Theorem 0.1.

$$G = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$$

Proof.

$$\begin{aligned} G^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dxdy \end{aligned}$$

Substitute $r = x^2 + y^2$, $r\sin\theta = y$, $r\cos\theta = x$, hence $dxdy = rdrd\theta$

$$\begin{aligned} G^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2}\,rdrd\theta \\ &= \left[\theta \int_0^{\infty} re^{-\frac{1}{2}r^2}\,dr\right]_{\theta=0}^{\theta=2\pi} \\ &= 2\pi\left[-e^{-\frac{1}{2}r^2}\right]_0^{\infty} \\ &= 2\pi \end{aligned}$$

Hence $G = \sqrt{2\pi}$ □

Theorem 0.2. *For a^2 real and positive.*

$$I = \int_{-\infty}^{\infty} e^{-a^2x^2+bx+c}\,dx = \frac{\sqrt{\pi}}{a}e^{c+\frac{b^2}{4a^2}}$$

Proof. First, "complete the square" in the exponent by defining r, s and t by the equation:

$$-a^2x^2 + bx + c = -\frac{(rx+s)^2}{2} + t$$

This gives us:

$$\begin{aligned} -r^2 &= -2a^2 & &\Rightarrow r = a\sqrt{2} \\ -rs &= b & &\Rightarrow s = \frac{-b}{a\sqrt{2}} \\ t - \frac{s^2}{2} &= c & &\Rightarrow t = \frac{b^2}{4a^2} \end{aligned}$$

Using the completed square in the integral I:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-\frac{(rx+s)^2}{2}+t}\,dx & (1) \\ &= e^t\int_{-\infty}^{\infty} e^{-\frac{(rx+s)^2}{2}}\,dx & (2) \end{aligned}$$

By assumption a is real and positive, hence r is also real and positive. We can therefore substitute $y = rx + s$ without changing the limits of the integration. With this substitution $\frac{dy}{r} = dx$.

$$= e^t \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{r} \tag{3}$$

$$= \frac{e^t}{r} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \tag{4}$$

Applying theorem 0.1 we have

$$= \frac{\sqrt{2\pi}}{r} e^t \tag{5}$$

Subsituting in for r and t

$$= \frac{\sqrt{\pi}}{a} e^{c + \frac{b^2}{4a^2}} \tag{6}$$

$\square$