Optimization in Economic Theory: Lagrange's Method

Francisco Treviño

April 22, 2016

Abstract

Exercises to practice Lagrange's optimization method.

Excersise 2.1: The Cobbs-Douglas Utility Function

Consider the consumer's problem with utility function U defined by

$$U(x, y) = x^{\alpha} y^{\beta} \tag{1}$$

Show that it yields the constant-budget-shared demand functions

$$x = \frac{\alpha I}{(\alpha + \beta)p}, \qquad y = \frac{\beta I}{(\alpha + \beta)q} \tag{2}$$

The budget constraint is

$$px + qy = I \tag{3}$$

We begin by constructing the Lagrangian L using the utility function and the constraint

$$L(x, y, \lambda) = x^{\alpha} y^{\beta} + \lambda(I - px - qy) \tag{4}$$

And now proceed with the partial derivatives, that involve implicit derivation, and optimize the results:

$$\frac{\partial L}{\partial x} = \alpha x^{\alpha-1} y^{\beta} - \lambda p = 0 \tag{5}$$

$$\frac{\partial L}{\partial y} = \beta x^{\alpha} y^{\beta-1} - \lambda q = 0 \tag{6}$$

$$\frac{\partial L}{\partial \lambda} = I - px - qy = 0 \tag{7}$$

If we clear for λ in the two first partial derivatives we have:

$$\lambda = \frac{\alpha x^{\alpha-1} y^{\beta}}{p} \tag{8}$$

$$\lambda = \frac{\beta x^{\alpha} y^{\beta-1}}{q} \tag{9}$$

So we can find a relationship between x and y doing some Algebra, as shown:

$$\frac{\alpha x^{\alpha-1} y^{\beta}}{p} = \frac{\beta x^{\alpha} y^{\beta-1}}{q} \tag{10}$$

Therefore, multiplying both sides by $x^{1-\alpha}$

$$\frac{\alpha y^{\beta}}{p} = \frac{\beta x y^{\beta-1}}{q} \tag{11}$$

and now, multiplying both sides by $y^{1-\beta}$

$$\frac{\alpha y}{p} = \frac{\beta x}{q} \tag{12}$$

From there we find x and y relationships:

$$x = \frac{\alpha q}{\beta p} y, \qquad y = \frac{\beta p}{\alpha q} x \tag{13}$$

Now lets substitute these, one at the time, in the Lagrange multiplier partial derivative 7, as follows:

$$I = px + qy \tag{14}$$

Lets begin with x:

$$I = px + q\frac{\beta p}{\alpha q} x \tag{15}$$

$$I = px + \frac{\beta p}{\alpha} x \tag{16}$$

$$I = px(1 + \frac{\beta}{\alpha}) \tag{17}$$

$$I = px \left(\frac{\alpha + \beta}{\alpha} \right) \tag{18}$$

So we lastly solve for x:

$$x = \frac{\alpha I}{(\alpha + \beta) p} \tag{19}$$

Now we follow the same process for y:

$$I = p\frac{\alpha q}{\beta p} y + qy \tag{20}$$

$$I = qy\left(\frac{\alpha}{\beta} + 1\right) \tag{21}$$

$$I = qy\left(\frac{\alpha+\beta}{\beta}\right) \tag{22}$$

Solving for y, we have:

$$y = \frac{\beta I}{(\alpha+\beta)q} \tag{23}$$

So we proved that the Utility function yields the constant-budget-shared demand functions 2

Lastly, we find the lagrange multiplier, lets use a previously obtained result, from the Lagrangian partial derivatives:

$$\frac{\partial L}{\partial x} = \alpha x^{\alpha-1}y^{\beta} - \lambda p = 0 \tag{24}$$

There we substitute the results obtained for x and y, 19 and 23 respectively, and so we have:

$$\alpha\left(\frac{\alpha I}{(\alpha+\beta)p}\right)^{\alpha-1}\left(\frac{\beta I}{(\alpha+\beta)q}\right)^{\beta} - \lambda p = 0 \tag{25}$$

$$\lambda = \frac{\alpha}{p}\left(\frac{\alpha I}{(\alpha+\beta)p}\right)^{\alpha-1}\left(\frac{\beta I}{(\alpha+\beta)q}\right)^{\beta} \tag{26}$$

And finally:

$$\lambda = \frac{\alpha^{\alpha}\beta^{\beta}I^{\alpha+\beta-1}}{(\alpha+\beta)^{\alpha+\beta-1}p^{\alpha}q^{\beta}} \tag{27}$$

.

Exercise 2.2: The Linear Expenditure System

Let the utility function be defined as:

$$U(x,y) = \alpha\ln(x-x_0) + \beta\ln(y-y_0) \tag{28}$$

where x_0, y_0 are given constants, and

$$\alpha + \beta = 1 \tag{29}$$

The budget constraint is

$$px + qy = I \tag{30}$$

Show that the optimal expenditures on the two goods are linear functions of income and prices.

The utility function brings with it a rich range of possible optimum choice. The budget shares of the two goods can now vary systematically with income and prices. One good can be a necessity and the other a luxury (but neither good can be inferior since α and β must be positive). But the expenditures still have a simple functional form. For these reasons, this specification was popular in the early empirical work on consumer demand.

Lets build the Langrangian:

$$L(x, y, z) = \alpha \ln(x - x_0) + \beta \ln(y - y_0) + \lambda(I - px - py) \tag{31}$$

And now calculate the partial derivatives, the optimal points that is:

$$\frac{\partial L}{\partial x} = \frac{\alpha}{x - x_0} - \lambda p = 0 \tag{32}$$

So we solve for x:

$$\alpha = \lambda p(x - x_0) \tag{33}$$

therefore:

$$x = \frac{\alpha}{\lambda p} + x_0 \tag{34}$$

Optimum for y:

$$\frac{\partial L}{\partial y} = \frac{\beta}{y - y_0} - \lambda q = 0 \tag{35}$$

We now solve for y:

$$\beta = \lambda q(y - y_0) \tag{36}$$

therefore:

$$y = \frac{\beta}{\lambda q} + y_0 \tag{37}$$

Optimum for λ:

$$\frac{\partial L}{\partial \lambda} = I - px - qy = 0 \tag{38}$$

Lets substitute the found expressions of x and y in the last equation:

$$I - p\left(\frac{\alpha}{\lambda p} + x_0\right) - q\left(\frac{\beta}{\lambda q} + y_0\right) = 0 \tag{39}$$

and do simplify:

$$I - \frac{\alpha}{\lambda} - px_0 - \frac{\beta}{\lambda} - qy_0 = 0 \tag{40}$$

$$\frac{1}{\lambda}(\alpha + \beta) = I - px_0 - qy_0 \tag{41}$$

Hence, due to restriction given above:

$$\lambda = 1/(I - px_0 - qy_0) \tag{42}$$

We now proceed to find two expressions for λ in terms of x and y and find the value of them in terms of the parameters.

From the partial derivative of the Lagrangian with respect to x we can easily see that

$$\lambda = \frac{\alpha}{p(x - x_0)} \tag{43}$$

we can equal this equation with 42:

$$\frac{\alpha}{p(x - x_0)} = \frac{1}{(I - px_0 - qy_0)} \tag{44}$$

$$px = \alpha I - \alpha px_0 - \alpha qy_0 + px_0 \tag{45}$$

$$px = \alpha I - \alpha qy_0 - px_0(\alpha - 1) \tag{46}$$

Remember the restriction 29, so we have:

$$px = \alpha I - \alpha qy_0 + \beta px_0 \tag{47}$$

From the partial derivative of the Lagrangian with respect to y we follow the same procedure as we did with x:

$$\lambda = \frac{\beta}{q(y - y_0)} \tag{48}$$

we can equal this equation with 42:

$$\frac{\beta}{q(y - y_0)} = \frac{1}{(I - px_0 - qy_0)} \tag{49}$$

$$qy = \beta I - \beta px_0 - \beta qy_0 + qy_0 \tag{50}$$

$$qy = \beta I - \beta px_0 - qy_0(\beta - 1) \tag{51}$$

Remember the restriction 29, so we have:

$$qy = \beta I - \beta px_0 + \alpha qy_0 \tag{52}$$

So we now have both Optimal expenditures for x and y