UNIVERSIDAD DEL PACÍFICO
Departamento Académico de Economía
Matemáticas III (130233) - Sección A
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3 | Complex numbers

- **Imaginary unit.** $i^2 = -1$ or $i = \sqrt{-1}$.

- **Powers of the imaginary unit.** The powers of i repeat in a cycle $1, i, -1, -i$:

$$\begin{array}{lll} i^0 = 1 & i^4 = i^3 i = -i^2 = 1 & i^8 = (i^4)^2 = 1 \\ i^1 = i & i^5 = (i^4) i = i & i^9 = (i^4)^2 i = i = i \\ i^2 = -1 & i^6 = (i^4) i^2 = -1 & i^{10} = (i^4)^2 i^2 = -1 \\ i^3 = i^2 i = -i & i^7 = (i^4) i^3 = -i & i^{11} = (i^4)^2 i^3 = -i \end{array}$$

  It follows that $i^{4k} = 1$ for $k \geq 0$.

  With this, any power of i can be easily computed. For instance, $i^{31} = i^{28} i^3 = (i^4)^7 i^3 = 1^7 i^3 = -i$. Also, $i^{100} = (i^4)^{25} = 1^{25} = 1$.

- **Complex number.** A number formed by a real part and an imaginary part: $z = p + qi$, where $p \in \mathbb{R}$ and $q \in \mathbb{R}$. p is the *real part* of the complex number, whereas q is the *imaginary part*. When $q = 0$, the complex number is a real number; when $p = 0$, the complex number is a purely imaginary number.

- **Addition.** If $z_1 = p_1 + q_1 i$ and $z_2 = p_2 + q_2 i$, then $z_1 \pm z_2 = (p_1 \pm p_2) + (q_1 \pm q_2) i$.

- **Multiplication.** If $z_1 = p_1 + q_1 i$ and $z_2 = p_2 \pm q_2 i$, then

$$z_1 z_2 = (p_1 + q_1 i)(p_2 \pm q_2 i) = p_1 p_2 + (p_2 q_1 \pm p_1 q_2) i \pm q_1 q_2 i^2 = (p_1 p_2 \mp q_1 q_2) + (p_2 q_1 \pm p_1 q_2) i \,.$$

- **Division.** If $z_1 = p_1 + q_1 i$ and $z_2 = p_2 + q_2 i$, then

$$\frac{z_1}{z_2} = \frac{p_1 + q_1 i}{p_2 + q_2 i} \left(\frac{p_2 - q_2 i}{p_2 - q_2 i} \right) = \frac{(p_1 p_2 + q_1 q_2) + (p_2 q_1 - p_1 q_2) i}{p_2^2 - q_2^2 i^2} = \frac{p_1 p_2 + q_1 q_2}{p_2^2 + q_2^2} + \left(\frac{p_2 q_1 - p_1 q_2}{p_2^2 + q_2^2} \right) i \,.$$

- **Complex conjugate.** If $z = p + qi$, its conjugate is $\bar{z} = p - qi$. Note that $z + \bar{z} = 2p \in \mathbb{R}$ and $z\bar{z} = p^2 + q^2 \in \mathbb{R}$.

- **Roots.** Let $P(x) = ax^2 + bx + c$. The roots of this polynomial satisfy $P(z) = 0$ and $P(\bar{z}) = 0$:

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \bar{z} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \,.$$

  If $b^2 > 4ac$, z and $\bar{z}$ are real and different. If $b^2 = 4ac$, $z = \bar{z} \in \mathbb{R}$. Finally, if $b^2 < 4ac$

$$z = p + qi \qquad \text{and} \qquad z = p - qi \,,$$

  are complex conjugate, with $p = -b/(2a)$ and $q = \sqrt{4ac - b^2}/(2a)$.

- **Degree n.** In general, a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ (with *real coefficients*) has n roots. If z is one such root, $P(z) = 0$, its conjugate will also be a root, $P(\bar{z}) = 0$. In other words, complex roots appear always as pairs with their corresponding conjugates.

  It follows that if n is odd, $P(x)$ must have *at least* one real root.

- **Modulus.** The "size" or "magnitude" (i.e., the distance from zero) of the complex number. It is the generalization of the notion of an absolute value. If $z = p + qi$, then

$$|\, z \,| = \sqrt{p^2 + q^2}\,.$$

- **Properties of the modulus.**
  (1) If z is a real number ($p = 0$), $|\, z \,| =$ Absolute value(z).
  (2) $|\, z \,| = |\, \bar{z} \,|$, where $\bar{z}$ is the conjugate of z.
  (3) $|\, z \,|^2 = z \cdot \bar{z}$.
  (4) For any two complex numbers z_1 and z_2, $|\, z_1 z_2 \,| = |\, z_1 \,| \cdot |\, z_2 \,|$.
  (5) For any two complex numbers z_1 and z_2, $|\, z_1 \div z_2 \,| = |\, z_1 \,| \div |\, z_2 \,|$.

- **Polar form.** The form $z = p + qi$ is the cartesian representation of z (the x-axis represents the real numbers, whereas the y-axis represents the purely imaginary numbers). An alternative representation, in *polar coordinates*, is the following

$$z = p + qi = r[\cos(\theta) + i\sin(\theta)] \quad \text{where} \quad r = \sqrt{p^2 + q^2} \text{ (modulus)}$$
$$\text{and } \theta = \arctan\left(\frac{q}{p}\right) = \arccos\left(\frac{p}{r}\right) = \arcsin\left(\frac{q}{r}\right) \text{ (argument)}\,.$$

- **Polar of the conjugate.** The modulus of $\bar{z}$ is the same of z, but the argument is $-\theta$. Thus,

$$\bar{z} = p - qi = r[\cos(-\theta) + i\sin(-\theta)] = r[\cos(\theta) - i\sin(\theta)]\,.$$

- **Trigonometric identities.** Recall that

$$\begin{aligned}
\cos(\theta_1 \pm \theta_2) &= \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2)\,,\\
\sin(\theta_1 \pm \theta_2) &= \sin(\theta_1)\cos(\theta_2) \pm \cos(\theta_1)\sin(\theta_2)\,.
\end{aligned}$$

- **Multiplication, polar form.** Let $z_j = r_j[\cos(\theta_j) + i\sin(\theta_j)]$ for $j = \{1,2\}$. Then,

$$\begin{aligned}
z_1 z_2 &= r_1 r_2[\cos(\theta_1) + i\sin(\theta_1)][\cos(\theta_2) + i\sin(\theta_2)]\\
&= r_1 r_2\{\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + [\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)]i\}\\
z_1 z_2 &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]\,.
\end{aligned}$$

- **Division, polar form.** Let $z_j = r_j[\cos(\theta_j) + i\sin(\theta_j)]$ for $j = \{1,2\}$. Then,

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1}{r_2}\frac{[\cos(\theta_1) + i\sin(\theta_1)]}{[\cos(\theta_2) + i\sin(\theta_2)]} = \frac{r_1}{r_2}\frac{[\cos(\theta_1) + i\sin(\theta_1)]}{[\cos(\theta_2) + i\sin(\theta_2)]}\frac{[\cos(\theta_2) - i\sin(\theta_2)]}{[\cos(\theta_2) - i\sin(\theta_2)]}\\
&= \frac{r_1}{r_2}\frac{\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + [\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2)]i}{\cos(\theta_2)^2 + \sin(\theta_2)^2}\\
\frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]\,.
\end{aligned}$$

- **De Moivre's formula for powers.** It follows from the workings of multiplication. For any integer $n > 0$,

$$z^n = r^n[\cos(\theta) + i\sin(\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]\,.$$

  De Moivre's formula also applies to negative powers. For any integer $n > 0$, recall that $r^2 = z \cdot \bar{z}$. Then,

$$z^{-n} = \frac{\bar{z}^{\,n}}{r^{2n}} = \frac{r^n}{r^{2n}}[\cos(\theta) - i\sin(\theta)]^n = r^{-n}[\cos(-\theta) + i\sin(-\theta)]^n = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)]\,.$$

- **Roots of a complex number.** Let w be a complex number such that $w^n = z$. Alternatively, $w = z^{1/n}$. Consider the polar form of each of these numbers:

  $$z = r[\cos(\theta) + i\sin(\theta)] \quad \text{and} \quad w = \rho[\cos(\alpha) + i\sin(\alpha)] .$$

  Using De Moivre's formula to determine w^n, it follows that

  $$\rho^n = r \rightarrow \quad \rho = r^{1/n} ,$$
  $$\cos(n\alpha) = \cos(\theta) \rightarrow \quad \alpha = \frac{\theta}{n} + \frac{2\pi k}{n} \quad \text{for} \quad k = 0, 1, \ldots, n-1 .$$

  Therefore,

  $$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \quad \text{for} \quad k = 0, 1, \ldots, n-1 .$$

- **Euler's formula.** Famous result (that we shall prove later in the course):

  $$e^{xi} = \cos(x) + i\sin(x) .$$

  It follows that

  $$e^{-xi} = \cos(x) - i\sin(x) .$$

- **Special cases.** $e^{\frac{1}{2}\pi i} = i$, $e^{\pi i} = -1$, $e^{\frac{3}{2}\pi i} = -i$ and $e^{2\pi i} = 1$.
- **Polar form, compact.** Using Euler's formula,

  $$p \pm qi = r[\cos(\theta) \pm i\sin(\theta)] \equiv re^{\pm\theta i} .$$

- **Multiplication and division, easier.** Using the exponential polar form,

  $$z_1 z_2 = \left(r_1 e^{\theta_1 i}\right)\left(r_2 e^{\theta_2 i}\right) = r_1 r_2 e^{(\theta_1+\theta_2)i} = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] .$$
  $$\frac{z_1}{z_2} = \frac{r_1 e^{\theta_1 i}}{r_2 e^{\theta_2 i}} = \frac{r_1}{r_2} e^{(\theta_1-\theta_2)i} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] .$$

- **Roots of a complex number, compact.** We have obtained a result for $z^{1/n}$. Using the exponential form,

  $$z^{1/n} = r^{1/n} e^{(\theta/n + 2\pi k/n)i} \equiv r^{1/n} e^{\theta i/n} e^{2\pi k i/n} \quad \text{for} \quad k = 0, 1, \ldots, n-1 .$$

  The term $r^{1/n}e^{\theta i/n}$ is called the *primitive root*. Each of the remaining $n-1$ roots is obtained by multiplying this primitive by $e^{2\pi k i/n}$ for $k = 1, 2 \ldots, n-1$. The term $e^{2\pi k i/n}$ is often known as *root of unity*.