Dynamic Allocation of Treasury and Corporate Bond Portfolios

Roger Walder[1]

University of Lausanne,
International Center FAME,
and Banque Cantonale Vaudoise,
Switzerland

First Version: November 2001
This Version: December 2002

JEL Classification: D9; G11
Keywords: Dynamic Asset Allocation, Portfolio Management, Credit Risk

Abstract

In this paper, we solve the intertemporal investment problem of an investor holding a portfolio of default-free and defaultable bonds. Default-risk is modeled in an intensity based framework with state variables following an affine diffusion. The structure of the optimal portfolio over time is investigated and compared to the static mean-variance portfolio. Furthermore, we describe the impact of time varying market prices of risk and interdependencies between interest rates and credit risk on the optimal portfolio structure.

[1]Postal address: Chemin de Sus-Craux 10B, CH-1166 Perroy, Switzerland.
Email: roger_walder@yahoo.com. I would like to thank Gianluca Cassese, Jean-Pierre Danthine, Jérôme Detemple, Thomas Domenig, Rajna Gibson, Xiangrong Jin, Olivier Scaillet, René Stulz, Carsten Sorensen, Jürg Tobler, Luigi Vignola and my former colleagues at Zurich Cantonalbank for their comments and their help. This work has benefited from a FAME Research Grant. I also gratefully acknowledge the financial support from Zurich Cantonalbank. All remaining errors are mine. Comments are welcome.

1 Introduction

Intertemporal consumption and investment problems in continuous time have been introduced in the financial literature by Merton (1971). He proposed a methodology to deal with the problem of a rational investor with time additive preferences who chooses how to allocate her wealth between consumption and existing securities (mainly stocks). The optimal portfolio policy turns out to be characterized by two components: A component that reflects the optimal portfolio in a single-period, mean-variance set-up and a component that reflects how the investor hedges against variations of the asset return and risk characteristics (or investment opportunity set). The computation of the hedge component requires the solution of a partial differential equation, which can be solved analytically only in some simple cases.

Intertemporal investment problems including default-free bonds in the investment universe have been solved by Poncet and Portait (1993), Bajeux-Besnainou and Portait (1998), Brennan, Schwartz and Lagnado (1997), Sorensen (1999), Campbell and Viceira (2001), Brennan and Xia (2000), Mougeot (2000), Munk and Sorensen (2001) and others. These papers show that the best hedge against stochastic variations of the interest rate is a zero-coupon bond with expiration at the investment horizon.

The optimal allocation for an investment universe containing defaultable bonds has received little attention in the literature.[2] Merton (1971) solved for the optimum consumption and portfolio structure in the case of an investor with constant relative risk aversion facing an investment universe composed of a common stock and a defaultable money market account. The default event is characterized by a Poisson process with constant intensity. He finds that the demand for stock is an increasing function of the intensity of the Poisson process. His setting has two main drawbacks: First, the assumption of a constant intensity of default is not compatible with time-varying credit spreads when the expected recovery rate is deterministic. Second, the interest rate dependent asset is not a bond because it bears no interest rate risk. Hou and Jin (2002) investigate the optimal allocation between equity and defaultable zero-coupon bonds in a framework where interest rates are constant. They show that a non-zero recovery rate induces a stochastic risk premium even when the spread risk is modeled with a constant risk premium. The optimal portfolio is then composed of the speculative demand and an additional term that shows how the investor hedges against this stochastic risk premium.

A whole pan of the literature deals with the bankruptcy *of the investor* in the context of optimal consumption and investment problems.[3] More precisely, they investigate the form of the utility function of the investor when her wealth tends to zero. In contrast, our paper considers the default *of the issuer of the assets* our investor is investing in and assumes that the investor herself does not default during the investment period.

[2] Although the volume of outstanding corporate bonds is higher than the one for government bonds in most countries.

[3] See Sethi (1998) for a survey.

The goal of this paper is to characterize the optimal allocation of a portfolio composed of default-free Treasury bonds and defaultable corporate bonds in an intertemporal setting. The investor's trade-off between risk and return is characterized by a constant relative risk aversion function for wealth at the end of the investment horizon. Equities or other asset classes are not included in the investment universe, mainly because it would imply an extended model, for which we do not have parameter estimates.[4] Nevertheless, this study provides useful insights on the portfolio structure for the cases where the investment universe is extended to other asset classes. In particular, qualitative statements on the portfolio structure for an investment universe including equities are provided. More specifically, the objective of this research is to:

- find the optimal weights of a portfolio composed of default-free and defaultable bonds and report on optimal bond portfolios for investors with different attitudes towards risk and varying investment horizon.
- evaluate the impact of time-varying interest rate and credit risk premia on optimal asset allocation. Several empirical studies (e.g. Chan, Karolyi, Longstaff and Sanders (1992) and Dahlquist (1996)) have detected time-varying interest rate risk premia in the bond market of most countries. Similarly, Liu, Longstaff and Mandell (2000) observe time-varying credit risk premia. Time-varying interest rate and credit risk premia are also observed in Walder (2002) between 1990 and 2000 on the Swiss market.
- look at the effect of interdependencies between market and credit risk on the optimal portfolio structure. Duffee (1998) and Düllmann et alii (1998) find a significant negative relation between credit spreads and Treasury yields using US and German data respectively. In Walder (2002), a positive dependence between interest rates and credit spreads is observed on the Swiss market.

The bonds in our investment universe are priced within an affine term structure framework as defined in Duffie and Kan (1996).[5] Default risk is modeled using the intensity based approach (see Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1999) and others). The approach treats default as an unpredictable event involving a sudden loss in market value. Default is assumed to occur at a risk-neutral hazard rate λ_t at any time t, meaning that the conditional risk-neutral probability of default at time t of a default over a small interval Δt, given no default before t is $\lambda_t \Delta t$. In our framework, the interest rate r_t as well as the intensity of default λ_t are modeled as linear functions of state variables following affine diffusions.

In the most interesting and relevant cases, namely when the interest rate and credit risk premia are time-varying, the investment opportunity set is stochastic and a hedging demand appears in the optimal portfolio policy. We use the approach developed in Detemple, Garcia and Rindisbacher (2002a, DGR in the

[4] Note that additional assets and risk factors pose no theoretical problem in our setup.

[5] See Dai and Singleton (2000) for a general classification of affine term structure models.

sequel) to represent the optimal investment strategy as a combination of the myopic, growth-optimal strategy (speculative demand) and two terms representing the hedge against changes in the short-term interest rate and the market prices of risk, respectively. The two hedge components are given as conditional expectations involving integrals of Malliavin derivatives (MD in the sequel) with respect to the short-term interest rate and the market price of risk vector.

As mentioned above, closed-form solutions for an intertemporal investment problem with changing investment opportunity set are obtained only for special cases.[6] We use the methodology developed in DGR to approximate the MD using Monte Carlo simulation and compute the two hedge components appearing in the formulation of the optimal portfolio. This method of computing the optimal weights shows clear advantages in terms of convergence speed and computation time when compared to alternative methods like PDE methods (Brennan, Schwartz and Lagnado (1997)) and other Monte Carlo estimators (Cvitanic, Goukasian and Zapatero (2000)). Beside its efficiency, the approach is also very flexible allowing to solve investment problems with a high number of assets and state variables following any diffusion process.

The results show that the optimal portfolio structure crucially depends on the specification of the market price of risk. The hedging needs are significantly higher when the risk premia vary over time, i.e. the market prices of risk are stochastic. The Treasury bond is used to hedge against interest rate risk and variations of the market price of interest rate risk. The market price of credit risk is hedged with a portfolio of Treasury bond (long) and corporate bonds (short). The ratio of default-free to defaultable bonds increases with the investment horizon and the risk aversion reflecting increased hedging needs. Positive (negative) interdependencies between the interest rate and the credit risk increasing (decreasing) the variability of the portfolio value increase (decrease) the hedging needs.

Our work differs from Hou and Jin (2002) in the following aspects. First, the investment universe of our investor is composed of default-free and defaultable bonds whereas it contains equity and defaultable bonds in Hou and Jin (2002). Second, the interest rate and the market price of interest rate risk are stochastic in our case. The interest rate is constant and the market price of interest rate risk is zero in Hou and Jin (2002). Third, the credit risk hedge is due to the stochastic default risk premium in our paper whereas it is due to a non-zero recovery rate in Hou and Jin (2002).

The paper is organized as follows: We present the investment problem in Section 2 and describe the complete market solution in Section 3. Section 4 is devoted to the description and the analysis of the optimal portfolio structure in various cases of the affine framework. The effect of time-varying risk premia and interdependencies between the interest rate and the credit risk on the optimal portfolio structure is also dealt with in this section. Section 5 concludes and proposes ideas for further research. The proofs are in Appendix A unless otherwise stated.

[6] Most of them are described in Merton (1990), Chapter 5.

2 The Investment Problem

This section describes the portfolio problem faced by the investor. It begins with a description of the information structure in our economy and presents the basic assets: the instantaneously risk-free money market account (MMA), Treasury bonds and corporate bonds.[7] There are basically two approaches for the pricing of corporate bonds. The structural approach pioneered by Merton (1974) explicitly models the evolution of firm value. The firm defaults when its market value falls below certain exogenously given threshold level or the value of its debt. The second approach, called reduced form approach or intensity based approach, is adopted by Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1997, 1999) and others. The intensity based approach defines default as the first jump of a Cox process with intensity λ_t. It does not try to identify the sources of default risk, but assumes that all relevant information about default risk is reflected in the observed credit spreads. The observed credit spreads or term structures form the basis for the estimation of all model parameters and for the determination of the state variable level. As a consequence, pricing errors are smaller compared to the structural approach, and the dynamics of the credit spreads are well captured.[8] This paper adopts the intensity based approach for the modeling of defaultable bonds, because the accurate modeling of the bond price dynamics is particularly important when allocating wealth in an intertemporal setting. In particular, a pricing model reflecting the empirically observed time-variation of the risk premia and the dependencies between yields will give valuable insights about the implications of these characteristics on the optimal portfolio structure.

After the description of the bond pricing approach, we derive the process of bond portfolios assuming that the investor is mainly interested in the allocation of wealth between well-diversified corporate and Treasury bond portfolios. We proceed by specifying the joint asset price process in the affine setting and the description of the investment problem with the investor's objective function.

2.1 The Economy

The economy is defined through its information structure and the assets traded in its financial market. The information structure of our economy is indexed by a time interval $[0, T^*]$. We start by representing the information uncertainty with a filtered probability space $\left(\Omega, \mathcal{F}^Y_{T^*}, \left\{\mathcal{F}^Y_t\right\}_{t=0}^{T^*}, P^Y\right)$ where $\mathcal{F}^Y_t$ is the filtration generated by some state variable vector Y and P^Y is the corresponding empirical probability measure.

We assume the existence of a progressively measurable and integrable short rate process r, so that the investor can place one unit of currency in a money

[7] Defaultable bonds and corporate bonds as well as default-free and Treasury bonds are equivalent in our terminology.

[8] See Walder (2002) for a more detailed discussion on the properties of both approaches.

market account (MMA) at time zero and roll over the proceeds until time t. The value $B(t)$ of the instantaneously risk-free MMA at time t is then

$$B(t) = \exp\left(\int_0^t r(u, Y_u)\, du\right).$$

In addition, there is a collection of default-free zero-coupon bonds with nominal value equal to one. Assuming the existence of an equivalent martingale measure Q^Y, the price at time t of the default-free bond maturing in T satisfies

$$B(t,T) = E^{Q^Y}\left[\exp\left(-\int_t^T r(u, Y_u)\, du\right) | \mathcal{F}_t^Y\right].$$

The third asset category in our economy are defaultable zero-coupon bonds. The defaultable bonds are priced using the intensity based approach proposed in Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1999) and others. In order to describe the payoff of defaultable bonds, we enlarge the filtration $\mathcal{F}_t^Y$ to include default information. The default information is generated by the particular default model. In the case of intensity models, the default of entity i is modeled through the default process $N_t^i = \mathbf{1}_{\{\tau^i \leq t\}}, i = 1, ..., I$ that jumps from zero to one at the time of default of entity i. The nonnegative process λ^i is called the intensity of the default process N_t^i if the compensated point process $M_t^i = N_t^i - \int_0^t \lambda_u^i du$ is a local martingale. The intensity process is adapted to the filtration $\mathcal{F}_t^Y$ generated by the state variable process Y. We denote by $\mathcal{G}_t^i$ the filtration generated by the i^{th} default process N^i. Our extended economy is now described by the filtered probability space $\left(- , \mathcal{F}_{T^*}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P\right)$ where the augmented filtration $\mathcal{F}_t = \mathcal{F}_t^Y \vee \mathcal{G}_t^1 \vee \mathcal{G}_t^2 \vee ... \vee \mathcal{G}_t^I$ contains the filtration generated by the state variables and the I default processes. The probability measure P is an extension of P^Y to $\mathcal{F}_{T^*}$.

Using the above information structure, we are able to price the defaultable bond by discounting its payoff under the risk-neutral measure Q which is an extension of Q^Y to $\mathcal{F}_{T^*}$. The price $D^i(t,T)$ at time t of a zero-coupon bond maturing in T and issued by entity i is

$$D^i(t,T) = E^Q\left[\exp\left(-\int_t^T r(u, Y_u)\, du\right) \mathbf{1}_{\{\tau^i > T\}} | \mathcal{F}_t\right] \tag{1}$$

when zero-recovery is assumed. The zero-recovery assumption is made in order to keep the portfolio problem simple. The dynamics of defaultable bond prices becomes mathematically involved when other recovery assumptions such as stochastic recovery, fractional recovery of Treasury or fractional recovery of face value are made. Moreover, the pricing and the estimation of models with stochastic recovery can be quite complex even for cases with a single stochastic factor (see Bakshi, Madan and Zhang (2001)). Therefore, this paper focuses on the effect of default risk on the optimal portfolio structure and keeps the analysis of the recovery assumption on the optimal portfolio for further research.

Through an appropriate change of filtration in (1) and the use of the properties of the conditional expectation operator,[9] we obtain the well-known expression

$$D^i(t,T) = \mathbf{1}_{\{\tau^i>t\}} E^Q\left[\exp\left(-\int_t^T \left(r(u,Y_u) + \widetilde{\lambda}^i(u,Y_u)\right) du\right) |\mathcal{F}_t^Y\right] \quad (2)$$

for the defaultable bond price based on the filtration $\mathcal{F}_t^Y$ generated by the state variables Y where $\widetilde{\lambda}^i$ is the default intensity under Q.[10] This formulation is appreciated because the point process has disappeared from the expectation operator and the resulting expression is similar to traditional default-free bond price formulas allowing the use of this knowledge to model the state variable process and solve the expectation. The probability measure Q equivalent to P on (- , $\mathcal{F}_t$) is defined by the $\mathcal{F}_t$-martingale $\eta_t^* := \frac{dQ}{dP}_{|\mathcal{F}_t}$. The Radon-Nikodym density process η^* is strictly positive and admits the integral representation

$$\eta_t^* = 1 + \int_{]0,t]} \eta_{u^-}\left(\varphi_u dW_u + \sum_{i=1}^I \zeta_u^i dM_u^i\right)$$

where W is a vector of independent standard Brownian motions. The market price of diffusive risk φ and the market price of default event risk $\zeta^i, i = 1, ..., I$ are $\mathcal{F}_{T^*}$-predictable processes. The dynamics of the defaultable bond price $D^i(t,T)$ under P will be of the form

$$dD^i(t,T) = D^i\left(t^-,T\right)\left[\left(r(t,Y_t) + \sigma^i(t,Y_t)\,\Theta(t,Y_t)\right)dt + \sigma^i(t,Y_t)\,dW_t - dM_t^i\right]$$

where Θ is a vector of market prices of risk and σ^i may be interpreted as the sensitivity of the bond price with respect to the sources of risk. Thus, a defaultable bond offers a premium $\sigma^i(t,Y_t)\,\Theta(t,Y_t)$ over the risk free rate to compensate the investor for taking interest rate and credit risk. The process experiences a sudden drop of dM_t^i in its value when the entity defaults.

2.2 Bond Portfolios

In the previous section, we have shown how default-free and defaultable bonds are priced in our economy. But typically, the portfolio allocation decision is not made at the bond level. Like in the stock market, the portfolio manager

[9] See for example Corrolary 1.1 in Jeanblanc and Rutkowski (2002).

[10] Note that the valuation formula (2) is consistent with the fractional recovery of market value assumption of Duffie and Singleton (1999) when $\widetilde{\lambda}$ is defined as the product of the default intensity and the loss rate in case of default. In this case $\widetilde{\lambda}$ is called the short credit spread. Under this assumption, the effect of default risk and recovery risk on the optimal portfolio cannot be disentangled. Decreasing the loss rate and increasing the default probability by the same ratio will result in approximately the same credit spread. Consequently, we do not follow this interpretation in the sequel.

first groups similar bonds into segments and then allocates between these segments. A typical group would be a rating class. The advantage of investing in a portfolio of bonds with the same rating is that the specific risk (in particular the idiosyncratic credit risk) of each bond is diversified away within the rating class. The portfolio manager obtains an exposure to the systematic credit risk of that rating class by investing in such a portfolio. In return, he expects a risk premium that will compensate for his risk taking. This diversification argument obviously does not hold for the Treasury market as there is no risk of default. In other words, a portfolio of zero-coupon Treasury bonds maturing in 5 years has the same risk as a 5 year zero-coupon Treasury bond with the same nominal.[11]

Thus, the relevant question in bond portfolio allocation is: How to allocate the wealth between a Treasury bond and a well diversified corporate bond portfolio of a given rating class? Consequently, the relevant assets will be the Treasury bond with price $B(t,T)$ and a portfolio of similar defaultable bonds with price $D(t,T)$. Based on Jarrow, Lando and Yu (2000), we will derive the value $D(t,T)$ of such a portfolio and compare it to the value of an individual bond in the following.

We assume that the I bonds are issued by I firms belonging to the same rating class, i.e. the firms have the same default intensity λ and sensitivities σ but different default times $\tau^i, i = 1,...,I$. We further assume that conditional on the state variable vector Y all default processes are independent. The terminal payoff of an equally weighted portfolio of the I bonds with maturity T is $\frac{1}{I}\sum_{i=1}^{I} 1_{\{\tau^i > T\}}$ assuming no recovery as in the previous section. The value $D^I(t,T)$ of this portfolio is:

$$D^I(t,T) = E^Q\left[\exp\left(-\int_t^T r_u du\right)\frac{1}{I}\sum_{i=1}^{I} 1_{\{\tau^i>T\}}|\mathcal{G}_t\right]$$

Jarrow, Lando and Yu (2000) show that the value of the diversified portfolio containing $I \to \infty$ securities is

$$D(t,T) \equiv \lim_{I\to\infty} D^I(t,T) = E_t^Q\left[\exp\left(-\int_t^T (r(u,Y_u) + \lambda(u,Y_u))\, du\right)|\mathcal{F}_t^Y\right]$$

by using the strong law of large numbers and dominated convergence. By price linearity, we observe that the value of the equally weighted portfolio $D(t,T)$ is equal to the value of the defaultable bond $D^i(t,T)$ conditional on no default by t. Jarrow, Lando and Yu (2000) further show that the statistical and the risk-neutral default intensities are identical if default-risk is diversifiable. Empirical support for this argument can be found in Chava and Jarrow (2001).

A significant difference between the equally weighted portfolio and the defaultable bond exists at the level of the stochastic differential equations (SDE)

[11] A diversification along the maturities may be reasonable for the Treasury and corporate bond markets, but this issue is not dealt with here. The interest rates of various maturities are highly correlated which substantially limits diversification benefits across maturities.

though. The SDE of the defaultable bond as shown in the previous section has the form

$$\frac{dD^i(t,T)}{D^i(t^-,T)} = \left(r(t,Y_t) + \sigma^i(t,Y_t)\,\Theta(t,Y_t)\right)dt + \sigma^i(t,Y_t)\,dW_t - dM_t^i, \quad i = 1, ..., I$$

where M_t^i is the compensated point process $M_t^i = N_t^i - \int_0^t \lambda_u^i du$. The last term represents the individual default risk, and drops by -1 at default. On the other hand, the portfolio of equally weighted bonds with identical default intensities follows the price dynamics

$$\frac{dD(t,T)}{D(t^-,T)} = \left(r(t,Y_t) + \sigma^I(t,Y_t)\,\Theta(t,Y_t)\right)dt + \sigma^I(t,Y_t)\,dW_t$$

where $\sigma^I(t,Y_t)$ is the sensitivity of the bond portfolio price with respect to the sources of risk. This portfolio has the same exposure to systematic risk as the bond with price $D^i(t,T)$ but is not exposed to the idiosyncratic default event risk. In other words, the specific default risk of the firm conditional on the default intensity has been fully diversified away. It also means that the idiosyncratic default risk is not rewarded in the limit economy.

Consequently, we assume for the rest of the paper to be in a setting where default risk is conditionally diversifiable in the sense of Jarrow, Lando and Yu (2000). The investor invests in a Treasury bond and a portfolio of infinitely many defaultable bonds with identical default intensity λ. This portfolio is exposed only to systematic risk factors whereas the idiosyncratic default event risk has been diversified away. The set of pricing measures is restricted when the number of instruments I in the portfolio is large as in this context.[12] In particular, the Radon-Nikodym density process $\eta_t := \frac{dQ}{dP}_{|\mathcal{F}_t}$ in this limit economy is $\mathcal{F}_t^Y$-adapted and we will assume that it satisfies

$$\eta_t = 1 + \int_0^t \eta_u \theta_u dW_u$$

for some market price of systematic risk vector θ. The purpose of the next section is to define the relation between the interest rate r, the default intensity λ, the market price of systematic risk vector θ and the state variable vector Y.

2.3 Affine Framework

For the rest of the article, we will restrict the framework to the so-called affine case. In this framework, the short rate r and the intensity λ as well as the conditional mean and variance of the state variable process Y are affine functions of Y.[13] These special assumptions imply that the yields of the bonds are affine

[12] See Jarrow, Lando and Yu (2000).

[13] In the sequel, we make no difference between the empirical and the risk-neutral default intensities because they are equal in the limit economy.

in the state variables. It is questionable, whether the interest rate, the intensity and the state variables follow such simple processes in reality. But more complex models imply more variables to estimate which is a real problem given the scarce data situation in many countries and the econometric identification difficulties inherent to term structure models. Deriving portfolio management implications based on a complex model with ad-hoc parameter values is shaky due to the lack of empirical foundation. For this reason, we prefer to base the allocation model on a simple framework for which parameters may be estimated.[14] In particular, we will use the specification and the estimates in Walder (2002) for the Swiss market.

The short rate r and the default intensity λ are affine functions of two state variables $Y_1(t)$ and $Y_2(t)$:

$$\begin{aligned} r(t, Y_t) &= \delta_0 + \delta_1 Y_1(t) \\ \lambda(t, Y_t) &= \delta_2 Y_2(t) \end{aligned}$$

The advantage of this specification is that the interpretation of the two state variables is made clear. The first state variable can be interpreted as the interest rate risk and the second state variable is the systematic credit risk factor. Moreover, the number of parameters to estimate are reasonable with only two state variables. The state variable vector $Y(t) = (Y_1(t), Y_2(t))$ follows an "affine diffusion" of the form[15]

$$\begin{aligned} dY(t) &= \widetilde{\mu}^Y(t, Y_t)\, dt + \sigma^Y(t, Y_t)\, d\widetilde{W}(t) \\ &= \widetilde{\kappa}\left(\widetilde{\vartheta} - Y(t)\right) dt + \Sigma\sqrt{S(t, Y_t)}\, d\widetilde{W}(t) \quad . \end{aligned} \tag{3}$$

$\widetilde{W}(t)$ is a 2-dimensional independent standard Brownian motion vector under the equivalent martingale measure $\underset{\sim}{Q}$, $\widetilde{\kappa}$ and Σ are 2×2 matrices, which may be non-diagonal and asymmetric,[16] $\widetilde{\vartheta} \in \mathbb{R}^2$, and $S(t, Y_t)$ is a diagonal matrix with the i^{th} element given by

$$[S(t, Y_t)]_{ii} = \alpha_i + \beta_i' Y(t) \quad i = 1, 2$$

where $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^2$.[17] In this setting, both the drift and the conditional covariance of the state variable vector is affine in $Y(t)$. The Vasicek and the

[14] See Walder (2002) for detailed discussion on the trade-off between model complexity and estimation quality.

[15] For simplicity, we do not consider the possibility of jumps. Duffie and Kan (1996) and Duffie, Pan and Singleton (2000) show that introducing jumps into affine term structure models preserves the affine dependence of yields on state variables provided the jump arrival intensity is an affine function of the state vector and the distribution of the jump sizes depends only on time.

[16] We will consider the case where the matrix $\widetilde{\kappa}$ is diagonal as well as the case where the matrix contains non-zero values in the off-diagonal.

[17] Models with more state variables allow for richer correlation structures (see for example Akgun (2001)), but are generally more difficult to estimate .

Cox-Ingersoll-Ross (CIR in the sequel) processes are special cases of this structure. In this paper, we will consider the cases where the state variables follow independent as well as dependent Vasicek and CIR processes.

Provided the parameterization is admissible, we know from Duffie and Kan (1996) that

$$\begin{aligned} B(t,T) &= \exp\left(a^B(\tau) + b^B(\tau)' Y(t)\right) \\ D(t,T) &= \exp\left(a^D(\tau) + b^D(\tau)' Y(t)\right) \end{aligned} \tag{4}$$

where $\tau = T - t$, and $a^k(\tau)$ as well as $b^k(\tau)$ for $k = B, D$ satisfy the ordinary differential equations (ODE) known as Ricatti equations:

$$\frac{\partial b^k(\tau)}{\partial \tau} = -\widetilde{\kappa}' b^k(\tau) + \frac{1}{2}\sum_{i=1}^{2}\left[\Sigma' b^k(\tau)\right]_i^2 \cdot \beta_i - \delta^k$$

$$\frac{\partial a^k(\tau)}{\partial \tau} = \widetilde{\vartheta}\widetilde{\kappa}' b^k(\tau) + \frac{1}{2}\sum_{i=1}^{2}\left[\Sigma' b^k(\tau)\right]_i^2 \cdot \alpha_i - \delta_0 \quad .$$

Here, $[H]_i$ denotes the i^{th} row of H, $\delta^B = \begin{pmatrix} \delta_1 & 0 \end{pmatrix}'$ and $\delta^D = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix}'$. The corresponding boundary conditions $b^k(0) = 0$ and $a^k(0) = 0$ for $k = B, D$ guarantee that the prices of the bonds equal one at maturity.

We assume that the market prices of risk are given by

$$\theta(t, Y_t) = \sqrt{S(t, Y_t)}\rho, \tag{5}$$

where ρ is a 2×1 vector of constants. The functional form has been used for example in Dai and Singleton (2000) and has the advantage that the state variable process keeps its affine structure under the natural probability measure. We obtain the process for the state variables under the natural probability measure P:

$$dY(t) = \kappa(\vartheta - Y(t))\,dt + \Sigma\sqrt{S(t, Y_t)}dW(t) \tag{6}$$

by substituting the Q-Brownian motion in (3) by the corresponding expression under P using Girsanov theorem and (5). Thus, $W(t)$ is a standard Brownian motion under P, $\kappa = \widetilde{\kappa} - \Sigma\phi$, $\vartheta = \kappa^{-1}\left(\widetilde{\kappa}\widetilde{\vartheta} + \Sigma\psi\right)$, where the i^{th} row of ϕ is given by $\rho_i\beta_i^T$, and ψ is a two-dimensional vector whose i^{th} element is $\rho_i\alpha_i$.

We are now in a situation to derive the joint process of the asset prices in our economy:

Proposition 1 *Assuming the affine framework described above, the joint process for the Treasury and the corporate bond prices*

$$P(t,T) = (B(t,T), D(t,T))'$$

is denoted by:

$$dP(t) = I_{P(t)}\left[\mu(t, Y_t)\,dt + \sigma(t, Y_t)\,dW(t)\right]$$

where $I_{P(t)}$ is a diagonal matrix with the elements of $P(t)$ in the diagonal. The drift of the asset price process is

$$\mu(t, Y_t) = r(t, Y_t) \cdot \iota_2 + b(T-t)\,\Sigma S(t, Y_t)\,\rho,$$

with

$$b(T-t) = \begin{pmatrix} b^B(T-t) & b^D(T-t) \end{pmatrix}',$$

and ι_2 being a 2-dimensional vector of ones. The diffusion part is

$$\sigma(t, Y_t) = b(T-t)\,\Sigma\sqrt{S(t, Y_t)}\quad.$$

Proof. In the Appendix.

The market price of risk

$$\theta(t, Y) = \sigma(t, Y_t)^{-1}\left[\mu(t, Y_t) - r(t, Y_t) \cdot \iota_2\right],$$

assuming $\sigma(.)$ is invertible, may be stochastic (CIR case) or constant (Vasicek case) depending on its concrete specification.[18] Markets are assumed to be complete in the limit economy, implying the existence of a unique state price density ξ_t, given by

$$\xi_t = B_t^{-1}\eta_t$$

where η_t denotes the $\mathcal{F}_t^Y$-adapted Radon-Nikodym derivative defined by

$$\eta_t = \exp\left(-\int_0^t \theta(u, Y_u)'\,dW_u - \frac{1}{2}\int_0^t \theta(u, Y_u)'\,\theta(u, Y_u)\,du\right).$$

Alternatively, we may write the state price density process as

$$\frac{d\xi_t}{\xi_t} = -r(t, Y_t)\,dt - \theta(t, Y_t)'\,dW(t); \qquad \xi_0 = 1.$$

Relative state prices are denoted $\xi_{t,s} = \xi_s/\xi_t$.

2.4 Investor's Objective Function

In order to meet her investment goals the investor chooses a portfolio consisting of the two – instantaneously – risky assets, a Treasury bond with price $B(t,T)$ and the well-diversified portfolio of corporate bonds with price $D(t,T)$, and the instantaneously risk-free MMA. We denote the proportions invested in the risky assets at time t by $\pi_t \in \mathbb{R}^2$. The proportion of wealth invested in the MMA at time t is $1 - \pi_t'\iota_2$. There are no short-sales restrictions. For any self-financing portfolio strategy, the wealth X_t of the investor follows the dynamics:

$$dX_t = r(t, Y_t)\,X_t dt + X_t\pi_t'\left[\left(\mu(t, Y_t) - r(t, Y_t)\,\iota_2\right)dt + \sigma(t, Y_t)\,dW(t)\right]; \qquad X_0 = x$$

[18] Both cases are dealt with more deeply in Section 4.

The investor seeks to maximize the expected utility of terminal wealth by choosing the optimal investment strategy $\{\widehat{\pi}_t\}_{t\in[0,\overline{T}]}$ where $\overline{T} \leq T$ is the end of the investment period. For an investor with power utility function implying a constant relative risk aversion γ (CRRA), the objective function is

$$\max_{\pi} U\left(X_{\overline{T}}\right) \equiv E\left[\frac{1}{1-\gamma}\left(X_{\overline{T}}\right)^{1-\gamma}\right]$$

such that

$$\begin{aligned} dX_t &= r\left(t, Y_t\right) X_t dt + X_t \pi_t^{'}\left[\left(\mu\left(t, Y_t\right) - r\left(t, Y_t\right) \iota_2\right) dt + \sigma\left(t, Y_t\right) dW\left(t\right)\right]; \qquad X_0 = x \\ X_t &\geq 0 \quad \text{for all } t \in [0, \overline{T}] \end{aligned}$$

The solution of this investment problem is the topic of the next section.

3 The Complete Market Solution

In the previous section, we have defined the optimization problem of the investor. She wants to maximize the utility of terminal wealth by investing in Treasury and corporate bonds as well as in the MMA. The solution to this investment problem is non-trivial for two main reasons. First, both the interest rate and the market prices of risk (interest rate and systematic credit risk) are stochastic in the more realistic cases where the risk premia vary over time. Stochastic market prices of risk induce a hedging demand in the investor's optimal portfolio that protects her optimal terminal wealth against variation of the investment opportunity set. The complexity of the solution comes from the evaluation of this hedging demand. Second, interdependencies between the state variables are also allowed, which hinders an analytical solution to the investment problem. In order to solve complex investment problems, various numerical techniques have been used in the literature. Brennan, Schwartz and Lagnado (1997) and Xia (2001) solve the partial differential equation appearing in the dynamic programming approach numerically. Campbell and Viceira (2001) and Barberis (2000) use discrete time modeling based on approximated Euler equations. Cvitanic, Goukasian and Zapatero (2000) compute numerically the covariance between wealth and uncertainty shocks and compare it to the conditional covariance of the wealth process to obtain the optimal weights. We follow the approach developed in Detemple, Garcia and Rindisbacher (2002a) because it has the following advantages: First, it proposes a nice representation of the optimal portfolio, which expresses the optimal investment strategy as a combination of the myopic, mean-variance optimal strategy and two terms representing the hedge against changes in the short-term interest rate and the market prices of risk respectively. Second, analytical solutions can be easily derived when they exist based on the representation of the optimal portfolio and using Malliavin calculus. And finally, DGR have developed a flexible and efficient numerical procedure to evaluate the Malliavin derivatives when analytical solutions are not available. This is the case when stochastic market prices of risk and interdependencies between the state variables are considered. We will need all these advantages to derive the composition of the optimal portfolio for various special cases of the affine framework.

The representation of the optimal portfolio developed in DGR allows us to write the solution to the investment problem described in Section 2 as:[19]

$$\widehat{\pi}_t = \widehat{\pi}_t^{MV} + \widehat{\pi}_t^{IR} + \widehat{\pi}_t^{MPR} \tag{7}$$

with[20]

$$\widehat{\pi}_t^{MV} = \frac{1}{\gamma} \left(\sigma \left(t, Y_t \right)' \right)^{-1} \theta \left(t, Y_t \right) \tag{8}$$

$$\widehat{\pi}_t^{IR} = \left(\frac{1}{\gamma} - 1 \right) \left(\sigma \left(t, Y_t \right)' \right)^{-1} E_t \left[\frac{\xi_{t,\overline{T}}^{1-1/\gamma} \int_t^{\overline{T}} \mathcal{D}_t r \left(s, Y_s \right) ds}{E_t \left[\xi_{t,\overline{T}}^{1-1/\gamma} \right]} \right]' \tag{9}$$

$$\widehat{\pi}_t^{MPR} = \left(\frac{1}{\gamma} - 1 \right) \left(\sigma \left(t, Y_t \right)' \right)^{-1} E_t \left[\frac{\xi_{t,\overline{T}}^{1-1/\gamma} \int_t^{\overline{T}} \left(d\widetilde{W}_s \right)' \mathcal{D}_t \theta \left(s, Y_s \right)}{E_t \left[\xi_{t,\overline{T}}^{1-1/\gamma} \right]} \right]' \tag{10}$$

This representation is very convenient for the analysis of the optimal portfolio structure. The optimal portfolio weights $\widehat{\pi}_t$ is composed of three parts, the mean-variance portfolio $\widehat{\pi}_t^{MV}$ and two hedging components $\widehat{\pi}_t^{IR}$ and $\widehat{\pi}_t^{MPR}$ which are given as conditional expectations involving integrals of MD with respect to the short-term interest rate and the market price of risk vector.

The mean-variance component $\widehat{\pi}_t^{MV}$ is equal to the optimal portfolio for a mean-variance optimizing investor with a short (single-period) investment horizon. This component is also called the speculative demand. The interest rate hedge (IRH) component $\widehat{\pi}_t^{IR}$ reflects the hedging behavior of the investor against stochastic variations of the interest rate. The conditional expectation in (9) represents the covariance between the present value of optimal terminal wealth and the cumulative sensitivity of the interest rate to shocks in the Brownian motions W_1 and W_2. The present value of optimal terminal wealth is represented as a functional of the state price density ξ_t in the case of a power utility function. The Malliavin derivative $\mathcal{D}_t r \left(s, Y_s \right)$ captures the sensitivity of the interest rate at time $s > t$ to past shocks in the Brownian motions W_1 and W_2 at time t.[21] The higher this covariance (high sensitivity of the interest rate to random shocks or high correlation between the optimal wealth and the interest rate sensitivity), the more the optimal wealth is exposed to interest rate risk. The investor will then make use of the optimal mix of available assets (through their sensitivities $\sigma \left(t, Y_t \right)$) to reduce the exposition of the optimal terminal wealth to interest rate risk.

[19] See DGR, Theorem 1.

[20] To simplify the notation, we denote by $E_t\left[.\right]$ the expectation conditional on the filtration $\mathcal{F}_t^Y$ generated by the state variable process Y_t.

[21] See DGR, Appendix D for an introduction to Malliavin calculus.

Similarly, the market price of risk hedge (MPRH) component $\widehat{\pi}_t^{MPR}$ reflects the hedging behavior of the investor faced with stochastic market prices of risk. In our case, the market price of interest rate risk and the market price of credit risk can be stochastic. The conditional expectation in (10) represents the covariance between the present value of the optimal terminal wealth and the future changes in the market prices of risk due to shocks in the Brownian motions W_1 and W_2. The higher the covariance, the higher the hedging needs. The MPRH is equal to zero when the market prices of risk are not stochastic.

The optimal portfolio for an investor with risk aversion $\gamma = 1$ (equivalent to an investor with logarithmic utility function) is equal to the mean-variance portfolio. Both hedge components are equal to zero in this case. In other words, the investor with logarithmic utility function cares about the return-risk trade-off of the assets through the mean-variance component, but does not care about future changes in the reward-to-risk ratio. When the risk aversion tends to infinity, the speculative demand is equal to zero and the investor hedges against variations of both the interest rate and the market price of risk completely.

In the following, we formulate the mean-variance component in the affine case more precisely and show how the two hedge components are determined based on DGR.

The mean-variance weights in (8) can be determined more precisely in the general affine case using the formulation of the market price of risk and the asset variance developed in Section 2.3:

$$\widehat{\pi}_t^{MV} = \frac{1}{\gamma}\left(b\left(T-t\right)'\right)^{-1}\left(\Sigma'\right)^{-1}\rho \tag{11}$$

The striking feature of this expression is that the mean-variance optimal weights do not depend on any state variable. This means that a myopic investor makes no market timing in the affine framework. The weights are deterministic and change only because the sensitivities $b\left(T-t\right)$ of the bonds with respect to the sources of risk decrease as the maturities of the bonds approach. The critical assumption for this result is that the market price of risk $\Lambda\left(t, Y_t\right)$ and the conditional variance $\sigma^Y\left(t, Y_t\right)$ of the state variable process have the same order of dependence with respect to the state variable. Both depend on the square-root of the state variables in the CIR case and both are constant in the Vasicek case.

We now devote the rest of this section to the computation of the hedging terms. Using basic rules of Malliavin calculus, we express the MD of the interest rate in (9) and the MD of the market prices of risk in (10) in terms of the MD of the state variable vector:

$$\begin{aligned} \mathcal{D}_t r\left(t, Y_t\right) &= \partial_2 r\left(t, Y_t\right)\mathcal{D}_t Y\left(s, Y_s\right) \\ &= \begin{pmatrix} \delta_1 & 0 \end{pmatrix}\mathcal{D}_t Y\left(s, Y_s\right) \end{aligned} \tag{12}$$

and

$$\mathcal{D}_t\theta\left(s, Y_s\right) = \partial_2\theta\left(t, Y_t\right)\mathcal{D}_t Y\left(s, Y_s\right) \tag{13}$$

where $\mathcal{D}_t Y(s)$ is a $\left(L^2([0,T])\right)^2$ −valued random variable. This expression shows that the computation of the MD of the interest rate and the market prices of risk can be reduced to the computation of the MD of the state variables.

DGR shows how Malliavin derivatives of diffusions like our state variables can be represented as the solution to a linear stochastic differential equation. More precisely, the solution $\mathcal{D}_t Y(s, Y_s)$ of this diffusion process has the form:[22]

$$\mathcal{D}_t Y(s, Y_s) = \sigma^Y(t, Y_t) \exp\left[\int_t^s dL_v\right] \tag{14}$$

with the 2×2 random variable dL_v defined by

$$dL_v \equiv \left[\partial_2 \mu^Y(v, Y_v) - \frac{1}{2}\sum_{j=1}^{2} \partial_2 \sigma_j^Y(v, Y_v)\, \partial_2 \sigma_j^Y(v, Y_v)'\right] dv + \sum_{j=1}^{2} \partial_2 \sigma_j^Y(v, Y_v)\, dW_{jv}$$

where $\sigma_j^Y(.)$ denotes the j^{th}−column of the matrix σ^Y. Thus, the MD $\mathcal{D}_t Y(s, Y_s)$ of the state variable vector can be computed analytically in the cases where the sensitivities $\partial_2 \sigma_j^Y(v, Y_v)$, $j = 1, 2$ of the diffusion of the state variable process with respect to the state variable vector itself has a simple form (zero or constant). We will use this property in the Vasicek case (see Section (4.2)).

In more general cases, it is still possible to evaluate the state variables Y_s and the MD $\mathcal{D}_t Y(s, Y_s)$ of the state variables for $s \in \left[t, \overline{T}\right]$ by simulation. The simulation is performed by applying a standard Euler scheme to (6) and the stochastic differential equation corresponding to (14). But DGR show that the speed of convergence of the simulation can be increased substantially by a change of variable that normalizes the volatility of the state variable process to a constant. This transformation allows to express the MD of the state variables in terms of Riemann-Stieltjes integrals of first and second derivatives of the coefficients of the state variable vector and eliminates the stochastic integral. The Euler scheme then converges weakly at speed $1/\varpi$, where ϖ is the number of time discretization points, compared to $1/\sqrt{\varpi}$ without the transformation. We use a multivariate extension of this variance stabilizing transformation to compute the MD of the state variables (see Theorem 2 in Detemple, Garcia and Rindisbacher (2002b)[23] which is the multivariate extension of the variance stabilizing transformation described in Detemple, Garcia and Rindisbacher (2002a)). To obtain values for the hedging terms (9) and (10), the joint system

$$\left(Y_s, \mathcal{D}_t Y_s, \xi_{t,s}, H_{t,s}^{IR}, H_{t,s}^{MPR}\right)$$

where

$$dH_{t,s}^{IR} = \partial_2 r(s, Y_s)\, \mathcal{D}_t Y(s, Y_s)\, ds$$

and

$$dH_{t,s}^{MPR} = \left(d\widetilde{W}_s\right)' \partial_2 \theta(s, Y_s)\, \mathcal{D}_t Y(s, Y_s) \tag{15}$$

[22] See Theorem 1 in Detemple, Garcia and Rindisbacher (2002a).

[23] This Theorem has been reproduced in Appendix B for the reader's convenience.

is then simulated simultaneously. This method of computing the optimal weights shows clear advantages in terms of convergence speed and computation time when compared to alternative methods like PDE methods (Brennan, Schwartz and Lagnado (1997)) and other Monte Carlo estimators (Cvitanic, Goukasian and Zapatero (2000)).[24] Beside its efficiency, the approach is also very flexible allowing to solve investment problems with high number of assets and state variables following any diffusion process.

[24] See DGR, Section VII.

4 Results

The previous section has shown how the work of DGR is used to solve our investment problem and quantify the optimal portfolio composition in the general affine case. In the present section, we will investigate the optimal portfolio for various special cases belonging to the general affine case. We are particularly interested in the structure of the optimal portfolio and its sensitivity to the investment horizon $\overline{T}$, the maturity T of the Treasury and corporate bonds, the level of the term structure and the risk aversion γ of the investor. The sensitivity of the weights to the process parameter values is also discussed.

We start in Section 4.1 with the Cox-Ingersoll-Ross (CIR) case where the state variables follow independent square-root processes. Beside the speculative demand, the optimal portfolio contains an interest rate and a market price of risk hedge component in the CIR case because both the interest rate and the market price of risk vector are stochastic. This contrasts to the so-called Vasicek case where the state variables follow a mean-reverting process with constant conditional covariance. By comparing the Vasicek case analyzed in Section 4.2 with the CIR case, we will learn about the implication of time-varying risk premia on the optimal portfolio composition. Interdependencies between the interest rate and the credit risk factor are introduced and analyzed in Section 4.3. We show that the investor also uses corporate bonds for hedging interest rate risk when interdependencies are allowed. Finally, we make some thoughts about the structure of the optimal portfolio when the investor cares about parameter uncertainty in Section 4.4.

The results of this section are based on the following basis case: Both the Treasury bond and the corporate bond portfolios have a maturity $T = 10$ years. The investor has an investment horizon $\overline{T} = 1$ year and a risk aversion of $\gamma = 2$. Brennan, Schwartz and Lagnado (1997) work with a risk aversion of $\gamma = 6$ for an asset allocation problem involving cash, a consol bond and an equity portfolio in a single currency. Brennan and Xia (2000) work with risk aversion parameters between $\gamma = 2$ and $\gamma = 6$ for an investment universe containing cash, an equity security and bonds in a single currency. When interdependencies between the state variables are introduced, we make sure that the resulting models are admissible in the sense of Dai and Singleton (2000).[25] The values for all the process parameters are the one estimated in Walder (2002) for the Swiss market.[26] The parameter estimates are reproduced in Table 1 for the reader's convenience.

[Insert Table 1: "Parameter Estimates" here]

[25] Dai and Singleton (2000) have shown that all possible d-factor affine term structure models (ATSM) can be classified in $d+1$ non-nested models. The classification depends on the number m of state variables that determine the conditional variance matrix of the state variables. For the case $d = 2$, we obtain 3 non-nested affine models $A_m(2), m = 0, 1, 2$.

[26] For the cases with independent state variables, we set the parameters representing the dependence to zero.

4.1 CIR Case

Several empirical studies (e.g. Chan, Karolyi, Longstaff and Sanders (1992) or Dahlquist (1996)) have found evidence for time-varying interest rate risk premia in the bond market of most countries. Similarly, Liu, Longstaff and Mandell (2000) observe time-varying credit risk premia by investigating the US swap-Treasury spreads between 1988 and 2000. Time-varying interest rate and credit risk premia are also observed in Walder (2002) between 1990 and 2000 on the Swiss market.

A straightforward way of taking account of time-varying risk premia in our affine context is to model the state variables with independent square-root processes. We refer to this model as the CIR case and model the state variable process is as in (3) with a mean-reverting drift and a stochastic conditional variance:[27]

$$\begin{aligned}\widetilde{\mu}^Y(t, Y_t) &= \widetilde{\kappa}\left(\widetilde{\vartheta} - Y(t)\right)\\ \sigma^Y(t, Y_t) &= \begin{pmatrix}\sqrt{Y_1(t)} & 0\\ 0 & \sqrt{Y_2(t)}\end{pmatrix}\end{aligned}$$

The mean reverting speed $\widetilde{\kappa}$ is a diagonal matrix so that the two state variable processes are independent. The vector of the interest rate and credit risk premia

$$\mu(t, Y_t) - r(t, Y_t) \cdot \iota_2 = b(T - t)\begin{pmatrix}Y_1(t) & 0\\ 0 & Y_2(t)\end{pmatrix}\rho$$

varies over time and is proportional to the level of the state variables. Following Section 2.3, the market price of risk vector $\theta(t, Y)$ (contains the market price of interest rate risk and the market price of systematic credit risk) is stochastic and equal to

$$\theta(t, Y) = \begin{pmatrix}\sqrt{Y_1(t)} & 0\\ 0 & \sqrt{Y_2(t)}\end{pmatrix} \cdot \rho$$

When the interest rate and the market prices of risk are stochastic as in the CIR case, the optimal portfolio is composed of the mean-variance optimal portfolio, the interest rate hedge component and the market price of risk hedge component. Unfortunately, no analytical solution for the interest rate hedge (IRH) and the market price of risk hedge (MPRH) can be found in this case. Thus, both hedge components are computed using the methodology based on Monte Carlo simulation described in Section 3.[28] In the following, we describe the structure of the optimal portfolio in the CIR case and its sensitivity with respect to the investment horizon $\overline{T}$, the maturity T of the Treasury and corporate bonds, the level of the term structure, the risk aversion γ of the investor and the most interesting process parameters.[29]

[27] The covariance matrix Σ has been dropped because it is equal to the identity matrix in the specification of Walder (2002).

[28] The results requiring simulations are based on a matrix of 1.09 Mio perturbations with Δt varying between 1/360 and 1/100 depending on the investment horizon.

[29] The parameter values for the CIR case are taken from the model $A_2(2)$ in Walder (2002) but with the matrix $\widetilde{\kappa}$ having off-diagonal elements equal to zero.

4.1.1 Mean-Variance Portfolio

Recalling the result of Section 3, the mean-variance optimal weights are deterministic and equal to:

$$\widehat{\pi}_t^{MV} = \frac{1}{\gamma}\left(b\left(T-t\right)'\right)^{-1}\rho$$

The weights are completely determined by the risk aversion parameter γ, the constant parameter ρ of the market price of risk vector $\theta\left(t, Y_t\right)$ and the sensitivities $b\left(T-t\right)$ of the two assets with respect to the state variables. The mean-variance optimal weights correspond to the linear combination of the two assets that synthesizes (some proportion of) the constant parameter of the market price of risk vector. In our case, the MV weights are composed of a long position in the corporate bond portfolio and a short position in the Treasury bond (see Table 2). A positive weight in the corporate bond portfolio implies a positive exposure to both interest rate and credit risk. Apparently, the implied interest rate exposure is then too high. Thus, the Treasury bond weight has to be negative in order to synthesize the market price of risk.

Table 2 and 4 confirm that the mean-variance weights are independent of the investment horizon $\overline{T}$ and the level of the term structures. The mean-variance weights decrease (in absolute value) when the investor's risk aversion increases (Table 5), but the ratio of Treasury bond to corporate bonds is not affected by the risk aversion parameter value. This corresponds to Tobin's separation theorem stipulating that all investors, independently of their risk aversion, hold the same mix of risky assets (market portfolio). As shown by Brennan and Xia (2000), this no longer holds when the investor hedges against stochastic variations of the interest rate with bonds. The mean-variance weights also decrease when the maturity of the bond increases (Table 4). Long bonds have higher sensitivities $b\left(T-t\right)$ to the state variables, i.e. are more risky. Therefore, less of them are necessary to synthesize the market price of risk vector.

The shortening of the time-to-maturity of the bonds is the only source of variation of the mean-variance component in the time space. Figure 1 shows the deterministic path of the mean-variance weights over the investment period for an investor having a risk aversion $\gamma = 2$, an investment horizon of $\overline{T} = 5$ years and investing in $T = 10$ year bonds. As time passes, the sensitivities $b\left(T-t\right)$ of the bonds decrease and, consequently, the mean-variance weights increase.

[Insert Table 2: "investment horizon" here]

[Insert Table 3: "bond maturity" here]

[Insert Table 4: "level of term structures" here]

[Insert Table 5: "risk aversion" here]

[Insert Figure 1: "MV a typical path" here]

Although we have not included equities in the investment universe for econometric reasons, the above results allow us to make qualitative statements on the optimal mean-variance portfolio for the case of an investment universe including equities. In general, the weight invested in equity will behave similarly to the corporate bond weight. The equity weight will be positive in order to synthesize the market price of equity risk. As the coefficient of risk aversion increases, the demand for corporate bonds and equities falls. The exact weight will depend on the parameter estimates and the dependence structure between the risk factors. The mean-variance weights will not stay deterministic except for very specific and unusual specifications of the market price of equity risk. The evolution of the mean-variance weights during the investment period would be interesting as the sensitivity of equity portfolios generally do not depend on some time-to-maturity. In such a case, a shift of wealth from equities to bonds would be observed during the investment period as the sensitivity of the bonds with respect to the sources of risk decreases.

4.1.2 Interest Rate Hedge

The IRH component protects the investor against future variations of the interest rate. The hedging needs are high when the present value of the optimal terminal wealth strongly depends on the interest rate and when the interest rate is highly sensitive to the sources of risk. In the independent CIR case, the interest rate depends only on the first Brownian motion W_1. Thus, the variance of the interest rate is high when the conditional variance of the first state variable is high and when the parameter δ_1, that reflects the sensitivity of the interest rate to the first state variable, is high (see equation (12)). The hedging needs are matched with the most effective combination of assets. The best way to protect the investor's wealth against variations of the short rate is to invest part of it in the Treasury bond. The best hedge would be a Treasury bond that matures at the investment horizon. The return of such a bond is known in advance and therefore bears no interest rate risk. Even if the bond maturity and the investment horizon are different, the Treasury bond performance is less subject to variations of the short rate compared to the money market account.

Consequently, the weights $\widehat{\pi}_t^{IR}$ of the IRH component reflect a transfer of the wealth from the risky (relative to the investment horizon) short rate to the less risky Treasury bond. This is a typical result for investment problems dealing with stochastic interest rates and including default-free bonds in the investment universe (Bajeux-Besnainou and Portait (1998), Campbell and Viceira (2001), Brennan, Schwartz and Lagnado (1997), Sorensen (1999), Brennan and Xia (2000), Wachter (2002) and others). Note that the weight of the corporate bond portfolio equals zero in the independent CIR case. Using the corporate bond portfolio to hedge against interest rate risk would imply an unnecessary exposure to credit risk. Similarly, when equities are added to the investment universe, the use of equity instead of Treasury bonds to hedge interest rate risk would not be necessary unless Treasury bonds depend on equity risk.

The more distant the investment horizon, the higher the hedging needs, although the weights increase less than proportionally to the investment horizon

(Table 2). The Treasury bond weight decreases as the maturity of the bonds in the investment universe increases (Table 3). This is due to the higher sensitivities (volatility) $\sigma(t, Y_t)$ of the long term bonds compared to the short term bonds. The interest rate level affects the variance of the interest rate because the state variables are square-root processes. Thus, an increase in the level of the term structure increases the sensitivity (Malliavin derivative) of the interest rate to the Brownian motions and therefore the hedging needs. But simultaneously, the increased variability of the bonds make them more efficient in hedging interest rate risk. Consequently, the Treasury bond weight of the IRH is relatively unaffected by the level of the term structures (Table 4). Not surprisingly, the IRH component increases with risk aversion (Table 5).

Figure 2 shows a typical path of the IRH weights over the investment period for an investor having a risk aversion $\gamma = 2$, an investment horizon of $\overline{T} = 5$ years and investing in $T = 10$ year bonds. The weight of the Treasury bond of the IRH declines over the investment period and reaches zero at the investment horizon. Recall that the corporate bond portfolio weight of the IRH is equal to zero at all times.

[Insert Figure 2: "IRH a typical path" here]

4.1.3 Market Price of Risk Hedge

The MPRH component $\widehat{\pi}_t^{MPR}$ protects the investor against stochastic variations of the market prices of risk. The hedging needs come from the fact that the optimal terminal wealth is exposed to variations of the market price of interest rate and credit risk. The hedging behavior depends on the sensitivities $\sigma(t, Y_t)$ of the assets to the sources of risk. Two sources of risk are priced in our economy and, therefore, there are two market prices of risk: the market price of interest rate risk and the market price of systematic credit risk. In order to ease interpretation, we decompose the weights $\widehat{\pi}_t^{MPR}$ in (10) into two components:

$$\widehat{\pi}_t^{MPR} = \widehat{\pi}_t^{MPIRH} + \widehat{\pi}_t^{MPCRH}$$

where the weights $\widehat{\pi}_t^{MPIRH}$ and $\widehat{\pi}_t^{MPCRH}$ are linked to the market price of interest rate risk hedge and the market price of credit risk hedge respectively.[30]

The market price of interest rate risk hedge is made of a short position in the Treasury bond. The short Treasury bond position reduces the positive exposure of the investor's wealth to changes in the market price of interest rate

[30] This is formally done by defining

$$\begin{aligned} \widehat{\pi}_t^{MPIRH} &= \left(\frac{1}{\gamma} - 1\right) \begin{pmatrix} \sigma_{11}^*(t, Y_t) \cdot h_1(t, Y_t) \\ \sigma_{21}^*(t, Y_t) \cdot h_1(t, Y_t) \end{pmatrix} \\ \widehat{\pi}_t^{MPCRH} &= \left(\frac{1}{\gamma} - 1\right) \begin{pmatrix} \sigma_{12}^*(t, Y_t) \cdot h_2(t, Y_t) \\ \sigma_{22}^*(t, Y_t) \cdot h_2(t, Y_t) \end{pmatrix} \end{aligned}$$

risk due to the mean-variance component (although the Treasury bond weight is negative in the mean-variance portfolio, the overall exposure to interest rate risk is positive). As in the IRH, the corporate bond portfolio has a weight equal to zero.

The market price of credit risk hedge is constructed through a short position in the corporate bond portfolio and a long position in the Treasury bond of the same magnitude. The short position in the corporate bond portfolio reduces the positive exposure of wealth to changes in the market price of credit risk due to the long position in defaultable bonds in $\widehat{\pi}_t^{MV}$. But this short position reduces simultaneously the interest rate risk exposure because corporate bonds are also sensitive to interest rate changes. To compensate for this effect, the investor takes a long position of the same magnitude in the Treasury bond leaving the aggregate interest rate risk exposure of the MPCRH component equal to zero. Thus, the effect of this long-short position is to extract the credit risk factor from the corporate bonds. Alternatively, the investor could have chosen to buy a credit derivative (for example a basket credit default swap), which implies the same net exposure as the long-short position in bonds.[31] Credit derivatives are characterized by having a small interest rate and a high credit risk exposure (credit risk is stripped). Their higher exposure to credit risk per unit dollar invested compared to corporate bonds would make them more effective in managing the market price of credit risk.[32] Similarly, Liu and Pan (2002) show how equity derivatives (straddles) are used in intertemporal portfolio strategies to disentangle diffusive from jump risk in the stock market and to hedge the jump risk.

The magnitude of the two market price of risk hedge components $\widehat{\pi}_t^{MPIRH}$ and $\widehat{\pi}_t^{MPCRH}$ increase with the investment horizon (Table 2). They make about 5% of investor's wealth for an investment horizon $\overline{T} = 1$ but increases to about 25% of wealth when $\overline{T} = 10$. This is consistent with the results in DGR, where the hedge components also increase in absolute value as the investment horizon

---

where

$$\begin{aligned} \sigma^*(t, Y_t) &= \left(\sigma(t, Y_t)'\right)^{-1} \\ h(t, Y_t) &= E_t \left[\frac{\xi_{t,\overline{T}}^{1-1/\gamma} \int_t^{\overline{T}} \left(d\widetilde{W}_s\right)' \mathcal{D}_t \theta(s, Y_s)}{E_t \left[\xi_{t,\overline{T}}^{1-1/\gamma}\right]} \right]'. \end{aligned}$$

and the subscripts denote the element of the matrix.

[31] The investment universe of our setting does not include credit derivatives. But, it would be interesting to test if the net exposure of the market price of credit risk hedge corresponds to a credit derivative maturing in $\overline{T}$ in the same way as the interest rate hedge component corresponds to a default-free bond maturing in $\overline{T}$ (see Vasicek case).

[32] An advantage of formulating the asset allocation problem with a Treasury bond and a credit derivative is that the returns of these assets display very low correlation. This alleviates the problem of obtaining weights that are highly sensitive to expected return differences as it is typical when using mean-variance optimization with highly correlated assets like in fixed income portfolios.

increases. As for the IRH, the bonds with longer maturities are more effective for hedging purpose because they have higher sensitivities to the risk factors (Table 3). As for the IRH component, higher levels of the term structures, implying an increase in the variations of the market prices of risk, increase the hedging needs on the one hand but increase the hedging efficiency of the bonds on the other hand. Thus, the weights of the MPRH components are relatively independent of the term structure levels (Table 4). The hedging needs increase with risk aversion (Table 5).

Typical paths for both market price of risk hedge components are shown in Figure 3 and 4 (risk aversion $\gamma = 2$, investment horizon of $\overline{T} = 5$ years and bond maturities $T = 10$ year). They start at a level close to the one in the Table (more precise) and decrease in magnitude until the investment horizon is reached.

[Insert Figure 3: "MPIRH a typical path" here]

[Insert Figure 4: "MPCRH a typical path" here]

The construction of a portfolio of default-free (long) and defaultable (short) bond to hedge against variations of the market price of credit risk is a main result of this study. The following sensitivity analysis will help us to gauge how the weights $\widehat{\pi}_t^{MPCRH}$ of the MPCRH component depend on the process parameter values of our model. The mean reversion speed κ_{22} of the second state variable (credit risk) determines the sensitivities $b^D(\tau)$ of the defaultable bonds with respect to the sources of risk. Higher values for the mean-reversion speed κ_{22} imply lower sensitivities of the bonds with respect to the sources of risk and hence, higher (absolute) weights are necessary to obtain the same hedge effect (see Figure 5). A higher mean-reversion level ϑ_2 increases the volatility, and hence the MD, of the state variable in (15). But at the same time, the magnitude of the derivative $\partial_2\theta(t, Y_t)$ decreases, leaving the weights $\widehat{\pi}_t^{MPCRH}$ relatively independent of the mean-reversion level ϑ_2 (see Figure 6).[33] Not surprisingly, the weights of the MPCRH component are positively dependent on the (absolute value of) the market price of risk parameter ρ_2 (see Figure 7). A higher level of the stochastic market price of risk induce higher hedging needs. Finally, the MPCRH component is independent of the value of δ in our specification.

[Insert Figure 5: "MPCRH Sensitivity to κ_{22}" here]

[Insert Figure 6: "MPCRH Sensitivity to ϑ_2" here]

[Insert Figure 7: "MPCRH Sensitivity to ρ_2" here]

Introducing equities in the investment universe would have two main effects. First, the equity weight in the MPIRH and the MPCRH would be different

[33] The mean-reversion level ϑ_2 does not affect the sensitivity $b^D(\tau)$ appearing in the MPCRH equation, it determines the coefficient $a^D(\tau)$. This also implies that the MV weight $\widehat{\pi}_t^{MV}$ is independent of the parameter ϑ_2.

from zero if Treasury and corporate bonds are modeled with dependencies with respect to equity risk. For example, if the dependence between corporate bonds and equities is positive, the negative corporate bond position in the MPCRH will be accompanied with a positive weight in equities like in the case of the Treasury bond. The second effect of introducing equities is to add a market price of equity risk component to the MPRH when the market price of equity risk is assumed stochastic. The equity weight will be negative if the stock returns and the market price of equity risk are positively dependent. This will reduce the positive exposure to equity risk in the mean-variance component (as assumed in Section 4.1.1). The weights of the Treasury and corporate bonds in the market price of equity risk should neutralize the exposure to interest rate and credit risk implied by this negative equity position. The sign of these weights will depend on the particular specification of the dependence between equities, interest rate and credit risk. For example, if the negative equity position implies a positive exposure to interest rate risk, then a Treasury bond should be shorted in the market price of equity risk hedge component.

Consequently, the introduction of equities should not alter the behavior of the hedging components described above. In particular, the weight of the Treasury bond in the MPIRH is negative and the MPCRH is composed of a portfolio with a long position in a Treasury bond and a short position in corporate bonds.

4.1.4 Asset Allocation

The previous three sections have shown how the optimal bond portfolio is structured and what determines its three components, the mean-variance optimal portfolio $\widehat{\pi}_t^{MV}$, the interest rate hedge $\widehat{\pi}_t^{IR}$, and the market price of risk hedge $\widehat{\pi}_t^{MPR}$. We now analyze the aggregated portfolio weights $\widehat{\pi}_t$ and its sensitivities to the investment horizon $\overline{T}$, the maturity T of the Treasury and corporate bonds, the level of the term structure and the risk aversion γ of the investor.

The Treasury bond weight increases with the investment horizon $\overline{T}$ for two reasons. First, the Treasury bond weight of the IRH is important in magnitude and increases with $\overline{T}$. Second, the long position in Treasury bond due to the MPCRH more than compensates the short position due to the MPIRH so that the MPRH in the aggregate has a long position in Treasury bond that increases with $\overline{T}$. The corporate bond weight decreases with $\overline{T}$ due to the MPCRH (Table 2). Thus, the ratio of Treasury bond to corporate bonds is increasing with the investment horizon $\overline{T}$. Analogously, Brennan and Xia (2000) find that the bond/stock ratio increases with the investment horizon whatever the risk aversion due to the hedging behavior of the investor.

The Treasury and corporate bond weights are smaller in magnitude when the bonds have high maturities (Table 3). This is due to the fact that long term bonds have higher sensitivities with respect to the risk factors compared to short term bonds. Thus, lower (absolute) weights are necessary to attain the same level of exposure or hedging needs.

The Treasury and corporate bond weights are independent of the level of the term structures (Table 4). This is due to the opposite effect of the state variable level on the hedging needs and the hedging efficiency of the assets: On one hand, the hedging needs increase with the level of the term structures due to direct impact of the state variable on the variance of the state variable, but on the other hand the bonds become more efficient in hedging when the level of the term structures increase.

The portfolio composition is very sensitive to the value of the investor's risk aversion. The logarithmic utility investor with risk aversion parameter $\gamma = 1$ does not hedge against variations of the interest rate nor against changes in the market prices of risk (Table 5). The portfolio composition reduces to the MV component which indicates that she should take a long position in corporate bonds and a short position in the Treasury bond. The risk averse investor ($\gamma > 1$) will hedge against interest rate risk and both market prices of risk. As the risk aversion increases, the importance of the MV component (or speculative demand) decreases and the two hedge components become more important. A strongly risk averse investor will invest the major part of her wealth in a Treasury bond and only a small amount in corporate bonds. The investor with a risk aversion $\gamma < 1$ increases the exposure towards the risk factors compared to the logarithmic utility investor by choosing more extreme mean-variance weights and making "negative hedging".

Consequently, our results show that the ratio of Treasury bond to risky asset (corporate bond portfolio) weights increases with the risk aversion parameter. Brennan and Xia (2000) and Campbell and Viceira (2001) obtain the same result when allocating between default-free bonds and stocks instead of default-free and defaultable bonds. This finding appears to be in contradiction to the Separation Theorem in Tobin (1958) , which postulates that the optimal mix of risky asset is independent of the investor's risk aversion. Brennan and Xia (2000) show that the separation argument is no more valid in a dynamic context. Wachter (2002) shows that the portfolio of an investor with utility over consumption at time $\overline{T}$ converges to the portfolio consisting entirely of a bond maturing at time $\overline{T}$ when risk aversion approaches infinity. This is because the riskless asset for a long-term investor is not a the MMA but the bond with maturity equal to the investment horizon.

Figure 8 shows a typical path of the total portfolio weights over the investment period for an investor having a risk aversion $\gamma = 2$, an investment horizon of $\overline{T} = 5$ years and investing in $T = 10$ year bonds. At the beginning of the investment period, the interest rate and the credit risk hedge components induce a high Treasury bond weight. The Treasury bond weight of the IRH and the MPRH component more than compensates the short position due the MV component. On the other side, the weight of the corporate bonds is reduced at the beginning of the investment period compared to the MV weights due to the MPCRH component. Overall, both weights are positive at the beginning of the investment period and some wealth is invested in the MMA. But, as the investment horizon approaches, the IRH and MPRH components vanish.[34] As a result, the optimal weights converge to the MV portfolio as the end of the investment period approaches. Thus, the behavior of a long-term investor is different to the one of a logarithmic or myopic investor who will go long in the corporate bond and short in the Treasury bond during the whole investment period.

Introducing equities in the investment horizon would not change the mechanism driving the asset allocation described above. Treasury bonds act as the riskless asset as soon as the portfolio is optimized intertemporally. As risk aversion increases, the investor will invest more in Treasury bonds and less in risky assets (equities and corporate bonds). The ratio of Treasury bonds to risky assets will increase with the investment horizon due to increased hedging needs. The weights are strongly influenced by the hedging needs at the beginning of the investment period, but converge to the mean-variance portfolio weights at the end of the investment horizon. The precise values for the optimal weights will depend on the particular model specification (in particular the risk factor dependence structure) and the parameter estimates in addition to all other variables analyzed in this paper.

Before ending this section, the following remark on our short-sales assumption should be made. Recall that our investor has no short-sales constraints. Thus, some of the optimal weights at the aggregate level are negative in the tables, although this does not happen in the majority of the cases. If short selling bonds may be a problem on some markets or linked to substantial costs, the investor may choose to replicate the negative exposures with derivatives. A short position in Treasury bonds is best replicated with a short position in Treasury bond futures which are liquid in most of the markets. Short corporate bond positions can be replicated with short positions in a corporate bond futures, a combination of a Treasury bond future and a credit derivative or with a portfolio of swaps and short term loans. Indeed, the combination of a (series of) short-term loans (at a variable rate) and a payer swap with maturity T is able to replicate the cash-flows of a short bond position with maturity T. If the investor has short selling constraints and is not allowed to use derivatives, the portfolio problem has to be solved with short selling constraints. Investment problems with short selling constraints are dealt with in Cvitanic and Karatzas (1993) and Cuoco (1997). The idea is to construct a related economy in which portfolio proportions are unconstrained, but such that the optimal portfolio weights process lies in the admissible set. In this related economy the drifts of the restricted assets include the Lagrange multiplier process linked to the constraint such that the short selling constraint acts as an additional dividend yield as soon as the constraint is binding. The optimal portfolio proportion in the

[34] This result appears also in DGR for an investment universe composed of a portfolio of "old economy" stocks, a portfolio of "new economy" stocks and a portfolio of long-term pure discount bonds.

constrained economy are then deduced from the ones in the related economy. Applying this methodology to our case would go beyond the scope of our paper.

4.2 Vasicek Case

In this section, we investigate the optimal portfolio composition when the two state variables Y_1 and Y_2 follow independent Gaussian processes. The main difference compared to the CIR case is that the conditional variance of the state variables and therefore the risk premia and the market prices of risk are constant and no more dependent on the level of the state variables.[35] The comparison of the Vasicek case (constant risk premia) with the CIR case (stochastic risk premia) will highlight the impact of time-varying interest rate and credit risk premia on the optimal portfolio composition.

The state variable process in the Vasicek case is as in (3) with a mean-reverting drift and a constant conditional covariance:

$$\begin{aligned} \widetilde{\mu}^{Y}(t, Y_t) &= \widetilde{\kappa}\left(\widetilde{\vartheta} - Y(t)\right) \\ \sigma^{Y}(t, Y_t) &= \Sigma \end{aligned}$$

The mean reverting speed $\widetilde{\kappa}$ and the conditional covariance matrix Σ are diagonal matrices so that the two state variable processes are independent. The market price of risk vector is constant and equal to $\theta = \rho$.[36] A constant market price of risk vector has two main implications: First, the investor does not have to hedge his portfolio against stochastic variations of the market prices of risk, i.e. the MPRH component is equal to zero. Second, the expression for the IRH component can be solved analytically so that simulations are not necessary for its computation. In the following, we investigate more precisely each component of the optimal portfolio in the Vasicek case and highlight the differences to the CIR case.

[Insert Table 6: "effect of time-varying risk premia..." here]

The MV component takes the same form as in the CIR case derived in (11). It is deterministic and its value changes only because the sensitivities $b(T-t)$ decrease as time passes. As in the CIR case, the mean-variance optimal weights consist of a long position in the corporate bond portfolio and a short position in the Treasury bond (Table 6). Differences in the values of the MV component in

[35] Modeling the state variables with Ornstein-Uhlenbeck processes is problematic in our context, because negative values for the default intensities are not excluded. The only purpose of the Vasicek case is to compare the optimal portfolio with the more realistic CIR case in order to gauge the effect of time-varying risk premia. Moreover, the issue of negative default intensities due to negative values of the second state variable is alleviated because the optimal weights are independent of the state variables as we will see.

[36] The parameter values for the Vasicek case are taken from the model $A_0(2)$ in Walder (2002) but with the matrix $\widetilde{\kappa}$ having off-diagonal elements equal to zero. The matrix Σ is an identity matrix.

the Vasicek case and the CIR case are only due to differences in the parameter estimates.

The IRH component can be solved analytically for the Vasicek case:[37]

$$\widehat{\pi}_t^{IR} = \left(\frac{1}{\gamma} - 1\right) \left(b\left(T-t\right)' \Sigma'\right)^{-1} h_1\left(t, Y_t\right) \tag{16}$$

with $h_1(t, Y_t) = -b^B \left(\overline{T} - t\right)' \Sigma$ and where $h_1(t, Y_t)$ corresponds to the sensitivity with respect to the sources of risk of a default-free bond maturing at the investment horizon $\overline{T}$.[38] This is a standard result reported in Poncet and Portait (1993), Brennan, Schwartz and Lagnado (1997), Bajeux-Besnainou and Portait (1998), Campbell and Viceira (2001), Sorensen (1999), Munk and Sorensen (2001) and Brennan and Xia (2000), Mougeot (2000) and others. Note that the IRH component is deterministic and no simulation is needed for its computation. Moreover, the IRH will be the same whatever the shape of the term structures because it does not depend on the state variable level. Interestingly, the weight of the Treasury bond of the IRH in the Vasicek case is lower compared to the CIR case for short investment periods but it is higher for investment horizons greater than five years.

The MPRH component is equal to zero because the market price of risk θ is constant. This result holds whatever the level and the shape of the term structures.

Consequently, the portfolio composition in the Vasicek case is completely deterministic and independent of the level of the term structures (level of the state variables). The whole path of the optimal portfolio until the investment horizon can be determined ex ante. The Treasury bond weight is composed of a negative speculative demand and a positive interest rate hedge component. The defaultable bond weight is composed only of a positive speculative demand.

The impact of time-varying market prices of risk is the following: First, the weights of the optimal portfolio are stochastic when the risk premia are time-varying (CIR case) whereas they are deterministic when the risk premia are constant (Vasicek case). Second, the corporate bond weight is lower in the CIR case because the investor hedges against variations of the market price of credit risk with a short position in corporate bonds. The MPRH component is equal to zero in the Vasicek case. And third, the optimal Treasury bond weight is higher in the CIR case because the MPIRH and MPCRH components imply in an increase in the Treasury bond weight compared to the Vasicek case.

4.3 Introducing Interdependencies

Until now, the two state variables representing the interest rate and the systematic credit risk were independent. But there is enough empirical evidence for

[37] The proof is in Appendix A.

[38] Values of $h_1(t, Y_t)' = [0.0075 \quad 0.0]$ correspond to a bond with a volatility of 0.75% and has to be compared to the volatility σ_Q of the hedging bond in Munk and Sorensen (2001).

interdependencies between interest rates and credit risk. For example, Duffee (1998) and Düllmann et alii (1998) have found negative dependence between the interest rate and the credit spread for the US and the German market respectively. Walder (2002) provides evidence for a positive relation between interest rates and credit spreads on the Swiss market in a different sample period. In the present section, we investigate the effect of positive and negative dependencies between the interest rate and the credit risk on the optimal portfolio composition.

The analysis of interdependencies in the conditional variance $\sigma^Y(t, Y_t)$ of the state variable process on the portfolio structure is problematic in our setting because the "volatility condition" for the efficient evaluation of the MD of the state variables (assumption (iv) in Detemple, Garcia and Rindisbacher (2002b), Theorem 2) is no more satisfied. For this reason, we will restrict ourselves to the analysis of interdependencies in the conditional mean of the state variables. More precisely, we will consider the affine model $A_2(2)$ in Walder (2002) with a mean-reverting speed $\widetilde{\kappa}$ having non-zero off-diagonal elements (see Table 1 for the parameter estimates).[39] The off-diagonal elements estimated on the Swiss market are negative in the sample period meaning that the two state variables representing the interest rate and the credit spread are positively dependent. The value $\widetilde{\kappa}_{12} = -0.38$ is greater in absolute terms than $\widetilde{\kappa}_{21} = -0.09$ implying that the interest rate depends more on the credit risk than vice versa. We will focus on these positive dependencies in our analysis but also consider the case of negative dependencies to gain some information for the markets and the sample periods displaying negative dependencies. For the case of negative dependencies, we will simply assume opposite signs for the off-diagonal elements of the mean-reversion speed, i.e. $\widetilde{\kappa}_{12} = 0.38$ and $\widetilde{\kappa}_{21} = 0.09$. Our interest for the remaining part of this section will be focused on the differences between the dependent (model $A_2(2)$) and the independent case (CIR case) presented in Section 4.1.

The effect of introducing positive (*negative*) interdependencies on the MV component is to reduce (*increase*) the extent of the long and the short position compared to the independent case (Table 7). The reason is the following: The sensitivities $b^B(T-t)$ and $b^D(T-t)$ of the Treasury bond and the corporate bonds with respect to the sources of risk have increased (*decreased*) with the introduction of positive (*negative*) interdependencies. Treasury bonds are now also dependent on the credit risk factor implying a non-zero value for the second element of the vector $b^B(T-t)$. Conversely, the sensitivity $b^D(T-t)$ of the defaultable bonds with respect to the interest rate factor (first element of the vector) increased (*decreased*) due to the positive (*negative*) interdependencies. The increased (*decreased*) sensitivities imply higher (*lower*) asset volatilities and, consequently, lower (*higher*) mean-variance weights (see equation (11)).

[Insert Table 7: "Interdependencies" here]

[39] The Vasicek case $A_0(2)$ with interdependencies in the conditional mean of the state variable is skipped because it is less interesting due to the lack of MPRH component.

The interest rate hedge component is also affected by the interdependencies. The MD of the interest rate with respect to the credit risk factor (the second Brownian motion W_2) is now different from zero, because the second state variable appears into the process of the first state variable. This implies a positive (*negative*) exposure of Treasury bonds to credit risk. To obtain a "pure" positive interest rate exposure, a long position in a Treasury bond inducing a positive (*negative*) credit risk exposure has to be combined with a short (*long*) corporate bond position. This is what happens with the IRH component where non-zero weights in corporate bonds appear (Table 7). The IRH is composed of a long Treasury bond position and a short (*long*) position in corporate bonds. It is interesting to note that the sum of the Treasury and the corporate bond weight in the interdependency cases of the IRH component sum to the Treasury bond weight of the independent case. This indicates that the net interest rate exposure of the positions in the dependent cases is approximately equal to the interest rate exposure in the independent case.

The effect of the interdependencies on the MPIRH is similar to the effect on the IRH. Due to the additional exposure of Treasury bonds to credit risk, the MPIRH is made of a combination of the Treasury bond and the corporate bond portfolio. Again, we infer from the addition of the Treasury and the corporate bond weights that the net exposure to interest rate risk in the dependent cases corresponds approximately to the independent case.

The MPCRH consists in a short position in corporate bonds and a long position in a Treasury bond in order to extract a negative pure credit risk exposure. Positive (*Negative*) interdependencies imply an increase (*decrease*) in the interest rate risk exposure of the corporate bond position implying an increase (*decrease*) in the Treasury bond weight in order to make the MPCRH independent of interest rate risk. But the higher (*lower*) Treasury bond weight comes with a positive (*negative*) credit risk exposure due to the modeled interdependence. This effect is compensated with a larger (*smaller*) negative weight of the corporate bond portfolio compared to the independent case.

At the aggregate level, we observe that positive (*negative*) interdependencies result in an increase (*decrease*) of the Treasury bond weight accompanied by a decrease (*an increase*) of the corporate bond portfolio weight compared to the independent case. The effect is similar to an increase (*decrease*) in the risk aversion of the investor or an increase (*decrease*) of the overall risk in the model. But the above results have shown that the portfolio structure changes in different way compared to a modification of the risk aversion or the overall risk in the model. In particular, corporate bonds are used for the IRH and the MPIRH of both interdependence cases, whereas their weight were equal to zero in the independent case.

What are the implications of our results for practitioners? The following conclusions are proposed to wealth managers based on the assumptions of this study: First, the risk-free asset is a Treasury bond maturing at the investment horizon when the portfolio is allocated intertemporally. Therefore, a strongly risk averse investor will invest most of his wealth in a Treasury bond instead of risky assets (corporate bonds, equities,...) or the instantaneously risk-free money market account. Second, risk averse investors who want to take risks will shift money from the Treasury bond to risky assets such as equities or defaultable bonds. Shifting wealth to risky assets should be made carefully not only because it represents an increase in the portfolio's instantaneous variance, but because it implies an exposure of wealth to the variation of the market prices of risk. This prudence is reflected by the mitigating role of the market price of risk hedge components on the risky asset weights. Third, the prudence is most important at the beginning of the investment period in order to avoid drifting away from the optimal wealth's path. As the end of the investment period approaches, the weights are less influenced by the hedging needs and converge to the mean-variance portfolio weights (speculative demand). Consequently, the ratio of Treasury bonds to risky assets increases with the investment horizon due to increased hedging needs. Fourth, the solution to the investment problem becomes more involved, when interdependencies between the risk factors exist. In general, positive (negative) dependencies between the risk factors increase (decrease) the risk in the portfolio and imply higher (lower) hedging needs. Ultimately, the precise values for the optimal weights will depend on the particular model specification (in particular the risk factor dependence structure) and the parameter estimates in addition to all other variables analyzed in this paper.

These implications are conditional on the assumptions of the paper. The most questionable assumption from a practitioners point of view is the choice of the investor's objective function. One problem is that the investment horizon for a pension plan is not fixed but progresses as time passes or is even expected to be infinite. Another problem of our utility function is that negative *and* positive deviations from the optimal terminal wealth are penalized. Our result may not hold if investors make the balance between expected return and some asymmetric risk measures such as value-at-risk or semi-variance in their objective function. Our choice for the objective function may explain the contradiction between our theoretical results and the investor wisdom saying that one should buy risky assets if the investment horizon is long. Another critical assumption is the affine framework for the bond pricing. Indeed, it is questionable whether bond prices follow such simple dynamics. The incorporation of further risk factors and nonlinearities in the drift and the diffusion of the processes should be made if the data situation and the econometric methods allow for it. Nevertheless, the main implications described above should be resistant to such refinements.

4.4 Robustness

A critical assumption of our work is that the model and its parameters are known with certainty. In fact, the specification and the estimation of the model are made with some error. Therefore, the use of this model for the determination of the optimal portfolio strategy contains some risk, which is called model risk. A complete analysis of model risk in our setting would go beyond the scope of our paper, but some useful insights can be found in the existing literature. One way to take model risk into account is to consider portfolio rules that are robust to a particular type of model misspecification, namely parameter uncertainty. Parameter uncertainty in intertemporal portfolio optimization problems has been treated using various approaches like the Bayesian approach (Barberis (2000)), continuous-time filtering (Detemple (1986), Genotte (1986), Brennan (1998), Xia (2001) and Mougeot (2000)) or the minimum entropy approach (Anderson, Hansen and Sargent (2000) and Maenhout (2002)).

Maenhout (2002) comes the closest to our setting by analyzing the effect of estimation risk on the optimal portfolio in the context of a stochastic opportunity set. Parameter uncertainty is taken into account by considering alternative parameter values in the optimization. How much these alternative values are allowed to deviate from the reference values depends on a parameter called the preference for robustness. Robust decisions are then designed to insure against the worst-case alternative values. In the context of a dynamic investment problem with stochastic investment opportunity set, the optimal demand for the risky asset changes in two ways. First, robustness increases the hedging demand due to the stochastic investment opportunity set. Second, it increases the risk aversion applied to the total asset demand. More precisely, the risk aversion parameter corrected for robustness corresponds to the sum of the original risk aversion and the preference for robustness. Applied to our setting, the first effect would lead to an increase the two hedge components, i.e. increase the negative exposure to corporate bonds and the positive exposure to the Treasury bond. The second effect would reduce the mean-variance demand. Therefore, the overall effect of introducing parameter uncertainty is to increase the ratio of Treasury to corporate bonds in a similar way as an increase in the risk aversion parameter would do it.

Mougeot (2000) is also interesting for our setting, although assuming constant market prices of risk, because the investor is faced with an investment universe containing default-free bonds beside the MMA and stocks. This work based on the filtering approach supports the idea that a risk averse investor faced with estimation risk hedges against this risk by increasing the weight of the Treasury bond and decrease the weight of the risky asset.[40]

To summarize, it seems safe to pretend that the effect of estimation risk on the optimal portfolio is to increase the wealth allocated in the Treasury bond and decrease the wealth allocated to corporate bonds in the same way as an increase in risk aversion. Therefore, the optimal portfolio structure corresponding to high risk aversion parameter values in Table 5 should be considered when the investor cares about estimation risk.

[40] Note that Brennan (1998) also finds that estimation risk reduces the allocation to the risky asset for a risk averse investor when the investment universe contains only the MMA and stocks.

5 Conclusion

The objective of this paper is to characterize the optimal allocation of a portfolio of Treasury and corporate bonds in an intertemporal setting. The defaultable bonds are priced based on an intensity based approach. The interest rate and the intensity of default are modeled as linear functions of state variables following affine diffusions. The parameter estimates are taken from Walder (2002). In the most interesting and relevant cases, namely when the state variables follow square-root processes, the market price of risk vector is stochastic which implies a time-varying investment opportunity set. Using results in Detemple, Garcia and Rindisbacher (2002a) on the representation and the evaluation of the optimal portfolio, we are able to characterize and evaluate the optimal allocation for an investor with a power utility function.

The optimal portfolio is composed of three components: the mean-variance portfolio, the interest rate hedge and the market price of risk hedge. The mean-variance portfolio turns out to be deterministic in the general affine setting considered here. It consists in a long position in corporate bonds and a short position in the Treasury bond. The interest rate hedge corresponds to the exposure of a default-free bond maturing at the investment horizon when the interest rate and the credit risk are independent. The market price of risk hedge has two components: the market price of interest rate risk and the market price of systematic credit risk. The market price of interest rate risk hedge is a short position in the Treasury bond that aims to reduce the positive exposure of the investor's wealth to changes in the market price of interest rate risk due to the mean-variance component. The market price of credit risk hedge consists in a short position in corporate bonds and a long position in the Treasury bond of similar amount. This yields a negative net exposure to the systematic credit risk factor and reduces the positive market price of credit risk exposure of the optimal wealth due to the mean-variance component. Note that the negative credit risk exposure in the market price of risk hedge could be realized at a possibly lower cost by going into a credit derivative. In the aggregate, the investor will optimally hold a long position in the Treasury bond due to both hedge components and a long position in corporate bonds that is lower than in the mean-variance optimal portfolio due to the market price of credit risk hedge. The hedging needs, and consequently the ratio of Treasury bond to corporate bonds, increase with the length of the investment period. Both hedge components vanish as the investment horizon approaches, implying that the optimal portfolio converges to the mean-variance portfolio at the investment horizon. The Treasury and corporate bond weights are smaller in magnitude when the bonds in the investment universe have high maturities, because they have higher sensitivities with respect to the risk factors compared to short term bonds. The portfolio composition is unaffected by the level of the initial default-free and defaultable term structures. With the risk aversion parameter, the investor determines the relative importance of the mean-variance and the hedge components. Not surprising, the ratio of Treasury bond to corporate bonds increases with risk aversion.

The paper further analyzes the impact of time-varying risk premia on the optimal portfolio structure by comparing the above results (CIR case) with the case of constant market prices of risk (Vasicek case). The corporate bond weight is lower in the case of stochastic risk premia because the investor hedges against variations of the market prices of risk with a short position in corporate bonds. The market price of risk hedge component is equal to zero when the risk premia are constant. The optimal Treasury bond weight is higher when the risk premia are time-varying because the market price of risk hedge, which does not exist when risk premia are constant, results in a positive Treasury bond weight.

The paper ends by investigating the effect of interdependencies between the interest rate and the credit risk on the optimal portfolio structure. Positive (*negative*) interdependencies reduce (*increase*) the magnitude of the long and the short positions in the mean-variance component because they increase (*reduce*) the risk exposure of the assets. Interestingly, corporate bonds are used for the interest rate hedge and the market price of interest rate risk hedge when interdependencies are taken into account. Indeed, a long position in a Treasury bond, implying a positive (*negative*) credit risk exposure through the dependence, has to be combined with a short (*long*) corporate bond position in order to obtain a "pure" interest rate hedge. The effect of positive (*negative*) interdependencies on the market price of credit risk hedge is to increase (*decrease*) the magnitude of the long and the short position in Treasury bond and corporate bonds respectively due to the increased (*decreased*) risk exposure of both assets. Considering all three components, the effect of positive (*negative*) interdependencies is to increase (*decrease*) the Treasury bond weight and decrease (*increase*) the corporate bond position relative to the independent case.

Based on a simple but empirically founded model, the paper provides useful implications for the management of bond portfolios in an intertemporal setting. This research has also highlighted at least two avenues for further research. First, the importance of the market price of risk for the optimal portfolio structure is obvious and should motivate further research on its specification and estimation. Second, the paper shows that the net exposure of the hedge components are similar to the exposure of financial derivatives. Including interest rate swaps, credit default swaps or other derivatives in the investment universe could help implementing the portfolio structure more efficiently or solving the portfolio problem when the underlying assets (bonds in our case) suffer from short-selling restrictions.

6 References

1. Akgun Aydin, 2001, "Defaultable Security Valuation and Model Risk", FAME Research Paper Nr. 28, Switzerland.

2. Anderson Evan, Hansen Lars, Sargent Thomas, 2000, "Robustness, Detection, and the Price of Risk", Working Paper, Stanford University.

3. Bajeux-Besnainou Isabelle, Portait Roland, 1998, "Dynamic Asset Allocation in a Mean-Variance Framework", *Management Science* **44**, 79-95.

4. Bakshi Gurdip, Madan Dilip, Zhang Frank, 2001, "Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates", Working Paper, University of Maryland and Federal Reserve Board (Washigton D.C.).

5. Barberis Nicholas, 2000, "Investing for the Long Run When Returns are Predictable", *The Journal of Finance* **55**, 225-264.

6. Brennan Michael, 1998, "The Role of Learning in Dynamic Portfolio Decisions", *European Finance Review* **1**, 295-306.

7. Brennan Michael, Schwartz Eduardo, Lagnado Ronald, 1997, "Strategic Asset Allocation", *Journal of Economic Dynamics and Control* **21**, 1377-1403.

8. Brennan Michael, Xia Yihong, 2000, "Stochastic Interest Rates and the Bond-Stock Mix", *European Finance Review* **4**, 197-210.

9. Campbell John, Viceira Luis, 2001, "Who Should Buy Long-Term Bonds?", *American Economic Review* **91**, 99-127.

10. Chan K C, Karolyi Andrew, Longstaff Francis, Sanders Anthony, 1992, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate", *The Journal of Finance* **47**, 1209-1227.

11. Chava Sudheer, Jarrow Robert, 2001, "Bankruptcy Prediction with Industry Effects, Market versus Accounting Variables, and Reduced Form Credit Risk Models", Working Paper, Cornell University.

12. Cuoco Domenico, 1997, "Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income", *Journal of Economic Theory* **72**, 33-73.

13. Cvitanic Jaksa, Goukasian Levon, Zapatero Fernando, 2000, "Monte Carlo Computation of Optimal Portfolios in Complete Markets", forthcoming: *Journal of Economic Dynamics & Control.*

14. Cvitanic Jaksa, Karatzas Ioannis, 1993, "Hedging Contingent Claims with Constrained Portfolios", *Annals of Applied Probability* **3**, 652-681.

15. Dahlquist Magnus, 1996, "On Alternative Interest Rate Processes", *Journal of Banking and Finance* **20**, 1093-1119.

16. Dai Qiang, Singleton Kenneth, 2000, "Specification Analysis of Affine Term Structure Models", *The Journal of Finance* **55**, 1943-1978.

17. Detemple Jérôme, 1986, "Asset Pricing in an Economy with Incomplete Information", *The Journal of Finance* **61**, 383-392.

18. Detemple Jérôme, Garcia René, Rindisbacher Marcel, 2002a, "A Monte Carlo Method for Optimal Portfolios", forthcoming: *Journal of Finance.*

19. Detemple Jérôme, Garcia René, Rindisbacher Marcel, 2002b, "Representation Formulas for Malliavin Derivatives of Diffusion Processes", Working Paper, CIRANO, Canada.

20. Duffee Gregory, 1998, "The Relationship between Treasury Yields and Corporate Bond Yield Spreads, *The Journal of Finance* **53**, 2225-2241.

21. Duffee Gregory, 1999, "Estimating the Price of Default Risk", *Review of Financial Studies* **12**, 197-226.

22. Duffie Darrell, Kan Rui, 1996, "A Yield Factor Model of Interest Rates", *Mathematical Finance* **6**, 379-406.

23. Duffie Darrell, Pan Jun, Singleton Kenneth, 2000, "Transform Analysis and Asset Pricing for Affine Jump Diffusions," *Econometrica* **68**, 1343-1376.

24. Duffie Darrell, Singleton Kenneth, 1997, "An Econometric Model of the Term Structure of Interest-Rate Swap Yields", *Journal of Finance* **52**, 1287-1321.

25. Duffie Darrell, Singleton Kenneth, 1999, "Modeling Term Structures of Defaultable Bonds", *Review of Financial Studies* **12**, 687-720.

26. Düllmann Klaus, Uhrig-Homburg Marliese, Windfuhr Marc, 1998, "Risk Structure of Interest Rates: An Empirical Analysis for Deutschemark-denominated Bonds", Working Paper, Mannheim University.

27. Gennotte G., 1986, "Optimal Portfolio Choice under Incomplete Information", *The Journal of Finance* **61**, 733-749.

28. Hou Yuanfeng, Jin Xiangrong, 2002, "Optimal Investment with Default Risk", FAME Research Paper Nr. 46, Switzerland.

29. Jarrow Robert, Lando David, Yu Fan, 2000, "Default Risk and Diversification: Theory and Applications", Working Paper.

30. Jarrow Robert, Turnbull Stuart, 1995, "Pricing Derivatives on Financial Securities Subject to Credit Risk", *The Journal of Finance* **50**, 53-85.

31. Jeanblanc Monique, Rutkowski Marek, 2002, "Default Risk and Hazard Processes", In: Mathematical Finance - Bachelier Congress 2000, H. Geman, D. Madan, S. Pliska, and T. Vorst, eds., Springer Verlag, Berlin, 281-312.

32. Liu Jun, Longstaff Francis, Mandell Ravit, 2000, "The Market Price of Credit Risk: An Empirical Analysis of Interest Rate Swap Spreads", Working Paper, The Anderson School at UCLA.

33. Liu Jun, Pan Jun, 2002, "Dynamic Derivative Strategies", forthcoming: *Journal of Financial Economics*.

34. Maenhout Pascal, 2002, "Robust Portfolio Rules, Hedging and Asset Pricing", Working Paper, Finance Department, INSEAD.

35. Madan Dilip, Unal Haluk, 1998, "Pricing the Risks of Default", *Review of Derivatives Research* **2**, 120-160.

36. Merton Robert, 1971, "Optimum Consumption and Portfolio Rules in a Continuous Time Model", *Journal of Economic Theory* **3**, 373-413.

37. Merton Robert, 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", *The Journal of Finance* **29**, 449-469.

38. Merton Robert, 1990, "Continuous-Time Finance", Blackwell Publishers, Oxford.

39. Mougeot Nicolas, 2000, "Interest Rate Risk and Learning in Dynamic Asset Allocation", Working Paper, University of Lausanne and FAME, Switzerland.

40. Munk Claus, Sorensen Carsten, 2001, "Optimal Consumption and Investment Strategies with Stochastic Interest Rates", Working Paper, Copenhagen Business School and Odense University.

41. Poncet Patrice, Portait Roland, 1993, "Investment and Hedging under a Stochastic Yield Curve", *European Economic Review* **37**, 1127-1147.

42. Sethi Suresh, 1998, "Optimal Consumption-Investment Decisions Allowing for Bankruptcy: A Survey " in: "Worldwide Asset and Liability Modeling", 397-426, William Ziemba, John Mulvey.

43. Sorensen Carsten, 1999, "Dynamic Asset Allocation and Fixed Income Management", *Journal of Financial and Quantitative Analysis* **34**, 513-531.

44. Tobin James, 1958, "Liquidity Preference as Behavior towards Risk", *Review of Economic Studies* **25**, 68-85.

45. Wachter Jessica, 2002, "Risk Aversion and Allocation to Long-Term Bonds", forthcoming: *Journal of Economic Theory*.

46. Walder Roger, 2002, "Interactions between Market and Credit Risk: Modeling the Joint Dynamics of Default-free and Defaultable Bond Term Structures", FAME Research Paper Nr. 56, Switzerland.

47. Xia Yihong, 2001, "Learning about Predictability", *The Journal of Finance* **56**, 205-246.

A Propositions and Proofs

Proof of Proposition 1. To obtain the process for the defaultable bond, apply Itô's formula to $F(t, Y_t) \equiv D(t, T)$ using (4) and (3):[41]

$$dF(t, Y_t) = \text{Đ}F(t, Y_t)\,dt + \partial_2 F(t, Y)\,\sigma_t^Y\,d\widetilde{W}_t$$

where

$$\text{Đ}F(t, Y) = \partial_1 F(t, Y) + \partial_2 F(t, Y) \cdot \mu_t^Y + \frac{1}{2} tr\left[\partial_{22} F(t, Y)\,\sigma_t^Y\,\left(\sigma_t^Y\right)'\right].$$

By Proposition 2, we also know that $\text{Đ}F(t, Y_t) = r_t D(t, T)$ under Q^Y. Using this result and $\partial_2 F(t, Y) = D(t, Y_t)\,b^D(T-t)'$ with the definition of σ_t^Y yields

$$dD(t, T) = D(t, T)\,r_t dt + D(t, T)\,b^D(T-t)'\,\Sigma\sqrt{S(t, Y_t)}d\widetilde{W}_t.$$

To obtain the process under the measure P^Y, it suffices to use the relation $d\widetilde{W}_t = dW_t + \Lambda_t dt = dW_t + \sqrt{S(t, Y_t)}\rho dt$ between the two probability measures following Girsanov's theorem:

$$dD(t, T) = D(t, T)\left[\left(r_t + b^D(T-t)'\,\Sigma S(t, Y_t)\,\rho\right)dt + b^D(T-t)'\,\Sigma\sqrt{S(t, Y_t)}\right]dW_t.$$

The default-free bond price is a special case of the defaultable bond with $\delta_2 = 0$:

$$dB(t, T) = B(t, T)\left[\left(r_t + b^B(T-t)'\,\Sigma S(t, Y_t)\,\rho\right)dt + b^B(T-t)'\,\Sigma\sqrt{S(t, Y_t)}\right]dW_t.$$

The joint process of the Treasury and corporate bond price is then obtained easily.□

Proposition 2 *Let* $m_t = E_t^Q\left[\frac{1}{\widehat{B}(T)}|\mathcal{F}_t^Y\right]$, *where* $\widehat{B}(t) = \exp\left(\int_0^t (r_s + \lambda_s)\,ds\right)$, *be a martingale under* Q^Y. *The dynamics of* m_t *under* P^Y *is then:*

$$dm_t = m_t\left(\sigma_t\theta_t dt + \sigma_t dW_t\right)$$

where θ_t *is the market price of risk and* σ_t *may be interpreted as the sensitivity of* m_t *with respect to the sources of risk. Both* θ_t *and* σ_t *are* $\mathcal{F}_t^Y-$*adapted predictable processes.*

Proof. We assume that the risk neutral measure is well defined and given by $\frac{dQ^Y}{dP^Y} = \eta_T$ where

$$\eta_t = \exp\left(-\int_0^t (\theta_s)'\,dW_s - \frac{1}{2}\int_0^t (\theta_s)'\,\theta_s ds\right)$$

[41] The processes r, θ, μ^Y and σ^Y depend on the state variable vector, but the notation has been simplified here.

and θ_t is the market price of risk. If m_t is a martingale under Q^Y, $m_t\eta_t$ is a martingale under P^Y. Using the Martingale Representation property, we have:

$$m_t\eta_t = 1 + \int_0^t (\gamma_u)' \, dW_u$$

Applying Ito's formula on the log of this expression and integrating again yields

$$m_t\eta_t = \exp\left(\int_0^t (m_s\eta_s)^{-1} (\gamma_s)' \, dW_s - \frac{1}{2}\int_0^t (m_s\eta_s)^{-2} \gamma_s^2 ds\right)$$

and thus:

$$m_t = \exp\left(\int_0^t \left((m_s\eta_s)^{-1} (\gamma_s)' + \theta_s\right)' dW_s - \frac{1}{2}\int_0^t \left((m_s\eta_s)^{-2} (\gamma_s)' \gamma_s - (\theta_s)' \theta_s\right) ds\right)$$

Differentiating m_t, we obtain

$$\begin{aligned}
\frac{dm_t}{m_t} &= -\frac{1}{2}\left((m_t\eta_t)^{-2} (\gamma_t)' \gamma_t - (\theta_t)' \theta_t\right) dt + \frac{1}{2}\left((m_t\eta_t)^{-1} \gamma_t + \theta_t\right)' \left((m_t\eta_t)^{-1} \gamma_t + \theta_t\right) dt \\
&\quad + \left((m_t\eta_t)^{-1} \gamma_t + \theta_t\right)' dW_t \\
&= -\frac{1}{2} (m_t\eta_t)^{-2} (\gamma_t)' \gamma_t dt + \frac{1}{2} (\theta_t)' \theta_t dt + \frac{1}{2} (m_t\eta_t)^{-2} (\gamma_t)' \gamma_t dt + \frac{1}{2} (m_t\eta_t)^{-1} (\gamma_t)' \theta_t dt \\
&\quad + \frac{1}{2} (m_t\eta_t)^{-1} (\theta_t)' \gamma_t dt + \frac{1}{2} (\theta_t)' \theta_t dt + \left((m_t\eta_t)^{-1} \gamma_t + \theta_t\right)' dW_t \\
&= (\theta_t)' \theta_t dt + (m_t\eta_t)^{-1} (\gamma_t)' \theta_t dt + \left((m_t\eta_t)^{-1} \gamma_t + \theta_t\right)' dW_t \\
&= \sigma_t\theta_t dt + \sigma_t dW_t
\end{aligned}$$

under P^Y with $\sigma_t = \left((m_t\eta_t)^{-1} \gamma_t + \theta_t\right)'$ and

$$\frac{dm_t}{m_t} = \sigma_t d\widetilde{W}_t$$

under Q^Y using the relation $\widetilde{W}_t = W_t + \int_0^t \theta_s ds$.$\square$

Proof of Equation 16

In the Vasicek case, the MD $\mathcal{D}_t r_s$ of the interest rate process is linear in the MD of the state variables

$$\mathcal{D}_t r_s = \left(\delta^{\underline{N}}\right)' \mathcal{D}_t Y_s$$

where $\delta^{\underline{N}}$ has the second element equal to zero. Using (14), the MD of the state variable simplifies to[42]

$$\mathcal{D}_t Y_s = \Sigma \exp\left[-\kappa (s-t)\right]$$

[42] Note that $\exp(.)$ is the matrix exponential function.

The term $a(t, Y_t)$ of the interest rate hedge component is then

$$
\begin{aligned}
a(t, Y_t)' &= \frac{E_t\left[\xi_{t,\overline{T}}^{1-1/R} \int_t^{\overline{T}} \left(\delta^{\underline{N}}\right)' \Sigma \exp\left[-\kappa(s-t)\right] ds\right]}{E_t\left[\xi_{t,\overline{T}}^{1-1/R}\right]} \\
&= \int_t^{\overline{T}} \left(\delta^{\underline{N}}\right)' \Sigma \exp\left[-\kappa(s-t)\right] ds \\
&= -b^B\left(\overline{T}-t\right)' \Sigma
\end{aligned}
$$

To prove the last line, it suffices to solve the Ricatti equations described in Section (2) for $b^B\left(\overline{T}-t\right)$. Consequently, the interest rate hedge corresponds to a long position in a default-free bond maturing at the investment horizon $\overline{T}$.□

B Variance Stabilizing Transformation (Multivariate Case)

This is the multivariate case of the variance stabilizing transformation described in Detemple, Garcia and Rindisbacher (2002a), Proposition 2 and corresponds to Theorem 2 in Detemple, Garcia and Rindisbacher (2002b). It is reproduced here for the reader's convenience.

Proposition 3 *If the following conditions hold:*
(i) differentiability of drift: $\mu^Y \in \mathcal{C}^1\left([0,T] \times \mathbb{R}^d\right)$
(ii) differentiability of volatility: $\sigma^Y \in \mathcal{C}^2\left([0,T] \times \mathbb{R}^d\right)$
(iii) growth condition: $\mu^Y(t,0)$ *and* $\sigma^Y(t,0)$ *are bounded for all* $t \in [0,T]$
(iv) volatility condition: rank $\left(\sigma^Y\right) = d$ *and the Lie algebra of the vector fields generated by the columns of* $\sigma^Y, \mathcal{L}\left\{\sigma_1^Y, ..., \sigma_d^Y\right\}$ *is Abelian, i.e.* $\left(\partial_2\sigma_i^Y\right)\sigma_j^Y = \left(\partial_2\sigma_j^Y\right)\sigma_i^Y$ *for all* $i,j = 1,...,d$ *where* $\partial_2\sigma_j^Y$ *is the* $d \times d$ *Jacobian matrix with respect to* y *of the* $d \times 1$ *vector function* σ_j^Y*. Then we have for* $t \leq s$ *that*

$$\mathcal{D}_t Y(s) = \sigma^Y(s, Y_s)\, Z_{t,s}$$

where the $d \times d$ *process* $Z_{t,s}$ *satisfies*

$$dZ_{t,s} = \left[\partial_2\left[\left(\sigma^Y\right)^{-1}\mu^Y + \frac{1}{2}H\right] + \partial_1\left(\sigma^Y\right)^{-1}\right](s, Y_s)\,\sigma^Y(s, Y_s)\, Z_{t,s}ds$$

subject to the boundary condition $Z_{t,t} = I_d$ *($d \times d$-identity matrix) where*

$$H = (I \quad \mathbf{1}')\left(K \odot \left(\mathbf{1} \quad \left(\sigma^Y\right)'\sigma^Y\right)\right)\mathbf{1}$$

with K *for the Jacobian matrix of* $\left(\sigma^Y\right)^{-1}$ *given by*

$$K = -\frac{1}{2}\left[\left(\sigma^Y \quad \left(\sigma^Y\right)'\right)^{-1}\partial_2\left(\left(\sigma^Y\right)'\right) + \left[\partial_2\left(\left(\sigma^Y\right)'\right)'\left(\left(\sigma^Y\right)' \quad \sigma^Y\right)^{-1}\right]_v\right]$$

The operators   and $\odot$ *represent, respectively, the Kronecker and Hadamard products, whereas the stack operator* $[.]_v$ *operates on a* $d \times d^2$ *matrix* $B = [B_1, ..., B_d]$*, where* B_i *are d-dimensional square matrices, as follows:* $[B]_v = \left[(B_1)', ..., (B_d)'\right]'$.

C Tables and Figures

	$A_0(2)$	$A_2(2)$
κ_{11}	0.13	0.64
	(2.62)	(35.63)
κ_{12}	-	-0.38
		(-14.46)
κ_{21}	0.0695	-0.09
	(2.61)	(-13.86)
κ_{22}	0.19	0.55
	(9.97)	(23.60)
θ_1	-	3.97
		(56.87)
θ_2	-	2.68
		(10.14)
β	-	-
δ_0	0.053	-
	(15.45)	
δ_1	0.008	0.007
	(5.50)	(62.96)
δ_2	0.009	0.002
	(74.09)	(14.70)
λ_1	-0.13	-0.13
	(-2.34)	(-6.10)
λ_2	-0.24	-0.24
	(-4.06)	(-8.57)
Function Value	1443.6	2662.2

Table 1: Estimation Results: The parameters of the models $A_0(2)$ and $A_2(2)$ are estimated with the efficient method of moments. t-statistics are in parentheses. See Walder (2002) for more details.

		1		3		5		7		10	
		Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate
MV		-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%
IRH		19.6%	0.0%	32.2%	0.0%	35.0%	0.0%	35.4%	0.0%	38.9%	0.0%
MPRH		4.8%	-6.4%	12.0%	-14.2%	15.9%	-18.1%	18.6%	-20.8%	23.0%	-26.0%
	MPIRH	-1.5%	0.0%	-2.3%	0.0%	-2.2%	0.0%	-2.2%	0.0%	-3.0%	0.0%
	MPCRH	6.4%	-6.4%	14.2%	-14.2%	18.1%	-18.1%	20.8%	-20.8%	26.0%	-26.0%
Total		7.1%	54.9%	26.8%	47.0%	33.5%	43.2%	36.7%	40.4%	44.5%	35.3%

Table 2: Investment Horizon and Portfolio Composition. The portfolio composition is shown for investment horizons $\overline{T} = 1, 3, 5, 7, 10$ years. MV: mean-variance optimal weights, IRH: interest rate hedge, MPRH: market price of risk hedge, MPIRH: market price of interest rate risk hedge, MPCRH: market price of credit risk hedge, Total: aggregated portfolio weights. The risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years, interest rate and credit risk are independent and the initial term structures are at a low level.

		1		3		5		7		10	
		Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate
MV		-79.5%	175.1%	-30.6%	82.3%	-21.7%	67.4%	-18.8%	63.1%	-17.4%	61.3%
IRH		45.0%	0.0%	22.6%	0.0%	21.5%	0.0%	20.0%	0.0%	19.6%	0.0%
MPRH		13.4%	-17.3%	6.5%	-8.2%	4.9%	-6.8%	4.7%	-6.3%	4.8%	-6.4%
	MPIRH	-3.9%	0.0%	-1.7%	0.0%	-1.8%	0.0%	-1.6%	0.0%	-1.5%	0.0%
	MPCRH	17.3%	-17.3%	8.2%	-8.2%	6.8%	-6.8%	6.3%	-6.3%	6.4%	-6.4%
Total		-21.1%	157.9%	-1.5%	74.1%	4.8%	60.7%	5.9%	56.8%	7.1%	54.9%

Table 3: Bond Maturity and Portfolio Composition. The portfolio composition is shown for bonds having maturities $T = 1, 3, 5, 7, 10$ years. MV: mean-variance optimal weights, IRH: interest rate hedge, MPRH: market price of risk hedge, MPIRH: market price of interest rate risk hedge, MPCRH: market price of credit risk hedge, Total: aggregated portfolio weights. The risk aversion is $\gamma = 2$, the investment horizon is $\overline{T} = 1$ year, interest rate and credit risk are independent and the initial term structures are at a low level.

		Low Level		Medium Level		High Level	
		Treasury	Corporate	Treasury	Corporate	Treasury	Corporate
MV		-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%
IRH		19.6%	0.0%	21.5%	0.0%	21.9%	0.0%
MPRH		4.8%	-6.4%	4.5%	-6.5%	4.5%	-6.6%
	MPIRH	-1.5%	0.0%	-2.0%	0.0%	-2.1%	0.0%
	MPCRH	6.4%	-6.4%	6.5%	-6.5%	6.6%	-6.6%
Total		7.1%	54.9%	8.6%	54.8%	9.0%	54.7%

Table 4: Term Structure Level and Portfolio Composition. The portfolio composition is shown for low, medium and high term structure levels. MV: mean-variance optimal weights, IRH: interest rate hedge, MPRH: market price of risk hedge, MPIRH: market price of interest rate risk hedge, MPCRH: market price of credit risk hedge, Total: aggregated portfolio weights. The risk aversion is $\gamma = 2$, the investment horizon is $\overline{T} = 1$ year, interest rate and credit risk are independent and the bond maturities are $T = 10$ years.

		0.5		1		2		5		10	
		Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate
MV		-69.5%	245.1%	-34.7%	122.6%	-17.4%	61.3%	-6.9%	24.5%	-3.5%	12.3%
IRH		-37.7%	0.0%	0.0%	0.0%	19.6%	0.0%	31.7%	0.0%	35.7%	0.0%
MPRH		-8.2%	10.9%	0.0%	0.0%	4.8%	-6.4%	8.0%	-10.5%	9.1%	-12.0%
	MPIRH	2.7%	0.0%	0.0%	0.0%	-1.5%	0.0%	-2.5%	0.0%	-2.9%	0.0%
	MPCRH	-10.9%	10.9%	0.0%	0.0%	6.4%	-6.4%	10.5%	-10.5%	12.0%	-12.0%
Total		-115.3%	256.0%	-34.7%	122.6%	7.1%	54.9%	32.7%	14.0%	41.3%	0.3%

Table 5: Risk Aversion and Portfolio Composition. The portfolio composition is shown for risk aversion levels $\gamma = 0.5, 1, 2, 5$ and 10. MV: mean-variance optimal weights, IRH: interest rate hedge, MPRH: market price of risk hedge, MPIRH: market price of interest rate risk hedge, MPCRH: market price of credit risk hedge, Total: aggregated portfolio weights. The investment horizon is $\overline{T} = 1$ year, the bond maturities are $T = 10$ years, interest rate and credit risk are independent and the initial term structures are at a low level.

Investment Horizon:		1		3		5		7		10	
		Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate	Treasury	Corporate
MV	Vasicek	-25.8%	51.8%	-25.8%	51.8%	-25.8%	51.8%	-25.8%	51.8%	-25.8%	51.8%
	CIR	-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%	-17.4%	61.3%
IRH	Vasicek	9.7%	0.0%	24.7%	0.0%	35.3%	0.0%	42.8%	0.0%	49.9%	0.0%
	CIR	19.6%	0.0%	32.2%	0.0%	35.0%	0.0%	35.4%	0.0%	38.9%	0.0%
Total	Vasicek	-16.1%	51.8%	-1.1%	51.8%	9.5%	51.8%	17.0%	51.8%	24.1%	51.8%
	CIR	7.1%	54.9%	26.8%	47.0%	33.5%	43.2%	36.7%	40.4%	44.5%	35.3%

Table 6: Effect of Time-Varying Risk Premia on Portfolio Composition. The portfolio composition for the Vasicek (constant risk premia) and CIR (stochastic risk premia) case is shown for investment horizons $\overline{T} = 1, 3, 5, 7, 10$ years. MV: mean-variance optimal weights, IRH: interest rate hedge, Total: aggregated portfolio weights. The market price of risk hedge (MPRH) is equal to zero in the Vasicek case and is not shown here for the CIR case. The risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years, interest rate and credit risk are independent and the initial term structures are at a low level.

		Independent		Positive Dependence		Negative Dependence	
		Treasury	Corporate	Treasury	Corporate	Treasury	Corporate
MV		-17.4%	61.3%	-8.3%	41.7%	-22.4%	76.7%
IRH		19.6%	0.0%	34.0%	-14.5%	7.1%	12.1%
MPRH		4.8%	-6.4%	6.1%	-6.9%	2.9%	-5.3%
	MPIRH	-1.5%	0.0%	-3.6%	1.7%	-0.4%	-1.0%
	MPCRH	6.4%	-6.4%	9.7%	-8.6%	3.3%	-4.3%
Total		7.1%	54.9%	31.8%	20.3%	-12.4%	83.6%

Table 7: Effect of Interdependencies on Portfolio Composition. The portfolio composition is shown for interest and credit risk being independent, positively dependent and negatively dependent. MV: mean-variance optimal weights, IRH: interest rate hedge, MPRH: market price of risk hedge, MPIRH: market price of interest rate risk hedge, MPCRH: market price of credit risk hedge, Total: aggregated portfolio weights. The risk aversion is $\gamma = 2$, the investment horizon is $\overline{T} = 1$ years, the bond maturities are $T = 10$ years and the initial term structures are at a low level.

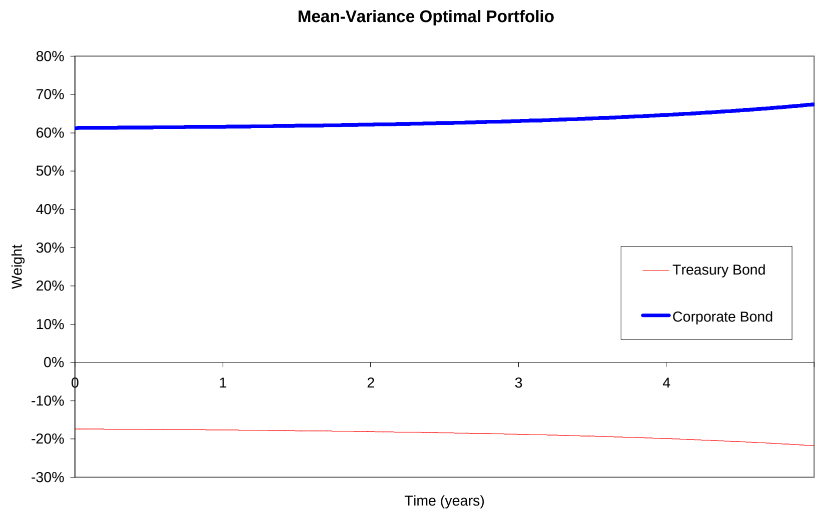

Figure 1: **Mean-Variance Component**. Typical path for the weights of the mean-variance component during the investment period in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 5$ years, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the initial term structures are at a low level.

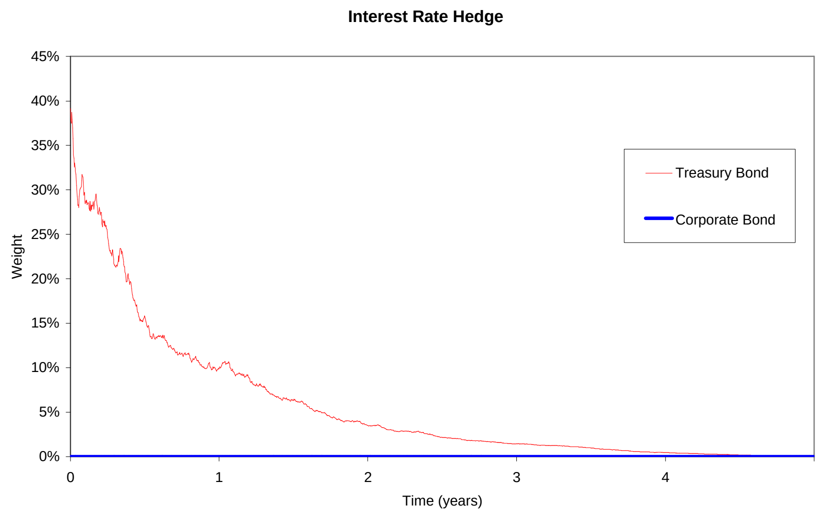

Figure 2: **Interest Rate Hedge**. Typical path for the weights of the interest rate hedge component during the investment period in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 5$ years, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the initial term structures are at a low level.

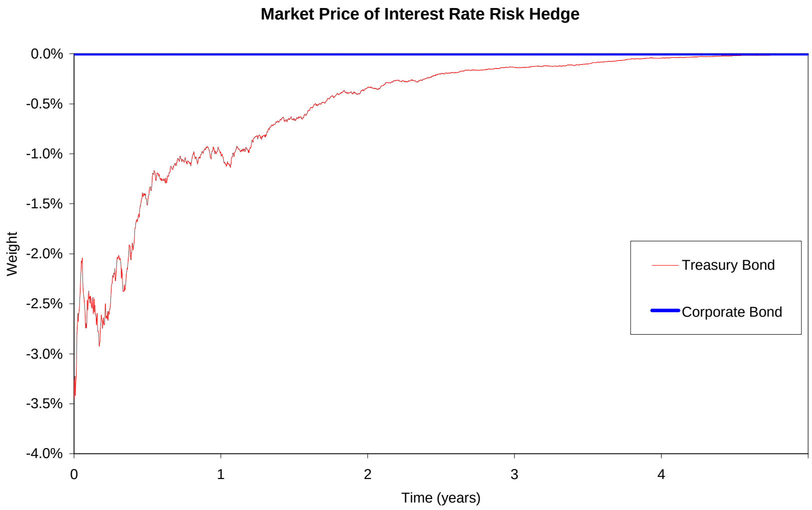

Figure 3: **Market Price of Interest Rate Risk Hedge**. Typical path for the weights of the market price of interest rate risk hedge component during the investment period in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 5$ years, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the initial term structures are at a low level.

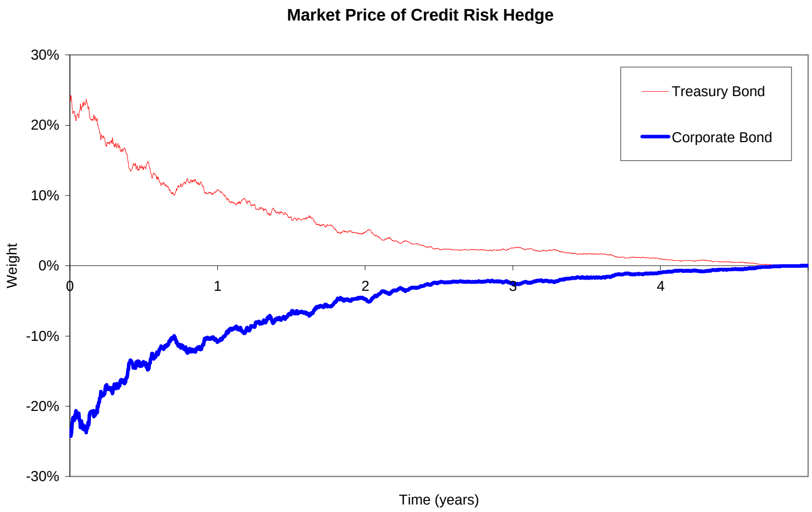

Figure 4: **Market Price of Credit Risk Hedge**. Typical path for the weights of the market price of credit risk hedge component during the investment period in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 5$ years, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the initial term structures are at a low level.

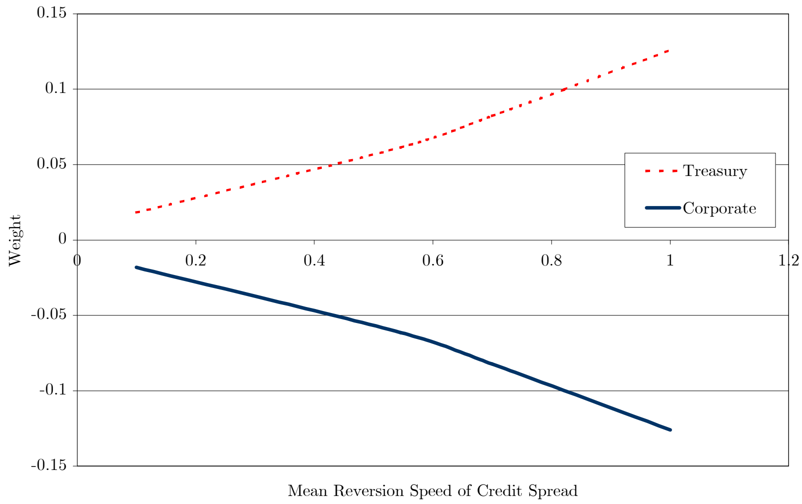

Figure 5: **Market Price of Credit Risk Sensitivity to Mean Reversion Speed.** The Treasury and corporate bond weights for various levels of the mean-reversion speed κ_{22} are shown in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 1$ year, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the initial term structures are at a low level.

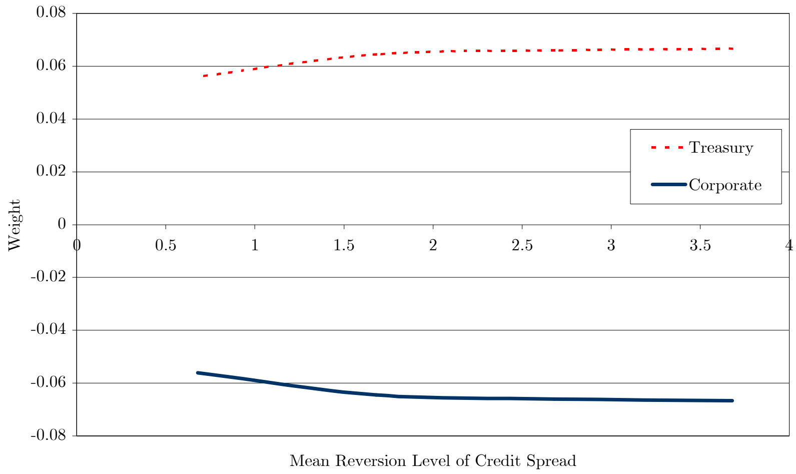

Figure 6: **Market Price of Credit Risk Sensitivity to Mean Reversion Level**. The Treasury and corporate bond weights for various levels of the mean-reversion level ϑ_2 are shown in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 1$ year, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the level of the state variables correspond to their mean-reversion level.

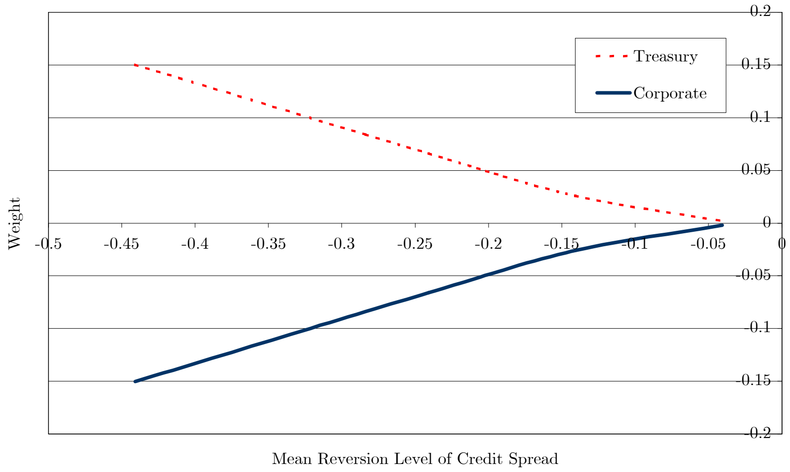

Figure 7: **Market Price of Credit Risk Sensitivity to Market Price of Risk Parameter**. The Treasury and corporate bond weights for various levels of the market price of credit risk parameter ρ_2 are shown in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 1$ year, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the level of the state variables correspond to their long term mean.

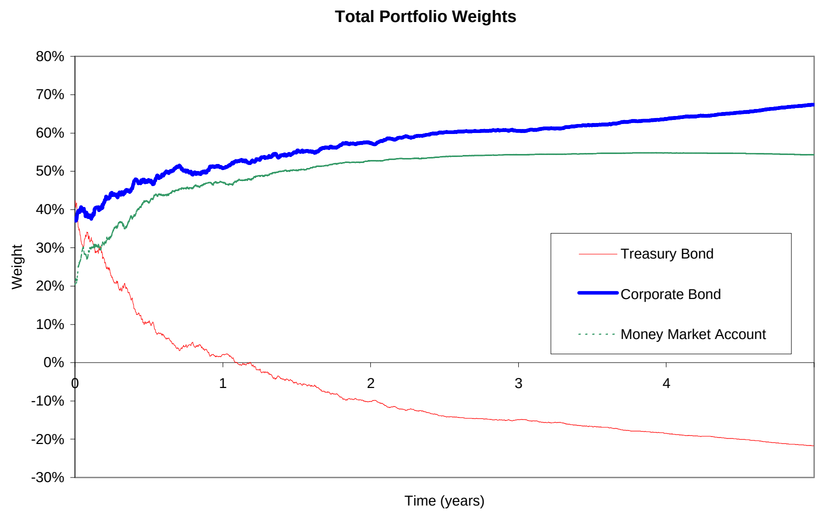

Figure 8: **Total Portfolio Weights**. Typical path for the total portfolio weights during the investment period in the CIR case (without interdependencies). The investment horizon is $\overline{T} = 5$ years, the risk aversion is $\gamma = 2$, the bond maturities are $T = 10$ years and the initial term structures are at a low level.