Mathematics III

Semester 2017-II

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Vectors in $\mathbb{R}^n$

Example 1.1 (Random Movement)

Simulate the path a player makes in the pitch. Build a simple model to accomplish the task.

Consider the case of a player who, for each moment t in time, decides

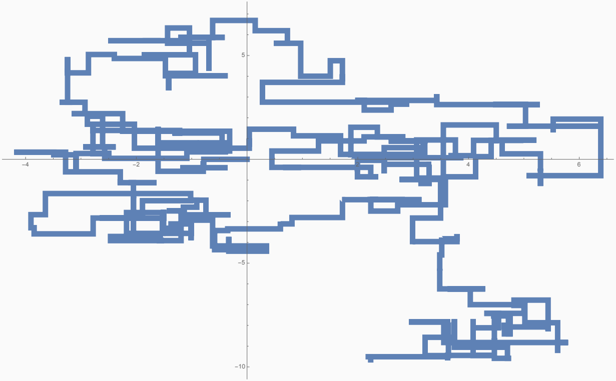
1. if he/she moves horizontally (left or right) and vertically (up or down)
2. how much he/she wants to move at time t (call this δ_t)

For example if he/she starts at the origin and moves horizontally in the first moment and vertically on the second, we obtain that:

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \delta_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

i.e a linear combination of selector vectors.

Sample path:



Matrices in $\mathbb{R}^{n \times m}$

Example 2.1 (Image Processing)

Say you have a picture as a $n \times n$ *matrix* $\boldsymbol{A}$ *in a grey scale, i.e. with entries* $a_{i,j} \in [0, 1]$*, where* $a_{i,j} = 0$ *means black and* $a_{i,j} = 1$ *means white.*

1. *How would you add light to the picture? (assume that for any transformed matrix* $\tilde{\boldsymbol{A}}$ *it holds that* $\tilde{a}_{i,j} \leq 0$ *means black and* $\tilde{a}_{i,j} \geq 0$ *means white.)*
2. *How can you rotate the picture horizontally by pre or post multiplying* $\boldsymbol{A}$ *by another matrix?*

- Define $\tilde{\boldsymbol{A}} = \lambda \boldsymbol{A}$, for $\lambda > 0$.

$1\boldsymbol{A}$

$1.5\boldsymbol{A}$

$2\boldsymbol{A}$

$2.5\boldsymbol{A}$

$3\boldsymbol{A}$

$3.5\boldsymbol{A}$

- Define $\tilde{\boldsymbol{A}} = \boldsymbol{AB}$, where $\boldsymbol{B}_j = \boldsymbol{I}_{n-j+1}$.

Vector Spaces

Example 3.1 (Spanning set of $\mathcal{P}_n(x)$)

1. *Every polynomial in $\mathcal{P}_n(t)$ can be expressed as a lc of $n+1$ polynomials of degree $\leq n$*

$$1, \quad x, \quad x^2, \quad x^3, \dots, x^n$$

Thus these monomials (powers of x), are a ss of $\mathcal{P}_n(x)$.

2. *For any scalar $c \in \mathbb{R}$, the following $n+1$ powers of $(x-c)$*

$$1, \quad (x-c), \quad (x-c)^2, \quad (x-c)^3, \dots, (x-c)^n$$

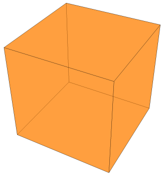
are also a ss of $\mathcal{P}_n(x)$.

Linear Transformations

Example 4.1 (Parallelotope)

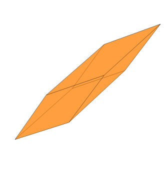
Consider $\mathcal{P}$ described by the canonical basis of $\mathbb{R}^3$; $\mathcal{P}'$ described by $\boldsymbol{A}_1 = (1,3,1)^\top$, $\boldsymbol{A}_2 = (2,1,1)^\top$ and $\boldsymbol{A}_3 = (1,1,2)^\top$; and $\mathcal{P}''$ described by $\boldsymbol{B}_1 = (1/2,1,1/2)^\top$, $\boldsymbol{B}_2 = (2,1,1)^\top$ and $\boldsymbol{B}_3 = (1,1/2,1/2)^\top$.

$\mathcal{P}$



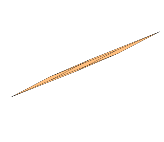
$\boldsymbol{I}(\boldsymbol{v})$

$\mathcal{P}'$



$\boldsymbol{A}(\boldsymbol{v})$

$\mathcal{P}''$

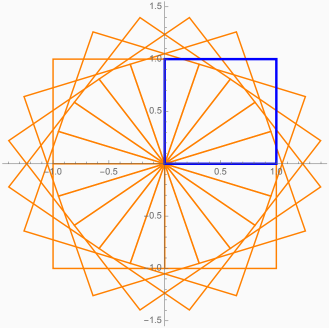


$(\boldsymbol{BA})(\boldsymbol{v})$

This interpretation makes immediate that $(\boldsymbol{BA})(\boldsymbol{v}) = \boldsymbol{B}(\boldsymbol{A}(\boldsymbol{v}))$

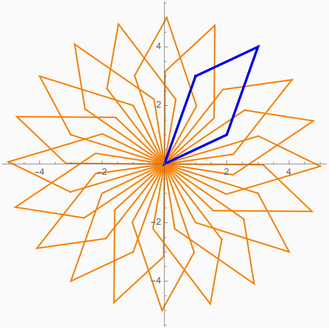
Example 4.2 (Rotation in $\mathbb{R}^{2\times 2}$)

Identity matrix



$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Arbitrary matrix



$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Diagonalization

Example 5.1 (Google's Search Algorithm)

Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix.

Example 5.2 (Augur's Consensus Algorithm)

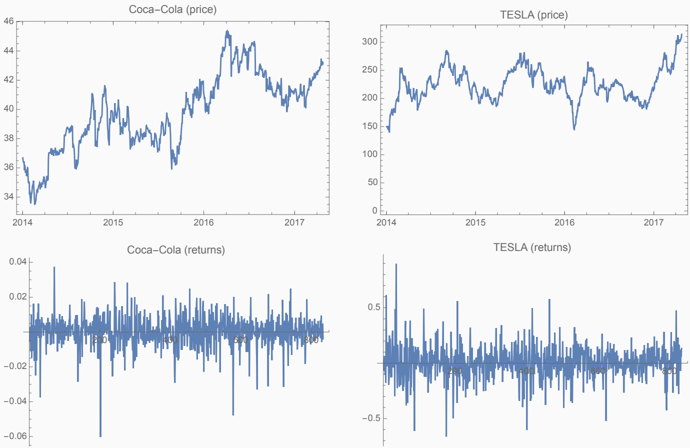
Augur's white paper use eigenvalue decomposition for its consensus algorithm.

source:youtube.com

Quadratic Forms

Example 6.1 (Portfolio Selection)

Consider an agent who invests his wealth in n risky assets.



BUYER'S GUIDE

2017 TESLA MODEL S P100D FIRST TEST: A NEW RECORD - 0-60 MPH IN 2.28 SECONDS!

The Model S P100D sets a new record (and accelerates like a real jerk)

We all understand acceleration. It's the rate of change of velocity. This 4,891-pound Tesla Model S P100D does it best, reaching 30, 40, 50, and 60 mph from a standstill more quickly than any other production vehicle we've ever tested, full stop. **In our testing, no production car has ever cracked 2.3 seconds from 0 to 60 mph. But Tesla has, in 2.275507139 seconds.**

The Tesla does not hold the advantage forever, though, because higher speeds give the advantage to horsepower over instant torque. The Ferrari LaFerrari hits 70 mph a tenth of a second quicker; the Porsche 918 and McLaren P1 pull ahead at 80 mph, and these hypercars all continue to pull away at higher speeds. But around town, everybody has long since lifted off the accelerator pedal.

Source:www.motortrend.com

- Denote $Z_{\mathrm{P}} = \boldsymbol{w}_{\mathrm{P}}^{\top}\boldsymbol{z}$ the return of portfolio P, where $\boldsymbol{w}$ represents the holdings of the N-dimensional vector of risky assets returns.
- Statistics of Z_{P}:

$$\begin{aligned}
\mathbb{E}\left[Z_{\mathrm{P}}\right] &= \mathbb{E}\left[\boldsymbol{w}_{\mathrm{P}}^{\top}\boldsymbol{z}\right] = \boldsymbol{w}_{\mathrm{P}}^{\top}\mathbb{E}\left[\boldsymbol{z}\right] = \boldsymbol{w}_{\mathrm{P}}^{\top}\boldsymbol{\mu} =: \mu_{\mathrm{P}} \\
\operatorname{var}\left[Z_{\mathrm{P}}\right] &= \mathbb{E}\left[\left(Z_{\mathrm{P}} - \mathbb{E}[Z_{\mathrm{P}}]\right)\left(Z_{\mathrm{P}} - \mathbb{E}[Z_{\mathrm{P}}]\right)^{\top}\right] \\
&= \mathbb{E}\left[\boldsymbol{w}_{\mathrm{P}}^{\top}\left(\boldsymbol{z} - \boldsymbol{\mu}\right)\left(\boldsymbol{z} - \boldsymbol{\mu}\right)^{\top}\boldsymbol{w}_{\mathrm{P}}\right] = \boldsymbol{w}_{\mathrm{P}}^{\top}\boldsymbol{\Sigma}\boldsymbol{w}_{\mathrm{P}} =: \sigma_{\mathrm{P}}^{2},
\end{aligned}$$

  with dist. $Z_{\mathrm{P}} \sim \mathcal{N}(\mu_{\mathrm{P}}, \sigma_{\mathrm{P}}^{2})$, and Sharpe ratio of portfolio: $\mathrm{SR}_{\mathrm{P}} = \frac{\mu_{\mathrm{P}}}{\sigma_{\mathrm{P}}}$.
- The investor's problem is to allocate

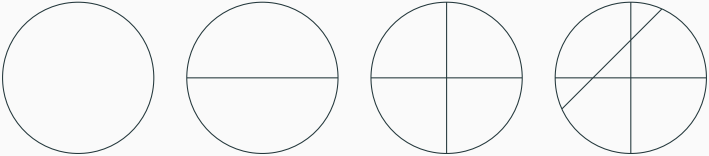
$$\hat{\boldsymbol{w}}_{\mathrm{P}} = \underset{\boldsymbol{w}_{\mathrm{P}}}{\arg\max}\left\{\mathcal{U}(\boldsymbol{w}_{\mathrm{P}}; \hat{\mu}, \hat{\boldsymbol{\Sigma}}) : \boldsymbol{w}_{\mathrm{P}} \in \mathcal{C}\right\},$$

  where $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are sample estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ resp; $\mathcal{U}(\cdot)$ is a convex function and $\mathcal{C}$ is a convex.

Taylor Polys & Taylor Series

Example 7.1 (Pizza Slices)

What is the maximum number of pizza slices that one can get by making n cuts?



cero cortes　　un corte　　dos cortes　　tres cortes

The recurrence is apparent:

$$x_n = x_{n-1} + n, \quad n > 0, x_1 = 2,$$

which is a linear first order DE with solution

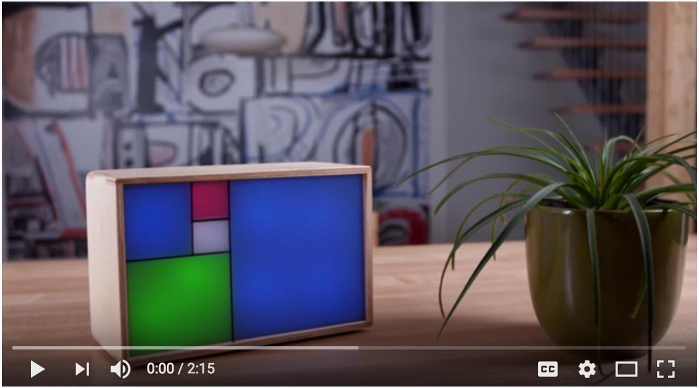
$$x_{c,n} = H, \quad \text{and} \quad x_{p,n} = \frac{n(n+1)}{2}.$$

Using the initial condition:

$$x_n = \frac{n(n+1)}{2} + 1.$$

Example 7.2

The recurrence $x_n = x_{n-1} + x_{n-2}$ with $x_1 = 1$ and $x_2 = 1$ provides the Fibonacci sequence. The numbers in the sequence are called Fibonacci numbers.

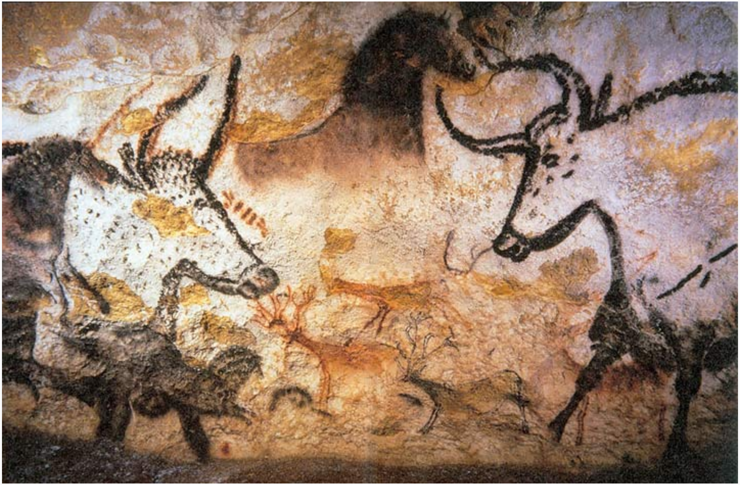


source:basbrum.com

Ordinary Differential Equations

Example 8.1 (Lascaux caves)

In the year 1940 a group of boys was hiking in the vicinity of a town in France called Lascaux. They suddenly became aware that their dog had disappeared. In the ensuing search he was found in a deep hole from he was unable to climb out. When one of the boys lowered himself into the hole to help extricate the dog, he made a startling discovery. The hole was in fact an ancient cave. On the walls of the cave there were marvellous paintings of stags, wild horses and cattle. In addition to the wall paintings, there were also found the charcoal remains of a fire. The problem of interest is to determine: how long ago the cave inhabitants lived there?



Source: https://en.wikipedia.org/wiki/Lascaux.

Example 8.2 (Solow's Growth Model)

Consider a macroeconomic model with equations

$$\dot{K} = sQ(K, L) \quad \text{and} \quad \frac{\dot{L}}{L} \;=\; \lambda, \quad \text{for} \quad s, \lambda > 0$$

i.e. capital increases prop to production, and labor increases at an exponential rate.

Given $k = K/L$, write an ODE for $\dot{k}$, considering a production func homogeneous of degree 1, and draw its phase diagram.

1. Write an ODE for k given $K = kL$:

$$\dot{K} = \dot{k}L + \dot{L}k \quad \Rightarrow \quad \dot{k} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k \quad \Rightarrow \quad \dot{k} = s\phi(k) - \lambda k$$

2. Write production in terms of $k = K/L$:

$$\frac{1}{L}Q = \frac{1}{L}f(K, L) = f\left(\frac{K}{L}, 1\right) = \phi(k), \quad k = K/L.$$

3. compute the sign of $\phi_k(k)$ and $\phi_{kk}(k)$:

$$\begin{aligned} \frac{dQ}{dK} &= \frac{d\phi(k)}{dK}L = \phi_k(k)\frac{dk}{dK}L = \phi_k(k) \quad \Rightarrow \phi_k(k) > 0 \\ \frac{d^2Q}{dK^2} &= \frac{d\phi_k(k)}{dK} = \phi_{kk}(k)\frac{dk}{dK} = \phi_{kk}(k)\frac{1}{L} \quad \Rightarrow \phi_{kk}(k) < 0 \end{aligned}$$

Difference Equations

Example 9.1 (Cobweb Model)

Consider the supply and demand model

$$D_t = \alpha - \beta P_t \quad y \quad S_t = -\gamma + \delta P_t^e,$$

where $\alpha, \beta, \gamma, \delta > 0$. *Find* P_t *implied by the equilibrium condition* $D_t = S_t$. *Describe the behaviour of this trayectory and indicate the conditions for it to be stable if*

1. *expectations are static, i.e.* $P_t^e = P_{t-1}$,
2. *expectations adapt, i.e.* $P_t^e = (1-\eta)P_{t-1}^e + \eta P_{t-1}$, *where* $\eta \in]0, 1]$.