Lecture Notes: Decision Theory

Decision theory is an interdisciplinary study involving psychology, economics, statistics, and philosophy, focusing on the nuances of decision-making, particularly in situations of uncertainty. The primary goal of this field is to dissect the decision-making process and enhance its effectiveness. At the heart of decision theory lies the concept of expected utility (EU), which represents the cumulative utility of all potential outcomes, each weighted by its probability. This is mathematically articulated as:

Definition 1 (Expected Utility)**.** *The Expected Utility (EU) is formulated by the equation:*

$$EU = \sum_{i=1}^{n} p_i u(x_i), \tag{1}$$

where p_i represents the probability of outcome x_i, and $u(x_i)$ is the utility correlated with that outcome.

The utility function $u(x)$, pivotal in decision theory, quantifies an individual's preferences among different outcomes. Consider a decision scenario such as choosing between receiving \$100 today or \$110 in a year. The utility function captures the individual's preference for immediate versus delayed rewards, thus influencing the decision-making process.

Example 1. *Consider an individual facing the decision to invest in a stock. The utility function $u(x)$ could represent the satisfaction gained from different monetary outcomes. Let's say the stock has a 50% chance of doubling the investment (gain) and a 50% chance of losing half of it (loss). If the individual's investment is \$100, the expected utility could be calculated as follows, assuming a simple utility function where the utility is equal to the monetary value:*

$$\begin{aligned} EU &= 0.5 \times u(\$200) + 0.5 \times u(\$50) \\ &= 0.5 \times \$200 + 0.5 \times \$50 \\ &= \$125. \end{aligned}$$

This example demonstrates how expected utility provides a framework to evaluate decisions under risk. The loss function, as the negative of utility, would mirror these values with negative signs. The concept of risk, as expected loss, would then be applied to assess the potential downsides of the investment decision.

Decision-making environments typically fall into one of three categories. In decisions under certainty, the outcome of each action is known. Under risk, the probabilities of outcomes are known, whereas under uncertainty, these probabilities are unknown. This classification helps in understanding and applying the concepts of utility and loss in various decision-making scenarios.

In this framework, we define loss as the inverse of utility.

Definition 2 (Loss)**.** *The loss associated with an outcome is the negative utility, defined as:*

$$L(x_i) = -u(x_i). \tag{2}$$

These foundational elements are crucial in statistical decision theory, guiding the pursuit of optimal decisions. They find applications across diverse fields, from public policy analysis to balancing potential benefits and costs in various sectors.

Pascal's Dilemma

One of the earliest applications of decision theory, which is closely related to game theory, is Pascal's Wager. The French mathematician Blaise Pascal proposed a decision problem to determine whether one should believe in God. The wager is simple: either God exists or He doesn't. Pascal argues that believing in God is the most rational choice, given the possible outcomes.

	God Exists	**God Doesn't Exist**
Believe in God	Infinite reward (Heaven)	Finite loss (wasted time in worship)
Don't Believe in God	Infinite loss (Hell)	Finite gain (time saved)

To quantify the utilities in Pascal's Wager, let's assign symbolic values to the outcomes. Let U_{Heaven} represent the utility of achieving Heaven, U_{Hell} for Hell, U_{Worship} for the finite loss due to worship, and $U_{\text{Time Saved}}$ for the finite gain of time saved by not worshipping. Assuming the existence and non-existence of God are equally probable, let p represent the probability, such that $p = 0.5$.

Calculating Expected Utilities

1. *Believing in God:* The expected utility for believing in God, denoted as EU_{Believe}, is a combination of the utility of going to Heaven and the loss due to worship. Thus, it is given by:

$$EU_{\text{Believe}} = p \times U_{\text{Heaven}} + (1 - p) \times U_{\text{Worship}}. \tag{3}$$

2. *Not Believing in God:* Similarly, the expected utility for not believing in God, $EU_{\text{Not Believe}}$, is the combination of the utility of Hell and the gain of time saved. This is expressed as:

$$EU_{\text{Not Believe}} = p \times U_{\text{Hell}} + (1 - p) \times U_{\text{Time Saved}}. \tag{4}$$

Assigning Values to Utilities

To proceed with the calculations, we need to assign values to the utilities. Given the nature of Pascal's Wager, we assign:

- $U_{\text{Heaven}} = +\infty$ (representing an infinite reward),
- $U_{\text{Hell}} = -\infty$ (representing an infinite loss),
- U_{Worship} and $U_{\text{Time Saved}}$ as finite values, where $U_{\text{Worship}} < U_{\text{Time Saved}}$ since the loss due to worship is considered less significant than the gain of time saved.

Evaluating the Decision

Given these values, the expected utilities can be evaluated:

- For EU_{Believe}, any term multiplied by $+\infty$ (representing the infinite utility of Heaven) will dominate the equation. Therefore, EU_{Believe} effectively becomes $+\infty$.
- Similarly, for $EU_{\text{Not Believe}}$, the term involving $-\infty$ (the infinite loss of Hell) dominates, making $EU_{\text{Not Believe}} = -\infty$.

In Pascal's Wager, from a purely utilitarian perspective, the decision to believe in God offers a higher expected utility due to the infinite reward of Heaven.

Much like hypothesis testing in statistics, the cost of making an incorrect decision (Type I or Type II errors) must be weighed. In hypothesis testing, the consequences of incorrectly rejecting a true null hypothesis (Type I error) versus failing to reject a false null hypothesis (Type II error) are considered.

In contemporary discussions, Pascal's Wager finds a parallel in the debate over climate change. The dilemma of whether to acknowledge and address climate change or to dismiss it can be examined through a similar analytical lens.

	Climate Change is Real	**Climate Change Isn't Real**
Act to Combat It	Avert catastrophic outcomes	Unnecessary expenditure of resources
Do Nothing	Risk grave consequences	Conservation of resources

This decision matrix contrasts the outcomes of action versus inaction on climate change, considering global impacts, economic ramifications, and the well-being of future generations.

In this scenario, the utilities are defined as follows:

- U_{Action} symbolizes the utility derived from combating climate change, potentially preventing dire consequences.
- U_{Inaction} represents the utility of not engaging in climate action, which might include short-term economic gains or convenience.
- U_{Adverse} stands for the negative utility associated with the severe impacts of unchecked climate change.
- $U_{\text{Redundant}}$ denotes the utility lost through futile actions if the severity of climate change is overestimated.

Considering the decision to act against climate change, the expected utility, EU_{Action}, is calculated. It balances the benefit of positive action, U_{Action}, against the cost of potentially unnecessary efforts, $U_{\text{Redundant}}$. This is mathematically expressed as:

$$EU_{\text{Action}} = p \cdot U_{\text{Action}} + (1 - p) \cdot U_{\text{Redundant}}, \tag{5}$$

where p is the probability of the adverse effects of climate change materializing.

Alternatively, the decision to refrain from climate action involves calculating the expected utility of inaction, EU_{Inaction}. This utility combines the immediate benefits of inaction, U_{Inaction}, with the long-term risks of adverse effects, U_{Adverse}. The formula for this utility is:

$$EU_{\text{Inaction}} = p \cdot U_{\text{Adverse}} + (1 - p) \cdot U_{\text{Inaction}}. \tag{6}$$

These formulations provide a structured approach to evaluating the complex decision-making process related to climate change, accounting for both immediate and future impacts.

While this model simplifies the multifaceted nature of climate change, it serves to demonstrate how decision theory can be applied to pressing global issues. The utilities assigned here reflect a trade-off between long-term global impacts and short-term considerations. The decision to act or not in the context of climate change will thus depend on the relative magnitudes of these utilities.

This analysis suggests that if the negative utility associated with the adverse effects of climate change is significantly high, it may overwhelmingly favor the decision to take proactive measures, mirroring the logic presented in Pascal's Wager. This perspective underscores the utility of decision theory in framing and analyzing complex and far-reaching issues like climate change, providing a rational foundation for decision-making amidst uncertainty.

In decision theory, the key elements include:

- **States of Nature:** Different scenarios that might occur.
- **Actions:** Choices available to the decision-maker.
- **Outcomes:** Results of taking certain actions under specific states of nature.
- **Payoffs:** Quantifiable returns or costs associated with outcomes.

Optimal Decision Making in Uncertainty

In statistical decision theory, we explore how to make the best choices under conditions of uncertainty. This involves understanding decisions d based on data x and considering unknown parameters θ. The key is to assess the effectiveness of these decisions, which is done using a loss function $L(\theta, d)$. This function measures the cost or penalty of making decision d when the true situation is θ.

Definition 3. *The* risk function *in decision theory, denoted as $R(\theta, d)$, is defined as the expected loss associated with a decision d when the true state of nature is θ. Mathematically, it is expressed as:*

$$R(\theta, d) = E_\theta[L(\theta, d)], \tag{7}$$

where E_θ represents the expectation under the probability distribution of the observed data x, given the parameter θ.

Our goal is to find a decision rule that reduces this risk function as much as possible, regardless of the actual value of θ. This concept is fundamental to many statistical techniques, like hypothesis testing, which involves selecting between different possibilities.

To properly set up a statistical decision problem, we need to consider:

- The probability model P_θ, describing data distribution.
- The decision space D, which includes all possible decisions.
- The loss function $L(\theta, d)$, representing the cost of each decision.

A decision rule is effective if there's no other rule that always results in a lower risk across all possible θ. This means the rule is rational and efficient under the given assumptions of the model.

Example - Medical Decision Table:

In the context of a medical decision-making scenario, where a doctor decides on treating a patient based on a diagnostic test, we can set up the following decision table:

Decision	**State (θ_1): Patient is sick**	**State (θ_2): Patient is not sick**	**Loss Function**
D_1: Treat	Low loss	Moderate loss	$L(\theta, D_1)$
D_2: Do not treat	High loss	No loss	$L(\theta, D_2)$

Table 1: Decision Table for Medical Treatment Scenario

In this table, the decisions D_1 and D_2 represent treating and not treating the patient, respectively. The loss associated with each decision depends on the actual health state of the patient (θ).

Choosing Effective Decision Rules:

A decision rule is effective if there's no other rule that always results in a lower risk across all possible θ. This means the rule is rational and efficient under the given assumptions of the model.

The loss function $L(\theta, D)$ measures the cost of decision D in the actual scenario θ:

$$\begin{aligned}
&L(\theta_1, D_1) : \text{Low loss for treating a sick patient.}\\
&L(\theta_1, D_2) : \text{High loss for not treating a sick patient.}\\
&L(\theta_2, D_1) : \text{Moderate loss for unnecessary treatment.}\\
&L(\theta_2, D_2) : \text{No loss when no treatment is needed for a healthy patient.}
\end{aligned}$$

Consider two decision rules:

1. Rule 1 (R1): Treat if the test is positive, otherwise don't.
2. Rule 2 (R2): Always treat, regardless of the test.

In terms of effectiveness:

- R1 could be effective if the test is reliable. It balances treating the sick and avoiding unnecessary treatments.
- R2 is less effective if the test is reliable, as it leads to unnecessary treatments. R1 is a better rule in this case.

An effective rule is one where no other rule is always better. R1 could be effective with a good diagnostic test, while R2 is less so due to its inefficiency.

Now consider a medical treatment model, where we consider a matrix $\mathbf{X}$ of explanatory variables, and a binary 'action' variable indicating treatment decisions (1 for treatment, 0 for no treatment). This action is separate from the variables in $\mathbf{X}$.

The matrix $\mathbf{X}$ and the action variable are arranged as follows:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \tag{8}$$

$$\text{action} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \tag{9}$$

where each a_i is either 0 or 1, indicating the treatment decision for each case.

We define a new observation vector $\mathbf{x_{new}}$ from $\mathbf{X}$ and an action value a_{new} for this new case:

$$\mathbf{x_{new}} = \begin{bmatrix} x_{\text{new}1} & x_{\text{new}2} & \cdots & x_{\text{new}n} \end{bmatrix}, a_{\text{new}} = \text{binary value (0 or 1)}. \tag{10}$$

In our regression model, we aim to predict the decision variable Y by considering both the new observations $\mathbf{x_{new}}$ and the impact of the treatment decision a_{new}. The model is expressed as:

$$\hat{y} = \mathbf{x_{new}}\hat{\beta} + a_{\text{new}}\hat{\beta}_{\text{action}}, \tag{11}$$

where $\hat{\beta}$ are the estimated coefficients for the explanatory variables $\mathbf{X}$, and $\hat{\beta}_{\text{action}}$ is the estimated coefficient reflecting the effect of the treatment decision. This model provides insights into how the combination of observed data and treatment decision influences the decision variable Y.

This model provides insights into how the combination of observed data and treatment decision influences the decision variable Y. By incorporating a_{new} in our model, we can directly assess the impact of the treatment on the predicted outcome.

Utility Calculation for Treatment Decisions:

To apply this model for a new patient case, represented by $\mathbf{x_{new}}$, we aim to calculate the utility for both treatment and no-treatment scenarios. This involves predicting the outcome $\hat{y}$ for each scenario and then evaluating the utility for each prediction.

- *No Treatment Scenario* ($a_{\text{new}} = 0$): In this case, the predicted outcome $\hat{y}_0$ is calculated without the treatment effect. It is given by:

  $$\hat{y}_0 = \mathbf{x_{new}}\hat{\beta} + \beta_0. \tag{12}$$

  The utility for this scenario is $U(\hat{y}_0) = -(\hat{y}_0 - y^*)^2$, where y^* is the desired outcome.

- *Treatment Scenario* ($a_{\text{new}} = 1$): Here, the predicted outcome $\hat{y}_1$ includes the effect of the treatment. It is given by:
$$\hat{y}_1 = \mathbf{x_{new}}\hat{\beta} + \hat{\beta}_{\text{action}} + \beta_0. \tag{13}$$
The utility in this case is $U(\hat{y}_1) = -(\hat{y}_1 - y^*)^2$.

Optimizing Treatment Decision:

The optimal treatment decision for $\mathbf{x_{new}}$ is then determined by comparing the utilities of the two scenarios. The decision rule can be summarized as follows:

- If $U(\hat{y}_1) > U(\hat{y}_0)$, administer the treatment ($a_{\text{new}} = 1$), as it leads to a higher utility.
- If $U(\hat{y}_0) \geq U(\hat{y}_1)$, do not administer the treatment ($a_{\text{new}} = 0$), as it either leads to a higher or equal utility.

This decision-making process is rooted in the principle of maximizing utility, which, in this context, translates to achieving the closest possible outcome to y^* for the patient.

Consider a study where doctors utilized a comprehensive dataset containing health-related variables to analyze the impact of medical treatments on patient outcomes. The core of this analysis involved a regression model, crucial for understanding how different factors, including treatment, affect patient health.

The regression model used in the study is described by the equation:

$$\begin{aligned} Y = \beta_0 + \beta_{\text{glucose}} \cdot \text{Glucose} + \beta_{\text{diastolic}} \cdot \text{Diastolic} \quad (14)\\ + \beta_{\text{triceps}} \cdot \text{Triceps} + \beta_{\text{insulin}} \cdot \text{Insulin} \quad (15)\\ + \beta_{\text{bmi}} \cdot \text{BMI} + \beta_{\text{age\_new}} \cdot \text{Age} \quad (16)\\ + \beta_{\text{treatment}} \cdot \text{Treatment} + \varepsilon, \quad (17) \end{aligned}$$

where each term represents the contribution of a specific health metric or treatment decision to the overall health outcome Y.

In our analysis of medical treatment decisions, two significant challenges are encountered: post-hoc analysis and the counterfactual problem. Understanding these challenges is crucial for interpreting the results of utility calculations, such as $u_x(a = 1)$ for treatment and $u_x(a = 0)$ for no treatment.

Post-hoc Analysis:

- Post-hoc analysis refers to making inferences after outcomes have been observed. It often relies on observational data, where treatment decisions are not randomized.
- This approach can lead to biases due to the data dependency and lack of randomization. The treatment effects observed may be confounded with patient-specific factors.
- Hindsight bias is another concern, where knowledge of the outcome influences the interpretation of the treatment's effectiveness.

Counterfactual Problem:

- The counterfactual problem deals with the question of what would have happened under a different treatment scenario that was not actually observed.
- For each patient, only one outcome is observable – either with or without treatment. The outcome under the alternative scenario remains unknown and speculative.
- Addressing this problem requires assumptions, as we cannot observe both outcomes for the same patient. This limits the certainty of causal inferences at the individual level.

To address the challenges of post-hoc analysis and the counterfactual problem in our study, we adopt a comparative approach. Instead of directly estimating individual utilities, we focus on comparing the expected utilities across different treatment decisions. The expected utility for a given treatment action is approximated as:

$$R_x(a) = \frac{1}{n_b} \sum_{b=1}^{n_b} \left[\left(\mathbf{x_{new}}^T \hat{\beta}_b + a \cdot \hat{\beta}_{\text{action},b} + \varepsilon_b - y^* \right)^2 \right], \tag{18}$$

where:

- $R_x(a)$ represents the expected utility for a particular treatment action a.
- $\hat{\beta}_b$ are the bootstrap estimators of the model coefficients.
- $\hat{\beta}_{\text{action},b}$ is the bootstrap estimator of the coefficient associated with the treatment action.
- ε_b is a random error term, as discussed previously.
- y^* is the known optimal health value.
- n_b is the number of bootstrap samples.

This approach allows us to estimate the average utility for each treatment action (a) across the bootstrap samples. By comparing these average utilities, we can infer which treatment action tends to be more beneficial on average, taking into account both the variability in the model estimations and the randomness in outcomes.

The bootstrap process not only provided a distribution of possible coefficient values but also highlighted the variability and stability of the predictions.

For each resampled dataset, the utility of treatment decisions was calculated. The utility function used in this analysis is defined as $U(y) = -(y - y^*)^2$, where y^* is the optimal or target health outcome. By substituting the predicted outcome $\hat{y}$ from the regression model into this utility function, the effectiveness of each treatment decision was quantified. Specifically, the utility was calculated for two scenarios: treatment ($a_{\text{new}} = 1$) and no treatment ($a_{\text{new}} = 0$).

The predicted outcome $\hat{y}$ for each scenario is determined by:

$$\hat{y}_{\text{treatment}} = \mathbf{x_{new}} \hat{\beta} + \hat{\beta}_{\text{action}} + \beta_0 + \text{error}, \tag{19}$$

$$\hat{y}_{\text{no treatment}} = \mathbf{x_{new}} \hat{\beta} + \beta_0 + \text{error}. \tag{20}$$

The utilities derived from these predictions provided two distributions, one for each treatment decision. These distributions were visualized in a plot, as shown below:

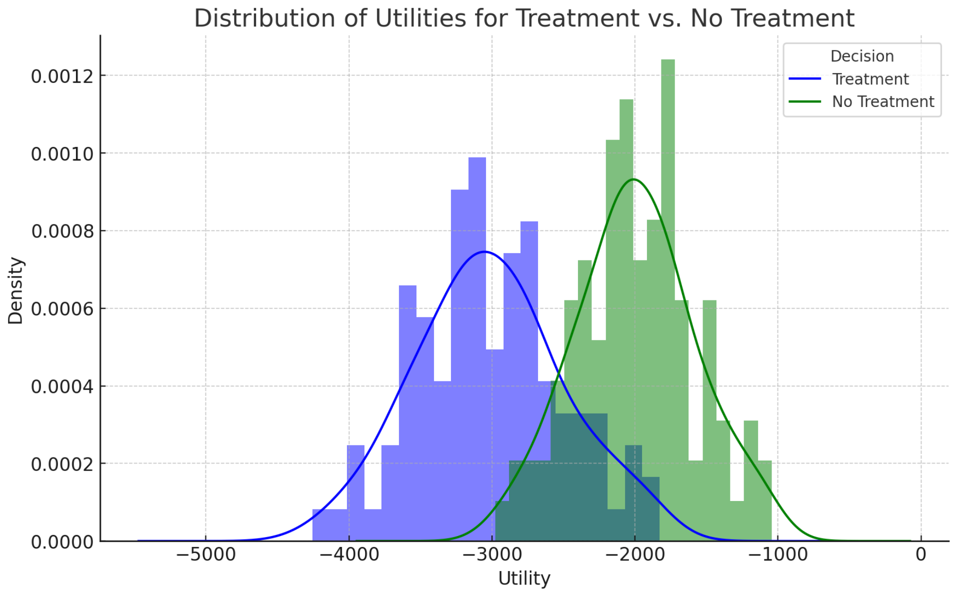

Figure 1: Distribution of Utilities for Treatment vs. No Treatment

This visualization offers a clear comparison of the utilities associated with each treatment decision. The insights gained from this analysis are invaluable, as they help doctors make more informed decisions about patient care, balancing statistical analysis with clinical judgment.

It's important to note that in the provided example, we used $y^* = 0$ for simplicity. However, in practical applications, the value of y^* would be specifically chosen to reflect the desired health outcomes in the context of the treatment being analyzed.

Prospect Theory in Behavioral Economics: Considering Financial Backgrounds

As we move from utility maximization in medical decision models to Prospect Theory, we encounter a rich framework developed by Daniel Kahneman and Amos Tversky that diverges from classical rational choice models. Prospect Theory suggests that people evaluate outcomes as gains or losses relative to a reference point, rather than in absolute terms. It emphasizes the principles of loss aversion and the certainty effect in decision-making under risk.

Example 2 (Monetary Gamble Across Different Financial Backgrounds)**.** *Consider a gamble where two individuals, Alex (wealthy) and Jamie (with limited resources), face the choice of a 50% chance to win or lose $1000. The table below illustrates their potential choices and outcomes:*

Decision	*Alex's Perception*	*Jamie's Perception*
Participate in Gamble	*Low Impact Loss or Gain*	*High Impact Loss or Gain*
Avoid Gamble	*Neutral Impact*	*Secure Current Financial Status*

In this scenario, Alex's decision is less influenced by the potential loss due to their wealth, while Jamie, whose financial situation is more precarious, perceives the risk of loss as more significant. Prospect Theory predicts that Jamie might be more averse to the gamble, illustrating how financial background affects the reference point in decision-making.

The utility function in Prospect Theory is expressed as:

$$U(x) = \begin{cases} (x - \text{reference point})^{\alpha} & \text{if } x \geq \text{reference point} \\ -\lambda \cdot (\text{reference point} - x)^{\beta} & \text{if } x < \text{reference point} \end{cases} \tag{21}$$

Here, x is the outcome, with α and β representing diminishing sensitivity, and λ capturing loss aversion.

This example underscores the psychological complexity and contextual factors inherent in real-world decision-making, as explained by Prospect Theory.

Game Theory

Game theory is a study of mathematical models of strategic interaction among rational decision-makers. It has applications in all fields of social science, as well as in logic, systems science and computer science. Decision-making is central to game theory, where choices made by players determine the outcome of the game.

Game Theory is crucial in analyzing situations of interdependence and strategic interaction. It's extensively used in economics, political science, psychology, and even biology. In medical decision-making, Game Theory can elucidate how various stakeholders, such as patients, doctors, healthcare providers, and insurance companies, interact and make decisions that affect each other.

Example 3 (Healthcare Decision Game)**.** *Imagine a scenario where a patient needs to choose a treatment option, and the decision is influenced by their insurance coverage. The patient's choice and the insurance company's policies form a strategic game:*

- *The patient wants to choose the treatment that maximizes their health outcome based on their condition and preferences.*
- *The insurance company strategizes its coverage to manage costs while maintaining patient satisfaction and health outcomes.*

The equilibrium of this game, where both the patient's and the insurance company's strategies converge, can be analyzed using Game Theory concepts like Nash Equilibrium or Pareto Optimality.

Similarly, doctors may engage in a strategic game with patients and healthcare systems. They must balance medical efficacy, patient preferences, and systemic guidelines or constraints in their treatment recommendations.

The Hawk-Dove Game

The Hawk-Dove game is a classic model in evolutionary game theory. The game represents a contest over a resource. Two strategies are considered: Hawk (aggressive) and Dove (peaceful). When two Hawks meet, they fight over the resource until one is injured. When a Hawk meets a Dove, the Hawk takes the resource without a fight. When two Doves meet, they share the resource.

The payoff matrix for the Hawk-Dove game is:

	Hawk	**Dove**
Hawk	$(V-C)/2$	V
Dove	0	$V/2$

Where V is the value of the resource and C is the cost of the fight.

For the Hawk-Dove game to be evolutionarily stable, certain conditions on V and C need to be met. If $V > C$, the Hawk strategy will dominate. If $V < C$, a mix of Hawk and Dove strategies will coexist in a population.

To further understand the dynamics of the game, one can analyze the replicator dynamics equations and study the stability of the equilibrium points. The Hawk-Dove game serves as a foundational model in understanding the evolution of conflict and cooperation in biological systems.

Minimax Criterion

The **minimax method** in the context of hypothesis testing is a decision-making strategy that aims to minimize the maximum possible loss (risk) associated with incorrect decisions. In statistical hypothesis testing, the two types of errors to consider are:

- Type I error (α): The error of rejecting a true null hypothesis (false positive).
- Type II error (β): The error of failing to reject a false null hypothesis (false negative).

The minimax criterion seeks a test that minimizes the worst-case (maximum) risk. The 'risk' here refers to a loss function which quantifies the severity of errors made by a decision rule or test.

In a hypothesis testing scenario, you typically have:

- **Null Hypothesis (H_0)**: A default statement that there is no effect or no difference.
- **Alternative Hypothesis (H_1)**: A statement that contradicts the null hypothesis, indicating an effect or a difference.

The minimax method involves:

1. **Define the Risk Function:** The risk function in hypothesis testing is typically a combination of the probabilities of Type I and Type II errors, weighted by the relative costs or losses associated with these errors.
2. **Determine the Worst-Case Scenario:** For each potential test, consider the maximum risk across all possible states of nature (i.e., all possible true parameters or distributions under the alternative hypothesis).
3. **Choose the Test with the Minimum Maximum Risk:** Select the test for which this worst-case risk is the smallest. This is the minimax test.

In a formal statistical setting, the minimax method can be expressed through a decision-theoretic framework:

- Let Θ be the parameter space, and θ a parameter that can take a value in Θ.
- The null hypothesis H_0 specifies that $\theta \in \Theta_0$, and the alternative hypothesis H_1 specifies that $\theta \in \Theta_1$, where Θ_0 and Θ_1 are disjoint subsets of Θ.
- A loss function $L(\theta, a)$ is defined, where a is the action taken based on the test result (e.g., reject H_0 or not).
- The risk function is then $R(\theta, a) = E[L(\theta, a)]$, which is the expected loss for taking action a when θ is the true state of nature.

The minimax test is designed such that it minimizes the maximum of this risk function over all θ in Θ:

$$\min_{a} \max_{\theta \in \Theta} R(\theta, a)$$

The minimax approach is conservative and guards against the worst possible scenario but may not always be the most efficient method, as it does not necessarily consider the actual probabilities of the different states of nature.

Hypothesis testing and game theory are distinct fields with conceptual similarities concerning decision-making under uncertainty. Here we explore their relationship and common themes:

Both hypothesis testing and game theory require decision-making under conditions of uncertainty. In hypothesis testing, the uncertainty concerns the validity of the null hypothesis given the data. Game theory deals with uncertainty regarding opponents' strategies or payoffs.

The minimax criterion illustrates a direct connection between the two fields:

- In **game theory**, a minimax strategy minimizes the maximum possible loss under the assumption of worst-case opponent strategies:

  $$\min_{s_i} \max_{s_{-i}} u_i(s_i, s_{-i})$$

  where s_i is the strategy of player i, s_{-i} represents the strategies of all other players, and u_i is the utility for player i.

- In **hypothesis testing**, a minimax approach aims to minimize the maximum risk of Type I and Type II errors:

  $$\min_{\text{test}} \max_{\theta \in \Theta} R(\theta, \text{test})$$

  where θ represents the true state of nature, Θ is the parameter space, and R is the risk function.