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5 | Ordinary differential equations

5.1 Generalities

An Ordinary Differential Equation (ODE) may be thought of as an equation that tells us how a variable x behaves through time. It is the relationship between a variable that implicitly depends on time $x = x(t)$, and its (total) derivatives. Solving an ODE implies finding the solution $x(t)$ that satisfies it. The purpose may be a predictive one: to say how, given current values of $x(t)$, the future values will evolve. Also, a goal is to understand the dynamics followed by x.

As a matter of convention, we call $\dot{x}$ to the first-derivative of x with respect to time, $\dot{x} \equiv \mathrm{d}x/\mathrm{d}t$, and $\ddot{x}$ to the second derivative, $\ddot{x} \equiv \mathrm{d}^2x/\mathrm{d}t^2$. Also, we keep the dependence of x on t implicit unless we refer to a solution, where we shall use $x(t)$. Since the independent variable t is interpreted as time, the solution $x(t)$ is often called *time path*.

Consider five examples of ODE:

$$\text{(a)}\ \dot{x} = 3t^2 \qquad \text{(b)}\ \dot{x} - a(t)x = b(t)$$
$$\text{(c)}\ \ddot{x} - a\dot{x} + bx = 0 \qquad \text{(d)}\ \dot{x} = (x-a)(x-b) \qquad \text{(e)}\ \dot{x} = f(x,t)$$

ODE are classified according to their type and structure, partly as an aid to using methods of solution of the equations. The **order** of an equation is determined by the highest derivative appearing in the equation. All the above examples are first-order equations, except for (c) which is a second order equation. is the highest power of a derivative.

An equation is described as **autonomous** if time does not appear explicitly in the equation, e.g. $\dot{x} = f(x)$, and **non-autonomous** otherwise, e.g. $\dot{x} = f(x,t)$. Examples (a), (b) and (e) are non-autonomous, while (c) and (d) are autonomous. An equation is **linear** if the terms in x, $\dot{x}$ and $\ddot{x}$ appear in linear form, and is **nonlinear** otherwise. Equations (a), (b) and (c) are linear, while (d) and (e) are nonlinear. A linear equation is termed **homogeneous** if only x and its derivatives appear in the equation, and **non-homogeneous** otherwise. Equation (c) is homogeneous, while equations (a) and (b) are non-homogeneous.

5.2 Separable first-order ODE

A first-order differential equation is separable if it can be written in the form

$$\dot{x} = f(x)g(t) \qquad \rightarrow \qquad \frac{dx}{f(x)} = g(t)dt\,. \tag{1}$$

Its solution involves the direct integration of this equality.

A leading example is what we call an homogenous linear equation $\dot{x} - a(t)x = 0$, where $a(t)$ is a function of t. In this case,

$$\dot{x} - a(t)x = 0 \quad \rightarrow \quad \frac{dx}{dt} = a(t)x \quad \rightarrow \quad \frac{dx}{x} = a(t)dt \quad \rightarrow \quad \ln(x) = \int a(t)dt + c \quad \rightarrow \quad x(t) = Ce^{\int a(t)dt}\,,$$

where c, and thus $C = e^c$, is an arbitrary constant of integration.

5.3 First-order linear ODE

Consider the general (non-separable) first-order non-homogeneous equation

$$\dot{x} - a(t)x = b(t)\,, \tag{2}$$

where $a(t)$ and $b(t)$ are two arbitrary functions of t. The function $a(t)$ is called the **coefficient**, whereas the function $b(t)$ is called the **term**. Next, we focus on a method to find a function $x(t)$ satisfying (2) that eases the derivation of further results.

5.3.1 Integrating factor

Solving (2) implies some sort of integration to get rid of $\dot{x}$. The following method works by turning the left-hand-side into the form of the derivative of a product. Hence, consider an unknown function of t, $\mu(t)$, such that if we multiply both sides of (2) by it we obtain

$$\mu(t)\dot{x} - \mu(t)a(t)x = \mu(t)b(t) \qquad \text{where } \mu(t) \text{ is such that} \qquad \frac{\mathrm{d}\mu(t)x}{\mathrm{d}t} = \mu(t)b(t)\,. \tag{3}$$

Consider the second equality in (3). The left-hand-side can be integrated directly by means of the fundamental theorem of calculus,

$$\mu(t)x = \int \mu(t)b(t)\,\mathrm{d}\,t + C\,,$$

where C is the arbitrary constant of integration. Solving for x gives

$$x(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,\mathrm{d}\,t + \frac{C}{\mu(t)}\,. \tag{4}$$

The function $\mu(t)$ is still unknown, so the problem of solving (2) has been translated into the easier problem of finding $\mu(t)$. Using first the **product rule** and then identifying terms by comparing both equalities in (3) gives

$$\frac{\mathrm{d}\mu(t)x}{\mathrm{d}t} = \mu(t)\dot{x} + \frac{\mathrm{d}\mu(t)}{\mathrm{d}t}x \qquad \text{implying} \qquad \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = -a(t)\mu(t)\,.$$

Thus, $\mu(t)$ obeys a very simple ODE. It follows that (recall the form of a logarithmic derivative)

$$\frac{1}{\mu(t)}\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = -a(t) \quad \rightarrow \quad \frac{\mathrm{d}\ln\mu(t)}{\mathrm{d}t} = -a(t) \quad \rightarrow \quad \ln\mu(t) = -\int a(t)\,\mathrm{d}\,t\,,$$

which finally renders

$$\mu(t) = e^{-\int a(t)\,\mathrm{d}\,t}\,. \tag{5}$$

Plugging, (5) into (4) gives the **general solution** to the ODE,

$$x(t) = C\,e^{\int a(t)\,\mathrm{d}\,t} + e^{\int a(t)\,\mathrm{d}\,t}\int e^{-\int a(t)\,\mathrm{d}\,t}\,b(t)\,\mathrm{d}\,t\,. \tag{6}$$

The function $\mu(t)$, given in (5), is called the **integrating factor**. It is clear from the derivations above that its role is to simplify the integration involved in the resolution of the ODE.

Two comments are worth-mentioning. First, the constant C in (6) appears due to integration and is arbitrary. This means, that for *any* value of C functions of the form (6) satisfy (2). To determine C we need further information, related not to the dynamics of x but to its position at a given point in time. For instance if we know that $x(\tau) = x_\tau$ for some τ, then evaluating (6) for $t = \tau$ gives a well-defined value of C satisfying the condition $x(\tau) = x_\tau$. Quite often, it is assumed that $\tau = 0$ and the determination of C is referred to as an **initial value problem**, since it involves the starting point of the path of $x(t)$. In other situations, what is specified is a **terminal value** (also known as a **transversality condition**) for the path $x(t)$, which implies that $\tau \to \infty$.

Second, the **stability** of $x(t)$ refers to the limiting behaviour of this path. If $x(t) \to x_{ss}$ as $t \to \infty$, where x_{ss} is a real value, then the solution $x(t)$ is said to be **stable** or convergent. Usually in Economics the value of x_{ss} is interpreted as a long-run equilibrium, and in many applications it is called the **steady state**. On the contrary, if $x(t) \to \pm\infty$ as $t \to \infty$, then the solution $x(t)$ is **unstable** or divergent (no "long-run" equilibrium exists).

5.3.2 The constant coefficient, constant term case

Consider a simpler version of (2) with $a(t) = a$ and $b(t) = b$,

$$\dot{x} - a\,x = b\,. \tag{7}$$

Applying the solution in (6) to this case gives the general solution

$$x(t) = C\,e^{at} + be^{at}\int e^{-at}\,\mathrm{d}\,t = C\,e^{at} - \frac{b}{a}\,. \tag{8}$$

The dynamic bevahiour of $x(t)$ is due to the first term, e^{at}. Hence, $x(t)$ will be stable if this function converges as $t \to \infty$. Clearly, $\lim_{t\to\infty} e^{at} = \infty$ if $a > 0$ and $\lim_{t\to\infty} e^{at} = 0$ if $a < 0$. Thus, a sufficient condition for the stability of the path implied by (7) is $a < 0$.

Let us focus now on the determination of C, when x_τ is the value of $x(t)$ at instant $t = \tau$. From the solution above it follows that

$$x_\tau = C\,e^{a\tau} - \frac{b}{a} \qquad \text{implying} \qquad C = \left(x_\tau + \frac{b}{a}\right)e^{-a\tau}\,,$$

so that a solution to (7) satisfying $x(\tau) = x_\tau$ is

$$x(t) = \left(x_\tau + \frac{b}{a}\right)e^{a(t-\tau)} - \frac{b}{a}\,.$$

When $x(t)$ is stable ($a < 0$) its steady state value is clearly $x_{ss} = -b/a$. Typically, the value of τ used to determine C is $\tau = 0$, i.e. an initial condition. Then,

$$x(t) = \left[x_0 - \left(-\frac{b}{a}\right)\right]e^{at} - \frac{b}{a} = (x_0 - x_{ss})\,e^{at} + x_{ss}\,. \tag{9}$$

The term in braces is known as the **initial disequilibrium**, and the path $x(t)$ shows how x evolves from the initial point x_0 to its long-run value x_{ss} when $a < 0$ or departs from it when $a > 0$.

5.3.3 A generalizable method with constant coefficients

The solution of $\dot{x} - a\,x = b(t)$ is

$$x(t) = C\,e^{at} + e^{at}\int e^{-at}\,b(t)\,\mathrm{d}\,t\,. \tag{10}$$

This solution can be thought of as the sum of two functions,

$$x(t) = x_c(t) + x_p(t)\,. \tag{11}$$

The first term $x_c(t) = C\,e^{at}$ is called the **complementary or homogenous solution**, whereas the second term involving the integral of the $b(t)$ function, $x_p(t)$, is called the **particular or inhomogeneous solution**.

The method, which generalizes to higher order ODE, is an application of a “guess and verify” procedure.

Complementary solution

This is the solution to the *homogenous equation* $\dot{x} - a\,x = 0$, i.e. the ODE resulting after setting $b(t) = 0$ (this equation involves only x and its derivative). Now, we *conjecture* (i.e., guess) that the complementary solution has the form $x_c(t) = C\,e^{rt}$ for arbitrary values of C and r. If this is the case, then $\dot{x}_c(t) = Cr\,e^{rt}$. Plugging this guess into the homogenous equation gives

$$\dot{x}_c - a\,x_c = 0 \qquad \to \qquad Cre^{rt} - aCe^{rt} = 0 \qquad \to \qquad Ce^{rt}(r-a) = 0\,.$$

For the nontrivial value of $C \neq 0$ this equation is only satisfied if $P(r) = r - a$ is equal to zero. Note that $P(r)$ is a polynomial in r, called the **characteristic polynomial** of the ODE. **The degree of this polynominal is the same as the order of the ODE**. The **characteristic equation** is $P(r) = 0$ and the value of r that solves the characteristic equation is the **root**. It is clear that the only value that solves $P(r) = 0$ is $r = a$, and so we conclude that

$$x_c(t) = C\, e^{at} ,$$

for *any* value of C (this value is akin to the integrating constat and so is determined from an initial or terminal condition). Note that the complementary solution *is stable if the root of the ODE is negative* (i.e., $r = a < 0$ since $e^{at} \to 0$ as t increases).

Particular solution

In the general solution, we have that

$$x_p(t) = e^{at} \int e^{-at}\, b(t)\, \mathrm{d}\, t .$$

Note that this solution does not involve an integrating constant. We may compute $x_p(t)$ by direct integration, which may be tedious. However, in many cases, especially when $b(t)$ takes certain forms, we can compute $x_p(t)$ by means of the **method of undetermined coefficients**. This method begins with a "guess" regarding the form of $x_p(t)$. The guess is not completely arbitrary, but depends on the form that $b(t)$ has. The guess will feature arbitrary coefficients to be determined (hence, the name of the method) by evaluating the ODE.

Let us see, through examples, the workings of the method. Of course, the same solutions can be obtained from direct integration of the general solution.

Consider that $b(t) = b$ is a **constant**, i.e.

$$\dot{x} - a\, x = b .$$

Our guess is, thus, that $x_p(t)$ is also a constant, say $x_p(t) = A$. This implies that $\dot{x}_p(t) = 0$. Plugging this guess into the ODE gives

$$\dot{x}_p - a\, x_p = b \qquad \to \qquad 0 - aA = b \qquad \to \qquad A = -\frac{b}{a} .$$

We have determined the undetermined coefficients and conclude that $x_p(t) = A = -b/a$.

Consider now that $b(t) = b_0 + b_1 t + b_2 t^2$ is a **polynomial in** t, i.e.

$$\dot{x} - a\, x = b_0 + b_1 t + b_2 t^2 .$$

Our guess is, thus, that $x_p(t)$ is also a polynomial in t (of the same degree as the term), say $x_p(t) = A_0 + A_1 t + A_2 t^2$. This implies that $\dot{x}_p(t) = A_1 + 2A_2 t$. Plugging this guess into the ODE gives

$$(A_1 + 2A_2 t) - a(A_0 + A_1 t + A_2 t^2) = b_0 + b_1 t + b_2 t^2 \quad \to \quad -aA_2 t^2 + (2A_2 - aA_1)t + (A_1 - aA_0) = b_2 t^2 + b_1 t + b_0 .$$

We obtain a three equation system for three unknowns. Firstly, $-aA_2 = b_2$ which gives $A_2 = -b_2/a$. Secondly, $2A_2 - aA_1 = b_1$ which gives $A_1 = (2A_2 - b_1)/a = -(2b_2 + ab_1)/a^2$. Finally, $A_1 - aA_0 = b_0$ which gives $A_0 = (A_1 - b_0)/a = -(2b_2 + ab_1 + a^2 b_0)/a^3$. We conclude that

$$x_p(t) = A_0 + A_1 t + A_2 t^2 = -\left(\frac{2b_2 + ab_1 + a^2 b_0}{a^3}\right) - \left(\frac{2b_2 + ab_1}{a^2}\right) t - \left(\frac{b_2}{a}\right) t^2 .$$

Consider now that $b(t) = b_1 \cos(\theta t) + b_2 \sin(\theta t)$ is a linear combination of **sines and cosines in** t, i.e.

$$\dot{x} - a\, x = b_1 \cos(\theta t) + b_2 \sin(\theta t) .$$

Our guess is, thus, that $x_p(t)$ is also a linear combination of sines and cosines polynomial in t (note that *both* sine and cosine are included), say $x_p(t) = A_1\cos(\theta t) + A_2\sin(\theta t)$. This implies that $\dot{x}_p(t) = -A_1\theta\sin(\theta t) + A_2\theta\cos(\theta t)$. Plugging this guess into the ODE gives

$$\begin{aligned}[-A_1\theta\sin(\theta t) + A_2\theta\cos(\theta t)] - a\,[A_1\cos(\theta t) + A_2\sin(\theta t)] &= b_1\cos(\theta t) + b_2\sin(\theta t)\\ \rightarrow\quad (A_2\theta - aA_1)\cos(\theta t) + (-A_1\theta - aA_2)\sin(\theta t) &= b_1\cos(\theta t) + b_2\sin(\theta t)\,.\end{aligned}$$

We obtain a two equation system for two unknowns, whose solution is

$$A_1 = -\frac{ab_1 + \theta b_2}{\theta^2 + a^2} \qquad \text{and} \qquad A_2 = \frac{\theta b_1 - ab_2}{\theta^2 + a^2}\,.$$

We have determined the undetermined coefficients and conclude that

$$x_p(t) = A_1\cos(\theta t) + A_2\sin(\theta t) = -\left(\frac{ab_1 + \theta b_2}{\theta^2 + a^2}\right)\cos(\theta t) + \left(\frac{\theta b_1 - ab_2}{\theta^2 + a^2}\right)\sin(\theta t)\,.$$

Note that the solution contains both sine and cosine functions, even though the term may contain only one of these functions, for instance if $b_1 = 0$ or $b_2 = 0$.

Consider now that $b(t) = be^{\theta t}$ is an **exponential function in** t, i.e.

$$\dot{x} - a\,x = be^{\theta t}\,.$$

Our guess is, thus, that $x_p(t)$ is also an exponential function in t, say $x_p(t) = Ae^{\theta t}$. This implies that $\dot{x}_p(t) = A\theta e^{\theta t}$. Plugging this guess into the ODE gives

$$A\theta e^{\theta t} - aAe^{\theta t} = be^{\theta t} \quad\rightarrow\quad A(\theta - a)e^{\theta t} = be^{\theta t} \quad\rightarrow\quad A = \frac{b}{\theta - a} \quad\rightarrow\quad x_p(t) = Ae^{\theta t} = \left(\frac{b}{\theta - a}\right)e^{\theta t}\,.$$

Note that if $\theta = a$, i.e. the term is of the form $b(t) = be^{at}$, our guess **does not work** since it gives $0 = be^{at}$. The reason is that the guess **coincides with the complementary solution** (more exactly, is linearly dependent to x_c). We require our guess to be linearly independent from x_c. The usual remedy is to simply multiply the guess by t: $x_p(t) = Ate^{at}$ so that $\dot{x}_p(t) = Ae^{at} + Aate^{at}$. Plugging this new guess into the ODE gives

$$\left(Ae^{at} + Aate^{at}\right) - aAte^{at} = be^{\theta t} \quad\rightarrow\quad Ae^{at} = be^{at} \quad\rightarrow\quad A = b \quad\rightarrow\quad x_p(t) = Ate^{at} = be^{at}\,.$$

5.3.4 Summary

Learn the methods, not the solutions!:

Equation	Solution
$\dot{x} - a\,x = b(t)$	$x(t) = C\,e^{at} + e^{at}\int e^{-at}\,b(t)\,\mathrm{d}\,t$
$\dot{x} - a\,x = 0$	$x(t) = C\,e^{at}$
$\dot{x} - a\,x = b$	$x(t) = C\,e^{at} - \dfrac{b}{a}$
$\dot{x} - a\,x = b_0 + b_1 t + b_2 t^2$	$x(t) = C\,e^{at} - \left(\dfrac{2b_2 + ab_1 + a^2 b_0}{a^3}\right) - \left(\dfrac{2b_2 + ab_1}{a^2}\right)t - \left(\dfrac{b_2}{a}\right)t^2$
$\dot{x} - a\,x = b_1\cos(\theta t) + b_2\sin(\theta t)$	$x(t) = C\,e^{at} - \left(\dfrac{ab_1 + \theta b_2}{\theta^2 + a^2}\right)\cos(\theta t) + \left(\dfrac{\theta b_1 - ab_2}{\theta^2 + a^2}\right)\sin(\theta t)$
$\dot{x} - a\,x = be^{\theta t} \quad (\theta \neq a)$	$x(t) = C\,e^{at} + \left(\dfrac{b}{\theta - a}\right)e^{\theta t}$
$\dot{x} - a\,x = be^{at}$	$x(t) = (C + bt)e^{at}$

5.4 Qualitative analysis of nonlinear first-order ODE

Many interesting applications involve a nonlinear ODE of the form $\dot{x} = f(x)$, where $f(\cdot)$ is an arbitrary function. Importantly, the ODE is **autonomous**, which means that it only depends on time implicitly. Quite often we will be unable to obtain a solution for the path $x(t)$, because $f(x)$ may not have a closed-form integral. However, we can still have an idea of many of the properties of this path by studying the dynamics implied by $\dot{x} = f(x)$.

A useful tool to analyse the dynamics of the nonlinear ODE is the **phase diagram**. The idea is simple: the function $f(\cdot)$ can be sketched on the $(x, \dot{x})$ plane to give a qualitative assessment of the behavior of x and $\dot{x}$ at any subinterval of the domain of $f(\cdot)$, or at any particular value of x. Usually the value or values of x that are of interest are the steady states, x_{ss}, that satisfy $f(x_{ss}) = 0$. The function depicted in the phase diagram is called the **phase curve** and, as we shall see, determining its slope around the steady state is crucial.

To motivate the discussion, let us study the linear equation $\dot{x} = ax + b$ by means of the phase diagram. In this case, $f(x) = ax + b$ and the shape of the curve drawn in the $(x, \dot{x})$ plane will depend on the value taken by a. Figure 1 displays the cases where $a > 0$ and $a < 0$.

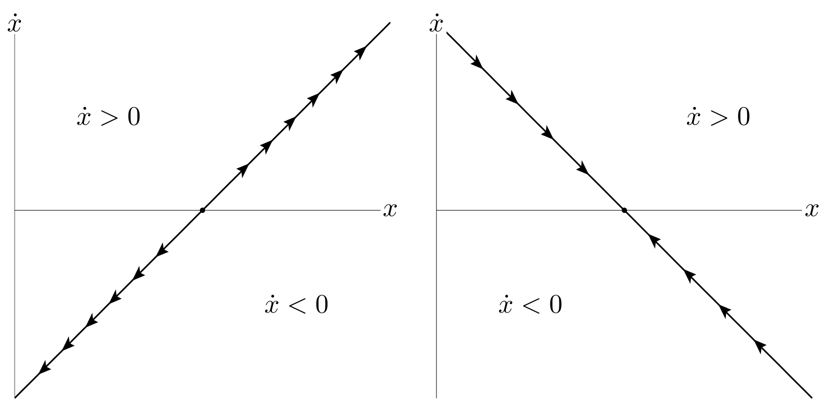
The $(x, \dot{x})$ plane is divided into three regions. First, the horizontal axes where, for any value of x, $\dot{x} = 0$. This is the familiar steady state condition and thus the value of x in which $f(x)$ intersects the horizontal axes is clearly the steady state in the ODE $\dot{x} = f(x)$. In this linear example, it is unique and equal to $x_{ss} = -b/a$ as we saw before. The second region is the quadrant above the horizontal axes where $\dot{x} > 0$. Since x is represented on the horizontal axes, any point on this quadrant will move eastwards, as x is increasing with time. This movement will be observed along the phase curve, and this is why we have drawn arrow heads pointing to the right in both Figures 1(a) and 1(b). Importantly, whether these arrowheads point *north*east- or *south*east-wards depends on the slope of the phase curve around the steady state. Finally, the third region corresponds to the quadrant below the horizontal, where $\dot{x} < 0$ and hence x moves westwards. Again, whether the dynamic behavior in the ODE will make the arrowheads point to the *north*west or to the *south*west depends on $f'(x_{ss})$.

It is clear that the underlying path in Figure 1(a) is unstable. If the initial condition is above the steady state, it will increase as time passes, getting further and further away from x_{ss}. Similarly, if the initial condition lies below the steady state, the path will be monotonically decreasing and will never reach x_{ss}. This is not surprising since we have already found that $x(t)$ is unstable if $a > 0$. The important conclusion here is that the slope of the phase curve is *positive*. The same reasoning applies to Figure 1(b), so we conclude that the steady state will be a stable point if the slope of the phase curve around it is *negative*.

Formally, using a Taylor expansion of the function $f(x)$ around x_{ss} we get

$$\dot{x} = f(x) \simeq f(x_{ss}) + f'(x_{ss})(x - x_{ss}) = f'(x_{ss})x - f'(x_{ss})x_{ss} \,, \tag{12}$$

Figure 1. *Phase diagram of the ODE* $\dot{x} = ax + b$



(a) Unstable equilibrium ($a > 0$) (b) Stable equilibrium ($a < 0$)

where we have used the fact that $f(x_{ss}) = 0$ by definition. The result is just a linear ODE of the form (7) with $a = f'(x_{ss})$ and $b = -f'(x_{ss})x_{ss}$. Its stability, therefore, depends on the sign of $f'(x_{ss})$. This analysis is local, on the neighborhood of x_{ss} and hence it must be carried out as many times as steady states are found in $\dot{x} = f(x)$.

5.5 Second-order linear differential equations

We now study the solution to the ODE

$$\ddot{x} - a_1\dot{x} - a_2x = b(t)\,.$$

Again, we think of the solution as the sum of a complementary (or homogenous) solution, and a particular (or inhomogeneous) solution.

Complementary solution

The *homogenous equation*, i.e. the ODE resulting after setting $b(t) = 0$ (involving only x and its derivatives), becomes $\ddot{x} - a_1\dot{x} - a_2x = 0$. Now, we *conjecture* (i.e., guess) that the complementary solution has the form $x_c(t) = C\,e^{rt}$ for arbitrary values of C and r. If this is the case, then $\dot{x}_c(t) = Cr\,e^{rt}$ and $\ddot{x}_c(t) = Cr^2\,e^{rt}$. Plugging this guess into the homogenous equation gives

$$\ddot{x}_c - a_1\dot{x}_c - a_2x_c = 0 \qquad \to \qquad Ce^{rt}(r^2 - a_1r - a_2) = 0\,.$$

For the nontrivial value of $C \neq 0$ this equation is only satisfied if $P(r) = r^2 - a_1r - a_2$ is equal to zero. Note that the characteristic polynomial $P(r)$ is now quadratic (since it is associated to a second-order equation), and as such there are **two** values of r that solve the characteristic equation is $P(r) = 0$ (i.e., two roots). Three possibilities emerge:

The roots are real and different. Here we have two real numbers, r_1 and r_2 such that $P(r_1) = P(r_2) = 0$. This means that e^{r_1t} and e^{r_2t} are *both* complementary solutions to the ODE. Since $r_1 \neq r_2$, these function are *linearly independent*. It turns out that any linear combination of them will also be a complementary solutions to the ODE. We conclude that

$$x_c(t) = C_1e^{r_1t} + C_2e^{r_2t}\,, \tag{13}$$

for *any* values of C_1 and C_2, which are related to integrating constats and so are determined from initial or terminal conditions. Note that the complementary solution *is stable if both roots of the ODE is negative* (i.e., $r_i < 0$ since $e^{r_it} \to 0$ as t increases).

The roots are real and equal. Here we a single real number r such that $P(r^*) = 0$ (twice). This means that e^{r^*t} is a complementary solution to the ODE. But it turns out that there is *another* solution, which is te^{r^*t}. Let us verify this. Note that the case arises because $a_1^2 = -4a_2$ which leads to $2r^* = a_1$. Now, let $z = te^{r^*t}$ so that $\dot{z} = e^{r^*t} + tr^*e^{r^*t}$ and $\ddot{z} = 2r^*e^{r^*t} + t(r^*)^2e^{r^*t}$. Evaluate the homogenous equation at $x_c = z$, to get

$$\ddot{z} - a_1\dot{z} - a_2z = \left[2r^* + (r^*)^2t\right]e^{r^*t} - a_1\left(1 + r^*t\right)e^{r^*t} - a_2te^{r^*t} = \left((r^*)^2 - a_1r^* - a_2\right)te^{r^*t} + (2r^* - a_1)e^{r^*t}\,.$$

The term multiplying te^{r^*t} is $P(r^*)$ which is equal to zero. Similarly, the term multiplying e^{r^*t} is also equal to zero. Thus, te^{r^*t} satisfies the homogenous equation and so is a solution. Moreover, te^{r^*t} is linearly independent from e^{r^*t}, and so any linear combination of these functions is also a complementary solution. We conclude that

$$x_c(t) = C_1e^{r^*t} + C_2te^{r^*t} = (C_1 + C_2t)e^{r^*t}\,, \tag{14}$$

for *any* values of C_1 and C_2. The complementary solution *is stable if the root is negative* (i.e., $r^* < 0$ since $e^{r^*t} \to 0$ and $e^{r^*t}t \to 0$ as t increases).

The roots are complex conjugates. It is possible that the roots of the system are complex conjugate of the form $r_1 = p + qi$ and $r_2 = p - qi$, where i is the imaginary unit. The complication here is the presence of i but we can deal with it using **Euler's formula**,

$$e^{rt} = e^{(p\pm qi)t} = e^{pt}[\,\cos(qt) \pm i\sin(qt)\,]\,.$$

This indicates that $e^{pt}\cos(qt)$ and $e^{pt}\sin(qt)$ are solutions to the homogenous equations (verifying this is not difficult). Moreover, these are linearly independent functions, and so any linear combination of them will also be a complementary solutions to the ODE. We conclude that

$$x_c(t) = e^{pt}\left[C_1\cos(qt) + C_2\sin(qt)\right] , \tag{15}$$

for arbitrary constants C_1 and C_2. The term in braces is the sum of two periodic functions and hence the path displays a cyclical, sinusoidal motion. Economists refer to this behavior as *fluctuations*, whereas in other disciplines this is known as *oscillations*. Besides, the stability of $x_c(t)$ depends on the exponential term, e^{pt} and thus on the value of p, the *real* part of the roots. The complementary solution *is stable if the real part of the root is negative* (i.e., $p < 0$ since $e^{pt} \to 0$ as t increases).

Particular solution

The treatment is identical to the first-order case. The method is that of **undetermined coefficients**, where we "guess" the form of $x_p(t)$ from the form of $b(t)$. It $b(t)$ happens to be linearly dependent with any complementary solution, we guarantee linear independence after multiplying the conjecture by t (as many times as necessary).

5.6 First-order linear systems

Sometimes called **coupled ODE**, we have a pair of ODE with mutual feedback between variables:

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_1(t) \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_2(t) .
\end{aligned}$$

Even though this is a special case, the dynamics embedded in such a simple system can describe a wide range of economic phenomena. Besides, most of the methods developed here can be easily extended to more general situations.

To solve the system, we replace the information in one equation into the other and obtain a second-order differential equation. From the first equation,

$$a_{12}x_2 = \dot{x}_1 - a_{11}x_1 - b_1(t) \qquad \text{and so} \qquad a_{12}\dot{x}_2 = \ddot{x}_1 - a_{11}\dot{x}_1 - \dot{b}_1(t) .$$

Multiply the second equation by a_{12} and replace these findings to get

$$\begin{aligned}
\left[\ddot{x}_1 - a_{11}\dot{x}_1 - \dot{b}_1(t)\right] &= a_{12}a_{21}x_1 + a_{22}\left[\dot{x}_1 - a_{11}x_1 - b_1(t)\right] + a_{12}b_2(t) \\
&\to \qquad \ddot{x}_1 - (a_{11} + a_{22})\dot{x}_1 + (a_{11}a_{22} - a_{12}a_{21})x_1 = \dot{b}_1(t) - a_{22}b_1(t) + a_{12}b_2(t) .
\end{aligned}$$

This equation can be solved using the techniques described before. Then, the path of $x_2(t)$ can be readily obtained from that of $x_1(t)$. In general, a first-order system with n variables can be reduced to a single n-th order ODE.

5.7 First-order linear systems, matrix approach

For simplicity, we assume that the terms in both equations are constant. In matrix terms

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \text{or, more compactly,} \qquad \dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B} , \tag{16}$$

where vectors $\boldsymbol{X}$ and $\boldsymbol{B}$ and matrix $\boldsymbol{A}$ are defined implicitly.

Let us consider a system simpler than (10), with a diagonal system matrix $\boldsymbol{\Lambda}$,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \text{or, more compactly,} \qquad \dot{\boldsymbol{Z}} = \boldsymbol{\Lambda}\boldsymbol{Z} + \boldsymbol{W} . \tag{17}$$

where vectors $\boldsymbol{Z}$ and $\boldsymbol{W}$ and matrix Λ are implicitly defined. Strictly speaking (11) is not really a system, but the mere collection of two independent equations, with no feedback whatsoever between the variables involved . As such, the solution to the system is just the collection of solutions to each equation. Using the results in (5) and expressing them in matrix form,

$$\boldsymbol{Z}(t)=\left[\begin{array}{c} z_1(t) \\ z_2(t) \end{array}\right]=\left[\begin{array}{c} C_1 e^{r_1 t} \\ C_2 e^{r_2 t} \end{array}\right]+\left[\begin{array}{c} z_{1p} \\ z_{2p} \end{array}\right], \tag{18}$$

where C_1 and C_2 are the arbitrary constants of integration associated to z_1 and z_2, respectively, and z_{1p} and z_{2p} are the particular solutions.

Provided that it is straightforward to obtain the solution like (11), is there a way to transform a more general system of the form (10) into the simpler form (11)? The answer is yes, and the key is to **diagonalize** matrix $\boldsymbol{A}$. Recall that $\boldsymbol{A}$ can be expressed as $\boldsymbol{A}=\boldsymbol{H}\Lambda\boldsymbol{H}^{-1}$, where Λ is a diagonal matrix whose entries are the **eigenvalues** of $\boldsymbol{A}$ and $\boldsymbol{H}$ is a non-singular matrix whose columns are the corresponding **eigenvectors**. Thus, if we define define $\boldsymbol{Z}=\boldsymbol{H}^{-1}\boldsymbol{X}$ so $\dot{\boldsymbol{Z}}=\boldsymbol{H}^{-1}\dot{\boldsymbol{X}}$ and $\boldsymbol{W}=\boldsymbol{H}^{-1}\boldsymbol{B}$, we move from (10) to (11).

Then, we can easily recover the paths of interest by using the equality $\boldsymbol{X}(t)=\boldsymbol{H}\boldsymbol{Z}(t)$. Therefore,[1]

$$\begin{array}{rcl} x_1(t) & = & C_1 H_{11} e^{r_1 t}+C_2 H_{12} e^{r_2 t}+x_{1p} \\ x_2(t) & = & C_1 H_{21} e^{r_1 t}+C_2 H_{22} e^{r_2 t}+x_{2p} \end{array} \tag{19}$$

where H_{ij} are the entries of $\boldsymbol{H}$, so $(H_{11},H_{21})'$ and $(H_{21},H_{22})'$ are the eigenvectors associated to the eigenvalues r_1 and r_2, respectively.

The paths $x_1(t)$ and $x_2(t)$ are linear combinations of $z_1(t)$ and $z_2(t)$. It follows that the complementary solutions of $x_1(t)$ and $x_2(t)$, and hence the source of dynamics in the system, are linear combinations of the functions $e^{r_1 t}$ and $e^{r_2 t}$. The coefficients of these combinations depend on the eigenvectors of $\boldsymbol{A}$ and the arbitrary constants of integration. The bottomline is that all the dynamic information of the system is contained in matrix $\boldsymbol{A}$, through its eigenvalues and eigenvectors.[2]

5.7.1 Dynamics with real roots

Let us focus on the complementary solutions

$$\begin{array}{rcl} x_{c1}(t) & = & C_1 H_{11} e^{r_1 t}+C_2 H_{12} e^{r_2 t} \\ x_{c2}(t) & = & C_1 H_{21} e^{r_1 t}+C_2 H_{22} e^{r_2 t} \end{array}, \tag{20}$$

whose equilibrium point is $(0,0)$. The general solutions are as (20) plus two constants, such that the equilibrium point becomes (x_{1p},x_{2p}) with identical (short-run) dynamics. It is convenient to illustrate the dynamic properties of these path on the (x_1,x_2) plane, as it is done in Figure 2. For this, note that from (20) it follows that

$$x_{c2}(t)=\left(\frac{C_1 H_{21} e^{r_1 t}+C_2 H_{22} e^{r_2 t}}{C_1 H_{11} e^{r_1 t}+C_2 H_{12} e^{r_2 t}}\right) x_{c1}(t)\,. \tag{21}$$

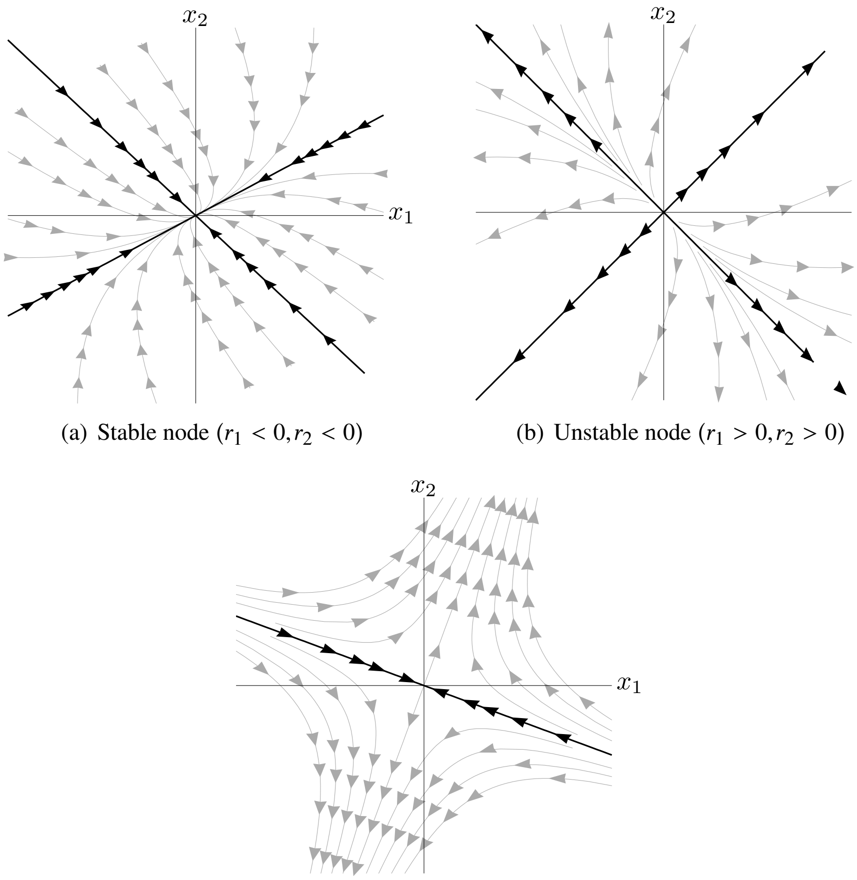
Both eigenvalues, r_1 and r_2 are called the **roots** of the system, and we assume in this section that they are real.

Suppose first that both roots are negative, $r_1<0$ and $r_2<0$. Clearly, both $e^{r_1 t}\to 0$ and $e^{r_2 t}\to 0$ as $t\to\infty$ so both paths are stable. If $C_1=0$, both paths approach zero along the $x_2=(H_{22}/H_{12})x_1$ line, whereas if $C_2=0$ they do so along the $x_2=(H_{21}/H_{11})x_1$ line, see (21). These curves are depicted in Figure 2(a) as the darkest, straight lines. In the more general case where $C_1\neq 0$ and $C_2\neq 0$, we observe from (21) that the slope of the implicit function

[1] Note that the particular solution to (10) is $\boldsymbol{X}_p=-\boldsymbol{A}^{-1}\boldsymbol{B}$.

[2] In what follows we focus on the case where $r_1\neq r_2$. The case where both eigenvalues are equal leads to the same qualitative conclusions, though some minor modifications in the analysis are required since the matrix $\boldsymbol{H}$ may become singular. For completeness, it is worth mentioning that the equilibria with $r_1\neq r_2$ are called **improper nodes** whereas the equilibria with $r_1=r_2$ are **proper nodes**. We make no distinction among the nodes in these notes.

Figure 2. *Dynamic equilibria*



(a) Stable node ($r_1 < 0, r_2 < 0$)

(b) Unstable node ($r_1 > 0, r_2 > 0$)

(c) Saddlepath ($r_1 < 0, r_2 > 0$)

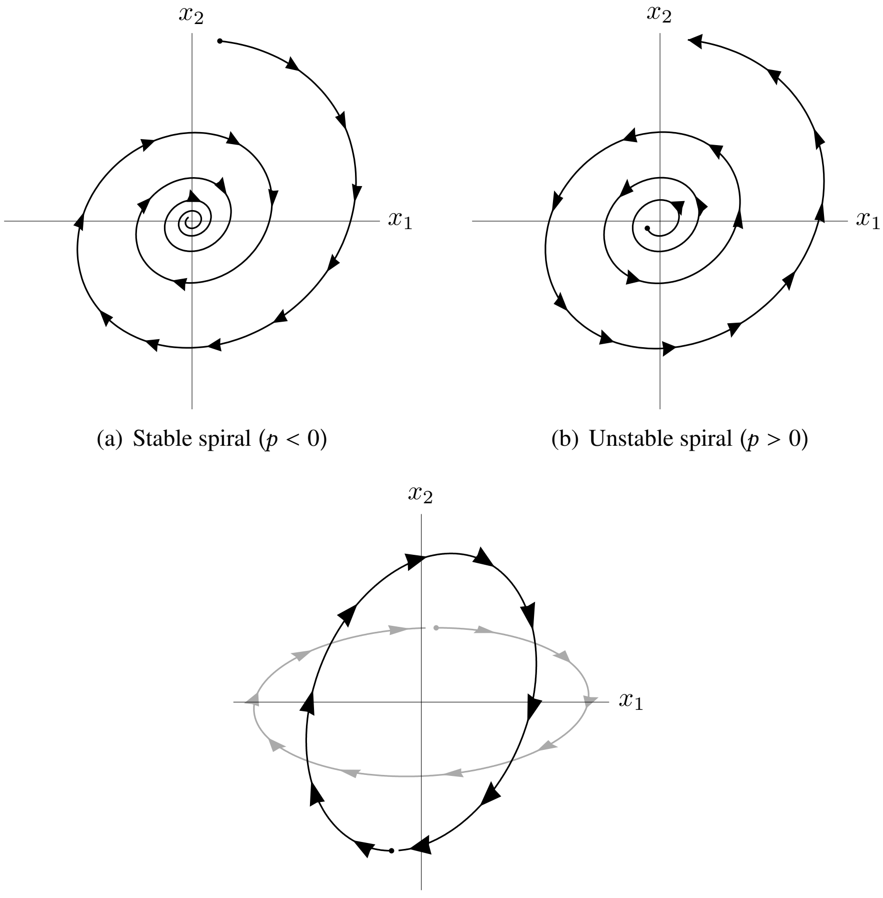
$x_2 = f(x_1)$ changes as time goes by, and is asymptotically tangent to one of the straight lines, depending on whether $r_1 < r_2 < 0$ or $r_2 < r_1 < 0$. These streamlines are sketched as gray curves in Figure 2(a). Notice that regardless of the initial conditions, $\boldsymbol{X}_c(t)$ converges to $(0,0)$. This equilibrium is called **stable node**.

The opposite case occurs where both roots are positive, $r_1 > 0$ and $r_2 > 0$, so that $e^{r_1 t} \to \infty$ and $e^{r_2 t} \to \infty$ as $t \to \infty$. Both paths are now unstable and the resulting equilibrium is called **unstable node**. In Figure 2(b) it can be seen that regardless of the initial condition, $\boldsymbol{X}_c(t)$ moves away from the equilibrium in a monotonic, explosive fashion. As in the previous case, each streamline in Figure 2(b) is associated with a combination (C_1, C_2), i.e. with a different initial condition.

A very interesting case happens when both roots are of *opposite sign*. With no loss of generality suppose $r_1 < 0$ and $r_2 > 0$, so that $e^{r_1 t} \to 0$ and $e^{r_2 t} \to \infty$ as $t \to \infty$. From (20) we can see that the paths $\boldsymbol{X}_c(t)$ are divergent whenever $C_2 \neq 0$, no matter what value C_1 takes. Nevertheless, there is one instance where the paths are convergent: if $C_2 = 0$, $x_{1c}(t) = C_1 H_{11} e^{r_1 t} \to 0$ and $x_{2c}(t) = C_1 H_{21} e^{r_1 t} \to 0$ as t goes to infinity. On the (x_1, x_2) plane this is captured by the points on the line $x_2 = (H_{21}/H_{11})x_1$ which is called the **stable arm** or **saddlepath**. This is represented by the black straight line Figure 2(c). The remaining streamlines are qualitatively similar to those in Figure 2(b).

Contrary to the previous cases, the stability of the equilibrium depends on the initial conditions, particularly on the value of C_2. The solution to the system will be stable only if the initial (or terminal) conditions are such that the system lies on the saddlepath. In many disciplines, the saddlepath equilibrium is regarded as unstable since $C_2 = 0$ is only one point of the real line. However, many economic models display such an equilibrium and the attention is centered precisely at the special case where $C_2 = 0$. The fascinating debate on rational expectations in Economics is directly related to whether economic agents are clever and forward-looking enough to locate the economy on the stable arm.

Figure 3. *Dynamic equilibria with oscillations*



(a) Stable spiral ($p < 0$) (b) Unstable spiral ($p > 0$)

(c) Vortex ($p = 0$)

5.7.2 Oscillations and complex roots

When the eigenvalues of $\boldsymbol{A}$ are complex conjugate of the form $r_{1,2} = p \pm qi$,

$$e^{rt} = e^{(p\pm qi)t} = e^{pt}[\,\cos(qt) \pm i \sin(qt)\,]\,,$$

Moreover, the fact that the eigenvalues are complex conjugates, implies that so will be the entries of the eigenvectors. It can be shown that $H_{11} = \alpha_1 + \alpha_2 i$, $H_{12} = \alpha_1 - \alpha_2 i$, $H_{21} = \beta_1 + \beta_2 i$ and $H_{22} = \beta_1 - \beta_2 i$. Upon replacing all these findings into (20), the following paths are obtained

$$\begin{array}{rcl} x_{c1}(t) & = & e^{pt}[\,\alpha_1(C_1 + C_2)\cos(qt) + \alpha_2(C_2 - C_1)\sin(qt)\,] \\ x_{c2}(t) & = & e^{pt}[\,\beta_1(C_1 + C_2)\cos(qt) + \beta_2(C_2 - C_1)\sin(qt)\,] \end{array}\,. \tag{22}$$

If $p < 0$, the paths will display damped oscillations. The term in braces fluctuates within a bounded range and this is multiplied by a exponentially decreasing function such that $\boldsymbol{X}_c(t) \to (0,0)$. The resulting equilibrium is a **stable spiral** or **stable focus**, see Figure 3(a).

On the other side, if $p > 0$ the oscillations are explosive since e^{pt} grows boundlessly as t increases. The equilibrium is in this case an **unstable spiral** or **unstable focus** and is depicted in Figure 3(b).

Finally, with $p = 0$, $e^{pt} = 1$ and the paths only exhibit the periodic behavior of the sine and cosine functions, and hence neither converge nor diverge. The equilibrium becomes a **center** or **vortex** and is never reached unless the initial conditions are such that $\boldsymbol{X}_c(t) = (0,0)$. The streamlines of two centres, each associated with a different initial condition, are shown is shown in Figure 3(c).

5.7.3 Classification of equilibria

From the last two sections it is evident that the eigenvalues of $\boldsymbol{A}$ play a key role in determining the dynamics of the system. If they are real, the equilibrium can be either a node or a saddle, whereas if they are complex it can be either a spiral or a vortex. In a similar fashion, the sign of (the real part of) these roots determine whether the equilibrium is stable.

Next, we will present criteria to qualify the equilibrium point without completely solving for the paths $x_1(t)$ and $x_2(t)$. These are useful if the main concern is to get a qualitative assessment on the dynamics of the system, and the actual paths themselves are not of importance. Besides, these criteria come on handy in the analysis of nonlinear systems.

The eigenvalues of matrix $\boldsymbol{A}$ are computed by solving the **characteristic equation** $P(r) = 0$, where

$$P(r) = |\boldsymbol{A} - r\boldsymbol{I}| = \begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) .$$

Note that solving $P(r) = 0$ is equal to finding r in the quadratic equation

$$r^2 - \operatorname{tr}(\boldsymbol{A})r + \det(\boldsymbol{A}) = 0 .$$

where $\operatorname{tr}(\boldsymbol{A})$ is the trace of $\boldsymbol{A}$ and $\det(\boldsymbol{A})$ is its determinant. Thus,

$$r_{1,2} = \frac{1}{2}\left[\operatorname{tr}(\boldsymbol{A}) \pm \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}\right] . \tag{23}$$

implying that $r_1 + r_2 = \operatorname{tr}(\boldsymbol{A})$ and $r_1 r_2 = \det(\boldsymbol{A})$. We can get the following conclusions, which are summarized in the Table 1 below:

- If $\det(\boldsymbol{A}) > 0$, both r_1 and r_2 are of the same sign. If also $\operatorname{tr}(\boldsymbol{A}) < 0$, we conclude that the real part of both roots is negative and hence that the equilibrium is stable.
- If $\det(\boldsymbol{A}) > 0$ and $\operatorname{tr}(\boldsymbol{A}) > 0$, the real part of both r_1 and r_2 is positive and hence the equilibrium is unstable.
- If $\det(\boldsymbol{A}) < 0$, r_1 and r_2 have opposite signs and therefore the equilibrium is a saddle.
- If $\operatorname{di}(\boldsymbol{A}) = \operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A}) > 0$, the roots are real and therefore the equilibrium is a node or a saddle point. If $\operatorname{di}(\boldsymbol{A}) < 0$, the roots are complex and the equilibrium is a spiral or a vortex.

Table 1. *Equilibria in the 2-variable system*

Equilibrium	Roots are...		$\det(\boldsymbol{A})$	$\operatorname{tr}(\boldsymbol{A})$	$\operatorname{di}(\boldsymbol{A})$
Saddle point	Real	$r_1 < 0, r_2 > 0$	< 0		> 0
Stable node	Real	$r_1, r_2 < 0$	> 0	< 0	> 0
Unstable node	Real	$r_1, r_2 > 0$	> 0	> 0	> 0
Vortex	Imaginary	$p = 0$	> 0	$= 0$	< 0
Stable spiral	Complex conjugate	$p < 0$	> 0	< 0	< 0
Unstable spiral	Complex conjugate	$p > 0$	> 0	> 0	< 0

5.7.4 An illustrative example

Let $a > 0$, $c > 0$ and $a \neq c$. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & c \\ c & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} \qquad \rightarrow \qquad \dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B} . \tag{24}$$

The particular solution to the system is given by the vector $\boldsymbol{X}_p$ such that $\dot{\boldsymbol{X}}_p = \boldsymbol{0}$,

$$\boldsymbol{X}_p = -\boldsymbol{A}^{-1}\boldsymbol{B} = \frac{1}{a^2-c^2}\begin{bmatrix} a & c \\ c & a \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix} = \frac{b}{a^2-c^2}\begin{bmatrix} a \\ c \end{bmatrix} .$$

On the other side, for the complementary solution we require to find the eigenvalues and eigenvectors of $\boldsymbol{A}$. Following (18), the eigenvalues satisfy

$$r_{1,2} = \frac{1}{2}\left[-2a \pm \sqrt{4a^2 - 4(a^2-c^2)}\right] \quad \rightarrow \quad \begin{array}{rcl} r_1 & = & -c-a \\ r_2 & = & c-a \end{array} .$$

Since $a > 0$ and $c > 0$, r_1 is always negative. If $a > c$, r_2 is also negative and the equilibrium $(x_{1p}, x_{2p})'$ is a stable node. On the other side, if $a < c$, r_2 is positive and we face a saddle point.

The eigenvectors are defined as $(\boldsymbol{A} - r_i\boldsymbol{I})\boldsymbol{H}_i = \boldsymbol{0}$ for $i = 1,2$:

$$\begin{bmatrix} c & c \\ c & c \end{bmatrix}\begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -c & c \\ c & -c \end{bmatrix}\begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies, after normalization, that $H_{11} = -H_{21} = H_{12} = H_{22} = 1$. Then, the general solution to the system is

$$\begin{array}{rcl} x_1(t) & = & C_1 e^{-(c+a)t} + C_2 e^{(c-a)t} + x_{1p} \\ x_2(t) & = & -C_1 e^{-(c+a)t} + C_2 e^{(c-a)t} + x_{2p} \end{array} .$$

For the case when the equilibrium is a saddle, $c > a$, the saddlepath corresponds to $C_2 = 0$ and is captured by the relationship

$$x_1(t) = -x_2(t) + (x_{1p} + x_{2p}) .$$

It is left as an exercise to perform the more qualitative analysis described in section 5.7.3.

5.7.5 The companion form

We finish this section by introducing the equivalence between higher order equations of scalar variables, and first-order equations of vector variables. Consider the ODE

$$\ddot{x} - a_1\dot{x} - a_2 x = b .$$

Define an auxiliary variable $y = \dot{x}$ so that $\dot{y} = \ddot{x}$. If we replace these definitions into the second-order equation we obtain $\dot{y} - a_1 y - a_2 x = b$. We can write these equations in matrix terms as

$$\begin{bmatrix} \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} , \tag{25}$$

so upon introducing a new artificial variable we moved form the second-order equation to a first-order system. The roots of this system satisfy $r^2 - a_1 r - a_2 = 0$, which resembles the homogenous version of the second-order equation. Additionally, it is easy to verify that the particular solution of x is $x_p = -b/a_2$ and corresponds to the value of x such that $\ddot{x} = \dot{x} = 0$. The complementary solution of x will have, therefore, the form (20) or (14).

5.8 Qualitative analysis of nonlinear first-order systems

As in the univariate case, many relevant applications involve nonlinear relationships and quite often we will not be able to find a closed-form solution. However, it is still possible to obtain qualitative conclusions on the nature of the equilibrium and the dynamics around it.

Consider the nonlinear **autonomous** system

$$\dot{x}_1 = f(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = g(x_1, x_2) , \tag{26}$$

where $f(.)$ and $g(.)$ are arbitrary functions. The strategy consists first to find a steady state $\boldsymbol{X}_{ss} = (x_{1ss}, x_{2ss})'$ such that the system is at rest $\dot{x}_1 = \dot{x}_2 = 0$, thus satisfying $f(\boldsymbol{X}_{ss}) = g(\boldsymbol{X}_{ss}) = 0$. Next, the functions $f(.)$ and $g(.)$ can be linearized around this point yielding

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \simeq \begin{bmatrix} \dfrac{\partial f(.)}{\partial x_1} & \dfrac{\partial f(.)}{\partial x_2} \\ \dfrac{\partial g(.)}{\partial x_1} & \dfrac{\partial g(.)}{\partial x_2} \end{bmatrix}_{X=X_{ss}} \begin{bmatrix} x_1 - x_{1ss} \\ x_2 - x_{2ss} \end{bmatrix} \qquad \text{or} \qquad \dot{\boldsymbol{X}} \simeq \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\,. \tag{27}$$

The matrix $\boldsymbol{A}$ is called the **Jacobian matrix** of the system and contains the partial derivatives evaluated at the steady state. The linearized system (27), of course, resembles the standard linear system (10) and hence all the tools presented in section 6.3 can be used to analyze it, specially the results in **Table 1**. Just as in the univariate case, the steady state may not be unique, in which case the system should be linearized around each of these points.

A complementary graphical analysis can be performed in the (x_1, x_2) plane, the two-dimensional phase diagram. Note that $\dot{x}_1 = 0$ provides a relationship $f(x_1, x_2) = 0$ that can be plotted in such a plane, and similarly with $\dot{x}_2 = 0$ that implies $g(x_1, x_2) = 0$. The steady states are the values at which the $\dot{x}_1 = 0$ curve or **locus** intersects with the $\dot{x}_2 = 0$ **locus**. Besides, each locus divide the plane into two regions (apart from the locus itself) denoting horizontal or vertical movements, rendering four regions around a given steady state containing combinations of these motions.

To fix ideas on the workings of the phase diagram, let us analyze the linear system in section 5.7.4,

$$\dot{x}_1 = -a\,x_1 + c\,x_2 + b \qquad \text{and} \qquad \dot{x}_2 = c\,x_1 - a\,x_2\,,$$

where, as we found before, the steady state is unique and given by the particular solution of (24).

Suppose first that $a > c$. The first locus is

$$\dot{x}_1 = 0 \qquad \rightarrow \qquad x_2 = \left(\frac{a}{c}\right) x_1 + \frac{b}{c}\,,$$

which is a line in the (x_1, x_2) plane with slope $a/c > 1$ and is plotted in Figure 4(a). This locus refers to *horizontal* movements as it involves $\dot{x}_1$, the direction of change of x_1. The points on the locus are such that $\dot{x}_1 = 0$, and hence the points lying on either of the two regions separated by the locus are such that $\dot{x}_1 \neq 0$.

There are many ways to sign $\dot{x}_1$ in these regions. For instance, one can take an arbitrary point on each region and evaluate the $f(.)$ function. Alternatively, the partial derivatives around the steady state can serve this purpose. We have that

$$\frac{\partial \dot{x}_1}{\partial x_1} = -a < 0 \qquad \text{and} \qquad \frac{\partial \dot{x}_1}{\partial x_2} = c > 0\,.$$

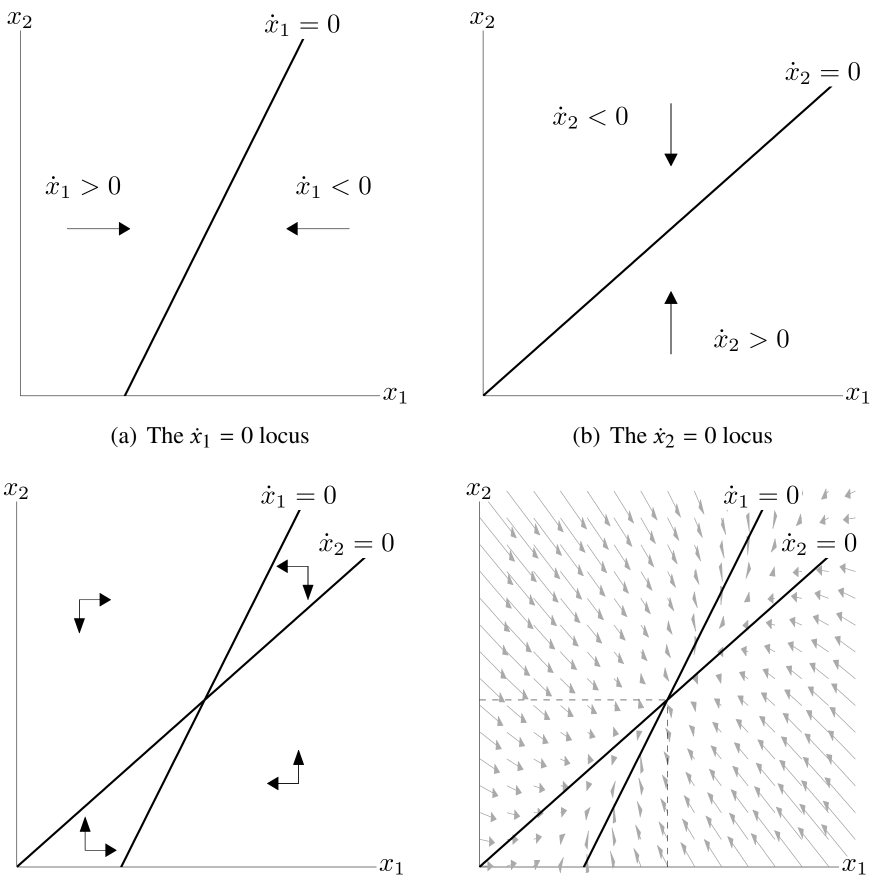
Let us first focus on $\partial \dot{x}_1/\partial x_1 < 0$. Consider a point on the locus satisfying $\dot{x}_1 = 0$ and a small increase in x_1. Hence the region involved after this perturbation is to the *right* of the locus, i.e. the direction at which x_1 increases. Since $\partial \dot{x}_1/\partial x_1 < 0$ we can conclude that $\dot{x}_1 < 0$ in this region and, following a similar reasoning, that $\dot{x}_1 > 0$ to the *left* of the locus. Figure 4(a) shows arrowheads pointing at the directions given by this derivative: eastwards in the region to the left of the locus and westwards otherwise.

The same conclusion is reached if we use $\partial \dot{x}_1/\partial x_2 > 0$ instead. Consider a point on the locus $\dot{x}_1 = 0$ and a small increase in x_2. Hence the region involved after this perturbation is *above* of the locus, i.e. the direction at which x_2 increases. Since $\partial \dot{x}_1/\partial x_2 > 0$ we can conclude that $\dot{x}_1 > 0$ in this region and that $\dot{x}_1 < 0$ in the region *below* the locus. Figure 4(a) shows arrowheads pointing at the directions given by this derivative: eastwards in the region above the locus and westwards otherwise.

The second locus is

$$\dot{x}_2 = 0 \qquad \rightarrow \qquad x_2 = \left(\frac{c}{a}\right) x_2\,,$$

Figure 4. *Example of the two-dimensional phase diagram (stable node)*



(a) The $\dot{x}_1 = 0$ locus (b) The $\dot{x}_2 = 0$ locus

(c) Both loci and arrowheads (d) System dynamics

which is a straight line with slope $c/a < 1$ that crosses the origin, as shown in Figure 4(b). This locus describes *vertical* movements as it involves $\dot{x}_2$, the direction of change of x_2. The points on the locus are such that $\dot{x}_2 = 0$, and hence the points lying on either of the two regions separated by the locus are such that $\dot{x}_2 \neq 0$. As before, we have that

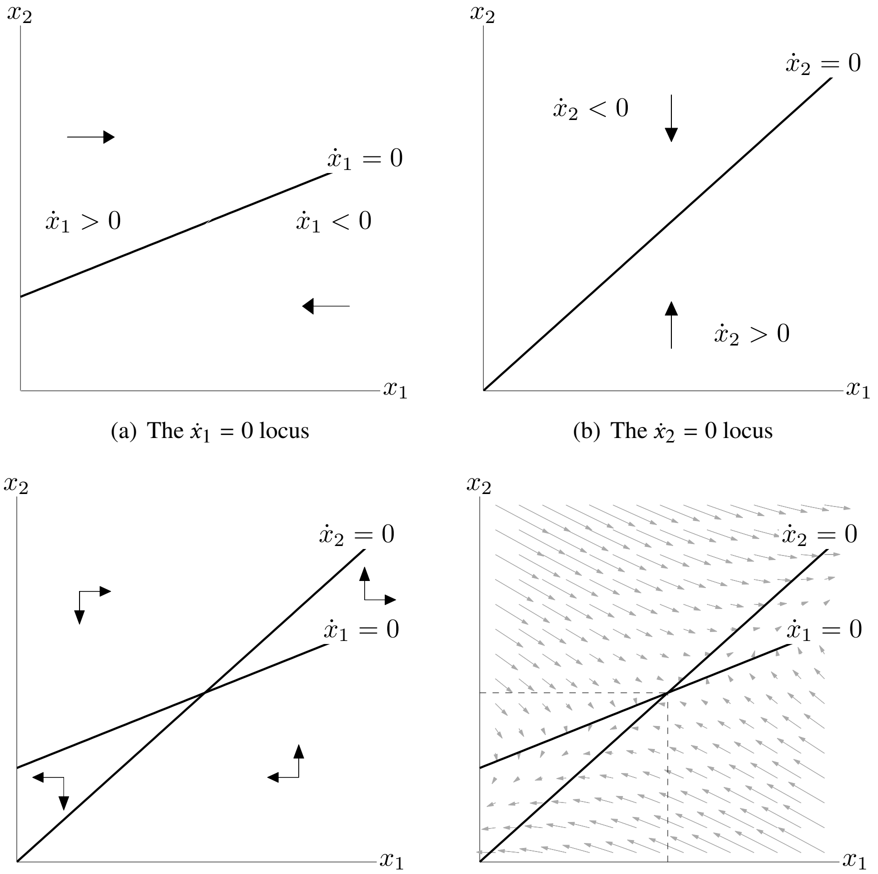
$$\frac{\partial \dot{x}_2}{\partial x_1} = c > 0 \qquad \text{and} \qquad \frac{\partial \dot{x}_2}{\partial x_2} = -a < 0\,.$$

Consider a point on the locus $\dot{x}_2 = 0$ and consider a small increase in x_1. Hence the region involved after this perturbation is to the *right* of the locus, i.e. the direction at which x_1 increases. Since $\partial \dot{x}_2/\partial x_1 > 0$ we can conclude that $\dot{x}_2 > 0$ in this region and, following a similar reasoning, that $\dot{x}_2 < 0$ to the *left* of the locus. Figure 4(b) shows arrowheads pointing at the directions given by this derivative: downwards in the region to the left of the locus and upwards to the right of the locus. Alternatively, consider a point on the locus $\dot{x}_2 = 0$ and an increase in x_2. Hence the region involved after this perturbation is *above* of the locus, i.e. the direction at which x_2 increases. Since $\partial \dot{x}_2/\partial x_2 < 0$ we can conclude that $\dot{x}_2 < 0$ in this region and that $\dot{x}_2 > 0$ in the region *below* the locus. Hence, the arrowheads point at the directions given by this derivative: downwards in the region above the locus and upwards below the locus.

Having defined the forces of motion implied by each locus, Figure 4(c) combine both loci and shows the determination of the steady state and its dynamics. A clear conclusion for this example is that the equilibrium is stable as all the arrowheads point directly at it. From Example 5.7.4, we know that the case $a > c$ corresponds to a stable node. An alternative illustration is given in Figure 4(d) where the direction of change is evaluated at many points on the plane. Compare panels (c) and (d) and note that they provide the same information.

Figure 5 repeats the analysis above but for the case where $c > a$. Panels (a) and (b) are qualitative the same as

Figure 5. *Example of the two-dimensional phase diagram (saddlepath equilibrium)*



before, though know the $\dot{x}_2 = 0$ locus is steeper than the $\dot{x}_1 = 0$ locus. The arrowheads remain the same since $a > 0$ and $c > 0$ in both cases. However, an important change can be observed when combining both loci in panels (c) or (d) of Figure 5. The relative position of the loci determine the nature of the equilibrium. We know from Example 5.7.4 that the case $c > a$ corresponds to a saddle point.

The principles of this analysis hold when the system is nonlinear.