Notes on Dynamics for cryptocurrency derivatives

August 15, 2024

Abstract

To be filled.

Contents

1. Ideas

1.1. Uncovered interest parity in the absence of bond

This section is mostly taken/copied from section 6.1 of Gudgeon *et al.* (2020) for idea generation. Uncovered interest parity (UIP) normally appear in the context of foreign exchange between two countries: domestic and foreign. An investor has the choice of whether to hold domestic or foreign assets. If the condition of UIP holds, a risk-neutral investor should be indifferent between holding the domestic or foreign assets because the exchange rate is expected to adjust such that returns are equivalent.

Example An investor starting with 1m GBP at $t = 0$ could either:

- receive an annual interest rate of $i_{\text{GBP}} = 3\%$, resulting in 1.03m GBP at $t = 1$
- or, immediately buy 1.23 USD at an exchange rate $S_{\text{GBP/USD}} = 0.813$, and receive an annual interest rate of $i_{\text{USD}} = 5\%$, resulting in 1.2915m USD at $t = 1$. Then, convert the USD with the exchange rate at $t = 1$, say $S_{\text{GBP/USD}} = 0.7974$, and get 1.03m GBP.

If UIP holds, despite the higher interest rate of the USD, the investor will be indifferent because the exchange rate between currencies offset the spread between interest rates. Mathematically, UIP is stated as

$$1 + R^{(i)} = (1 + R^{(j)}) \frac{\mathsf{E} S_{t+k}}{S_t},$$

where $R^{(i/j)}$ is the interest rate payable on asset i/j from time t to $t + k$, and S_t is the exchange rate ar time t.

Now the question is both $R^{(i/j)}$ are not known in advance due to the lack of a liquid bond market that investors can secure the future payoff by holding a cryptocurrency. However, the good news is that we have the observable historical short rate, $r_t^{(i)}$, and exchange rate $S_t^{(i/j)}$. The UIP condition in this case have to be adjusted to incorporate the fact that the investor consider also the uncertainty of the domestic and foreign short rate, i.e.[1]

$$\mathsf{E}\left(\exp\int_0^T r_t^{(i)}\mathrm{d}t\right) = \mathsf{E}\left(\exp\left(\int_0^T r_t^{(j)}\mathrm{d}t\right)\frac{S_T}{S_t}\right).$$

The above UIP condition open quite some questions

1. It seems necessary to model the joint dynamics of the foreign short rate and the exchange rate, such that the R.H.S. of the above equation can be evaluated.

2. Under which measure should we take the expectation of both sides? Any criteria of choosing the measure? No-arbitrage?

3. When will the above equation hold? When will not?

4. If the condition does not hold, are there any arbitrage opportunities?

5. What is the dynamics of $r^{(i/j)}$? For further development, we might want a parametrised stochastic model such that we can (i) perform measure change, (ii) price bonds, swaps, or any other derivatives easily, (iii) capture the interest rate dynamics nicely.

6. Another idea: Can we use machine learning method to get a good enough estimate of the L.H.S. and R.H.S. separately, and make a statistical argument over the difference between L.H.S. and R.H.S.?

$$\hat{\mathsf{E}}\left(\exp\int_0^T r_t^{(i)}\mathrm{d}t\right) = \hat{\mathsf{E}}\left(\exp\left(\int_0^T r_t^{(j)}\mathrm{d}t\right)\frac{S_{t+k}}{S_t}\right) + \varepsilon_{t,k}.$$

[1] The AAVE and Compound interest rate are compounded every second, which is close enough to model the short rate payoff in a continuous compounding scheme.

1.2. A crypto interest rate model

This idea is motivated by a conversation in an interview with a team of quant working for a centralised crypto exchange (CEX). The pricing of crypto derivatives requires a ”risk-free” rate for each underlying, e.g. a ”risk-free” rate of holding BTC, another ”risk-free” rate of holding ETH.

I put quotation marks for the term risk-free rate because there is no consensus to what is the risk-free rate in the crypto market. In practice, there are four ways of getting a risk-free rate by viewing the risk-free rate as the opportunity cost of investing the holdings to the derivative market instead of earning a risk-free rate in another market. The four ways of getting a risk-free are referring to calibrating the risk-free rate to three different markets:

1. DeFi lending protocols, e.g. AAVE and Compound,
2. DeFi staking protocols, e.g. ETH POS, AQRU,
3. Futures traded on CEX,
4. Perpetual futures traded on CEX.

The first two markets are not exactly risk-free since trading on DeFi protocols expose oneself to platform risk. Although DeFi lending protocols enforce over-collateralisation, lenders still expose themselves to gap risk since there is a chance that the collateral value jump through the value of the borrowings and there is no way to force the borrower to pay back.

The third and forth markets suffer from the platform risk. However, if a trader manage/hedge her exposure on CEX with other instruments traded on the same CEX, then we can argue that the platform risk is somehow offset. The perpetual futures market does not exempt carry trade traders from losses since funding rate can go negative. It turns out, after considering the pros and cons of the markets, the futures market traded on CEX is commonly used by the team of quant to get the risk-free rate to price their derivatives, which is also a common practice in the traditional market.

The key idea here is to have a CEX futures market centric view, i.e. considering the CEX futures market as risk-free, and to study the differences of the rate dynamics among the markets. In order to do so, a common interest rate model for crypto is needed. It will be a challenge to have a single interest rate model that fits the rate dynamics in all the markets. For example, the DeFi lending/staking rate is always positive, but the funding rate of perpetual futures can go negative. A possible solution is to discount the DeFi lending/staking rates by the corresponding default intensity.

1.3. A model for perpetual futures

Suppose the price of the perpetual futures, F_t, is reverting to its underlying, S_t, with some Brownian noise involved, i.e.

$$\mathrm{d}F_t = \kappa \left(\delta S_t - F_t\right) \mathrm{d}t + \epsilon \mathrm{d}W_t,$$

where $\kappa > 0$ is the mean reverting speed, $\delta \in \mathbb{R}$ is the spot-futures spread, and $\epsilon > 0$ is the volatility.
We derive the solution of the perpetual futures by applying Ito's lemma to $e^{\kappa t} F$:

$$\begin{aligned}\mathrm{d}e^{\kappa t} F_t &= \left(\kappa e^{\kappa t} F_t + \kappa \left(\delta S_t - F_t\right) e^{\kappa t}\right) \mathrm{d}t + \epsilon e^{\kappa t} \mathrm{d}W_t \\ &= \kappa \delta S_t e^{\kappa t} \mathrm{d}t + \epsilon e^{\kappa t} \mathrm{d}W_t.\end{aligned}$$

The solution is obtained by integrating both sides from time 0 to T,

$$e^{\kappa T} F_T = F_0 + \kappa \delta \int_0^T S_t e^{\kappa t} \mathrm{d}t + \int_0^T \epsilon e^{\kappa t} \mathrm{d}W_t.$$

The expected value of F_T under a risk-neutral measure $\mathbb{Q}$ is

$$\begin{aligned}
e^{\kappa T}\mathsf{E}_{\mathbb{Q}}\left(F_T\right) &= F_0 + \kappa\delta\int_0^T \mathsf{E}_{\mathbb{Q}}\left(S_t\right)e^{\kappa t}\mathrm{d}t \\
&= F_0 + \kappa\delta e^{rT}S_0\int_0^T e^{\kappa t}\mathrm{d}t \\
&= F_0 + \delta e^{rT}S_0\left(e^{\kappa T}-1\right) \\
\mathsf{E}_{\mathbb{Q}}\left(F_T\right) &= e^{-\kappa T}\left(F_0 - \delta e^{rT}S_0\right) + \delta e^{rT}S_0.
\end{aligned}$$

Suppose $e^{-rt}F_t$ is a martingale under $\mathbb{Q}$ (i.e. $\mathsf{E}_{\mathbb{Q}}\left(F_T\right) = e^{rT}F_0$),

$$e^{rT}F_0 = e^{-\kappa T}\left(F_0 - \delta e^{rT}S_0\right) + \delta e^{rT}S_0.$$

By such, we work out the formula for the spread δ:

$$\begin{aligned}
\delta &= \frac{F_0}{S_0}\frac{\left(e^{rT}-e^{-\kappa T}\right)}{e^{rT}\left(1-e^{-\kappa T}\right)} \\
&= \frac{F_0}{S_0}\frac{\left(1-e^{-(\kappa+r)T}\right)}{1-e^{-\kappa T}}
\end{aligned}$$

Observations

1. The spread δ depends on κ, r (risk-free rate), and T (can be considered as maturity)
2. If $r > 0$, then δ is a time-increasing function

$$\frac{\partial\delta}{\partial T} = \frac{F_0}{S_0}\frac{Te^{-rT}}{e^{\kappa T}-1} > 0$$

3. If $r = 0$, then $\delta = F_0/S_0$, meaning the spread is no longer depending on maturity

4. If we send $T \to \infty$, then $\delta = F_0/S_0$ asymptotically.

Now, say the spread is a time-dependant deterministic function with a form

$$\delta(t) = Ce^{dt},$$

where $C > 0$ and $d \in \mathbb{R}$ are constants to be determined. One can think of d as the rate of divergence between the spot and futures.
The price dynamics of the perpetual futures is

$$\mathrm{d}F_t = \kappa \left(\delta(t) S_t - F_t\right) \mathrm{d}t + \epsilon \mathrm{d}W_t.$$

Again, by apply Ito's lemma to $e^{\kappa t} F_t$ and integration, we get the solution of F_T

$$e^{\kappa T} F_T = F_0 + \kappa C \int_0^T S_t e^{(\kappa+d)t} \mathrm{d}t + \int_0^T \epsilon e^{\kappa t} \mathrm{d}W_t.$$

Taking expectation under $\mathbb{Q}$ of both sides yields,

$$\begin{aligned} e^{\kappa T} \mathsf{E}_{\mathbb{Q}}\left(F_T\right) &= F_0 + \kappa C S_0 e^{rT} \int_0^T e^{(\kappa+d)t} \mathrm{d}t \\ e^{(\kappa+r)T} F_0 &= F_0 + \kappa C S_0 e^{rT} \frac{e^{(\kappa+d)T} - 1}{\kappa + d} \end{aligned}$$

2. Literatures

2.1. Uncovered Interest Parity and its variants

Cappiello and De Santis (2007)

1. This paper proposes an extension of UIP called the Uncovered Return Parity (URP)

2. The URP condition is

$$\mathsf{E}\left(R_{t+1}\frac{S_{t+1}}{S_t}m_{t+1}\middle|\mathcal{F}_t\right) = 1,$$

where R_{t+1} is the gross return on a foreign asset denominated in a foreign currency, and S_{t+1} is the spot exchange rate, defined as the number of units of domestic currency exchanged for one unit of foreign currency.

3. The R.H.S. (=1) of the above equation stemmed from definition of stochastic discount factor, see Section 3.1 of Back (2010).

4. Then the authors assume that there exist a foreign risk-free bond (which we do not have that in the cryptomarket) and yield the following

$$\mathsf{E}\left(\frac{S_{t+1}}{S_t}m_{t+1}\middle|\mathcal{F}_t\right) = \frac{1}{R_{f,t}}.$$

5. The remaining paper is about estimation of URP. The authors estimate the SDF via GMM.

2.2. Affine term structure models

Anderson *et al.* (2010)

1. The paper extends the affine class of term structure models to describe the joint dynamics of exchange rates and interest rates

2.3. Interest rate derivatives in the crypto space

Inter-Protocol Offered Rate (IPOR) `https://docs.ipor.io/`

1. The IPOR company offers an interest rate benchmark (weighted average of DeFi interest rate) with the same name that summarizes the lending and borrowing interest rates of crypto loan platforms.

2. The company offers trading of fixed income derivatives, e.g. interest rate swap. The pricing and transaction are based on the IPOR rate and automated market maker.

3. The main derivatives traded on IPOR is cancellable interest rate swaps.

4. IPOR uses Hull-White jump-diffusion model for rate simulation and Longstaff-Schwartz method for pricing (the cancellable part)

5. Criticisms

   (a) Complex product designs: cancellable swaps + Hull-White-jump-model + Longstaff-Schwartz

   (b) Lack of market-involved pricing mechanism: the AMM takes the spread calculated only by the Hull-White model + Longstaff-Schwartz.

   (c) Spread calculation often results in high spread that prohibits transactions.

   (d) Max tenor is too short (90-day longest)

   (e) See https://scapital.medium.com/ipor-a-postmortem-for-the-interest-rate-swap-pioneer-5dc8492c2f7c

3. Backgrounds

3.1. Short rate models

A quote from Section 10.1 of Shreve (2004) nicely summarise what is short rate traditionally:

> "The interest rate (sometimes called the short rate) is an idealization corresponding to the shortest maturity yield or perhaps the overnight rate offered by the government, depending on the particular application."

Remark 1. In the crypto world, instead of using liquidly traded bonds and fixed-income products to infer the short rate (mainly its dynamics) and price more complex products, the short rate itself is directly observable and impacts the growth rate of borrowing and lending accounts, i.e. the growth of crypto borrowing and lending accounts is a realization of the crypto short rate. Beware that *all* borrowings in a lending pool, disregard of the identity of borrower and starting date of the borrowings (there is no maturity in crypto loans), are growing at the *same* crypto borrowing rate.

The simplest model for fixed income markets begin with a stochastic differential equation for the interest rate

$$\mathrm{d}r_t = \beta(t, r_t)\mathrm{d}t + \gamma(t, r_t)\mathrm{d}W_t.$$

The zero-coupon bond pricing formula is

$$B(t,T) = \mathsf{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{F}_t\right],$$

provided that $B(T,T) = 1$.
Since r_t is governed by a SDE, it is a Markov process and we must have

$$B(t,T) = f(t, r_t)$$

By Feynman-Kac, we obtain the partial differential equation

$$\partial_t f(t,r) + \beta(t,r)\partial_r f(t,r) + \frac{1}{2}\gamma^2(t,r)\partial_{rr} f(t,r) = r f(t,r),\ \ f(T,r) = 1.$$

Hull-White model
In the Hull-White model, the evolution of the interest rate is given by

$$\mathrm{d}r_t = (a(t) - b(t)r_t)\,\mathrm{d}t\sigma(t)\mathrm{d}W_t,$$

where $a(t)$, $b(t)$, and $\sigma(t)$ are non-random positive functions of time. The PDE for the zero-coupon bond price becomes

$$\partial_t f(t,r) + (a(t) - b(t))\,\partial_r f(t,r) + \frac{1}{2}\sigma^2(t)\partial_{rr} f(t,r) = r f(t,r),\ \ f(T,r) = 1.$$

By ansatz, the solution of the above PDE has a form

$$f(t,r) = \exp\left(-rC(t,T) - A(t,T)\right).$$

Let's work out the partial derivatives:

$$\begin{aligned}
\partial_t f &= \left(-r\partial_t C(t,T) - \partial_t A(t,T)\right) f \\
\partial_r f &= -C(t,T) f \\
\partial_{rr} f &= C^2(t,T) f.
\end{aligned}$$

Substitute into the PDE gives

$$\begin{aligned}
0 = \Big[&\left(-\partial_t C + b(t)C - 1\right) r \\
&- \partial_t A - a(t)C + \frac{1}{2}\sigma^2(t)C\Big] f(t,r).
\end{aligned}$$

Since f is nonzero and this equation must hold for all r,

$$\begin{aligned}
-\partial_t C + b(t)C - 1 = 0 \\
-\partial_t A - a(t)C + \frac{1}{2}\sigma^2(t)C = 0.
\end{aligned}$$

Since $f(r,T) = 1$, $C(T,T) = A(T,T) = 0$. The solution of the above equations is

$$\begin{aligned}
C(t,T) &= \int_t^T \exp\left(-]int_t^s b(v)\mathrm{d}v\right) \mathrm{d}s \\
A(t,T) &= \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T)\right) \mathrm{d}s.
\end{aligned}$$

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