Chapter 1

Elliptic Functions and integrals

Prerequisites: **c**omplex analysis

1.1 Introduction

The fact that sine and cosine are periodic, with period 2π should be familiar
to you. That fact turns out to be enormously useful all over mathematics.
Wherever anything periodic turns up such as waves or relationships between
circular and linear motion turns up we found it useful.

But it turns out that sine and cosine are very interesting when viewed from
the point of view of complex numbers. In one sense, they are both aspects
of the exponential function, the most famous transcendental function. In
another sense they are the hyperbolic functions sine and cosine sinh and
cosh "rotated" through $\pi/2$.

But sine and cosine are periodic in one direction - parallel to the real line.
What about functions that have two independent periods? Such functions
are known as "elliptic functions".

By "independent" periods, I mean "linearly independent". Sine has a pe-
riod of 2π but also of $4\pi, 6\pi, 8\pi, \dots$ but the additional periodos aren't very
interesting. They are all real and so, if considered as 2D vectors in the
complex plane, they all point in the same direction.

To put it formally, an elliptic function is one with two non-zero[1] periods ω_1
and ω_2 such that there is no real number r with $\omega_1 = r\omega_2$.

Just this amount of information (and no more) allows us to deduce lots
of properties that elliptic functions must have. That is where I will start.
There are then three ways to get to concrete examples of elliptic functions,
all of which I shall try to do;

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[1]All functions have a "period" of zero

- We could discover them by realising that functions we already know about, the Jacobi elliptic functions are elliptic functions (the clue is in the name, but proving it requires some work). In fact what we will start with Jacobi elliptic functions and discover elliptic integrals.

- Then we will realise that elliptic integrals define elliptic functions.

- As a different tactic, we start from the definition of elliptic function and build a function from scratch that matches that definition. This is the approach of Weierstrass.

Elliptic things include:

- Elliptic Functions

- Elliptic Integrals

- Elliptic Curves

There is an odd history to these names. Elliptic integrals were discovered first[2]. Elliptic functions were then defined in terms of elliptic integrals and their properties derived from the integrals. Viewed from the perspective of complex analysis, elliptic curves are the curves associated with elliptic functions, so they inherited the name, even though they are not in fact ellipses.

We have, so far, taken a different approach. We met Jacobi Elliptic functions to help us solve some problems with ellipses and non-linear differential equations.

1.2   Jacobi Elliptic Functions and Integrals

Point P has position $(a\cos(\phi), b\sin(\phi))$. Point P' is vertically above P and is known as the "corresponding point". It has position, $(a\cos(\phi), a\sin(\phi))$. First we obtain a useful formula relating α and ϕ.

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[2]is this true?

$$\tan\alpha = \frac{b}{a}\tan\phi \qquad\qquad \text{elementary trigonometry} \quad (1.1)$$

$$\tan^2\alpha = \frac{b^2}{a^2}\tan^2\phi \qquad\qquad\qquad \text{squaring} \quad (1.2)$$

$$\sec^2\alpha = 1 + \frac{b^2}{a^2}(\sec^2\phi - 1) \qquad \text{using trig identities} \quad (1.3)$$

$$\sec^2\alpha = (1 - \frac{b^2}{a^2}) + \frac{b^2}{a^2}(\sec^2\phi - 1) \qquad \text{rearranging} \quad (1.4)$$

$$\sec^2\alpha = k^2 + \frac{b^2}{a^2}(\sec^2\phi - 1) \text{substituting definition of k} \quad (1.5)$$

$$(1.6)$$

Next we need a formula defining the differential of α in terms of ϕ.

$$\sec^2\alpha\, d\alpha = \frac{b}{a}\sec^2\phi\, d\phi \qquad\qquad \text{differentiating (1.1)} \quad (1.7)$$

$$d\alpha = \frac{b}{a}\frac{\sec^2\phi}{\sec^2\alpha}\, d\phi \qquad\qquad\qquad (1.8)$$

$$= \frac{b}{a}\frac{\sec^2\phi}{k^2 + \frac{b^2}{a^2}(\sec^2\phi - 1)}\, d\phi\, \text{substituting (1.5)} \quad (1.9)$$

$$= \frac{b}{a}\frac{d\phi}{k^2\cos^2\phi + \frac{b^2}{a^2}}\text{multiply through by } cos^2\phi (1.10)$$

$$= \frac{b}{a}\frac{d\phi}{(1 - k^2\sin^2\phi)}\text{multiply through by } cos^2\phi (1.11)$$

$$(1.12)$$

Now to calculate the arc length, we start with an easy formula in α and then transform it into an integral involving ϕ.

$$ds = \int r \, d\alpha \tag{1.13}$$

$$= \int \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} \, d\alpha \tag{1.14}$$

$$= \int a \sqrt{1 - k^2 \sin^2 \phi} \, d\alpha \tag{1.15}$$

$$= \int \frac{b}{a} \frac{a \sqrt{1 - k^2 \sin^2 \phi}}{(1 - k^2 \sin^2 \phi)} \, d\phi \tag{1.16}$$

$$= \int b \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \tag{1.17}$$

$$\tag{1.18}$$