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Decision Support

# An analytic derivation of admissible efficient frontier with borrowing

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**Abstract**

The admissible efficient portfolio selection problem for risky assets has been discussed by Zhang and Nie. In this paper, the admissible efficient portfolio model is proposed under the assumption that there exists the borrowing (money or a risk free asset) case. The admissible efficient frontiers are developed by the spreads of expected return and risk from admissible errors. The analytic forms of the admissible efficient frontiers when short sales are not allowed on all risky assets are derived from two cases: the borrowing with an upper bound constraint, or without an upper bound constraint. The influence on the admissible efficient frontier is explained under the different interest rates of the borrowing. The differences between the results with the borrowing and the results without the borrowing is revealed by a real numerical example.


*Keywords:* Portfolio selection; Admissible error; Borrowing; Efficient frontier

## 1. Introduction

The mean-variance methodology for the portfolio selection problem, posed originally by Markowitz, has played an important role in the development of modern portfolio selection theory. The Markowitz's mean-variance model to portfolio selection involves tracing out an efficient frontier [1,2]. To formulate this model, it is necessary to estimate the probability distribution, strictly speaking, a mean vector and a covariance matrix. In the mean-variance portfolio selection problem, previous research includes Sharpe [3], Merton [4], Szegö [5], Perold [6], Vörös [7], Pang [8], Best and Grauer [9], Best and Jaroslava [10], Yoshimoto [13] and Kawadai and Konno [15]. Their researches usually assume that the mean vector and the covariance matrix are known, it means that all mean returns, variances, covariances of risky assets can be accurately estimated by an investor. Furthermore, the basic assumption for using Markowitz's model is that the future state of the assets can be correctly reflected by asset data in the past, that is, the means, variances, covariances in the future

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are similar to the past one. However, it is hard to ensure this kind of assumption for real ever-changing asset markets. Best and Grauer [19] discussed the sensitivity of the efficient portfolios to changes in the means both from an analytical and a computational point of view. Chopra and Ziemba [20] discussed the effects of estimation errors, in means and covariances, on the optimal portfolio choice. There exists a number of studies about efficient algorithms for finding efficient portfolios from the solution of the mean-variance model which still plays a fundamental role in portfolio optimization, but it would be very difficult to obtain the efficient portfolios in closed form under general constraints such as non-negativity constraints on correlated assets (precluding short sales).

Tanaka et al. [11] proposed the portfolio selection model based on fuzzy probabilities, which can be regarded as a natural extension of Markowitz's model because it extends probability into fuzzy probability. This approach permits the incorporation of expert knowledge by means of a possibility grade, to reflect the degree of similarity between the future state of assets and the state of previous periods. However, the expected return and risk of asset cannot be predicted accurately because of uncertain factors that affect the financial markets. Zhang and Nie [12] discussed the admissible efficient portfolio selection for risky assets under the assumption that the expected return and risk of asset have admissible errors to reflect the uncertainty in real investment actions and gave an analytic derivation of admissible efficient frontier when short sales are not allowed on all risky assets. In the mature market, investors do not only purchase risk assets, but also they can borrow money to buy some risky assets. How do the investors make a correct decision of borrowing? It is very important for a real portfolio selection problem.

The contribution of this paper is as follows. We present the admissible efficient portfolio model when there exists the borrowing (money or a risk free asset) case in real investment actions. We formulate the analytic forms of the admissible efficient frontiers for two cases: the borrowing with an upper bound constraint, or without an upper bound constraint. We illustrate the differences between the results with the borrowing and the results without the borrowing. We give the relationships between the admissible efficient frontiers in the optimistic, pessimistic and intermediate cases. We explain the influence on the admissible efficient frontier under the different interest rates of the borrowing.

## 2. Admissible efficient portfolio model with borrowing

Let $w$ be the investor's initial capital and let $v$ be the maximum amount of money that the investor is able to borrow. A finite $v$ imposes an upper bound on borrowing, otherwise borrowing is unbounded. In order to describe conveniently, we use the following notations:

$r_j$, the expected return rate of risky asset $j, j = 1, \ldots, n$;
$r^*$, the interest rate of borrowing;
$x_j$, the investment amount in risky asset $j, j = 1, \ldots, n$;
$\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, the investment amount vector of risky assets;
$\mathbf{r} = (r_1, r_2, \ldots, r_n)'$, the vector of expected return rates;
$\mathbf{V} = (\sigma_{ij})_{n\times n}$, the covariance matrix of returns;
$\mathbf{l} = (1, 1, \ldots, 1)'$,

where prime ($'$) denotes matrix transposition and all non-primed vectors are column vectors.

If the borrowing is allowed on the risk free asset, then the expected return and variance associated with portfolio $\mathbf{x}$ are given as $\mathbf{r}'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x})$ and $\mathbf{x}'\mathbf{V}\mathbf{x}$, respectively.

Thus, the portfolio selection problem may be described by the following quadratic programming:

$$\begin{aligned} \min \quad & \mathbf{x}'\mathbf{V}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{r}'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) = \mu, \\ & w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v, \\ & \mathbf{x} \geqslant \mathbf{0}, \end{aligned} \tag{1}$$

where $\mu$ is a required return of portfolio, and the quantities $\mu$ and $x_i$ are homogeneous for $i = 1, 2, \ldots, n$.

Let the observation data on returns of assets over $m$ periods be given. At the discrete time $k(k = 1, \ldots, m)$, $n$ kinds of return rates are denoted as a vector $\mathbf{r}_k = (r_{k1}, \ldots, r_{kn})'$. In Tanaka et al. [11], the fuzzy weighted average to estimate $r_j$ and the fuzzy covariance to estimate $\sigma_{ij}$ for the given data are, respectively, defined by

$$\bar{r}_j = \sum_{k=1}^{m} h_k r_{kj} \Big/ \sum_{k=1}^{m} h_k, \quad j = 1, \ldots, n \tag{2}$$

and

$$\bar{\sigma}_{ij} = \sum_{k=1}^{m} (r_{ki} - \bar{r}_i)(r_{kj} - \bar{r}_j) h_k \Big/ \sum_{k=1}^{m} h_k, \quad i, j = 1, \ldots, n, \tag{3}$$

where $h_k$ is a possibility grade to reflect a similarity degree between the future state of asset markets and the $k$th sample offered by experts.

Since $r_j, j = 1, \ldots, n$ are in a fuzzy uncertain economic environment and vary from time to time, the future states of returns and risks of $n$ assets cannot be predicted accurately. In Zhang and Nie [12], the admissible average returns and covariances are, respectively, defined as

$$r_j^* = \bar{r}_j + \phi_j, \quad \phi_{jl} \leqslant \phi_j \leqslant \phi_{jh}, \quad j = 1, \ldots, n \tag{4}$$

and

$$\sigma_{ij}^* = \bar{\sigma}_{ij} + \varepsilon_{ij}, \quad \varepsilon_{ijl} \leqslant \varepsilon_{ij} \leqslant \varepsilon_{ijh}, \quad i, j = 1, \ldots, n, \tag{5}$$

where $\phi_j$ denotes the admissible error for $\bar{r}_j$, $\phi_{jl}$ denotes the lower bound of $\phi_j$, $\phi_{jh}$ denotes the upper bound of $\phi_j$, $\varepsilon_{ij}$ denotes the admissible error for $\bar{\sigma}_{ij}$, $\varepsilon_{ijl}$ denotes the lower bound of $\varepsilon_{ij}$, $\varepsilon_{ijh}$ denotes the upper bound of $\varepsilon_{ij}$.

Combining the future information on assets' return with experts' opinion, $\phi_{jl}$, $\phi_{jh}$, $\varepsilon_{ijl}$ and $\varepsilon_{ijh}$ can be estimated. Correspondingly, the intervals $[\bar{r}_j + \phi_{jl}, \bar{r}_j + \phi_{jh}]$ and $[\bar{\sigma}_{ij} + \varepsilon_{ijl}, \bar{\sigma}_{ij} + \varepsilon_{ijh}]$ are determined. $\phi_j$ and $\varepsilon_{ij}$ can be selected by an investor based on his attitude to return and risk.

The admissible average return vector can be denoted by

$$\bar{\mathbf{r}}^* = \mathbf{a} + \Phi, \quad \Phi_l \leqslant \Phi \leqslant \Phi_h, \tag{6}$$

where $\mathbf{a} = (\bar{r}_1, \ldots, \bar{r}_n)'$, $\Phi = (\phi_1, \ldots, \phi_n)'$, $\Phi_l = (\phi_{1l}, \ldots, \phi_{nl})'$, $\Phi_h = (\phi_{1h}, \ldots, \phi_{nh})'$.

Similarly, the admissible covariance matrix can be denoted by

$$\mathbf{V}^* = \mathbf{\Sigma} + \mathbf{V}_\varepsilon, \quad \mathbf{V}_{\varepsilon l} \leqslant \mathbf{V}_\varepsilon \leqslant \mathbf{V}_{\varepsilon h}, \tag{7}$$

where $\mathbf{\Sigma} = (\bar{\sigma}_{ij})_{n\times n}$, $\mathbf{V}_\varepsilon = (\varepsilon_{ij})_{n\times n}$, $\mathbf{V}_{\varepsilon l} = (\varepsilon_{ijl})_{n\times n}$ and $\mathbf{V}_{\varepsilon h} = (\varepsilon_{ijh})_{n\times n}$.

The admissible expected value and variance of the return associated with portfolio $\mathbf{x}$ are given by

$$\hat{r} = (\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \tag{8}$$

and

$$\hat{\sigma}^2 = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} + \mathbf{x}'\mathbf{V}_\varepsilon\mathbf{x}, \tag{9}$$

respectively.

For any $\mathbf{x} \geqslant \mathbf{0}$, it follows that

$$(\mathbf{a} + \Phi_l)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \leqslant \hat{r} \leqslant (\mathbf{a} + \Phi_h)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x})$$

and

$$\mathbf{x}'\mathbf{\Sigma}\mathbf{x} + \mathbf{x}'\mathbf{V}_{\varepsilon l}\mathbf{x} \leqslant \hat{\sigma}^2 \leqslant \mathbf{x}'\mathbf{\Sigma}\mathbf{x} + \mathbf{x}'\mathbf{V}_{\varepsilon h}\mathbf{x}.$$

Thus, we construct the following admissible portfolio model:

$$\begin{aligned} \min \quad & \mathbf{x}'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)\mathbf{x} \\ \text{s.t.} \quad & (\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \geqslant \mu, \\ & w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{10}$$

If $(\Phi, \mathbf{V}_\varepsilon) = (\Phi_h, \mathbf{V}_{\varepsilon l})$, then (10) can be rewritten as:

$$\begin{aligned} \min \quad & \mathbf{x}'(\mathbf{\Sigma} + \mathbf{V}_{\varepsilon l})\mathbf{x} \\ \text{s.t.} \quad & (\mathbf{a} + \Phi_h)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \geqslant \mu, \\ & w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{11}$$

If $(\Phi, \mathbf{V}_\varepsilon) = (\Phi_l, \mathbf{V}_{\varepsilon h})$, then (10) can be rewritten as:

$$\begin{aligned} \min \quad & \mathbf{x}'(\mathbf{\Sigma} + \mathbf{V}_{\varepsilon h})\mathbf{x} \\ \text{s.t.} \quad & (\mathbf{a} + \Phi_l)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \geqslant \mu, \\ & w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{12}$$

The model (11) means that the investor estimates the return and risk optimistically. The model (12) means that the investor estimates the return and risk pessimistically. The model (10) covers the scenario where the investor makes his portfolio selection neither too optimistically nor pessimistically. The admissible efficient portfolio models of risky assets in [12] can be obtained by the models (10)–(12) when $v = 0$ and $w = 1$. It is obvious that the model (10) is extensions of previous models for portfolio selection problem, such as the models in [11,12].

## 3. The admissible efficient frontier with an upper bound on borrowing

In this section, we formulate an explicit solution to the problem (10) based on some essential assumptions. In the following discussion $(\Phi, \mathbf{V}_\varepsilon)$ is always fixed. Our results about the problem (10) will require the following assumptions to be satisfied.

**Assumption 1.** (i) $\bar{\mathbf{r}} + \Phi \neq k\mathbf{l}$, for any $k \in \Re$, (ii) $\mathbf{\Sigma} + V_\varepsilon$ is a positive definite matrix.

Assumption 1(i) is essential. Assumption 1(ii) is easily satisfied by a proper selection of $\mathbf{V}_\varepsilon$. Define

$$\begin{aligned} & e = (\mathbf{a} + \Phi)'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi), \quad f = \mathbf{l}'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}, \quad d = (\mathbf{a} + \Phi)'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}, \\ & \delta = ef - d^2, \quad \alpha^* = (e - dr^*)w/(d - fr^*), \\ & \beta^* = 1/(fr^{*2} - 2dr^* + e), \\ & \gamma = [v(fr^{*2} - 2dr^* + e) + (e - dr^*)w]/(d - fr^*), \\ & \eta = (w + v)e - vr^*d, \quad \theta = vr^*f - (w + v)d. \end{aligned} \tag{13}$$

Using Lagrangian multiplier method, we obtain the following lemma.

**Lemma 1.** *Let Assumption 1(ii) be satisfied. Then* $\mathbf{x} = t(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}/f$ *is the unique optimal solution of the problem* $\min\{\mathbf{x}'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)\mathbf{x} | \mathbf{l}'\mathbf{x} = t\}$ *and the minimum value of objective function is* $t^2/f$.

By Lemma 1, $\mathbf{x} = w(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}/f$ is the unique optimal solution of the problem $\min\{\mathbf{x}'(\Sigma + \mathbf{V}_\varepsilon)\mathbf{x} | \mathbf{l}'\mathbf{x} \geqslant w\}$ and satisfies $(\mathbf{a} + \Phi)'\mathbf{x} = wd/f$. Therefore, the following assumption is natural.

**Assumption 2.** $r^* < d/f$.

Some properties of $e$, $f$, $d$, $\beta^*$, $\alpha^*$, $\gamma$ and $\delta$ are given in the following proposition.

**Proposition 1.** *Let Assumptions 1 and 2 be satisfied. Then the following results hold*:

(i) $e > 0, \quad f > 0$,
(ii) $\delta > 0, \quad \beta^* > 0$,
(iii) $wd/f \leqslant \alpha^* \leqslant \gamma$.

**Proof.** From Zhang and Nie [12], parts (i) and (ii) hold.

According to $(e - dr^*)f > (d - fr^*)d$ and $r^* < d/f$, it follows that

$$wd/f \leqslant (e - dr^*)w/(d - fr^*) = \alpha^*.$$

Furthermore,

$$\gamma = \frac{v}{\beta^*(d - fr^*)} + \alpha^* \geqslant \alpha^*.$$

Thus, part (iii) holds. □

The following Lemma 2 is obvious.

**Lemma 2.** $\max\{(\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) | w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v, \mathbf{x} \geqslant \mathbf{0}\} = (w + v)\max\{\bar{r}_i + \phi_i, 1 \leqslant i \leqslant n\} - vr^*$.

Lemma 2 means that the maximum return of the portfolio that an investor can obtain is $(w + v)\max\{\bar{r}_i + \phi_i, 1 \leqslant i \leqslant n\} - vr^*$ under constraints $w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v$ and $\mathbf{x} \geqslant \mathbf{0}$. This result can be obtained when the investment amount in the risky asset with the maximum admissible expected return $\bar{r}_i + \phi_i$ is $w + v$ and others are 0.

Before solving the problem (10), it is helpful to introduce the following problem

$$\begin{aligned} \min \quad & \mathbf{x}'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)\mathbf{x} \\ \text{s.t.} \quad & (\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \geqslant \mu, \\ & \mathbf{l}'\mathbf{x} \geqslant w, \\ & \mathbf{l}'\mathbf{x} \leqslant w + v. \end{aligned} \tag{14}$$

The problem (14) means that short sales are allowed on all risky assets. The optimal solution to (14) is formulated in the following theorem.

**Theorem 1.** *Let Assumptions 1 and 2 be satisfied. Then the optimal solution to (14) is*:

(i) *if* $\mu \leqslant wd/f$, *then* $\mathbf{x} = wf^{-1}(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}$,
(ii) *if* $wd/f < \mu \leqslant \alpha^*$, *then* $\mathbf{x} = (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[(e\mathbf{l} - d\mathbf{a} - d\Phi)w + (f\mathbf{a} + f\Phi - d\mathbf{l})\mu]/\delta$,
(iii) *if* $\alpha^* < \mu \leqslant \gamma$, *then* $\mathbf{x} = \beta^*(\mu - wr^*)(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi - r^*\mathbf{l})$,
(iv) *if* $\mu > \gamma$, *then* $\mathbf{x} = (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[\eta\mathbf{l} + \theta(\mathbf{a} + \Phi) + (f(\mathbf{a} + \Phi) - d\mathbf{l})\mu]/\delta$.

**Proof.** Since $\mathbf{\Sigma} + \mathbf{V}_\varepsilon$ is a positive definite matrix, the K-T conditions are both necessary and sufficient for optimality (see [14] or [16,17]). Solving the problem (14) is equal to choose $\mathbf{x} \in \Re^n$ and $\lambda_i \in \Re$ $(i = 1, 2, 3)$ such that

$$\begin{cases} (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)\mathbf{x} = \lambda_1(\mathbf{a} + \Phi - r^*\mathbf{l}) + \lambda_2\mathbf{l} + \lambda_3(-\mathbf{l}), \\ (\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \geqslant \mu, \\ w \leqslant \mathbf{l}'\mathbf{x} \leqslant w + v, \\ \lambda_1[(\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) - \mu] = 0, \\ \lambda_2(\mathbf{l}'\mathbf{x} - w) = 0, \\ \lambda_3(-\mathbf{l}'\mathbf{x} + w + v) = 0, \\ \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_3 \geqslant 0. \end{cases} \tag{15}$$

If $\mu \leqslant wd/f$, then there are multipliers $\lambda_2 = w/f, \lambda_1 = \lambda_3 = 0$ and $\mathbf{x} = wf^{-1}(\mathbf{\Sigma} + V_\varepsilon)^{-1}\mathbf{l}$ such that (15) holds.

Thus, part (i) is true.

If $wd/f < \mu \leqslant \alpha^*$, then $f\mu - dw > 0$ and $(e - dr^*)w - (d - fr^*)\mu \geqslant 0$.

There are multipliers $\lambda_1 = (f\mu - dw)/\delta$, $\lambda_2 = [(e - dr^*)w - (d - fr^*)\mu]/\delta$, $\lambda_3 = 0$ and

$$\mathbf{x} = (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[(e\mathbf{l} - d\mathbf{a} - d\Phi)w + (f\mathbf{a} + f\Phi - d\mathbf{l})\mu]/\delta$$

such that (15) holds.

Thus, part (ii) is true.

If $\alpha^* < \mu \leqslant \gamma$, then $\mu > wd/f > wr^*$.

There are multipliers $\lambda_1 = \beta^*(\mu - wr^*)$, $\lambda_2 = \lambda_3 = 0$ and

$$\mathbf{x} = \beta^*(\mu - wr^*)(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi - r^*\mathbf{l})$$

such that (15) holds.

Hence, part (iii) is true.

If $\mu > \gamma$, then $\theta + \mu f > \theta + \gamma f = (w + v)\delta/(d - fr^*) > 0$.

There are multipliers $\lambda_1 = (\theta + \mu f)/\delta$, $\lambda_2 = 0$, $\lambda_3 = (\mu - \gamma)(d - fr^*)/\delta$ and

$$\mathbf{x} = (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[\eta\mathbf{l} + \theta(\mathbf{a} + \Phi) + (f(\mathbf{a} + \Phi) - d\mathbf{l})\mu]/\delta$$

such that (15) holds.

The proof of the theorem is completed. □

We show the following relation to the optimal solutions between the problem (10) and the problem (14).

**Lemma 3.** *Let Assumption 1(ii) be satisfied and let $\mu$ be a constant. If $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)'$ is the optimal solution of the problem (14) with $x_{i_1}^* < 0, \ldots, x_{i_k}^* < 0$ and others $x_j^* \geqslant 0$, and if $\mathbf{x}^{1*} = (x_1^{1*}, \ldots, x_n^{1*})'$ is the optimal solution of the problem (10), then $\mathbf{x}^{1*} \in \bigcup_{s=1}^{k}\{\mathbf{x} = (x_1, \ldots, x_n)' | x_{i_s} = 0, x_i \geqslant 0, i \neq i_s\}$.*

**Proof.** With the help of reduction to absurdity. Let $x_{i_s}^{1*} \neq 0$ for all $s = 1, \ldots, k$.

Then $x_{i_s}^{1*} > 0$ for all $s = 1, \ldots, k$ and others $x_j^{1*} \geqslant 0$.

Assumption 1(ii) is satisfied, so $\sigma^2(\mathbf{x}) = \mathbf{x}'\,(\Sigma + V_\varepsilon)\mathbf{x}$ is a strictly convex function of $\mathbf{x}$.

We set $F(t) = \sigma^2(t\mathbf{x}^{1*} + (1 - t)\mathbf{x}^*)$ and $\mathbf{x}^{**}(t) = t\mathbf{x}^{1*} + (1 - t)\mathbf{x}^*$ for $t \in [0,1]$.

According to Theorem 3.2.4 in [18], $F(t)$ is a strictly convex function of $t$ for $t \in [0,1]$.

It can be shown that $\mathbf{x}^{**}(t) = (x_1^{**}(t), \ldots, x_n^{**}(t))'$ for all $t \in [0,1]$ is a feasible solution of the problem (14), so $F(0) = \sigma^2(\mathbf{x}^*) < \sigma^2(t\mathbf{x}^{1*} + (1 - t)\mathbf{x}^*) = F(t)$ for all $t \in (0,1)$.

Thus, $F(t)$ is a strictly increasing function of $t$ for $t \in (0,1)$.

Setting $t_{i_s} = -x_{i_s}^*/(x_{i_s}^{1*} - x_{i_s}^*)$ for $s = 1, \ldots, k$ and Using $x_{i_s}^* < 0$, we obtain $t_{i_s} \in (0, 1)$ and $x_{i_s}^{**}(t_{i_s}) = 0$ for $s = 1, \ldots, k$.

Define $t_0 = max\{t_{i_s} | s = 1, \ldots, k\}$ and take $t_0 < t_1 < 1$. Since $x_{i_1}^{**}(t), \ldots, x_{i_k}^{**}(t)$ are strictly increasing, we have $\mathbf{x}^{**}(t_1) \geqslant 0$. This implies that $\mathbf{x}^{**}(t_1)$ is also a feasible solution of the problem (10).

Thus, $\sigma^2(\mathbf{x}^{**}(t_1)) = F(t_1) < F(1) = \sigma^2(\mathbf{x}^{1*})$ is in contradiction with the assumption of $\mathbf{x}^{1*}$.

This concludes the proof of the lemma. □

For $\mu > \gamma$, the optimal solution of the problem (14) can be rewritten as

$$x_k = [\eta g_k + \theta a_k + \theta\varphi_k + (fa_k + f\varphi_k - dg_k)\mu]/\delta, k = 1, \ldots, n, \tag{16}$$

where

$$(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l} = (g_1, g_2, \ldots, g_n)', \quad (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{a} = (a_1, a_2, \ldots, a_n)', \quad (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n)'.$$

Since $\delta > 0$ and (16) satisfies $\sum_{k=1}^{n} x_k = \mathbf{l}'\mathbf{x} = w + v$, there exist $h$, $p$ such that

$$fa_h + f\varphi_h - dg_h < 0 \quad \text{and} \quad fa_p + f\varphi_p - dg_p > 0.$$

Define

$$\beta_0 = \min\left\{\frac{\eta g_k + \theta a_k + \theta\varphi_k}{dg_k - fa_k - f\varphi_k} | fa_k + f\varphi_k - dg_k < 0, k = 1, \ldots, n\right\},$$

$$\alpha_0 = \max\left\{\frac{\eta g_k + \theta a_k + \theta\varphi_k}{dg_k - fa_k - f\varphi_k} | fa_k + f\varphi_k - dg_k > 0, k = 1, \ldots, n\right\}.$$

**Lemma 4.** *Let Assumptions 1 and 2 be satisfied. If* $(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}(\mathbf{a}+\Phi)\ngeqslant r^*(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}\mathbf{l}\geqslant\mathbf{0}$. *Then the optimal solution of the problem (10) for* $\mu>\alpha^*$ *contains zero coordinate.*

**Proof.** There exists at least one $k\in I$ such that $a_k+\varphi_k<r^*g_k$.

According to Assumption 2, it follows that $a_k+\varphi_k<r^*g_k<dg_k/f$, i.e., $fa_k+f\varphi_k-dg_k<0$.

For $\alpha^*<\mu\leqslant\gamma$, the optimal solution of the problem (14) satisfies

$$x_k=\beta^*(\mu-wr^*)(a_k+\varphi_k-r^*g_k)<0.$$

For $\mu>\gamma$, the optimal solution of the problem (14) satisfies

$$\begin{aligned}x_k&=\tfrac{1}{\delta}[\eta g_k+\theta a_k+\theta\varphi_k+(fa_k+f\varphi_k-dg_k)\mu]\\&\leqslant\tfrac{1}{\delta}[\eta g_k+\theta a_k+\theta\varphi_k+(fa_k+f\varphi_k-dg_k)\gamma]\\&=\tfrac{w+v}{d-r^*f}(a_k+\varphi_k-r^*g_k)<0.\end{aligned}$$

From Lemma 3, it implies that the optimal solution of the problem (10) for $\mu>\alpha^*$ satisfies $x_k=0$. □

Now, we discuss the optimal solution of (10) under the assumption $(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}(\mathbf{a}+\Phi)\geqslant r^*(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}\mathbf{l}$. In order to describe conveniently, we introduce the following notations:

$$\begin{cases}I=\{1,2,\ldots,n\},\quad I_i=\{i_1,\ldots,i_l\}\subseteq I,\\ \mathbf{a}_i+\Phi_i=(\bar r_{i_1}+\phi_{i_1},\ldots,\bar r_{i_l}+\phi_{i_l})',\quad \mathbf{\Sigma}_i=(\bar\sigma_{pq})_{l\times l}(p,q\in I_i),\quad \mathbf{V}_{i,\varepsilon}=(\varepsilon_{pq})_{l\times l}(p,q\in I_i),\\ (\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon})^{-1}\mathbf{a}_i=(a_{i,i_1},\ldots,a_{i,i_l})',\quad (\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon})^{-1}\Phi_i=(\varphi_{i,i_1},\ldots,\varphi_{i,i_l})',\\ (\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon})^{-1}\mathbf{l}_i=(g_{i,i_1},\ldots,g_{i,i_l})',\quad \mathbf{l}_i=(1,\ldots,1)',\\ e_i=(\mathbf{a}_i+\Phi_i)'(\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon})^{-1}(\mathbf{a}_i+\Phi_i),\quad f_i=\mathbf{l}_i'(\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon})^{-1}\mathbf{l}_i,\\ d_i=(\mathbf{a}_i+\Phi_i)'(\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon})^{-1}\mathbf{l}_i,\quad \delta_i=e_if_i-d_i^2,\\ \eta_i=(w+v)e_i-vr^*d_i,\theta_i=vr^*f_i-(w+v)d_i,\end{cases}\tag{17}$$

where $I$ denotes the set of $n$ assets, $I_i$ denotes a subset of $I$, $\mathbf{a}_i+\Phi_i$ and $\mathbf{\Sigma}_i+\mathbf{V}_{i,\varepsilon}$ denote the admissible expected return vector and covariance matrix of $I_i$, respectively.

**Theorem 2.** *Let Assumptions 1 and 2 be satisfied. If* $(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}(\mathbf{a}+\Phi)\geqslant r^*(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}\mathbf{l}$ *and* $\mu>\alpha^*$, *then there exist positive numbers* $\alpha^*\leqslant\beta_i\leqslant(w+v)\max_{1\leqslant j\leqslant n}\{\bar r_j+\phi_j\}-vr^*$ *and subsets* $I_i\subseteq I(i=1,\ldots,t,t\leqslant n)$ *such that the optimal solution of the problem (10) is*:

(i) *if* $\alpha^*<\mu\leqslant\gamma$, *then*

$$\mathbf{x}=\beta^*(\mu-wr^*)(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}(\mathbf{a}+\Phi-r^*\mathbf{l})$$

(ii) *if* $\gamma<\mu\leqslant\beta_0$, *then*

$$\mathbf{x}=(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}[\eta\mathbf{l}+\theta(\mathbf{a}+\Phi)+(f(\mathbf{a}+\Phi)-d\mathbf{l})\mu]/\delta$$

(iii) *if* $\beta_{i-1}<\mu\leqslant\beta_i(i=,1,\ldots,t)$, *then*

$$x_k=\begin{cases}[\eta_ig_{i,k}+\theta_ia_{i,k}+\theta_i\varphi_{i,k}+(f_ia_{i,k}+f_i\varphi_{i,k}-d_ig_{i,k})\mu]/\delta_i & \textit{for all } k\in I_i\\ 0 & \textit{for all } k\in I\setminus I_i.\end{cases}$$

*where* $a_{i,k}$, $\varphi_{i,k}$, $g_{i,k}$, $e_i$, $f_i$, $d_i$, $\delta_i$, $\eta_i$, $\theta_i$ *are defined by (17)*,

$$\begin{aligned}&\beta_i=\min\{(\eta_ig_{i,k}+\theta_ia_{i,k}+\theta_i\varphi_{i,k})/(d_ig_{i,k}-f_ia_{i,k}-f_i\varphi_{i,k})|f_ia_{i,k}+f_i\varphi_{i,k}-d_ig_{i,k}<0,k\in I_i\},\\&\beta_t=(w+v)\max_{1\leqslant j\leqslant n}\{\bar r_j+\phi_j\}-vr^*,\beta_i\neq(w+v)\max_{1\leqslant j\leqslant n}\{\bar r_j+\phi_j\}-vr^*\textit{for all } i=1,\ldots,t-1.\end{aligned}$$

**Proof.** From $(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}(\mathbf{a}+\Phi)\geqslant r^*(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}\mathbf{l}$, it follows that

$$\beta^*(\mu-wr^*)(\mathbf{\Sigma}+\mathbf{V}_\varepsilon)^{-1}(\mathbf{a}+\Phi-r^*\mathbf{l})\geqslant\mathbf{0}\quad\text{for }\mu\geqslant wr^*.$$

For $\alpha^* \leqslant \mu \leqslant \gamma$, using Theorem 1(iii) and Proposition 1, the optimal solution of the problem (14) is also one of the problem (10), that is, part (i) holds.

When $\mu = \gamma$, we obtain

$$(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[\eta\mathbf{l} + \theta(\mathbf{a} + \Phi) + (f(\mathbf{a} + \Phi) - d\mathbf{l})\mu]/\delta = \beta^*(\mu - wr^*)(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi - r^*\mathbf{l}) \geqslant \mathbf{0}.$$

For all $\gamma \leqslant \mu \leqslant \beta_0$, according to the definition of $\beta_0$ and considering the continuity of variables it is sufficient to see that the optimal solution of the problem (10) is given by (16).

Without loss of generality we assume that there exists the unique $h \in I$ such that

$$\beta_0 = \frac{\eta g_h + \theta a_h + \theta\varphi_h}{dg_h - fa_h - f\varphi_h}.$$

If $\beta_0 < \mu \leqslant (w + v)\max_{1\leqslant i\leqslant n}\{\bar{r}_i + \phi_i\} - vr^*$, then $x_h < 0$.

By Lemma 3, the optimal solution of the problem (10) satisfies that $x_h = 0$ for $\beta_0 < \mu \leqslant (w + v)\max_{1\leqslant i\leqslant n}\{\bar{r}_i + \phi_i\} - vr^*$.

We erase $x_h$ in (10) and consider the new model with $n - 1$ assets:

$$\begin{aligned} \min \quad & \mathbf{x}_1'(\mathbf{\Sigma}_1 + \mathbf{V}_{1\varepsilon})\mathbf{x}_1 \\ \text{s.t.} \quad & (\mathbf{a}_1 + \Phi_1)'\mathbf{x}_1 + r^*(w - \mathbf{l}_1'\mathbf{x}_1) \geqslant \mu, \\ & w \leqslant \mathbf{l}_1'\mathbf{x}_1 \leqslant w + v, \\ & \mathbf{x}_1 \geqslant \mathbf{0}, \end{aligned} \tag{18}$$

where $\mathbf{x}_1 = (x_1, \ldots, x_{h-1}, x_{h+1}, \ldots, x_n)'$, $\mathbf{a}_1 = (\bar{r}_1, \ldots, \bar{r}_{h-1}, \bar{r}_{h+1}, \ldots, \bar{r}_n)'$,
$\Phi_1 = (\phi_1, \ldots, \phi_{h-1}, \phi_{h+1}, \ldots, \phi_n)'$, $\mathbf{l}_1 = (1,1, \ldots, 1)'$, $\mathbf{\Sigma}_1 = (\bar{\sigma}_{pq})_{(n-1)\times(n-1)}(p, q \neq h)$,
$\mathbf{V}_{1,\varepsilon} = (\varepsilon_{pq})_{(n-1)\times(n-1)}(p,q \neq h)$.

We still solve the problem (18) to adopt the same way as shown above. Clearly, there exists

$$\beta_1 = \min\left\{\frac{\eta_1 g_{1,k} + \theta_1 a_{1,k} + \theta_1\varphi_{1,k}}{d_1 g_{1,k} - f_1 a_{1,k} - f_1\varphi_{1,k}} | f_1 a_{1,k} + f_1\varphi_{1,k} - d_1 g_{1,k} < 0, k \in I_1\right\}$$

such that the optimal solution of the problem (10) for $\beta_0 < \mu \leqslant \beta_1$ is

$$x_k = \begin{cases} [\eta_1 g_{1,k} + \theta_1 a_{1,k} + \theta_1\varphi_{1,k} + (f_1 a_{1,k} + f_1\varphi_{1,k} - d_1 g_{1,k})\mu]/\delta_1 & \text{for all } k \in I_1 \\ 0 & \text{for all } k \in I \setminus I_1, \end{cases}$$

where

$$\begin{cases} (\mathbf{\Sigma}_1 + \mathbf{V}_{1,\varepsilon})^{-1}\mathbf{l}_1 = (g_{1,1}, \ldots, g_{1,h-1}, g_{1,h+1}, \ldots, g_{1,n})', \\ (\mathbf{\Sigma}_1 + \mathbf{V}_{1,\varepsilon})^{-1}\mathbf{a}_1 = (a_{1,1}, \ldots, a_{1,h-1}, a_{1,h+1}, \ldots, a_{1,n})', \\ (\mathbf{\Sigma}_1 + \mathbf{V}_{1,\varepsilon})^{-1}\Phi_1 = (\varphi_{1,1}, \ldots, \varphi_{1,h-1}, \varphi_{1,h+1}, \ldots, \varphi_{1,n})', \\ e_1 = (\mathbf{a}_1 + \Phi_1)'(\mathbf{\Sigma}_1 + \mathbf{V}_{1,\varepsilon})^{-1}(\mathbf{a}_1 + \Phi_1), \quad f_1 = \mathbf{l}_1'(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_{1,\varepsilon})^{-1}\mathbf{l}_1, \\ d_1 = (\mathbf{a}_1 + \Phi_1)'(\mathbf{\Sigma}_1 + \mathbf{V}_{1,\varepsilon})^{-1}\mathbf{l}_1, \quad \delta_1 = e_1 f_1 - d_1^2, \eta_1 = (w + v)e_1 - vr^* d_1, \theta_1 = vr^* f_1 - (w + v)d_1, \\ I_1 = I \setminus \{h\} = \{1, \ldots, h - 1, h + 1, \ldots, n\}. \end{cases}$$

If $\beta_1 < (w + v)\max_{1\leqslant i\leqslant n}\{\bar{r}_i + \phi_i\} - vr^*$, then we repeat the same procedure used above until $\beta_t = (w + v)\max_{1\leqslant i\leqslant n}\{\bar{r}_i + \phi_i\} - vr^*$ and $I_{t-1} = \{q|(m + v)(\bar{r}_q + \phi_q) - vr^* = \beta_t, q \in I\}$.

Thus, we can obtain all optimal solutions of the problem (10) for $\alpha^* \leqslant \mu \leqslant (w + v)\max_{1\leqslant i\leqslant n}\{\bar{r}_i + \phi_i\} - vr^*$.

This concludes the proof of the theorem. □

**Theorem 3.** *Let Assumptions 1 and 2 be satisfied. If* $(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi) \geqslant r^*(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}$ *and* $\mu \leqslant \alpha^*$, *then there exist positive numbers* $w\min_{1\leqslant j\leqslant n}\{\bar{r}_j + \phi_j\} \leqslant \alpha_j \leqslant \alpha^*$ *and subsets* $I_j \subseteq I (j = 1, 2, \ldots, s, s \leqslant n)$ *such that the admissible efficient portfolio to (10) is*

(i) *if* $\alpha_0 < \mu \leqslant \alpha^*$, *then*

$$\mathbf{x} = (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[(e\mathbf{l} - d\mathbf{a} - d\Phi)w + (f\mathbf{a} + f\Phi - d\mathbf{l})\mu]/\delta$$

(ii) *if* $\alpha_j < \mu \leqslant \alpha_{j-1} (j = 1, \ldots, s)$, *then*

$$x_k = \begin{cases} [(e_j g_{j,k} - d_j a_{j,k} - d_j \varphi_{j,k})w + (f_j a_{j,k} + f_j \varphi_{j,k} - d_j g_{j,k})\mu]/\delta_j & \text{for all } k \in I_j \\ 0 & \text{for all } k \in I \setminus I_j. \end{cases}$$

(iii) *if* $\mu \leqslant \alpha_s$, *then*

$$x_k = \begin{cases} w g_{s,k}/f_s & \text{for all } k \in I_s \\ 0 & \text{for all } k \in I \setminus I_s \end{cases}$$

*where* $a_{j,k}$, $\varphi_{j,k}$, $g_{j,k}$, $e_j$, $f_j$, $d_j$, $\delta_j$, $\eta_j$, $\theta_j$ *are defined by* (17),

$$\alpha_j = \max\left\{ \frac{(d_j a_{j,k} + d_j \varphi_{j,k} - e_j g_{j,k})w}{f_j a_{j,k} + f_j \varphi_{j,k} - d_j g_{j,k}} | f_j a_{j,k} + f_j \varphi_{j,k} - d_j g_{j,k} > 0, k \in I_j \right\},$$

$\alpha_s = wd_s/f_s, \alpha_j \neq wd_j/f_j$ *for all* $j = 0, 1, \ldots, s-1$.

**Proof.** When $\mu = \alpha^*$, it follows that

$$\begin{aligned}(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[(e\mathbf{l} - d\mathbf{a} - d\Phi)w + (f\mathbf{a} + f\Phi - d\mathbf{l})\mu]/\delta &= w(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[(e\mathbf{l} - d\mathbf{a} - d\Phi)(d - fr^*) \\ &\quad + (f\mathbf{a} + f\Phi - d\mathbf{l})(e - dr^*)]/(d - fr^*)\delta \\ &= w(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[\mathbf{a} + \Phi - r^*\mathbf{l}]/(d - fr^*) \geqslant 0.\end{aligned}$$

For $\alpha_0 < \mu \leqslant \alpha^*$, according to the definition of $\alpha_0$ and considering the continuity of variables it is sufficient to see that the optimal solution of the problem (14) is also one of the problem (10).

For $\mu \leqslant \alpha_0$, the proof of the theorem is similar to case 1 of [12]. □

The principal results for Theorems 2 and 3 are as follows. For $\mu < \alpha^*$, the admissible efficient portfolios consist of only risky assets, that is, it isn't needed to borrow money (the risk free asset). For $\alpha^* \leqslant \mu \leqslant (w + v)\max_{1\leqslant j\leqslant n}\{\bar{r}_j + \phi_j\} - vr^*$, the borrowing is needed in order to obtain the admissible efficient portfolios. As $\mu$ is increased, the borrowing is increased from zero. At $\mu = \gamma$, the borrowing is increased to its upper bound $v$ and remains there for all $\gamma \leqslant \mu \leqslant (w + v)\max_{1\leqslant j\leqslant n}\{\bar{r}_j + \phi_j\} - vr^*$. Furthermore, for $\alpha_0 < \mu < \beta_0$, all the risky asset are held. For $\mu \leqslant \alpha_0$ or $\mu \geqslant \beta_0$, part of the risky assets is held. As $\mu$ is increased beyond $\beta_0$, the risky assets except the risky asset with the maximum expected return are, respectively, reduced to zero. At $\mu = \beta_t$, the risky asset with the maximum expected return is increased to $w + v$ and all other risky assets are reduced to zero. As $\mu$ is decreased below $\alpha_0$, part of the risky assets is reduced to zero and the process is stopped at $\mu = \alpha_s$. For $\alpha^* \leqslant \mu \leqslant \gamma$, the admissible efficient frontier for the problem (10) in $(\sigma, \mu)$ space is a straight line. The remainder of the admissible efficient frontier is piece-wise hyperbolic.

It should be seen that $\alpha^*$ is a monotone increasing function of $r^*$ for $r^* < d/f$. It means that the interval of borrowing, $[\alpha^*, (w + v)\max_{1\leqslant j\leqslant n}\{\bar{r}_j + \phi_j\} - vr^*]$, becomes narrow when the cost of borrowing is increased. If $r^* \to d/f$, then $\alpha^* \to +\infty$. Specially, if $r^*$ satisfies the condition that $\alpha^* \geqslant (w + v)\max_{1\leqslant j\leqslant n}\{\bar{r}_j + \phi_j\} - vr^*$, then the investor does not need to borrow money (the risk free asset). In other words, the borrowing cannot improve the return of efficient portfolio. In this case, the admissible efficient frontier with borrowing is as same as the admissible efficient frontier without borrowing. If $v \to +\infty$, then $\gamma \to +\infty$ and $(w + v)\max_{1\leqslant j\leqslant n}\{\bar{r}_j + \phi_j\} - vr^* \to +\infty$. It shows that the greater $v$, the greater expected return of admissible efficient portfolio that the investor can obtain.

## 4. The admissible efficient frontier without an upper bound on borrowing

If there is not an upper bound on borrowing, then the problem (10) can be written as

$$\begin{aligned} \min \quad & \mathbf{x}'(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)\mathbf{x} \\ \text{s.t.} \quad & (\mathbf{a} + \Phi)'\mathbf{x} + r^*(w - \mathbf{l}'\mathbf{x}) \geqslant \mu, \\ & \mathbf{l}'\mathbf{x} \geqslant w, \\ & \mathbf{x} \geqslant 0. \end{aligned} \tag{19}$$

The optimal solution of the problem (19) is formulated in the following theorem.

**Theorem 4.** *Let Assumptions 1 and 2 be satisfied. If* $(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi) \geqslant r^*(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}\mathbf{l}$, *then there exist positive numbers* $w \min_{1\leqslant i\leqslant n}\{\bar{r}_i + \phi_i\} \leqslant \alpha_i \leqslant \alpha^*$ *and subsets* $I_i \subseteq I (i = 1, \ldots, s, s \leqslant n)$ *such that the optimal solution to the problem (19) is*:

(i) *if* $\mu > \alpha^*$, *then*

$$\mathbf{x} = \beta^*(\mu - wr^*)(\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}(\mathbf{a} + \Phi - r^*\mathbf{l}),$$

(ii) *if* $\alpha_0 < \mu \leqslant \alpha^*$, *then*

$$\mathbf{x} = (\mathbf{\Sigma} + \mathbf{V}_\varepsilon)^{-1}[(e\mathbf{l} - d\mathbf{a} - d\Phi)w + (f\mathbf{a} + f\Phi - d\mathbf{l})\mu]/\delta,$$

(iii) *if* $\alpha_j < \mu \leqslant \alpha_{j-1} (j = 1, \ldots, s)$, *then*

$$x_k = \begin{cases} [(e_j g_{j,k} - d_j a_{j,k} - d_j\varphi_{j,k})w + (f_j a_{j,k} + f_j\varphi_{j,k} - d_j g_{j,k})\mu]/\delta_j & \text{for all } k \in I_j \\ 0 & \text{for all } k \in I \setminus I_j, \end{cases}$$

(iv) *if* $\mu \leqslant \alpha_s$, *then*

$$x_k = \begin{cases} wg_{s,k}/f_s & \text{for all } k \in I_s \\ 0 & \text{for all } k \in I \setminus I_s, \end{cases}$$

*where* $a_{j,k}$, $\varphi_{j,k}$, $g_{j,k}$, $e_j$, $f_j$, $d_j$, $\delta_j$, $\eta_j$, $\theta_j$ *are defined by (17)*,

$$\alpha_j = \max\left\{\frac{(d_j a_{j,k} + d_j\varphi_{j,k} - e_j g_{j,k})w}{f_j a_{j,k} + f_j\varphi_{j,k} - d_j g_{j,k}} | f_j a_{j,k} + f_j\varphi_{j,k} - d_j g_{j,k} > 0, k \in I_j\right\},$$

$\alpha_s = wd_s/f_s, \alpha_j \neq wd_j/f_j$ *for all* $j = 1, \ldots, s - 1$.

**Proof.** The proof is similar to the proof of Theorems 2 and 3. □

The principal result for Theorem 4 is as follows. For $\mu < \alpha^*$, the admissible efficient portfolio to the problem (19) is the same as the problem (10), that is, it is not needed to borrow money. The admissible efficient frontier in $(\sigma, \mu)$ space is piece-wise hyperbolic. For $\mu \geqslant \alpha^*$, the borrowing is needed in order to obtain the admissible efficient portfolios for the problem (19). As $\mu$ is increased, the borrowing is increased from zero. The admissible efficient frontier in $(\sigma, \mu)$ space is a straight line.

The optimal solutions to the problem (11) and the problem (12) can be obtained by Theorems 1–3 as $(\Phi, \mathbf{V}_\varepsilon) = (\Phi_h, \mathbf{V}_{\varepsilon l})$ and $(\Phi, \mathbf{V}_\varepsilon) = (\Phi_l, \mathbf{V}_{\varepsilon h})$, respectively. Correspondingly, the upper and lower admissible efficient frontiers can also be indicated clearly. The upper admissible efficient frontier is constructed based on the maximum of the admissible returns and the minimum of the admissible risks to assets, so it is consistent with the investor who makes his portfolio selection optimistically. Similarly, the lower admissible efficient frontier is constructed based on the minimum of the admissible returns and the maximum of the admissible risks to assets, so it is consistent with the investor who makes his portfolio selection pessimistically. The upper admissible efficient frontier is above the lower admissible efficient frontier. For any $(\Phi, \mathbf{V}_\varepsilon) \in \{(\Phi, \mathbf{V}_\varepsilon) | \Phi_l \leqslant \Phi \leqslant \Phi_h, \mathbf{V}_{\varepsilon l} \leqslant \mathbf{V}_\varepsilon \leqslant \mathbf{V}_{\varepsilon h}\}$, the problems (10) and (19) cover the scenario where the investor makes his portfolio selection neither too pessimistically nor optimistically. Its admissible efficient frontier is between the upper and lower admissible efficient frontiers. Specifically, Tnanka's fuzzy efficient frontier in [11] is obtained when $(\Phi, \mathbf{V}_\varepsilon) = (\mathbf{0}, \mathbf{0})$ and $v = 0$. Zhang and Nie's admissible efficient frontier in [12] is obtained when $v = 0$.

## 5. Numerical example

In order to illustrate the proposed algorithms and analytic formulas of the admissible efficient portfolios in this paper, we consider a practical example shown in Table 1 introduced by Markowitz [2] and Tanaka et al. [11]. The columns 2–10 represent American Tobacco, A.T.&T., United States Steel, General Motors, Atche-

Table 1
Returns on nine securities and importance grades

| $h_t$ | Year | Am.T. | A.T.&T. | U.S.S. | G.M. | A.T.&S. | C.C. | Bdn. | Frstn. | S.S. |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 1937(1) | −0.305 | −0.173 | −0.318 | −0.477 | −0.457 | −0.065 | −0.319 | −0.4 | −0.435 |
| 0.118 | 1938(2) | 0.513 | 0.098 | 0.285 | 0.714 | 0.107 | 0.238 | 0.076 | 0.336 | 0.238 |
| 0.135 | 1939(3) | 0.055 | 0.2 | −0.047 | 0.165 | −0.424 | −0.078 | 0.381 | −0.093 | −0.295 |
| 0.153 | 1940(4) | −0.126 | 0.03 | 0.104 | −0.043 | −0.189 | −0.077 | −0.051 | −0.09 | −0.036 |
| 0.171 | 1941(5) | −0.28 | −0.183 | −0.171 | −0.277 | 0.637 | −0.187 | 0.087 | −0.194 | −0.24 |
| 0.188 | 1942(6) | −0.003 | 0.067 | −0.039 | 0.476 | 0.865 | 0.156 | 0.262 | 1.113 | 0.126 |
| 0.206 | 1943(7) | 0.428 | 0.3 | 0.149 | 0.225 | 0.313 | 0.351 | 0.341 | 0.58 | 0.639 |
| 0.224 | 1944(8) | 0.192 | 0.103 | 0.26 | 0.29 | 0.637 | 0.233 | 0.227 | 0.473 | 0.282 |
| 0.241 | 1945(9) | 0.446 | 0.216 | 0.419 | 0.216 | 0.373 | 0.349 | 0.352 | 0.229 | 0.578 |
| 0.259 | 1946(10) | −0.088 | −0.046 | −0.078 | −0.272 | −0.037 | −0.209 | 0.153 | −0.126 | 0.289 |
| 0.276 | 1947(11) | −0.127 | −0.071 | 0.169 | 0.144 | 0.026 | 0.355 | −0.099 | 0.009 | 0.184 |
| 0.294 | 1948(12) | −0.015 | 0.056 | −0.035 | 0.107 | 0.153 | −0.231 | 0.038 | 0 | 0.114 |
| 0.312 | 1949(12) | 0.305 | 0.038 | 0.133 | 0.321 | 0.067 | 0.246 | 0.273 | 0.223 | −0.222 |
| 0.329 | 1950(14) | −0.096 | 0.089 | 0.732 | 0.305 | 0.579 | −0.248 | 0.091 | 0.65 | 0.327 |
| 0.347 | 1951(15) | 0.016 | 0.09 | 0.021 | 0.195 | 0.04 | −0.064 | 0.054 | −0.131 | 0.333 |
| 0.365 | 1952(16) | 0.128 | 0.083 | 0.131 | 0.39 | 0.434 | 0.079 | 0.109 | 0.175 | 0.062 |
| 0.382 | 1953(17) | −0.01 | 0.035 | 0.006 | −0.072 | −0.027 | 0.067 | 0.21 | −0.084 | −0.048 |
| 0.4 | 1945(18) | 0.154 | 0.176 | 0.908 | 0.715 | 0.469 | 0.077 | 0.112 | 0.756 | 0.185 |

son & Topeka & Santa Fa, Coca-Cola, Borden, Firestone and Sharon Steel securities data, respectively. In this example, $\Phi_h$ and $\Phi_l$ are given by

$$\Phi_h = (0.014, 0.014, 0.039, 0.041, 0.047, 0.01, 0.027, 0.043, 0.028)', \Phi_l = -\Phi_h.$$

In order to compare with Tanaka et al. [11], Zhang and Nie [12], we take $\mathbf{V}_\varepsilon = \mathbf{0}$ and the possibility grade $h_t$ is still defined by

$$h_{kij} = h_{ki} = h_k = 0.1 + 0.3(k-1)/17, \quad k = 1, \ldots, 18.$$

Using (3), we obtain $\hat{\mathbf{a}}, \hat{\mathbf{a}} + \Phi_h$ and $\hat{\mathbf{a}} + \Phi_l$ as follows:

$$\hat{\mathbf{a}} = (0.07099, 0.07012, 0.19645, 0.20610, 0.23390, 0.05317, 0.13619, 0.21603, 0.14386)',$$
$$\hat{\mathbf{a}} + \Phi_h = (0.08499, 0.08412, 0.23545, 0.24710, 0.28090, 0.06317, 0.16319, 0.25903, 0.17186)',$$
$$\hat{\mathbf{a}} + \Phi_l = (0.05699, 0.05612, 0.15745, 0.16510, 0.18690, 0.04317, 0.10919, 0.17303, 0.11586)'.$$

Using (4), we obtain the admissible covariance matrix $\mathbf{V}^*$ as follows:

$$\mathbf{V}^* = \widehat{\Sigma} = \begin{pmatrix} 0.04042 & 0.01593 & 0.01933 & 0.03277 & 0.01046 & 0.02665 & 0.0183 & 0.02816 & 0.02077 \\ 0.01593 & 0.01119 & 0.01701 & 0.01875 & 0.00771 & 0.00705 & 0.00912 & 0.02106 & 0.01518 \\ 0.01933 & 0.01701 & 0.09852 & 0.06287 & 0.04545 & 0.00694 & 0.0035 & 0.07752 & 0.03114 \\ 0.03277 & 0.01875 & 0.06287 & 0.07954 & 0.04465 & 0.02072 & 0.00904 & 0.07894 & 0.01964 \\ 0.01046 & 0.00771 & 0.04545 & 0.04465 & 0.09713 & 0.00585 & 0.01096 & 0.086 & 0.0285 \\ 0.02665 & 0.00705 & 0.00694 & 0.02072 & 0.00585 & 0.0405 & 0.00953 & 0.01952 & 0.01095 \\ 0.0183 & 0.00912 & 0.0035 & 0.00904 & 0.01096 & 0.00953 & 0.01986 & 0.01784 & 0.0079 \\ 0.02816 & 0.02106 & 0.07752 & 0.07894 & 0.086 & 0.01952 & 0.01784 & 0.13343 & 0.03455 \\ 0.02077 & 0.01518 & 0.03114 & 0.01964 & 0.0285 & 0.01095 & 0.0079 & 0.03455 & 0.06211 \end{pmatrix}.$$

Let the investor's initial capital be one hundred thousand USD, i.e. $w = \$100{,}000$. We assume that the interest rate of borrowing is 4% and the maximum amount that the investor is able to borrow money is one hundred thousand USD, i.e. $r^* = 4\%$ and $v = \$100{,}000$. Using algorithms for finding admissible efficient portfolios, we obtain the investment amount of the upper and lower admissible efficient portfolios shown in

Table 2
The investment amount of the upper admissible efficient portfolios when borrowing is allowed

| $\mu$($10^4$) | $x_2$($10^4$) | $x_3$($10^4$) | $x_4$($10^4$) | $x_5$($10^4$) | $x_6$($10^4$) | $x_7$($10^4$) | $x_9$($10^4$) |
|---|---|---|---|---|---|---|---|
| [0, 0.97] | 7.61 | 0 | 0 | 0.29 | 0.94 | 1.16 | 0 |
| [0.97, 1.24] | $14.81-7.43\mu$ | 0 | 0 | $-1.77+2.13\mu$ | $2.52-1.62\mu$ | $-5.56+6.93\mu$ | 0 |
| [1.24, 1.39] | $16.4-8.71\mu$ | $-2.85+2.30\mu$ | 0 | $0.31+0.45\mu$ | $1.65-0.92\mu$ | $-5.51+6.89\mu$ | 0 |
| [1.39, 1.58] | $17.55-9.54\mu$ | $-2.45+2.01\mu$ | 0 | $0.92+0.01\mu$ | $1.55-0.85\mu$ | $-5.27+6.72\mu$ | $-2.30+1.65\mu$ |
| [1.58, 1.74] | $16.51-8.88\mu$ | $0.05+0.43\mu$ | $-3.92+2.48\mu$ | $2.04-0.70\mu$ | $2.18-1.25\mu$ | $-4.10+5.98\mu$ | $-2.76+1.94\mu$ |
| [1.74, 1.84] | $19.02-10.33\mu$ | $-0.47+0.73\mu$ | $-3.57+2.28\mu$ | $1.97-0.66\mu$ | 0 | $-4.17+6.02\mu$ | $-2.78+1.96\mu$ |
| [1.84, 1.95] | 0 | $1.58-0.38\mu$ | $-5.65+3.41\mu$ | $-11.24+6.52\mu$ | 0 | $18.76-6.44\mu$ | $6.55-3.11\mu$ |
| [1.95, 3.65] | 0 | $0.54(\mu-0.4)$ | $0.64(\mu-0.4)$ | $0.95(\mu-0.4)$ | 0 | $4.00(\mu-0.4)$ | $0.31(\mu-0.4)$ |
| [3.65, 3.81] | 0 | $3.12-0.38\mu$ | $-9.99+3.41\mu$ | $-19.99+6.52\mu$ | 0 | $35.46-6.44\mu$ | $11.88-3.11\mu$ |
| [3.81, 4.77] | 0 | $7.70-1.61\mu$ | $-13.02+4.16\mu$ | $-19.99+6.52\mu$ | 0 | $45.56-9.07\mu$ | 0 |
| [4.77, 4.96] | 0 | 0 | $-7.38+3\mu$ | $-19.31+6.38\mu$ | 0 | $46.69-9.38\mu$ | 0 |
| [4.96, 5.22] | 0 | 0 | $154.38-29.58\mu$ | $-134.38+29.58\mu$ | 0 | 0 | 0 |

the following Tables 2 and 3. Correspondingly, we easily get the upper and lower admissible efficient frontiers shown in the following Tables 4 and 5. Moreover, Tanaka's efficient frontier in [11] is shown in the following Table 6. All efficient portfolios do not contain security 1 and security 8, i.e., $x_1 = x_8 = 0$.

Table 2, representing the upper admissible efficient portfolio under assumption that borrowing is allowed, shows that the investor does not need to borrow for $u \leqslant \$1.95\times 10^4$. He begins to borrow at $u = \$1.95\times 10^4$

Table 3
The investment amount of the lower admissible efficient portfolios when borrowing is allowed

| $\mu$($10^4$) | $x_2$($10^4$) | $x_3$($10^4$) | $x_4$($10^4$) | $x_5$($10^4$) | $x_6$($10^4$) | $x_7$($10^4$) | $x_9$($10^4$) |
|---|---|---|---|---|---|---|---|
| [0, 0.65] | 7.61 | 0 | 0 | 0.29 | 0.94 | 1.16 | 0 |
| [0.65, 0.83] | $14.93-11.26\mu$ | 0 | 0 | $-1.77+3.17\mu$ | $2.47-2.37\mu$ | $-5.63+10.45\mu$ | 0 |
| [0.83, 0.92] | $16.46-13.10\mu$ | $-2.87+3.46\mu$ | 0 | $0.33+0.64\mu$ | $1.61-1.33\mu$ | $-5.53+10.33\mu$ | 0 |
| [0.92, 1.05] | $17.57-14.31\mu$ | $-2.46+3.01\mu$ | 0 | $0.94-0.02\mu$ | $1.52-1.23\mu$ | $-5.27+10.05\mu$ | $-2.30+2.50\mu$ |
| [1.05, 1.18] | $16.50-13.28\mu$ | $0.02+0.64\mu$ | $-3.89+3.70\mu$ | $2.04-1.07\mu$ | $2.15-1.83\mu$ | $-4.10+8.93\mu$ | $-2.72+2.91\mu$ |
| [1.18, 1.23] | $18.93-15.35\mu$ | $-0.5+1.08\mu$ | $-3.58+3.44\mu$ | $1.96-1.00\mu$ | 0 | $-4.05+8.89\mu$ | $-2.76+2.94\mu$ |
| [1.23, 1.33] | 0 | $1.58-0.61\mu$ | $-5.86+5.30\mu$ | $-11.34+9.82\mu$ | 0 | $19.18-9.99\mu$ | $6.44-4.52\mu$ |
| [1.33, 2.33] | 0 | $0.85(\mu-0.4)$ | $1.22(\mu-0.4)$ | $1.78(\mu-0.4)$ | 0 | $6.46(\mu-0.4)$ | $0.45(\mu-0.4)$ |
| [2.33, 2.44] | 0 | $3.01-0.61\mu$ | $-9.67+5.30\mu$ | $-18.81+9.82\mu$ | 0 | $34.46-9.99\mu$ | $11.01-4.52\mu$ |
| [2.44, 3.07] | 0 | $7.34-2.39\mu$ | $-12.27+6.37\mu$ | $-18.69+9.77\mu$ | 0 | $43.62-13.75\mu$ | 0 |
| [3.07, 3.17] | 0 | 0 | $-6.98+4.64\mu$ | $-18.02+9.57\mu$ | 0 | $45.00-14.21\mu$ | 0 |
| [3.17, 3.34] | 0 | 0 | $153.13-45.87\mu$ | $-133.13+45.87\mu$ | 0 | 0 | 0 |

Table 4
The upper admissible efficient frontier

| Borrowing is not allowed | | Borrowing is allowed | |
|---|---|---|---|
| $\mu$($10^4$) | $\sigma$($10^4$) | $\mu$($10^4$) | $\sigma$($10^4$) |
| [0, 0.97] | 1.0461 | [0,0.97] | 1.0461 |
| [0.97, 1.24] | $(2.1475-2.2724\mu+1.1721\mu^2)^{1/2}$ | [0.97, 1.24] | $(2.1475-2.2724\mu+1.1721\mu^2)^{1/2}$ |
| [1.24, 1.39] | $(1.5145-1.2511\mu+0.7601\mu^2)^{1/2}$ | [1.24,1.39] | $(1.5145-1.2511\mu+0.7601\mu^2)^{1/2}$ |
| [1.39, 1.58] | $(1.2866-0.9230\mu+0.6420\mu^2)^{1/2}$ | [1.39,1.58] | $(1.2866-0.9230\mu+0.6420\mu^2)^{1/2}$ |
| [1.58, 1.74] | $(0.7338-0.2234\mu+0.4206\mu^2)^{1/2}$ | [1.58,1.74] | $(0.7338-0.2234\mu+0.4206\mu^2)^{1/2}$ |
| [1.74, 1.84] | $(0.9060-0.4208\mu+0.4772\mu^2)^{1/2}$ | [1.74,1.84] | $(0.9060-0.4208\mu+0.4772\mu^2)^{1/2}$ |
| [1.84, 2.11] | $(18.3634-19.3662\mu+5.6178\mu^2)^{1/2}$ | [1.84, 1.95] | $(18.3634-19.3662\mu+5.6178\mu^2)^{1/2}$ |
| [2.11, 2.58] | $(20.7034-21.5857\mu+6.1435\mu^2)^{1/2}$ | [1.95, 3.65] | $0.9(\mu-0.4)$ |
| [2.58, 2.68] | $(21.6567-22.3491\mu+6.2984\mu^2)^{1/2}$ | [3.65, 3.81] | $(58.84-34.23\mu+5.62\mu^2)^{1/2}$ |
| [2.68, 2.81] | $(524.1-397.2\mu+76.2\mu^2)^{1/2}$ | [3.81,4.77] | $(66.52-38.24\mu+6.14\mu^2)^{1/2}$ |
| | | [4.77, 4.96] | $(69.71-39.65\mu+6.29\mu^2)^{1/2}$ |
| | | [4.96,5.22] | $(1797.15-736.03\mu+76.47\mu^2)^{1/2}$ |

Table 5
The lower admissible efficient frontier

| Borrowing is not allowed | | Borrowing is allowed | |
|---|---|---|---|
| $\mu$($10^4$) | $\sigma$($10^4$) | $\mu$($10^4$) | $\sigma$($10^4$) |
| [0, 0.65] | 1.0461 | [0,0.65] | 1.0461 |
| [0.65, 0.83] | $(2.1534 - 3.4176\mu + 2.6375\mu^2)^{1/2}$ | [0.65, 0.83] | $(2.1534 - 3.4176\mu + 2.6375\mu^2)^{1/2}$ |
| [0.83, 0.92] | $(1.5121 - 1.8657\mu + 1.6982\mu^2)^{1/2}$ | [0.83,0.92] | $(1.5121 - 1.8657\mu + 1.6982\mu^2)^{1/2}$ |
| [0.92, 1.05] | $(1.2808 - 1.3632\mu + 1.4251\mu^2)^{1/2}$ | [0.92,1.05] | $(1.2808 - 1.3632\mu + 1.4251\mu^2)^{1/2}$ |
| [1.05, 1.18] | $(0.7324 - 0.3247\mu + 0.9335\mu^2)^{1/2}$ | [1.05,1.18] | $(0.7324 - 0.3247\mu + 0.9335\mu^2)^{1/2}$ |
| [1.18, 1.23] | $(0.9019 - 0.6115\mu + 1.0549\mu^2)^{1/2}$ | [1.18, 1.23] | $(0.9019 - 0.6115\mu + 1.0549\mu^2)^{1/2}$ |
| [1.23, 1.42] | $(18.9264 - 29.8940\mu + 12.9502\mu^2)^{1/2}$ | [1.23, 1.33] | $(18.9264 - 29.8940\mu + 12.9502\mu^2)^{1/2}$ |
| [1.42, 1.73] | $(21.0269 - 32.9829\mu + 14.0293\mu^2)^{1/2}$ | [1.33,2.33] | $1.5492(\mu - 0.4)$ |
| [1.73, 1.78] | $(22.1053 - 34.1316\mu + 14.3868\mu^2)^{1/2}$ | [2.33,2.44] | $(53.80 - 49.39\mu + 12.94\mu)^{1/2}$ |
| [1.78, 1.87] | $(587.5002 - 662.5012\mu + 190.5013\mu^2)^{1/2}$ | [2.44, 3.07] | $(60.37 - 54.75\mu + 14.04\mu^2)^{1/2}$ |
| | | [3.07, 3.17] | $(63.26 - 56.73\mu + 14.38\mu^2)^{1/2}$ |
| | | [3.17, 3.34] | $(1762.32 - 1128.72\mu + 183.46\mu^2)^{1/2}$ |

Table 6
Tanaka's efficient frontier

| $\mu$($10^4$) | $\sigma$($10^4$) |
|---|---|
| [0, 0.81] | 1.0461 |
| [0.81, 1.03] | $(2.1539 - 2.7295\mu + 1.6880\mu^2)^{1/2}$ |
| [1.03, 1.15] | $(1.5166 - 1.4981\mu + 1.0915\mu^2)^{1/2}$ |
| [1.15, 1.32] | $(1.2818 - 1.1010\mu + 0.9195\mu^2)^{1/2}$ |
| [1.32, 1.46] | $(0.7339 - 0.2647\mu + 0.6024\mu^2)^{1/2}$ |
| [1.46, 1.54] | $(0.9045 - 0.4989\mu + 0.6824\mu^2)^{1/2}$ |
| [1.54, 1.76] | $(18.5539 - 23.4925\mu + 8.1623\mu^2)^{1/2}$ |
| [1.76, 2.16] | $(20.9302 - 26.0775\mu + 8.8961\mu^2)^{1/2}$ |
| [2.16, 2.23] | $(21.8698 - 27.0284\mu + 9.1269\mu^2)^{1/2}$ |
| [2.23, 2.34] | $(561.5385 - 509.2308\mu + 117.2308\mu^2)^{1/2}$ |

and is able to do it until $u = \$3.65 \times 10^4$. Here the investor has to change the structure of the portfolio because the borrowing is over. Security 9 begins to be cancelled at $u = \$3.81 \times 10^4$ until $u = \$5.22 \times 10^4$ because $x_9 < 0$. Security 3 begins to be cancelled at $u = \$4.77 \times 10^4$ until $u = 5.22 \times 10^4$ because $x_3 < 0$. Security 7 begins to be cancelled at $u = \$4.96 \times 10^4$ until $u = \$5.22 \times 10^4$ because $x_7 < 0$. Finally all capital is invested at Security 5 and the maximum admissible expected return is $\$5.22 \times 10^4$.

Table 3, representing the lower admissible efficient portfolio under assumption that borrowing is allowed, shows that the investor does not need to borrow for $u \leqslant \$1.33 \times 10^4$. He begins to borrow at $u = \$1.33 \times 10^4$ and is able to do it until $u = \$2.33 \times 10^4$. Here the investor has to change the structure of the portfolio because the borrowing is over. Security 9 begins to be cancelled at $u = \$2.44 \times 10^4$ until $u = \$3.34 \times 10^4$ because $x_9 < 0$. Security 3 begins to be cancelled at $u = \$3.07 \times 10^4$ until $u = \$3.34 \times 10^4$ because $x_3 < 0$. Security 7 begins to be cancelled at $u = \$3.17 \times 10^4$ until $u = \$3.34 \times 10^4$ because $x_7 < 0$. Finally all capital is invested at Security 5 and the maximum admissible expected return is $\$3.34 \times 10^4$.

The upper and lower admissible efficient frontiers with borrowing are shown in Fig. 1. The curve $\widehat{AMP}$ represents the upper admissible efficient frontier and the curve $\widehat{CNQ}$ represents the lower admissible efficient frontier under the assumption that borrowing is allowed. The upper admissible efficient frontier is above the lower admissible efficient frontier. Clearly, the upper admissible efficient frontier with borrowing shifts upward from point $M$ ($u = \$1.95 \times 10^4$) to point $P$ ($u = \$5.22 \times 10^4$), and the lower admissible efficient frontier with borrowing shifts upward from point $N$ ($u = \$1.33 \times 10^4$) to point $Q$ ($u = \$3.34 \times 10^4$).

Tanaka's efficient frontier, the upper and lower admissible efficient frontiers without borrowing are shown in Fig. 2. The curve $\widehat{AE}$ represents the upper admissible efficient frontier and the curve $\widehat{CG}$ represents the lower admissible efficient frontier under the assumption that borrowing is not allowed. The curve $\widehat{BF}$ represents

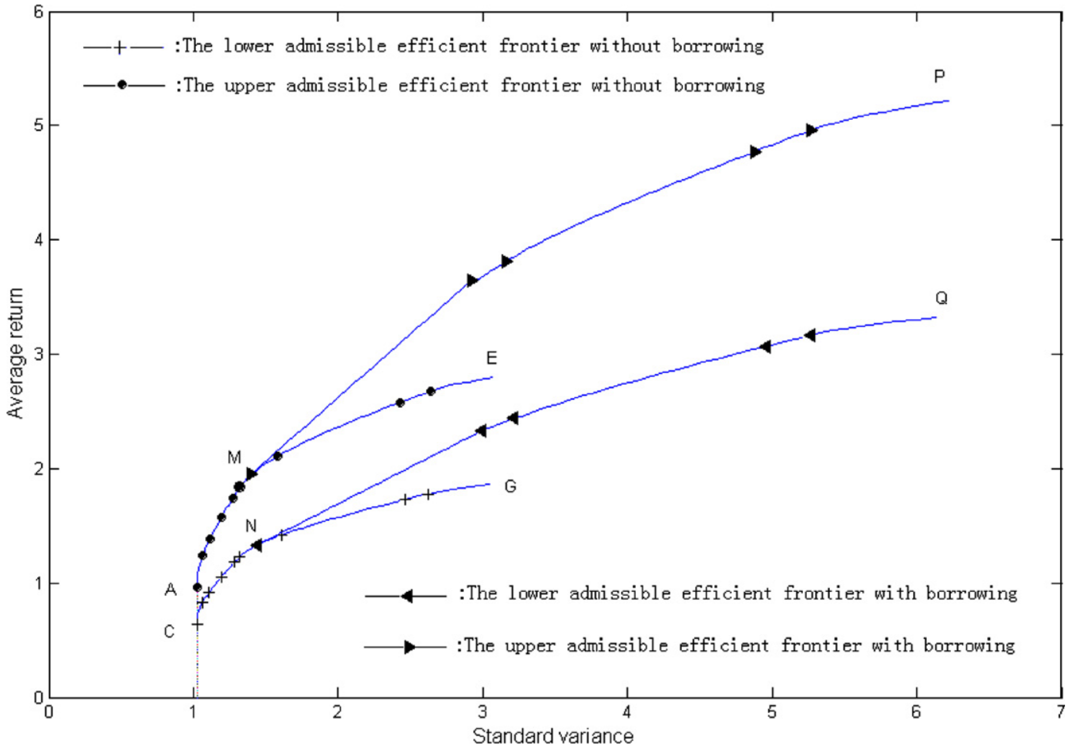


Fig. 1. The admissible efficient frontier with an upper bound on borrowing.

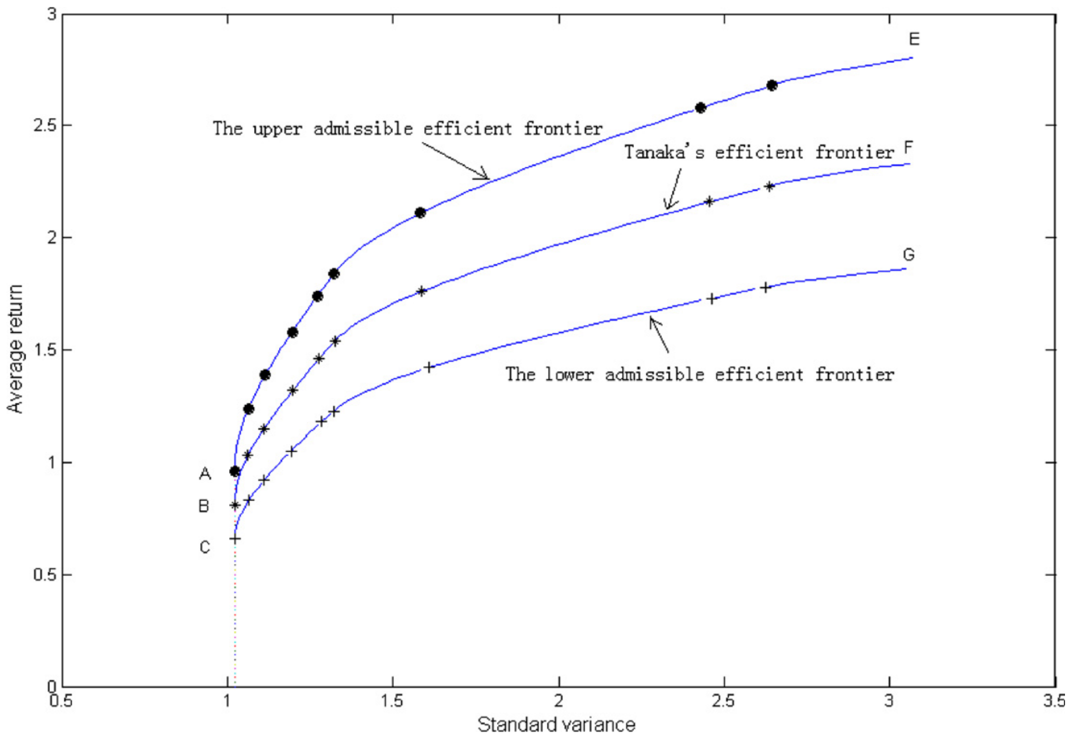


Fig. 2. The admissible efficient frontier without borrowing.

Tanaka's efficient frontier. The upper admissible efficient frontier is above the lower admissible efficient frontier. Tanaka's efficient frontier is middle of the upper and lower admissible efficient frontiers.

Comparing Fig. 1 with Fig. 2, it is found that the return of the upper admissible efficient portfolios with borrowing is bigger than the ones without borrowing when $u \geqslant \$1.95 \times 10^4$ and the return of the lower admissible efficient portfolios with borrowing is bigger than the ones without borrowing when $u \geqslant \$1.33 \times 10^4$. By borrowing, the maximum return of the upper admissible efficient portfolios is increased from $u = \$2.81 \times 10^4$ to $u = \$5.22 \times 10^4$ and the maximum return of the lower admissible efficient portfolios is increased from $u = \$1.87 \times 10^4$ to $u = \$3.34 \times 10^4$.

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