1 Physical Motivation

In this section I aim to motivate the use of eigen vectors in spectral community detection by taking of from interpreting the laplacian as the second derivative operator $(\nabla \cdot \nabla) = \nabla^2$:

First lets start of with a vector $\underline{u}(t)$ where each component $u_i(t)$ represents the amount of energy or heat in a node. Additionaly we allow edges to permit heat flow in between nodes thus if one attempts to model the evolution of the system (of N nodes) with respect to determine the rate of $u_i(t)$ as the following:

$$\frac{d}{dt}u_i(t) = -\kappa \sum_{j=1}^{N} A_{ij}(u_i(t) - u_j(t))$$

Where :

$$A_{ij} = \begin{cases} 1 & when\ i \text{ is coneccted to } j \\ 0 & otherwise \end{cases}$$

Distributing the sum operator:

$$\frac{d}{dt}u_i(t) = -\kappa \left(u_i(t) \sum_{j=1}^{N} A_{ij} - \sum_{j=1}^{N} A_{ij} u_j(t) \right)$$

We know that if we sum up row 1 in matrix A we find out the number of nodes incident on node 1 which we define as the degree of node 1. Generlazing this to i follows trivially:

$$\frac{d}{dt}u_i(t) = -\kappa \left(u_i(t) deg(i) - \sum_{j=1}^{N} A_{ij} u_j(t) \right)$$

Since $A_{ii} = 0$ (A node connected to itself is trivial in our system and thus it is not accounted/allowed) we can use the kronecker delta symbol to rewrite this more compactly:

$$\delta_{ij} = \begin{cases} 1 & when\ i = j \\ 0 & otherwise \end{cases}$$

$$\frac{d}{dt}u_i(t) = -\kappa \sum_{j=1}^{N} (\delta_{ij} deg(i) - A_{ij} u_j(t))$$

Which we can compact in to the vector matrix form (Where D is the diagonal degree matrix):

$$\frac{d}{dt}\underline{u}(t) = -\kappa \underbrace{(D - A)}_{L} \underline{u}(t)$$

This gives us a first order linear system of differential equations (similar to the heat equation if $L = \nabla^2$). Due to basic properties of matrix algebra solving this equation is rather trivial. I will provide a run through solution given that we ignore initial conditions for simplicity.

$$\frac{d}{dt}\underline{u}(t) = -\kappa L \underline{u}(t)$$

Lets guess the following ansazt to this ODE (common method in physics) as $\underline{v}e^{-\kappa\lambda t}$ where $\underline{v}$ is some constant vector. Plugging in to the heat equation (and cancelling out equal terms) we now have:

$$\lambda \underline{v} = L \underline{v}$$

Which we recognize as a typical eigen value eigen vector equation. Which we can solve easily but the important thing I would like to comment on is what the solution means about our system. The different eigenvalues λ (also known as the spectrum of L) desribe how every different solution (mode) to our heat equation changes dynamically since our solutions are of the form : $\underline{v}_i e^{-\kappa\lambda_i t}, \quad i \in N$ so bigger lambdas may imply that our system is experiencing a decaying transfer of energy in time the eigen vector itself also tells us about the dynamics of the energy transfer since each of its component represents a node and in contrast to the others it tells us how the coupled system is simultaneously transfering energy (i.e. One node giving the other loosing , energy transfer in synch etc..). Thus in a sense the eigen vectors by describing the different possible dynamics of energy transfer in the system tell us different things about connectivity in the graph and thus can help us find elements that may be clique like meaning that they all like to be conected to each other like a clique of friends.