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Optimal Leveraged Portfolio Selection Under Quasi-Elastic Market Impact

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Abstract. We study optimal portfolio choice under leveraging to improve portfolio performance when trade execution faces market impact. We consider a quasi-elastic market with continuous trading in which temporary liquidity costs are sufficiently large relative to permanent impact. The resulting convex optimization model is used to show analytically that an unlevered portfolio maximizing the Sharpe ratio is no longer a tangency portfolio, and increasing the portfolio target mean leads to severely undermining the risk-adjusted returns and requiring increased portfolio leverage. This paper develops theoretical properties underlying the relationships among target mean, leverage, and Sharpe ratio in optimal portfolio selection under market impact. The Sharpe-leverage efficient frontiers under market impact are consistently dominated when setting higher return targets. Moreover, leverage-constrained and less risk-averse investors ignoring liquidity costs ex ante suffer the most losses in expected utility. Detailed computational analyses are provided using real-world data to support and highlight our analytical findings.

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Keywords: portfolio optimization • liquidity risk • trading impact on price • portfolio leverage • risk-adjusted returns

1. Introduction

In this paper, we analyze the economic performance of a risky portfolio, whose target return is enhanced via portfolio leveraging when portfolio execution faces market illiquidity[1] in the form of temporary and permanent trading costs. Temporary price impact refers to situations in which intense trading activity makes it difficult to find counterparties within a short time period, hence leading to adverse price impact that dissipates once the trading pressure eases—the hallmark of elasticity. In contrast, permanent price impact refers to situations in which large trades associated with, for example, the revelation of new information on the fundamentals of the asset cause permanent adverse impact, a trait of plastic deformation; see Schoneborn and Schied (2007). Our focus is on a market with both elastic and plastic price impacts of trading; however, the price effect on an asset because of temporary liquidity shortages is sufficiently large in relation to a permanent price effect, that is, the impact is sufficiently elastic, a condition herein referred to as a quasi-elastic market.

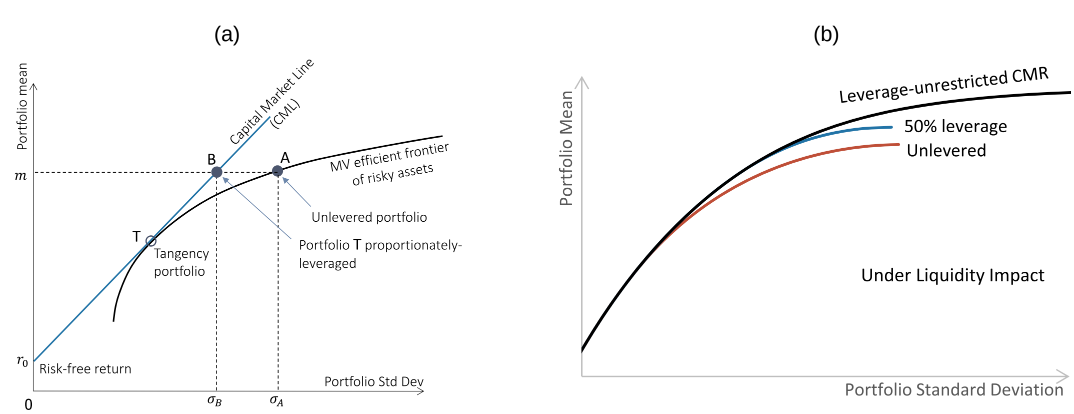
Market liquidity costs can be significant, especially in illiquid assets and during market downturns; see the vast literature on market microstructure, including the seminal work in Roll (1984) and Kyle (1985) and the excellent review in Hasbrouck (2007). The sensitivity of asset returns to market liquidity is well-studied empirically; see, for example, Pastor and Staumbaugh (2003), the survey in Amihud et al. (2005), or Rahi and Zigrand (2008). Acharya and Pedersen (2005) derive a liquidity-adjusted capital asset pricing model to explain the empirical findings within a unified framework but without any impact of leveraging. In contrast, Jacobs and Levy (2012, 2013) propose a mean-variance (MV) utility function augmented with an additional term for leverage risk aversion but without liquidity costs to obtain MV-optimal portfolios having leverage levels more consistent with the real world. As in these studies, we are only concerned with assets' market liquidity rather than traders' funding liquidity, which measures the ease of obtaining funding.[2]

Portfolio leveraging is a standard approach for enhancing fund performance, generally practiced through the so-called risk parity (RP) asset-allocation strategies that gained attention in the aftermath of the global financial crisis in 2008.[3] In RP, an unlevered long/short portfolio of an acceptable (low) risk profile (but having no debt) is further invested in by borrowing cash to create a levered portfolio. Whereas leveraging amplifies any risks or imbalances that exist in the unlevered portfolio, it can also lead to a gain in the portfolio's return in excess of the interest payable. Consequently, by overweighting safer assets, a portfolio is achieved with the highest risk-adjusted returns (Asness et al. 2012). To exemplify using the standard MV framework (Markowitz 1952) in Figure 1(a), to meet a given return target m, the risk-diversified tangency portfolio T may be proportionately levered up to obtain the efficient portfolio B rather than using a risk-concentrated unlevered portfolio A. That is, proportionate leveraging of an unlevered efficient portfolio is an optimal strategy leading to the capital market line (CML) with no degradation of risk-adjusted performance (Sharpe ratio) if liquidity impact of trading is ignored. Therefore, the RP portfolio B preserves the long-short diversification profile and the Sharpe ratio of T, whereas the variance of B is reduced relative to A although B is (cash) debt-ridden. However, when liquidity risk is considered ex ante, our theoretical work and the accompanying numerical analysis show that the theory of leveraged portfolio selection is quite different, and the implications on portfolio design are nontrivial. The linearity of the efficient frontier, that is, CML under leveraging, in Figure 1(a) is completely changed to the nonlinear capital market relationship (CMR) in Figure 1(b) when facing liquidity costs (see case study in Section 6). Our result is consistent with Li et al. (2020), which finds the MV-efficient frontier of a factor model being strictly concave in the presence of price-impact costs. As such, applying the RP strategy of "proportionate leveraging" becomes an inferior and (ex post) costly strategy with degraded risk-adjusted performance because of the increased average trading costs of leveraged positions. Moreover, the adverse impact on portfolio performance magnifies as the target return is increased, thus requiring even higher levels of leverage. Consequently, ignoring market impact becomes quite costly when forming ex ante optimal portfolios to meet higher target returns, especially at lower risk aversion, in terms of risk-adjusted returns and expected utility.

The focus of the paper is to develop insights on setting portfolio leverage and target mean levels in a quasi-elastic market with liquidity costs. We extend the standard MV model with liquidity parameters representing permanent and temporary impacts on asset prices because of the volume and intensity of trading (Carlin et al. 2007, Brown et al. 2010), two components primarily responsible for the inability to establish or liquidate portfolio positions in a timely manner at acceptable prices.[4] Specifically, we consider a linear permanent impact function and a general power-law temporary impact function with an exponent between zero and one, consistent with theoretical and empirical findings; see, for example, Huberman and Stanzl (2005), Almgren et al. (2005), Frazzini et al. (2018), and Tóth et al. (2011).[5] Relaxing the instantaneous trading assumption in the MV model, we incorporate a trade execution short time window facing liquidity issues but without price uncertainty relative to the investment horizon that captures the uncertainties in shaping the portfolio risk–return trade-off. The MV liquidity-impacted and leverage-restricted (MVLL) portfolio model with continuous trading during the execution window is presented in Section 2. If trade execution is assumed instantaneous with no market impact, our model collapses to the standard MV model. We prove in Section 3 that the optimal trading strategy corresponds to a constant trading rate.

In Section 4, we analyze the structure of the MVLL portfolio, which is a combination of the traditional MV portfolio and an "aim" portfolio impacted by the leverage constraint and the liquidity cost. We prove that the Sharpe-maximizing unlevered portfolio does not correspond to a tangency portfolio of the MV frontier

Figure 1. (Color online) Portfolio Behavior When Leveraging with and Without Liquidity Costs



Notes. (a) Leveraging low-risk funds ignoring market impact. (b) Optimal leveraging under market impact.

contrary to the classical portfolio theory under no market impact. Moreover, a feasible portfolio that can achieve a desired target return may not exist under strict leverage restrictions and when the constituent assets exhibit significant trading costs. We show that there exists a minimum required leverage to attain a given portfolio target, and this minimum leverage grows at an increasing rate with target return; moreover, there exists a maximum leverage level beyond which target return cannot be increased further—concepts not germane to the classic portfolio theory. Accordingly, portfolio choice is bounded by these minimum and maximum leverage levels for a specified target return when liquidity impact is present. Therefore, when setting portfolio targets, the investor must consider the interactions among acceptable leverage level, the market impact of trading because of illiquidity, and the parameters of the portfolio assets, simultaneously.

Given a fixed leverage level, in Section 5, we claim analytically that an increase in target mean is accompanied by a reduced risk-adjusted performance. This reduction in portfolio Sharpe ratio can be reversed to some extent with an increase in leverage; however, it can never reach the maximum Sharpe ratio attainable at a lower target mean; see Figure 2(a) for a graphic illustration (obtained from the case study in Section 6). That is, Sharpe leverage frontiers are separated by portfolio target mean, and frontiers of lower targets strictly dominate those of higher targets. This is a significant departure from the classic portfolio theory that ignores market impact, in which the maximum Sharpe ratio of a lower target mean can be easily attained even at a higher target mean provided sufficient portfolio leverage is applied; see Figure 2(b).

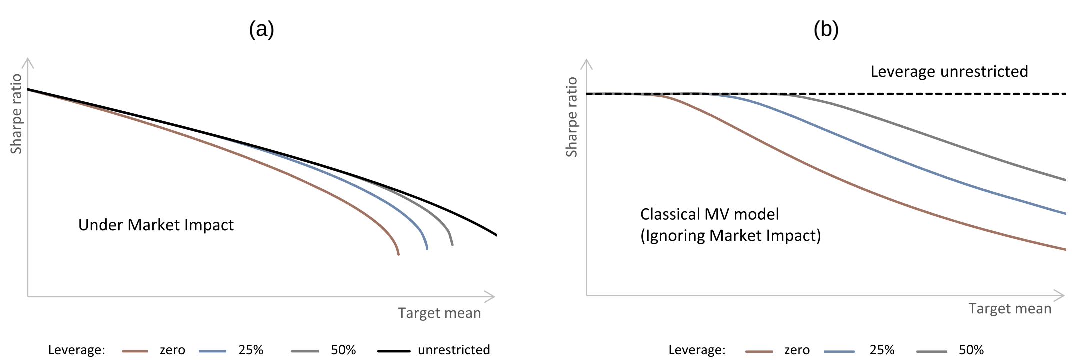
We perform computational experiments using New York Stock Exchange–traded assets to highlight the underlying properties of our ex ante analytical findings.[6] In the case study reported in Section 6, we obtain portfolio choice boundaries and illustrate the differences that exist in portfolio frontiers under liquidity impact relative to a scenario in which liquidity costs are ignored. It also turns out that portfolio performance shortfall when ignoring trading impact can be substantial, thus providing a measure of ex ante liquidity risk. An optimal leveraged portfolio maximizing the Sharpe ratio undergoes changes to its composition (risk profile) from that of an optimal unlevered portfolio when liquidity impact is incorporated ex ante, thus arguing against the usual rationale for RP strategies that leveraging preserves risk profile, that is, proportionate leveraging in RP is suboptimal. Section 7 concludes the paper, and the online appendix includes proofs of all analytical results.

The three major contributions of this paper are emphasized for easy access: first, we extend the standard MV model incorporating liquidity impact and leverage conditions to obtain the MVLL portfolio. The generality of the temporary impact function we employ allows for varying market conditions when constructing optimal portfolios. Second, we analyze the structure of the MVLL portfolio and interactions that exist among portfolio target return, leverage level, Sharpe ratio, and trading impact to develop insightful theoretical properties that are valuable in setting portfolio parameters. Third, we conduct computational experiments to fully illustrate the implications of our analytical results. Our major findings contrast with the implications of the classic portfolio theory that ignores trading impact on prices, and they are summarized as follows:

• Similar to the portfolio strategy in Gârleanu and Pedersen (2013), the MVLL portfolio strategy also has the property of "aiming in front of target." The aim portfolio is the standard MV portfolio adjusted by the leverage requirement, and the liquidity impact further imposes a shrinkage effect on the aim portfolio.

• Because of the market impact of trading, the Sharpe-maximizing portfolio departs from the tangency portfolio of the MV frontier even without leveraging, complementing

Figure 2. (Color online) Optimal Sharpe Ratio as Target Mean Varies Under Leveraging



Notes. (a) Liquidity impact on Sharpe ratio under leveraging. (b) Sharpe ratio ignoring market impact.

the result by Frazzini and Pedersen (2014) that investors who face funding or leverage constraints construct a portfolio different from the tangency portfolio.

- There exists a minimum required leverage to attain a given portfolio target, and this minimum leverage grows at an increasing rate with target return; there exists a maximum leverage level beyond which target return cannot be increased further.
- Given a fixed leverage level, an increase in target mean is accompanied by a reduced risk-adjusted performance, which can be reversed with an increase in leverage but can never reach the maximum Sharpe ratio attainable at a lower target mean. In other words, Sharpe leverage frontiers are separated by portfolio target mean, and frontiers of lower targets strictly dominate those of higher targets.

2. Liquidity-Impacted Pricing and Portfolio Model

Following the standard MV framework, consider a one-period economy in which a portfolio of n risky assets is traded at date 0 and the portfolio performance is observed at the end of the (first) investment period, say, date T. Denote the initial (date 0) share position in asset j by x_{0j}, $j = 1, \ldots, n$. Suppose all trade executions are completed by date 1 to the new portfolio positions vector, $x_1 \in \mathbb{R}^n$.[7] In the standard MV model, the portfolio transition $x_0 \to x_1$ is assumed instantaneous at date 0, and the rebalanced portfolio's performance is evaluated at date T. However, in the real world, the transition $x_0 \to x_1$ faces liquidity issues in the underlying assets during the execution interval $(0, 1)$, which invariably leads to trading losses, particularly for large funds.[8] For an active institutional portfolio manager of such a fund, trade execution should be nonconcentrated in order to mitigate the price impact because of potentially large trades. To account for this, the positions x_1 in our model are formed following a continuous-time trading trajectory during time $(0, 1)$.

Assumption 1. *Market uncertainty in the short execution period* $(0, 1)$ *is assumed negligible relative to uncertainties of the period* $[1, T]$, *where* $T \gg 1$. *Moreover, risk-free discounting during the period* $(0, 1)$ *is negligible relative to the period* $[1, T]$.

The justification for Assumption 1 is that, when the portfolio holding period is much longer than the trade execution period, price uncertainty and the effect of discounting during the execution window can be ignored. Accordingly, at any time $t \in (0, 1)$, asset prices during the execution period progress from p_{0j} to p_{tj}, $j = 1, \ldots, n$, deterministically, whereas portfolio performance as evaluated at the terminal date T captures uncertainties outside the execution period.[9]

Frazzini et al. (2018) consider a unique data set of \$1.7 trillion of live executed trades from a large money manager and find that, by placing limit orders for execution, the average trading horizon is slightly less than one day. There is a trade-off between using market and limit orders, the latter of which incurs a lower price impact but longer execution time, leading to higher price uncertainty; see Hautsch and Huang (2012) and Mitchell and Chen (2020) for instance. In our model, we consider market orders to ensure immediate execution as in Almgren and Chriss (2000) and Gatheral and Schied (2012). The empirical statistics in Frazzini et al. (2018) can guide investors to determine the appropriate time length of execution in our model.

If the total trade volume in an asset is large, then a significant permanent price impact may be expected during the execution period, and if the position transition occurs at a high intensity, that is, speed or rate, then trading may also face temporary liquidity shortages that cause a temporary price impact that is instantaneous and reversible. Such a partitioning of trading impact is consistent with that in Carlin et al. (2007), which follows the early work by Kraus and Stoll (1972), Holthausen et al. (1990), and several others. This approach contrasts with models that add an impact cost function that depends on $\|x_1 - x_0\|$ in an otherwise instantaneous trading paradigm. In the latter case, using quadratic transaction costs, Gârleanu and Pedersen (2013) obtain a closed-form optimal portfolio policy under predictable returns; Olivares-Nadal and DeMiguel (2018) consider the Markowitz MV problem with a broader class of transaction costs, that is, the p-norm transaction cost $\|x_1 - x_0\|_p$. However, with borrowing constraints as considered in this paper, portfolio optimization with transaction costs is analytically intractable, and properties of the optimal portfolios are unknown.

We employ both permanent and temporary liquidity impact during the execution period under the condition that market impact is symmetric on buy and sell sides.[10] Denote the position in asset j at time $t \in [0, 1)$ by x_{tj} and let the instantaneous rate of trading be $y_{tj} = \frac{\mathrm{d}x_{tj}}{\mathrm{d}t}$ along an absolutely continuous trading trajectory $t \to x_{tj}$. A positive rate indicates buying, and a negative rate indicates selling. Note that the total trading volume until time t leading to a permanent impact is $x_{tj} - x_{0j} = \int_0^t y_{sj}\mathrm{d}s$. Let γ_j and λ_j denote the (nonnegative) permanent and temporary impact coefficients, respectively. Following the extension of the single asset price model of Carlin et al. (2007) to a portfolio of multiple assets by Brown et al. (2010), the total liquidity impact on price since date 0 until time $t < 1$ is

$$I_{tj} = \gamma_j(x_{tj} - x_{0j}) + \lambda_j y_{tj}^k,\ j = 1, \ldots, n, \tag{1}$$

where $k = \frac{2p+1}{2q+1}$ with p, q being two nonnegative integers and $p \leq q$. Denote y_t^k as the n-dimensional vector with the j^{th} element being y_{tj}^k. We set $k = \frac{2p+1}{2q+1}$ for the ease of mathematical exposition as it preserves the sign of y_{tj} with a positive (negative) sign indicating buying (selling) in asset j. Note that our choice for the form of k does not limit the model's applicability because, for an arbitrary fraction $\frac{a}{b} \in (0, 1]$, there exists a nonnegative δ such that $|\frac{2\delta a+1}{2\delta b+1} - \frac{a}{b}| < \epsilon$, $\forall \epsilon > 0$.

Remark 1. The functional form of the temporary impact need not be restricted to the power-law format in (1). Instead, suppose the temporary impact function is concave and nondecreasing over $y_{tj} \in \mathbb{R}_+$, represented by $g(y_{tj}, \lambda_j)$ and satisfying $g(-y_{tj}, \lambda_j) = -g(y_{tj}, \lambda_j)$, $\forall y_{tj}$. We show that all major theoretical results developed in this article are preserved under $g(y_{tj}, \lambda_j)$; see Online Appendix K for details. However, in order to define and interpret the quasi-elastic market condition in this paper, we employ the power-law form in (1).

The power-law temporary impact function of the form $\lambda_j y_{tj}^k$ is well-supported by empirical evidence, in which the exponent k is in the range from zero to one. For instance, Almgren et al. (2005) report an exponent of 0.6 based on a large data sample from Citigroup Equity Trading that contains 700,000 U.S. stock orders. Using Capital Fund Management proprietary trading data of nearly 500,000 trades in futures markets, Tóth et al. (2011) adopt an exponent of 0.5 for small tick contracts and 0.6 for large tick contracts. Frazzini et al. (2018) reveal the square root impact law using data from AQR Capital, which is also supported by Bouchaud et al. (2009) and Bucci et al. (2019). However, for the permanent impact, Huberman and Stanzl (2004) show that the linear model must be used to exclude arbitrage, which is confirmed empirically in Almgren et al. (2005).

At the end of trading when $t = 1$, any temporary impact associated with the trading rate y_{tj} disappears, and thus, $I_{1j} = \gamma_j(x_{1j} - x_{0j})$. It follows that the asset prices resulting from the impact from the investor's trading action against other market participants are given by $p_{tj} = p_{0j} + I_{tj}$, $t \in (0, 1]$.

2.1. Random Asset Returns and Portfolio Parameters

Following Assumption 1, during the period $[1, T]$, asset prices react randomly to exogenous information that may be macroeconomic and/or firm-specific. The random adjustment to price of asset j to reflect this new information is denoted by the random variable Δ_j so that the asset price at the end of the investment period is $p_{Tj} = p_{1j} + \Delta_j$, that is,

$$p_{Tj} = p_{0j} + I_{1j} + \Delta_j = p_{0j} + \gamma_j(x_{1j} - x_{0j}) + \Delta_j. \tag{2}$$

To set the notation, consider the overall return on the asset over the investment period given by

$$R_j = \frac{p_{Tj} - p_{0j}}{p_{0j}} = \frac{\Delta_j + I_{1j}}{p_{0j}} = \frac{\Delta_j}{p_{0j}} + \frac{\gamma_j(x_{1j} - x_{0j})}{p_{0j}}. \tag{3}$$

Hence, R_j consists of two components: first the random return realized if no trading is initiated by the investor in the asset, denoted by the unaffected return $r_j := \frac{\Delta_j}{p_{0j}}$, and second, the permanent return adjustment because of trading in the asset. Hence, the observed data on asset returns in the absence of trading are generated by the distribution of r_j, whereas the second component is based on the investor's own trading activity. We assume the unaffected returns to be normally distributed, that is, $r \sim \mathcal{N}(\mu, V)$, where $\mu = \mathbb{E}[r] \in \mathbb{R}^n$ is the mean vector and $V = \mathrm{Var}[r] \in \mathbb{R}^{n \times n}$ is the covariance matrix. We denote the diagonal matrices $\mathbf{P}_0 = \mathrm{diag}(p_{01}, \ldots, p_{0n})$, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where γ_j and λ_j are the coefficients of liquidity impact in (1).

Recall that the positions vector x_1 is created through a continuous trading trajectory $t \to x_{tj}$ as specified by the trading rate y_{tj}, where $x_{1j} - x_{0j} = \int_0^1 y_{tj} \mathrm{d}t$. Financing of the trading strategy $t \to x_{tj}$ may require borrowing or lending cash, which would make the net portfolio return at date T different from $R^\top x_1$. We assume cash (as a risk-free asset) can be borrowed or lent at a continuously compounded risk-free rate of $r_0(> 0)$ for the rebalancing trading period $T - 1$. Because of Assumption 1, discounting during the execution time period (0, 1) is negligible.

Let $K_y(x_1)$ be the net cash generated by the trading strategy y by time 1, and thus,

$$K_y(x_1) = -\int_0^1 p_t^\top y_t \mathrm{d}t. \tag{4}$$

Observe that $K_y(x_1)$ allows the possibility of borrowing exogenous cash if required to leverage in the process of constructing the portfolio x_1 along the chosen trading trajectory.

Denote the portfolio initial (cash) liability at date 0 by L_0. A positive or negative L_0 indicates an initial debt level or surplus cash position, respectively. The initial portfolio net wealth at date 0 is $w_0 := p_0^\top x_0 - L_0$, which we assume to be positive. The total dollar return at date T consists of the net cash (generated or required), plus the random asset value and the initial net asset value. Denoting the portfolio rate of return random variable for the period by $\mathcal{R}_y(x_1)$,

$$\mathcal{R}_y(x_1) := \frac{1}{w_0}[e^{r_0} K_y(x_1) + [p_0 + \Delta + \Gamma(x_1 - x_0)]^\top x_1 - e^{r_0} L_0] - 1, \tag{5}$$

where $\Delta = \mathbf{P}_0 r$, and it is assumed that the initial liability (or surplus cash) grows at the risk-free rate. The portfolio expected return at date T under the trading strategy

$y_t : t \to x_{tj}$ is

$$\mathbb{E}[\mathcal{R}_y(x_1)] = \frac{1}{w_0}[e^{r_0}K_y(x_1) + x_1^\top \Gamma x_1 + (p_0 - \Gamma x_0 + \mathbf{P}_0\mu)^\top x_1] - \left(\frac{e^{r_0}L_0}{w_0} + 1\right). \tag{6}$$

Moreover, the variance of the portfolio return at date T is

$$\mathrm{Var}[\mathcal{R}_y(x_1)] = \frac{1}{(w_0)^2}\mathrm{Var}[\Delta^\top x_1] = \frac{1}{(w_0)^2} x_1^\top \mathbf{P}_0 V \mathbf{P}_0 x_1, \tag{7}$$

which depends on the trading strategy y in creating positions x_1.

2.2. Portfolio Leverage

In order to improve portfolio expected return in (6), x_1 may be allowed to incur portfolio debt under risk-free borrowing, but up to an "acceptable" level of leveraging in portfolio x_1 at the end of trading. Define portfolio leverage ratio by total liability divided by total net assets. Then, the leverage ratio at date 0 is $\rho_0 := \frac{1}{w_0}\max\{L_0, 0\}$. Consider the liability after trading at date 1:

$$L_y(x_1) = L_0 - K_y(x_1). \tag{8}$$

If $L_y > 0$, it is a liability, and $L_y < 0$ indicates an excess cash position in the portfolio. The net asset position at date 1 is

$$A_y(x_1) = p_1^\top x_1 - L_y(x_1). \tag{9}$$

The net asset $A_y(x_1)$ is required to be nonnegative to ensure financial solvency. We define the leverage ratio after trading at date 1 provided $A_y(x_1) > 0$:

$$\mathcal{L}_y(x_1) := \frac{L_y^+(x_1)}{A_y(x_1)}, \quad \text{where} \quad L_y^+(x_1) := \max\{L_y(x_1), 0\}. \tag{10}$$

Restrictions on financial leverage are imposed because of either margin requirements or regulatory constraints. Given an investor-specified allowable maximum leverage level $\rho\,(\geq 0)$ on $\mathcal{L}_y$ in (10), a feasible portfolio must satisfy the leverage constraint

$$L_y^+(x_1) \leq \rho A_y(x_1). \tag{11}$$

Note that, when $A_y(x_1) \geq 0$, the leverage requirement $L_y^+(x_1) \leq \rho A_y(x_1)$ is equivalent to $L_y(x_1) \leq \rho A_y(x_1)$.[11] The leverage constraint imposed at time 1 when the portfolio is constructed can be violated during the portfolio holding period, and deleveraging is needed if the leverage ratio becomes too high.[12] How to unwind positions to deleverage is another research question beyond the scope of this study, but it is considered in the existing literature; see Brown et al. (2010) for example.

2.3. Liquidity-Impacted and Leverage-Restricted MV Portfolio Model

Under the liquidity-impacted asset price model under Assumption 1, for specified maximum leverage level $\rho > 0$, the variance-minimizing portfolio that achieves a desired return target $m\,(\geq 0)$ is determined by solving the model

$$\min_{x_1, y}\left\{\mathrm{Var}[\mathcal{R}_y(x_1)] \;:\; \mathbb{E}[\mathcal{R}_y(x_1)] \geq m,\, L_y(x_1) \leq \rho A_y(x_1),\; A_y(x_1) \geq 0,\; x_t = x_0 + \int_0^t y_s \mathrm{d}s, \quad t \in [0,1]\right\}. \tag{12}$$

Observe that (12) is a generalization of the classic MV-optimal portfolio model to the case when portfolio rebalancing is subject to liquidity costs, hence noninstantaneous trade execution and controlling maximum portfolio leverage. Ignoring liquidity costs and setting $\Lambda = 0 = \Gamma$ along with dropping the restriction on leverage, (12) simplifies to the standard MV model.

3. Optimal Trading Policy

Observe that Model (12) is specified over all possible trade execution strategies, $y_t : t \to x_{tj}$. Without knowing the optimal execution strategy, the net cash function $K_y(.)$ in (4) cannot be determined ex ante. In the following, we prove that a (static) constant-rate trading (CRT) strategy is optimal for Model (12). To show this, we present Model (12) in the following equivalent format:

$$\min_{x_1, y}\left\{\mathbb{U}_y(x_1) \equiv \mathbb{E}[\mathcal{R}_y(x_1)] - \pi\mathrm{Var}[\mathcal{R}_y(x_1)] \;:\; L_y(x_1) \leq \rho A_y(x_1),\, A_y(x_1) \geq 0,\; x_t = x_0 + \int_0^t y_s \mathrm{d}s, \quad t \in [0,1]\right\}, \tag{13}$$

where $\pi(> 0)$ is the risk aversion corresponding to the target mean m.

Proposition 1 (Optimal Trading Policy). *Under Assumption 1, a CRT strategy is optimal in Model (13), that is,* $y_t^* \equiv$ constant, *for* $t \in (0, 1)$.

Thus, a CRT strategy is optimal in the generalized MV model (12). For the problem of pure asset liquidation to maximize the terminal (constant absolute risk aversion) utility over a finite time horizon, Schied et al. (2010) show that a deterministic (static) strategy is optimal when no randomness is present in the trade execution period. Proposition 1 claims for the more general case of multiasset leveraged long-short MV portfolio selection problem in (12) that the static constant-rate trade execution, henceforth CRT strategy, is optimal over adaptive (dynamic) strategies under Assumption 1.[13]

We remark that, if Assumption 1 is not held, for example, the presence of uncertainty or discounting during execution, the optimal trading strategy might not be a CRT or a more general static policy. However, for the special case of a single stock liquidation, Almgren and Chriss (2000) show the optimality of a static strategy under an arithmetic Brownian motion of price uncertainty, whereas Gatheral and Schied (2012) use a geometric Brownian motion to obtain an optimal dynamic strategy.

The Sharpe ratio is a widely adopted metric to evaluate a portfolio's risk-adjusted performance. Let the portfolio Sharpe ratio be $\Psi_y(x_1) := \frac{\mathbb{E}[\mathcal{R}_y(x_1)]-\hat{r}_0}{\sigma[\mathcal{R}_y(x_1)]}$, where $\sigma[\mathcal{R}_y(x_1)] = \sqrt{\mathrm{Var}[R_y(x_1)]}$ and $\hat{r}_0$ is a given risk-free rate. The Sharpe ratio maximization model for $\rho > 0$ is given by

$$\max_{x_1, y} \left\{ \Psi_y(x_1) : \mathbb{E}[\mathcal{R}_y(x_1)] \geq m, L_y(x_1) \leq \rho A_y(x_1), A_y(x_1) \geq 0, x_t = x_0 + \int_0^t y_s \mathrm{d}s, t \in [0,1]. \right\}. \tag{14}$$

Following Proposition 1, we show that Model (14) also admits a constant optimal trading strategy.

Proposition 2. *A CRT strategy is optimal in Model* (14).

3.1. MV Model with CRT

The remainder of the paper is developed using the optimal CRT strategy for the trade execution period as claimed in Proposition 1. The resulting optimal model components are identified by the subscript s. First, the constant rate of trading is

$$y_j \equiv y_{tj} = \frac{x_{1j} - x_{0j}}{1} = x_{1j} - x_{0j}, \tag{15}$$

because the length of the execution period is one. Thus, y is linear in x_1. Denote $\phi_s := e^{r_0}\left(p_0 - \frac{1}{2}\Gamma x_0\right)^\top x_0 - e^{r_0} L_0$. Under the CRT strategy, the portfolio expected return in (6) becomes

$$\mathbb{E}[\mathcal{R}_s(x_1)] = \frac{1}{w_0}\left\{ \left(1 - \frac{e^{r_0}}{2}\right) x_1^\top \Gamma x_1 - e^{r_0}(x_1 - x_0)^\top \Lambda (x_1 - x_0)^k + [\mathbf{P}_0 \mu - (e^{r_0} - 1)(p_0 - \Gamma x_0)]^\top x_1 \right\} + \frac{\phi_s}{w_0} - 1. \tag{16}$$

Moreover, the portfolio liability (8) and the net asset position (9) at date 1, respectively, become

$$L_s(x_1) = \frac{1}{2} x_1^\top \Gamma x_1 + (x_1 - x_0)^\top \Lambda (x_1 - x_0)^k + (p_0 - \Gamma x_0)^\top x_1 - e^{-r_0}\phi_s; \tag{17}$$

$$A_s(x_1) = \frac{1}{2} x_1^\top \Gamma x_1 - (x_1 - x_0)^\top \Lambda (x_1 - x_0)^k + e^{-r_0}\phi_s. \tag{18}$$

We extend the concept of market elasticity in Schoneborn and Schied (2007) and introduce the notion of a quasi-elastic market for portfolio trading in which price impact because of temporary liquidity shortages is sufficiently large in relation to a permanent price effect in each asset. Let C_j be the maximum number of shares traded in asset j, determined, for example, by an allowable limit on the fraction of market total traded volume in asset j during the trading window.

Assumption 2 (Quasi-Elastic Market). *Market illiquidity during the execution time period* $(0,1)$ *is such that* $\lambda_j > \frac{(C_j)^{1-k}}{k(k+1)}\gamma_j > 0$ *holds for all assets* $j = 1, 2, \ldots, n$, *that is, the market is sufficiently elastic for portfolio trading, in which the constant* C_j *depends on the fraction of the portfolio traded volume in asset j relative to the total market traded volume in asset j.*[14]

Remark 2. It can be easily shown that, for $r_0 \geq 0$, $\mathbb{E}[\mathcal{R}_s(x_1)]$ and $A_s(x_1)$ are concave in the asset positions under Assumption 2 and the quotient k specified in (1). This implies that portfolio mean return obeys the economic law of diminishing returns under liquidity impact. Therefore, excessive leveraging of a given portfolio x_1 can lead to even negative portfolio return.

The portfolio leverage constraint in (11) under the optimal CRT strategy is $L_s^+(x_1) \leq \rho A_s(x_1)$. To ensure financial solvency, we focus on portfolios with nonnegative net asset value, identified by the convex set $\Omega := \{x_1 : A_s(x_1) \geq 0\}$. We assume that the liquidated portfolio $x_1 = 0$ satisfies $A_s(0) = w_0 - \frac{1}{2}x_0^\top \Gamma x_0 - x_0^\top \Lambda x_0^k \geq 0$; otherwise, execution is too costly to be considered. Hence, $0 \in \Omega$.

3.2. Portfolio Sharpe Ratio and Proportionate Leveraging

Instead of investing in risky securities, suppose the initial portfolio x_0 is liquidated continuously over the execution period under CRT while investing the proceeds at the risk-free rate r_0. This way, the investor obtains the effective risk-free rate for comparison with risky investments. That is, the position target $x_1 = 0$ is set for liquidation by date 1, satisfying $\int_0^1 y_j \mathrm{d}t = -x_{0j}$. It follows that the effective risk-free rate becomes $r_s := \mathbb{E}[\mathcal{R}_s(0)] = \frac{\phi_s - e^{r_0} x_0^\top \Lambda x_0^k}{w_0} - 1$. Then, the portfolio Sharpe ratio is $\Psi_s(x_1) := \frac{\mathbb{E}[\mathcal{R}_s(x_1)] - r_s}{\sigma[\mathcal{R}_s(x_1)]}$.

Proposition 3. *In a quasi-elastic market, the Sharpe ratio* Ψ_s *is pseudo-concave on* $\{x_1 : \mathbb{E}[\mathcal{R}_s(x_1)] > r_s\}$. *For a given scalar* $\nu \in \mathbb{R}$, $\Psi_s(\nu x_1)$ *is decreasing in* $\nu > 0$. *Moreover,* $\Psi_s(\nu x_1) < \Psi_s(x_1)$ *if* $\nu > 1$ *and* $\Psi_s(\nu x_1) > \Psi_s(x_1)$ *if* $0 < \nu < 1$.

In the standard practice of proportionate leveraging to enhance portfolio mean, a given portfolio x_1 is levered to a portfolio $\tilde{x}_1$ with a nonzero factor (or multiplier) ν such that $\tilde{x}_{1j} = \nu x_{1j}, \ \forall j$. As Proposition 3 claims proportionate leveraging (with $\nu > 1$) of a portfolio under liquidity impact leads to a strictly worsened risk-adjusted return (whereas trimming positions proportionately leads to an increase in Sharpe ratio). In contrast, if liquidity impact is ignored, that is, $\Gamma = 0 = \Lambda$, then Ψ_s is positively homogeneous (of degree 0) in asset positions. That is, under no liquidity costs, the portfolio Sharpe ratio is invariant to proportionate increases in portfolio positions, implying that proportionate leveraging is an optimal strategy for leveraging. However, under liquidity impact, one should seek an optimal reallocation of the debt according to Model (12).

Proposition 3 holds in a quasi-elastic market under Assumption 2. When the market is sufficiently inelastic, the permanent impact is the dominant effect and may lead to favorable price movements that outweigh the temporary liquidity costs. In this case, proportionate leveraging of a portfolio can result in an increase in the portfolio Sharpe ratio.

4. MVLL Portfolio Model

Because of the optimal CRT strategy, the liquidity-impacted MV optimization model in (12) is simplified to the following MVLL portfolio optimization model:

$$\text{(MVLL)}: F(m,\rho) := \min_{x_1\in\Omega}\{\sigma^2[\mathcal{R}_s(x_1)] \;:\; \mathbb{E}[\mathcal{R}_y(x_1)] \geq m, \\ L_y(x_1) - \rho A_y(x_1) \leq 0\} \tag{19}$$

given a specified level of leverage ρ and a desired return target m. We denote an optimal portfolio of (19) by $x_1^*(m,\rho)$. Under Assumption 2, because $\mathbb{E}[\mathcal{R}_s(x_1)]$ and $A_s(x_1)$ are concave, and $L_s(x_1)$ and $\sigma^2[\mathcal{R}_s(x_1)]$ are convex, it follows that (19) is a convex optimization model.

In the following, we analyze the structure of the liquidity-impacted optimal portfolio in (19) and show its superiority over the standard MV portfolio that ignores liquidity impact. The trade-off curves considering portfolio target return, leverage, and Sharpe ratio are studied in the sequel, and they turn out to be quite different from the modern portfolio theory (MPT).

4.1. MVLL Portfolio Structure

With quadratic transaction costs added to the mean-variance framework, Gârleanu and Pedersen (2013) obtain a dynamic portfolio policy that has the property of "aiming in front of the target," the core idea of which is that investors aim to achieve a target portfolio but trade only partially toward this aim portfolio because of transaction costs. In our MVLL model incorporating a leverage requirement and liquidity impact, we show that there also exists an aim portfolio. For the ease of mathematical exposition, we consider the special case when the risk-free rate $r_0 = 0$, the market is highly elastic such that the permanent impact can be ignored (i.e., $\Gamma = 0$), and the temporary impact function is linear (i.e., $k = 1$). Note that these simplifications are only limited to Section 4.1 in order to facilitate our analysis of the MVLL portfolio structure.

Let δ_1, δ_2, and δ_3 be the nonnegative Lagrangian multipliers associated with constraints $\mathbb{E}[\mathcal{R}_s(x_1)] \geq m$, $L_s(x_1) \leq \rho A_s(x_1)$, and $A_s(x_1) \geq 0$ in Model (19), respectively. Using the first order Karush–Kuhn–Tucker optimality conditions, we obtain the structure of the optimal portfolio policy.

Proposition 4. *Assume that $r_0 = 0$, $\Gamma = 0$, and $k = 1$. Then, the optimal MVLL portfolio of* Model (19) *is*

$$x_1^* = x_0 + \alpha_1(\delta_1,\delta_2,\delta_3)[\alpha_2(\delta_1,\delta_2) - x_0],$$

where $\alpha_1(\delta_1,\delta_2,\delta_3) = \{\mathbf{P}_0 V \mathbf{P}_0 + [\delta_1 + \delta_2(1+\rho) + \delta_3]\Lambda\}^{-1}\mathbf{P}_0 V \mathbf{P}_0$ *and* $\alpha_2(\delta_1,\delta_2) = \frac{1}{2}(\mathbf{P}_0 V \mathbf{P}_0)^{-1}[\delta_1 \mathbf{P}_0\mu - \delta_2 p_0]$.

Proposition 4 implies that, starting from the initial portfolio x_0, it is optimal to trade toward the aim portfolio $\alpha_2(\delta_1,\delta_2)$, which depends on the leverage requirement ρ implicitly via δ_2. In the absence of the leverage requirement, the aim portfolio becomes $\alpha_2(\delta_1) = \frac{\delta_1}{2}(\mathbf{P}_0 V\mathbf{P}_0)^{-1}\mathbf{P}_0\mu$, which is the classic MV portfolio. It can be seen from the expression of $\alpha_1(\delta_1,\delta_2,\delta_3)$ that incorporating the liquidity impact Λ has a shrinkage effect on the aim portfolio, and the degree of shrinkage is associated with the level of leverage restriction ρ. When we ignore the asset correlations, $\alpha_1(\delta_1,\delta_2,\delta_3)$ reduces to a diagonal matrix that captures asset trading rates toward the aim positions.

Corollary 1. *Assume that $V = diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$. Then, $\alpha_1(\delta_1,\delta_2,\delta_3)$ becomes a diagonal matrix with the jth element being $\alpha_{1j}(\delta_1,\delta_2,\delta_3) = \frac{\sigma_j^2 p_{0j}^2}{\sigma_j^2 p_{0j}^2 + [\delta_1+\delta_2(1+\rho)+\delta_3]\lambda_j}$, $j = 1, 2, \ldots, n$. $\alpha_{1j}(\delta_1,\delta_2,\delta_3)$ is increasing with respect to σ_j and $p_{0,j}$ but decreasing with respect to λ_j.*

Intuitively, trading volatile assets aggressively helps avoid adverse price movements and protects the portfolio's stability; trading illiquid assets smoothly helps reduce the price impacts and preserve the portfolio's wealth. Corollary 1 confirms our intuition from a theoretical perspective that volatility σ_j and liquidity impact λ_j exert opposite effects on an asset's trading rate toward the aim position: a high volatility accelerates trading, whereas a high liquidity impact slows down the process.

4.2. Unlevered Maximum-Sharpe Ratio and Leverage Impact

Consider the case when portfolio leverage is not allowed, that is, $\rho = 0$, and thus, $L_s(x_1) \leq 0$. With no restriction on target mean, would the portfolio that maximizes the Sharpe ratio allow "lending," that is, $L_s(x_1) < 0$? It turns out that an unlevered portfolio achieves a maximum Sharpe ratio when it (asymptotically) becomes the liquidating portfolio.

Proposition 5. *The maximum Sharpe ratio in $\sup_{x_1\in\Omega}\Psi_s(x_1)$ is attained when $x_1 \to 0$, that is, the liquidating portfolio with investment only in the risk-free asset. Furthermore, for fixed liability l, define the maximum Sharpe ratio*

$$\zeta_{SH}(l) := \sup_{x_1\in\Omega}\{\Psi_s(x_1) : L_s(x_1) = l\}. \tag{20}$$

Then, the maximum Sharpe ratio $\zeta_{SH}(l)$ is nonincreasing for $l > L_s(0) = x_0^\top \Lambda x_0^k - e^{-r_0}\phi_s$.

Proposition 5 states that increased portfolio lending (with no return requirement) improves the optimal Sharpe ratio, which is the highest when (liability) $l \to L_s(0)$, achieved by the liquidating portfolio. Liquidating a portfolio implies closing all risky asset positions and holding only risk-free assets, earning returns typically below what most investors desire in practice. With a target return higher than the risk-free return, the optimal portfolio with the maximum Sharpe ratio is different from the liquidating portfolio, which indeed is no longer a feasible option.

As the specified liability approaches zero, the portfolio lending disappears, remaining unlevered, a case that corresponds to the "worst" Sharpe ratio among all unlevered portfolio solutions in (20) with $l \le 0$. We denote this Sharpe ratio by $\zeta_{SH}^{\max} \equiv \zeta_{SH}(0)$ because it is shown to be no less than that of any levered portfolio; see Proposition 6. The portfolio achieving $\zeta_{SH}^{\max}$, denoted x_1^0, is invested only in the risky assets, determined by

$$\zeta_{SH}^{\max} = \sup_{x_1\in\Omega}\{\Psi_s(x_1) : L_s(x_1) = 0\}. \tag{21}$$

If liquidity impact is ignored, that is, $\Lambda = 0 = \Gamma$ in (21), the "tangency" portfolio of the MV-efficient frontier of the risky assets determines the maximum Sharpe ratio. Under $\Lambda = 0 = \Gamma$, the value of (21) is the slope $\frac{d\mathbb{E}[\mathcal{R}_s]}{d\sigma[\mathcal{R}_s]}$ evaluated at the optimal portfolio (with no lending and no borrowing), vis-á-vis CML. However, incorporating liquidity impact with nonzero Λ and Γ in (21) leads to a significant departure from the preceding standard result. Let θ_1 and θ_2 be the Lagrangian multipliers associated with the constraints $L_s(x_1) = 0$ and $A_s(x_1) \ge 0$ in (21), respectively.

Proposition 6 (Unlevered Sharpe-Maximizing Portfolio). *Denote an optimal solution of (21) by x_1^0. Then, the maximum Sharpe ratio under no leveraging satisfies*

$$\zeta_{SH}^{\max} = \left.\frac{d\mathbb{E}[\mathcal{R}_s]}{d\sigma[\mathcal{R}_s]}\right|_{x_1^0} - \theta_1\sigma[\mathcal{R}_s(x_1^0)]\left.\frac{dL_s}{d\sigma[\mathcal{R}_s]}\right|_{x_1^0} + \theta_2\sigma[\mathcal{R}_s(x_1^0)]\left.\frac{dA_s}{d\sigma[\mathcal{R}_s]}\right|_{x_1^0} \tag{22}$$

with $\theta_1 < 0$ and $\theta_2 \ge 0$. Suppose the portfolio asset value after trading is positive, that is, $A_s(x_1^0) > 0$. Then, the optimal Sharpe ratio strictly decreases if sufficiently small portfolio debt is applied.

The first term in (22) is the slope of the tangent to the MV frontier under liquidity impact at the optimal portfolio x_1^0 that is invested only in the risky assets with no borrowing. Hence, (22) implies that the Sharpe-maximizing portfolio is not a tangency portfolio of the MV frontier in the usual sense, but the result is a generalization under liquidity risk of the well-known linear CML relationship in MV theory. That is, in the immediate vicinity of the Sharpe-maximizing unlevered portfolio x_1^0, (22) provides the capital market relationship under liquidity impact.

When the investor consuming the portfolio x_1^0 contemplates leverage (to improve portfolio return), the investor allocates the liability L_s optimally such that (22) is satisfied to obtain a new portfolio with the best Sharpe ratio possible instead of proportionately leveraging the portfolio x_1^0. Indeed, when liquidity impact is ignored by setting $\Lambda = 0 = \Gamma$, it can be easily shown that $\zeta_{SH}^{\max} = \frac{d\mathbb{E}[\mathcal{R}_s]}{d\sigma[\mathcal{R}_s]}|_{x_1^0}$, which is the slope of the tangency portfolio as the MPT suggests.

Frazzini and Pedersen (2014) show that investors with funding or leverage constraints construct a portfolio different from the tangency portfolio. Proposition 6 tells that, even without leveraging, the optimal portfolio departs from the tangency portfolio because of the liquidity impact. Note that the maximum possible Sharpe ratio under sufficiently small portfolio debt strictly decreases from the no leverage Sharpe ratio, $\zeta_{SH}^{\max}$, so long as the net asset value remains positive after trading. That is, given the lending- and debt-free portfolio x_1^0 with a maximum Sharpe ratio of $\zeta_{SH}^{\max}$, any leveraging under liquidity risk results in a strict decrease in the Sharpe ratio. This conclusion is fundamentally different from the corresponding result ignoring market impact, in which the Sharpe ratio is known to remain invariant under risk-free borrowing, that is, when leveraging the tangency portfolio. With liquidity impact, in contrast, increased portfolio variance under leveraging is not sufficiently offset by portfolio return increases because of losses from trading impact.

Claim 1. *Sharpe ratio $\zeta_{SH}^{\max}$ under liquidity impact is achieved when target return is $m_0 \equiv \mathbb{E}[\mathcal{R}_s(x_1^0)]$, corresponding to the unlevered and lending-free portfolio x_1^0 that solves (21); any leveraging of this portfolio strictly decreases the risk-adjusted returns to below $\zeta_{SH}^{\max}$.*

It is already claimed in Proposition 3 that proportionate leveraging leads to a worsening Sharpe ratio. What is concluded in Claim 1 is that a marginal increase in debt that is optimally allocated in an otherwise unlevered Sharpe-maximized portfolio also leads to a worsening Sharpe ratio when the portfolio value after rebalancing is positive. Indeed, an optimal allocation of debt is still preferable to the proportionate-leveraging strategy. These results hold for a quasi-elastic market described by Assumption 2. However, in an inelastic market in which the permanent impact associated with information revelation far dominates the temporary impact caused by liquidity shortage, referred to as plastic markets, $\theta_0 > 0$ is possible, implying that leveraging may lead to favorable price changes that outweigh unfavorable trading costs, leading to an increase in the Sharpe ratio.

4.3. Limits on Target Return and Leverage

Consider the problem of determining the maximum target return, $m_{\max}(\rho)$, under a prescribed maximum leverage level ρ:

$$m_{\max}(\rho) = \max_{x_1 \in \Omega}\{\mathbb{E}[\mathcal{R}_s(x_1)] : L_s(x_1) \le \rho A_s(x_1)\}. \tag{23}$$

First, because the portfolio x_1^0 that solves (21) is feasible in (23) for $\rho = 0$, $m_0 \le m_{\max}(0)$ holds. If $m_0 < m_{\max}(0)$, then the Sharpe ratio of an unlevered portfolio $x_1^{\max}(0)$ that solves (23) for $\rho = 0$ is inferior to that of the unlevered portfolio x_1^0; that is, $\zeta_{SH}^{\max} > \Psi_s(x_1^{\max}(0))$. Second, for a desired target return $m > m_{\max}(\rho)$, there does not exist any feasible portfolio; that is, the investor has to decrease the required return to form a levered portfolio. Third, $\mathbb{E}[\mathcal{R}_s(x_1)]$ is a strictly concave function in quasi-elastic markets, see Remark 2, and thus, (23) is a solvable convex program and $m_{\max}(\rho)$ is finite. It also follows from (23) that, when the investor is leverage unrestricted and, hence, (23) is unconstrained, the resulting target mean $m_\infty \equiv m_{\max}(\infty)$ is still finite. This is in contrast to the MPT that ignores market impact, in which a portfolio satisfying any target mean is deemed possible using appropriate long/short asset positions under leveraging. In practice, this is certainly unrealistic because of liquidity costs, and one must utilize (23) instead of the classic model. Consider

$$x_1^\infty := \arg\max_{x_1 \in \Omega}\{\mathbb{E}[\mathcal{R}_s(x_1)]\} \quad \text{and} \quad \rho_\infty := \mathcal{L}_s(x_1^\infty). \tag{24}$$

x_1^∞ is the portfolio that attains the highest possible return m_∞ under liquidity impact and ρ_∞ denotes the leverage level corresponding to the maximum return.

Next, consider the problem of determining the minimum leverage level, $\rho_{\min}$, needed before a feasible portfolio can be constructed for a prescribed target return $m\,(< m_\infty)$, formulated as

$$\rho_{\min}(m) := \min_{x_1 \in \Omega}\{\mathcal{L}_s(x_1^\infty) : \mathbb{E}[\mathcal{R}_s(x_1)] \ge m\}. \tag{25}$$

The following properties hold for $\rho_{\min}(m)$ and $m_{\max}(\rho)$.[15]

Proposition 7. *Suppose $L_s(x_1^\infty) > 0$ holds. Then,*

i. $\rho_{\min}(m) = 0$ *holds for* $m \in [r_s, m_{\max}(0)]$. *Moreover, $\rho_{\min}(m)$ is positive, nondecreasing, and quasi-convex in m for $m > m_{\max}(0)$.*

ii. *$m_{\max}(\rho)$ is nondecreasing and quasi-concave in $\rho > 0$.*

iii. *$\rho_{\min}(m_\infty) = \rho_\infty$. Moreover, $m_{\max}(\rho) = m_\infty$ for $\rho \ge \rho_\infty$. Conversely, if $L_s(x_1^\infty) \le 0$, then $\rho_{\min}(m) = 0$ holds for $m \in [r_s, m_\infty]$.*

The monotonicity of $\rho_{\min}(m)$ and $m_{\max}(\rho)$ in Proposition 7 holds in a generic market regardless of the market elasticity. When portfolio debt is not allowed, target mean cannot exceed $m_{\max}(0)$, that is, an unlevered portfolio can be constructed so long as the target mean is no more than $m_{\max}(0)$. However, if target mean m is increased beyond $m_{\max}(0)$, an accompanying leverage (minimal) increase to $\rho_{\min}(m)$ is required, which results in a more concentrated risk profile; further increases of leverage beyond $\rho_{\min}(m)$ yield more diversified risk profiles at the expense of additional leverage risk. The (quasi) convex increase of $\rho_{\min}(m)$ in m under liquidity risk may particularly be unsettling for an investor with higher return targets. Also, the highest attainable target return follows the law of (almost) diminishing returns for incurring leverage risk because of the quasi-concavity.

5. MVLL Portfolio Model Properties and Insights

For an investor specifying a sufficiently large target return, leveraging is unavoidable when facing asset liquidity impact, and thus, such an investor is faced with potential trade-off between portfolio risk-adjusted performance and portfolio leverage. As argued earlier, for target mean $m \in [m_0, m_{\max}(0)]$, an unlevered portfolio can be constructed with an inferior Sharpe ratio that can be improved via leveraging; when $m > m_{\max}(0)$, portfolio leveraging is inevitable because a feasible portfolio exists only when leverage level $\rho \ge \rho_{\min}(m) > 0$.

Consider $m \in (m_0, m_\infty)$ and $\rho \ge \rho_{\min}(m)$. In this case, the MVLL model (19) is feasible, and its optimal portfolio is denoted by $x_1^*(m,\rho)$. For the pair (m,ρ), the Sharpe-maximizing portfolio is determined by the model

$$\zeta_{SH}^*(m,\rho) := \max_{x_1 \in \Omega}\{\Psi_s(x_1) : \mathbb{E}[\mathcal{R}_s(x_1)] \ge m, \; L_s(x_1) \le \rho A_s(x_1)\}. \tag{26}$$

Proposition 8 (Sharpe Leverage Liquidity Theorem). *Suppose the target mean satisfies $m \in (m_0, m_\infty)$ and the leverage level satisfies $\rho \ge \rho_{\min}(m)$. Let $x_1^* \equiv x_1^*(m,\rho)$ be optimal in the MVLL model (19). Then,*

i. *$\mathbb{E}[\mathcal{R}_s(x_1^*(m,\rho))] = m$ holds.*

ii. *$x_1^*(m,\rho)$ also solves the Sharpe-maximizing model (26), that is, the optimal value of (26) is $\zeta_{SH}^*(m,\rho) := \Psi_s(x_1^*(m,\rho))$.*

iii. *Sharpe ratio $\zeta_{SH}^*(m,\rho)$ is nondecreasing in ρ and nonincreasing in m.*

iv. *Sharpe ratio $\zeta_{SH}^*(m,\rho)$ is pseudo-concave in ρ and m.*

Several remarks follow from Proposition 8 and its proof. For specified target mean $m \in (m_0, m_\infty)$, with the leverage level chosen such that $\rho \geq \rho_{\min}(m)$, a family of mean-variance-leverage efficient portfolios $x_1^*(m,\rho)$ are generated under the MVLL model (19). In particular, $x_1^*(m,\rho_{\min}(m))$ is a more risk-concentrated efficient portfolio with the least-possible level of leveraging, whereas $x_1^*(m,\rho)$ for $\rho > \rho_{\min}(m)$ is more risk diversified. We note that Proposition 8(i) may not hold in plastic markets because price movements caused by the permanent impact can outweigh trading losses because of the temporary impact, thus leading to portfolio returns higher than the specified target. On the other hand, Proposition 8(iii) holds in a generic market, in which increasing leverage in $[\rho_{\min}(m),\rho_\infty)$ increases the Sharpe ratio but at a decreasing rate of growth in a quasi-elastic market as implied by the (pseudo) concavity in Proposition 8(iv).

Claim 2. *The most risk-concentrated portfolio $x_1^*(m,\rho_{\min}(m))$ has the lowest risk-adjusted returns for the fixed target m. Increasing ρ above $\rho_{\min}(m)$ leads to improved risk-adjusted returns in $x_1^*(m,\rho)$ but at the expense of increased leverage risk.*

On the other hand, at any fixed leverage level, increasing the target mean leads to worsening the Sharpe ratio as claimed in Proposition 8(iii). Can sufficient leverage be applied at a higher target mean to obtain a Sharpe ratio level available at a lower target mean? The answer turns out to be negative, unfortunately, in the presence of liquidity risk as concluded in the next section.

5.1. Maximum Sharpe Ratio for Fixed Target Mean Under Liquidity Impact

Because increasing leverage leads to increasing the Sharpe ratio, consider a portfolio that yields the maximum Sharpe ratio under unrestricted leveraging for fixed $m \in (m_0, m_\infty)$ determined by

$$\zeta_{SH}^{**}(m) = \max_{\rho \geq \rho_{\min}(m)} \zeta_{SH}^*(m,\rho). \tag{27}$$

Note from Proposition 8 that $\zeta_{SH}^*(m,\rho)$ is nonincreasing pseudo-concave and nondecreasing pseudo-concave, respectively, in m and ρ. Although optimizing $\zeta_{SH}^*(m,\rho)$ appears complicated, we show that the solution of the maximization in (27) is accomplished directly from (19) by setting $\rho = +\infty$. Toward this, consider the MVLL model by dropping the leverage constraint, that is, selecting a levered portfolio with minimum variance under liquidity impact given the target mean m:

$$(\text{CMR}): \qquad F^*(m) := \min_{x_1 \in \Omega} \{\sigma^2[\mathcal{R}_s(x_1)] : \mathbb{E}[\mathcal{R}_s(x_1)] \geq m\}. \tag{28}$$

Observe that Model (28) represents the CMR for a quasi-elastic market with liquidity impact, the counterpart to the usual linear CML in a frictionless market. We develop the insight that CMR is not linear (in contrast to CML), and its implications for portfolio risk-adjusted returns are not straightforward.

To see this, let the unique optimal solution of (28) be denoted by $x_1^{**}(m)$, where the uniqueness is due to the objective function being strictly convex. Define $\rho_{\max}(m)$ as the corresponding portfolio leverage level, that is,

$$\rho_{\max}(m) := \mathcal{L}_s(x_1^{**}(m)) \geq \rho_{\min}(m). \tag{29}$$

Proposition 9. *The leverage-unrestricted Sharpe-maximizing portfolio of (27) for fixed target return $m \in (m_0, m_\infty)$ is given by the unique solution $x_1^{**}(m)$ of (28) with*

$$\zeta_{SH}^{**}(m) = \Psi_s(x_1^{**}(m)) = \frac{m - r_s}{\sqrt{F^*(m)}}, \tag{30}$$

and $x_1^{**}(m)$ solves (26) for the pair $(m, \rho_{\max}(m))$. Moreover, $\zeta_{SH}^{**}(m)$ is nonincreasing and pseudo-concave in $m \in (m_0, m_\infty)$.

It follows from Propositions 8 and 9 for fixed m that $\mathbb{E}[\mathcal{R}_s(x_1^{**}(m))] = m$ and $L_s(x_1^{**}(m)) - \rho_{\max}(m) A_s(x_1^{**}(m)) = 0$. Moreover, setting $\rho > \rho_{\max}(m)$ in (26) does not improve the Sharpe ratio for the fixed target m, and thus, $\zeta_{SH}^*(m,\rho) = \zeta_{SH}^{**}(m)$ for $\rho > \rho_{\max}(m)$. On the other hand, for leverage smaller than $\rho_{\max}(m)$, the maximum attainable Sharpe ratio is smaller than $\zeta_{SH}^{**}(m)$, that is, $\zeta_{SH}^*(m,\rho) < \zeta_{SH}^{**}(m)$ holds for $\rho_{\min}(m) \leq \rho < \rho_{\max}(m)$.

Noting the uniqueness of the solution $x_1^{**}(m)$ in (28), as m increases, the increase of $\mathbb{E}[\mathcal{R}_s(x_1^{**}(m))]$ needs to be compensated with extra borrowing, associated with an increase in $\mathcal{L}_s(x_1^{**}(m))$. Thus, $\rho_{\max}(m)$ is nondecreasing; see the illustration in Figure 3(a) in the case study. Then, define

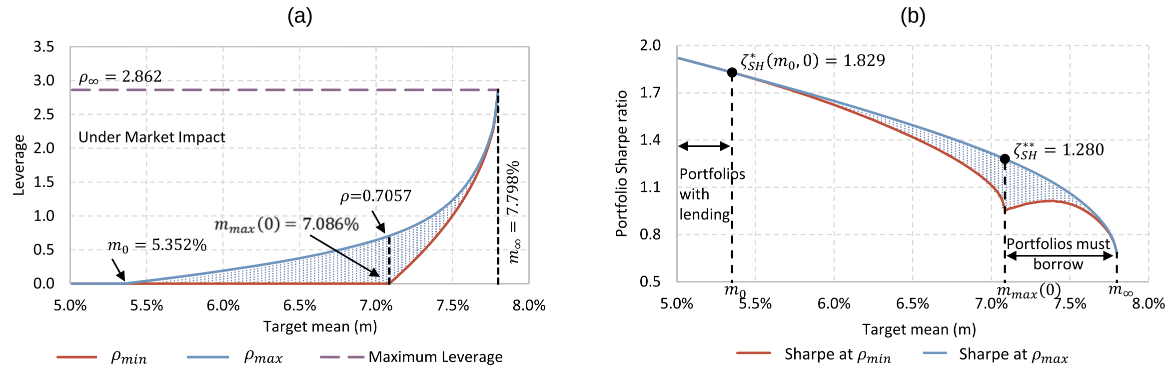
$$m^*(\rho) := \inf_m \{m : \rho_{\max}(m) = \rho\}, \tag{31}$$

which is nondecreasing.

Summarizing, under liquidity impact, a minimum leverage level of $\rho_{\min}(m)$ is required to form a feasible portfolio with target return m, and as ρ increases from $\rho_{\min}(m)$ to $\rho_{\max}(m)$, the portfolio Sharpe ratio increases and reaches a maximum of $\zeta_{SH}^{**}(m)$ at $\rho = \rho_{\max}(m)$. Moreover, as Proposition 9 claims, as m increases, the maximum attainable Sharpe ratio declines (even with no restriction on leveraging). The fact that this decline can be strict is demonstrated in the case study in Section 6, a differentiating feature of CMR under liquidity impact in comparison with the CML that ignores market impact.

Because $\zeta_{SH}^{**}(m)$ is nonincreasing in m, as target return is decreased closer to the level that can be obtained by an unlevered portfolio, that is, $m = m_{\max}(0) + \varepsilon$ (for $\varepsilon > 0$), the portfolio Sharpe ratio attains the highest level among all levered portfolios. As illustrated in Figure 4(a) from the case study, the Sharpe leverage frontier is dominated when the target return is increased. In contrast, Figure 4(b) depicts the case of ignoring liquidity

Figure 3. (Color online) Minimum and Maximum Leverages as Target Mean Varies Under Liquidity Impact



Notes. Leverage ($\rho_{\max}$) required for maximum possible Sharpe ratio appears to be convex and increasing in m. Its difference with the minimum leverage ($\rho_{\min}$) required for portfolio feasibility, that is, $[\rho_{\max}(m) - \rho_{\min}(m)]$, is strictly decreasing in m, and the difference approaches zero at the highest mean, m_∞. (a) Minimum and maximum leverages on target mean. (b) Maximum possible Sharpe ratio on target mean.

impact, that is, the classic portfolio theory, in which case the highest Sharpe ratio (the slope of CML) is achieved at any target return under sufficient leverage.

5.2. Maximum Sharpe Ratio for Fixed Leverage Under Liquidity Impact

The case of the maximum Sharpe ratio under no lending or borrowing is discussed in Proposition 6, which is attained at portfolio target mean, m_0. Given a fixed leverage level ρ, what is the maximum attainable Sharpe ratio and the corresponding portfolio mean? Consider the problem

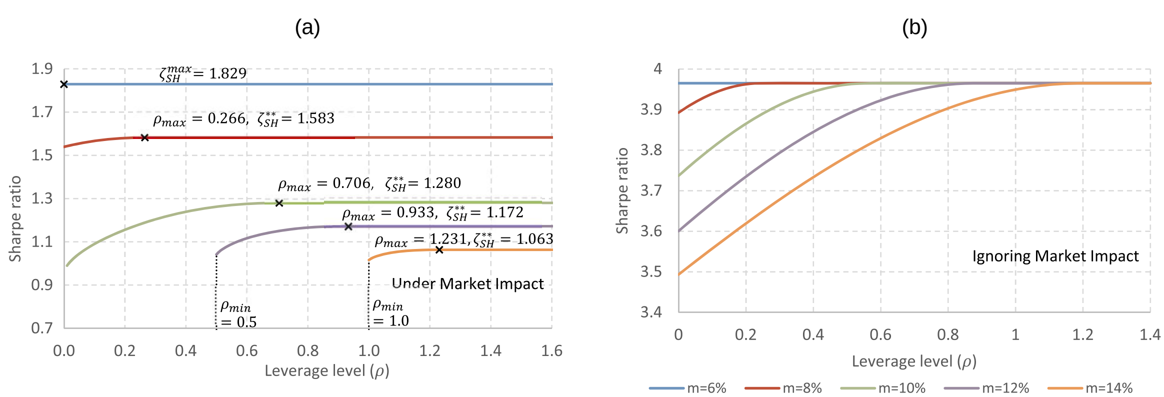
$$\zeta^{*U}_{SH}(\rho) := \max_{x_1 \in \Omega}\{\Psi_s(x_1) : L_s(x_1) - \rho A_s(x_1) = 0\}. \tag{32}$$

Model (32) is nonconvex, and its solution is computationally intensive. Instead, we follow an indirect approach to obtain its solution using the preceding results.

Proposition 10. *An optimal portfolio solution of* (32) *for fixed* ρ *is* $x_1^{**}(m^*(\rho)) \equiv x_1^*(m^*(\rho), \rho)$, *where* $x_1^{**}(.)$ *solves* (28), $x_1^*(.,.)$ *solves the MVLL model* (19), *and* $m^*(\rho)$ *is given by* (31). *Moreover, the optimal value* $\zeta^{*U}_{SH}(\rho)$ *of* (32) *is nonincreasing in* ρ.

That is, under liquidity impact, increasing leverage can never increase the maximum attainable Sharpe ratio of the optimal portfolio, and it can possibly decrease. The fact that a strict decrease is possible is illustrated in our numerical calculations. In contrast, when liquidity impact is ignored, the MPT suggests that the maximum Sharpe ratio remains unchanged (which is the slope of the CML involving the tangency portfolio) as additional portfolio leverage is employed to increase mean return, the basic tenet of risk parity portfolio construction. In passing, we also comment that even if the constraint of (32) is expressed as $L_s(x_1) - \rho A_s(x_1) \geq 0$, which is still a

Figure 4. (Color online) Effect of Leverage Level and Target Mean on Sharpe Ratio



Notes. For a fixed target mean, under liquidity impact, whereas risk-adjusted returns improve with increased leverage, they can never reach the maximum Sharpe ratio attained at a lower target mean. When market impact is ignored, such a phenomenon does not exist. (a) Case of market impact. (b) Case of ignoring market impact.

Table 1. Forecasts of Asset Return and Market Parameters for January 7, 2015

Parameter	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY	SPY
Mean, %	−0.576	−4.817	2.054	2.107	1.463	2.544	3.670	2.693	1.988	1.384
Standard deviation, %	4.804	6.362	3.835	5.380	4.830	2.788	2.596	4.571	4.178	4.200
Daily volume, millions	3.40	29.58	23.63	11.28	6.05	8.87	12.68	13.21	6.48	—
Asset correlations										
XLB	1.000	0.842	0.905	0.903	0.942	0.867	0.241	0.852	0.939	0.947
XLE	0.842	1.000	0.799	0.886	0.801	0.761	0.571	0.706	0.711	0.854
XLF	0.905	0.799	1.000	0.931	0.951	0.868	0.430	0.919	0.936	0.971
XLI	0.903	0.886	0.931	1.000	0.944	0.900	0.532	0.918	0.875	0.975
XLK	0.942	0.801	0.951	0.944	1.000	0.949	0.327	0.939	0.944	0.987
XLP	0.867	0.761	0.868	0.900	0.949	1.000	0.383	0.900	0.846	0.935
XLU	0.241	0.571	0.430	0.532	0.327	0.383	1.000	0.355	0.196	0.421
XLV	0.852	0.706	0.919	0.918	0.939	0.900	0.355	1.000	0.897	0.948
XLY	0.939	0.711	0.936	0.875	0.944	0.846	0.196	0.897	1.000	0.942

Note. Asset return (for 20 trading days) parameters for January 7, 2015, are estimated using a trailing 60-day historical time window (up to January 6, 2015).

nonconvex model, its optimal solution has the property that the leverage constraint is binding.

Remark 3. For $\rho = 0$, Model (21) is identical to (32), that is, $\zeta_{SH}^{\max} = \zeta_{SH}^{*U}(0)$. Hence, for the optimal solution of (21), we have $x_1^0 = x_1^{**}(m^*(0))$, and its target return is $m_0 = m^*(0)$. Therefore, equivalently, the solution of Model (21) can be obtained by solving the MVLL model (19) for $m = m^*(0)$ and $\rho = 0$; that is, $x_1^0 = x_1^*(m^*(0), 0)$ solves (21).

6. Case Study with Multiple Risky Assets

We use a case study with multiple risky assets to illustrate the preceding analytical results. Consider the nine select sector Exchange-traded fund (ETF) assets given by the ticker symbols XLB, XLE, XLF, XLI, XLK, XLP, XLU, XLV, and XLY. These ETFs cover the full breadth of the S&P 500 market index and are chosen because they are rather liquid so that an overemphasis of our analytical results is avoided. First, the market impact parameters are estimated using the millisecond trade-and-quote data for the nine ETF assets for the time period January 1–31, 2015. Online Appendix L provides the methodology used to estimate the impact parameters λ_j and γ_j in (1) for which we set the exponent $k = 0.6$ as in Almgren et al. (2005), corresponding to $p = 1 < 2 = q$. The resultant estimates are given in Online Table L.3.1. We set the trade execution period as one day. For the quasi-elastic market condition in Assumption 2 to hold, trade volumes of assets must be no more than the limits shown in the last column of Online Table L.3.1. These limits are well in excess of the typical daily trade volumes in these assets so that position limiting constraints are unnecessary.

We consider the (trade-holding) investment horizon as 20 trading days, that is, $T = 20$, and set the annual risk-free rate to 2%. Trade execution day is set to January 7, 2015 (arbitrarily), for the analysis. The required asset return parameters (for 20 trading days) are estimated using a trailing 60-day historical time window (up to January 6, 2015), including the S&P 500 index-tracking fund, SPY. Optimal portfolios are constructed at the beginning of January 7, 2015, with an initial portfolio of $x_0 = 0$, $w_0 = \$1$ million and no initial liability (hence, $L_0 = -\$1$ million), using the market impact parameters in Online Table L.3.1 and asset return parameters in Table 1. Target portfolio mean over $T = 20$ days, m, is specified such that $m \le m_\infty$.

6.1. Unlevered Maximum Sharpe and Maximum Target Return

As Claim 1 indicates, the unlevered and lending-free portfolio x_1^0 attains the maximum Sharpe ratio $\zeta_{SH}^{\max}$ under liquidity impact, and its target return is m_0. We employ the approach discussed under Remark 3 to solve Model (21), which yields $m_0 = 5.352\%$, associated with the Sharpe ratio, $\zeta_{SH}^{\max} = 1.829$. When liquidity risk is ignored (by setting $\lambda_j = \gamma_j = 0,\ \forall j$), $\zeta_{SH}^{\max} = 3.9653$.

The maximum attainable portfolio mean under market impact is computed using (24) as $m_\infty = \mathbb{E}[\mathcal{R}_s(x_1^\infty)] = 7.798\%$. The liability $L_s(x_1^\infty)$ associated with m_∞, as a percentage of the initial wealth, is 249.468%, which corresponds to a maximum leverage level of $\rho_\infty = 286.2\%$. Moreover, the maximum possible return for an unlevered portfolio is $m_{\max}(0) = 7.086\%$; see (23). That is, to obtain a portfolio return in excess of 7.086% (but no more than 7.798%), leverage must be applied with $\rho \ge \rho_{\min}(m)$; see (25). In contrast, for a target mean in $[m_0, m_{\max}(0)] \equiv [5.352\%, 7.086\%]$, there exist portfolios that are borrowing and lending free.

6.2. Leveraging Under Increased Mean Targets

The optimal portfolios that correspond to a range of target means are reported in Table 2, in which, for each specified m, the MVLL model is solved with $\rho = \rho_{\min}(m)$.[16] Thus, the Sharpe ratios in Table 2 are for the least possible leverage for each given target mean, hence representing

Table 2. Optimal Portfolio Choice Under Liquidity Costs for Fixed Target Mean and Minimum Leverage

Asset	m_0 5.352%	$m_{\max}(0)$ 7.086%	m_∞ 7.798%	Portfolio mean target (m) 5.000%	5.540%	6.080%	6.620%	7.160%	7.700%
	Optimal portfolio (dollar) weights at beginning of period, relative to initial wealth								
XLB	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
XLE	−12.74%	−34.71%	−14.06%	−10.68%	−14.79%	−20.72%	−26.73%	−33.41%	−20.16%
XLF	−21.19%	−51.19%	−43.73%	−18.92%	−22.71%	−27.90%	−35.20%	−51.09%	−46.00%
XLI	35.29%	63.35%	115.39%	30.88%	36.96%	42.13%	48.87%	66.35%	96.30%
XLK	20.52%	32.68%	57.85%	18.41%	22.77%	28.45%	32.49%	33.86%	48.62%
XLP	2.39%	17.65%	45.96%	1.41%	3.30%	6.37%	10.40%	19.15%	35.19%
XLU	40.64%	49.62%	78.92%	37.85%	40.13%	39.57%	41.08%	51.41%	68.18%
XLV	97.23%	92.08%	124.82%	91.00%	96.72%	94.67%	92.26%	93.84%	112.68%
XLY	14.87%	72.29%	111.59%	10.77%	18.07%	28.99%	43.78%	75.01%	97.60%
CASH	0.00%	0.00%	−249.5%	8.8%	0.00%	0.00%	0.00%	−13.3%	−160.5%
Max relative trade size	0.116%	0.128%	0.204%	0.109%	0.116%	0.113%	0.110%	0.133%	0.176%
	Liquidity-impacted MVLL-optimal risk-adjusted returns under minimal leverage								
Leverage, $\rho = \rho_{\min}$	0	0	2.862	0	0	0	0	0.149	1.816
Sharpe ratio, $\zeta^*_{SH}(m, \rho)$	1.829	0.962	0.684	1.923	1.776	1.593	1.361	0.977	0.868
	Liquidity-ignored MV-optimal portfolio's Sharpe ratio accounting for market impact								
Sharpe ratio	0.889	0.310	0.090	1.016	0.823	0.638	0.460	0.287	0.120
Percentage loss in Sharpe ratio	51.4	67.7	86.8	47.2	53.6	60.0	66.2	70.6	86.2

Notes. Portfolios are obtained (on January 7, 2015) for the least possible level of leverage for each target mean: for $m \le m_{\max}(0)$, $\rho_{\min}(m) = 0$, that is, the portfolio is unlevered, and for $m > m_{\max}(0)$, portfolios are leveraged at $\rho = \rho_{\min}(m) > 0$. The resulting Sharpe ratio generally decreases as target mean increases. The Sharpe ratio of the pure MV-optimal portfolio is 3.9653 when ignoring market impact, but with trading costs accounted for, the Sharpe ratio becomes substantially weaker. The pure MV-optimal portfolio remains unlevered in the range of target mean reported.

more risk-concentrated portfolios that are not necessarily designed to maximize the Sharpe ratio.

Note that a positive leverage is required only when $m > m_{\max}(0) = 7.086\%$. At $m = m_0 = 5.352\%$, $\zeta^{\max}_{SH} = 1.829$ is the highest possible Sharpe ratio for a portfolio with no lending/borrowing. When the target mean is strictly less than m_0, the optimal portfolio holds CASH (i.e., portfolio lending) and the Sharpe ratio is strictly greater than $\zeta^{\max}_{SH}$, for example, $\zeta^*_{SH}(5.0\%, 0) = 1.923 > \zeta^{\max}_{SH}$, agreeing with the conclusions in Section 4.2. When leverage is held fixed (at $\rho_{\min}(m) = 0$ in Table 2), the portfolio Sharpe ratio is nonincreasing as m increases, agreeing with Proposition 8(iii); however, for cases when m increases above 7.086%, because the leverage setting $\rho = \rho_{\min}(m)$ also increases, Proposition 8(iii) cannot predict the direction of change of the portfolio Sharpe ratio. To exemplify, when m increases from 7.086% to 7.160%, associated with a leverage increase from 0 to 0.149, respectively, the Sharpe ratio increases from 0.962 to 0.977. However, when m increases further to 7.700%, associated with leverage increasing to 1.816, the Sharpe ratio decreases to 0.868—see the depiction in Figure 3(b).

In Table 2, we also report the performance of pure MV-optimal portfolios that ignore liquidity costs for the indicated target mean levels. All pure MV portfolios incur zero leveraging and result in a liquidity-ignored ex ante Sharpe ratio of 3.9653. However, when these portfolios are evaluated under liquidity costs ex post, the resulting Sharpe ratios are quite weak. For example, when $m = 6.080\%$, the Sharpe ratio drops to 0.638 (from the liquidity-ignored 3.9653). However, the ex ante liquidity-impacted MVLL-optimal portfolio, although requiring no leveraging, achieves a Sharpe ratio of 1.593. Thus, ignoring liquidity costs ex ante results in a 60.0% decline in portfolio risk-adjusted performance. Also, note that this performance shortfall worsens as the target mean increases.

6.3. Insights on Choosing Target Mean Under Liquidity Risk

For a given target m, the highest Sharpe ratio is attained when leveraging reaches the level $\rho_{\max}(m)$; see (27), Proposition 9, and the depiction in Figure 3(a). We also plot the minimum leverage, $\rho_{\min}$, required for portfolio feasibility at a given target m in the same graph in which both plots indicate monotonic increases with target mean. Figure 3(a) speaks for convex growth of $\rho_{\min}$ that is steeper than the predicted quasi-convexity in Proposition 7(i), implying that an increase in target mean requires a substantial increase in leverage. The leverage $\rho_{\max}$ also appears to have a convex-growth in target return, and its gap with $\rho_{\min}$ monotonically decreases for $m \ge m_{\max}(0)$, and they converge to each other as $m \to m_\infty = 7.798\%$ with leverage at $\rho_\infty = 2.862$.

To elucidate, at the maximum possible target mean under no leveraging, that is, $m_{\max}(0) = 7.086\%$, the MV-optimal Sharpe ratio is $\zeta^*_{SH}(7.086\%, 0) = 0.962$; see

Table 2. However, because $\rho_{\max}(7.086\%) = 0.7057$, see Figure 3(a), the Sharpe ratio can be increased further to a maximum of $\zeta_{SH}^{**}(7.086\%) = \zeta_{SH}^{*}(7.086\%, 0.7057) = 1.280$; see Figure 3(b). However, this level of leverage (risk) may not be acceptable to an investor. For instance, if the investor's maximum acceptable leverage level is only $\rho = 25\%$, the investor's optimal Sharpe ratio portfolio is obtained by solving the MVLL model (19), which yields $\zeta_{SH}^{*}(7.086\%, 25\%) = 1.1821$. This represents a decline of roughly 7.65% in Sharpe ratio from the maximum achievable accompanied with a 64.57% reduction in leverage level. An investor is not at liberty to choose the target mean $\hat{m}$ and the acceptable leverage level $\hat{\rho}$ independently of each other in the presence of liquidity risk; indeed, the choice must satisfy the boundary conditions $\hat{m} \le m_{\max}(\hat{\rho})$ and $\rho_{\min}(\hat{m}) \le \hat{\rho} \le \rho_{\max}(\hat{m})$ as illustrated by the shaded region in Figure 3(a).

Furthermore, Figure 3(b) shows that the maximum possible Sharpe ratio $\zeta_{SH}^{**}(m)$, attained with leveraging to $\rho_{\max}(m)$, is strictly decreasing in target return, and the rate of decline becomes steeper as target mean increases, worse than the pseudo-concave decrease predicted in Proposition 9. Therefore, an investor focused on risk-adjusted portfolio returns has to exercise caution in demanding higher target returns even when leverage risk is not a major concern in the face of a quasi-elastic market with sufficient liquidity risk. On the same figure, the Sharpe ratio corresponding to the least-possible leverage $\rho = \rho_{\min}(m)$, that is, $\zeta_{SH}^{*}(m, \rho_{\min}(m))$, is shown as target mean varies, which satisfies $\zeta_{SH}^{*}(m, \rho_{\min}(m)) \le \zeta_{SH}^{**}(m)$. For $m \le m_0$ and at $m = m_\infty$, we have $\zeta_{SH}^{*}(m, \rho_{\min}(m)) = \zeta_{SH}^{**}(m)$. Note that $\zeta_{SH}^{*}(m, \rho_{\min}(m))$ is decreasing for $m \le m_{\max}(0)$ because $\rho_{\min}(m) = 0$, steeper than the pseudo-concavity predicted in Proposition 8(iv); however, for $m > m_{\max}(0)$, the Sharpe ratio is nonmonotonic as both m and $\rho_{\min}(m)$ are increasing simultaneously, noting the discontinuity at $m_{\max}(0)$. The shaded region in Figure 3(b) signifies attainable portfolio performance as a guide on setting target return by trading off the Sharpe ratio with an aversion to leverage risk under liquidity impact.

6.4. Sharpe Leverage Frontiers

Figure 4(a) depicts the monotonic behavior of the Sharpe ratio as leverage is increased under liquidity impact. In particular, given a fixed target mean, increasing leverage increases the risk-adjusted return as claimed in Proposition 8(iii), but it can never reach that attainable at a lower target mean. That is, as target mean increases, not only does the required minimum portfolio leverage increase at an increasing rate, also the Sharpe leverage frontiers are progressively dominated.

The MPT counterpart that ignores liquidity impact is plotted in Figure 4(b), in which the maximum Sharpe ratio of a lower target mean can be attained by a higher target mean, provided sufficient leverage is applied, and its maximum Sharpe ratio corresponds to the slope of the CML. Comparison between Figure 4(a) and (b), underscores the pronounced loss in Sharpe ratio when an increased target return is desired under liquidity risk in a quasi-elastic market.

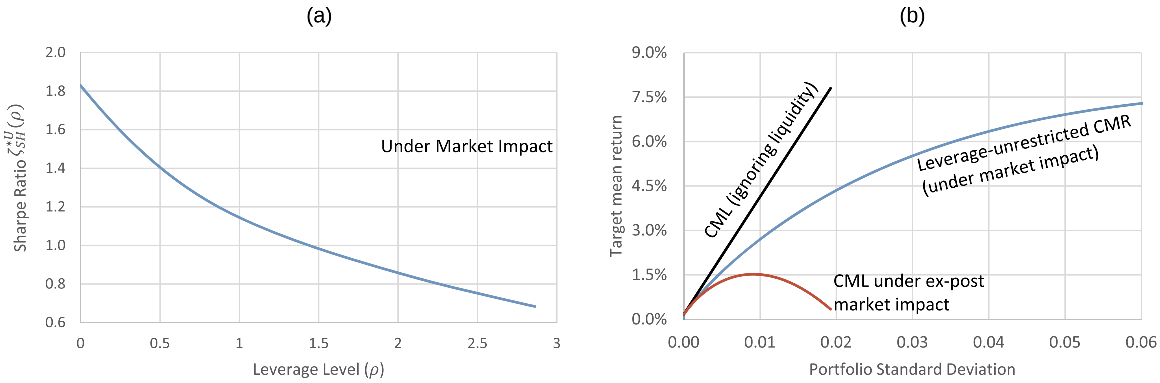
6.5. CMR and MV Frontiers Under Leverage

The leverage-fixed highest Sharpe ratio under liquidity impact, $\zeta_{SH}^{*U}(\rho)$, decreases (nonlinearly) in leverage ρ, see Proposition 10, whereas under the MPT without liquidity risk, the maximum Sharpe ratio is a constant (at 3.9653) for any fixed leverage as given by the slope of the CML. To determine $\zeta_{SH}^{*U}(\rho)$, (31) is used to obtain $m^*(\rho)$, which is then used in the MVLL model (19) to obtain the optimal portfolio $x_1^*(m^*(\rho), \rho)$. Figure 5(a) plots ζ_{SH}^{*U}, which is strictly decreasing in leverage.

The leverage-unrestricted optimal portfolio with the maximum Sharpe ratio for a fixed target mean m is given by Model (28). For the optimal portfolio $x_1^{**}(m) \equiv x_1^*(m, \rho_{\max}(m))$ so determined, the MV frontier is plotted in Figure 5(b), termed the CMR. Therefore, an optimal CMR portfolio employs a maximal amount of leverage $\rho_{\max}(m)$ as indicated in Figure 3(a), to achieve the mean return m. An efficient CMR fund for given target m corresponds to the highest Sharpe portfolio that is optimally leveraged to the level $\rho_{\max}(m)$. In contrast, when liquidity impact is ignored, the MV frontier is the linear CML (with a slope of 3.9653) because of proportionate leveraging being optimal; see Figure 5(b). Whereas the liquidity-ignored CML appears above the ex ante CMR under market impact, when the CML is evaluated ex post under market impact, it produces substantially weak performance. As Figure 5(b) shows, the latter ex post CML has a concave shape indicative of progressively inefficient performance as risk aversion decreases.

For a fixed-leverage $\rho = \hat{\rho}$, the MVLL-optimal portfolio corresponding to the mean-leverage combination, $(m^*(\hat{\rho}), \hat{\rho})$, is on the CMR curve in Figure 5(b). What if leverage is held constant at $\hat{\rho}$ but target mean is increased to $m > m^*(\hat{\rho})$? Then, the resulting leverage-fixed MV frontier branches off from the CMR curve at the target mean $m^*(\hat{\rho})$, and the frontier falls below the CMR as shown in Figure 6(a). For example, the MV frontier for $\rho = 0.25$ shares the CMR up to a target mean of $m^*(0.25) = 6.27\%$, and it separates from the CMR for $m > 6.27\%$. That is, optimal portfolios on the MV frontier for $\rho = 0.25$ and $m > 6.27\%$ have lower Sharpe ratios than those on the CMR with the same target mean. Moreover, the MV frontier for $\rho = 0.25$ is defined only for a maximum target mean of $m_{\max}(0.25) = 7.207\%$. In fact, MV frontiers for fixed ρ exist only for target mean $m \le m_{\max}(\rho)$; see Figure 6(a). Observe that the maximum attainable target mean for a given leverage increases monotonically, see Figure 6(b), only as a concave function instead of the predicted quasi-concavity in Proposition 7(ii). Moreover, the optimal standard deviation of the portfolio with target mean $m_{\max}(\rho)$ is

Figure 5. (Color online) Maximum Sharpe Ratio Under Leverage and CMR



Notes. (a) Maximum Sharpe ratio on leverage. (b) Sharpe-maximized mean-variance CMR.

not monotonic in ρ though $m_{\max}(\rho)$ increases with leverage.

6.6. Portfolio Performance Under Leverage Restrictions

In Figure 7(a), frontiers separated by leverage under liquidity impact are plotted for target mean up to $m_{\max}(\rho)$ with $\rho = 0$, 25%, 50%, and leverage unconstrained; that is, $m_{\max}(0) = 7.086\%$, $m_{\max}(0.25) = 7.207\%$, $m_{\max}(0.5) = 7.316\%$, and $m_\infty = 7.798\%$. Note that the Sharpe ratio at the maximum target mean (for given leverage) increases initially as ρ increases, but it significantly worsens at elevated leverage. In contrast, when liquidity impact is ignored, see Figure 7(b), the drop in Sharpe ratio is somewhat subdued at higher target means because of the pseudo-concave shape, which clearly overestimates the actual Sharpe ratio. Figure 7 underscores the importance of incorporating liquidity impact in building an optimal portfolio, especially at higher target returns.

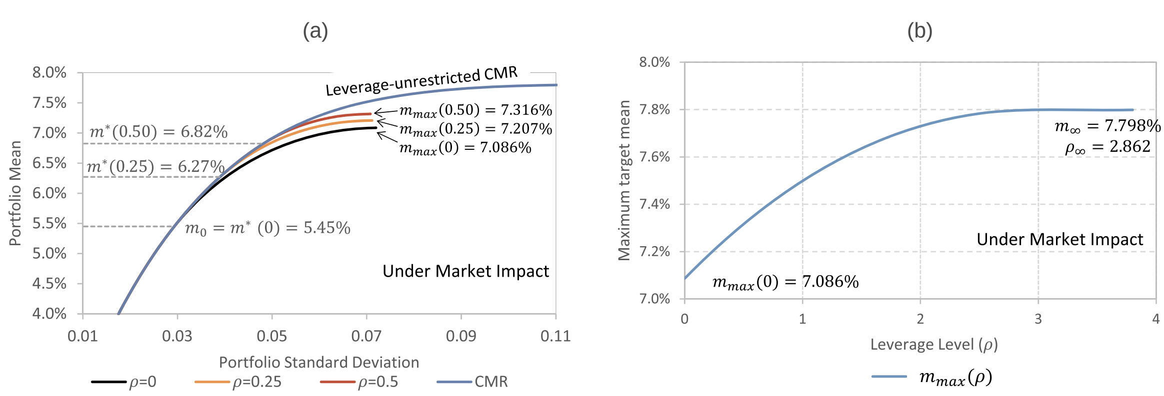
Performance impact using expected utility when leverage restrictions are imposed in a quasi-elastic market is studied next. Given (m, ρ), we evaluate the optimal portfolio $x_1^*(m, \rho)$ by solving the MVLL model (19), in which the optimal Lagrange multiplier of the return constraint is denoted by $\theta^*(m, \rho)$. It can be shown that $x_1^*(m, \rho)$ also solves the expected utility maximization model:

$$\max_{x_1 \in \Omega} \{\mathbb{U}_s(x_1, \pi) \equiv \mathbb{E}[\mathcal{R}_s(x_1)] - \pi\sigma^2[\mathcal{R}_s(x_1)] \\ : L_s(x_1) - \rho A_s(x_1) \le 0\}, \tag{33}$$

when risk aversion is set to $\pi = \hat{\pi}(m, \rho) := 1/\theta^*(m, \rho)$; see Online Appendix J for a proof of this result. The resulting expected utility, $\mathbb{U}_s^*(m, \rho) = \mathbb{E}[\mathcal{R}_s(x_1^*(m, \rho))] - \hat{\pi}(m, \rho)\sigma^2[\mathcal{R}_s(x_1^*(m, \rho))]$, is plotted against the risk aversion $\hat{\pi}(m, \rho)$ in Figure 8 by varying the leverage ρ.

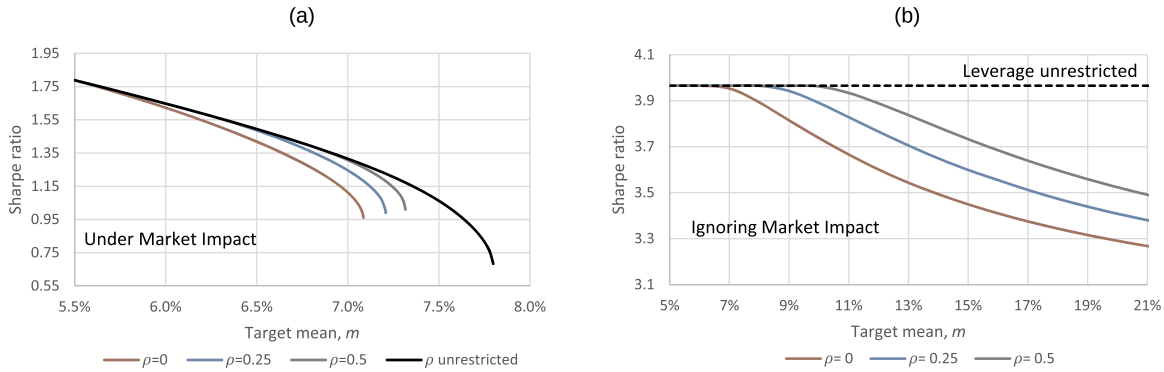
As Figure 8(a) indicates, with liquidity costs incorporated ex ante, for less risk-averse investors, expected utility declines as leverage restrictions are imposed. For example, at $\pi = 2$, going from an optimal leverage-unrestricted portfolio to a completely unlevered optimal portfolio results in a loss of 4.7% in expected utility. Figure 8(b) depicts the case with the standard MV

Figure 6. (Color online) Mean-Variance-Leverage Efficient Frontiers Under Liquidity Impact



Notes. In (a), at fixed leverage ρ, the frontier exists only up to a target mean of $m_{\max}(\rho)$, which increases in ρ; see (b). For $m \le m_0$, all frontiers coincide. A frontier for fixed ρ branches off from the CMR at target mean, $m^*(\rho)$. (a) MV frontiers branch off from the CMR. (b) Maximum portfolio mean on leverage.

Figure 7. (Color online) Effect of Target Mean on Optimal Risk-Adjusted Performance



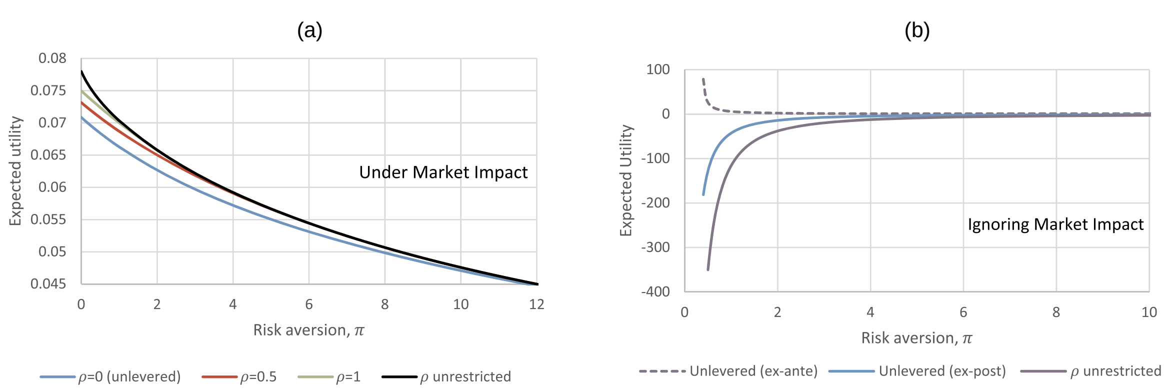
Notes. At any fixed leverage, optimal Sharpe declines at increased target mean at a more steep concave rate under liquidity impact in contrast to a more subdued pseudo-concave drop when market impact is ignored. Increased leverage can recover some losses in risk-adjusted performance. (a) Liquidity impact on Sharpe ratio under leveraging. (b) Ex ante Sharpe ratio ignoring market impact.

model that ignores liquidity impact ex ante but the utility being evaluated under trading costs ex post. Whereas the MV model yields positive expected utility ex ante, the ex post utility turns out negative at low or high levels of risk aversion regardless of the leverage level. Observe that less risk-averse agents ignoring liquidity costs suffer significant losses in expected utility. As the investor becomes more risk-averse, however, the utility losses resulting from ignoring liquidity costs are more contained because highly risk-averse portfolios have low trading intensity given the initial all-cash portfolio.

7. Concluding Remarks

This paper presents an analysis of the optimal trade-off between portfolio performance and leverage level in the presence of liquidity costs in a quasi-elastic market. The direct consideration of liquidity impact within a trade execution period significantly alters our understanding of the interplay among risk, return, and leverage. The Sharpe-maximizing portfolio under market impact, without leveraging, is no longer a tangency portfolio. Furthermore, an optimal leveraged portfolio under market impact does not correspond to proportionately leveraging a lower risk portfolio composition. Consequently, the argument made for risk parity portfolio construction is flawed under liquidity risk. Moreover, market illiquidity causes significant deterioration in portfolio performance as target return is increased (equivalently, less risk averse) although some of the adverse effects can be countered with increased leverage risk; this is true with both portfolio Sharpe ratio and expected utility. In this sense, the insights developed for the trade-off that exists between leverage risk and portfolio performance are paramount when return targets are set for market scenarios facing substantial liquidity risk. Note that the quasi-elastic market condition allows us to develop the theoretical predictions on portfolio performance. Without the latter condition, the MVLL

Figure 8. (Color online) Effect of Risk Aversion on Optimal Expected Utility Under (Ex Post) Liquidity Impact



Notes. Expected utility $\mathbb{U}_s^*(m, \rho)$ on risk aversion $\hat{\pi}(m, \rho)$ is plotted for optimal MVLL and pure-MV portfolios evaluated under trading impact. At any fixed leverage, optimal MVLL utility declines with risk aversion and remains positive; however, ignoring market impact leads to negative expected utility at any risk aversion or leverage level. (a) Liquidity/leverage impact on utility (MVLL). (b) Impact on utility ignoring liquidity costs.

model is nonconvex, and its computational solution is tedious. Such a case is considered in Edirisinghe et al. (2021) for the problem of portfolio deleveraging. In this paper, we consider a predetermined portfolio execution horizon. Because the time length of execution is closely related to the trading cost and risk, modeling the execution horizon as a decision variable that depends on the investor's cost/risk aversion and the market condition is an important direction for future research.

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Endnotes

[1] Market illiquidity may be thought of conceptually as the difficulties or costs associated with reversing a trade instantaneously after a trade is executed, and it depends on market depth, volume of the trade, bid–ask spread, intermediation, and transaction costs.

[2] Brunnermeier and Pedersen (2009) claim that these two forms of liquidity have compounding effects on one another to produce a downward spiral of illiquidity.

[3] The total assets under management in RP strategies is estimated at about \$1.5 trillion. Whereas RP approaches are found in large hedge funds in 1996, for example, the All Weather Fund by Bridgewater Associates, early research can be traced to Qian (2006); also, see Choueifaty and Coignard (2008), Maillard et al. (2010), Chaves et al. (2011), and Carvalho et al. (2012).

[4] Gârleanu and Pedersen (2016) consider transitory (temporary) and persistent (permanent) transaction costs in a portfolio choice model in continuous time to address the trading behavior and inventory dynamics of market makers.

[5] Indeed, we show that our analytical findings are still valid under a general concave temporary impact function.

[6] Any ex post empirical and statistical analyses quantifying a potential premium gained by considering market impact is outside the scope of this paper.

[7] Whereas Almgren et al. (2005) normalize each asset position by its trading volume during the execution period, our estimation method of impact parameters (see Online Appendix L) already incorporates the volume-based effects, and thus, no further volume-based normalization is performed.

[8] In our numerical application, a monthly rebalancing model with a one-day trade execution period is considered.

[9] Brown et al. (2010) consider the rapid portfolio liquidation under a high leverage, which also assumes no uncertainty during execution because of a short trading period. Ignoring uncertainty in the short execution window allows us to develop a tractable optimization model to determine optimal portfolio positions and trade quantities. When trade volumes are given up front and the objective is to decide the trading trajectories, stochastic price dynamics can be incorporated in the execution. For example, for the case of a single asset execution, Almgren and Chriss (2000) and Gatheral and Schied (2012) employ arithmetic and geometric Brownian motion models, respectively, to model asset price dynamics; Chen et al. (2023) extend to the multi-asset case with a multidimensional geometric Brownian motion for price dynamics.

[10] There is evidence that buyer-initiated trades in bear markets or seller-initiated trades in bull markets face increased liquidity, implying higher permanent impact in block trades for buyers in bull markets and sellers in bear markets (Chiyachantana et al. 2004). Extensions of this information asymmetry in the permanent price impact of block trades using S&P ETF and index futures in bull and bear markets are discussed in Frino et al. (2017).

[11] To see this, note that $L_y(x_1) \le L_y^+(x_1)$, and thus, $L_y^+(x_1) \le \rho A_y(x_1)$ implies that $L_y(x_1) \le \rho A_y(x_1)$. On the other hand, if $L_y(x_1) \le \rho A_y(x_1)$, then either $L_y(x_1) \le 0$ or $0 < L_y(x_1) \le \rho A_y(x_1)$. In the former case, $L_y^+(x_1) = 0 \le \rho A_y(x_1)$; in the latter case, $L_y^+(x_1) = L_y(x_1) \le \rho A_y(x_1)$. Therefore, in both cases, $L_y^+(x_1) \le \rho A_y(x_1)$.

[12] The value of the leverage ratio at the terminal date T is not explicitly controlled as in the work of Edirisinghe and Jeong (2022) in which a probabilistic constraint is imposed on the random leverage ratio at time T.

[13] CRT is applied with multiple assets for portfolio deleveraging in Brown et al. (2010) and Chen et al. (2014) without a formal proof.

[14] Similar assumptions, such as $\lambda_j > 0.5\gamma_j$, are made in Almgren and Chriss (2000) and Brown et al. (2010), corresponding to the case of $k = 1$. For the empirical estimates of the price impact parameters of ETF assets reported in Online Table L.3.1, Assumption 2 is satisfied for each asset with $k = 0.6$ even if the entire market volume is traded in each asset. Accordingly, without loss of generality, position limits are not explicitly considered for ease of exposition.

[15] Equations (23) and (25) admit the equivalent (dual) representation $\rho_{\min}(m) := \min\{\rho : m_{\max}(\rho) \ge m\}$ and $m_{\max}(\rho) := \max\{m : \rho_{\min}(m) \le \rho\}$. This yields $m_{\max}(\rho_{\min}(m)) = m$.

[16] Observe that the optimal trade size in any asset is less than $\frac{1}{4}$% of its daily traded volume for any target return setting considered; also, see Online Table L.3.1. Relative trade is an asset's portfolio trade size relative to an asset's total daily volume, and the maximum is taken over all assets.

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