Week #7: Stochastic Simulation

March 10, 2025

1 Stochastic Simulation

Stochastic simulation provides a robust framework for modeling and analyzing systems that evolve probabilistically over time. Markov chains play a central role in this domain, as they capture the evolution of system states based on local probabilistic rules. Despite their simplicity, Markovian models can reproduce complex patterns observed in nature, ranging from ecological processes to economic and social dynamics.

We explore two fundamental stochastic simulation paradigms:

- **Stochastic Cellular Automata**: Grid-based models where local probabilistic rules govern state transitions.
- **Agent-Based Models**: Systems where autonomous agents make decisions based on their own states and interactions.

1.1 Stochastic Cellular Automaton

A **stochastic cellular automaton** is a discrete system consisting of a grid of cells, where each cell evolves according to probabilistic rules that depend on its own state and the states of its neighbors. Formally, we define a cellular automaton as follows:

Definition.

A stochastic cellular automaton is a tuple $(\mathcal{G}, S, \mathcal{N}, T)$, where:

- $\mathcal{G}$ is a discrete lattice (e.g., a 2D grid).
- S is a finite set of possible states for each cell.
- $\mathcal{N} : \mathcal{G} \to 2^{\mathcal{G}}$ is a neighborhood function assigning each cell $C \in \mathcal{G}$ a set of neighboring cells.
- $T : S^{|\mathcal{N}|} \to \mathcal{P}(S)$ is a probabilistic transition function determining the next state distribution of a cell based on its current state and its neighbors.

The state of each cell at time $t + 1$ is determined by:

$$S_C(t+1) \sim T\big(S_C(t), \mathcal{N}(S_C(t))\big),$$

where the right-hand side represents a stochastic transition governed by T. The Markovian nature of this model ensures that the evolution of the system depends only on the present configuration.

Example. Forest Fire Model

The forest fire model is a probabilistic cellular automaton that simulates wildfire spread across a two-dimensional landscape. The environment is a grid where each cell represents a patch of land and can be in one of four states:

$$S_C \in \{\text{Grass}, \text{Tree}, \text{Burning}, \text{Empty}\}.$$

The system follows the stochastic transition rules:

- A **Grass** cell transitions into a **Tree** with probability p_{growth}.
- A **Tree** cell ignites and becomes **Burning** with probability p_{ignite} if any neighboring cell is Burning.
- A **Tree** may spontaneously ignite with probability $p_{\text{spontaneous}}$.
- A **Burning** cell transitions to **Empty**, indicating the depletion of combustible material.

Mathematically, we define the transition probabilities for a given cell C as:

$$P(S_C(t+1) = s \mid S_C(t), \mathcal{N}(S_C(t))) = \begin{cases} p_{\text{growth}}, & \text{if } S_C(t) = \text{G}, s = \text{T} \\ p_{\text{ignite}}, & \text{if } S_C(t) = \text{T}, s = \text{B}, \exists C' \in \mathcal{N}(C), S_{C'}(t) = \text{B} \\ p_{\text{spontaneous}}, & \text{if } S_C(t) = \text{T}, s = \text{B} \\ 1, & \text{if } S_C(t) = \text{B}, s = \text{E} \\ 0, & \text{otherwise.} \end{cases}$$

This stochastic model enables exploration of wildfire propagation under different environmental conditions, providing insights into forest management and disaster mitigation strategies.

1.2 Agent-Based Modeling

Agent-Based Modeling (ABM) is a computational approach that simulates the actions and interactions of autonomous agents, revealing emergent system-wide patterns. Unlike cellular automata, which rely on uniform local rules, ABMs allow agents to have heterogeneous characteristics and adaptive behaviors.

1.2.1 Mathematical Representation of Agents

An **agent** is an autonomous entity characterized by states and decision rules. Formally, an agent A is represented as:

$$A = (S, R, G, T, M, D),$$

where:

- S is the **state** space of the agent.

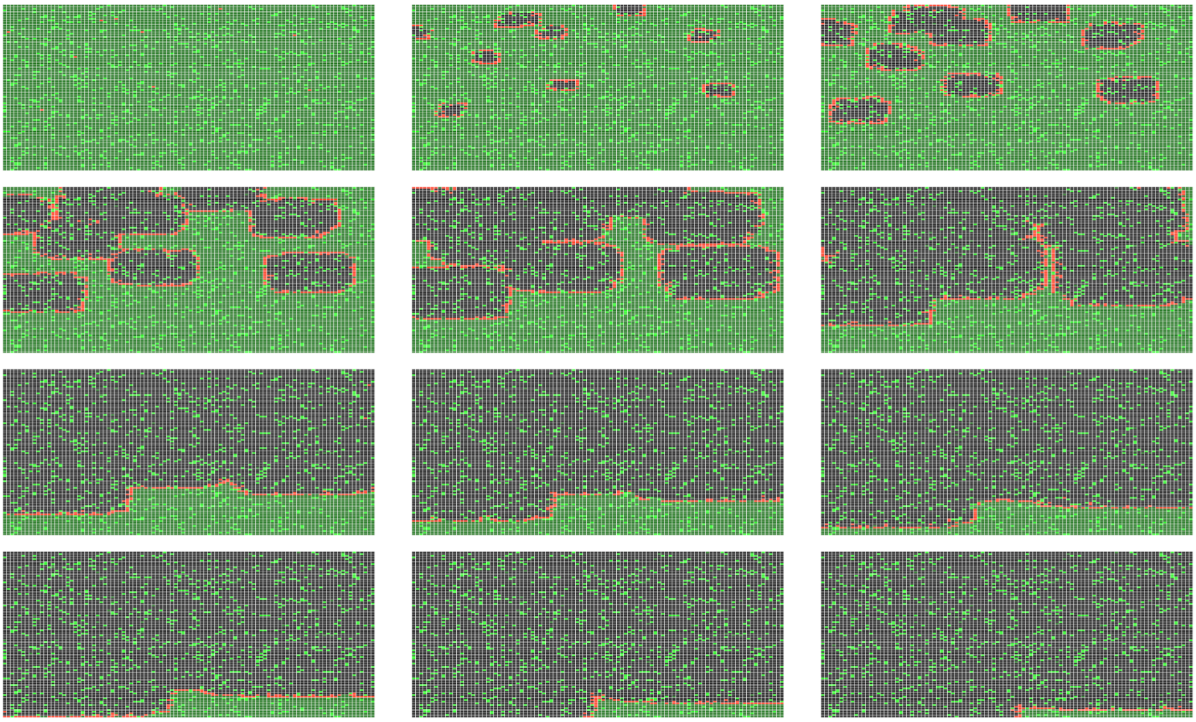

Figure 1: Snapshots from the forest fire simulation, illustrating the spread and aftermath of the fire.

- R denotes its **role** in the system.
- G is the **goal function**, defining its objectives.
- T specifies **transition rules**, governing how its state evolves.
- M represents its **memory** of past states.
- D defines the **decision-making process**, determining how the agent acts.

The state evolution of agent i is given by:

$$S_i(t+1) = f\big(S_i(t), \mathcal{N}_i(t)\big),$$

where $\mathcal{N}_i(t)$ represents the states of neighboring agents.

Example. Ant Foraging Simulation

A classic example of ABM is the simulation of ant foraging behavior, where a colony of ants searches for food on a two-dimensional grid.

Each ant is modeled as an agent with state:

$$S_a = (X, Y, F),$$

where:

- X, Y denote the ant's position on the grid.
- F is a binary variable: $F = 1$ if carrying food, $F = 0$ otherwise.

The system follows these behavioral rules:

1. **Food Acquisition:** An ant picks up food if it enters a food-containing cell ($F = 0 \rightarrow 1$).
2. **Returning to Nest:** An ant carrying food moves toward the nest.
3. **Pheromone Deposition:** While carrying food, an ant leaves a pheromone trail.
4. **Following Pheromones:** Ants without food are attracted to higher pheromone concentrations.
5. **Random Exploration:** If no food or pheromones are nearby, the ant moves randomly.

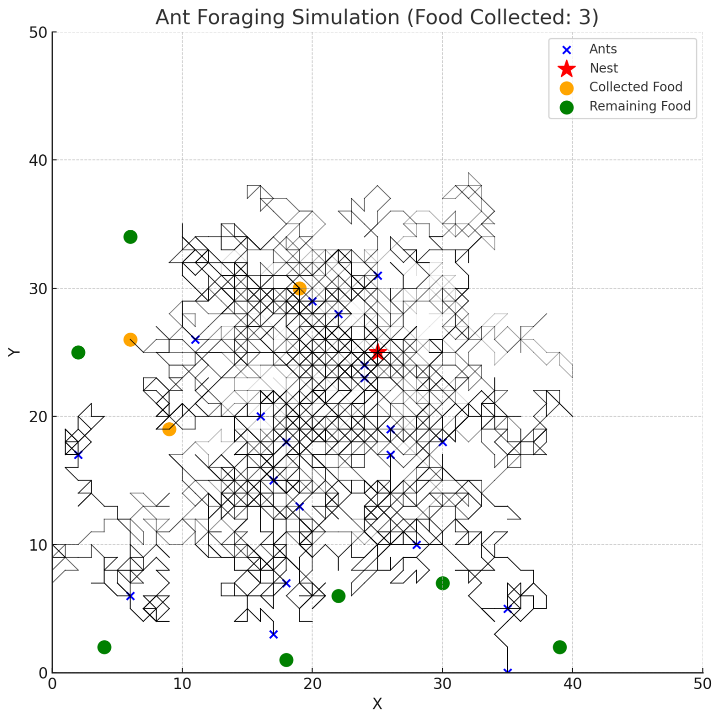

Figure 2: Ant Foraging Simulation. The plot illustrates ant trajectories in a 50×50 grid. The nest is marked by a red star, with food locations in orange.

Both stochastic cellular automata and agent-based models demonstrate how simple probabilistic rules and local interactions can generate complex emergent behaviors. Cellular automata provide a structured framework for modeling grid-based dynamics, while ABMs allow for heterogeneous, adaptive behavior, making them powerful tools in studying ecological, economic, and social systems.