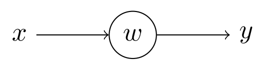
1 Gradient Descent

If we keep decreasing the ϵ in our Finite Difference approach we effectively get the Derivative of the Cost Function.

$$C'(w) = \lim_{\epsilon \to 0} \frac{C(w+\epsilon) - C(w)}{\epsilon} \tag{1}$$

Let's compute the derivatives of all our models. Throughout the entire paper n means the amount of samples in the training set.

1.1 Linear Model



$$y = x \cdot w \tag{2}$$

1.1.1 Cost

$$C(w) = \frac{1}{n}\sum_{i=1}^{n}(x_i w - y_i)^2 \tag{3}$$

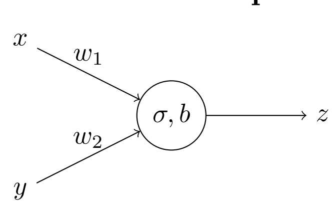
$$C'(w) = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i w - y_i)^2\right)' = \tag{4}$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i w - y_i)^2\right)' \tag{5}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left((x_i w - y_i)^2\right)' \tag{6}$$

$$= \frac{1}{n}\sum_{i=1}^{n}2(x_i w - y_i)x_i \tag{7}$$

1.2 One Neuron Model with 2 inputs

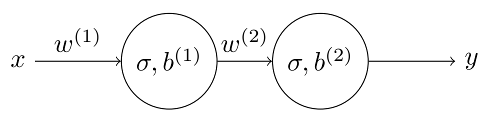


$$z = \sigma(xw_1 + yw_2 + b) \tag{8}$$
$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{9}$$
$$\sigma'(x) = \sigma(x)(1 - \sigma(x)) \tag{10}$$

1.2.1 Cost

$$a_i = \sigma(x_i w_1 + y_i w_2 + b) \tag{11}$$
$$\partial_{w_1} a_i = \partial_{w_1}(\sigma(x_i w_1 + y_i w_2 + b)) = \tag{12}$$
$$= a_i(1 - a_i)\partial_{w_1}(x_i w_1 + y_i w_2 + b) = \tag{13}$$
$$= a_i(1 - a_i)x_i \tag{14}$$
$$\partial_{w_2} a_i = a_i(1 - a_i)y_i \tag{15}$$
$$\partial_b a_i = a_i(1 - a_i) \tag{16}$$
$$C = \frac{1}{n}\sum_{i=1}^{n}(a_i - z_i)^2 \tag{17}$$
$$\partial_{w_1} C = \frac{1}{n}\sum_{i=1}^{n}\partial_{w_1}\left((a_i - z_i)^2\right) = \tag{18}$$
$$= \frac{1}{n}\sum_{i=1}^{n}2(a_i - z_i)\partial_{w_1} a_i = \tag{19}$$
$$= \frac{1}{n}\sum_{i=1}^{n}2(a_i - z_i)a_i(1 - a_i)x_i \tag{20}$$
$$\partial_{w_2} C = \frac{1}{n}\sum_{i=1}^{n}2(a_i - z_i)a_i(1 - a_i)y_i \tag{21}$$
$$\partial_b C = \frac{1}{n}\sum_{i=1}^{n}2(a_i - z_i)a_i(1 - a_i) \tag{22}$$

1.3 Two Neurons Model with 1 input



$$a^{(1)} = \sigma(xw^{(1)} + b^{(1)}) \tag{23}$$
$$y = \sigma(a^{(1)}w^{(2)} + b^{(2)}) \tag{24}$$

The superscript in parenthesis denotes the current layer. For example $a_i^{(l)}$ denotes the activation from the l-th layer on i-th sample.

1.3.1 Feed-Forward

$$a_i^{(1)} = \sigma(x_i w^{(1)} + b^{(1)}) \tag{25}$$

$$\partial_{w^{(1)}} a_i^{(1)} = a_i^{(1)}(1 - a_i^{(1)}) x_i \tag{26}$$

$$\partial_{b^1} a_i^{(1)} = a_i^{(1)}(1 - a_i^{(1)}) \tag{27}$$

$$a_i^{(2)} = \sigma(a_i^{(1)} w^{(2)} + b^{(2)}) \tag{28}$$

$$\partial_{w^{(2)}} a_i^{(2)} = a_i^{(2)}(1 - a_i^{(2)}) a_i^{(1)} \tag{29}$$

$$\partial_{b^{(2)}} a_i^{(2)} = a_i^{(2)}(1 - a_i^{(2)}) \tag{30}$$

$$\partial_{a_i^{(1)}} a_i^{(2)} = a_i^{(2)}(1 - a_i^{(2)}) w^{(2)} \tag{31}$$

1.3.2 Back-Propagation

$$C^{(2)} = \frac{1}{n}\sum_{i=1}^{n}(a_i^{(2)} - y_i)^2 \tag{32}$$

$$\partial_{w^{(2)}} C^{(2)} = \frac{1}{n}\sum_{i=1}^{n}\partial_{w^{(2)}}((a_i^{(2)} - y_i)^2) = \tag{33}$$

$$= \frac{1}{n}\sum_{i=1}^{n}2(a_i^{(2)} - y_i)\partial_{w^{(2)}} a_i^{(2)} = \tag{34}$$

$$= \frac{1}{n}\sum_{i=1}^{n}2(a_i^{(2)} - y_i)a_i^{(2)}(1 - a_i^{(2)})a_i^{(1)} \tag{35}$$

$$\partial_{b^{(2)}} C^{(2)} = \frac{1}{n}\sum_{i=1}^{n}2(a_i^{(2)} - y_i)a_i^{(2)}(1 - a_i^{(2)}) \tag{36}$$

$$\partial_{a_i^{(1)}} C^{(2)} = \frac{1}{n}\sum_{i=1}^{n}2(a_i^{(2)} - y_i)a_i^{(2)}(1 - a_i^{(2)})w^{(2)} \tag{37}$$

$$e_i = a_i^{(1)} - \partial_{a_i^{(1)}} C^{(2)} \tag{38}$$

$$C^{(1)} = \frac{1}{n}\sum_{i=1}^{n}(a_i^{(1)} - e_i)^2 \tag{39}$$

$$\partial_{w^{(1)}} C^{(1)} = \partial_{w^{(1)}}\left(\frac{1}{n}\sum_{i=1}^{n}(a_i^{(1)} - e_i)^2\right) = \tag{40}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\partial_{w^{(1)}}\left((a_i^{(1)} - e_i)^2\right) = \tag{41}$$

$$= \frac{1}{n}\sum_{i=1}^{n}2(a_i^{(1)} - e_i)\partial_{w^{(1)}} a_i^{(1)} = \tag{42}$$

$$= \frac{1}{n}\sum_{i=1}^{n}2(\partial_{a_i^{(1)}} C^{(2)})x_i \tag{43}$$

$$\partial_{b^1} C^{(1)} = \frac{1}{n}\sum_{i=1}^{n}2(\partial_{a_i^{(1)}} C^{(2)}) \tag{44}$$

1.4 Arbitrary Neurons Model with 1 input

Let's assume that we have m layers.

1.4.1 Feed-Forward

Let's assume that $a_i^{(0)}$ is x_i.

$$a_i^{(l)} = \sigma(a_i^{(l-1)} w^{(l)} + b^{(l)}) \tag{45}$$

$$\partial_{w^{(l)}} a_i^{(l)} = a_i^{(l)}(1 - a_i^{(l)}) a_i^{(l-1)} \tag{46}$$

$$\partial_{b^{(l)}} a_i^{(l)} = a_i^{(l)}(1 - a_i^{(l)}) \tag{47}$$

$$\partial_{a_i^{(l-1)}} a_i^{(l)} = a_i^{(l)}(1 - a_i^{(l)}) w^{(l)} \tag{48}$$

1.4.2 Back-Propagation

Let's denote $a_i^{(m)} - y_i$ as $\partial_{a_i^{(m)}} C^{(m+1)}$.

$$C^{(l)} = \frac{1}{n} \sum_{i=1}^{n} (\partial_{a_i^{(l)}} C^{(l+1)})^2 \tag{49}$$

$$\partial_{w^{(l)}} C^{(l)} = \frac{1}{n} \sum_{i=1}^{n} 2(\partial_{a_i^{(l)}} C^{(l+1)}) a_i^{(l)}(1 - a_i^{(l)}) a_i^{(l-1)} = \tag{50}$$

$$\partial_{b^{(l)}} C^{(l)} = \frac{1}{n} \sum_{i=1}^{n} 2(\partial_{a_i^{(l)}} C^{(l+1)}) a_i^{(l)}(1 - a_i^{(l)}) \tag{51}$$

$$\partial_{a_i^{(l-1)}} C^{(l)} = \frac{1}{n} \sum_{i=1}^{n} 2(\partial_{a_i^{(l)}} C^{(l+1)}) a_i^{(l)}(1 - a_i^{(l)}) w^{(l)} \tag{52}$$