Exo-Snowballs: 1D Spherical Model

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1 Model Variables

ϕ	longitude
θ	colatitude
z	vertical coordinate
σ_{ij}	stress tensor
τ_{ij}	deviatoric stress tensor
p	ice shelf pressure
p_w	hydrostatic water pressure
$\hat{n}_\theta$	normal vector at terminus
ρ_i	density of ice
ρ_w	density of water
h	ice shelf thickness
s	ice shelf surface elevation
b	ice shelf bottom elevation
B	ice effective viscosity
$\tilde{B}_N$	means to an end?
$A(T)$	temperature-dependence of ice viscosity
$\dot{\epsilon}_{ij}$	rate of strain
$\dot{\epsilon}$	symmetric rate of strain tensor
u	zonal ice shelf velocity
v	meridional ice shelf velocity
$\Delta\theta$ or dθ	distance b/t grid cells in degrees
r	radius of the Earth

2 Variables and Approximations in the 1D Spherical Model

2.1 Variables in the 1D Spherical Model

$$
\begin{gathered}
h = s - b \\
s = (1 - \mu)h \\
b = -\mu h \\
\frac{\mathbf{r}}{\mathbf{h}}\mathbf{B} = \langle \mathbf{A}(\mathbf{T})^{-\frac{1}{3}} \rangle \dot{\boldsymbol{\epsilon}}^{\frac{1}{3}-1} \\
\tau_{\mathbf{ij}} = \mathbf{A}(\mathbf{T})^{-\frac{1}{3}} \dot{\boldsymbol{\epsilon}}^{\frac{1}{3}-1} \dot{\epsilon}_{\mathbf{ij}} \\
\sigma_{ij} = \tau_{ij} - p\delta_{ij} \\
\nabla = \left(\frac{1}{r\sin(\theta)}\partial_\phi, \frac{1}{r}\partial_\theta, \partial_r \right) \\
p = -\frac{1}{3}\sigma_{kk} \\
\mu = \frac{\rho_i}{\rho_w} \\
p_w = \rho_w g z \\
\dot{\epsilon}^2 = \frac{1}{2}(\dot{\epsilon}^2_{\phi\phi} + \dot{\epsilon}^2_{\theta\theta} + \dot{\epsilon}^2_{zz} + 2\dot{\epsilon}^2_{\phi\theta}) \\
\dot{\epsilon}_{zz} = -(\dot{\epsilon}^2_{\phi\phi} + \dot{\epsilon}^2_{\theta\theta})
\end{gathered}
$$

2.2 Approximations in the 1D Spherical Model

$$
\begin{gathered}
u = 0 \\
\tau_{z\theta} \approx 0 \\
\tau_{z\phi} \approx 0 \\
\partial_\phi \to 0 \\
\partial_z \to 0
\end{gathered}
$$

from which it follows that

$$
\begin{gathered}
\sigma_{z\theta} = 0 \\
\sigma_{\phi\theta} = 0
\end{gathered}
$$

The Ice Shelf Approximation allows us to neglect $\dot{\epsilon}_{\theta z}$ and $\dot{\epsilon}_{\phi z}$.

Thus, it follows that $\sigma_{z\theta} = \tau_{z\theta} - p\cancelto{0}{\delta_{z\theta}} \approx 0$.

Starting with $\tau_{ij} = A(T)^{-\frac{1}{3}}\dot{\epsilon}^{\frac{1}{3}-1}\dot{\epsilon}_{ij}$, it may easily be shown from $\partial_\phi \to 0$ and $u = 0$ that $\dot{\epsilon}_{\phi\theta} = 0$ and therefore $\sigma_{\phi\theta} = 0$.

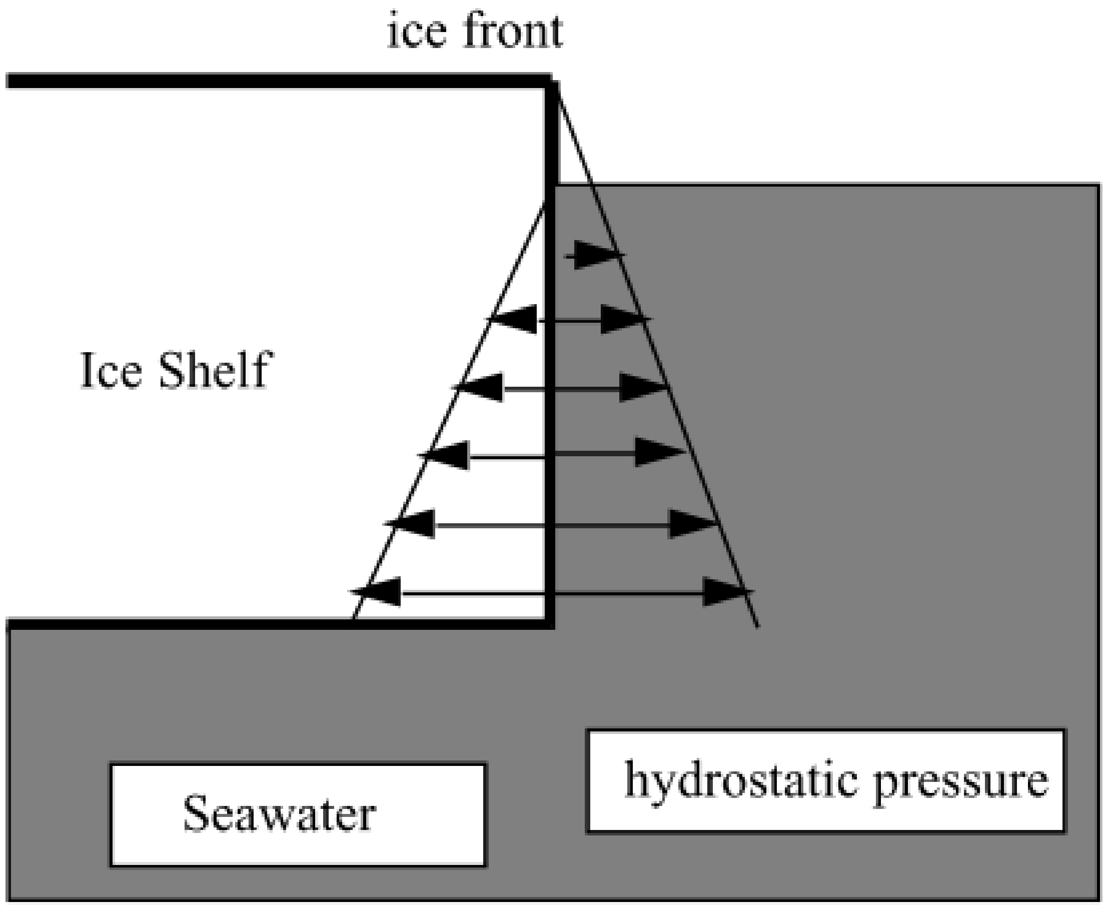

Figure 1: Force Balance on the Ice Shelf Terminus (*MacAyeal*, 1997).

3 Boundary Condition at the Ice Shelf Terminus

$$\int_b^s \sigma \cdot \hat{n}_\theta dz = \int_b^0 p_w \hat{n}_\theta dz, \tag{1}$$

Equation (1) indicates that the vertically-integrated stress in the $\hat{\theta}$ direction at the ice shelf terminus is equal to the vertically-integrated hydrostatic water pressure.

$$\sigma \cdot \hat{n}_\theta = \begin{bmatrix} \sigma_{\phi\phi} & \sigma_{\phi\theta} & \sigma_{\phi z} \\ \sigma_{\theta\phi} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{z\phi} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{\phi\theta}, \sigma_{\theta\theta}, \sigma_{z\theta} \end{bmatrix}$$

$$\int_b^s \left[\sigma_{\phi\theta}, \sigma_{\theta\theta}, \sigma_{z\theta}\right] dz = \int_b^0 p_w \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} dz$$

where $\hat{n}_\theta = (0, 1, 0)$. Using the thin-ice approximation, r is assumed to be constant in z. Following *Tziperman et al.* (2012), $\sigma_{\phi\theta} = \sigma_{z\theta} = 0$. Therefore, we approximate that there are are no stresses perpendicular to the direction of flow within the 1D ice shelf.

The left-hand side of Equation (1) can now be rewritten for the 1D spherical case.

$$\int_b^s \sigma_{\theta\theta} dz = \int_b^0 p_w dz, \tag{2}$$

Substituting $\sigma_{\theta\theta} = \tau_{\theta\theta} - p\delta_{ij}$,

$$\int_b^s (\tau_{\theta\theta} - p) dz = \int_b^0 p_w dz \tag{3}$$

3.1 Depth-Integrated Force Balance: RHS of Equation (3)

First, we plug an expression for the hydrostatic water pressure

$$p_w = \rho_w g z$$

into the RHS of Equation (3), obtaining:

$$\int_b^0 \rho_w g z dz = -\frac{1}{2}\rho_w g b^2 \tag{4}$$

Since $h = s - b$ and $s = (1 - \mu)h$, then $b = -\mu h$. We can plug this expression for b into Equation 4, at the same time substituting $\mu = \frac{\rho_i}{\rho_w}$.

$$-\frac{1}{2}\rho_w g(-\mu h)^2 = -\frac{1}{2}\rho_w g(\frac{\rho_i}{\rho_w})^2 h^2 = -\frac{1}{2}\rho_i g \mu h^2$$

$$\int_b^0 \rho_w g z dz = -\frac{1}{2}\rho_i g \mu h^2 \tag{5}$$

Equation (5) is the force integrated balance of the hydrostatic pressure of the water column on the ice shelf.

3.2 Depth-Integrated Force Balance: LHS of Equation (3)

$$\int_b^s \sigma_{\theta\theta} dz = \int_b^s (\tau_{\theta\theta} - p) dz = \int_b^s \tau_{\theta\theta} dz - \int_b^s p dz \tag{6}$$

Tziperman et al. (2012) provides an expression for the vertically-integrated pressure of the ice shelf.

$$-\int_b^s p = -\frac{1}{2}g\rho_i h^2 + \int_b^s (\tau_{\phi\phi} + \tau_{\theta\theta}) - \frac{1}{r}\int_b^s \int_z^s (\tau_{\phi\phi} + \tau_{\theta\theta}) \tag{7}$$

Following the method of *Tziperman et al.* (2012), terms of $\Omega(\frac{h}{r}) \approx 0$ and are therefore neglected. Since $\tau_{\phi\phi}$ and $\tau_{\theta\theta}$ have no z dependence, the third term on the righthand side of Equation **??** will drop out. Plugging Equation (7) into Equation (6), we get

$$\int_b^s (\tau_{\theta\theta} - p)dz = \int_b^s \tau_{\theta\theta}dz - \frac{1}{2}g\rho_i h^2 + \int_b^s (\tau_{\phi\phi} + \tau_{\theta\theta})dz = \int_b^s (\tau_{\phi\phi} + 2\tau_{\theta\theta}) - \frac{1}{2}g\rho_i h^2 \tag{8}$$

Noting that the deviatoric stresses $\tau_{\phi\phi}$ and $\tau_{\theta\theta}$ are independent of z, Equation 8 becomes

$$\int_b^s (\tau_{\theta\theta} - p)dz = h(\tau_{\phi\phi} + 2\tau_{\theta\theta}) - \frac{1}{2}g\rho_i h^2$$

To proceed, we need to put τ_{ij} in terms of the strain rate $\dot{\epsilon}_{ij}$. See Appendix A or *Tziperman et al.* (2012) for the formulas.

$$\tau_{\theta\theta} = \frac{r}{h}B\dot{\epsilon}_{\theta\theta} = \frac{r}{h}B[\frac{1}{r}\partial_\theta v] = \frac{1}{h}B[\partial_\theta v]$$

Since $\partial_\phi \to 0$ in the case of a 1D sphere,

$$\tau_{\phi\phi} = \frac{r}{h}B\dot{\epsilon}_{\phi\phi} = \frac{r}{h}B[\frac{1}{rsin(\theta)}(\cancelto{0}{\partial_\phi u} + v\cos(\theta))] = \frac{1}{h}B[v\cot(\theta)]$$

Finally, we substitute $\tau_{\phi\phi}$ and $\tau_{\theta\theta}$.

$$\int_b^s (\tau_{\theta\theta} - p)dz = h(\frac{1}{h}Bv\cot(\theta) + 2\frac{1}{h}B\partial_\theta v) - \frac{1}{2}g\rho_i h^2 = B(v\cot(\theta) + 2\partial_\theta v) - \frac{1}{2}g\rho_i h^2 \tag{9}$$

Equation 9 is the depth-integrated stress of the ice shelf.

3.3 Depth-Integrated Force Balance: LHS and RHS

Combining the LHS and RHS of the depth-integrated force balance, we get

$$B[v\cot(\theta) + 2\partial_\theta v] - \frac{1}{2}g\rho_i h^2 = -\frac{1}{2}\rho_i g\mu h^2 \tag{10}$$

We can simplify the above equation to get our desired boundary condition at the ice shelf terminus in the 1D spherical case.

$$B[v\cot(\theta) + 2\partial_\theta v] = \frac{1}{2}g\rho_i h^2[1 - \mu] \tag{11}$$

3.4 Discrete Form of Equation (11) in the Northern Hemisphere

Denote the index of the last fully-filled ice grid cell before the open ocean by N.

$$B_N\Big[v_N \cot\theta_N + 2\big(\frac{v_{N+1} - v_N}{\Delta\theta}\big)\Big] = \frac{1}{2}\rho_I g h_N^2[1-\mu] \tag{12}$$

3.5 Discrete Form of Equation (11) in the Southern Hemisphere

Denote the index of the last fully-filled ice grid cell before the open ocean by N.

$$B_N\Big[v_N \cot\theta_N + 2\big(\frac{v_N - v_{N-1}}{\Delta\theta}\big)\Big] = \frac{1}{2}\rho_I g h_N^2[1-\mu] \tag{13}$$

4 The Ice Effective Viscosity: B_j

We begin the derivation by stating the boundary condition.

$$\tilde{B}_N[v_N \cot(\theta_N) + 2\partial_\theta v_N] = \frac{1}{2} g\rho_i h_N^2[1-\mu] \tag{14}$$

At the last fully-filled ice grid cell, we solve for $\tilde{B}_N$.

Using the axisymmetric one-dimensional model from *Tziperman et al.* (2012), we state the expression for B

$$B = \frac{1}{r} h\langle A(T)^{-\frac{1}{3}}\rangle \dot{\epsilon}^{\frac{1}{3}-1}$$

and for $\dot{\epsilon}$

$$\dot{\epsilon}^2 = \frac{1}{2}(\dot{\epsilon}_{\phi\phi}^2 + \dot{\epsilon}_{\theta\theta}^2 + \dot{\epsilon}_{zz}^2 + 2\dot{\epsilon}_{\phi\theta}^2)$$

where

$$\dot{\epsilon}_{zz} = -(\dot{\epsilon}_{\phi\phi}^2 + \dot{\epsilon}_{\theta\theta}^2)$$

and $\dot{\epsilon}_{\phi\theta}^2 = 0$.

In the 1D case, $\cancelto{0}{\partial_\phi}$ and the components of $\dot{\epsilon}$ simplify to the following.

$$\dot{\epsilon}_{\phi\phi} = \frac{1}{r}\cot\theta v$$

$$\dot{\epsilon}_{\theta\theta} = \frac{1}{r}\partial_\theta v$$

$$\dot{\epsilon}_{zz} = -(\frac{1}{r}\cot\theta v + \frac{1}{r}\partial_\theta v)$$

4.1 The Effective Ice Viscosity in the Northern Hemisphere

We will consider the location of the last fully-filled grid cell to be $j = N$, the location of the transition from the fully-filled to the partially-filled grid cell to be $j = N + \frac{1}{2}$, and the location of the partially-filled grid cell to be $j = N + 1$.

Substituting these expressions into $\tilde{B}$ and setting $j = N$, we obtain:

$$\tilde{B}_N = \frac{h_N}{r_N}\langle A(T)^{-\frac{1}{3}}\rangle \Big[\frac{1}{2}\left((\frac{1}{r_N}\cot\theta_N v_N)^2 + (\frac{1}{r_N}\partial_\theta^{bd} v_N)^2 + (\frac{1}{r_N}\cot\theta_N v_N + \frac{1}{r_N}\partial_\theta^{bd} v_N)^2 + \cancelto{0}{2\dot{\epsilon}^2_{\phi\theta}} \right)\Big]^{-\frac{1}{3}}$$

It is appropriate to use a backward finite differencing approximation for $\partial_\theta v_N$ in the expression for $\tilde{B}_N$.

$$\partial_\theta^{bd} v_N = \frac{v_N - v_{N-1}}{d\theta}$$

Consequently,

$$\tilde{B}_N = \frac{h_N}{r_N}\langle A(T)^{-\frac{1}{3}}\rangle \left[\frac{1}{2}\left(\left(\frac{1}{r_N}\cot\theta_N v_N\right)^2 + \left(\frac{1}{r_N}\frac{v_N - v_{N-1}}{d\theta}\right)^2 + \left(\frac{1}{r_N}\cot\theta_N v_N + \frac{1}{r_N}\frac{v_N - v_{N-1}}{d\theta}\right)^2 \right)\right]^{-\frac{1}{3}}$$

We turn next to Equation (14), which is restated just below..

$$\tilde{B}_N[v_N\cot(\theta_N) + 2\partial_\theta^{fd} v_N] = \frac{1}{2} g\rho_i h_N^2[1-\mu]$$

In Equation (14), it is appropriate to use a forward differencing approximation for $\partial_\theta v_N$.

$$\partial_\theta^{fd} v_N = \frac{v_{N+1} - v_N}{d\theta}$$

$$\tilde{B}_N\left[v_N\cot\theta_N + 2\frac{v_{N+1} - v_N}{d\theta}\right] = \frac{1}{2} g\rho_i h_N^2[1-\mu] \tag{15}$$

Our derived expression for $\tilde{B}_N$ can now be plugged into Equation (15). I won't do so here for readability.

We can rearrange Equation (15) to solve for $\partial_\theta^{fd} v_N = (v_{N+1} - v_N)/d\theta$.

$$\left[v_N\cot\theta_N + 2\frac{v_{N+1} - v_N}{d\theta}\right] = \frac{1}{2} g\rho_i h_N^2[1-\mu]\tilde{B}_N^{-1}$$

Upon further simplification,

$$\partial_\theta^{fd} v_N = \frac{v_{N+1} - v_N}{d\theta} = \frac{1}{4} g\rho_i h_N^2[1-\mu]\tilde{B}_N^{-1} - \frac{1}{2} v_N\cot\theta_N \tag{16}$$

Now, we plug Equation (16) into Equation (17) for a final solution to B_N.

$$B_N = \frac{h_N}{r_N}\langle A(T)^{-\frac{1}{3}}\rangle \left[\frac{1}{2}\left((\frac{1}{r_N}\cot\theta_N v_N)^2 + (\frac{1}{r_N}\partial_\theta^{fd} v_N)^2 + (\frac{1}{r_N}\cot\theta_N v_N + \frac{1}{r_N}\partial_\theta^{fd} v_N)^2 \right)\right]^{-\frac{1}{3}} \tag{17}$$

4.2 The Effective Ice Viscosity in the Southern Hemisphere

We will consider the location of the last fully-filled grid cell to be $j = N$, the location of the transition from the fully-filled to the partially-filled grid cell to be $j = N - \frac{1}{2}$, and the location of the partially-filled grid cell to be $j = N - 1$.

Substituting these expressions into $\tilde{B}$ and setting $j = N$, we obtain:

$$\tilde{B}_N = \frac{h_N}{r_N}\langle A(T)^{-\frac{1}{3}}\rangle \Big[\frac{1}{2}\left((\frac{1}{r_N}\cot\theta_N v_N)^2 + (\frac{1}{r_N}\partial_\theta^{fd} v_N)^2 + (\frac{1}{r_N}\cot\theta_N v_N + \frac{1}{r_N}\partial_\theta^{fd} v_N)^2 + \cancelto{0}{2\dot{\epsilon}^2_{\phi\theta})} \right)\Big]^{-\frac{1}{3}}$$

It is appropriate to use a forward finite differencing approximation for $\partial_\theta v_N$ in the expression for $\tilde{B}_N$.

$$\partial_\theta^{fd} v_N = \frac{v_{N+1} - v_N}{d\theta}$$

Consequently,

$$\tilde{B}_N = \frac{h_N}{r_N}\langle A(T)^{-\frac{1}{3}}\rangle \left[\frac{1}{2}\left(\left(\frac{1}{r_N}\cot\theta_N v_N\right)^2 + \left(\frac{1}{r_N}\frac{v_{N+1} - v_N}{d\theta}\right)^2 + \left(\frac{1}{r_N}\cot\theta_N v_N + \frac{1}{r_N}\frac{v_{N+1} - v_N}{d\theta}\right)^2 \right)\right]^{-\frac{1}{3}}$$

We turn next to Equation (14), which is restated for the Southern Hemisphere.

$$\tilde{B}_N[v_N \cot(\theta_N) + 2\partial_\theta^{bd} v_N] = \frac{1}{2} g\rho_i h_N^2 [1 - \mu]$$

In Equation (14), it is appropriate to use a backward differencing approximation for $\partial_\theta v_N$.

$$\partial_\theta^{bd} v_N = \frac{v_N - v_{N-1}}{d\theta}$$

$$\tilde{B}_N \left[v_N \cot\theta_N + 2\frac{v_N - v_{N-1}}{d\theta} \right] = \frac{1}{2} g\rho_i h_N^2 [1 - \mu] \tag{18}$$

Our derived expression for $\tilde{B}_N$ can now be plugged into Equation (18). I won't do so here for readability.

We can rearrange Equation (18) to solve for $\partial_\theta^{bd} v_N = (v_N - v_{N-1})/d\theta$.

$$\left[v_N \cot\theta_N + 2\frac{v_N - v_{N-1}}{d\theta} \right] = \frac{1}{2} g\rho_i h_N^2 [1-\mu] \tilde{B}_N^{-1}$$

Upon further simplification,

$$\partial_\theta^{bd} v_N = \frac{v_N - v_{N-1}}{d\theta} = \frac{1}{4} g\rho_i h_N^2 [1-\mu] \tilde{B}_N^{-1} - \frac{1}{2} v_N \cot\theta_N \tag{19}$$

Now, we plug Equation (19) into Equation (20) for a final solution to B_N.

$$B_N = \frac{h_N}{r_N} \langle A(T)^{-\frac{1}{3}} \rangle \left[\frac{1}{2} \left((\frac{1}{r_N} \cot\theta_N v_N)^2 + (\frac{1}{r_N} \partial_\theta^{bd} v_N)^2 + (\frac{1}{r_N} \cot\theta_N v_N + \frac{1}{r_N} \partial_\theta^{bd} v_N)^2 \right) \right]^{-\frac{1}{3}} \tag{20}$$

5 Solving for the Velocity outside the Boundary

5.1 Solving for v_{N+1} using the Boundary Condition in the Northern Hemisphere

First, we'll manipulate Equation (16) to get an expression for v_{N+1}.

$$v_{N+1} = v_N + \frac{d\theta}{4} g\rho_i h_N^2 [1-\mu] \tilde{B}_N^{-1} - \frac{d\theta}{2} v_N \cot\theta_N$$

We can simplify one step further for the final expression for v_{N+1}.

$$v_{N+1} = v_N \left[1 - \frac{d\theta}{2} \cot\theta_N \right] + \frac{d\theta}{4} g\rho_i h_N^2 [1-\mu] \tilde{B}_N^{-1} \tag{21}$$

5.2 Solving for v_{N-1} using the Boundary Condition in the Southern Hemisphere

First, we'll manipulate Equation (19) to get an expression for v_{N-1}.

$$v_{N-1} = v_N - \frac{d\theta}{4} g\rho_i h_N^2 [1-\mu] \tilde{B}_N^{-1} + \frac{d\theta}{2} v_N \cot\theta_N$$

We can simplify one step further for the final expression for v_{N-1}.

$$v_{N-1} = v_N \left[1 + \frac{d\theta}{2} \cot\theta_N \right] - \frac{d\theta}{4} g\rho_i h_N^2 [1-\mu] \tilde{B}_N^{-1} \tag{22}$$

6 The Tridiagonal Form of the Ice Shelf Momentum Equation at the Terminus

6.1 Northern Hemisphere

Next, we restate the tridiagonal form of the ice shelf momentum equations.

$$
\begin{aligned}
&v_{N+1}\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big] \\
&+v_N\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2 - B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N/d\theta^2 \\
&\quad - s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2 - \cot^2\theta_N B_N\Big] \\
&+v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2 + B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}/d\theta^2\Big] \\
&= g\rho_I(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)
\end{aligned}
$$

The following approximations are then made.

- $B_{N+\frac{1}{2}} \approx B_N$
- $s_{N+\frac{1}{2}} \approx s_N$
- $s_{N+\frac{1}{2}}^{-1} \approx s_N^{-1}$

We now wish to plug an expression for v_{N+1} into the tridiagonal form of the 1D ice shelf momentum equations.

$$
\begin{aligned}
&\Bigg[v_N\Big[1-\frac{d\theta}{2}\cot\theta_N\Big] + \frac{d\theta}{4}g\rho_i h_N^2[1-\mu]\tilde{B}_N^{-1}\Bigg]\Big[B_N/d\theta^2 + B_N s_N^{-1}s_{N+1}/d\theta^2\Big] \\
&\qquad +v_N\Big[-2B_N/d\theta^2 - B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N/d\theta^2 \\
&\qquad\quad - s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2 - \cot^2\theta_N B_N\Big] \\
&\qquad +v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2 + B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}/d\theta^2\Big] \\
&\qquad = g\rho_I(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)
\end{aligned}
$$

We perform a small rearrangement of the above equation.

$$
\begin{aligned}
&v_N\Big[1-\frac{d\theta}{2}\cot\theta_N\Big]\Big[B_N/d\theta^2+B_Ns_N^{-1}s_{N+1}/d\theta^2\Big]\\
&+v_N\Big[-2B_N/d\theta^2-B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N/d\theta^2\\
&\quad -s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2-\cot^2\theta_NB_N\Big]\\
&+v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2+B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}/d\theta^2\Big]\\
&=g\rho_I(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)-\frac{d\theta}{4}g\rho_ih_N^2[1-\mu]\tilde{B}_N^{-1}\Big[B_N/d\theta^2+B_Ns_N^{-1}s_{N+1}/d\theta^2\Big]
\end{aligned}
$$

Next, we collect like-terms and make a few cancellations.

$$
\begin{aligned}
&v_N\Big[B_N\frac{1}{d\theta^2}+B_Ns_N^{-1}s_{N+1}\frac{1}{d\theta^2}-\frac{1}{2d\theta}\cot\theta_NB_N-\frac{1}{2d\theta}\cot\theta_NB_Ns_N^{-1}s_{N+1}\\
&\quad -2B_N\frac{1}{d\theta^2}-B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N\frac{1}{d\theta^2}-s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}\frac{1}{d\theta^2}-\cot^2\theta_NB_N\Big]\\
&+v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}\frac{1}{d\theta^2}+B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}\frac{1}{d\theta^2}\Big]\\
&=g\rho_I(1-\mu)h_N\frac{h_{N+1}-h_{N-1}}{2d\theta}-\frac{d\theta}{4}g\rho_ih_N^2[1-\mu]\tilde{B}_N^{-1}\Big[B_N\frac{1}{d\theta^2}+B_Ns_N^{-1}s_{N+1}\frac{1}{d\theta^2}\Big]
\end{aligned}
\tag{23}
$$

In this equation, the pressure balance term on the RHS of the equation uses a partial derivative approximation. At the boundary though, we will use a backward finite difference approximation in order to evaluate the derivative.

$$
\begin{aligned}
&v_N\Big[B_N\frac{1}{d\theta^2}+B_Ns_N^{-1}s_{N+1}\frac{1}{d\theta^2}-\frac{1}{2d\theta}\cot\theta_NB_N-\frac{1}{2d\theta}\cot\theta_NB_Ns_N^{-1}s_{N+1}\\
&\quad -2B_N\frac{1}{d\theta^2}-B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N\frac{1}{d\theta^2}-s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}\frac{1}{d\theta^2}-\cot^2\theta_NB_N\Big]\\
&+v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}\frac{1}{d\theta^2}+B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}\frac{1}{d\theta^2}\Big]\\
&=g\rho_I(1-\mu)h_N\frac{h_N-h_{N-1}}{d\theta}-\frac{d\theta}{4}g\rho_ih_N^2[1-\mu]\tilde{B}_N^{-1}\Big[B_N\frac{1}{d\theta^2}+B_Ns_N^{-1}s_{N+1}\frac{1}{d\theta^2}\Big]
\end{aligned}
\tag{24}
$$

Simplifying the RHS,

$$
\begin{aligned}
&v_N\Big[B_Ns_N^{-1}s_{N+1}\frac{1}{d\theta^2}-\frac{1}{2d\theta}\cot\theta_NB_N-\frac{1}{2d\theta}\cot\theta_NB_Ns_N^{-1}s_{N+1}\\
&\quad -B_N\frac{1}{d\theta^2}-B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N\frac{1}{d\theta^2}-s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}\frac{1}{d\theta^2}-\cot^2\theta_NB_N\Big]\\
&+v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}\frac{1}{d\theta^2}+B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}\frac{1}{d\theta^2}\Big]\\
&=g\rho_I(1-\mu)h_N\left[\frac{h_N-h_{N-1}}{d\theta}-\frac{h_N}{4d\theta}\frac{B_N}{\tilde{B}_N}(1+s_N^{-1}s_{N+1})\right]
\end{aligned}
\tag{25}
$$

6.2 Southern Hemisphere

Next, we restate the tridiagonal form of the ice shelf momentum equations.

$$\begin{aligned}
&v_{N+1}\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big] \\
&+v_N\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2 - B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_N/d\theta^2 \\
&\quad - s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2 - \cot^2\theta_N B_N\Big] \\
&+v_{N-1}\Big[s_N^{-1}B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}/d\theta^2 + B_{N-\frac{1}{2}}s_{N-\frac{1}{2}}^{-1}s_{N-1}/d\theta^2\Big] \\
&= g\rho_I(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)
\end{aligned}$$

The following approximations are then made.

- $B_{N-\frac{1}{2}} \approx B_N$
- $s_{N-\frac{1}{2}} \approx s_N$
- $s_{N-\frac{1}{2}}^{-1} \approx s_N^{-1}$

$$\begin{aligned}
&v_{N+1}\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big] \\
&+v_N\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2 \\
&\quad - 2B_N/d\theta^2 - \cot^2\theta_N B_N\Big] \\
&+v_{N-1}\Big[B_N/d\theta^2 + B_N s_N^{-1}s_{N-1}/d\theta^2\Big] \\
&= g\rho_I(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)
\end{aligned}$$

We now wish to plug an expression for v_{N-1} into the tridiagonal form of the 1D ice shelf momentum equations.

$$\begin{aligned}
&v_{N+1}\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big] \\
&+v_N\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2 \\
&\quad - 2B_N/d\theta^2 - \cot^2\theta_N B_N\Big] \\
+&\left[v_N\Big[1+\frac{d\theta}{2}\cot\theta_N\Big] - \frac{d\theta}{4}g\rho_i h_N^2[1-\mu]\tilde{B}_N^{-1}\right]\Big[B_N/d\theta^2 + B_N s_N^{-1}s_{N-1}/d\theta^2\Big] \\
&= g\rho_I(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)
\end{aligned}$$

We perform a small rearrangement of the above equation.

$$
\begin{aligned}
v_{N+1}&\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big]\\
+v_N&\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2\\
&-2B_N/d\theta^2 - \cot^2\theta_N B_N\\
&+B_N/d\theta^2 + B_N s_N^{-1}s_{N-1}/d\theta^2 + B_N\cot\theta_N/(2d\theta)\\
&+B_N\cot\theta_N s_N^{-1}s_{N-1}/(2d\theta)\Big]\\
&-\frac{1}{4d\theta}\frac{B_N}{\tilde{B}_N}g\rho_I h_N^2(1-\mu)[1+s_N^{-1}s_{N-1}]\\
=g\rho_I&(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)
\end{aligned}
$$

Next, we collect like-terms and make a few cancellations.

$$
\begin{aligned}
v_{N+1}&\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big]\\
+v_N&\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2\\
&-B_N/d\theta^2 - \cot^2\theta_N B_N\\
&+B_N s_N^{-1}s_{N-1}/d\theta^2 + B_N\cot\theta_N/(2d\theta)\\
&+B_N\cot\theta_N s_N^{-1}s_{N-1}/(2d\theta)\Big]\\
=g\rho_I&(1-\mu)h_N(h_{N+1}-h_{N-1})/(2d\theta)\\
&+\frac{1}{4d\theta}\frac{B_N}{\tilde{B}_N}g\rho_I h_N^2(1-\mu)[1+s_N^{-1}s_{N-1}]
\end{aligned}
$$

In this equation, the pressure balance term on the RHS of the equation uses a partial derivative approximation. At the boundary though, we will use a forward finite difference approximation in order to evaluate the derivative.

$$
\begin{aligned}
v_{N+1}&\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big]\\
+v_N&\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2\\
&-B_N/d\theta^2 - \cot^2\theta_N B_N\\
&+B_N s_N^{-1}s_{N-1}/d\theta^2 + B_N\cot\theta_N/(2d\theta)\\
&+B_N\cot\theta_N s_N^{-1}s_{N-1}/(2d\theta)\Big]\\
=g\rho_I&(1-\mu)h_N(h_{N+1}-h_N)/d\theta\\
&+\frac{1}{4d\theta}\frac{B_N}{\tilde{B}_N}g\rho_I h_N^2(1-\mu)[1+s_N^{-1}s_{N-1}]
\end{aligned}
$$

Simplifying the RHS,

$$
\begin{aligned}
&v_{N+1}\Big[s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 + B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_{N+1}/d\theta^2\Big] \\
&+v_N\Big[-s_N^{-1}B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}/d\theta^2 - B_{N+\frac{1}{2}}s_{N+\frac{1}{2}}^{-1}s_N/d\theta^2 \\
&\qquad - B_N/d\theta^2 - \cot^2\theta_N B_N \\
&\qquad + B_N s_N^{-1}s_{N-1}/d\theta^2 + B_N\cot\theta_N/(2d\theta) \\
&\qquad + B_N\cot\theta_N s_N^{-1}s_{N-1}/(2d\theta)\Big] \\
&= g\rho_I(1-\mu)h_N\Big[\frac{h_{N+1}-h_N}{d\theta} + \frac{h_N}{4d\theta}\frac{B_N}{\tilde{B}_N}(1+s_N^{-1}s_{N-1})\Big]
\end{aligned}
$$

7 The Ice Shelf 1D Spherical Mass Continuity Equation

7.1 Evaluated at the NH Terminus

In it's continuous form, the ice shelf 1D spherical mass continuity equation is written as follows.

$$\frac{\partial h}{\partial t} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v h) = \kappa\nabla^2 h + S(\theta)$$

We begin with the height advection term, which uses a backward finite difference.

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v h) = \frac{1}{r_j\sin\theta_j}\Big[\cancelto{|_j}{\sin\theta_{j+1}v_{j+1}h_{j+1}} - \sin\theta_{j-1}v_{j-1}h_{j-1}\Big]\frac{1}{\cancelto{1}{2}d\theta}$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v h) = \frac{1}{r_j\sin\theta_j}\Big[\sin\theta_j v_j h_j - \sin\theta_{j-1}v_{j-1}h_{j-1}\Big]\frac{1}{d\theta}$$

Next, a boundary condition usually employed in the diffusion term is that there is no diffusive flux out of the ice edge into the ocean. This would imply:

$$\frac{h_{j+1}-h_j}{d\theta} = 0$$

the full diffusion expression then can be discretized as follows.

$$
\begin{aligned}
\kappa\nabla^2 h &= \kappa\frac{1}{r_j^2\sin\theta_j}\Big[\sin\theta_{j+\frac{1}{2}}\frac{h_{j+1}-h_j}{d\theta} - \sin\theta_{j-\frac{1}{2}}\frac{h_j-h_{j-1}}{d\theta}\Big]\frac{1}{d\theta} \\
&= \kappa\frac{1}{r_j^2\sin\theta_j}\Big[-\sin\theta_{j-\frac{1}{2}}(h_j-h_{j-1})\Big]\frac{1}{d\theta^2}
\end{aligned}
$$

We end with the partial time derivative of the ice shelf height.

$$\frac{\partial h}{\partial t} = \frac{h_j^{n+2}-h_j^{n+1}}{dt}$$

Now, to use the adams bashforth method, we define

$$RHS(t_{n+1}, h_{n+1}) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta vh)|_j^{n+1} + \kappa\nabla^2 h|_j^{n+1} + S(\theta)|_j^{n+1}$$

$$RHS(t_n, h_n) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta vh)|_j^{n} + \kappa\nabla^2 h|_j^{n} + S(\theta)|_j^{n}$$

$$h_j^{n+2} = h_j^{n+1} + \frac{dt}{2}[3RHS(t_{n+1}, h_{n+1}) - RHS(t_n, h_n)]$$

7.2 Evaluated at the SH Terminus

In it's continuous form, the ice shelf 1D spherical mass continuity equation is written as follows.

$$\frac{\partial h}{\partial t} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta vh) = \kappa\nabla^2 h + S(\theta)$$

We begin with the height advection term, which uses a forward finite differencing approximation.

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta vh) = \frac{1}{r_j\sin\theta_j}\left[\sin\theta_{j+1}v_{j+1}h_{j+1} - \cancel{\sin\theta_{j-1}v_{j-1}h_{j-1}}\right]\Big|_j \frac{1}{\overset{1}{\cancel{2}}d\theta}$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta vh) = \frac{1}{r_j\sin\theta_j}\left[\sin\theta_{j+1}v_{j+1}h_{j+1} - \sin\theta_j v_j h_j\right]\frac{1}{d\theta}$$

Next, a boundary condition usually employed in the diffusion term is that there is no diffusive flux out of the ice edge into the ocean. This would imply:

$$\frac{h_j - h_{j-1}}{d\theta} = 0$$

the full diffusion expression then can be discretized as follows.

$$\begin{aligned}\kappa\nabla^2 h &= \kappa\frac{1}{r_j^2\sin\theta_j}\left[\sin\theta_{j+\frac{1}{2}}\frac{h_{j+1}-h_j}{d\theta} - \sin\theta_{j-\frac{1}{2}}\frac{h_j - h_{j-1}}{d\theta}\right]\frac{1}{d\theta} \\ &= \kappa\frac{1}{r_j^2\sin\theta_j}\left[\sin\theta_{j+\frac{1}{2}}(h_{j+1}-h_j)\right]\frac{1}{d\theta^2}\end{aligned}$$

We end with the partial time derivative of the ice shelf height.

$$\frac{\partial h}{\partial t} = \frac{h_j^{n+2} - h_j^{n+1}}{dt}$$

Now, to use the adams bashforth method, we define

$$RHS(t_{n+1}, h_{n+1}) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta vh)|_j^{n+1} + \kappa\nabla^2 h|_j^{n+1} + S(\theta)|_j^{n+1}$$

$$RHS(t_n, h_n) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v h)|_j^n + \kappa\nabla^2 h|_j^n + S(\theta)|_j^n$$

$$h_j^{n+2} = h_j^{n+1} + \frac{dt}{2}[3RHS(t_{n+1}, h_{n+1}) - RHS(t_n, h_n)]$$

8 Ice Shelf Advance and Retreat: the Grid-Subgrid Parameterization

8.1 Terminology

8.1.1 GC

A grid cell or GC is any cell where the ice height exceeds H_{cr}.

8.1.2 SC

A subgrid cell or SC is any cell immediately adjacent to the ice shelf terminus where $R \in (0, 1)$. By definition, no more than 2 SC can exist in the model domain at a given time.

8.1.3 eSC

An extraneous subgrid cell or eSC is any subgrid cell, SC, that forms more than one cell away from the terminus in a respective hemisphere.

8.1.4 OC

An ocean cell is any cell immediately adjacent to the ice shelf terminus where R = 0. No more than 2 OC can exist in the model domain at a given time.

8.2 Description of the Subgrid Parameterization

8.2.1 Field of Fractional Ice Coverage, R

In order to paramerize subgrid advance and retreat, a field of fractional ice cover denoted R is introduced on the A-grid, where $R \in [0, 1]$.

$$R = \begin{cases} 0 & h_j = 0 \\ \frac{h_j}{H_{cr}} & 0 < h_j < H_{cr} \\ 1 & h_j \geq H_{cr} \end{cases}$$

R takes on values of 0 in open ocean, 1 for grid cell ice thicknesses greater than the critical height, and values between 0 and 1 for grid cell ice thicknesses less than the critical height.

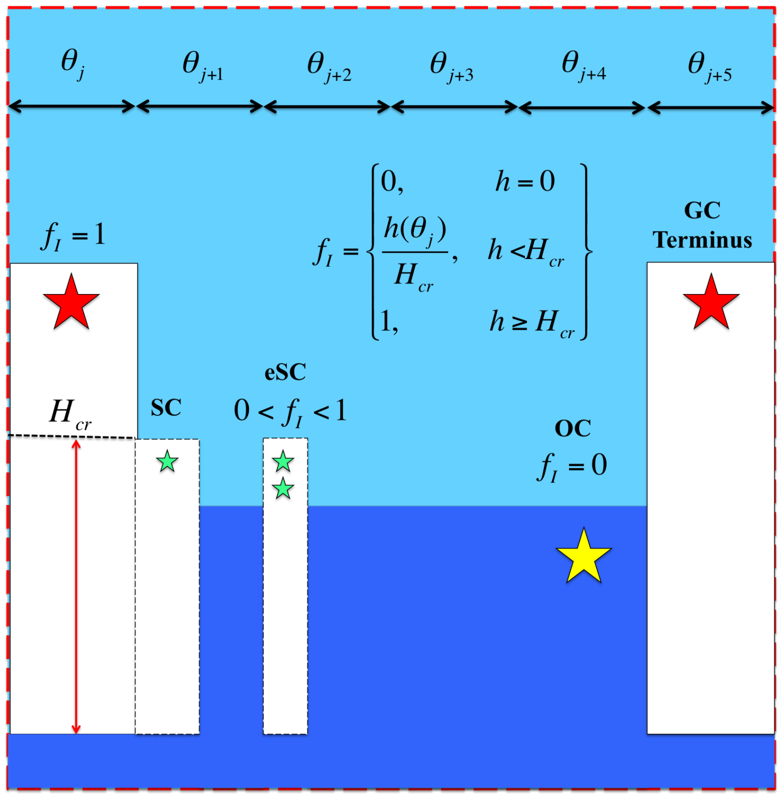

Figure 2: Example Scenario of Advance and Retreat. In the Northern Hemisphere, advance and retreat will occur in a SC. In the Southern Hemisphere, ice shelf advance will occur in the OC. By definition, ice shelf retreat does not occur in an OC because there is no ice to melt.

When the ice thickness in a grid cell drops below a critical height, H_{cr}, the cell height is set to zero, and $R \in (0, 1)$ in that cell.

8.3 Description of Ice Shelf Advance and Retreat

8.3.1 Mass Balance in a Subgrid Cell

In a SC in the Northern Hemisphere, the continuity equation simplifies to the following.

$$\frac{\partial h}{\partial t}|_{j+1} = S(\theta_{j+1}) - \frac{h_j}{R}\frac{\partial^{fd} v_j}{\partial \theta} = S(\theta_{j+1}) - \frac{h_j}{R}\frac{\cancelto{0}{v_{j+1}} - v_j}{\partial \theta} = S(\theta_{j+1}) + \frac{v_j h_j}{R \Delta \theta_{rad}}$$

where $j+1$ is the index of the SC, and j is the index of the terminus. If the advective term in the above equation is negative, then the term is set to zero.

In a SC in the Southern Hemisphere, the continuity equation simplifies to the following.

$$\frac{\partial h}{\partial t}|_{j-1} = S(\theta_{j-1}) - \frac{h_j}{R}\frac{\partial^{bd} v_j}{\partial \theta} = S(\theta_{j-1}) - \frac{h_j}{R}\frac{v_j - \cancelto{0}{v_{j-1}}}{\partial \theta} = S(\theta_{j-1}) - \frac{v_j h_j}{R\Delta\theta_{rad}}$$

where j is the index of the terminus, and $j-1$ is the index of the SC. If the advective term in the above equation is positive, then the term is set to zero.

8.3.2 Mass Balance in an Extraneous Subgrid Cell

In an eSC, the continuity equation simplifies to the following in both hemispheres.

$$\frac{\partial h}{\partial t}|_{j} = S(\theta_j)$$

where j is the index of the eSC. Thus, mass balance in an eSC is set by the source term.

8.3.3 Mass Balance in an Ocean Cell

In an OC in the Northern Hemisphere, the continuity equation simplifies to the following.

$$\frac{\partial h}{\partial t}|_{j+1} = S(\theta_{j+1}) - \frac{h_j}{R}\frac{\partial^{fd} v_j}{\partial \theta} = S(\theta_{j+1}) - \frac{h_j}{R}\frac{\cancelto{0}{v_{j+1}} - v_j}{\partial \theta} = S(\theta_{j+1}) + \frac{v_j h_j}{R\Delta\theta_{rad}}$$

where $j+1$ is the index of the OC, and j is the index of the terminus. If the advective term in the above equation is negative, then the term is set to zero.

In an OC in the Southern Hemisphere, the continuity equation simplifies to the following.

$$\frac{\partial h}{\partial t}|_{j-1} = S(\theta_{j-1}) - \frac{h_j}{R}\frac{\partial^{bd} v_j}{\partial \theta} = S(\theta_{j-1}) - \frac{h_j}{R}\frac{v_j - \cancelto{0}{v_{j-1}}}{\partial \theta} = S(\theta_{j-1}) - \frac{v_j h_j}{R\Delta\theta_{rad}}$$

where j is the index of the terminus, and $j-1$ is the index of the OC. If the advective term in the above equation is positive, then the term is set to zero.

Please note, **only subgrid advance is allowed in an OC**.

9 *Pollard and Kasting* (2005) Energy Balance Model

9.1 Prognostic Equation for Ocean Mixed-Layer Temperature:

$$\rho_o c_o h_o \frac{\partial T_o}{\partial t} = \frac{\rho_o c_o h_o D_o}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial T_o}{\partial \theta}\right] + Q_o + I_o + H_o - LE_o + G$$

where ρ_o is the uniform density of water, c_o is the specific heat of water, h_o is the ocean slab thickness, T_o is the ocean mixed layer temp, θ is the colatitude, D_o is the latitudinal diffusivity of ocean heat, Q_o is the net solar radiative fluxes absorbed by ocean, I_o is the net thermal radiative fluxes absorbed at the ocean surface, H_o is the sensible heat flux from atmosphere to the ocean, L is the appropriate latent heat constant, E_o is the evaporation rates from the ocean to atmosphere, G is the geothermal heat flux.

Note that ocean-ice interactions aren't included in this formulation.

Now, to use the adams bashforth method, we define

$$RHS(t_{n+1}, h_{n+1}) = \frac{1}{\rho_o c_o h_o}\left[\frac{\rho_o c_o h_o D_o}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial T_o}{\partial\theta}\right]|_j^{n+1} + Q_o|_j^{n+1} + I_o|_j^{n+1} + H_o|_j^{n+1} - LE_o|_j^{n+1} + G|_j^{n+1}\right]$$

$$RHS(t_n, h_n) = \frac{1}{\rho_o c_o h_o}\left[\frac{\rho_o c_o h_o D_o}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial T_o}{\partial\theta}\right]|_j^{n} + Q_o|_j^{n} + I_o|_j^{n} + H_o|_j^{n} - LE_o|_j^{n} + G|_j^{n}\right]$$

$$T_o|_j^{n+2} = T_o|_j^{n+1} + \frac{dt}{2}[3RHS(t_{n+1}, h_{n+1}) - RHS(t_n, h_n)]$$

9.2 Prognostic Equation for Atmospheric (near-surface) air temperature T_a,

$$\rho_a c_a h_a \frac{\partial T_a}{\partial t} = \frac{c_a h_a D_a}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\rho_a \sin\theta \frac{\partial T_a}{\partial\theta}\right] + Q_a + I_a + H_a + LP$$

where ρ_a is the density of the atmosphere, c_a is the specific heat of air, h_a is the atmospheric layer thickness for heat, D_a is the latitudinal diffusivity of atmospheric heat, I_a is the net thermal radiative flux absorbed by the atmosphere, H_a is the Sensible Heat Fluxes from the overall surface to the Atmosphere, and r is Earth's radius.

$$I_a = \sigma T_s^4 - 2\epsilon\sigma T_a^4$$

Now, to use the adams bashforth method, we define

$$RHS(t_{n+1}, h_{n+1}) = \frac{1}{c_a h_a}\left[\frac{c_a h_a D_a}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\rho_a \sin\theta \frac{\partial T_a}{\partial\theta}\right]|_j^{n+1} + Q_a|_j^{n+1} + I_a|_j^{n+1} + H_a|_j^{n+1} + LP|_j^{n+1}\right]$$

$$RHS(t_n, h_n) = \frac{1}{\rho_a c_a h_a}\left[\frac{c_a h_a D_a}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\rho_a \sin\theta \frac{\partial T_a}{\partial\theta}\right]|_j^{n} + Q_a|_j^{n} + I_a|_j^{n} + H_a|_j^{n} + LP|_j^{n}\right]$$

$$T_a|_j^{n+2} = T_a|_j^{n+1} + \frac{dt}{2}[3RHS(t_{n+1}, h_{n+1}) - RHS(t_n, h_n)]$$

We make the approximation that the atmosphere absorbs no shortwave radiation, i.e. $Q_a = 0$. When the ocean is exposed, $I_a = -I_o$. Similarly, with H_a and H_o.

9.3 Prognostic Equation for Atmospheric Specific Humidity (q_a)

$$\rho_a h_q \frac{\partial q_a}{\partial t} = \frac{h_q D_q}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left[\rho_a \sin\theta \frac{\partial q_a}{\partial \theta}\right] - (P - E)$$

where h_q is the atmospheric layer thickness for moisture, q_a is the specific humidity (nondimensional), D_q is the latitudinal diffusivity of atmospheric moisture, P is the precipitation rate and E is the evaporation rates from the overall surface to atmosphere.

Now, to use the adams bashforth method, we define

$$RHS(t_{n+1}, h_{n+1}) = \frac{1}{\rho_a h_q}\left[\frac{h_q D_q}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left[\rho_a \sin\theta \frac{\partial q_a}{\partial \theta}\right]|_j^{n+1} - (P - E)|_j^{n+1}\right]$$

$$RHS(t_n, h_n) = \frac{1}{\rho_a h_q}\left[\frac{h_q D_q}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left[\rho_a \sin\theta \frac{\partial q_a}{\partial \theta}\right]|_j^{n} - (P - E)|_j^{n}\right]$$

$$q_a|_j^{n+2} = q_a|_j^{n+1} + \frac{dt}{2}[3RHS(t_{n+1}, h_{n+1}) - RHS(t_n, h_n)]$$

10 Appendix A: Normal Vector to the Vertical Face of the Ice Shelf Terminus

We require a vector everywhere normal to the vertical face of the ice front. Following *Tziperman et al.* (2012), the normal vector to the vertical face of the terminus is given by the gradient of $G(\phi, \theta) = \theta - f(\phi)$, where $f(\phi)$ is a function that returns the colatitude of the terminus as a function of latitude.

$$\nabla = \left(\frac{1}{rsin(\theta)}\partial_\phi, \frac{1}{r}\partial_\theta, \partial_r\right)$$

$$\hat{n}_f = \frac{\nabla G}{\|\nabla G\|} = \frac{\left(-\frac{1}{rsin(\theta)}\partial_\phi f(\phi), \frac{1}{r}, 0\right)}{\| -\frac{1}{rsin(\theta)}\partial_\phi f(\phi), \frac{1}{r}, 0\|} \tag{26}$$

This formulation assumes a sheer vertical surface at the ice front to avoid a z-dependence in $f(\phi)$. Like *MacAyeal* (1997), this normal vector is required to lie in the horizontal plane.

$$\hat{n}_\theta = \left[0, 1, 0\right] \tag{27}$$

11 Appendix B: Tensor and Vector Operations

The dot product of a tensor with a vector is:

$$\bar{\bar{T}} \cdot \bar{v} = \bar{\delta}_1(T_{11}v_1 + T_{12}v_2 + T_{13}v_3) \\ + \bar{\delta}_2(T_{21}v_1 + T_{22}v_2 + T_{23}v_3) \\ + \bar{\delta}_3(T_{31}v_1 + T_{32}v_2 + T_{33}v_3)$$

where $\bar{\delta}_i$ are unit vectors. This treatment holds in spherical coordinates, with the exception of differential operators. These must be handled explicitly because in those cases $\bar{\delta}_i$ are not constant in direction.

Consider the above formula where $\phi = 1$, $\theta = 2$, $z = 3$. If $\bar{v} = \hat{n}_\theta$, then $v_1 = 0$, $v_2 = 1$, $v_3 = 0$.

$$\bar{\bar{\sigma}} \cdot \hat{n}_\theta = \bar{\delta}_\phi(T_{\phi\theta}v_\theta) \\ + \bar{\delta}_\theta(T_{\theta\theta}n_\theta) \\ + \bar{\delta}_z(T_{z\theta}n_\theta)$$

$$\bar{\bar{\sigma}} \cdot \hat{n}_\theta = \bar{\delta}_\phi(\sigma_{\phi\theta}) \\ + \bar{\delta}_\theta(\sigma_{\theta\theta}) \\ + \bar{\delta}_z(\sigma_{z\theta})$$

From the Approximations of the 1D Spherical Model, we note that $\sigma_{\phi\theta} = 0$ and $\sigma_{z\theta} = 0$.

$$\bar{\bar{\sigma}} \cdot \hat{n}_\theta = \bar{\delta}_\theta(\sigma_{\theta\theta})$$

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