6 | Difference equations

6.1 Comparison to ODE and Generalities

Even though we could study the theory of Difference Equations (DE) as an independent topic, it is very instructive to establish a parallel with the theory of Ordinary Differential Equations (ODE). It is worth mentioning that economic dynamic phenomena can be stated either as an ODE or DE, the choice depends mostly on convenience. For instance, theoretical models often employs ODE, whereas dynamic models studied in Econometrics rather use DE.

The key difference is that the independent variable "time" is *continuous* in ODE Theory, so it can take any value on the real line (usually constrained to be positive if $t = 0$ is taken as an initial point), whereas time is *discrete* in DE analysis and hence often takes positive integer values only, e.g. $t = 0, 1, 2, \ldots$. To elaborate further on this point let us introduce the *relative rate of change* of a variable x at two different points in time, t and $t - h$, defined as

$$\Delta_h x_t = \frac{x_t - x_{t-h}}{t - (t-h)} = \frac{x_t - x_{t-h}}{h} ,$$

where x_s denotes the value of x at period s. In ODE theory the rate of change corresponds to infinitesimal (continuous) changes in t, so that $h \to 0$ and $\dot{x} = \lim_{h\to 0} \Delta_h x_t = \mathrm{d}x/\mathrm{d}t$. On the other hand, for DE theory we interpret each integer value of t as a period and focus the attention to changes occurring from period to period, implying $h = 1$. The resulting operator Δ_h is the **difference operator** $\Delta x_t = x_t - x_{t-1}$, the main unit of change for DE analysis.

Functions of the form $e^{\lambda t}$ feature in the solution to ODE, where the constant λ is the *root* of the ODE. Just as the derivative can be obtained as a limit, note that

$$\lim_{h\to 0} (1 + \lambda h)^{\frac{t}{h}} = e^{\lambda t} ,$$

so that after setting $h = 1$, we shall see that functions of the form $(1 + \lambda)^t$ or more compactly $(r)^t$ feature in the closed-form solutions of DE.

We may define a DE as the relationship between a variable that depends on time and its discrete rates of change. Solving a DE implies finding the function of time that satisfies it. As a matter of convention, we call Δx_t to the first difference of x and $\Delta^2 x_t$ to the second difference, or to the result of applying the difference operator twice $\Delta^2 x_t \equiv \Delta(\Delta x_t)$. Also, since the independent variable t is interpreted as time, the solution is often called a *path*. We use x_t to denote a variable that is an unknown function of time, and we use $x(t)$ to refer to the solution to the DE, i.e. the path.

An equivalent and more common definition states that a DE is the relationship between a variable x_t and its **lags** $(x_{t-1}, x_{t-2}, \ldots)$ and/or its **leads** $(x_{t+1}, x_{t+2}, \ldots)$. This equivalence is due to the fact that the difference operator of arbitrary order n, $\Delta^n = \Delta \cdot \Delta \cdots \Delta$ applied n times, is linear and hence $\Delta^n x_t$ is the linear combination between x_t and its n lags, implying $F(\Delta^n x_s) = F(x_s, x_{s-1}, \ldots, x_{s-n})$ for any value of s. To illustrate this point consider the following three DE:

$$\text{(a)}\ \ \Delta x_t - q x_{t-1} = b, \qquad \text{(b)}\ \ \Delta x_{t+1} = f(x_t), \qquad \text{(c)}\ \ \Delta^2 x_t - a_1 \Delta x_t - a_0 x_t = b_t \ .$$

Equation (a) resembles inevitably the linear ODE $\dot{x} - qx = b$, equation (b) can be thought of as the ODE $\dot{x} = f(x)$, whereas equation (c) resembles the linear ODE $\ddot{x} - a_1\dot{x} - a_0 x = b(t)$. Note that if we apply the definition $\Delta x_t = x_t - x_{t-1}$ to (a) and $\Delta x_{t+1} = x_{t+1} - x_t$ to (b), we obtain

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$$(\text{a}')\ x_t - (1+q)x_{t-1} = b \qquad \text{and} \qquad (\text{b}')\ x_{t+1} = x_t + f(x_t) = f^*(x_t)\ .$$

Thus, in (a) we had a relationship between Δx_t and x_{t-1} that could be rewritten as an equation featuring x_t and x_{t-1}, whereas in case (b) we began with a relationship between Δx_{t+1} and x_t that was eventually expressed as a relationship between x_t and x_{t+1}.

For (c) use also $\Delta^2 x_t = \Delta x_t - \Delta x_{t-1} = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = x_t - 2x_{t-1} + x_{t-2}$ to get

$$(\text{c}')\ (1 - a_1 - a_0)x_t - (2 - a_1)x_{t-1} + x_{t-2} = b_t\ ,$$

so the equation relating x_t to Δx_t and $\Delta^2 x_t$ is equivalent to an equation relating x_t to x_{t-1} and x_{t-2}.

DE are classified according to their type and structure. The **order** of an equation is determined by the highest difference appearing in the equation. If this difference is Δ^n, the DE is of order n. Following our previous discussion the order can be determined in an alternative way. If x_t and lags of it appear in the equation, then the order is given by the highest lag. If x_t and leads of it appear in the equation, then the order is given by the highest lead. Examples (a) and (b) above are first-order DE, whereas (c) is a second order equation.

If both lags and leads feature in the DE, then the order is given by the distance in terms of time between the furthest lead and the furthest lag. For example, $x_{t+1} - x_{t-1} = b$ is a second order equation, an so is $x_{t+2} - x_t = b$. Note that what is important is *not* the value of t, but the time span among leads, lags and x_t.

An equation is described as **autonomous** if time does not appear explicitly in the equation, e.g. $\Delta x_t = f(x_t)$, and **non-autonomous** otherwise, e.g. $x_t = f(x_{t-1}, t)$. Examples (a) and (b) are autonomous, while (c) is non-autonomous due to the presence of b_t. An equation is **linear** if the terms in x_t, Δx_t and $\Delta^2 x_t$ appear in linear form, and is **nonlinear** otherwise. Alternatively, an equation is linear if all x_t, lags and leads appear in linear form. Examples (a) and (c) are linear, while (b) is nonlinear if $f(.)$ displays a non-linearity. A linear equation is termed **homogeneous** if only x_t and its lags or leads appear in the equation, and **non-homogeneous** otherwise. Equations (a) and (c) are non-homogeneous unless $b = 0$.

Finally, recall that the **stability** of $x(t)$ refers to its limiting behaviour. If $x(t) \to x_{ss}$ as $t \to \infty$, where x_{ss} is a real value, then the solution $x(t)$ is said to be **stable** or convergent. On the contrary, if $x(t) \to \pm\infty$ as $t \to \infty$, then the solution $x(t)$ is **unstable** and no “long-run” equilibrium exists.

6.2 First-order linear DE

Consider the general first-order non-homogeneous equation

$$x_t - a x_{t-1} = w_t\ , \tag{1}$$

where w_t is an arbitrary function of t and for simplicity we focus on the case where a is a constant. This DE is equivalent to its lags and leads, in the sense that the solution to (1) is the same as the solution to $x_{t+s} - a x_{t+s-1} = w_{t+s}$ for any integer value, positive or negative, of s. We emphasize again that in general the actual value of t is not relevant. What is important for the mathematical structure of the problem is the span between the furthest lead and lag.

6.2.1 Successive substitution

Suppose that at some point τ the value of x_t is known to be x_τ. Furthermore, suppose that τ occurs in the “past” so $t > \tau$ and x_τ can be regarded as an **initial condition**. We can exploit the recursive nature of (1) to obtain x_t by successive substitution from τ onwards (which can be thought of as the discrete time equivalent of integration):

$$\begin{aligned}
x_{\tau+1} &= a x_\tau + w_{\tau+1} \\
x_{\tau+2} = a x_{\tau+1} + w_{\tau+2} &= a^2 x_\tau + w_{\tau+2} + a w_{\tau+1} \\
x_{\tau+3} = a x_{\tau+2} + w_{\tau+3} &= a^3 x_\tau + w_{\tau+3} + a w_{\tau+2} + a^2 w_{\tau+1} \\
x_{\tau+4} = a x_{\tau+3} + w_{\tau+4} &= a^4 x_\tau + w_{\tau+4} + a w_{\tau+3} + a^2 w_{\tau+2} + a^3 w_{\tau+1} \\
x_{\tau+5} = a x_{\tau+4} + w_{\tau+5} &= a^5 x_\tau + w_{\tau+5} + a w_{\tau+4} + a^2 w_{\tau+3} + a^3 w_{\tau+2} + a^4 w_{\tau+1}\ ,
\end{aligned}$$

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and so on. A pattern clearly arises such that the solution to the DE is:

$$x(t) = a^{t-\tau} x_\tau + \sum_{j=\tau+1}^{t} a^{t-j} w_j \,. \tag{2}$$

Since $t > \tau$ above so x_τ is a value observed in the past, (2) is a **backward solution**.

On the other hand, suppose that τ occurs in the "future" so $t < \tau$ and x_τ can be thought of as a **terminal condition**. Let $\alpha = 1/a$ so (1) can be written as $x_{t-1} = \alpha x_t - \alpha w_t$. Successive substitution from τ backwards gives:

$$\begin{array}{lcl}
x_{\tau-1} & = & \alpha x_\tau - \alpha w_\tau \\
x_{\tau-2} = \alpha x_{\tau-1} - \alpha w_{\tau-1} & = & \alpha^2 x_\tau - \alpha w_{\tau-1} - \alpha^2 w_\tau \\
x_{\tau-3} = \alpha x_{\tau-2} - \alpha w_{\tau-2} & = & \alpha^3 x_\tau - \alpha w_{\tau-2} - \alpha^2 w_{\tau-1} - \alpha^3 w_\tau \\
x_{\tau-4} = \alpha x_{\tau-3} - \alpha w_{\tau-3} & = & \alpha^4 x_\tau - \alpha w_{\tau-3} - \alpha^2 w_{\tau-2} - \alpha^3 w_{\tau-1} - \alpha^4 w_{\tau-1} \\
x_{\tau-5} = \alpha x_{\tau-4} - \alpha w_{\tau-4} & = & \alpha^5 x_\tau - \alpha w_{\tau-4} - \alpha^2 w_{\tau-3} - \alpha^3 w_{\tau-2} - \alpha^4 w_{\tau-1} - \alpha^5 w_\tau \,,
\end{array}$$

and so on. A pattern clearly arises such that the solution to the DE is:

$$x(t) = \alpha^{\tau-t} x_\tau - \sum_{j=t}^{\tau-1} \alpha^{j-t+1} w_{j+1} = a^{t-\tau} x_\tau - \sum_{j=t}^{\tau-1} a^{t-j-1} w_{j+1} \,. \tag{3}$$

Since $t < \tau$ above so x_τ is a value observed in the future, (3) is a **forward solution**.

6.2.2 A particular case

Consider a simpler version of (1) with $w_t = w$ (a constant),

$$x_t - a x_{t-1} = w \,. \tag{4}$$

Applying the backward solution in (2) gives (recall the properties of geometric series)

$$x(t) = a^{t-\tau} x_\tau + \sum_{j=\tau+1}^{t} a^{t-j} w = a^{t-\tau} x_\tau + w a^t \left(\frac{a^{-\tau-1} - a^{-t-1}}{1 - a^{-1}} \right) = \left(x_\tau - \frac{w}{1-a} \right) a^{-\tau} a^t + \frac{w}{1-a} \,,$$

whereas applying the forward solution in (3) renders

$$x(t) = a^{t-\tau} x_\tau - \sum_{j=t}^{\tau-1} a^{t-j-1} w = a^{t-\tau} x_\tau - w a^t \left(\frac{a^{-t-1} - a^{-\tau-1}}{1 - a^{-1}} \right) = \left(x_\tau - \frac{w}{1-a} \right) a^{-\tau} a^t + \frac{w}{1-a} \,.$$

The backward solution is exactly the same as the forward solution and both take the form

$$x(t) = C\, a^t + \frac{w}{1-a} \,, \tag{5}$$

where C is an arbitrary constant that resembles the constant of integration in ODE theory.

The dynamic bevahiour of $x(t)$ is due to the term a^t. Hence, $x(t)$ will be stable if this function converges as $t \to \infty$. Clearly, $\lim_{t\to\infty} a^t = \pm\infty$ if $|\,a\,| > 1$ and $\lim_{t\to\infty} a^t = 0$ if $|\,a\,| < 1$. Thus, a sufficient condition for the stability of the path implied by (4) is $|\,a\,| < 1$. This condition holds *regardless of the value of* τ and thus we confirm that the distinction between the backward and forward solutions is unimportant in this case.

The solution in (5) can be thought of as the sum of two functions. The first term $x_c(t) = C\, a^t$ is the **complementary or homogenous solution**, and it is easy to note that this is the solution to the homogenous equation $x_t - a x_{t-1} = 0$, i.e. the DE resulting after setting $w = 0$. The second term, $x_p(t) = x_{ss} = w/(1-a)$ is the **particular or inhomogeneous solution** and we can verify that it corresponds to the value of x such that $\Delta x_t = 0$ or $x_t = x_{t-1} = x_{ss}$, a "steady state" condition.

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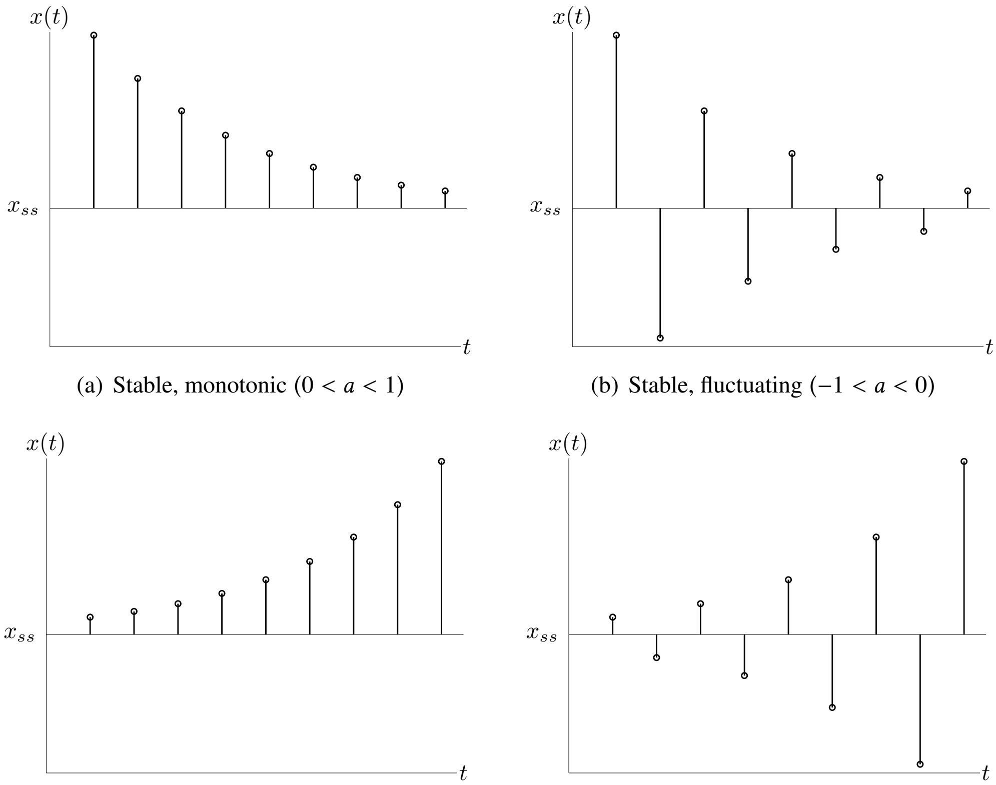
Figure 1. *Solution path to the DE* $x_t - ax_{t-1} = w$



(a) Stable, monotonic ($0 < a < 1$)

(b) Stable, fluctuating ($-1 < a < 0$)

(c) Unstable, monotonic ($a > 1$)

(d) Unstable, fluctuating ($a < -1$)

The coefficient a is sometimes called the **root** of the DE. The reason is that one can conjecture that the complementary solution has the form $x_c(t) = C\,r^t$ for arbitrary values of C and r. Plugging this guess into the homogenous equation gives

$$x_{ct} - a\,x_{ct-1} = 0 \qquad \rightarrow \qquad Crr^{t-1} - aCr^{t-1} = 0 \qquad \rightarrow \qquad Cr^{t-1}(r-a) = 0\,.$$

For a nontrivial case where $C \neq 0$ and $r \neq 0$, this equation is satisfied if $r = a$. The term $P(r) = r - a$ is the **characteristic equation** associated to the ODE, and the value of its zero is the **root**. Since the dynamics of $x(t)$ are completely due to the complementary solution, it is often said that $x(t)$ *is stable if the modulus (or absolute value) of the root of the DE is less than one* (i.e., $|\,r\,| = |\,a\,| < 1$).

Typically, the value of τ used to determine C is $\tau = 0$, i.e. an initial condition. Then,

$$x(t) = \left(x_0 - \frac{w}{1-a}\right) a^t + \frac{w}{1-a} = (x_0 - x_{ss})\,a^t + x_{ss}\,. \tag{6}$$

The term in braces is known as the **initial disequilibrium** and the path $x(t)$ shows how x evolves from the initial point x_0 to its long-run value x_{ss} when $|\,a\,| < 1$ or departs from it when $|\,a\,| > 1$. Examples are shown in Figure 1 for different values of a and for a positive initial disequilibrium, $x_0 > x_{ss}$.

It is evident that the *magnitude* of a determines the stability of $x(t)$ ($|\,a\,| < 1$ is associated with stable paths), whereas the *sign* of a determines whether the path is monotonic ($a > 0$) or fluctuating ($a < 0$). This source of fluctuations is not a higher order phenomenon (as the sinusoidal paths in section 6.3) but the result of even powers of $a < 0$ being positive and odd powers being negative.

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6.2.3 Another case

Consider a simpler version of (1) with $w_t = w^t$ (a constant to the power of t),

$$x_t - a x_{t-1} = w^t \, . \tag{7}$$

Applying the backward solution in (2) yields

$$\begin{aligned} x(t) &= a^{t-\tau} x_\tau + \sum_{j=\tau+1}^{t} a^{t-j} w^j \\ &= a^{t-\tau} x_\tau + a^t \left(\frac{w^{\tau+1} a^{-\tau-1} - w^{t+1} a^{-t-1}}{1 - w a^{-1}} \right) \\ &= a^{t-\tau} x_\tau - a^t \left(\frac{w^\tau a^{-\tau} - w^t a^{-t}}{1 - w^{-1} a} \right) = \left(x_\tau - \frac{w^{\tau+1}}{w - a} \right) a^{-\tau} a^t + \left(\frac{w}{w-a} \right) w^t \, , \end{aligned}$$

whereas the forward solution as (3) is

$$\begin{aligned} x(t) &= a^{t-\tau} x_\tau - \frac{w}{a} \sum_{j=t}^{\tau-1} a^{t-j} w^j \\ &= a^{t-\tau} x_\tau - a^t \left(\frac{w}{a} \right) \left(\frac{w^t a^{-t} - w^\tau a^{-\tau}}{1 - w a^{-1}} \right) = \left(x_\tau - \frac{w^{\tau+1}}{w - a} \right) a^{-\tau} a^t + \left(\frac{w}{w-a} \right) w^t \, . \end{aligned}$$

Again, the backward and forward solutions coincide and are the sum of two functions. The complementary solution $x_c(t) = C\,a^t$ and the particular solution who has the form $x_p(t) = H w^t$ where H is a constant to be determined. This form of the particular solution comes from an educated guess and responds to the nonhomogeneous term of the DE. Since in this case, $w_t = w^t$ the guess is that $x_p(t)$ belongs to the family of power functions $H w^t$. In section 6.2.2 the non-homogenous term was a constant and thus the guess was that the particular solution was a constant as well. This is an example of the workings of the *method of undetermined coefficients*. To obtain a unique or "particular" solution we need to determine the constant H by replacing our guess in the DE:

$$x_{pt} - a x_{pt-1} = w^t \quad \rightarrow \quad H w^t - H \left(\frac{a}{w} \right) w^t = w^t \quad \rightarrow \quad H = \frac{w}{w-a} \, .$$

As before, the constant C can be set to satisfy an initial or terminal condition. The stability of $x(t)$ is now determined by the values of a and $w \neq a$ and it is easy to see that $|\,a\,| < 1$ and $|\,w\,| < 1$ are sufficient conditions for convergence.

The reason why in this section and in section 6.2.2 the distinction between the backward and forward solution was irrelevant is because w_t was in both cases a function of time that could be forecasted or "backcasted" with arbitrary accuracy (a constant in section 6.2.2 and a power function in this section). The reader can check that the same holds true for polynomials, e.g. $w_t = p_0 + p_1 t$, or trigonometric functions, e.g $w_t = \cos(\eta\, t)$.

6.2.4 When the difference matters (Optional)

In many economics applications w_t is either a deterministic function of time that does not have a well-established pattern, or it is an stochastic term that cannot be perfectly predicted. Hence, the distinction between the backward and forward solutions can be of importance. Let us consider the stability of these solutions for limiting values of τ. The only restriction we impose to w_t is that it is a **bounded sequence** such that for a constant $|\,c\,| < 1$, $\lim_{s\to\infty} c^s w_{t\,\pm\,s} = 0$.

Consider first the far past $\tau \to -\infty$. If $|\,a\,| < 1$, the first term in (2) vanishes, whereas the second term has a well-defined limit given the assumption that w_t is bounded. Thus, if $|\,a\,| < 1$ the *backward solution is stable* an equals

$$x(t) = \sum_{j=-\infty}^{t} a^{t-j} w_j = \sum_{j=0}^{\infty} a^j w_{t-j} \qquad \text{for} \qquad |\,a\,| < 1 \, , \tag{8}$$

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i.e., $x(t)$ is a weighted average with exponentially decreasing weighs of the lags of w_t .

On the other side, consider the far future $\tau \to \infty$. If $|\,a\,| > 1$, the first term in (3) vanishes, whereas the second term has a well-defined limit given the assumption that w_t is bounded. Hence if $|\,a\,| > 1$ the *forward solution is stable* an equals

$$x(t) = -\sum_{j=t}^{\infty}\left(\frac{1}{a}\right)^{j-t+1} w_{j+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{a}\right)^{j} w_{t+j} \qquad \text{for} \qquad |\,a\,| > 1\,. \tag{9}$$

i.e., $x(t)$ is a weighted average with exponentially decreasing weighs of the leads of w_t .

In a nutshell, for limiting values of τ, the stability of the *particular* solution to the DE depends on the value of a: if $|\,a\,| < 1$ the backward solution is stable, whereas if $|\,a\,| > 1$ the forward solution is stable. The complementary solution in both cases has vanished, and so we can interpret the limits of τ as conditions leading to $C = 0$.

6.3 Linear systems

To motivate the discussion we first focus on the first-order linear system with two variables and constant coefficients. Then we show how this construct can be used to generalize the analysis to higher order linear DE and systems. As usual, the emphasis is on path stability.

6.3.1 First-order linear system with two variables

Consider the first-order linear system with two variables with constant coefficients

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

or, in matrix form,

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{W}\,, \tag{10}$$

where vectors $\boldsymbol{X}_t$ and $\boldsymbol{W}$ and matrix $\boldsymbol{A}$ are defined implicitly. Consider also a system simpler than (10), with a diagonal system matrix Λ,

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \qquad \text{or, in matrix form,} \qquad \boldsymbol{Z}_t = \Lambda\boldsymbol{Z}_{t-1} + \boldsymbol{K}\,. \tag{11}$$

where vectors $\boldsymbol{Z}_t$ and $\boldsymbol{K}$ and matrix Λ are defined implicitly. System (11) is just the collection of two independent equations, with no feedback between the variables involved. As such, the solution to the system is just the collection of the solutions to each equation. Using the results in (5) and expressing them in matrix form,

$$\boldsymbol{Z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} (r_1)^t & 0 \\ 0 & (r_2)^t \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} z_{1p} \\ z_{2p} \end{bmatrix}, \tag{12}$$

where C_1 and C_2 are arbitrary constants and z_{1p} and z_{2p} are the particular solutions.

We can transform a system of the form (10) into the simpler form (11) by diagonalizing matrix $\boldsymbol{A}$. A square matrix like $\boldsymbol{A}$ can be expressed as $\boldsymbol{A} = \boldsymbol{H}\Lambda\boldsymbol{H}^{-1}$, where Λ is a diagonal matrix whose entries are the **eigenvalues** of $\boldsymbol{A}$ and $\boldsymbol{H}$ is a non-singular matrix whose columns are the corresponding **eigenvectors**. Thus, if we define $\boldsymbol{Z}_t = \boldsymbol{H}^{-1}\boldsymbol{X}_t$ and $\boldsymbol{K} = \boldsymbol{H}^{-1}\boldsymbol{W}$, we move from (10) to (11).

Then, we can easily recover the paths of interest by using the equality $\boldsymbol{X}(t) = \boldsymbol{H}\boldsymbol{Z}(t)$. Therefore,[1]

$$\begin{aligned} x_1(t) &= C_1H_{11}(r_1)^t + C_2H_{12}(r_2)^t + x_{1p} \\ x_2(t) &= C_1H_{21}(r_1)^t + C_2H_{22}(r_2)^t + x_{2p} \end{aligned} \tag{13}$$

[1] Note that the particular solution to (10) is $\boldsymbol{X}_p = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{W}$.

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where H_{ij} are the entries of $\boldsymbol{H}$, so $(H_{11}, H_{21})'$ and $(H_{21}, H_{22})'$ are the eigenvectors associated to the eigenvalues r_1 and r_2, respectively.

The paths $x_1(t)$ and $x_2(t)$ are **linear combinations** of $z_1(t)$ and $z_2(t)$. Hence, the complementary solutions of $x_1(t)$ and $x_2(t)$ are linear combinations of the functions $(r_1)^t$ and $(r_2)^t$. The coefficients of these combinations depend on the eigenvectors of $\boldsymbol{A}$ and the arbitrary constants related to the initial conditions. Thus, all the dynamic information of the system is contained in matrix $\boldsymbol{A}$, through its eigenvalues and eigenvectors.

Both eigenvalues, r_1 and r_2 are called the **roots** of the system, and we first assume that they are **real**. If both are less than one in absolute value, $|\, r_1 \,| < 1$ and $|\, r_2 \,| < 1$, then $(r_1)^t \to 0$ and $(r_2)^t \to 0$ as $t \to \infty$ and both paths in (13) are stable, regardless of the values of C_1 and C_2. The opposite case occurs when both roots are greater than one in absolute value, $|\, r_1 \,| > 1$ and $|\, r_2 \,| > 1$, so that $(r_1)^t \to \pm\infty$ and $(r_2)^t \to \pm\infty$ as $t \to \infty$. Both paths are now unstable regardless of the initial condition. An intermediate case occurs when $|\, r_1 \,| < 1$ and $|\, r_2 \,| > 1$, so that $(r_1)^t \to 0$ but $(r_2)^t \to \pm\infty$ as $t \to \infty$. From (13) we observe that the paths $\boldsymbol{X}(t)$ are explosive if $C_2 \neq 0$ and that a saddlepath is obtained by setting $C_2 = 0$, rendering a convergent solution $\boldsymbol{X}(t)$. Clearly, the stability of this equilibrium depends on the initial condition.

It is tempting to classify the various cases as nodes or saddle points as we did in ODE analysis. This is possible when the paths are monotonic, i.e. ever increasing or ever decreasing. In other words, when the roots are *positive* so we avoid the fluctuations due to changes in sign as saw in Figure 1 (remember: stability is related to the *magnitude* of the roots in DE analysis, not to their sign). Thus, we may refer to a **stable node** if $0 < r_1, r_2 < 1$, to an **unstable node** if $r_1 > 1$ and $r_2 > 1$, and to a **saddlepath equilibrium** if $0 < r_1 < 1$ and $r_2 > 1$. The cases when some roots are negative often display dynamics that are heavily dependent on the numerical configuration of the problem, and hence it is of no use to classify or name such equilibria.

Consider now the case when the roots are complex conjugate $r_{1,2} = p \pm qi$, where i is the imaginary unit $i = \sqrt{-1}$. We can deal with the presence of i by using the **polar representation of a complex number** $p \pm qi = \rho[\cos(\theta) \pm i \sin(\theta)\,]$, where ρ is the *modulus* of the complex number $\rho = \sqrt{p^2 + q^2}$ and θ is the *argument* of the complex number $\tan(\theta) = q/p$. Then, using **De Moivre's formula**

$$(p \pm qi)^t = \rho^t [\,\cos(\theta t) \pm i \sin(\theta t)\,]\,,$$

and the fact that the eigenvectors will have elements of the form $H_{11} = \alpha_1 + \alpha_2 i$, $H_{12} = \alpha_1 - \alpha_2 i$, $H_{21} = \beta_1 + \beta_2 i$ and $H_{22} = \beta_1 - \beta_2 i$, the following paths are obtained

$$\begin{array}{rcl} x_1(t) & = & \rho^t [\,\alpha_1 (C_1 + C_2) \cos(\theta t) + \alpha_2 (C_2 - C_1) \sin(\theta t)\,] + x_{1p} \\ x_2(t) & = & \rho^t [\,\beta_1 (C_1 + C_2) \cos(\theta t) + \beta_2 (C_2 - C_1) \sin(\theta t)\,] + x_{2p} \end{array} . \tag{14}$$

In both cases, the term in braces is the sum of two periodic functions and hence the paths display a sinusoidal motion. The stability of (14) depends on ρ^t and thus on the magnitude of ρ, the *modulus* of the roots. By definition, the modulus is always positive and thus we do not have the "sign" issue of Figure 1. If $\rho < 1$, the paths display damped oscillations (a **stable spiral**), if $\rho > 1$ the oscillations are explosive (an **unstable spiral**) and finally if $\rho = 1$, the paths exhibit the periodic behavior of the sine and cosine functions, and hence neither converge nor diverge (a **center**).

6.3.2 First-order linear system with many variables

Consider the first-order linear system with n variables and constant coefficients

$$\begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \\ \vdots \\ x_{nt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \\ x_{3t-1} \\ \vdots \\ x_{nt-1} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} \tag{15}$$

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or, in matrix form,

$$X_t = AX_{t-1} + W\,, \tag{16}$$

where vectors X_t and W and matrix A are defined implicitly. Note that regardless of the dimension of this system, the matrix representation of (16) is identical to (10). Therefore, all the analysis involving the eigenvalues and eigenvectors of A remain unchanged. The paths of interest are given by $X(t) = HZ(t)$, where H is the matrix containing the eigenvectors of A and

$$Z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} (r_1)^t & 0 & 0 & \cdots & 0 \\ 0 & (r_2)^t & 0 & \cdots & 0 \\ 0 & 0 & (r_3)^t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (r_n)^t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} + \begin{bmatrix} z_{1p} \\ z_{2p} \\ z_{3p} \\ \vdots \\ z_{np} \end{bmatrix}, \tag{17}$$

where (for $j = 1, 2, \ldots, n$) r_j are the eigenvalues of A and C_j are arbitrary constants. If follows that the complementary solutions to the variables of the system are linear combinations of the terms $(r_1)^t, (r_2)^t, \ldots, (r_n)^t$. If some of these roots are complex, then the linear combinations involve terms of the form $(r)^t$ and, by pairs, $\rho^t[\,\cos(\theta t) + \sin(\theta t)\,]$ as in equations (13) and (14), respectively.

How can we assess the stability of the system (16)? The eigenvalues of A are computed by solving the **characteristic equation** $P(r) = |\,A - rI\,| = 0$. If the modulus of all eigenvalues of A are be less than one, then A is **sequentially stable** and implies that it permits a geometric series representation of the form $\sum_{j=0}^{\infty} A^j = (I - A)^{-1}$. This is a matrix generalization of $|\,a\,| < 1$.

Recall that $n = 2$ we already know that the characteristic equation is $r^2 - \text{tr}(A)r + \det(A) = 0$ and thus,

$$r_{1,2} = \frac{1}{2}\left[\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}\right]. \tag{18}$$

6.3.3 Higher order linear DE and higher order systems

To fix ideas, consider the linear DE of second order

$$x_t - a_1 x_{t-1} - a_2 x_{t-2} = w\,. \tag{19}$$

By introducing an auxiliary variable we can express this DE as a first-order system. Let $y_t = x_{t-1}$ so the original DE can be written as $x_t - a_1 x_{t-1} - a_2 y_{t-1} = w$. Stacking these two equation renders

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix} \qquad \text{or, in matrix form,} \qquad X_t = AX_{t-1} + W\,.$$

Matrix A is called in this context the **companion matrix** of the DE. Since this is a two-variable system we conclude that the complementary solution to the DE (19) is the linear combination of two terms, either both based on real roots or based on a pair of complex conjugate roots. Note also that the characteristic equation to this system (and hence to the second-order DE) is

$$r^2 - a_1 r - a_2 = 0\,, \tag{20}$$

and resembles the homogenous equation associated with (19), i.e. when $w = 0$.

To generalize, consider a linear DE of order n

$$x_t - a_1 x_{t-1} - a_2 x_{t-2} - a_3 x_{t-3} - \ldots - a_{n-1} x_{t-n+1} - a_n x_{t-n} = w\,. \tag{21}$$

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Under the same reasoning as above, the **companion form** of this DE is

$$\begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-n+2} \\ x_{t-n+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ \vdots \\ x_{t-n+1} \\ x_{t-n} \end{bmatrix} + \begin{bmatrix} w \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

so, again, we managed to write a higher order DE as a first-order system $\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{W}$. One can check that the characteristic polynomial of the companion matrix $\boldsymbol{A}$ is

$$r^n - a_1 r^{n-1} - a_2 r^{n-2} - a_3 r^{n-3} - \ldots - a_{n-1} r - a_n = 0 \,. \tag{22}$$

thus, resembling the homogenous equation associated with (21). The sufficient conditions for stability are (1) $|\, a_1 \,| < n$, (2) $|\, a_n \,| < 1$, and (3) $\sum_{j=1}^{n} |\, a_j \,| < 1$.

Finally, the very same principle can be applied to higher-order systems. Consider a system of n variables and of order p

$$\boldsymbol{X}_t = \boldsymbol{A}_1 \boldsymbol{X}_{t-1} + \boldsymbol{A}_2 \boldsymbol{X}_{t-2} + \boldsymbol{A}_3 \boldsymbol{X}_{t-3} + \ldots + \boldsymbol{A}_{p-1} \boldsymbol{X}_{t-p+1} + \boldsymbol{A}_p \boldsymbol{X}_{t-p} + \boldsymbol{W} \,,$$

which in Econometrics is called a vector autoregression or VAR. By stacking the various vectors and matrices the companion form of this system is

$$\begin{bmatrix} \boldsymbol{X}_t \\ \boldsymbol{X}_{t-1} \\ \boldsymbol{X}_{t-2} \\ \vdots \\ \boldsymbol{X}_{t-p+2} \\ \boldsymbol{X}_{t-p+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots & \boldsymbol{A}_{p-1} & \boldsymbol{A}_p \\ \boldsymbol{I}_n & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{I}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \boldsymbol{0}_n & \boldsymbol{0}_n & \boldsymbol{0}_n & \ddots & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{I}_n & \boldsymbol{0}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{t-1} \\ \boldsymbol{X}_{t-2} \\ \boldsymbol{X}_{t-3} \\ \vdots \\ \boldsymbol{X}_{t-p+1} \\ \boldsymbol{X}_{t-p} \end{bmatrix} + \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{n\times 1} \\ \vdots \\ \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{n\times 1} \end{bmatrix}$$

or more compactly $\tilde{\boldsymbol{X}}_t = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{X}}_{t-1} + \tilde{\boldsymbol{W}}$. Each matrix $\boldsymbol{A}_j$ is of dimension $n \times n$, $\boldsymbol{I}_n$ is an identity matrix of order n and $\boldsymbol{0}_n$ denotes an $n \times n$ matrix full of zeroes. Hence, the companion matrix $\tilde{\boldsymbol{A}}$ is of dimension $np \times np$ and vectors $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{W}}$ are $np \times 1$.

6.4 Qualitative analysis of nonlinear first-order DE

Consider a general nonlinear first-order difference equation of the form $\Delta x_t = g(x_{t-1})$ where the function $g(\cdot)$ does not allow us to rewrite this equation as a linear DE. As shown in section 6.1 this equation can be rewritten as

$$x_t = f(x_{t-1}) \,. \tag{23}$$

In this more general case, we might not be able to find an explicit solution. However, we might still be able to characterize the solution qualitatively and graphically. The strategy consists first to find "steady state" values, i.e. values of x that satisfy $x_t = x_{t-1} = x_{ss}$. In the nonlinear ODE, $x_{ss} = f(x_{ss})$. Using a linear approximation of the function $f(\cdot)$ around x_{ss} delivers

$$x_t \simeq f'(x_{ss}) x_t + (1 - f'(x_{ss})) x_{ss} \,, \tag{24}$$

when the equality $f(x_{ss}) = x_{ss}$ has been used. The result is just a linear DE of the form (4) with $a = f'(x_{ss})$ and $w = (1 - f'(x_{ss})) x_{ss}$. Its stability, therefore, depends on the magnitude of $f'(x_{ss})$.[2]

[2] Notice that this analysis is local, on the neighborhood of x_{ss}, and hence it must be carried out as many times as steady states are found.

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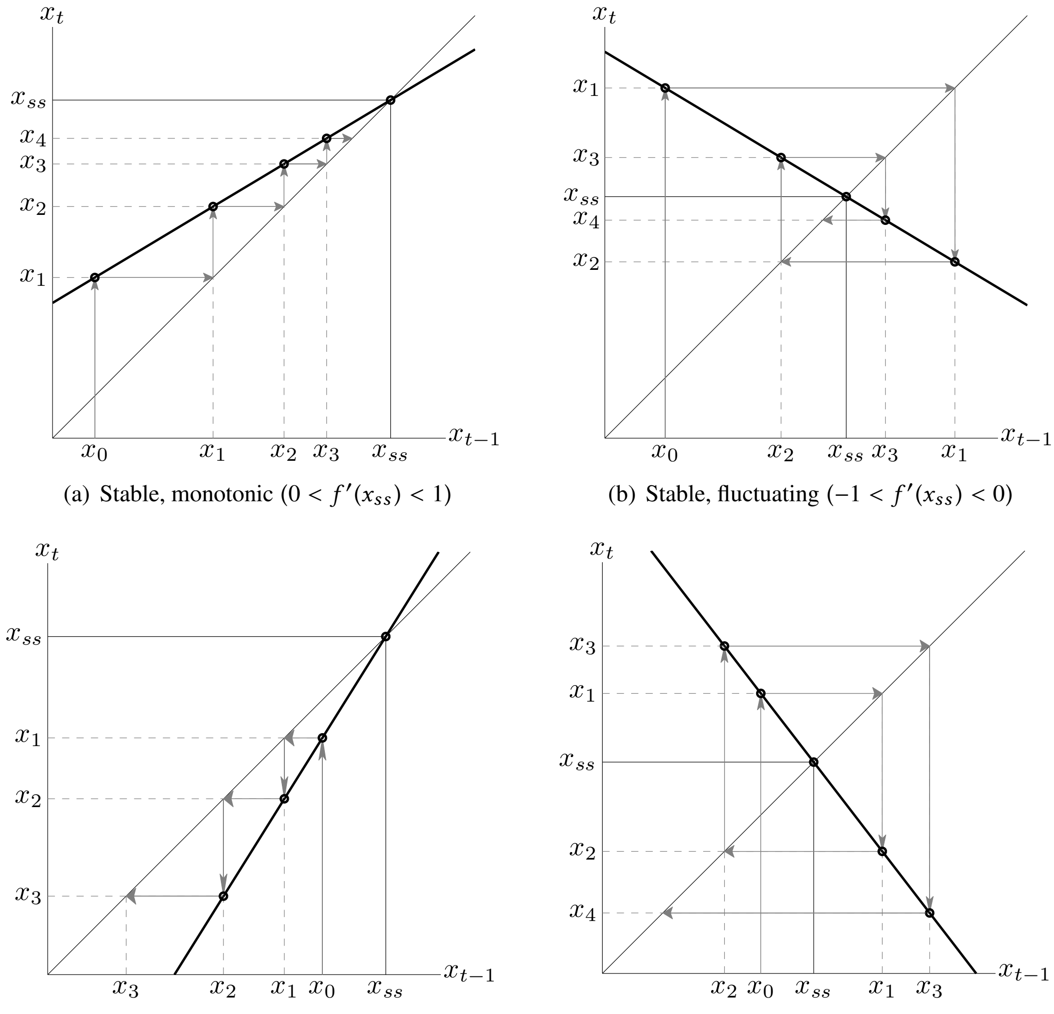
Figure 2. *Recursion diagrams of the equation $x_t = f(x_{t-1})$*



(a) Stable, monotonic ($0 < f'(x_{ss}) < 1$)

(b) Stable, fluctuating ($-1 < f'(x_{ss}) < 0$)

(c) Unstable, monotonic ($f'(x_{ss}) > 1$)

(d) Unstable, fluctuating ($f'(x_{ss}) < -1$)

A useful tool to analyze the dynamics of the nonlinear DE is the **recursion diagram** or **phase diagram**. The idea is simple: the function $f(\cdot)$ can be sketched on the (x_{t-1}, x_t) plane to give a qualitative assessment of the behavior of x_t and x_{t-1} at any subinterval of the range of $f(\cdot)$, or at any particular value of x.[3] Usually the values of x that are of interest are the steady states, x_{ss}. The function depicted in the recursion diagram is called the **phase curve**. Graphically, the steady state values can be found on the intersections between the phase curve and the 45° line whose points satisfy the steady state condition $x_t = x_{t-1}$. Determining $f'(x_{ss})$, the slope around the steady state, is crucial. From our knowledge on linear equations, we can conclude that if $f'(x_{ss}) < 0$ the path fluctuates around the steady state and if $f'(x_{ss}) > 0$ the path is monotonic. Also, if $|f'(x_{ss})| < 1$ the equilibrium is stable and if $|f'(x_{ss})| > 1$ the equilibrium is unstable.

Figure 2 presents the recursion diagram for various cases (the phase curves are depicted as dark bold lines). Let us focus first in panel (a) of Figure 2 where $0 < f'(x_{ss}) < 1$. Consider that the initial value of x_t lies below the steady state, $x_0 < x_{ss}$. Using the DE we can compute $x_1 = f(x_0)$ and this value can be located on the vertical axes by the value on the phase curve that corresponds to x_0. Next, we can locate x_1 on the horizontal axes by projecting this value from the vertical axes to the 45° line. Graphically, this is the horizontal arrow that departs from $f(x_0)$ to the 45° line. The result is that $x_1 > x_0$. Having found the value of x_1, we can determine $x_2 = f(x_1)$ and find this value

[3] Often the analysis is based on the DE $x_{t+1} = f(x_t)$, so the recursion diagram is based on the (x_t, x_{t+1}) plane instead. The rest of the analysis remains the same.

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on the vertical axes. Upon projecting this value to the horizontal axes through the 45° line we see that $x_2 > x_1$. We can proceed further using this logic and hence determine the behavior of $x(t)$. From the figure it is clear that this path is *monotonic*, $x_0 < x_1 < x_2 < x_3 < \ldots < x_{ss}$, and *stable* since $x(t)$ gets closer to x_{ss} at each iteration.

Figure 2(b) presents the case where $-1 < f'(x_{ss}) < 0$. Here the initial value is also $x_0 < x_{ss}$. The value of x_1 satisfies $x_1 = f(x_0)$ and after projecting it to the horizontal axes we observe that $x_1 > x_0$. Next, we determine the value of $x_2 = f(x_1)$ and we realize that $x_2 < x_1$ and $x_2 > x_0$. Proceeding recursively it can be establish that the path is *fluctuating* and *stable*.

The unstable cases are given in Figure 2(c) and Figure 2(d). Panel (c) deals with the $f'(x_{ss}) > 1$ case and the result is a *monotonic* sequence, $x_{ss} > x_0 > x_1 > x_2 > \ldots$. Finally, panel (d) displays the $f'(x_{ss}) < -1$ situation when the underlying path is *fluctuating*.

6.5 The lag operator

Let x_t be a sequence of real numbers or random variables. The lag operator L is defined as

$$Lx_t = x_{t-1} \, . \tag{25}$$

Thus, a single application of the lag operator to a sequence x_t defines another sequence y_t such that $y_t = x_{t-1}$. The applications of the lag operator several times can be thought of as raising L to arbitrary integer powers, e.g. $L^2x_t = L(Lx_t) = Lx_{t-1} = x_{t-2}$ or more generally

$$L^k x_t = x_{t-k} \, ,$$

where k is any integer. If $k = 0$ then $L^0 = 1$ ($L^0 x_t = x_t$) whereas if $k < 0$ the lag operator is actually a *lead operator*, e.g. $L^{-2}x_t = x_{t+2}$.

The lag operator is commutative and distributive so that for two constants a and b,

$$L(ax_t + by_t) = L(ax_t) + L(by_t) = a(Lx_t) + b(Ly_t) = ax_{t-1} + by_{t-1} \, .$$

Also and trivially, the lag operator applied to a constant renders that very constant, $La = a$.

Even though this is an operator, it can be treated very conveniently as an algebraic quantity. For this, it is convenient to define **polynomials in** L that can be of finite and infinite order. Hence if

$$B(L) = b_0 + b_1 L + b_2 L^2 + \ldots + b_q L^q \, ,$$

then $B(L)x_t = b_0 x_t + b_1 x_{t-1} + b_2 x_{t-2} + \ldots + b_q x_{t-q}$. The difference operator is a particular case $\Delta x_t = (1 - L)x_t = x_t - x_{t-1}$ and so is the second differences operator $\Delta^2 = (1-L)^2 = 1-2L+L^2$ so $\Delta^2 x_t = (1-L)^2 x_t = x_t - 2x_{t-1} + x_{t-2}$. In fact, one of the benefits of introducing the lag operator is that it provides us with a compact notation for writing and handling these polynomials. Define $B(L)$ as above and $A(L) = a_0 + a_1 L + a_2 L^2 + \ldots + a_p L^p$, where with no loss of generality $p \leq q \leq \infty$. The algebra of polynomials in L closely resembles the algebra of standard polynomials:

- **Addition**: If $C(L) = A(L) + B(L)$ then $C(L) = c_0 + c_1 L + c_2 L^2 + \ldots + c_q L^q$ where $c_j = a_j + b_j$ for $j \leq p$ and $c_j = b_j$ if $p < j \leq q$.
- **Multiplication**: If $C(L) = A(L) \cdot B(L) = B(L) \cdot A(L)$, a polynomial said to be the *convolution* of $A(L)$ and $B(L)$, then

$$\begin{aligned} C(L) &= (a_0 + a_1 L + a_2 L^2 + \ldots + a_p L^p)(b_0 + b_1 L + b_2 L^2 + \ldots + b_q L^q) \\ &= a_0 b_0 + (a_1 b_0 + a_0 b_1)L + (a_0 b_2 + a_1 b_1 + a_2 b_0)L^2 + \ldots \\ &= c_0 + c_1 L + c_2 L^2 + \ldots + c_{p+q} L^{p+q} \, . \end{aligned}$$

- **Inversion**: We define $[A(L)]^{-1}$ to be the polynomial in L such that $[A(L)]^{-1} \cdot A(L) = 1$. This allows also to define the division of two polynomials as $C(L) = B(L)/A(L) = B(L) \cdot [A(L)]^{-1}$.

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6.5.1 First-order DE

We can use the lag operator to analyze the DE in (1)

$$x_t - a x_{t-1} = w_t \qquad \rightarrow \qquad (1 - aL)x_t = w_t \,,$$

solving for x_t yields

$$x_t = \left(\frac{1}{1 - aL}\right) w_t \,.$$

If $|\,a\,| < 1$ the polynomial $1/(1 - aL)$ can be expanded using geometric series as

$$\frac{1}{1 - aL} = 1 + aL + a^2L^2 + a^3L^3 + \ldots = \sum_{j=0}^{\infty} a^j L^j \,, \tag{26}$$

which is the inverse of $A(L) = 1 - aL$. Hence, the solution to the DE is given by

$$x(t) = \sum_{j=0}^{\infty} a^j L^j w_t = \sum_{j=0}^{\infty} a^j w_{t-j} \,.$$

This is the same **backward solution** that we obtained by successive substitution, see equation (8) in section 6.2.4. Note that we have set $C = 0$ for the complementary solution.

If $|\,a\,| > 1$ the polynomial $1/(1 - aL)$ cannot be expanded as in (26). Let $\alpha = 1/a$ so $|\,\alpha\,| < 1$. Then,

$$\frac{1}{1 - aL} = -\left(\frac{\alpha L^{-1}}{1 - \alpha L^{-1}}\right) = -(\alpha L^{-1} + \alpha^2 L^{-2} + \alpha^3 L^{-3} + \ldots) = -\sum_{j=1}^{\infty} \left(\frac{1}{a}\right)^j L^{-j} \,. \tag{27}$$

The solution to the DE is given now by

$$x(t) = -\sum_{j=1}^{\infty} \left(\frac{1}{a}\right)^j L^{-j} w_t = -\sum_{j=1}^{\infty} \left(\frac{1}{a}\right)^j w_{t+j} \,.$$

This is the same **forward solution** that we got by successive substitution, see equation (9) in section 6.2.4, after setting $C = 0$ for the complementary solution.

6.5.2 Second-order DE

We now use the lag operator to obtain the particular solution to the second-order DE

$$x_t - a_1 x_{t-1} - a_2 x_{t-2} = w_t \qquad \rightarrow \qquad (1 - a_1 L - a_2 L^2) x_t = w_t \,.$$

The polynomial $A(L) = 1 - a_1 L - a_2 L^2$ can be factorized as $A(L) = (1 - r_1 L)(1 - r_2 L)$, where r_1 and r_2 are the characteristic roots of the equation, i.e. those who solve (20) and satisfy $r_1 + r_2 = a_1$ and $r_1 r_2 = a_2$.

We assume that $r_1 \neq r_2$.[4] Solving for x_t yields

$$x_t = \left(\frac{1}{1 - r_1 L}\right)\left(\frac{1}{1 - r_2 L}\right) w_t \,.$$

The polynomial in L above can be expressed, using partial fractions, as

$$\left(\frac{1}{1 - r_1 L}\right)\left(\frac{1}{1 - r_2 L}\right) = \frac{1}{r_1 - r_2}\left(\frac{r_1}{1 - r_1 L} - \frac{r_2}{1 - r_2 L}\right) ,$$

[4] The case when $r_1 = r_2$ is qualitatively similar but needs some further manipulation.

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yielding two polynomials $1/(1-r_1L)$ and $1/(1-r_2L)$ that can be expanded using the geometric series approach. Consider the **real roots** case. If $|\,r_1\,| < 1$ and $|\,r_2\,| < 1$ we obtain the backward solution, see (26):

$$x(t) = \frac{r_1}{r_1-r_2}\sum_{j=0}^{\infty} r_1^j L^j w_t - \frac{r_2}{r_1-r_2}\sum_{j=0}^{\infty} r_2^j L^j w_t = \frac{1}{r_1-r_2}\sum_{j=0}^{\infty}\left(r_1^{j+1}-r_2^{j+1}\right) w_{t-j}\,.$$

If $|\,r_1\,| > 1$ and $|\,r_2\,| > 1$, we have that (see equation (27) above)

$$\frac{1}{r_1-r_2}\left(\frac{r_1}{1-r_1L}-\frac{r_2}{1-r_2L}\right) = \frac{1}{r_1-r_2}\left(\frac{-L^{-1}}{1-(1/r_1)L^{-1}}+\frac{L^{-1}}{1-(1/r_2)L^{-1}}\right),$$

and we can obtain the forward solution:

$$x(t) = \frac{1}{r_1-r_2}\sum_{j=1}^{\infty}\left(\frac{1}{r_2}\right)^{j-1} L^{-j} w_t - \frac{1}{r_1-r_2}\sum_{j=1}^{\infty}\left(\frac{1}{r_1}\right)^{j-1} L^{-j} w_t = \frac{1}{r_1-r_2}\sum_{j=1}^{\infty}\left[\left(\frac{1}{r_2}\right)^{j-1}-\left(\frac{1}{r_1}\right)^{j-1}\right] w_{t+j}\,.$$

In the intermediate case when $|\,r_1\,| < 1$ and $|\,r_2\,| > 1$, we have that

$$\frac{1}{r_1-r_2}\left(\frac{r_1}{1-r_1L}-\frac{r_2}{1-r_2L}\right) = \frac{1}{r_1-r_2}\left(\frac{r_1}{1-r_1L}+\frac{L^{-1}}{1-(1/r_2)L^{-1}}\right),$$

and we obtain the saddlepath solution with both a backward and a forward component:

$$x(t) = \frac{r_1}{r_1-r_2}\sum_{j=0}^{\infty}(r_1)^j w_{t-j} + \frac{r_2}{r_1-r_2}\sum_{j=1}^{\infty}\left(\frac{1}{r_2}\right)^j w_{t+j}\,.$$

When the roots are **complex conjugate**, we should evoke De Moivre's formula

$$(r_1)^s = \rho^s[\,\cos(\theta s) + i\sin(\theta s)\,] \qquad \text{and} \qquad (r_2)^s = \rho^s[\,\cos(\theta s) - i\sin(\theta s)\,]\,.$$

Hence $r_1 - r_2 = 2i\rho\sin(\theta)$, $r_1^{j+1} - r_2^{j+1} = 2i\rho^{j+1}\sin(\theta(j+1))$ and $(r_2)^{-j} - (r_1)^{-j} = -2i\rho^{-j}\sin(\theta j)$. Replacing these findings into the solutions above we obtain:

$$x(t) = \sum_{j=0}^{\infty}\rho^j\frac{\sin(\theta(j+1))}{\sin(\theta)}w_{t-j} \quad \text{if } \rho < 1\,, \quad \text{and} \quad x(t) = -\sum_{j=1}^{\infty}\left(\frac{1}{\rho}\right)^{j-1}\frac{\sin(\theta(j-1))}{\sin(\theta)}w_{t+j} \quad \text{if } \rho > 1\,.$$

To conclude this section, the above analysis provide a basis to analyze higher order equations. Consider (21) with a time-dependent inhomogeneous term

$$x_t - a_1x_{t-1} - a_2x_{t-2} - \ldots - a_nx_{t-n} = w_t \qquad \to \qquad (1 - a_1L - a_2L^2 - \ldots - a_nL^n)x_t = w_t\,.$$

The polynomial $A(L)$ in the above equation can be factorized as $A(L) = (1-r_1L)(1-r_2L)\cdots(1-r_nL)$, so solving for x_t leads to

$$x_t = \left(\frac{1}{1-r_1L}\right)\left(\frac{1}{1-r_2L}\right)\cdots\left(\frac{1}{1-r_nL}\right)w_t\,.$$

Thus, we need to expand the polynomials $1/(1-r_jL)$ for $j = 1, 2, \ldots, n$, which can be done using the expansions in (26) for stable backward components ($|\,r_j\,| < 1$) and (27) for stable forward components ($|\,r_j\,| > 1$).

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