Laboratory Manual: Chaos in a dripping faucet

Marissa B. Adams

University of Rochester
Department of Physics & Astronomy

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Contents

1 Purpose

The purpose of this experiment is to observe a display of attractors and the transition to chaos through a variety of plotting methods. These measurements will aid in the illustration of chaos in a dripping faucet (CIDF).

2 Background

Chaotic dynamics placed its seedlings in the mind of Henri Poincaré in the late 1800s where he discovered that the three-body problem had non-periodic orbits. However this area of research burgeoned and gained popularity in the 1980s[1] with the help of Otto Rössler, Robert Shaw, and a multitude of other scientists, when they discovered that nonlinear phenomena can be observed in day-to-day occurrences [5]. From flipping a coin to an inverted pendulum, many mechanical systems exhibit nonlinear, or chaotic, behavior.

The dripping water faucet is one such system. Drop formation from a faucet is an incredibly pervasive phenomena that finds its application in ink-jet printing as well as annoying your roommate late at night. It was suggested in 1977 by Rössler and then experimentally confirmed a few years later by Shaw where he carried out a study involving the formation of thousands of drops in succession [6], [4]. Given the nature of Shaw's experiment, it is clear that a time interval between drops, $t_1, t_2, ..., t_i, ...,$ where t_i is the time interval between the ith and the $(i-1)$th drops, is necessary. This sequence of drip intervals in a leaky faucet naturally describes a discrete mapping. So for any given value of the drip rate, we can plot the next drip interval versus the previous one. Plotting any discrete data like so can give one a clear idea of the behavior (in this case, the dripping) at hand and of the possible existence of attractors (for definition, see §2.1). This type of data representation is called a *return mapping*, or *Poincaré map*[2], and will

[1] This is even evident in popular culture: Ian Malcolm, the Jurassic Park character, was a mathematician who specialized in Chaos theory.

[2] Poincaré definitely has left his mark, hasn't he? The $\{t_i\}$ discretizes a continuous system, a trademark of Poincaré, as it approximates the flow in the system state space[1]. To learn more, prove the Poincaré conjecture.

be what we plan to create in the undergraduate advanced laboratory[5].

We can describe this iterative process mathematically in the form

$$x_{n+1} = f(x_n) \tag{1}$$

where $f(x)$ is a continuous function defined by a one-dimensional interval.

Some call into question how restrictive the one-dimensional mapping is as it can be viewed as a discrete time version of a continuous by dissipative dynamical system. However these dissipative terms will shrink the volume of the phase space occupied by the system until it becomes effectively one-dimensional, therefore we may assume our function to be such a simple mapping [5].

One does not necessarily need a precise definition for $f(x)$, however to analyze the dynamics of a logistic map (Figure 1), we need a single quadratic maximum. Therefore $f(x)$ in equation (1) will take the form of (2),

$$f(x) = \mu x(1 - x) \tag{2}$$

where μ is a parameter that measures the strength of the nonlinearity of the system. With this choice of $f(x)$, equation (1) now describes a nonlinear and singular (i.e. non-invertible) map of the unit interval one itself. In other words, the time evolution is either multivalued or undefined, implying that our dynamical system, governed by non-linear equations, is chaotic.

We can see this more so in Figure 1, where as we increase from $\mu = 0$, which is trivial, it approaches an attractor. The larger the value of μ the more interesting the dynamics of the system. At $\mu \approx 3.0$ we observe the first bifurcation (period-2 attractor), then the second at $\mu \approx 3.4$, and so on until we eventually reach chaos. This cascade of bifurcations continues indefinitely. Although the interval values of μ in which it acts as an attractor shrinks very quickly, as illustrated in Figure 1, by a rate governed by what is called the "universal parameter,"

$$\delta = \lim_{n\to} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = 4.6692... \tag{3}$$

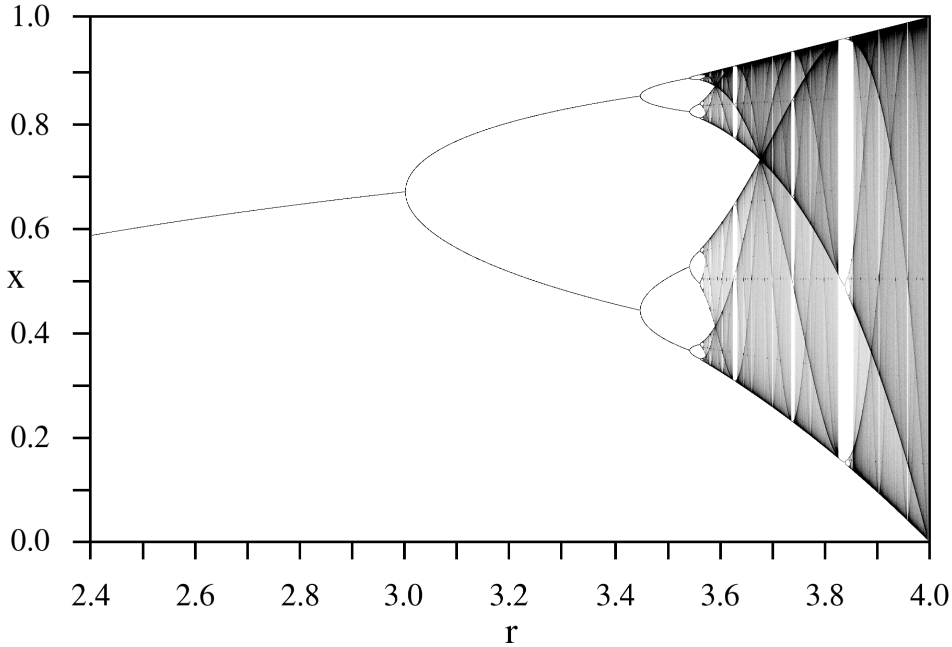

Figure 1: A bifurcation diagram of the logistic map, which shows the asymptotic behavior of x_n for values of $\mu = r$ between 2.4 and 4. The x on the dependent axes refers to an attractor.

One can go about empirically calculating the value for δ in the Chaos in an RLC Circuit laboratory provided in this course. A more complete discussion of the properties of the logistic map can be found in the account given by Feigenbaum[3].

Ultimately nonlinear differential equations that display erratically varying behavior based on small changes in the initial conditions are the crux to chaotic behavior. In the case of water dripping from a faucet, each droplet sets an initial condition for the next. At particular "drop rates," or drops per unit time, we expect to see chaos ensue. In the aforementioned chaos in a circuit laboratory, the parameter analog to drop rate is voltage and is varied continuously in a controlled way. In this laboratory, the drop rate is controlled by an uncalibrated valve that controls the flow of water to some nozzle, or faucet. Given this

[3] Feigenbam, M.J., 1983, Physica **7D** 16

circumstance, which we will discuss later on in this laboratory manual, the drop rate for any data set is to be determined after the data has been collected. We do this by calculating the average time between each successive drop. Therefore it is impractical to collect "continuous" data sets that might be used to make the bifurcation diagram illustrated by Figure 1. We go about observing chaos in a dripping faucet another way.

2.1 Terminology

As many students who are taking PHY243W (Advanced Lab) are not required to take MTH215 (Fractals and Chaotic Dynamics) or any of the graduate level physics and chemistry courses on the topic of chaotic dynamics, it might be useful to discuss some chaos terminology.

- As one could assume in the second before googling the term, **bifurcation** is the division of something into two branches or parts. We see this in Figure 1, above, where one experiences period doubling, quadrupling, etc. that accompanies the onset of chaos. So as a system approaches chaos, it will *bifurcate* by splitting.
- **Period doubling bifurcation** in a discrete dynamical system is a bifurcation in which the system switches to a new behavior with twice the period of the original system.
- A **logistic map**, an example of period-doubling bifurcation, is a polynomial mapping of degree 2 (hence period doubling), as stated above in equation (2).
  - Mathematical constants that express ratios in the bifurcation diagram (Figure 1) for a nonlinear map are called **Feigenbaum constants**. These constants were discovered by their namesake, mathematician Mitchell Feigenbaum, in the late 1970s. These constants can be calculated using the ratio stated in equation (3) for period $(2, 4, 8, ...)$ or $n = 1, 2, ...$, with their bifurcation parameter, x_n.
  - The **universal parameter** is one of these constants, and is considered to be the first F-constant.

- An **attractor** is a value that the system attains with regularity. So as a system bifurcates, it will exhibit more and more separate attractors, oscillating with regularity. We see this as the white space in Figure 1 between each bifurcation.
  - A good example to understand what an attractor *does* is to consider a harmonically driven pendulum. Given the frequency and the strength of the driving force, the motion of the system is to be determined by the angle θ and angular speed $\dot{\theta}$ of the pendulum.
  - It turns out that $(\theta, \dot{\theta})$ are the coordinates of the phase space of the pendulum as it swings back and forth.
  - If the strength of the driving force vanishes, then no matter how the pendulum's motion starts, it will eventually come to rest at a point of stable equilibrium after a number of oscillations.
  - These stationary motions in which the pendulum system settles after the oscillations, or transients, have died are examples of attractors.
  - This name, "attractor," comes from the perspective of the phase space for this pendulum, where it many nearby orbits are attracted to these stable points. So an attractor is a set of physical properties toward which a system tends to evolve regardless of the initial conditions.
  - An **orbit** is a collection of points related by the "evolution function" of the dynamical system, which is a subset of the phase space.
  - If these attractors are fractal, they are then called **Strange Attractors**, which usually correspond to a chaotic orbit being attracting. When we observe chaos in this laboratory, it will be in the form of a strange attractor.
- As the system descends into chaos in Figure 1 we see these **islands of regularity** in which the chaos gives away to several attractors and then goes back into chaos.

3 Experimental Apparatus & Equipment

3.1 Experimental Set Up

To get ameliorate the discussion on the experimental apparatus, view Figures 2-4. The overall process is like such:

1. The water flows from a reservoir into a secondary bucket. The water level in the secondary bucket is kept constant by use of the same apparatus that maintains the water level in a toilet. You can see this in Figures 2[4].

Figure 2: *From left to right*: Within the secondary bucket is a toilet valve, as pictured. This secondary bucket is the rightmost bucket pictured here.

- There are a variety of bathrooms and faucets to refill the buckets with. Find a bucket/receptacle nearby. Do not dismantle the apparatus to do this.
- **Warning:** In order to refill the the apparatus, do not place water into the secondary bucket, it will disrupt the pressure and water level. Therefore refill through the first bucket.
- If one were to accidentally refill through the secondary bucket, then run all of the water through the apparatus to clear the reservoirs.

[4]Photos taken by D. Allan, et. al.

2. Water will flow out the secondary bucket into a tube. At this junction, the water flow rate is controlled by an uncalibrated valve.

    - There will be two valves one can use to aid in changing the drop rate. The lower most valve will drastically change the drip rate, where as the first, or highest valve is for finer tuning.

3. The water flows through the droppers illustrated in Figure 3[5] that you might have placed within the tubing. When the valve is set within a certain range, the water emerges in discrete droplets at a variety of rates.

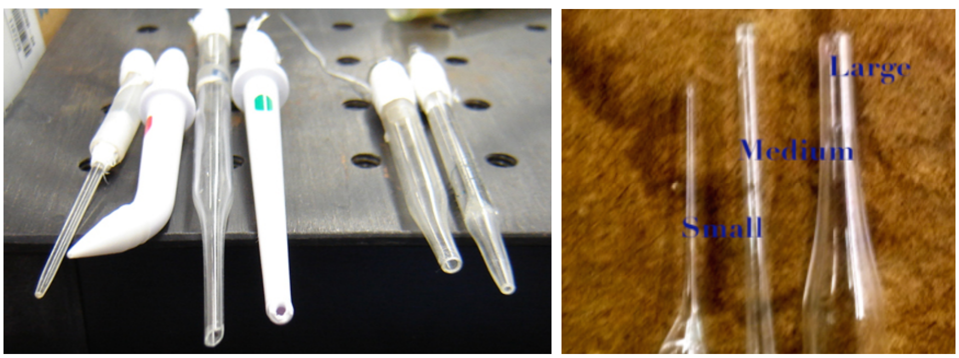

Figure 3: The droppers, nozzles, or faucets, whatever one prefers to call them. Designated from left to right, syringe-like, red dot, D, green dot, A, and B. In the rightmost photo, A is the "large" nozzle, and D is the "medium."

    - It may be difficult to pull the nozzle out of the tubing. When changing nozzles, use the shoe-horn type object that is near the experimental set up. This instrument will help you to pull the nozzle out of the tubing if you insert it to one side.
    - The nozzles act as a variable in this experiment, as they vary in diameter and shape.

4. The droplets intercept the path of a beam from a HeNe laser.

[5]Left photo taken by D. Allan, et. al.

5. A photosensitive detector, at which the laser is aimed, registers the brief interruption by the droplet.

6. A LabView GUI allows you to record the time lapsed between each successive interruption of the beam (i.e. between each droplet of water).

   - The LabView program, and the laser, will help you to align the droplets with the sensor. Check the program and ensure that there are drops being recorded. You can do this by adjusting the position of the nozzle in its placeholder.

Overall the experimental set up involves a series of buckets, one which has a toilet valve, some tubing and valves to control water flow through the nozzle. From this system, drops will form, and be highlighted be an He-Ne laser, which will be picked up by a photodiode, which sends information to LabView. In its totality, one can see the set up in Figure 4.

3.2 Equipment Specifications

3.2.1 He-Ne Laser

The laser used in this experiment is a 3mW laser diode that emits visible red light as a wavelength of 635nm (He-Ne). Laser goggles should not be necessary, but one should avoid looking directly into the light beam. The laser can be powered by a supply of around 3V. The laser is manufactured by Quarton, Part Number VLM-635-02-LPA, and is comparable to the red lasers used in other experiments in the advanced laboratory.

Note: Turn off the laser after use. Simply unplug it.

3.2.2 Detector

(I am not sure if this detector is the same as the one from the Faraday Rotation Lab, but for not I use that lab-manual's description as a filler. I also don't know the manufacturer.) The detector is a photodiode sensor connected in series to three resistors; 10K, 3K, and 1K (Ω).

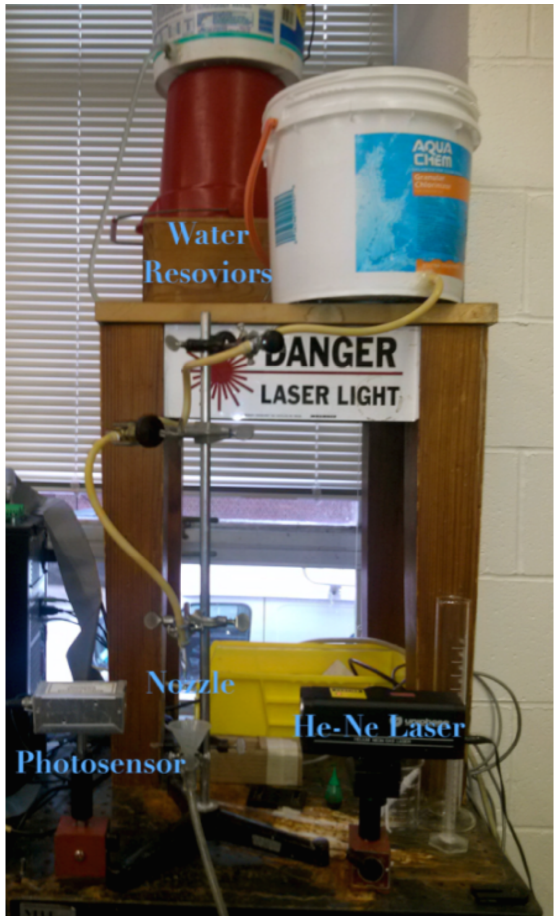

Figure 4: The entire experimental set up as described by steps 1-6. The computer that holds the LabView code is to the left, out of view of the photograph.

It will convert the light sensed from the drop into voltage, and send it to the computer.

Note: Turn off the sensor after use. One can flick a switch on the back of the sensor's housing to the "off" position.

3.3 On the LabView Code

(Here I want to provide a discussion on the code, and where the files end up. I plan to elaborate more on this during the rest of break with screen captures, etc.)

3.4 Video Camera

(Insert Specs on Mike's camera here...)

4 Procedure

1. Pour water into the primary reservoir bucket.

2. Using the food coloring provided, put a few drops into the reservoir. It has been said that this helps the droplet pick up the laser light.

3. Using the valves affixed on the tubing, twist them such that the water starts to flow to the nozzle.

4. After adjusting the nozzle such that it is aligned with the laser and photodiode, hit start in the LabView GUI (it would be nice to have screen captures here).

5. Initially one will want to run a few trials to get an estimate of their drop rate. The drop rate will be indicated by the time between each drop in the program. As previously stated, one will calculate the average drop rate after acquiring the data set.

6. Aim to acquire data for "slow," "some-what fast" and "fast" drop rates. One will see a general trend, as we'll discuss in §4, that a low drop rate corresponds to a few stable attractors. However increasing the drop rate will lead to multiple attractors and chaos. Thus, you expect to observe strange attractors in some middle regime, before the drops become a stream.

4.1 Problems; Sources of Error

1. **Missing droplets?** Some of the droplets take slightly varying trajectories, resulting in a few of them missing the laser beam and going unregistered. This can be seen by example in Figure 5 (a)-(c). Some attribute this to an asymmetry in the nozzle (the shape), or even if it is dirty, causing unnecessary friction between the water and the nozzle, hence affect the initial conditions between drops.

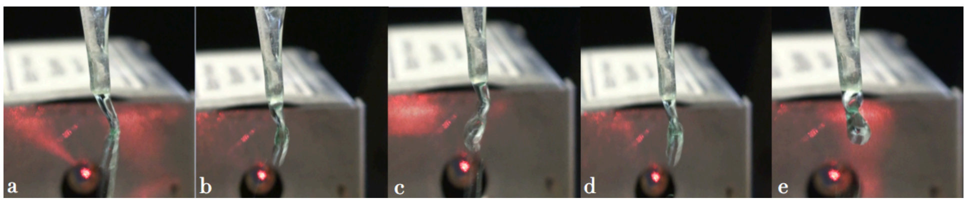

Figure 5: A stop motion visual of the some of the affects we see when the water is dropping at around 15 drops per second. In frame (a)-(c) we see that the droplet's do not necessarily fall straight down into the path of the laser. In frames (d) & (e) we start to see the desired affect, where one droplet affects the initial conditions of the other.

Solution: Firstly, clean the nozzles. There are a variety of solutions that people have tried to alleviate this issue in the past. One solution is to increase the observed coverage with two lasers instead of one. How one could do this is to have one beam travel perpendicular to the other.

Another, more simpler solution might be to change the height of the dropper end (either closer to the beam's path, our farther way, depending on the dropper and drop rate. The length of the nozzle might also be a contributing factor. The longer the nozzle, the more it might provide a direct path for the droplet. However the length of the nozzle might also hinder one in estimating a desired drop rate. Ultimately the solution to this issue is to experiment with what works best.

One can see the magnitude of missed droplets in the LabView program as the t_n versus t_{n+1} plot will look something like Figure

6[6], where we see a period-1 attractor, with these strange lines that are hugging the axes. These lines are symptoms that indicate missed drops as a result of our set up. When the sensor misses a drop, but then observes the next drop, it will make up for the missed drop in the LabView program by creating these lines.

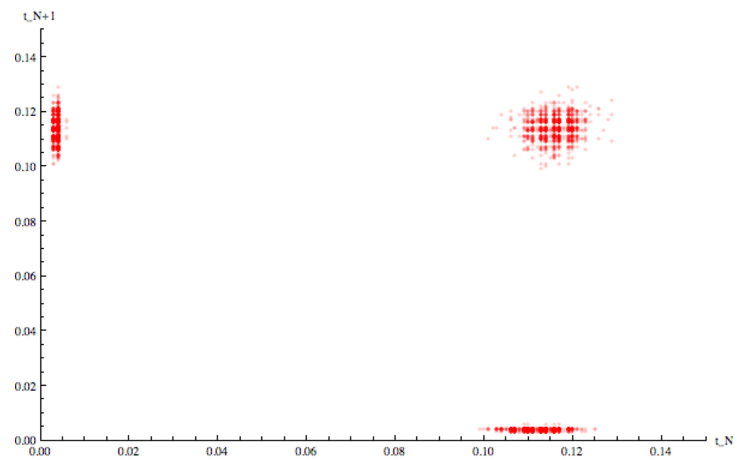

Figure 6: A t_n versus t_{n+1} plot for one of the slower drip rates. This is clear as we see a period-1 attractor. The phenomena that hugs the axes is a symptom of the experimental apparatus, as discussed above.

2. **Computer lag?** The computer lag can also result in "dropped" data that will corrupt results. One can see this on the default clock rate in the LabView GUI (screen capture here?).

   Solution: What one can do is overhaul the code for efficiency and allow the user to adjust data taking rate (or time resolution). We can do this by adjusting the amplitude. Too little resolution will miss some drops, where as too much resolution will lead to a lag in the computer.

3. **Sudden "shifts"?** When graphing the data verses an independent variable (time), one can observe sudden "shifts" that aren't necessarily prominent in other data sets, we can see this in some of the features in Figure 7.

[6] Data from D. Allan

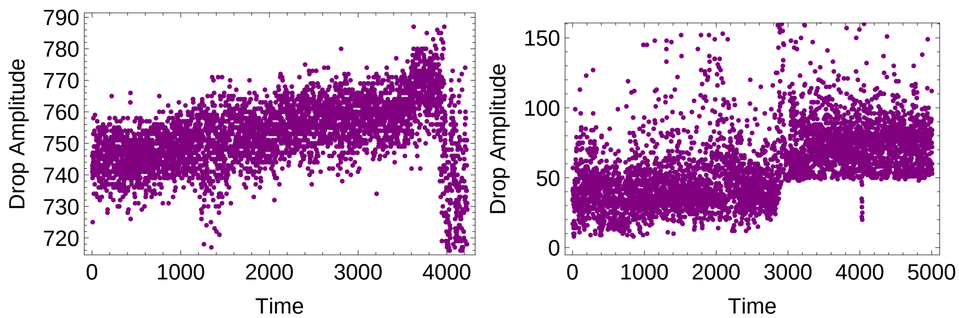

Figure 7: Two plots for different data sets, both of which illustrate the drop amplitude versus time. Notice the linear affect in the leftmost graph with a sudden shift toward the end. The rightmost graph has a more prominent shift in the middle, albeit around the same number of drops as the left (∼3500 drops).

> **Solution:** This affect illustrate in Figure 7, is generally attributed to a changing pressure. Despite adding a toilet valve, it still seems to be an issue. Perhaps letting all of the water run through the system and then dumping it back into the primary bucket might help this issue. Despite the "shifts," one can still observe this linear effect as shown in the left most graph. This implies that the toilet valve might not always be maintaining constant water level over time. This could be due to friction between the valve and it's hinge. It might be helpful to ensure that the valve is "stable" in the bucket by lubricating it.

5 Analysis

Ultimately this experiment is relatively qualitative in the analysis. There are a variety of different plots and graphs one can produce in order to better understand their data. One can create the previously mentioned *return map* (t_n versus t_{n+1}), which is also produced in the LabView GUI. This type of plot is the most revealing with regard to chaos, as each "cluster" indicates an attractor. One could also produce 3D plots of one's data in this form.

One can also create a *amplitude* versus *time* graph, where one simply plots the data versus some independent variable (which is time). This perspective might yield an understanding of how well the experimental set up is working. Is there a linearity in the data? Do we see any shifts?

Another type of graph one could make is a histogram, which might help one to ascribe weight to each cluster on the data's respective return map. Some clusters will seem very "deep," or that there are several data points occupying the same dot, while others are more sparsely populated and should be given less weight. This analysis is dependent on the density of the clusters.

Overall one will want to sort their plots for each data set based on the type of dropper used and the other of the drop rate calculated. After acquiring and plotting all the data, then one can come to conclusions about the experiment.

The data acquired from the LabView program will be in a .txt format. This format can be used directly in MatLab, however if one would prefer to do plotting in Excel or *Mathematica*, then one must convert the files to .csv, which can be easily done in Excel and then imported in *Mathematica*.

If one desires to do t_n versus t_{n+1} plots in MatLab, one might execute a code comparable to:

```
mat = dlmread('/Users/YourName/Desktop/filename');
x = mat(:, 2);
y = x;

x(1:6) = [];
y(1:5) = [];

x(size(x) - 4:size(x)) = [];
y(size(y) - 5:size(y)) = [];
scatter(x, y, '.');
xlabel('Time between drops (ms)');
ylabel('Time between next drops (ms)');
```

This code spits out plots that look comparable to Figure 8. However one does not need to use this code, but it may act like a guide for those who are unfamiliar with MatLab.

If one would prefer to use *Mathematica*, then one can create an *amplitude* versus *time* plot like so,

```
ListPlot[Flatten[Drop[DATANAME, {}, {1}]],
Frame -> True,
FrameLabel -> {"Time", "Drop Amplitude"},
LabelStyle -> {FontFamily -> "Arial",
FontSize -> 14},
Axes -> False,
PlotStyle -> {Purple}]
```

which will produce plots that look like those in Figure 7. Note like how we refer to the location of a file in the MatLab code, we do so similarly in the *Mathematica* code, where,

```
DATANAME = Import[
   "/Users/YourName/Desktop/filename"];
```

one names the imported data before plotting it.

Now that you've acquired your data and have plotted it, the real analysis begins. Recall the terminology we previously discussed in §2.1. In your results you'll hopefully observe a series of attractors, corresponding to different periods. Some data sets might seem as if they are harder to distinguish. If you're observing a weird set of attractors, they are most likely strange attractors.

For instance let us use the graphs shown in Figure 8 as an example. These data sets were collected for the "largest" nozzle (see Figure 3) for three different drop rates, increasing from approximately 10 drops per second to 20 drops per second. The leftmost plot, we can clearly observe a period-1 attractor. We can see this because it forms a single cluster. In the rightmost plot, we see about four major clusters, despite three of them making an "L" shape. One might think this to be a period-4 attractor, however these plots are incredibly high in symmetry. The clusters adjacent over the $y = x$ line are conjugate pairs. This indicates that this is a period-2 attractor. In the middle graph we see 5 clusters. For similar reasons to the graph previously discussed, this plot

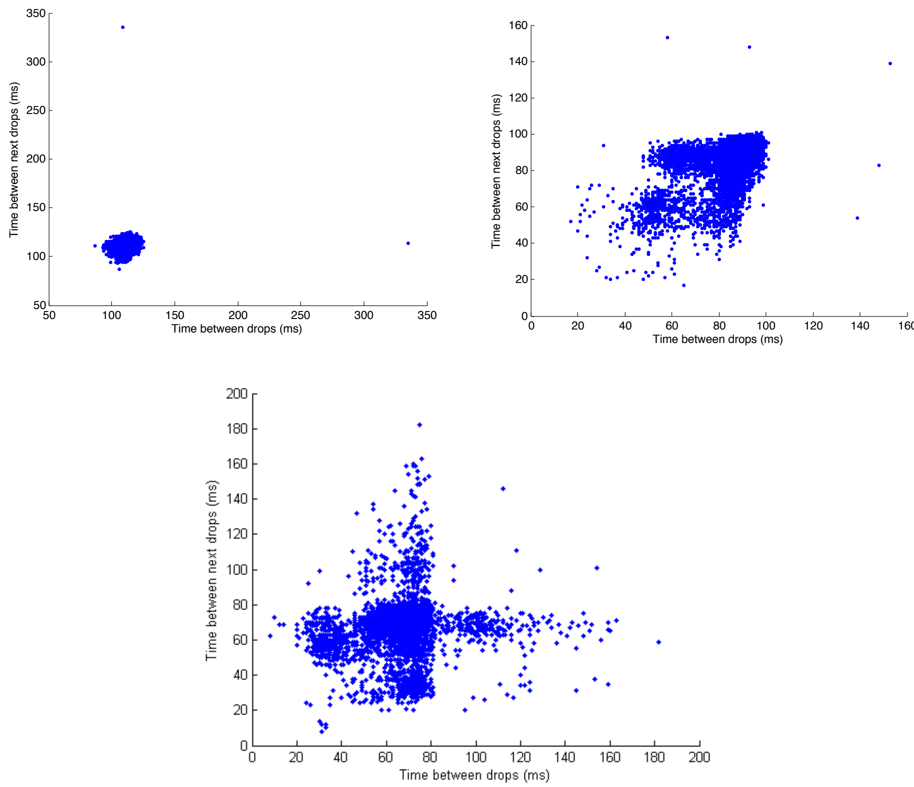

Figure 8: Data sets collected for the "large" nozzle. From left to right we have 10, 15, and 20 drops per second illustrated in a return mapping.

demonstrates a period-3 attractor. As the period number increases, the more likely we are to see chaos.

So what does chaos look like? One can observe in Figure 9[7] that we so not see any regular attractors as previously illustrated. In fact, these are strange attractors, where we have descent into chaos. Both of these plots were taken in a drop rate regime that was generally "in the middle," or somewhere between 1 drop per second and 30 drops per second (steady stream), where we would expect to see chaos occur.

Despite seeing a series of period attractors, whether they are regular or strange. We can observe other weird behavior. As we can see Figure

[7]Data from D. Allan.

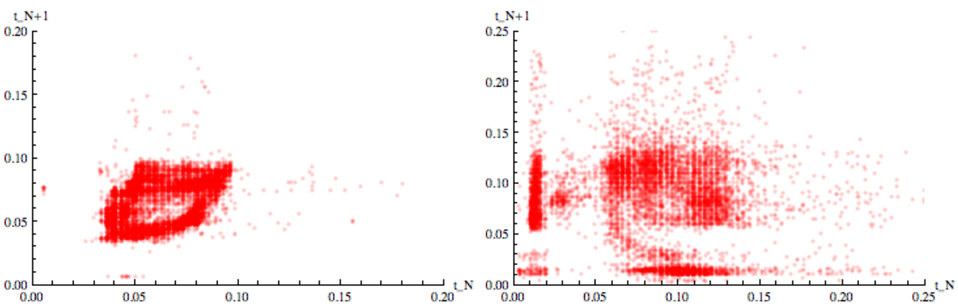

Figure 9: The leftmost return map is refers to dropper A (Figure 3) at around 15 drops per second. The rightmost return map refers to the data collected from the green dot dropper at around 12 drops per second.

10[8], one can observe a strange cross shape. Sometimes a variety of these cross crop up, perhaps indicating the existence of attractors. This type of result seems to stem from the smaller nozzles, perhaps "cross"-ness is a function of diameter. It might also be an effect that is the result of asymmetry in the nozzle, perhaps it also has jagged edges.

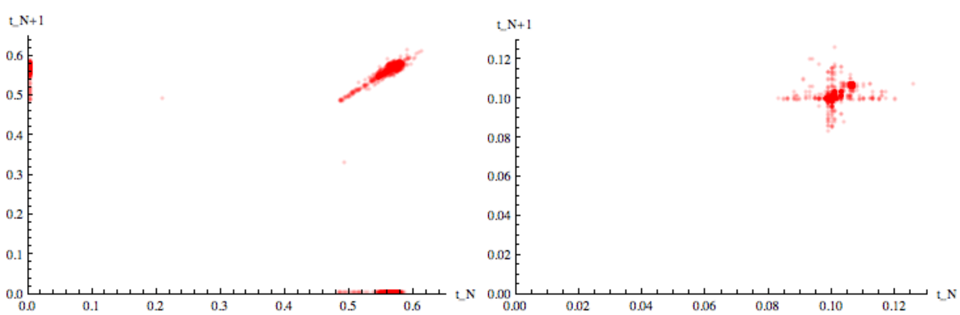

Figure 10: In the rightmost plot we see a strange cross-shape. In the leftmost plot we see a linear progression.

One can also see a "tail" or linear progression in their t_n versus t_{n+1}. This "tail" is a result of pressure changing in the reservoir, or that the drop rate is changing. When one sees this type of result, pressure change is mostly likely the cause. One can observe this in the LabView

[8]Data from D. Allan.

GUI directly while changing the valves, and thusly increasing the drop rate.

Ultimately the analysis of the data collected is attempting to understand what each set is illustrating, a progression toward chaos, chaos, or something wrong with the experimental set up. From there we try to make sense of our conclusions based on what could cause error, and the drop rate. Overall the general trend should be clear, that for lower drop rates, we see regular attractors, in the middle regimes we see chaos, and we revisit period attractors in the higher regimes, as the drops create a stream.

6 Future Work

(I plan to add more to this section later) There are a variety of aspects in this laboratory experiment that can be further explored and specified.

- Are the cross shaped pattern a result of certain droppers? To what extent are these significant and how might they be further explained?

- How might one could about further controlling the flow rate in the valve such that it can be more precise and continuously adjustable? Is it possible to calibrate the valves?

- With a calibrated value, is the point where one experiences the transition to chaos in a dripping faucet regular across a series of trials? How is this affected by the radius of the dropper?

- How can one could about measuring the diameter of the droppers, as they are so small? Is it possible to get a standard set of droppers?

- How does viscosity affect the transition to chaos? Can this laboratory be replicated for different fluids?

7 Conclusion

Using water, a series of buckets, tubes, and a variety of nozzle types, as well as a laser, photodiode and LabView program, we are able to illustrate chaos in a dripping faucet. We are able to do this by plotting return maps with our acquired data for different drop rates. These maps illustrate the period attractors that indicate the beginning bifurcations in the standard chaotic logistic map. They also might illustrate strange attractors within the middle regimes, thusly allowing one to observe the chaotic dynamics in a dripping faucet.

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