incorporate other sources of information).[25]

Forecasters wish to provide accurate estimates of the aggregate outcome (We will refer to this simply as the outcome from now on). If there are many forecasters, each individual expects their prediction to have only a small effect on overall expectations.[26] However they recognize that overall expectations will be used by the policy maker to take an action. These two factors imply that forecasters' private information will shape their expectations of policy decisions, which in turn will affect their forecasts.

This situation is easiest to model if we assume that forecasters observe each others' forecasts, and can make revisions based on what others say. We won't get into why might still "agree to disagree" even when they observe eachothers' forecasts. The consistency condition is that each forecaster doesn't want to change their forecast given those of the others, and the announced policy rule. Assume that for any fixed action by the policy makers such a fixed point exists. Then we are back to the original situation.

Formally, this model is very similar to the market price model. Assume there is a continuum of forecasters $\mathcal{F}$. Each forecaster $i \in \mathcal{F}$ receives a signal σ_i about the state. Signals are conditionally independent across forecasters. Forecasters make predictions about the value of some variable v, which will not be realized until after the principal has taken an action. Forecastes may have different models of the world, i.e. ways to map their information to a prediction, but assume for simplicity that there are only finitely many models in the population.

The principal bases their decision on some real-valued function of the profile of forecasts, the forecast aggregate. Forecasters iteratively revise their predictions based on their observations of the forecast aggregate. Thus we are looking for a rational expectations equilibrium conditional on the principal's announced decision rule M. In this context, assumption $A1$ can be restated as follows.

Fixed any principal action a, $\mathcal{I} \subseteq \Theta$ and $\theta \in \Theta$, and value of the forecast aggregate

[25] Another example is inflation expectations. Forward guidance could destroy the informativeness of the signal.

[26] Bloomberg surveys around 80 economists for predictions on the monthly unemployment rate.

f. Assume all forecasters know that the principal will take action a, that the state is in $\mathcal{I}$, and that the value of the forecast aggregate is f (in addition the their private signals). Let the $X(f|a, \mathcal{I}, \theta)$ be the new value of the forecast aggregate after forecasters have a chance to revise their predictions. This is a deterministic function since there are a continuum of forecasters with i.i.d. signals. Then forecasts reach a fixed point when

$$X(f|a, \mathcal{I}, \theta) = f.$$

Assume that there is unique fixed point for any $a, \mathcal{I}, \theta$ (which will be the case, for example, when individual forecasts, as well as the aggregator, are monotone in f), and denote this fixed point by $F^*(a, \mathcal{I}, \theta)$.

A1'. For any $a \in \mathcal{A}$ and $\mathcal{I} \subseteq \Theta$, if $F^*(a, \mathcal{I}, \theta) = F^*(a, \mathcal{I}, \theta')$ for all $\theta, \theta' \in \mathcal{I}$ then $F^*(a, \mathcal{I}, \theta) = F^*(a, \theta, \theta)$ for all $\theta \in \mathcal{I}$.

Assumption A1' is satisfied, for example, when the distribution of beliefs induced in the population is monotone (in an FOSD sense, with an appropriate order on beliefs) in the state, individuals forecasts are monotone in their beliefs, and the forecast aggregate is monotone in individual forecasts (in an FOSD sense).

The existence of R in this setting follows from Proposition 15

C.2.2 Policy decision ex-post

Some policy decisions may condition on realized outcomes, rather than expectations. For example, Congress extended the time frame for spending PPP funds after observing that companies had difficulty re-hiring employees. Congress also approved a second tranche of PPP funds after the first was exhausted. In this case companies will condition their payroll decisions or loan applications on expectations of future aggregate outcomes. Forecasters make predictions knowing that *i*) expectations will shape business decisions, and *ii*) business decisions will shape the policy response. Again, assume forecasters observe each others' forecasts. Then we need a fixed point that takes into account the feedback of forecasts on policy through business decisions.

Formally this case is similar to that discussed above. There are two periods. Some variable v will realize in the second period, and the principal will take an action in the second period conditional on v. Assume that the principal commits to a rule M mapping v to an action (We will discuss why later).

A unit mass of economic agents, call them individuals, care about the principal's future action, as well as some underlying state θ. In order to predict what the principal's action will be, individuals rely on the predictions of a set $\mathcal{F}$ of forecasters. Individuals are fairly simplistic: they aggregate forecaster predictions in some way, and assume that this forecast aggregate f will be the true value. They choose their actions based on the action implied by the principal's decision rule, as well as their own private information. Assume that individuals do not infer anything about the state from the forecasters' predictions.[27] The actions off all individuals, along with the state, jointly determine the outcome v. When all individuals expect the principal to take action $a \in \mathcal{A}$ and the state is θ, the aggregate outcome in the second period will be given by $J(a, \theta)$.

As before, there is a unit mass of forecasters, each of whom receives a private signal about the state. Forecasters observe the current value of the forecast aggregate and revise their decisions. As before a fixed point is reached when $X(f|a, \mathcal{I}, \theta) = f$. This function incorporates the fact that a affects the aggregate outcome through $J(a, \theta)$. Assuming $A1'$ holds, we have an R function by Proposition 15.

The interesting part of the ex-post decision model is that principal is not intending to use the equilibrium variable, in this case the forecast, to make a decision. The principal may not even be able to commit to a mapping M from the aggregate outcome to an action. It could just be that agents anticipate the principal to behave in a certain way ex-post. Nonetheless, the forecast will be determined as a fixed point, and this will impact the aggregate outcome, and thus the principal's decision.

C.2.3 Alternative model

Suppose that there is a single forecaster who gets a signal σ. The forecaster is aware of the effect that their prediction will have on individual behavior. The forecaster simply reports their expectation of the outcome v, when this is well defined. This will be well defined iff there is a fixed point to the function $f \mapsto \mathbb{E}[J(M(p), \theta)|\sigma]$. Let $R(a, \sigma) = \mathbb{E}[J(M(p), \theta)|\sigma]$. The analysis of the paper applies, with σ replacing R.

C.2.4 Adding forecast uncertainty

The fact that forecasters receive conditionally independent signals may seem unrealistic. It is straightforward to generalize to a situation in which signals are correlated. Assume that the state consists of a pair (κ, θ). As before, θ is the payoff-relevant state. κ simply determines the joint distribution of signals. Signals are conditionally independent given (κ, θ).

[27]This type of inference can be added without too much complication, with a suitable with a version of assumption A1.

Let Σ be the space of population signal profiles $\{\sigma_i\}_{i\in\mathcal{F}}$. Assume that there is a complete order on the space of signal profiles, which can be represented by a bijection $b : \Sigma \mapsto [0,1]$.[28] Since b is a bijection, $b(\{\sigma_i\}_{i\in\mathcal{F}})$ contains the same information as $\{\sigma_i\}_{i\in\mathcal{F}}$. Then when all forecasters expect the principal to take action a and know that the current forecast aggregate is f, and know that $b(\{\sigma_i\}_{i\in\mathcal{F}}) \in \mathcal{I} \subseteq [0,1]$, then the updated forecast aggregate will be $X(f|a,\mathcal{I},b)$. Then the analysis proceeds as before, except that b replaces θ. The principal will have to account for the residual uncertainty when choosing a decision rule.

The discussion in this section applies whether the set $\mathcal{F}$ of forecasters is finite on infinite. However the assumption of a continuum of forecasters remains convenient for two reasons. First, the assumption that such a bijection b exists makes more sense when there is a continuum of forecasters (see the example in the footnotes). Second, when there are finitely many forecasters they will behave strategically. For example, there in general no reason to expect that forecasts should reach a fixed point when forecasters take into the effect that their forecasts have on the forecast aggregate. For example, with a single forecaster trying to minimize the expected difference between their prediction and the actual outcome, unless they perfectly observe the state, it may be optimal for them to make a forecast that they know cannot be correct.

[28] For example, $\sigma_i = \varepsilon_i + \kappa + \theta$, where ε_i are i.i.d. random variables with a common bounded-support distribution. In this case the order on the set of signal profiles is given by the population mean $\int_{i\in\mathcal{F}} \sigma_i di$.

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