3. Leyes de cerradura (5 ptos)

Considere dos vectores linealmente independientes $\boldsymbol{a} \in \mathbb{R}^n$ y $\boldsymbol{b} \in \mathbb{R}^n$ y el siguiente conjunto:

$$\mathcal{B}(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{0} \qquad \text{ó} \qquad \theta_{x,a} = \theta_{x,b}\} \ ,$$

donde $\theta_{x,y} = \theta_{y,x}$ denota el ángulo que forman los vectores $\boldsymbol{x}$ e $\boldsymbol{y}$.

a) **(3 ptos)** Muestre que $\mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$ cumple con la ley de cerradura interna.

b) **(2 ptos)** Muestre que $\mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$ cumple con la ley de cerradura externa.

Solución:

a) Tomemos vectores $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$. Si alguno de ellos es nulo, claramente la suma $\boldsymbol{x}_1 + \boldsymbol{x}_2$ pertenece a $\mathcal{B}(a, b)$. Supongamos que ambos son no nulos. Entonces:

$$\begin{aligned}
\cos(\theta_{x_1+x_2,a}) &= \frac{(\boldsymbol{x}_1 + \boldsymbol{x}_2)'\boldsymbol{a}}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|\|\boldsymbol{a}\|} = \frac{\boldsymbol{x}_1{}'a}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|\|\boldsymbol{a}\|} + \frac{\boldsymbol{x}_2{}'\boldsymbol{a}}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|\|\boldsymbol{a}\|} \\
&= \frac{1}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|}\left(\frac{\boldsymbol{x}_1{}'\boldsymbol{a}}{\|\boldsymbol{a}\|} + \frac{\boldsymbol{x}_2{}'\boldsymbol{a}}{\|\boldsymbol{a}\|}\right) \\
&= \frac{1}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|}\left(\|\boldsymbol{x}_1\|\frac{\boldsymbol{x}_1{}'\boldsymbol{a}}{\|\boldsymbol{x}_1\|\|\boldsymbol{a}\|} + \|\boldsymbol{x}_2\|\frac{\boldsymbol{x}_2{}'\boldsymbol{a}}{\|\boldsymbol{x}_2\|\|\boldsymbol{a}\|}\right) \\
&= \frac{1}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|}\left(\|\boldsymbol{x}_1\|\cos(\theta_{x_1,a}) + \|\boldsymbol{x}_2\|\cos(\theta_{x_2,a})\right) \\
&= \frac{1}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|}\left(\|\boldsymbol{x}_1\|\cos(\theta_{x_1,b}) + \|\boldsymbol{x}_2\|\cos(\theta_{x_2,b})\right) \\
&= \frac{1}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|}\left(\|\boldsymbol{x}_1\|\frac{\boldsymbol{x}_1{}'\boldsymbol{b}}{\|\boldsymbol{x}_1\|\|\boldsymbol{b}\|} + \|\boldsymbol{x}_2\|\frac{\boldsymbol{x}_2{}'\boldsymbol{b}}{\|\boldsymbol{x}_2\|\|\boldsymbol{b}\|}\right) = \frac{(\boldsymbol{x}_1 + \boldsymbol{x}_2)'\boldsymbol{b}}{\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|\|\boldsymbol{b}\|} = \cos(\theta_{x_1+x_2,b}) \,.
\end{aligned}$$

Por lo tanto $\theta_{x_1+x_2,a} = \theta_{x_1+x_2,b}$ y $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$.

b) Sea $t \in \mathbb{R}$ y $\boldsymbol{x} \in \mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$. Si $\boldsymbol{x}$ es nulo, entonces $t \cdot \boldsymbol{x} = \boldsymbol{0} \in \mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$. Supongamos que $\boldsymbol{x}$ no es nulo. Entonces:

$$\begin{aligned}
\cos(\theta_{t\cdot x,a}) &= \frac{(t \cdot \boldsymbol{x})'\boldsymbol{a}}{\|t \cdot \boldsymbol{x}\|\|\boldsymbol{a}\|} = \frac{t}{|t|}\frac{\boldsymbol{x}'\boldsymbol{a}}{\|\boldsymbol{x}\|\|\boldsymbol{a}\|} \\
&= \frac{t}{|t|}\cos(\theta_{tx,a}) = \frac{t}{|t|}\cos(\theta_{tx,b}) \\
&= \frac{t}{|t|}\frac{\boldsymbol{x}'\boldsymbol{b}}{\|\boldsymbol{x}\|\|\boldsymbol{b}\|} = \frac{(t \cdot \boldsymbol{x})'\boldsymbol{b}}{\|t \cdot \boldsymbol{x}\|\|\boldsymbol{b}\|} = \cos(\theta_{t\cdot x,b}) \,.
\end{aligned}$$

Por lo tanto $\theta_{tx,a} = \theta_{tx,b}$ y $t \cdot \boldsymbol{x} \in \mathcal{B}(\boldsymbol{a}, \boldsymbol{b})$.

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