Formulário Transformadas de Laplace

$F(s)=\mathcal{L}\{f(t)\}(s)$, $s>s_f$; $G(s)=\mathcal{L}\{g(t)\}(s)$, $s>s_g$. $\mathcal{L}\{f(t)\}(s)=\int_0^{+\infty} e^{-st} f(t)dt$

$f(t)$	$F(s)$
1	$\frac{1}{s}$, $s>0$
t^n $(n\in\mathbb{N}_0)$	$\frac{n!}{s^{n+1}}$, $s>0$
e^{at} $(a\in\mathbb{R})$	$\frac{1}{s-a}$, $s>a$
$sen(at)$ $(a\in\mathbb{R})$	$\frac{a}{s^2+a^2}$, $s>0$
$cos(at)$ $(a\in\mathbb{R})$	$\frac{s}{s^2+a^2}$, $s>0$
$senh(at)$ $(a\in\mathbb{R})$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
$cosh(at)$ $(a\in\mathbb{R})$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
$f(t)+g(t)$	$F(s)+G(s)$, $s>s_f, s_g$
$\alpha f(t)$ $(\alpha\in\mathbb{R})$	$\alpha F(s)$, $s>s_f$
$e^{\lambda t}f(t)$ $(\lambda\in\mathbb{R})$	$F(s-\lambda)$, $s>s_f+\lambda$
$H_a(t)f(t-a)$ $(a>0)$	$e^{-as}F(s)$, $s>s_f$
$f(at)$ $(a>0)$	$\frac{1}{a}F(\frac{s}{a})$, $s>as_f$
$t^n f(t)$ $(n\in\mathbb{N})$	$(-1)^n F^{(n)}(s)$, $s>$ ordem exp. de f
$f'(t)$	$s\,F(s)-f(0)$, $s>$ ord. exp. de f
$f''(t)$	$s^2\,F(s)-s\,f(0)-f'(0)$, $s>$ ordens exp. de f, f'
$f^{(n)}(t)$	$s^n\,F(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$, onde $f^{(0)}\equiv f$, $s>$ ordens exp. de $f, f', \ldots, f^{(n-1)}$
$f'''(t)$	$s^3\mathcal{L}\{f(t)\}(s)-s^2f(0)-sf'(0)-f''(0)$
$f(t)*g(t)=\int_0^t f(u)g(t\text{-}u)du$	$F(s)G(s)$

$f(t)=\mathcal{L}^{-1}\{F(s)\}(t)$; $g(t)=\mathcal{L}^{-1}\{G(s)\}(t)$.

Aula 22: Prop. 3.2 (Deslocamento na transformada)

Prop. 3.2: Sejam $f : [0, +\infty[\to \mathbb{R}$ integrável em todo o intervalo $[0, b]$, com $b > 0$, e $\lambda \in \mathbb{R}$. Se $F(s) = \mathcal{L}\{f\}(s)$ existe para $s > s_f$, então também existe

$$\mathcal{L}\{e^{\lambda t} f(t)\}(s) \;=\; \mathcal{L}\{f(t)\}(s - \lambda)\,, \quad \text{para } s > \lambda + s_f.$$

Invertendo o deslocamento na transformada temos:

$$\boxed{\mathcal{L}^{-1}\{F(s-\lambda)\}(t) = e^{\lambda t}\,\mathcal{L}^{-1}\{F(s)\}(t).}$$

Ex. 1: $\mathcal{L}\{e^{-2t}\operatorname{sen}(6t)\}(s) = \mathcal{L}\{\operatorname{sen}(6t)\}(s+2) = \dfrac{6}{(s+2)^2 + 6^2}\,, \quad s > -2.$

Ex. 2: $\mathcal{L}^{-1}\{\frac{1}{(s-5)^2-4}\}(t) = e^{5t}\,\mathcal{L}^{-1}\{\frac{1}{s^2-4}\}(t) = e^{5t}\,\frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^2-2^2}\}(t) = \frac{e^{5t}}{2}\operatorname{senh}(2t), \quad (s > 5).$

Exercício: 1: Mostre que:

(1) $\mathcal{L}\{e^{t}\operatorname{sen}(3t)\}(s) = \dfrac{3}{s^2 - 2s + 10}$, $s > 1$.

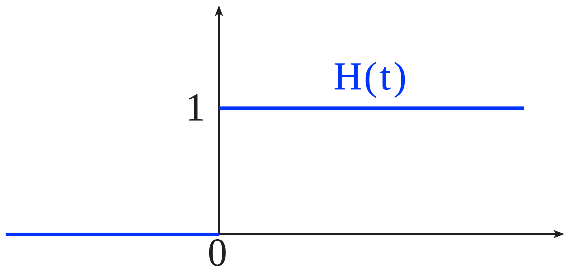
(2) $\mathcal{L}\{e^{\beta t}\cosh(\alpha t)\}(s) = \dfrac{s-\beta}{(s-\beta)^2 - \alpha^2}$, $s > |\alpha| + \beta$.

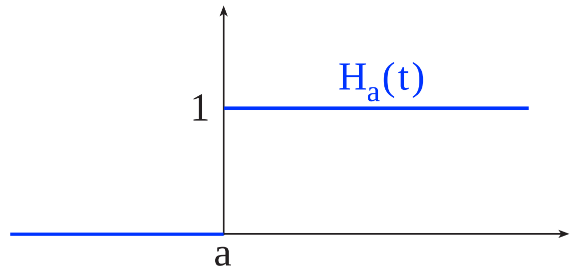
(3) $\mathcal{L}\{e^{2t}\operatorname{senh}(-\sqrt{3}\,t)\}(s) = \dfrac{-\sqrt{3}}{(s-2)^2 - 3}$, $s > \sqrt{3} + 2$.

(4) $\mathcal{L}^{-1}\{\dfrac{3}{s^2 - 2s + 10}\}(t) = e^{t}\operatorname{sen}(3t)$.

(5) $\mathcal{L}^{-1}\{\dfrac{s-\beta}{(s-\beta)^2 - \alpha^2}\}(t) = e^{\beta t}\cosh(\alpha t)$.

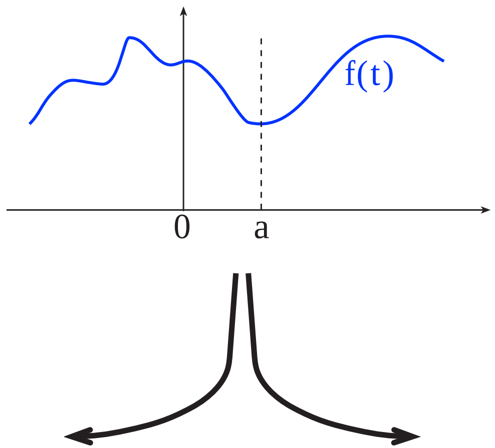
Aula 23: Função de Heaviside

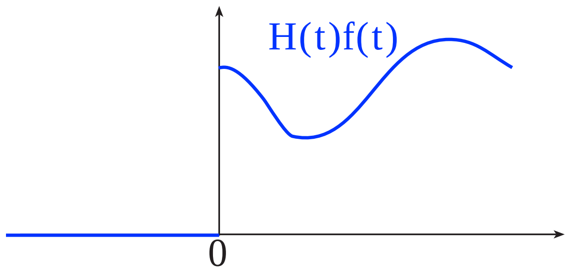


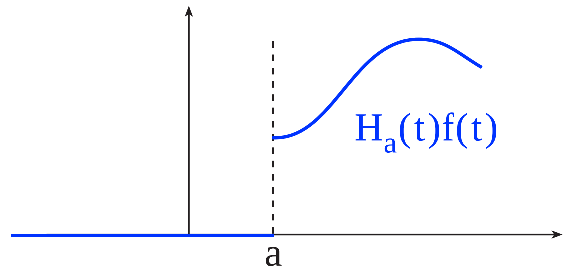


$$H(t) = \begin{cases} 0 \quad , & t < 0 \\ 1 \quad , & t \geq 0 \end{cases}$$

$$H_a(t) = H(t-a) = \begin{cases} 0 \quad , & t < a \\ 1 \quad , & t \geq a \end{cases}$$







Aula 23: Prop. 3.3 (Transformada do deslocamento)

Prop. 3.3: Sejam $f:\mathbb{R}\to\mathbb{R}$ integrável em todo o intervalo $[0,b]$, com $b>0$. Se $F(s)=\mathcal{L}\{f(t)\}(s)$ existe para $s>s_f$, então $\forall\, a\in\mathbb{R}^+$,

$$\mathcal{L}\{H_a(t)f(t-a)\}(s)=e^{-as}\mathcal{L}\{f(t)\}(s)$$

também existe para $s>s_f$.

Nota: $\forall\, f:\mathbb{R}\longrightarrow\mathbb{R}$, $\qquad H(t)f(t)=f(t)H(t)=\begin{cases}0 & ,\quad t<0\\ f(t) & ,\quad t\geq 0\,,\end{cases}$

pelo que $\boxed{\mathcal{L}\{H(t)f(t)\}(s)=\mathcal{L}\{f(t)\}(s)}$ $\qquad$ (se esta existir)

Nota: $$\mathcal{L}\{H_a(t)f(t)\}(s) \;=\; \mathcal{L}\{H_a(t)f(\underbrace{t+a}_{t}-a)\}(s) \;=\; e^{-as}\mathcal{L}\{f(t+a)\}(s). \qquad (a>0)$$

e^{-as}

Invertendo a transformada do deslocamento temos $(a>0)$:

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) \;=\; H_a(t)\,\mathcal{L}^{-1}\{F(s)\}(t-a)}$$

Aula 23: Exercícios 1(Transformada do deslocamento)

Mostre que:

(1) $\mathcal{L}\{H_a(t)f(t)\}(s) = e^{-as}\mathcal{L}\{f(t+a)\}(s)$
$(s > s_f)$

(2) $\mathcal{L}\{H_a(t)e^t\}(s) = \dfrac{e^{a(1-s)}}{s-1}$
$(s > 1)$

(3) $\mathcal{L}\{H_{\frac{\pi}{2}}(t)\operatorname{sen}(t)\}(s) = \dfrac{e^{-\frac{\pi}{2}s}}{s^2+1}$
$(s > 0)$

(4) $\mathcal{L}\{H_\pi(t)\cos(t)\}(s) = -\dfrac{s\,e^{-\pi s}}{s^2+1}$
$(s > 0)$

(5) $\mathcal{L}\{t^2 H_a(t)\}(s) = e^{-as}(\dfrac{a^2}{s} + \dfrac{2a}{s^2} + \dfrac{2}{s^3})$
$(s > 0)$

(6) $\mathcal{L}\{H_\pi(t)\operatorname{sen}(t)\}(2s) = -\dfrac{e^{-2\pi s}}{4s^2+1}$
$(s > 0)$

(7) $\mathcal{L}\{H_{-\frac{\pi}{4}}(t)\cos(2t)\}(s-1) = \dfrac{2e^{\frac{\pi}{4}(s-1)}}{(s-1)^2+4}$ $\quad (s > 1)$.

(8) $\mathcal{L}\{H_2(t)\cosh(\sqrt{2}t)\}(2s+1) = \frac{e^{-4s-2}}{2}(\frac{e^{2\sqrt{2}}}{2s+1-\sqrt{2}} + \frac{e^{-2\sqrt{2}}}{2s+1+\sqrt{2}})$ $\quad (s > \frac{\sqrt{2}-1}{2})$.

Calcule:

(9) $\mathcal{L}^{-1}\{\frac{e^{-as}}{s}\}(t)$ **(10)** $\mathcal{L}^{-1}\{\frac{12e^{5s}}{s^4}\}(t)$ **(11)** $\mathcal{L}^{-1}\{\frac{5e^{-6s}}{s^2+4}\}(t)$ **(12)** $\mathcal{L}^{-1}\{\frac{12e^{4s}(2+s)}{s^2-4}\}(t)$

$\cosh(a+b) = \cosh(a)\cosh(b) + \operatorname{senh}(a)\operatorname{senh}(b)$

$\operatorname{senh}(a+b) = \operatorname{senh}(a)\cosh(b) + \cosh(a)\operatorname{senh}(b)$

Prop. 3.4: Sejam $f : [0, +\infty[\to \mathbb{R}$ integrável em todo o intervalo $[0, b]$, com $b > 0$, e $a \in \mathbb{R}^+$. Se $F(s) = \mathcal{L}\{f\}(s)$ existe para $s > s_f$, então também existe

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\mathcal{L}\{f(t)\}(\frac{s}{a}), \quad \text{para } s > a\, s_f.$$

Exemplo:

$$\mathcal{L}\{(at)^n\}(s) = a^n \frac{n!}{s^{n+1}}, \quad \text{para } s > 0 \quad (a > 0).$$

Aula 23: Derivada da transformada

Prop. 3.5: Se $f : [0, +\infty[\to \mathbb{R}$ é seccionalmente contínua e de ordem exponencial a (i.e. $|f(t)| \leq Me^{at}$), então, para todo $n \in \mathbb{N}_0$, existe

$$\mathcal{L}\{t^n\, f(t)\}(s) = (-1)^n \mathcal{L}\{f(t)\}^{(n)}(s)\,, \quad \text{para } s > a.$$

Nota 1: $\displaystyle\int_0^{+\infty} |e^{-st} t^n f(t)| dt \leq M \int_0^{+\infty} e^{-(s-a)t} t^n dt = \mathcal{L}\{t^n\}(s-a).$

Nota 2: $\displaystyle\mathcal{L}\{f(t)\}'(s) = \frac{d}{ds} \int_0^{+\infty} e^{-st} f(t) dt = \int_0^{+\infty} \frac{d}{ds} e^{-st} f(t) dt = -\mathcal{L}\{t f(t)\}(s).$

Exercício 3: Mostre que:

(1) $\displaystyle\mathcal{L}\{t^2 e^{2t}\}(s) = \frac{2}{(s-2)^3}, \ \ s > 2.$

(2) $\displaystyle\mathcal{L}\{t \operatorname{sen}(2t)\}(s) = \frac{4s}{(s^2+4)^2}, \ \ s > 0.$

(3) $\displaystyle\mathcal{L}\{t^2 \cos(3t)\}(s) = \frac{2s^3 - 54s}{(s^2+9)^3}, \ \ s > 0.$ $\quad \big(\text{para } s > a, \forall\, a > 0 \Leftrightarrow s > 0.\big)$

Invertendo a Derivada da transformada: $\boxed{t^n\, \mathcal{L}^{-1}\{F(s)\}(t) = (-1)^n\, \mathcal{L}^{-1}\{F(s)^{(n)}\}(t)}$

$$\mathcal{L}^{-1}\{F(s)\}(t) = (-1)^n\, t^{-n}\, \mathcal{L}^{-1}\{F(s)^{(n)}\}(t)$$

Aula 23: Exercícios 2

1. Determine as transformadas inversas de Laplace das seguintes funções $F = F(s)$, consideradas em domínios adequados:

(a) $F(s) = \frac{e^{-\pi s}}{s^2+16}$.

(b) $F(s) = \operatorname{arctg}\left(\frac{4}{s}\right)$.

(c) $F(s) = \ln\left(1 + \frac{1}{s^2}\right)$.

(d) $F(s) = \frac{d}{ds}\frac{1-e^{-5s}}{s}$.

(e) $F(s) = \frac{2}{s^3-4s^2+5s}$.

2. Determine o par de soluções $y = y(t)$, $z = z(t)$, $t \geq 0$, do sist. de equações:

$$\begin{cases} 3y' + z' + 2y = 1 \\ y' + 4z' + 3z = 0 \end{cases} \quad y(0) = z(0) = 0\,. \ \left(\text{ Tome } Y = \mathcal{L}\{y\}(s) \ \text{ e } \ Z = \mathcal{L}\{z\}(s)\right)$$

Sol. intercalar: $Z = -\frac{1}{(s+1)(s+\frac{6}{11})} = \frac{11}{5}\frac{1}{s+1} - \frac{11}{5}\frac{1}{s+\frac{6}{11}}$ e $Y = \frac{4s+3}{s(s+1)(s+\frac{6}{11})} = \frac{11}{2}\frac{1}{s} - \frac{11}{5}\frac{1}{s+1} - \frac{33}{10}\frac{1}{s+\frac{6}{11}}$

Sol: $z = \frac{11}{5}e^{-t} - \frac{11}{5}e^{-\frac{6}{11}t}$ e $y = \frac{11}{2} - \frac{11}{5}e^{-t} - \frac{33}{10}e^{-\frac{6}{11}t}$.

3. Usando transformadas de Laplace, justifique que:

(a) $\int_0^\infty te^{-2t}\cos(t)dt = \frac{3}{25}$.

(b) $\int_0^\infty t^3e^{-t}\operatorname{sen}(t)dt = 0$.

Aula 23: Transformada da derivada

Prop. 3.6: Se as funções f, f', f'',..., $f^{(n-1)}$ $(n \in \mathbb{N})$ são todas de ordem exponencial s_0, para algum $s_0 \in \mathbb{R}$, e se $f^{(n)}$ existe e é seccional/ contínua em $[0, +\infty[$, então existe $\mathcal{L}\{f^{(n)}(t)\}(s)$ para $s > s_0$ e

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - p_{n-1}(s),$$

onde p_{n-1} é o seguinte polinómio de grau $n-1$ em s:

$$p_{n-1}(s) = s^{n-1} f(0) + s^{n-2} f'(0) + \cdots + s\, f^{(n-2)}(0) + f^{(n-1)}(0).$$

Por exemplo:

$$\mathcal{L}\{f'(t)\}(s) = s\,\mathcal{L}\{f(t)\}(s) - f(0).$$

$$\mathcal{L}\{f''(t)\}(s) = s^2\,\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0).$$

$$\mathcal{L}\{f'''(t)\}(s) = s^3\,\mathcal{L}\{f(t)\}(s) - s^2 f(0) - sf'(0) - f''(0).$$

Aula 23: Exercícios 3 (transformada da derivada)

1. Seja $f : [0, \infty[\to \mathbb{R}$ de ordem exponencial à direita e tal que f' existe e é contínua. Mostra que

$$\begin{aligned} \mathcal{L}\{f'(t)\}(s) &= \left[e^{-st} f(t)\right]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st} f(t)\,dt && (1) \\ &= s\mathcal{L}\{f(t)\}(s) - f(0), && (2) \end{aligned}$$

desde que os valores de s sejam tomados suficientemente grandes (tenta ser preciso aqui).

2. Aplica por duas vezes o resultado do exercício 1 acima para concluir, no caso de f, f' serem ambas de ordem exponencial à direita e de f'' existir e ser contínua, que

$$\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0),$$

desde que s seja tomado suficientemente grande (tenta ser mais preciso aqui).

3. Convence-te de que, iterando o processo, no caso de $f, f', \ldots, f^{(n-1)}$ serem todas de ordem exponencial à direita e de $f^{(n)}$ existir e ser contínua, então

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0),$$

desde que s seja tomado suficientemente grande (tenta ser mais preciso aqui).

4. Mostra que $\mathcal{L}\{\cos(at)\}(s)$ se pode obter de $\mathcal{L}\{\sin(at)\}(s)$ usando a propriedade anterior.

Aula 23: Exercícios 4

1. Determine:

(a) $\mathcal{L}\{f'(t)\}(s)$ em que $f(t) = senh(-\frac{3}{2}t)$.

(b) $\mathcal{L}\{f''(t)\}(s)$ em que $f(t) = t^2 e^{2t}$.

(c) $\mathcal{L}\{f'''(t)\}(s)$ em que $f(t) = e^{-t} - sen(2t)$.

2. Determine a transformada de Laplace das seguintes funções e o respectivo domínio:

(a) $24\, cos(8t) + t^2 - 48\, e^{-2t}$.

(b) $e^{6t}\, sen(3t)$.

(c) $t^2\, e^{4t}\, cosh(6t)$.

(d) $4 + t + 5t^2 - \pi\, e^{-2t} t^{30}$.

(e) $(10 - H_\pi) sen(t)$.

(f) $(t-8)^3\, e^{4(t-8)} H_8$.

3. Seja $f(t) = \text{arctg}\,(t)$.

(a) Mostre que existe transformada de Laplace da função $f(t)$.

(b) Mostre que $\mathcal{L}\{\frac{1}{1+t^2}\}(s) = s\mathcal{L}\{\text{arctg}\,(t)\}(s)$.

Formulário Transformadas de Laplace

$F(s)=\mathcal{L}\{f(t)\}(s)$, $s>s_f$; $\quad G(s)=\mathcal{L}\{g(t)\}(s)$, $s>s_g$. $\qquad \mathcal{L}\{f(t)\}(s)=\int_0^{+\infty} e^{-st}f(t)dt$

$f(t)$	$F(s)$
1	$\frac{1}{s}$, $\quad s>0$
t^n $(n\in\mathbb{N}_0)$	$\frac{n!}{s^{n+1}}$, $\quad s>0$
e^{at} $(a\in\mathbb{R})$	$\frac{1}{s-a}$, $\quad s>a$
$sen(at)$ $(a\in\mathbb{R})$	$\frac{a}{s^2+a^2}$, $\quad s>0$
$cos(at)$ $(a\in\mathbb{R})$	$\frac{s}{s^2+a^2}$, $\quad s>0$
$senh(at)$ $(a\in\mathbb{R})$	$\frac{a}{s^2-a^2}$, $\quad s>\|a\|$
$cosh(at)$ $(a\in\mathbb{R})$	$\frac{s}{s^2-a^2}$, $\quad s>\|a\|$
$f(t)+g(t)$	$F(s)+G(s)$, $\quad s>s_f, s_g$
$\alpha\, f(t)$ $(\alpha\in\mathbb{R})$	$\alpha\, F(s)$, $\quad s>s_f$
$e^{\lambda t}f(t)$ $(\lambda\in\mathbb{R})$	$F(s-\lambda)$, $\quad s>s_f+\lambda$
$H_a(t)f(t-a)$ $(a>0)$	$e^{-as}F(s)$, $\quad s>s_f$
$f(at)$ $(a>0)$	$\frac{1}{a}F(\frac{s}{a})$, $\quad s>as_f$
$t^n f(t)$ $(n\in\mathbb{N})$	$(-1)^n F^{(n)}(s)$, $\quad s>$ ordem exp. de f
$f'(t)$	$s\,F(s)-f(0)$, $\quad s>$ ord. exp. de f
$f''(t)$	$s^2\,F(s)-s\,f(0)-f'(0)$, $\quad s>$ ordens exp. de f, f'
$f^{(n)}(t)$	$s^n\,F(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$, onde $f^{(0)}\equiv f$, $\quad s>$ ordens exp. de $f, f', \ldots, f^{(n-1)}$
$f'''(t)$	$s^3\mathcal{L}\{f(t)\}(s)-s^2f(0)-sf'(0)-f''(0)$
$f(t)*g(t)=\int_0^t f(u)g(t\text{-}u)du$	$F(s)G(s)$

$f(t)=\mathcal{L}^{-1}\{F(s)\}(t)$; $\quad g(t)=\mathcal{L}^{-1}\{G(s)\}(t)$.

Aula 23: Resolução de EDOS lineares com transformadas de Laplace

Note-se que a transformada de Lapace pode ser usada para resolver equações diferenciais lineares de coeficientes constantes em problemas de Cauchy (ou mesmo o integral geral).

Exercício 5: Resolva os seguintes problemas de Caychy envolvendo EDOS lineares de coeficientes constantes:

1. $y' + 2y = e^t$, $y(0) = 1$.

2. $y'' + 2y' + 10y = 1$, $y'(0) = 0$, $y(0) = 1$.

3. $y'' - 6y' + 5y = 0$, $y(0) = 1$, $y'(1) = -3$.

4. $y' + 2y = 4te^{-2t}$, $y(0) = -3$.

5. $y'' - 3y' + 2y = e^{3t}$, $y'(0) = 0$, $y(0) = 1$.

6. $y' = 1 - \operatorname{sen} t - \displaystyle\int_0^t y(u)du \ , \qquad y(0) = 0.$

Aula 23: Resolucao do exercicio 5.2 com as Cond^s. Iniciais modificadas

Exercício 5:

Condições iniciais:

2) $y'' + 2y' + 10y = 1$, $y'(0) = 1$, $y(0) = 0$

$y = ?$

Aplicar a transformada de Laplace:

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = \mathcal{L}\{1\}$ a propondo $\underline{\underline{s}}$

$s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1} + 2\, s\mathcal{L}\{y\} - \overset{0}{y(0)} + 10\mathcal{L}\{y\} = \frac{1}{s}$

Designemos por $Y = \mathcal{L}\{y\}(s)$ $y = \mathcal{L}^{-1}\{Y\}(t)$

Temos: $s^2 Y - 1 + 2sY + 10Y = \frac{1}{s}$

$\Leftrightarrow Y(s^2 + 2s + 10 - 1) = \frac{1}{s}$

$\Leftrightarrow Y = \frac{1}{s(s^2+2s+9)}$

Então a solução $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s(\underline{s^2+2s+9})}\right\}(t) =$

$b^2 - 4ac < 0$
raízes complexas

$\frac{1}{s(s^2+2s+9)}$ = frações simples $= \frac{A}{s} + \frac{Bs+C}{s^2+2s+9}$ A, B, C = por determinar (no fim)

$y = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{s^2+2s+9}\right\}(t)$

$= A\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t)}_{1} + B\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+8}\right\} + C\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+8}\right\}$

$F(s+1) = F(s-a)$, $a = -1$

$= A + Be^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+8}\right\}(t) + \frac{Ce^{-t}}{\sqrt{8}}\mathcal{L}^{-1}\left\{\frac{1\times\sqrt{8}}{s^2+(\sqrt{8})^2}\right\}(t)$

$= A + Be^{-t}\cos(\sqrt{8}t) + \frac{Ce^{-t}}{\sqrt{8}}\,\mathrm{sen}(\sqrt{8}t)$

$A = \frac{1}{9}$ $B = -\frac{1}{9}$ $C = -\frac{2}{9}$

Aula 23: Resolucao do exercicio 5.6

Exercício 5

6) $y' = 1 - \operatorname{sen} t - \int_0^t y(u)\,du \quad , y(0) = 0$

$y'(0) = 1 - 0 - 0 = 1$

$y'' = -\cos t - y$

logo temos: $y'' + y = -\cos t \quad , y(0) = 0, \; y'(0) = 1$

Aplicando transformadas de Laplace:

$\mathcal{L}\{y'' + y\}(s) = -\mathcal{L}\{\cos t\}(s)$ vou omitir s

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = -\dfrac{s}{s^2+1}$

$s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1} + \mathcal{L}\{y\} = -\dfrac{s}{s^2+1}$

seja $Y = \mathcal{L}\{y\}$.

$s^2 Y + Y = 1 - \dfrac{s}{s^2+1} \iff Y(s^2+1) = 1 - \dfrac{s}{s^2+1}$

$Y = \dfrac{1}{s^2+1} - \dfrac{s}{s^2+1}\dfrac{1}{s^2+1}$

Solução da eq. dif. com condições iniciais:

$y = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}(t) - \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\dfrac{1}{s^2+1}\right\}(t)$

$= \operatorname{sen} t - \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\}(t) * \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}(t)$

$= \operatorname{sen} t - \cos t * \operatorname{sen} t$

$*$ = produto de convolução (não sai)

Aula 23: Exercícios 6 (TL)

Determine a solução do seguinte problema de valores iniciais:

1. $y'' + t = 0$, $y(0) = 1$ e $y'(0) = 0$. Sol: $y = 1 - \frac{t^3}{6}$
2. $y'' - 6y' + 5y = 0$, $y(0) = 1$ e $y'(0) = -3$. Sol: $y = 2e^t - e^{5t}$
3. $y' + 2y = 4te^{-2t}$, $y(0) = -3$. Sol: $y = (2t^2 - 3)e^{-2t}$
4. $y'' - 2y' + 2y = \cos(t)$, $y(0) = 1$ e $y'(0) = 0$. Sol: $\frac{1}{5}(\cos(t) - 3\,\mathrm{sen}\,(t) + 4e^t\cos(t) - 2e^t\,\mathrm{sen}\,(t))$
5. $y' - y = e^{-t}$, $y(1) = 0$. Sol: $z(x) = e^{-1}\mathrm{senh}\,(x)$, $\rightsquigarrow$ $y = e^{-1}\mathrm{senh}\,(t-1)$

   [**Sugestão:** faça a mudança de variáveis $t - 1 = x \rightsquigarrow z(x) = y(x+1)$, $z' = \frac{dz}{dx} = \frac{dy}{dt}\frac{dt}{dx} = y'$, $z(0) = y(1) = 0$]
6. $y'' - y = \cosh(t)$, $y(2) = 0$, $y'(2) = 0$.

7. Determine um integral geral da equação diferencial $y'' - y' + y = H_2(t)$.

[**Sugestão:** faça $y(0) = C_1$ e $y'(0) = C_2$]

Formulário Derivadas e Primitivas quase imediatas

$(u^p)' = p\, u^{p-1}\, u'$

$(\operatorname{arcsen}(u))' = \dfrac{u'}{\sqrt{1-u^2}}$

$(\ln u)' = \dfrac{u'}{u}$

$(\operatorname{arctg}(u))' = \dfrac{u'}{1+u^2}$

$(\cos u)' = -u' \operatorname{sen} u$

$(\sec u)' = u' \sec(u) \operatorname{tg}(u)$

$(\operatorname{sen} u)' = u' \cos u$

$(\operatorname{cosec} u)' = -u' \operatorname{cosec}(u) \operatorname{cotg}(u)$

$(\operatorname{tg} u)' = u' \sec^2 u$

$(e^u)' = u'\, e^u$

$(\operatorname{cotg} u)' = -u' \operatorname{cosec}^2 u$

$(a^u)' = \dfrac{u' a^u}{\ln a}\ , \ a \in \mathbb{R}^+ \backslash \{1\}$

$(\operatorname{senh}^{-1} u)' = \dfrac{u'}{\sqrt{1+u^2}}$

$(uv)' = u'v + uv'$

$(\operatorname{tgh} u)' = u' \operatorname{sech}^2 u$

$(\operatorname{sech} u)' = -u' \operatorname{sech} u \operatorname{tgh} u$

$(\operatorname{senh}^{-1} u)' = \frac{u'}{\sqrt{1+u^2}}$

$(\operatorname{tgh}^{-1} u)' = \frac{u'}{1-u^2}$

$\int u'\, u^p\, dx = \frac{u^{p+1}}{p+1} + C,$
$(p \neq -1)$

$\int \frac{u'}{\sqrt{1-u^2}} dx = \arcsin(u) + C$

$\int \frac{u'}{u}\ \mathrm{d}x = \ln|u| + C$

$\int \frac{u'}{1+u^2}\ \mathrm{d}x = \arctan(u) + C$

$\int u' \sin u\ \mathrm{d}x = -\cos u + C$

$\int u' \sec u \tan u\ \mathrm{d}x = \sec u + C$

$\int u' \cos u\, dx = \sin u + C$

$\int u' \operatorname{cosec} u \mathrm{cotgudx} = -\mathrm{cosec\, u} + \mathrm{C}$

$\int u' \sec^2 u\ \mathrm{d}x = \tan u + C$

$\int u' e^u\ \mathrm{d}x = e^u + C$

$\int u' \operatorname{cosec}^2 u\ \mathrm{d}x = -\mathrm{cotgu} + \mathrm{C}$

$\int u' a^u\ \mathrm{d}x = \frac{a^u}{\ln a} + C, \quad a \in \mathbb{R}^+ \backslash \{1\}$

$\int \frac{u'}{\sqrt{1+u^2}}\ \mathrm{d}x = \operatorname{senh}^{-1} u + C$

$\int u'v + uv'\ \mathrm{d}x = uv + C$

$\int u' \operatorname{sech}^2 u\ \mathrm{d}x = \operatorname{tgh} u + C$

$\int u' \operatorname{sech} u \operatorname{tgh} u\ \mathrm{d}x = -\operatorname{sech} u + C$

$\int \frac{u'}{\sqrt{1+u^2}}\ \mathrm{d}x = \operatorname{senh}^{-1}(u) + C$

$\int \frac{u'}{1-u^2}\ \mathrm{d}x = \operatorname{tgh}^{-1} u + C$

$\int u' \sec u\ \mathrm{d}x = \ln|\sec u + \operatorname{tg} u| + C$