Book I

Easy Concepts

Chapter 1

Introduction

What physics is about. All physics is wrong. Not using calculus. Concepts. Depth. Pre-requisites. Circularity.

Part I

kinematics

Chapter 2

Numbers

Chapter 3

Space (Distance)

Volume. Area. Area of a circle (without calculus). Volume of a sphere. Pi. PIck's Theorem (in the extras). Pythagoras's theorem. Working out some distances.

Chapter 4

Angles

Meaning of an angle. Radians [other measures, degrees etc]. Solid angles.

Chapter 5

Time

Chapter 6

Dimensions

Chapter 7

Change: Speed etc

Jerk. Jounce. [snap, crackle, pop] Look forward: differential calculus.

Chapter 8

Spacetime

Part II

Dynamics

Chapter 9

Introduction

Chapter 10

Equations

Chapter 11

Momentum

Chapter 12

Force

Pressure. Gravity. Electrostatics. Springs.

Chapter 13

Mass

Chapter 14

Work

Part III

Thermodynamics

Chapter 15

Temperature

Chapter 16

Heat

Chapter 17

Energy

Chapter 18

Entropy

Why things go. Melting of ice.

Chapter 19

Heat Engines

19.1 The physics

Our scientists

Set up situation.

Train drivers passing by each other. Really very fast trains. Actually had better be spaceships. With velocity v. Velocity could be measured in anything so we may need a conversion factor.

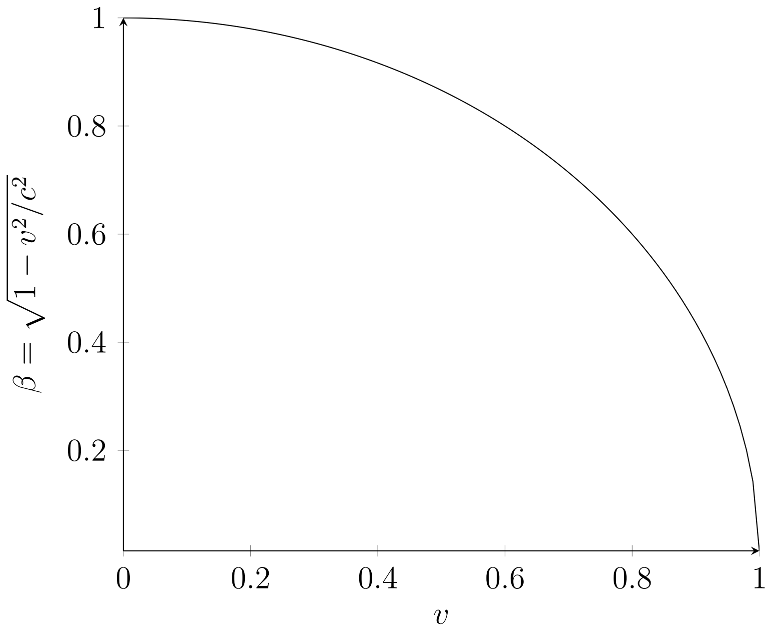
Length contraction

Each sees the other's measuring rods shrunk along the axis of movement by a factor of $\sqrt{1 - \frac{v^2}{c^2}}$. This shrinkage factor is so important that it has its own name β (it is also much easier to write β in what follows. Here c stands for a conversion factor $299792458ms^{-1}$, which makes $\frac{v^2}{c^2}$ dimensionless. It did not matter what we used. What c means will make sense in a moment (forward ref).

(Fitzgerald contraction)

Note that because β depends on v^2 it is the same for positive and negativ velocities. In other words both scientists see the same shrinkage of the other. You very occasionally hear it said "rods contract if you move faster" but that *s*ounds like it would be one-sided. How can both scientists think the other one's rod is contracting, who is right (neither?), what is going on. We aren't seeing the whole picture.

As velocity increases and gets closer to c then β tends to zero. (see graph).



Time dilation

In 1971, four atomic clocks were flown around the world: two Westwards and two Eastwards. On their return they were compared with two clocks that had been left at the US Naval Observatory. The Eastward clocks – flying with the rotation of the Earth had lost about 59ns (59 x 10⁻9s). By contrast the Westward clocks – flying against the rotation of the Earth had gained about 273 nanoseconds and so, on average, had run faster.

The second discrepancy the two scientists notice is one of time.

Book II

Grown-up Physics

Chapter 20

Astronomy

Sphere. Stars. Spherical model. No distance yet. Sun. Orbits.

Chapter 21

Elementary trigonometry

We have met the ideas of angles and distances. The subject of trigonometry allows us to convert between the two. For example, if we know that the wheel of a steam engine has turned one eight of a complete turn ($\pi/4$ radians) then how much forward or backward movement will the piston of the engine have moved?

"Trigonometry" literally means (from Greek) the measurement of triangles ("trigons" though we don't call them that). In a way the name undersells its usefulness. In a sense you can use trigonometry even if there are no triangles around, although of course if there are at least three points in any problem you can always draw a triangle in to connect them.

Let us start with a simple example of the problem that we want to solve. Let us imagine that the circle depicted in figure 21 is vertical. O is the centre of the circle. The line passing horizontally through the circle we can take, arbitrarily, as our zero height line. If it helps to think of it as "ground level" then that is fine. The line OP is something that is rotating in that vertical plane. It might be an abstraction of (say) the arm of a crane, or a rocket launching vehicle.

Let us suppose that the arm has rotated through an angle θ. How high will the end of the arm (point P) be above the baseline.

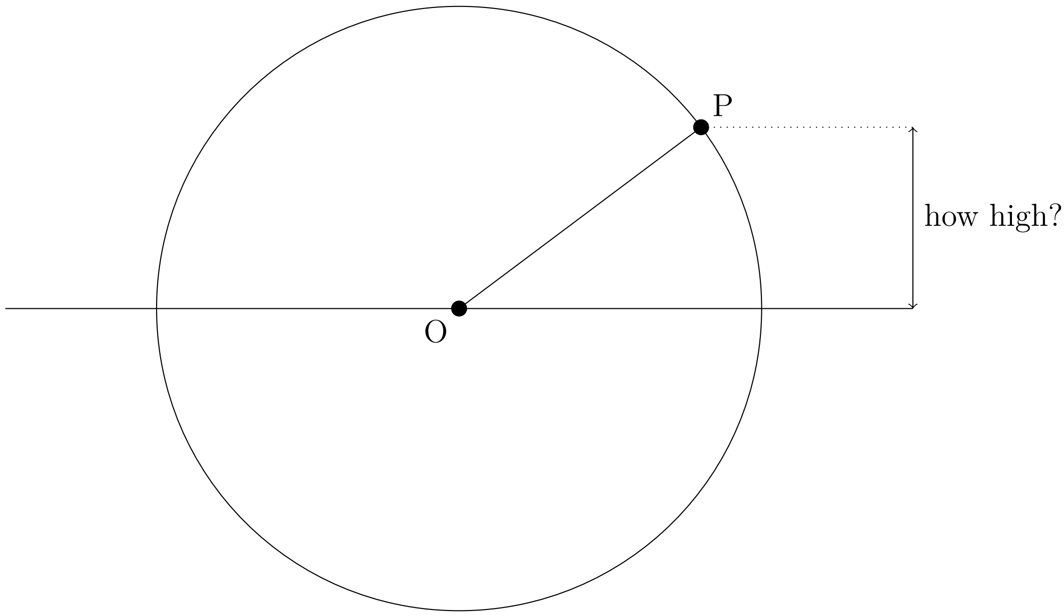


Figure 21.1: How high is P?

Well without more information this question is unanswerable because it depends on the size of the circle as well as the angle of OP from the "horizontal".

As it happens the size of the circle is not really critical, because if we scale up the size of the circle, we scale the height of P by the same amount. We might just as well work out the example for a circle with radius of 1. Once we have the answer, we can just multiply it by the radius if the circle we are interested in.

If you aren't happy with the idea that everything just scales up, I will prove it in the end notes of this chapter using similar triangles.

Even though we don't (yet) know exactly how the height of P behaves as it moves around the circle, there are a few things we do know. Let us draw a picture with four points (A, B, C, D) marked around it at the obvious four cardinal points of the circle and consider what happens to P's height as it travels around the circle. Mathematicians traditionally measure angles anticlockwise[1].

[1] why?

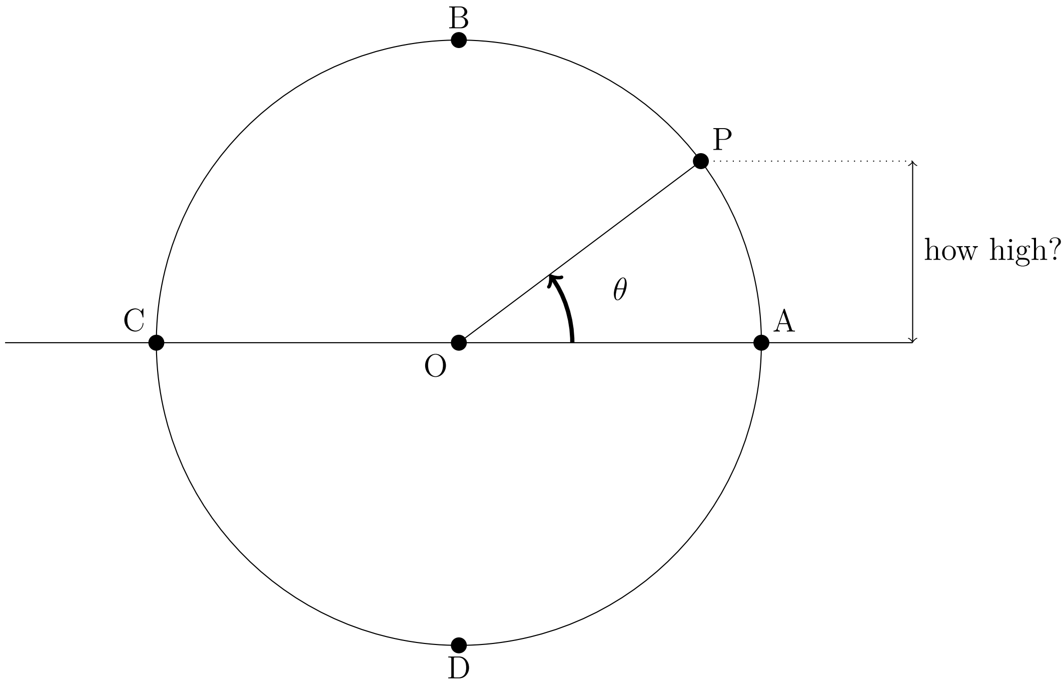


Figure 21.2: P's journey around the circle

When P starts out at A, it clearly has a height of zero. As it rotates anticlockwise, the height will increase until it reaches a maximum at B. At this point $\theta = \pi/2$radians. Since the circle has radius 1 (known in the trade as a "unit circle"), that maximum is 1.

Further round the circle, between $\pi/2$ and π radians, the height declines until at C (where $\theta = \pi$) the height is zero.

Further around the circle the arm is going to go below ground level. We can imagine that it is able to do so because someone has dug a ditch (not illustrated). Maybe it is the digging arm of a digger doing just that. The sensible thing to do here is treat the height of P as negative, because it is below our chosen zero.

Thus P then becomes negative from $\theta = \pi$ until $\theta = 3\pi/2$ at which point it has reached its minimum height -1. After this P begins to move back up until it arrives back at A. It should be clear that if θ grows beyond 2π the whole process will simply be repeated.

If we draw a graph of the height of P against the angle, this pattern becomes clearer.

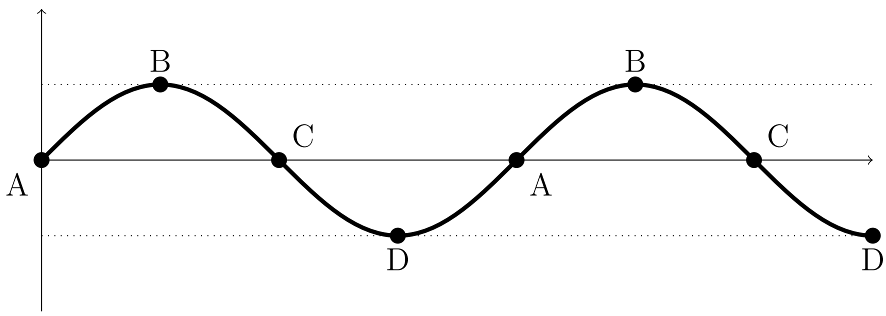


Figure 21.3: The true sine - a folding line

Figure (21) shows a graph of height against angle. The rather beautiful shape of the line inspired an Arabic name for the function *j*aib meaning a fold. Translated into Latin by mediaeval writers this became *s*inus. In modern mathematical writing this has become "sine" and the sine of the angle θ is usually written $sin\theta$.

Many graphs of sine (in textbooks too I am afraid) deliberately stretch the y-axis so as to butch up the curve to make it look like a steeper wave, but figure (21) is as near to scale as I can make it and illustrates just how smoothly the curve rises and falls. It is one of the most beautiful shapes in mathematics or physics.

Armed with this new name, we can answer the question in figure 21.

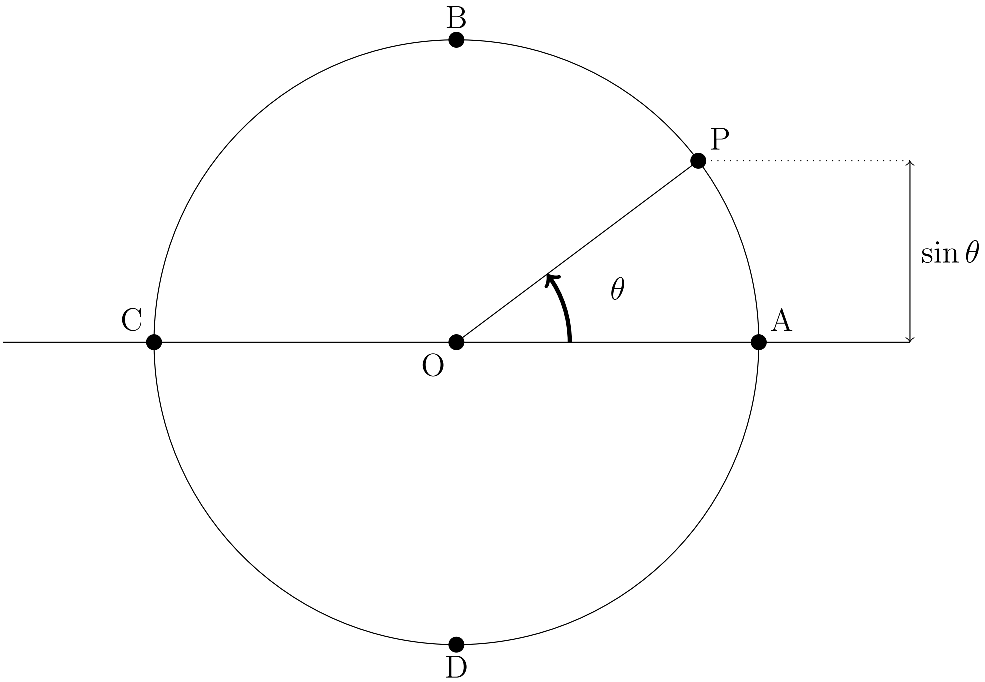


Figure 21.4: The definition of sine

Another equally interesting distance to work out would be how far P has stretched out horizontally. So useful was this value that it was given a name the "cosine" because it is a companion of the sine.

Cosine is in fact our friend since in disguise. If you turn figure ???? on its side you can see that cosine is, in some sense, the same function as sine only $\pi/2$ ahead of it. This becomes clear if we plot cosine and sine on the same graph;

It should be clear from the graph (and the definition) that

$$\sin(\theta) = \cos(\theta + \pi/2)$$

The Tangent

$$\sec^2 + 1 = \tan^2$$

Obscure trigonometrical functions

Ptolemy. Hipparchian. Copernican. Kepler.

Chapter 22

Differential Calculus

Chapter 23

Dynamics

Chapter 24

Universal Gravitation

24.1 Elementary Elliptic Functions

Introduction

This is advertising: elliptic functions are a class of function that turn up in many places in mathematics and physics. To a mathematician, they are very beautiful: to a physicist they are also potentially useful.

The full definition of elliptic functions requires complex numbers.[1] However, one important subclass of elliptic functions known as the "Jacobi elliptic functions" (named after Carl Gustav Jakob Jacobi) can be explained quite simply with some geometry.

Roughly speaking, Jacobi elliptic functions do for ellipses what trigonometrical functions like sine and cosine do for circles. In the same way that all periodic functions are, in some sense, made of sines and cosines (see Fourier Series). For our purposes ellipses are slightly wonky circles and so Jacobi elliptic functions will seem like slightly wonky trigonometrical functions. A lot more clever stuff can be found in the article on advanced Elliptic functions.

The Jacobi Elliptic Functions

When we defined the elementary trigonometrical functions, we drew a picture which related the x and y positions of a rotating point (the end of a mechanical arm, or a planet rotating in a circular orbit) to the angle the point moved through, as illustrated in figure 24.1.

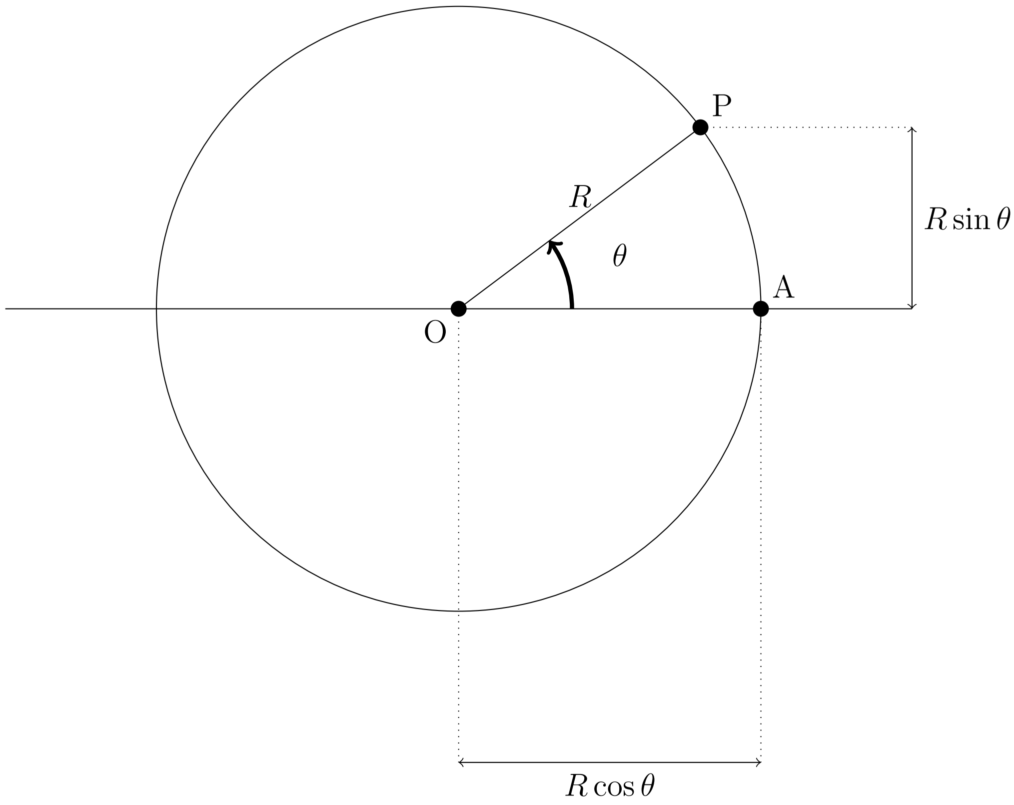


Figure 24.1: The definition of sine and cosine

Drawing a similar picture for an ellipse is not quite so easy. For a point P (representing, perhaps, a planet in a circular orbit) moving around a cricle, the angle through which is has moved θ is always proportional to the length of the arc along which it has travelled.

$$arc(AP) = R\theta$$

.

It doesn't really matter whether we think about the arc or the angle, they are both essentially the same thing. For a unit circle, they are identical except that the angle is dimensionless.

For an ellipse distance OP changes as P goes around the ellipse. This means that the relationship between the arc AP and the angle through which P has turned becomes a more complicated one. It turns out to be easier to start with the arc length AP rather than the angle θ.

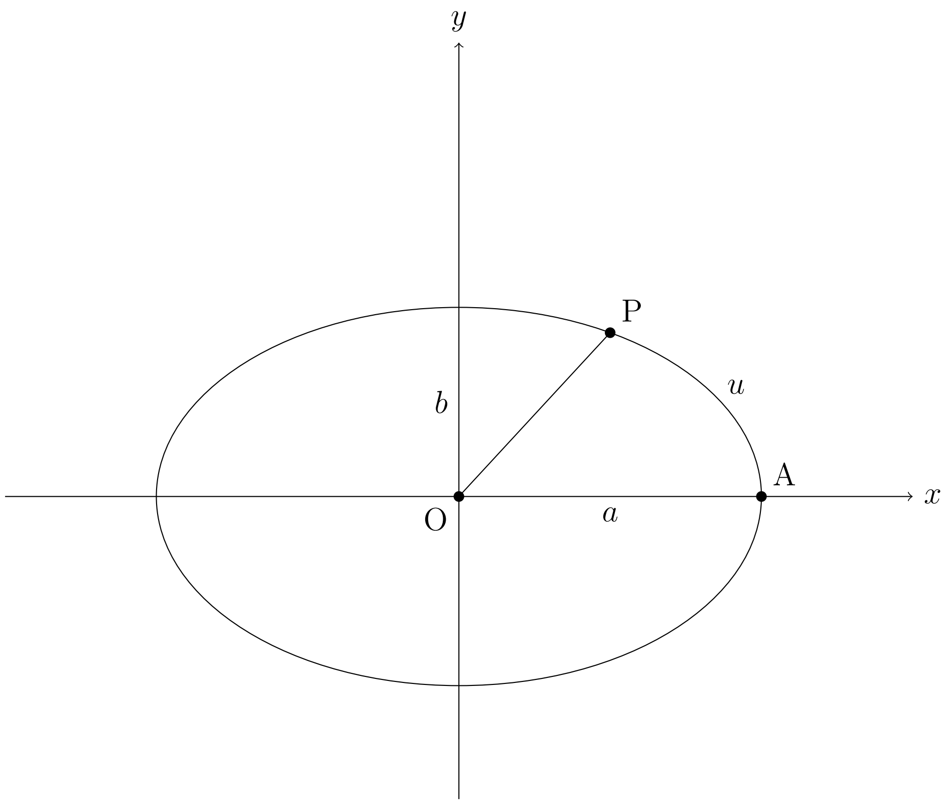


Figure 24.2: The ellipse

In the field of elliptic functions the arc length is usually written u. It is in terms of this arc length that we define our first elliptic functions as illustrated in figure (24.1).

Parametrising the ellipse

Recalling that a circle of radius R may be parametrised using sine and cosine by $(Rcos(\phi), Rsin(\phi))$. Can we do something like that with an ellipse, such as the one depicted in figure 24.1, with semi-minor axis a and semi-major axis b? To keep the mathematics simple, let us use Cartesian co-ordinates with an origin at the centre of the ellipse.

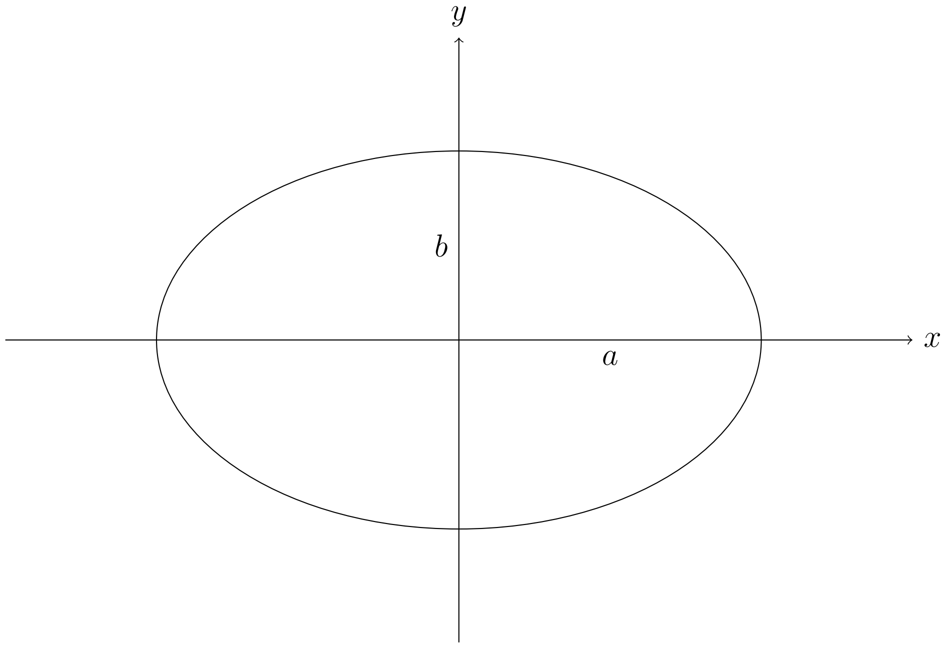


Figure 24.3: Our typical ellipse

Recall from the section on conic sections that such an ellipse is defined by the formula:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A little bit of algebra (and recalling some trigonometrical identities) then gives us a parametrisation for our ellipse:

$$(a\cos(\phi), b\sin(\phi))$$

It is tempting to think there must be a right angle-triangle with ... But what is ϕ?

24.2 End notes

Notes

[1]If you do know about complex numbers, you should know about the complex plane. An "elliptic function" is one which is periodic in two different directions in the complex plane. Elliptic functions are "doubly periodic" by contrast with trigonometrical functions like sine and cosine which are singly periodic.

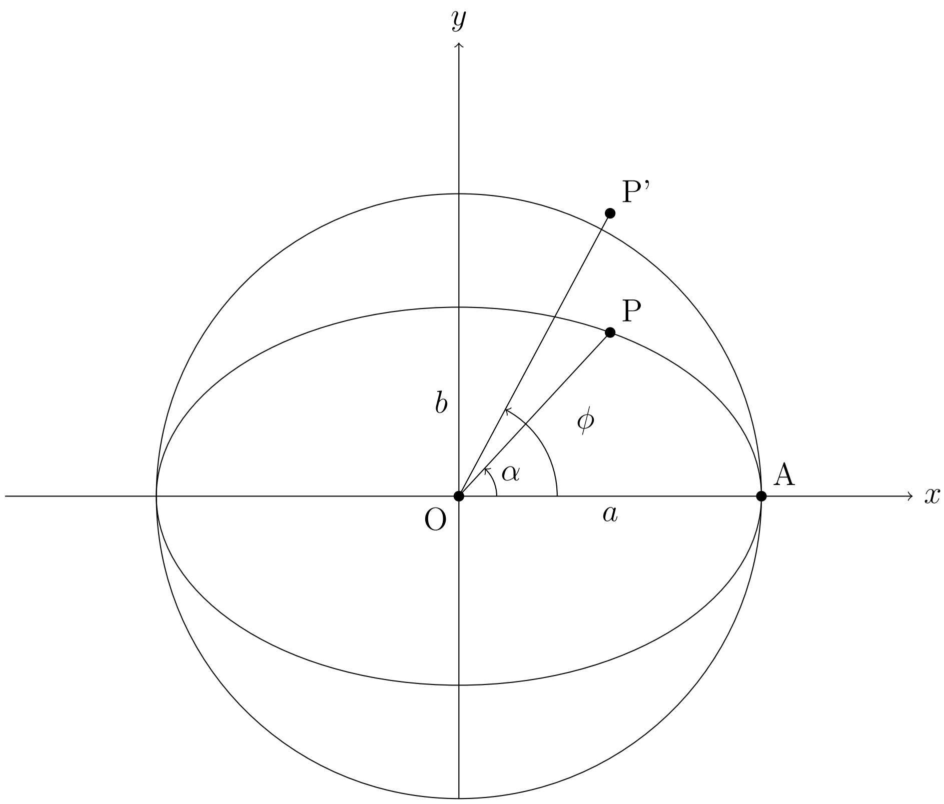


Figure 24.4: The ellipse

Trajectories. Orbits. Rockets (variable mass). Interplanetary network. Lots of fun with calculus. Zarg's law of space combat.

Chapter 25

Integral Calculus

Part IV

Level II

Chapter 26

potential theory

Chapter 27

exterior algebra

Chapter 28

electrodynamics

Circuit Theory. Optics

Chapter 29

phase space

Chapter 30

lagrangian dynamics

Chapter 31

hamiltonian dynamics

Chapter 32

Hamilton-Jacobi

Chapter 33

Chaos

Chapter 34

Fractals

Chapter 35

Special Relativity

Chapter 36

Quanum Mechanics

Chapter 37

Quantum Chemistry

Chapter 38

Quantum Electronics

Chapter 39

Statistical Thermodynamics

Chapter 40

Fluid Dynamics

Chapter 41

Elasticity

Chapter 42

Elliptic Functions and integrals

Prerequisites: **c**omplex analysis

42.1 Introduction

The fact that sine and cosine are periodic, with period 2π should be familiar to you. That fact turns out to be enormously useful all over mathematics. Wherever anything periodic turns up such as waves or relationships between circular and linear motion turns up we found it useful.

But it turns out that sine and cosine are very interesting when viewed from the point of view of complex numbers. In one sense, they are both aspects of the exponential function, the most famous transcendental function. In another sense they are the hyperbolic functions sine and cosine sinh and cosh "rotated" through $\pi/2$.

But sine and cosine are periodic in one direction - parallel to the real line. What about functions that have two independent periods? Such functions are known as "elliptic functions".

By "independent" periods, I mean "linearly independent". Sine has a period of 2π but also of $4\pi, 6\pi, 8\pi, \ldots$ but the additional periodos aren't very interesting. They are all real and so, if considered as 2D vectors in the complex plane, they all point in the same direction.

To put it formally, an elliptic function is one with two non-zero[1] periods ω_1 and ω_2 such that there is no real number r with $\omega_1 = r\omega_2$.

[1] All functions have a "period" of zero

Just this amount of information (and no more) allows us to deduce lots of properties that elliptic functions must have. That is where I will start. There are then three ways to get to concrete examples of elliptic functions, all of which I shall try to do;

- We could discover them by realising that functions we already know about, the Jacobi elliptic functions are elliptic functions (the clue is in the name, but proving it requires some work). In fact what we will start with Jacobi elliptic functions and discover elliptic integrals.
- Then we will realise that elliptic integrals define elliptic functions.
- As a different tactic, we start from the definition of elliptic function and build a function from scratch that matches that definition. This is the approach of Weierstrass.

Elliptic things include:

- Elliptic Functions
- Elliptic Integrals
- Elliptic Curves

There is an odd history to these names. Elliptic integrals were discovered first[2]. Elliptic functions were then defined in terms of elliptic integrals and their properties derived from the integrals. Viewed from the perspective of complex analysis, elliptic curves are the curves associated with elliptic functions, so they inherited the name, even though they are not in fact ellipses.

We have, so far, taken a different approach. We met Jacobi Elliptic functions to help us solve some problems with ellipses and non-linear differential equations.

[2] is this true?

42.2 Jacobi Elliptic Functions and Integrals

Point P has position $(a\cos(\phi), b\sin(\phi))$. Point P' is vertically above P and is known as the "corresponding point". It has position, $(a\cos(\phi), a\sin(\phi))$. First we obtain a useful formula relating α and ϕ.

$$
\begin{aligned}
\tan\alpha &= \frac{b}{a}\tan\phi && \text{elementary trigonometry} && (42.1)\\
\tan^2\alpha &= \frac{b^2}{a^2}\tan^2\phi && \text{squaring} && (42.2)\\
\sec^2\alpha &= 1 + \frac{b^2}{a^2}(\sec^2\phi - 1) && \text{using trig identities} && (42.3)\\
\sec^2\alpha &= (1 - \frac{b^2}{a^2}) + \frac{b^2}{a^2}(\sec^2\phi - 1) && \text{rearranging} && (42.4)\\
\sec^2\alpha &= k^2 + \frac{b^2}{a^2}(\sec^2\phi - 1) && \text{substituting definition of k} && (42.5)\\
& && && (42.6)
\end{aligned}
$$

Next we need a formula defining the differential of α in terms of ϕ.

$$
\begin{aligned}
\sec^2\alpha\, d\alpha &= \frac{b}{a}\sec^2\phi\, d\phi && \text{differentiating (42.1)} && (42.7)\\
d\alpha &= \frac{b}{a}\frac{\sec^2\phi}{\sec^2\alpha}\, d\phi && && (42.8)\\
&= \frac{b}{a}\frac{\sec^2\phi}{k^2 + \frac{b^2}{a^2}(\sec^2\phi - 1)}\, d\phi && \text{substituting (42.5)} && (42.9)\\
&= \frac{b}{a}\frac{d\phi}{k^2\cos^2\phi + \frac{b^2}{a^2}} && \text{multiply through by } cos^2\phi && (42.10)\\
&= \frac{b}{a}\frac{d\phi}{(1 - k^2\sin^2\phi)} && \text{multiply through by } cos^2\phi && (42.11)\\
& && && (42.12)
\end{aligned}
$$

Now to calculate the arc length, we start with an easy formula in α and then transform it into an integral involving ϕ.

$$
\begin{aligned}
ds &= \int r\, d\alpha && (42.13)\\
&= \int \sqrt{a^2 \cos^2\phi + b^2 \sin^2\phi}\, d\alpha && (42.14)\\
&= \int a\sqrt{1 - k^2 \sin^2\phi}\, d\alpha && (42.15)\\
&= \int \frac{b}{a} \frac{a\sqrt{1 - k^2 \sin^2\phi}}{(1 - k^2 \sin^2\phi)}\, d\phi && (42.16)\\
&= \int b \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} && (42.17)\\
& && (42.18)
\end{aligned}
$$

Book III

Advanced Physics

Part V

General Relativity

Part VI

Cosmology

Book IV

Cutting-Edge Physics?

Part VII

Reference

42.3 The Circle

42.4 Circular Functions (trigonometry)

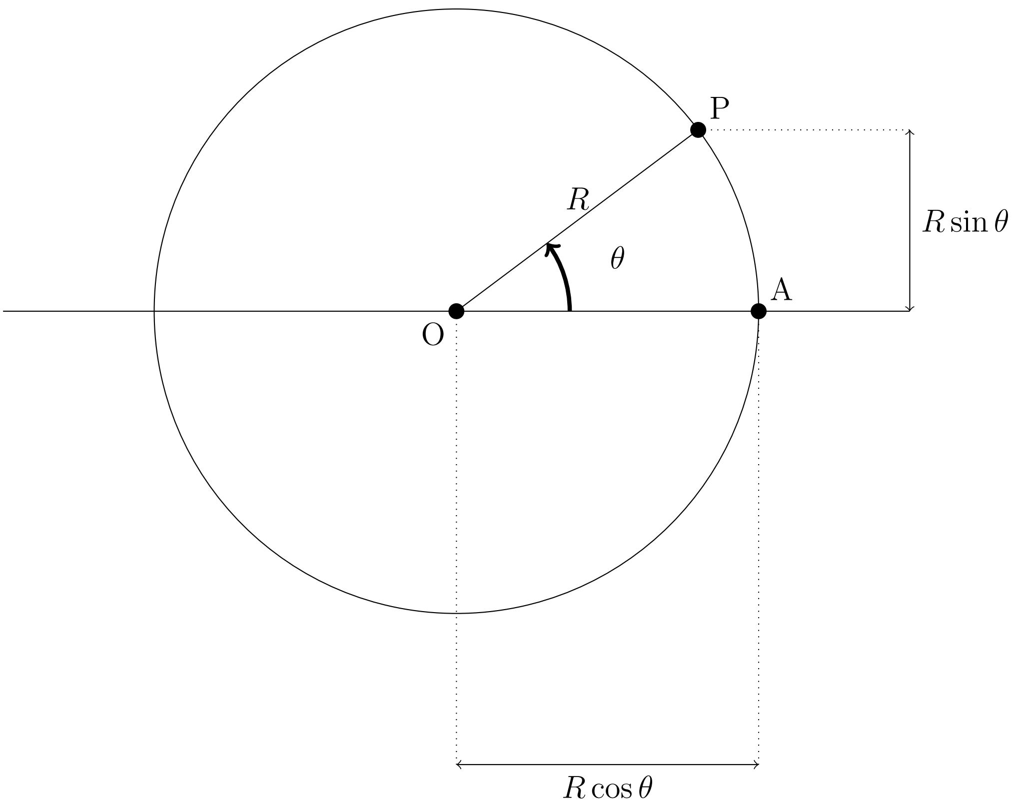


Figure 42.1: The definition of sine and cosine

Trigonometric functions may be written with or without parentheses $sin\theta$, $sin(\theta)$ and so on.

42.5 The Ellipse

The two foci are marked as F_1 and F_2. The two points at the ends of the majoraxis (V_1 and V_2) are sometimes refeerred to as "vertices" while the two points at the ends of the minor axis (V_3 and V_4) are referred to as "co-vertices".

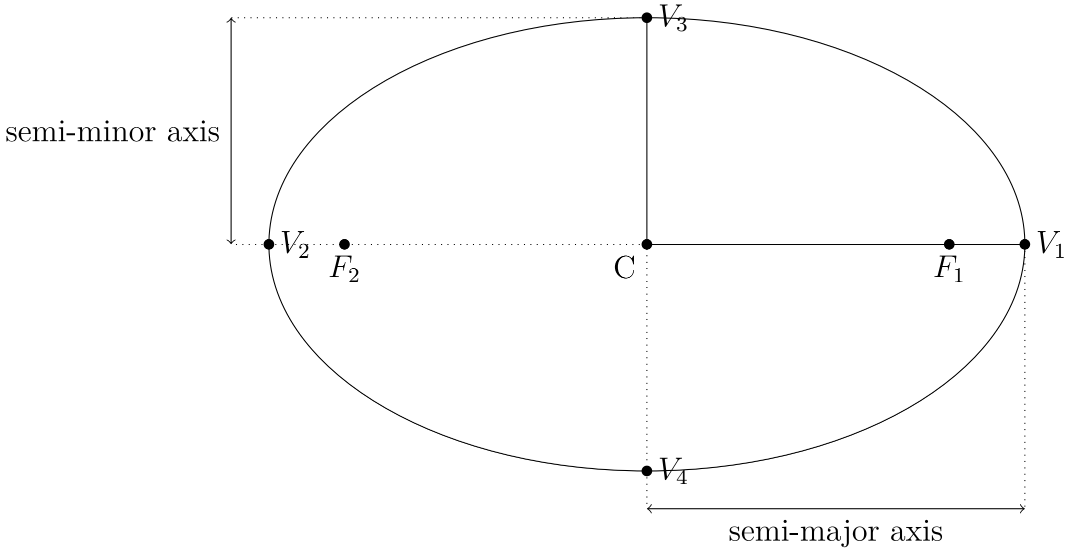


Figure 42.2: Ellipse dimensions

$CF_1 = CF_2$	linear eccentricity	c	half-focal separation (e, f)
$CV_1 = CV_2$	semi-major axis	a	
$CV_3 = CV_4$	semi-minor axis	b	
$\frac{c}{a}$	eccentricity	e	first eccentricity, mathematical eccentricity (ϵ)

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

The *t*rue anomaly of a point P on an ellipse is the angle between the major axis and the line from a focus to that point – $\angle VCP$ in the diagram labelled c. The true anomaly is also written θ or ν in the literature.

The *e*ccentric anomaly is constructed by using an “auxiliary” circle of radius a. The point P is projeted (?) “up” to the auxiliary circle onto point P'. The eccentric anomaly is then $\angle VCP'$, in other words the angle between the semi-major axis and the line connecting the centre to P'.

The standard ellipse

The standard ellipse is given by the cartesian coordinate formula:

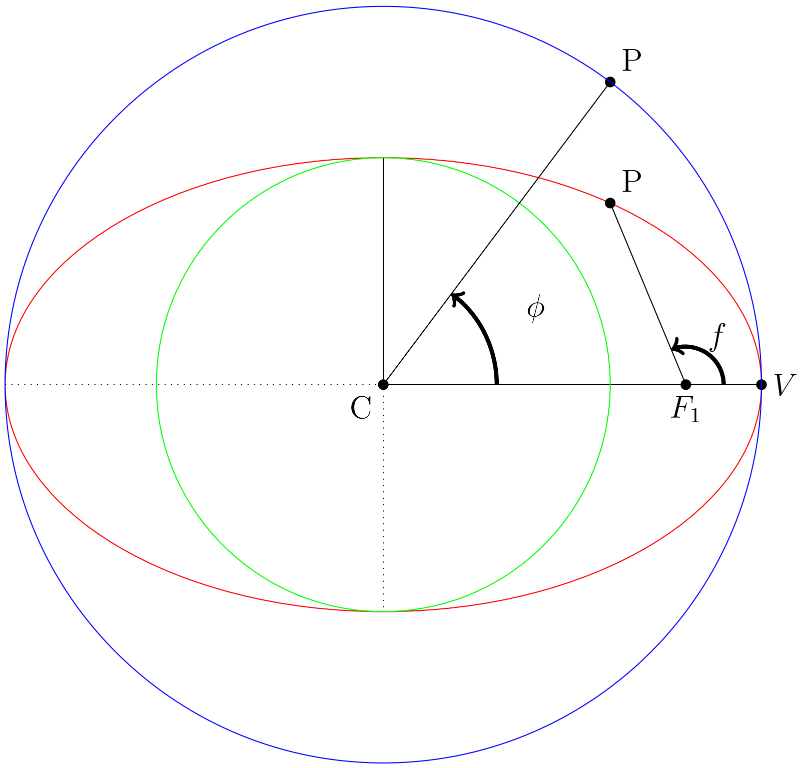


Figure 42.3: Ellipse angles

$$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$$