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Multivariate Subordination, Selfdecomposability and Stability

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Abstract

Multivariate subordinators are multivariate Lévy processes that are increasing in each component. Various examples of multivariate subordinators, of interest for applications, are given. Subordination of Lévy processes with independent components by multivariate subordinators is defined. Multiparameter Lévy processes and their subordination are introduced so that the subordinated processes are multivariate Lévy processes. The relations between the characteristic triplets involved are established. It is shown that operator selfdecomposability and the operator version of the class L_m property are inherited from the multivariate subordinator to the subordinated process under the condition of operator stability of the subordinand.

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1 Introduction

Subordination of a Lévy process $\{X(t)\}$ by a positive Lévy process, or subordinator, $\{T(t)\}$ is an extensively studied area, see in particular Bertoin (1996,1999) and Sato (1999). From the viewpoint of theory as well as applications it is of interest to extend the study to multivariate subordination, i.e.

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subordination of multivariate Lévy processes $\{X(t)\}$ by multivariate subordinators $\{T(t)\}$, i.e. multivariate positive (in the sense of positivity of each coordinate) Lévy processes. In the present paper we take up a systematic investigation of multivariate subordination.

The subordination of a one-dimensional Lévy process $\{X(t)\}$ by a one-dimensional increasing process, or subordinator, $\{T(t)\}$ means introducing a new process $\{Y(t)\}$ defined by composition as $Y(t) = X(T(t))$, where $\{X(t)\}$ and $\{T(t)\}$ are assumed to be independent. To go to higher dimensions one may keep a one-dimensional subordinator $\{T(t)\}$ and let $\{X(t)\} = \{(X_1(t), \ldots, X_d(t))^\top\}$ be a d-dimensional Lévy process with the superscript $\top$ denoting the transpose, again defining the subordinated process by composition

$$Y(t) = X(T(t)) = \big(X_1(T(t)), \ldots, X_d(T(t))\big)^\top. \tag{1.1}$$

Alternatively, we may take $\{T(t)\}$ as a d-dimensional subordinator, in the sense of being a d-dimensional Lévy process $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ that is increasing in each coordinate, and, letting $\{X(t)\} = \{(X_1(t), \ldots, X_d(t))^\top\}$ consist of d independent Lévy processes $\{X_1(t)\}, \ldots, \{X_d(t)\}$ independent of $\{T(t)\}$, introduce a new process $\{Y(t)\} = \{(Y_1(t), \ldots, Y_d(t))^\top\}$ by

$$Y(t) = \big(X_1(T_1(t)), \ldots, X_d(T_d(t))\big)^\top \quad \text{(a process on } \mathbf{R}^d\text{)}. \tag{1.2}$$

In both cases (1.1) and (1.2), $\{Y(t)\}$ is itself a Lévy process. The introduction of $\{Y(t)\}$ by (1.1) is called subordination of a multivariate Lévy process. It is extensively studied in Chapter 6 of Sato (1999). The present paper concerns the extension of the concept of subordination in the direction that induces the type (1.2)[1].

More generally, we consider $\{X(t)\} = \{(X_1(t), \ldots, X_d(t))^\top\}$, an n-dimensional process with d independent components, as a subordinand. The jth component is n_j-dimensional; thus $n = n_1 + \ldots + n_d$. Assume that each $\{X_j(t)\}$ is a Lévy process on $\mathbf{R}^{n_j}$. Given a d-dimensional subordinator (in the sense above) $\{T(t)\}$ which is independent of $\{X(t)\}$, define the subordinated process $\{Y(t)\}$ by the formula

$$Y(t) = \big(X_1(T_1(t)), \ldots, X_d(T_d(t))\big)^\top \quad \text{(a process on } \mathbf{R}^n\text{)}. \tag{1.3}$$

The types (1.1) and (1.2) are both special cases. We call the procedure to introduce $\{Y(t)\}$ in (1.3) *multivariate subordination.*

[1]To the best of our knowledge, this type has not been considered previously in the literature.

From one point of view it appears more natural to consider $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$, an $\mathbf{R}^n$-valued process with multiparameter $\mathbf{s} \in \mathbf{R}_+^d$ and extend the concept of subordination to substitution of $\mathbf{s}$ by a d-dimensional subordinator $\{T(t)\}$ independent of $\{X(\mathbf{s})\}$. Here we do not assume $n \geq d$. In order to guarantee the resulting process $Y(t) = X(T(t))$ to be a Lévy process we assume $\{X(\mathbf{s})\}$ to be a multiparameter Lévy process, a notion which we will define in Section 4. But this extension of the concept of subordination is, essentially, not more general than our multivariate subordination (1.3) with n replaced by nd, as we will show in that section.

In Section 2 we discuss a number of ways to construct multivariate subordinators, by superposition or by subordination, and illustrate these by examples. Section 3 establishes the relations between the various characteristic triplets involved in multivariate subordination, and Section 4 extends the perspective to multiparameter Lévy processes.

Once we establish the concept of multivariate subordination of $\{X(t)\}$ by $\{T(t)\}$, a general question is what properties of $\{X(t)\}$ and $\{T(t)\}$ give certain desirable properties of $\{Y(t)\}$. The second half of the paper takes up this question in relation to selfdecomposability and stability. Our objects are multidimensional and we are led to the area where it is natural to employ the notions of operator selfdecomposability and operator stability, which have been studied during the last three decades. In Section 5 we recall a number of results concerning these concepts as well as the operator extension $L_m(Q)$ of the subclasses L_m of the class L of selfdecomposable laws. A main result of Section 6 is that strict operator stability of the subordinand $\{X(t)\}$ coupled with operator selfdecomposability of the subordinator $\{T(t)\}$ gives operator seldecomposability of the subordinated process $\{Y(t)\}$. An extension of this to the class $L_m(Q)$ property is also made. The prototype result is that subordination of the Brownian motion by selfdecomposable subordinators gives selfdecomposable processes. This is connected with distributions of type G, and we will formulate a special case of our result as a generalization of the concept of type G.

2 Multivariate subordinators: Examples

First we introduce some notation. Throughout the paper d denotes a positive integer. Elements of $\mathbf{R}^d$ are considered as column vectors. The distribution of a random vector is denoted by $\mathcal{L}(X)$. For two vectors X and Y, $X \stackrel{d}{=} Y$ means $\mathcal{L}(X) = \mathcal{L}(Y)$. The characteristic function of a distribution μ on $\mathbf{R}^d$ is denoted by $\widehat{\mu}(z)$ for $z \in \mathbf{R}^d$. When $z = (z_1, \ldots, z_d)^\top$ we also write $\widehat{\mu}(z_1, \ldots, z_d)$ as an alternative to $\widehat{\mu}((z_1, \ldots, z_d)^\top)$. Finally, we often need to stack vectors. Hence, when $x_1, \ldots, x_d$ are vectors with $x_j \in \mathbf{R}^{n_j}$ for $j = 1, \ldots, d$, we use the notation $(x_1, \ldots, x_d)^\top$ to denote the stacked vector

$$(x_1, \ldots, x_d)^\top = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}. \tag{2.1}$$

Notice that $(x_1, \ldots, x_d)^\top \in \mathbf{R}^n$ with $n = n_1 + \ldots + n_d$.

There are various general ways of constructing interesting examples of multivariate subordinators $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$, such as by superposition or through subordination of subordinators.[2] Often, in applications, the character of an example may be usefully described in terms of an underlying graph.

2.1 Construction via superposition

Let $\mathbf{T}$ denote a rooted tree with d leaves and let each edge ε of $\mathbf{T}$ be equipped with a subordinator U^ε, these subordinators being independent. Then define $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ by

$$T_i(t) = \sum_{\varepsilon \in b_i} U^\varepsilon(t) \tag{2.2}$$

for $i = 1, \ldots, d$, the summation being over the edges that belong to the branch b_i ending in leaf i.

Examples of some particular interest in mathematical finance occur for the U^ε being all gamma distributed with common scale parameter α or all inverse Gaussian with common drift parameter γ. In the setting of mathematical finance, for a group of stocks one may, for instance, think of a tree of height 2 with the root edge process determining a volatility common to all stocks and the U^ε processes at height 1 of the tree as corresponding to volatility characteristic of type of firm, while the U^ε at level 2 are individual to the single stocks.

Next, let $\mathbf{G}$ be a finite directed and acyclic graph with d terminal vertices (an terminal vertex of such a graph is a vertex from which no arrows emanate). Associate to each vertex v a subordinator U^v, let $U = \{U^v : v \in V\}$ where V denotes the set of vertices of $\mathbf{G}$ and assume that the U^v are independent. We may then define a d-dimensional subordinator $\{T(t)\} =$

[2] Yet another approach, that we do not pursue here, consists in defining $\{T(t)\}$ by choosing the Lévy measure.

$\{(T_1(t), \ldots, T_d(t))^\top\}$ by

$$T_i(t) = \sum_{v \in V_i} U^v(t), \tag{2.3}$$

where V_i denotes the set of all vertices v of $\mathbf{G}$ for which there is some route in the graph containing v and leading from some initial vertex to the ith terminal vertex. (We return to this type of graph in the next subsection.)

Still a third way is to associate with each subset a of $\{1, ..., d\}$ a subordinator $\{U^a(t)\}$ with the $\{U^a(t)\}$ being independent and then to define $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ by letting

$$T_i(t) = U^\emptyset(t) + \sum_{a \ni i} U^a(t). \tag{2.4}$$

Example 2.1 *Multivariate negative binomial subordinator.* Let $\kappa > 0$ and $0 < \pi < 1$. The negative binomial distribution $b^-(\kappa, \pi)$ with point probabilities

$$(1-\pi)^\kappa \binom{x+\kappa-1}{x} \pi^x \tag{2.5}$$

($x = 0, 1, ...$) is infinitely divisible and hence generates a *negative binomial Lévy process* B^- such that $B^-(t)$ has law (2.5) with $\kappa = t$. With notation as in (2.4), suppose that $U^\emptyset(t) = t$ while $U^a(t) \overset{d}{=} B^-(\rho^a t)$ where the ρ^a are positive and satisfy

$$\sum_{a \ni i} \rho^a = 1$$

for $i = 1, ..., d$. Then $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ is, what we shall call, a *multivariate negative binomial subordinator* such that, for each i,

$$T_i(t) \overset{d}{=} t + B^-(t).$$

We take up this example again in the next subsection. $\square$

2.2 Construction via subordination

By $\Gamma(\lambda, \alpha)$ we denote the gamma distribution with probability density function

$$\frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha x} \tag{2.6}$$

Example 2.2 *Multivariate gamma subordinators.* Let $\{Z_1(t)\}, \ldots, \{Z_d(t)\}$ be independent gamma Lévy processes with $Z_i(t)$ having law $\Gamma(\lambda t, \alpha)$, and define $\{T_i(t)\}$ by

$$T_i(t) = Z_i(t + \lambda^{-1} B^-(\lambda t)),$$

where $\{B^-(t)\}$ is a negative binomial Lévy process, as defined in Example 2.1, and is assumed to be independent of $\{Z_1(t)\}, \ldots, \{Z_d(t)\}$. Then $\{T_i(t)\}$ is again a gamma process, with $\mathcal{L}(T_i(t)) = \Gamma(\lambda t, (1-\pi)\alpha)$, and $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ is a d-dimensional *gamma subordinator.* In the two-dimensional case, the probability density of $\{T(t)\} = \{(T_1(t), T_2(t))^\top\}$ can be expressed for $x_1, x_2 > 0$ as

$$\begin{aligned} f_t(x_1, x_2) &= (1-\pi)^{\lambda t} \alpha^{2\lambda t} (x_1 x_2)^{\lambda t - 1} e^{-\alpha(x_1 + x_2)} \frac{1}{\Gamma(\lambda t)} \sum_{i=0}^{\infty} \frac{(\alpha^2 \pi x_1 x_2)^i}{i! \Gamma(\lambda t + i)} \\ &= (1-\pi)^{\lambda t} \pi^{(-\lambda t + 1)/2} \alpha^{\lambda t + 1} \frac{1}{\Gamma(\lambda t)} \\ &\times (x_1 x_2)^{(\lambda t - 1)/2} e^{-\alpha(x_1 + x_2)} I_{\lambda t - 1}(2\alpha \sqrt{\pi x_1 x_2}), \end{aligned}$$

where $I_{\lambda t-1}$ is the modified Bessel function of the first kind with order $\lambda t - 1$. The fact that this density is infinitely divisible was shown by Vere-Jones (1967), see also Moran and Vere-Jones (1969), Griffiths (1984) and Barndorff-Nielsen (1980).

There is here a notable relation to selfdecomposability. For $d = 1$ we have $T(t) \stackrel{d}{=} Z(t) + Z'(B^-(t))$ where $\{Z'(t)\}$ denotes an independent copy of $\{Z(t)\}$. Furthermore, assuming $\lambda = 1$, the fact that $\mathcal{L}(T(t)) = \Gamma(t, (1-\pi)\alpha)$ and $\mathcal{L}(Z(t)) = \Gamma(t, \alpha)$ implies $T(t) \stackrel{d}{=} (1-\pi)^{-1} Z(t)$ and since the gamma distribution is selfdecomposable one sees that the law of $Z'(B^-(t))$ is that of an innovation term in a gamma OU process (process of Ornstein-Uhlenbeck type).

Next, consider the d-dimensional *negative multinomial Lévy process* M^- defined by letting $M^-(1)$ having the negative multinomial law $m^-(\kappa, \pi)$ with point probabilities

$$(2.7) \qquad (1-\pi_+)^\kappa \frac{\Gamma(x_+ + \kappa)}{x_1! \cdots x_d! \Gamma(\kappa)} \pi_1^{x_1} \cdots \pi_d^{x_d},$$

where $x_1, \ldots, x_d$ are nonnegative integers, $x_+ = x_1 + ... + x_d$ and where π is the d-dimensional probability parameter $\pi = (\pi_1, ..., \pi_d)^\top$ satisfying $\pi_1 > 0, ..., \pi_d > 0$ and $\pi_+ = \pi_1 + ... + \pi_d < 1$. We recall that the process M^- may be considered as a subordinated process, of d independent Poisson processes by a $\Gamma(\kappa, \alpha)$ Lévy process.

Letting again $\{Z_1(t)\}, \dots, \{Z_d(t)\}$ be independent gamma Lévy processes as above and defining $\{T(t)\} = \{(T_1(t), \dots, T_d(t))^\top\}$ by

$$T_i(t) = Z_i(t + M_i^-(t))$$

with $M_i^-(t)$ being the ith coordinate of $M^-(t)$, we obtain another type of d-dimensional gamma subordinator.

A third type is obtained by subordination of the independent gamma Lévy processes $\{Z_1(t)\}, \dots, \{Z_d(t)\}$ by the multivariate negative binomial Lévy process discussed in Example 2.1, through letting

$$T_i(t) = Z_i(B_i^-(t)),$$

where now B_i^- stands for the ith component of the multivariate negative binomial Lévy process. This type of construction introduces a useful correlation structure among the components of $(T_1(t), \dots, T_d(t))^\top$. $\square$

Consider again a finite directed and acyclic graph $\mathbf{G}$ with subordinators associated to the vertices of $\mathbf{G}$, as in the previous subsection. But now build the multivariate subordinator T as follows. From the given set of independent subordinators $U = \{U^v : v \in V\}$ define a new set $T = \{T^v : v \in V\}$, by recursive specification such that for any $v \in V$ the process T^v is subordinated by the sum of all T^u with $u \prec v$. More specifically, define the *order* of a vertex v to be the length of the longest path in $\mathbf{G}$ leading from an initial vertex to v. For a vertex of order 0 we take $T^v = U^v$; for vertices of order 1 we let T^v be the subordination of U^v by $U^{v'}$ where v' denotes the initial vertex associated with v; for vertices of order 2, T^v is the subordination of U^v by $\sum_{w \prec v} T^w$; etc.

For use in the next example, recall that the inverse Gaussian distribution with parameters δ and γ has probability density function

$$h(x; \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} e^{-(\delta^2 x^{-1} + \gamma^2 x)/2}. \tag{2.8}$$

Example 2.3 *Inverse Gaussian subordinator with graph* $\mathbf{G}$. Define the law of U^v, $v \in V$, to be inverse Gaussian with parameters δ_v and γ_v. Then the joint law of the T^v at time 1 has probability density

$$p(x; \delta, \gamma) = \prod_{v \in V} h(x^v; x^{[v]}\delta_v, \gamma_v), \tag{2.9}$$

where $x = \{x^v : v \in V\}$, $\delta = \{\delta_v : v \in V\}$, $\gamma = \{\gamma_v : v \in V\}$ and

$$x^{[v]} = \sum_{u \prec v} x^u.$$

The model (2.9) has a number of important statistical properties, see Barndorff-Nielsen and Blæsild (1988). (Another example from that paper would lead to a *gamma - inverse Gaussian subordinator with graph* $\mathbf{G}$.) $\square$

3 Multivariate subordination

In this section we consider multivariate subordination of $\{X(t)\}$ by $\{T(t)\}$. In subsection 3.3 we determine the characteristic triplet of the subordinated process, but first we state the Lévy-Khintchine formula for the subordinator $\{T(t)\}$ in 3.1 and we note some elementary properties of the subordinand $\{X(t)\}$ in 3.2.

3.1 Characterization of multivariate subordinators

Let $\{T(t)\}$ be a stochastic process on $\mathbf{R}^d$. Let the coordinate processes of $\{T(t)\}$ be $\{T_1(t)\}, \ldots, \{T_d(t)\}$. We say that $\{T(t)\}$ is a multivariate subordinator (or a d-dimensional subordinator) if $\{T(t)\}$ is a Lévy process on $\mathbf{R}_+^d = [0, \infty)^d$. Thus, trajectories of multivariate subordinators are increasing in each coordinate.

Skorohod (1991, Theorem 3.21) discusses Lévy processes on cones. Since $\mathbf{R}_+^d$ is a particular example of a cone in $\mathbf{R}^d$ we get from his result the Lévy-Khintchine formula for multivariate subordinators.

Proposition 3.1 (i) *Let $\{T(t)\}$ be a d-dimensional subordinator, and let $\lambda = \mathcal{L}(T(1))$. The characteristic function $\widehat{\lambda}$ can be represented as*

$$\widehat{\lambda}(z) = \exp\Big[\int_{\mathbf{R}_+^d} (e^{\mathrm{i}\langle z,\mathbf{s}\rangle} - 1)\,\rho(\mathrm{d}\mathbf{s}) + \mathrm{i}\langle z, c\rangle\Big], \quad z \in \mathbf{R}^d, \tag{3.1}$$

where $c \in \mathbf{R}_+^d$ and ρ is a σ-finite measure on $\mathbf{R}^d$ which is concentrated on $\mathbf{R}_+^d \backslash \{0\}$ and satisfies $\int_{\mathbf{R}_+^d} |\mathbf{s}| \wedge 1\,\rho(\mathrm{d}\mathbf{s}) < \infty$.
(ii) *Conversely, let $c \in \mathbf{R}_+^d$ and ρ be a σ-finite measure on $\mathbf{R}^d$ which is concentrated on $\mathbf{R}_+^d \backslash \{0\}$ and satisfies $\int_{\mathbf{R}_+^d} |\mathbf{s}| \wedge 1\,\rho(\mathrm{d}\mathbf{s}) < \infty$. Then there exists a d-dimensional subordinator $\{T(t)\}$ such that $\mathcal{L}(T(1)) = \lambda$ with $\widehat{\lambda}$ satisfying* (3.1).

The measure ρ appearing in (3.1) is the Lévy measure and c is the drift of $\{T(t)\}$.

In Section 2 we have described general ways of construction of interesting multivariate subordinators. Already in the Wiener-Hopf factorization theory of one-dimensional Lévy processes, special bivariate subordinators play an important role. Namely, given a general Lévy process $\{X(t)\}$ on $\mathbf{R}$ let $S(t) = \sup_{0\le s\le t} X(s)$, the supremum process, and $R(t) = S(t) - X(t)$, the reflected process. The latter is a Markov process on $\mathbf{R}_+$ having a local time $\{L(t)\}$ at 0. Assume that $L(\infty) = \infty$ a.s. The right-continuous inverse, $\{L^{-1}(t)\}$, of the local time is called the ladder process. Let $H(t) = S(L^{-1}(t))$. Then the pair $\{(L^{-1}(t), H(t))^\top\}$ is a bivariate subordinator, known as the ladder process. The expression for the bivariate Laplace transform of its distribution is called Fristedt's formula. See Bertoin (1996) pp. 157 and 165 for details.

3.2 Lévy processes with independent components

Consider d independent processes $\{X_1(t)\}, \dots, \{X_d(t)\}$ such that $\{X_j(t)\}$ is a Lévy process on $\mathbf{R}^{n_j}$ for $j = 1, \dots, d$. The stacked process $\{X(t)\}$ defined by $X(t) = (X_1(t), \dots, X_d(t))^\top$ is then a Lévy process on $\mathbf{R}^n$, where $n = n_1 + \dots + n_d$. We call $X_1(t), \dots, X_d(t)$ the components of $X(t)$. Denote the coordinates by $X_{jk}(t)$, which means that $X_j(t) = \big(X_{j1}(t), \dots, X_{jn_j}(t)\big)^\top$. Consider a multiparameter time $\mathbf{s} = (s_1, \dots, s_d)^\top \in \mathbf{R}^d_+$ and define a multiparameter process by $X(\mathbf{s}) = (X_1(s_1), \dots, X_d(s_d))^\top$. We also use the notation $X(s_1, \dots, s_d)$ for the variable $X((s_1, \dots, s_d)^\top)$. Introduce the partial ordering $\preceq$ of $\mathbf{R}^d_+$ where $\mathbf{s}^1 = (s^1_1, \dots, s^1_d)^\top$ and $\mathbf{s}^2 = (s^2_1, \dots, s^2_d)^\top$ are ordered by

$$\mathbf{s}^1 \preceq \mathbf{s}^2 \;\Leftrightarrow\; s^1_j \le s^2_j, \quad \text{for all } j = 1, \dots, d.$$

We have the following properties of the process $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}^d_+\}$.

Lemma 3.2 *Let $\mathbf{s}^k = (s^k_1, \dots s^k_d)^\top \in \mathbf{R}^d_+,\ k = 1, \dots, m,$ with $\mathbf{s}^1 \preceq \dots \preceq \mathbf{s}^m$.*
(i) *For any bounded measurable functions $f^k : \mathbf{R}^n \to \mathbf{R},\ k = 1, ..., m-1$, we have*

$$E\Big[\prod_{k=1}^{m-1} f^k\big(X(\mathbf{s}^{k+1}) - X(\mathbf{s}^k)\big)\Big] = \prod_{k=1}^{m-1} E\Big[f^k\big(X(\mathbf{s}^{k+1} - \mathbf{s}^k)\big)\Big], \tag{3.2}$$

where $\mathbf{s}^{k+1} - \mathbf{s}^k = (s^{k+1}_1 - s^k_1, \dots, s^{k+1}_d - s^k_d)^\top \in \mathbf{R}^d_+$.
(ii) *The variables $X(\mathbf{s}^1), X(\mathbf{s}^2) - X(\mathbf{s}^1), \dots, X(\mathbf{s}^m) - X(\mathbf{s}^{m-1})$ are independent.*
(iii) *For $k = 1, \dots, m-1$, it holds that $\mathcal{L}(X(\mathbf{s}^{k+1}) - X(\mathbf{s}^k)) = \mathcal{L}(X(\mathbf{s}^{k+1} - \mathbf{s}^k))$.*

Proof. Note that (ii) and (iii) follow from (i) so we need only prove the first part. Moreover, it is enough to prove the equality (3.2) in the case where

$$f^k(x_1, \dots, x_d) = f^k_1(x_1) \cdots f^k_d(x_d), \quad k = 1, \dots, m-1,$$

with $f_j^k : \mathbf{R}^{n_j} \to \mathbf{R}$ bounded and measurable. The left-hand side of (3.2) is then equal to

$$
\begin{aligned}
& E\Big[\prod_{k=1}^{m-1}\prod_{j=1}^{d} f_j^k\big(X_j(s_j^{k+1}) - X_j(s_j^k)\big)\Big] = \prod_{j=1}^{d} E\Big[\prod_{k=1}^{m-1} f_j^k\big(X(s_j^{k+1}) - X(s_j^k)\big)\Big] \\
= \;& \prod_{j=1}^{d}\prod_{k=1}^{m-1} E\Big[f_j^k\big(X_j(s_j^{k+1}) - X_j(s_j^k)\big)\Big] = \prod_{j=1}^{d}\prod_{k=1}^{m-1} E\Big[f_j^k\big(X_j(s_j^{k+1} - s_j^k)\big)\Big] \\
= \;& \prod_{k=1}^{m-1} E\Big[\prod_{j=1}^{d} f_j^k\big(X_j(s_j^{k+1} - s_j^k)\big)\Big],
\end{aligned}
$$

which is the right-hand side of (3.2). Here we have used independence of the components and independence and stationarity of the increments of each component process. $\square$

Let $\mu_j = \mathcal{L}(X_j(1))$, $\mu = \mathcal{L}(X(1))$, and for $\mathbf{s} = (s_1, \ldots, s_d)^\top \in \mathbf{R}_+^d$ let $\mu^{\mathbf{s}} = \mathcal{L}(X(\mathbf{s}))$. For $z = (z_1, \ldots, z_d)^\top \in \mathbf{R}^n$ with $z_j \in \mathbf{R}^{n_j}$ we hence have the following relations between the characteristic functions:

$$
\begin{aligned}
&(3.3) & \widehat{\mu}(z) &= \widehat{\mu}_1(z_1)\widehat{\mu}_2(z_2)\cdots\widehat{\mu}_d(z_d), \\
&(3.4) & \widehat{\mu}^{\mathbf{s}}(z) &= \widehat{\mu}_1(z_1)^{s_1}\widehat{\mu}_2(z_2)^{s_2}\cdots\widehat{\mu}_d(z_d)^{s_d}, \\
&(3.5) & \log\widehat{\mu}^{\mathbf{s}}(z) &= \sum_{j=1}^{d} s_j \log\widehat{\mu}_j(z_j) = \langle \operatorname{Log}\widehat{\mu}(z), \mathbf{s}\rangle,
\end{aligned}
$$

where

$$
(3.6) \qquad \operatorname{Log}\widehat{\mu}(z) = \big(\log\widehat{\mu}_1(z_1), \ldots, \log\widehat{\mu}_d(z_d)\big)^\top.
$$

Denote the characteristic triplet of $\{X_j(t)\}$, $j = 1, \ldots, d$, by $(\Sigma_j, \nu_j, \gamma_j)$. This means that Σ_j (Gaussian covariance matrix) is a nonnegative definite $n_j \times n_j$ matrix, ν_j (Lévy measure) is a measure on $\mathbf{R}^{n_j}$ satisfying $\nu_j(\{0\}) = 0$ and $\int_{\mathbf{R}^{n_j}} |x_j|^2 \wedge 1\, \nu_j(\mathrm{d}x_j) < \infty$, and $\gamma_j \in \mathbf{R}^{n_j}$, and the characteristic function of μ_j is

$$
\widehat{\mu}_j(z_j) = \exp\Big[-\frac{1}{2}\langle\Sigma_j z_j, z_j\rangle + \int_{\mathbf{R}^{n_j}} \big(e^{\mathrm{i}\langle z_j, x_j\rangle} - 1 - \mathrm{i}\langle z_j, x_j\rangle 1_{\{|x_j|\le 1\}}\big)\, \nu_j(\mathrm{d}x_j) + \mathrm{i}\langle\gamma_j, z_j\rangle\Big],
$$

for $z_j \in \mathbf{R}^{n_j}$. By using (3.3) we find that the characteristic triplet of $\{X(t)\}$ is

$$\Sigma = \mathrm{diag}(\Sigma_1, \ldots, \Sigma_d), \tag{3.7}$$
$$\nu(\mathrm{d}x) = 1_{\mathcal{A}_1}(x)\nu_1(\mathrm{d}x_1) + \ldots + 1_{\mathcal{A}_d}(x)\,\nu_d(\mathrm{d}x_d), \tag{3.8}$$
$$\gamma = (\gamma_1, \ldots, \gamma_d)^\top, \tag{3.9}$$

where $\mathrm{diag}(\Sigma_1, \ldots, \Sigma_d)$ denotes the $n \times n$ matrix with diagonal blocks as indicated and all other blocks zero, and for $j = 1, .., d$ we let

$$\mathcal{A}_j = \{\ x = (x_1, \ldots, x_d)^\top \in \mathbf{R}^n \ : x_k \in \mathbf{R}^{n_k}, k = 1, \ldots, d, \text{ and } x_k = 0 \text{ for } k \neq j\}. \tag{3.10}$$

3.3 Characteristic triplet of the subordinated process

Let $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ be a d-dimensional subordinator with $\lambda^t = \mathcal{L}(T(t))$ as considered in Subsection 3.1. Let $\{X(t)\} = \{(X_1(t), \ldots, X_d(t))^\top\}$ and $\{X(\mathbf{s})\}$ be the processes considered in Subsection 3.2. We use all the notation from these subsections. Assume that $\{X(t)\}$ and $\{T(t)\}$ are independent. Define a process $\{Y(t)\}$ on $\mathbf{R}^n$ by replacing the parameter $\mathbf{s}$ in $X(\mathbf{s})$ by $T(t)$, i.e.

$$Y(t) = \big(X_1(T_1(t)), \ldots, X_d(T_d(t))\big)^\top, \quad t \geq 0, \tag{3.11}$$

where $X_j(T_j(t))$ is the variable on $\mathbf{R}^{n_j}$ given by

$$X_j(T_j(t)) = \big(X_{j1}(T_j(t)), \ldots, X_{jn_j}(T_j(t))\big)^\top. \tag{3.12}$$

We say that $\{Y(t)\}$ appears by *multivariate subordination of* $\{X(t)\}$ *by* $\{T(t)\}$.

Of particular interest is the case $n = d$ and $n_j = 1$ for each j. We then have that each $\{X_j(t)\}$ is a Lévy process on $\mathbf{R}$. Moreover, Σ_j is a nonnegative number, $\gamma_j \in \mathbf{R}$ and ν_j is a measure on $\mathbf{R}$. The set $\mathcal{A}_j$ in (3.10) is the jth coordinate axis on $\mathbf{R}^d$. Another case is when $d = 1$ and $n = n_1$. Then $\{X(t)\} = \{X_1(t)\}$ and $\{T(t)\} = \{T_1(t)\}$. Thus, $\{Y(t)\}$ appears by usual (univariate) subordination of an n_1-dimensional Lévy process as discussed in Theorem 30.1 in Sato (1999).

In the general case with $\{Y(t)\}$ appearing by multivariate subordination of $\{X(t)\}$ by $\{T(t)\}$ we have the following analogue to Theorem 30.1 in Sato (1999), where we recall that $\mathrm{Log}\,\widehat{\mu}$ is defined by (3.6) and that $\mu = \mathcal{L}(X(1))$, $\mu^{\mathbf{s}} = \mathcal{L}(X(\mathbf{s}))$.

Theorem 3.3 *The process $\{Y(t)\}$ is a Lévy process on $\mathbf{R}^n$ and*

$$P(Y(t) \in B) = \int_{\mathbf{R}_+^n} \mu^{\mathbf{s}}(B)\, \lambda^t(\mathrm{d}\mathbf{s}), \quad B \in \mathcal{B}(\mathbf{R}^n), \tag{3.13}$$

$$E\big[e^{\mathrm{i}\langle z, Y(t)\rangle}\big] = \exp\big(t\Psi(\operatorname{Log}\hat{\mu}(z))\big), \quad z \in \mathbf{R}^n, \tag{3.14}$$

where for any $w = (w_1, \ldots, w_d)^\top \in \mathbf{C}^d$ with $\operatorname{Re} w_j \le 0,\ j = 1, \ldots, d$, we let

$$\Psi(w) = \int_{\mathbf{R}_+^d} \big(e^{\langle w, \mathbf{s}\rangle} - 1\big)\, \rho(\mathrm{d}\mathbf{s}) + \langle w, c\rangle. \tag{3.15}$$

Moreover, the characteristic triplet $(\Sigma^\#, \nu^\#, \gamma^\#)$ of $\{Y(t)\}$ is as follows:

$$\Sigma^\# = \operatorname{diag}(c_1\Sigma_1, \ldots, c_d\Sigma_d), \tag{3.16}$$

$$\nu^\#(B) = \nu^\#_{(1)}(B) + \nu^\#_{(2)}(B), \tag{3.17}$$

$$\gamma^\# = \int_{\mathbf{R}_+^d} \rho(\mathrm{d}\mathbf{s}) \int_{|x|\le 1} x\, \mu^{\mathbf{s}}(\mathrm{d}x) + (c_1\gamma_1, \ldots, c_d\gamma_d)^\top, \tag{3.18}$$

with $\nu^\#_{(1)}$ and $\nu^\#_{(2)}$ defined by $\nu^\#_{(1)}(\{0\}) = \nu^\#_{(2)}(\{0\}) = 0$ and for $B \in \mathcal{B}(\mathbf{R}^n \backslash \{0\})$ by

$$\nu^\#_{(1)}(B) = \int_{\mathbf{R}_+^d} \mu^{\mathbf{s}}(B)\rho(\mathrm{d}\mathbf{s}), \tag{3.19}$$

$$\nu^\#_{(2)}(B) = \int_B \big(c_1 1_{\mathcal{A}_1}(x)\, \nu_1(\mathrm{d}x_1) + \ldots + c_d 1_{\mathcal{A}_d}(x)\, \nu_d(\mathrm{d}x_d)\big), \tag{3.20}$$

where $x = (x_1, \ldots, x_d)^\top$ with $x_j \in \mathbf{R}^{n_j}$. Finally, if $c = 0$ and $\int_{\mathbf{R}_+^d} |\mathbf{s}|^{\frac{1}{2}}\, \rho(\mathrm{d}\mathbf{s}) < \infty$, then $\Sigma^\# = 0$, $\int_{|x|\le 1} |x|\, \nu^\#(\mathrm{d}x) < \infty$, and $\{Y(t)\}$ has zero drift and is of bounded variation on any finite time interval a.s.

Proof. We follow the proof of Theorem 30.1 in Sato (1999) closely. Let $m \ge 2$ and $f^1, \ldots, f^{m-1} : \mathbf{R}^n \to \mathbf{R}$ be measurable and bounded, and let $0 \le t_1 \le \ldots \le t_m$. When $\mathbf{s}^k = (s^k_1, \ldots, s^k_d)^\top \in \mathbf{R}_+^d,\ k = 1, \ldots, m$, with $\mathbf{s}^1 \preceq \mathbf{s}^2 \preceq \ldots \preceq \mathbf{s}^m$ we let

$$G(\mathbf{s}^1, \ldots, \mathbf{s}^m) = E\Big[\prod_{k=1}^{m-1} f^k\big(X(\mathbf{s}^{k+1}) - X(\mathbf{s}^k)\big)\Big]. \tag{3.21}$$

By Lemma 3.2 it follows that

$$G(\mathbf{s}^1, \ldots, \mathbf{s}^m) = \prod_{k=1}^{m-1} g^k(\mathbf{s}^{k+1} - \mathbf{s}^k),$$

where, for $\mathbf{s} \in \mathbf{R}_+^d$, $g^k(\mathbf{s}) = E[f^k(X(\mathbf{s}))]$. As $\{X(t)\}$ and $\{T(t)\}$ are independent we obtain

$$
\begin{aligned}
(3.22)\ E\Big[\prod_{k=1}^{m-1} f^k(Y(t_{k+1}) - Y(t_k))\Big] &= E\big[G(T(t_1), \ldots, T(t_m))\big] \\
&= \prod_{k=1}^{m-1} E\big[g^k\big(T(t_{k+1}) - T(t_k)\big)\big].
\end{aligned}
$$

Choosing $f^{k'} = 1$ for all $k \neq k'$ and using that $\{T(t)\}$ has stationary increments, we see that

$$
\begin{aligned}
(3.23) \qquad & E\Big[f^k\big(Y(t_{k+1}) - Y(t_k)\big)\Big] = E\Big[g^k\big(T(t_{k+1}) - T(t_k)\big)\Big] \\
&= E\Big[g^k\big(T(t_{k+1} - t_k)\big)\Big] = E\Big[f^k\big(Y(t_{k+1} - t_k)\big)\Big]
\end{aligned}
$$

for each k. We get (3.13) as a special case. Equations (3.22) and (3.23), together with the obvious property of right-continuity with left-limits, show that $\{Y(t)\}$ is a Lévy process.

Next we prove (3.14). Note that for $w \in \mathbf{C}^d$ with $\mathrm{Re} w_j \le 0,\ j = 1, \ldots, d$, we have $E[e^{\langle w, T(t)\rangle}] = e^{t\Psi(w)}$. Thus, by (3.5),

$$
\begin{aligned}
(3.24) \qquad E\big[e^{\mathrm{i}\langle z, Y(t)\rangle}\big] &= \int_{\mathbf{R}_+^d} \widehat{\mu}^{\mathbf{s}}(z)\, \lambda^t(\mathrm{d}\mathbf{s}) = \int_{\mathbf{R}_+^d} e^{\langle \mathrm{Log}\, \widehat{\mu}(z), \mathbf{s}\rangle}\, \lambda^t(\mathrm{d}\mathbf{s}) \\
&= \exp\big(t\Psi(\mathrm{Log}\, \widehat{\mu}(z)\big),
\end{aligned}
$$

which yields the result.

To find the characteristic triplet of $\{Y(t)\}$ the following two results are needed. First, for $\epsilon > 0$ there exists a constant $C > 0$ such that, for any $\mathbf{s} \in \mathbf{R}_+^d$,

$$
\begin{aligned}
(3.25) \qquad && P\Big[\big|X(\mathbf{s})\big| > \epsilon\Big] &\le C|\mathbf{s}|, \\
(3.26) \qquad && E\Big[\big|X(\mathbf{s})\big|^2;\ \big|X(\mathbf{s})\big| \le 1\Big] &\le C|\mathbf{s}|, \\
(3.27) \qquad && \Big|E\Big[X(\mathbf{s});\ \big|X(\mathbf{s})\big| \le 1\Big]\Big| &\le C|\mathbf{s}|, \\
(3.28) \qquad && E\Big[\big|\, X(\mathbf{s})\big|;\ \big|X(\mathbf{s})\big| \le 1\Big] &\le C|\mathbf{s}|^{\frac{1}{2}}.
\end{aligned}
$$

These estimates are easy consequences of Lemma 30.3 in Sato (1999). Secondly, by (3.5) and (3.15),

$$
(3.29) \qquad \Psi\big(\mathrm{Log}\, \widehat{\mu}(z)\big) = \int_{\mathbf{R}_+^d} (\widehat{\mu}^{\mathbf{s}}(z) - 1)\, \rho(\mathrm{d}\mathbf{s}) + \langle \mathrm{Log}\, \widehat{\mu}(z), c\rangle.
$$

Let $D = \{x \in \mathbf{R}^n : |x| \le 1\}$ and define $\nu^{\#}_{(1)}$ by (3.19). Then (3.25), (3.26) and (3.27) yield

$$\begin{aligned} \int_{|x|>1} \nu^{\#}_{(1)}(\mathrm{d}x) &= \int_{\mathbf{R}^d_+} P\Big[\big|X(\mathbf{s})\big| > 1\Big]\,\rho(\mathrm{d}\mathbf{s}) \quad < \quad \infty, \\ \int_D |x|^2\,\nu^{\#}_{(1)}(\mathrm{d}x) &= \int_{\mathbf{R}^d_+} E\big[|X(\mathbf{s})|^2; |X(\mathbf{s})| \le 1\big]\,\rho(\mathrm{d}\mathbf{s}) \quad < \quad \infty, \\ & \int_{\mathbf{R}^d_+} \rho(\mathrm{d}\mathbf{s}) \Big| \int_D x\,\mu^{\mathbf{s}}(\mathrm{d}x)\Big| \quad < \quad \infty. \end{aligned} \tag{3.30}$$

With $g(z,x) = e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_D(x)$ we thus find

$$\begin{aligned} &\int_{\mathbf{R}^d_+} (\widehat{\mu}^{\mathbf{s}}(z) - 1)\,\rho(\mathrm{d}\mathbf{s}) = \int \rho(\mathrm{d}\mathbf{s}) \int (e^{\mathrm{i}\langle z,x\rangle} - 1)\mu^{\mathbf{s}}(\mathrm{d}x) \\ = \quad & \int \rho(\mathrm{d}\mathbf{s}) \int g(z,x)\,\mu^{\mathbf{s}}(\mathrm{d}x) + \mathrm{i} \int \rho(\mathrm{d}\mathbf{s}) \int \langle z,x\rangle 1_D(x)\,\mu^{\mathbf{s}}(\mathrm{d}x) \\ = \quad & \int g(z,x)\nu^{\#}_{(1)}(\mathrm{d}x) + \mathrm{i}\Big\langle z, \int \rho(\mathrm{d}\mathbf{s}) \int_{|x|\le 1} x\,\mu^{\mathbf{s}}(\mathrm{d}x)\Big\rangle. \end{aligned}$$

Combining this with (3.24) and (3.29) we see that the characteristic triplet of $\{Y_t\}$ is as stated above.

Finally, assume $c = 0$ and $\int_{\mathbf{R}^d_+} |\mathbf{s}|^{\frac{1}{2}}\,\rho(\mathrm{d}\mathbf{s}) < \infty$. Then $\nu^{\#}_{(2)} = 0$ and by (3.28)

$$\int_{|x|\le 1} |x|\nu^{\#}_{(1)}(\mathrm{d}x) = \int \rho(\mathrm{d}\mathbf{s}) \int_{|x|\le 1} |x|\,\mu^{\mathbf{s}}(\mathrm{d}x) < \infty.$$

This finishes the proof. $\square$

4 Multiparameter Lévy processes

In the proof of Theorem 3.3 a key point is that the multiparameter process $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}^d_+\}$ defined in Subsection 3.2 satisfies (ii) and (iii) in Lemma 3.2. In this section we analyze the general class of multiparameter processes satisfying these two conditions. We need two positive integers n and d; it is not assumed that $n \ge d$. Note that we are using 0 also for the vector $(0,\ldots,0)^\top$.

Definition 4.1 *Let $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ be a process with parameter in $\mathbf{R}_+^d$ and values in $\mathbf{R}^n$. We call it an $\mathbf{R}_+^d$-parameter Lévy process if the following four conditions are satisfied:*
(i) *for any $m \geq 3$ and for any choice of $\mathbf{s}^1 \preceq \ldots \preceq \mathbf{s}^m$, $X(\mathbf{s}^j) - X(\mathbf{s}^{j-1})$, $j = 2, \ldots, m$, are independent;*
(ii) *for any $\mathbf{s}^1 \preceq \mathbf{s}^2$ and $\mathbf{s}^3 \preceq \mathbf{s}^4$ satisfying $\mathbf{s}^2 - \mathbf{s}^1 = \mathbf{s}^4 - \mathbf{s}^3$, $X(\mathbf{s}^2) - X(\mathbf{s}^1) \stackrel{d}{=} X(\mathbf{s}^4) - X(\mathbf{s}^1)$;*
(iii) $X(0) = 0$ *a.s.*
(iv) *Almost surely, $X(\mathbf{s})$ is right-continuous with left limits in $\mathbf{s}$ in the partial ordering of $\mathbf{s} \in \mathbf{R}_+^d$.*

Example 4.2 Let $\{Z(t) : t \geq 0\}$ be a Lévy process on $\mathbf{R}^n$. Fix $c = (c_1, \ldots, c_d)^\top \in \mathbf{R}_+^d$ and define

$$X(\mathbf{s}) = Z(\langle c, \mathbf{s} \rangle) = Z(c_1 s_1 + \ldots + c_d s_d) \quad \text{for } \mathbf{s} = (s_1, \ldots, s_d)^\top \in \mathbf{R}_+^d.$$

Then $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ is an $\mathbf{R}_+^d$-parameter Lévy process on $\mathbf{R}^n$. The proof is straightforward. □

Example 4.3 Given independent $\mathbf{R}_+^d$-parameter Lévy processes $\{X_j(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$, $j = 1, \ldots, m$, on $\mathbf{R}^n$, let

$$X(\mathbf{s}) = X_1(\mathbf{s}) + \ldots + X_m(\mathbf{s}).$$

Then, obviously, $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ is again an $\mathbf{R}_+^d$-parameter Lévy process. □

Example 4.4 Let $\{Z_j(t) : t \geq 0\}$, $j = 1, \ldots, d$, be independent Lévy processes on $\mathbf{R}^n$ and define

$$V(\mathbf{s}) = Z_1(s_1) + \ldots + Z_d(s_d) \quad \text{for } \mathbf{s} = (s_1, \ldots, s_d)^\top \in \mathbf{R}_+^d. \tag{4.1}$$

Then $Z_j(s_j) = Z_j(\langle \delta_j, \mathbf{s} \rangle)$, where $\delta_j = (\delta_{j1}, \ldots, \delta_{jd})^\top$ with Kronecker's δ_{jk}. Hence,by combination of Examples 4.2 and 4.3, $\{V(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ is seen to be an $\mathbf{R}_+^d$-parameter Lévy process on $\mathbf{R}^n$. □

The following *structure theorem for $\mathbf{R}_+^d$-parameter Lévy processes on $\mathbf{R}^n$* shows that Example 4.4 represents the general class to some degree.

Theorem 4.5 *Let $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ be an $\mathbf{R}_+^d$-parameter Lévy process on $\mathbf{R}^n$. Then there exist independent Lévy processes $\{Z_j(t) : t \geq 0\}$, $j = 1, \ldots, d$, on $\mathbf{R}^n$ such that the $\mathbf{R}_+^d$-parameter Lévy process $\{V(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ defined by* (4.1) *has the property that, for any choice of $m \geq 1$ and $\mathbf{s}^1 \preceq \ldots \preceq \mathbf{s}^m$,*

$$\big(X(\mathbf{s}^1), \ldots, X(\mathbf{s}^m)\big)^\top \stackrel{d}{=} \big(V(\mathbf{s}^1), \ldots, V(\mathbf{s}^m)\big)^\top. \tag{4.2}$$

Proof. Let $\delta_j = (\delta_{j1}, \ldots, \delta_{jd})^\top$ and let $X_j(t) = X(t\delta_j)$. Then $\{X_j(t) : t \geq 0\}$ is a Lévy process on $\mathbf{R}^n$. Choose $\{Z_j(t) : t \geq 0\}$, $j = 1, \ldots, d$, as independent Lévy processes such that $\{Z_j(t)\} \overset{d}{=} \{X_j(t)\}$ for each j. Define an $\mathbf{R}_+^d$-parameter Lévy process $\{V(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ by (4.1). We also use the notation $V(s_1, \ldots, s_d)$ for $V((s_1, \ldots, s_d)^\top)$. We claim that

$$X(\mathbf{s}) \overset{d}{=} V(\mathbf{s}) \quad \text{for } \mathbf{s} \in \mathbf{R}_+^d. \tag{4.3}$$

Indeed, for $\mathbf{s} = (s_1, \ldots, s_d)^\top$,

$$\begin{aligned} X(\mathbf{s}) \;&=\; X(s_1, 0, \ldots, 0) + \big(X(s_1, s_2, 0, \ldots, 0) - X(s_1, 0, \ldots, 0)\big) \\ &\quad + \ldots + \big(X(s_1, \ldots, s_{d-1}, s_d) - X(s_1, \ldots, s_{d-1}, 0)\big). \end{aligned}$$

The right-hand side is the sum of d independent terms by the definition of a $\mathbf{R}_+^d$-parameter Lévy process. Further, by the condition (ii) in the same definition,

$$\begin{aligned} X(s_1, 0, \ldots, 0) &= X_1(s_1), \\ X(s_1, s_2, 0, \ldots, 0) - X(s_1, 0 \ldots, 0) \;&\overset{d}{=}\; X(0, s_2, 0, \ldots, 0) = X_2(s_2), \end{aligned}$$

and so on. Hence we obtain (4.3). Now we claim that (4.2) holds for any $\mathbf{s}^1 \preceq \ldots \preceq \mathbf{s}^m$. In order to show this, it is enough to prove

$$\begin{aligned} &\big(X(\mathbf{s}^1) - X(\mathbf{s}^0), X(\mathbf{s}^2) - X(\mathbf{s}^1), \ldots, X(\mathbf{s}^m) - X(\mathbf{s}^{m-1})\big)^\top \\ \overset{d}{=}\; &\big(V(\mathbf{s}^1) - V(\mathbf{s}^0), V(\mathbf{s}^2) - V(\mathbf{s}^1), \ldots, V(\mathbf{s}^m) - V(\mathbf{s}^{m-1})\big)^\top, \end{aligned} \tag{4.4}$$

where $\mathbf{s}^0 = 0$. The components of each side of (4.4) are independent, and

$$\begin{aligned} X(\mathbf{s}^j) - X(\mathbf{s}^{j-1}) \;&\overset{d}{=}\; X(\mathbf{s}^j - \mathbf{s}^{j-1}) \overset{d}{=} V(\mathbf{s}^j - \mathbf{s}^{j-1}) \\ &\overset{d}{=}\; V(\mathbf{s}^j) - V(\mathbf{s}^{j-1}) \quad \text{for } j = 1, \ldots, m \end{aligned}$$

by virtue of (4.3). This proves (4.4). Hence (4.2) is true. $\square$

Remark 4.6 The two $\mathbf{R}_+^d$-parameter Lévy processes $\{X(\mathbf{s})\}$ and $\{V(\mathbf{s})\}$ satisfy (4.2) whenever $\mathbf{s}^1 \preceq \ldots \preceq \mathbf{s}^m$, but the joint distributions are not necessarily identical. That is, if $\mathbf{s}^1, \ldots, \mathbf{s}^m \in \mathbf{R}_+^d$ do not satisfy $\mathbf{s}^1 \preceq \ldots \preceq \mathbf{s}^m$, then (4.2) is not necessarily true. Thus the system of joint distributions of $\big(X(\mathbf{s}^1), \ldots, X(\mathbf{s}^m)\big)^\top$ for $\mathbf{s}^1 \preceq \ldots \preceq \mathbf{s}^m$ does not determine the whole system of joint distributions of $\{X(\mathbf{s})\}$. For example, consider $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ of Example 4.2 with $c = (1, \ldots, 1)^\top$. Then $X(\mathbf{s}^1) = X(\mathbf{s}^2)$ if $\mathbf{s}^1$ and $\mathbf{s}^2$ have the same sum of coordinates, but $V(1, 0, \ldots, 0), \ldots, V(0, \ldots, 0, 1)$ are independent.

The following theorem formulates *subordination of* $\mathbf{R}_+^d$*-parameter Lévy processes by d-dimensional subordinators* and specifies the relation between the characteristic triplets involved.

Theorem 4.7 *Let* $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$ *be an* $\mathbf{R}_+^d$*-parameter Lévy process on* $\mathbf{R}^n$ *and let* $\{T(t) : t \geq 0\}$ *be a d-dimensional subordinator independent of* $\{X(\mathbf{s}) : \mathbf{s} \in \mathbf{R}_+^d\}$*. Define*

$$Y(t) = X(T(t)), \quad t \geq 0. \tag{4.5}$$

Then $\{Y(t) : t \geq 0\}$ *is a Lévy process on* $\mathbf{R}^n$ *satisfying* (3.13) *and* (3.14), *where* $\mu^{\mathbf{s}} = \mathcal{L}(X(\mathbf{s}))$, $\lambda^t = \mathcal{L}(T(t))$, Ψ *is the function* (3.15), *and*

$$\operatorname{Log} \widehat{\mu}(z) = \big(\log \widehat{\mu}_1(z), \ldots, \log \widehat{\mu}_d(z) \big)^\top \tag{4.6}$$

with $\mu_j = \mathcal{L}(X_j(\delta_j))$, $\delta_j = (\delta_{j1}, \ldots \delta_{jd})^\top$*. Let* $(\Sigma_j, \nu_j, \gamma_j)$ *be the characteristic triplet of the Lévy process* $\{X_j(t\delta_j)\}$ *and let* ρ *and* $c = (c_1, \ldots, c_d)^\top$ *be the Lévy measure and drift of* $\{T(t)\}$*. Then the characteristic triplet* $(\Sigma^\#, \nu^\#, \gamma^\#)$ *of* $\{Y(t)\}$ *is expressed as follows:*

$$\Sigma^\# = \sum_{j=1}^d c_j \Sigma_j, \tag{4.7}$$

$$\nu^\#(B) = \int_{\mathbf{R}_+^d} \mu^{\mathbf{s}}(B)\, \rho(\mathrm{d}\mathbf{s}) + \sum_{j=1}^d c_j \nu_j(B), \quad B \in \mathcal{B}(\mathbf{R}^n \backslash \{0\}), \tag{4.8}$$

$$\gamma^\# = \int_{\mathbf{R}_+^d} \rho(\mathrm{d}\mathbf{s}) \int_{|x| \leq 1} x \mu^{\mathbf{s}}(\mathrm{d}x) + \sum_{j=1}^d c_j \gamma_j. \tag{4.9}$$

Proof. The proof that $\{Y(t)\}$ is a Lévy process is similar to that of Theorem 3.3. The formula (3.13) comes from the definition of $\{Y(t)\}$. Let $\{Z_j(t)\}$ be the Lévy processes in Theorem 4.5. Then

$$E\big[e^{\mathrm{i}\langle z, X(\mathbf{s})\rangle}\big] = \prod_{j=1}^d E\big[e^{\mathrm{i}\langle z, Z_j(s_j)\rangle}\big] = \prod_{j=1}^d \widehat{\mu}_j(z)^{s_j} = e^{\langle \operatorname{Log} \widehat{\mu}(z), \mathbf{s}\rangle}$$

for $z \in \mathbf{R}^n$ and $\mathbf{s} \in \mathbf{R}_+^d$, and

$$\begin{aligned} E\big[e^{\mathrm{i}\langle z, Y(t)\rangle}\big] &= E\Big[\Big(E\big[e^{\mathrm{i}\langle z, X(\mathbf{s})\rangle}\big]\Big)_{\mathbf{s}=T(t)}\Big] \\ = \quad & E\big[e^{\langle \operatorname{Log} \widehat{\mu}(z), T(t)\rangle}\big] = \exp\big(t\Psi(\operatorname{Log} \widehat{\mu}(z))\big) \end{aligned}$$

for $z \in \mathbf{R}^n$.

To find the characteristic triplet of $\{Y(t)\}$ introduce a process $\{\widetilde{Y}(t)\}$ on $\mathbf{R}^{nd}$ by $\widetilde{Y}(t) = \big(Z_1(T_1(t)), \ldots, Z_d(T_d(t))\big)^\top$, where we recall from Theorem 4.5 that $\{Z_j(t)\}, j = 1, \ldots, d$, are independent. Note that $\{\widetilde{Y}(t)\}$ appears by multivariate subordination of $\{(Z_1(t), \ldots, Z_d(t))^\top\}$ by $\{T(t)\}$. From Theorem 3.3 we therefore find that the characteristic triplet $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$ of $\{\widetilde{Y}(t)\}$ is

$$\widetilde{\Sigma} = \operatorname{diag}(c_1\Sigma_1, \ldots, c_d\Sigma_d), \tag{4.10}$$

$$\begin{aligned} \widetilde{\nu}(B) = & \int_{\mathbf{R}^n} \widetilde{\mu}^{\mathbf{s}}(B)\, \rho(\mathrm{d}\mathbf{s}) \\ & + \int_B \big(c_1 1_{\mathcal{A}_1}(y)\, \nu_1(\mathrm{d}y_1) + \ldots + c_d 1_{\mathcal{A}_d}(y)\, \nu_d(\mathrm{d}y_d)\big), \\ & B \in \mathcal{B}(\mathbf{R}^{nd}\backslash\{0\}), \end{aligned} \tag{4.11}$$

$$\widetilde{\gamma} = \int_{\mathbf{R}^d_+} \rho(\mathrm{d}\mathbf{s}) \int_{|y|\le 1} y\widetilde{\mu}^{\mathbf{s}}(\mathrm{d}y) + (c_1\gamma_1, \ldots, c_d\gamma_d)^\top, \tag{4.12}$$

where, for $\mathbf{s} = (s_1, \ldots, s_d)^\top \in \mathbf{R}^d_+$, $\widetilde{\mu}^{\mathbf{s}} = \mathcal{L}((Z_1(s_1), \ldots, Z_d(s_d))^\top)$ and for $j = 1, .., d$ we let

$$\mathcal{A}_j = \{y = (y_1, \ldots, y_d)^\top \in \mathbf{R}^{nd} : y_k \in \mathbf{R}^n \text{ and } y_k = 0 \text{ for } k \neq j\}.$$

Further, in (4.11) we have decomposed $y \in \mathbf{R}^{nd}$ as $y = (y_1, \ldots, y_d)^\top$, where the components are n-dimensional. Let U be the $nd \times n$ matrix $U = (I_n, \ldots, I_n)$, where I_n is the $n \times n$ identity matrix. By Theorem 4.5 we have for all $\mathbf{s} \in \mathbf{R}^d_+$ that $X(\mathbf{s}) \stackrel{d}{=} Z_1(s_1) + \ldots + Z_d(s_d) = U\big(Z_1(s_1), \ldots, Z_d(s_d)\big)^\top$. Replacing $\mathbf{s}$ by $T(t)$ we also get

$$\begin{aligned} Y(t) &= X(T(t)) \stackrel{d}{=} Z_1(T_1(t)) + \ldots + Z_d(T_d(t)) \\ &= U\big(Z_1(T_1(t)), \ldots, Z_d(T_d(t))\big)^\top = U\widetilde{Y}(t). \end{aligned}$$

Thus, $Y(t)$ is, in distribution, a linear transformation of $\widetilde{Y}(t)$. From Proposition 11.10 in Sato (1999) it follows that the characteristic triplet $(\Sigma^\#, \nu^\#, \gamma^\#)$ of $\{Y(t)\}$ is expressed in terms of $(\widetilde{\Sigma}, \widetilde{\nu}, \widetilde{\gamma})$ in the following way:

$$\Sigma^\# = U\widetilde{\Sigma}U^\top, \tag{4.13}$$

$$\nu^\# = [\widetilde{\nu}U^{-1}]_{\mathbf{R}^n\backslash\{0\}}, \tag{4.14}$$

$$\gamma^\# = U\widetilde{\gamma} + \int_{\mathbf{R}^{nd}} Uy(1_{\{|Uy|\le 1\}} - 1_{\{|y|\le 1\}})\widetilde{\nu}(\mathrm{d}y), \tag{4.15}$$

where $(\widetilde{\nu}U^{-1})(B) = \widetilde{\nu}(\{y : Uy \in B\})$ and $[\widetilde{\nu}U^{-1}]_{\mathbf{R}^n\backslash\{0\}}$ is the restriction of the measure to $\mathbf{R}^n\backslash\{0\}$. Since the three terms $\Sigma^{\#}$, $\nu^{\#}$ and $\gamma^{\#}$ are handled in a similar way we only consider $\nu^{\#}$. From (4.11) and (4.14) we find that

$$\nu^{\#}(B) = (\widetilde{\nu}U^{-1})(B) \;\; = \;\; \int_{\mathbf{R}_+^d} (\widetilde{\mu}^{\mathbf{s}}U^{-1})(B)\rho(\mathrm{d}\mathbf{s}) + \sum_{j=1}^{d} c_j\nu_j(B),$$

for $B \in \mathcal{B}(\mathbf{R}^n\backslash\{0\})$. Moreover, as $\mu^{\mathbf{s}} = \mathcal{L}(X(\mathbf{s}))$, $\widetilde{\mu}^{\mathbf{s}} = \mathcal{L}((Z_1(s_1), \ldots, Z_d(s_d))^\top)$ and $X(\mathbf{s}) \stackrel{d}{=} U((Z_1(s_1), \ldots, Z_d(s_d))^\top$, we have $\widetilde{\mu}^{\mathbf{s}}U^{-1} = \mu^{\mathbf{s}}$. Therefore

$$\nu^{\#}(B) \;\; = \;\; \int_{\mathbf{R}_+^d} \mu^{\mathbf{s}}(B)\rho(\mathrm{d}\mathbf{s}) + \sum_{j=1}^{d} c_j\nu_j(B), \quad B \in \mathcal{B}(\mathbf{R}^n\backslash\{0\})),$$

which is (4.8). This concludes the proof. $\square$

Notice from the above proof that a process identical in law with the process $\{Y(t)\}$ can be constructed in two steps, where the first step is to get the process $\{\widetilde{Y}(t)\}$ by multivariate subordination and the second step is to perform a linear transformation of $\{\widetilde{Y}(t)\}$. In this sense the subordination of an $\mathbf{R}_+^d$-parameter Lévy process by a d-dimensional subordinator can be seen as a transformation of a multivariate subordination. We therefore concentrate on multivariate subordination in this paper.

5 Prerequisites on operator selfdecomposability and operator stability

We have established the concept of multivariate subordination. Now an important general question is what properties of the subordinator $\{T(t)\}$ and the subordinand $\{X(t)\}$ give certain desirable properties to the subordinated process $\{Y(t)\}$. In particular we are concerned with selfdecomposability and stability. In this section we recall various known facts about operator selfdecomposability and stability. We will mention the connection with processes of Ornstein-Uhlenbeck type on $\mathbf{R}^d$. Although we do not use this connection in this paper explicitly, this is important in the understanding of these concepts.

For a set $J \subset \mathbf{R}$ let $M_J(d)$ be the set of real $d \times d$ matrices all of whose eigenvalues have real parts in J. For $a > 0$ and a $d \times d$ matrix Q let

$$a^Q = e^{(\log a)Q} = \sum_{n=0}^{\infty} \frac{1}{n!}(\log a)^n Q^n.$$

5.1 Operator selfdecomposability

A random vector on $\mathbf{R}^d$ is called selfdecomposable if, for every $b > 1$, there is a Y_b such that

$$X \stackrel{d}{=} b^{-1}X + Y_b, \tag{5.1}$$

where X and Y_b are independent. Correspondingly a probability measure μ on $\mathbf{R}^d$ is called selfdecomposable if, for every $b > 1$, there is a μ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z). \tag{5.2}$$

This notion appears in the theory of limit distributions of sums of independent (not necessarily identically distributed) random variables. See e.g. Loève (1977,1978). Its extension in the theory of limit distributions involving operator (i.e. matrix) normalizations was made by Urbanik (1972a). Let $Q \in M_{(0,\infty)}(d)$. A random vector X on $\mathbf{R}^d$ or its distribution μ is called Q-selfdecomposable if, for every $b > 1$, there is Y_b such that

$$X \stackrel{d}{=} b^{-Q}X + Y_b, \tag{5.3}$$

where X and Y_b are independent, that is, if

$$\widehat{\mu}(z) = \widehat{\mu}\big(b^{-Q^\top}z\big)\widehat{\mu}_b(z) \tag{5.4}$$

with some μ_b. In the case when $Q = I$, the $d \times d$ identity matrix, this is the usual selfdecomposability.

The class of all Q-selfdecomposable distributions on $\mathbf{R}^d$ is denoted $L_0(Q)$. For $m = 1, 2, \ldots$ the class $L_m(Q)$ is defined to be the class of distributions on $\mathbf{R}^d$ such that, for every $b > 1$, there exists $\mu_b \in L_{m-1}(Q)$ satisfying (5.4). Then

$$L_0(Q) \supset L_1(Q) \supset \ldots \supset L_\infty(Q),$$

where we define $L_\infty(Q) = \bigcap_{m<\infty} L_m(Q)$. Moreover, $L_0(Q)$ is a subclass of the class of infinitely divisible distributions. In the case $Q = I$, these nested classes were introduced by Urbanik (1972b, 1973) and reformulated by Sato (1980). These are the classes L_m in the ordinary sense. For general Q, they were studied by Jurek (1983) and Sato and Yamazato (1984,1985). We say that a Lévy process $\{X(t)\}$ on $\mathbf{R}^d$ is of class $L_m(Q)$ if $\mathcal{L}(X(1))$ is of class $L_m(Q)$. Distributions and processes of class $L_m(Q)$ are sometimes called $m+1$ times Q-selfdecomposable if $m < \infty$, or completely Q-selfdecomposable if $m = \infty$.

Remark 5.1 For any $c > 0$ we have $L_m(cQ) = L_m(Q)$ for $m = 0, 1, \ldots, \infty$. This is shown by using $b^{-cQ} = (b^c)^{-Q}$.

Remark 5.2 Let $X = (X_1, \ldots, X_d)^\top$ be a random vector on $\mathbf{R}^d$ with independent coordinates. If $\mathcal{L}(X_j)$ is selfdecomposable for each j, then, for any diagonal matrix $Q = \mathrm{diag}(q_1, \ldots, q_d)$ with diagonal entries $q_j > 0$, $\mathcal{L}(X)$ is Q-selfdecomposable. Indeed, let $\mu = \mathcal{L}(X)$ and $\mu_j = \mathcal{L}(X_j)$. We have for every $b > 1$

$$\begin{aligned} \widehat{\mu}(z_1, \ldots, z_d) &= \widehat{\mu}_1(z_1) \cdots \widehat{\mu}_d(z_d) \\ &= \widehat{\mu}_1(b^{-q_1} z_1)\widehat{\rho}_1(z_1) \cdots \widehat{\rho}_d(b^{-q_d} z_d)\widehat{\rho}_d(z_q) \end{aligned}$$

with some distributions $\rho_1, \ldots, \rho_d$. It follows that

$$\begin{aligned} \widehat{\mu}(z_1, \ldots, z_d) &= \widehat{\mu}(b^{-q_1} z_1, \ldots, b^{-q_d} z_d)\widehat{\mu}_b(z_1, \ldots, z_d) \\ &= \widehat{\mu}\big(b^{-Q^\top}(z_1, \ldots, z_d)^\top\big)\widehat{\mu}_b(z_1, \ldots, z_d) \end{aligned}$$

with some μ_b.

Let μ be an infinitely divisible distribution on $\mathbf{R}^d$ with characteristic triplet (Σ, ν, γ). To describe the condition for Q-selfdecomposability it is convenient to introduce the class $\mathcal{B}_0(\mathbf{R}^d)$ which consists of all Borel sets in $\mathbf{R}^d$ having positive distance to the origin. Then, as discussed in Sato and Yamazato (1985), μ is Q-selfdecomposable if and only if the following two conditions (i) and (ii) are satisfied:
(i) for every $z \in \mathbf{R}^d$, $\langle \Sigma e^{-tQ^\top} z, e^{-tQ^\top} z\rangle$ is decreasing in $t > 0$;
(ii) for every $B \in \mathcal{B}_0(\mathbf{R}^d)$ and $b > 1$, $\nu(B) - \nu(b^Q B) \geq 0$, where $b^Q B = \{b^Q x : x \in B\}$.
We always use the word 'decreasing' allowing flatness. The condition (ii) above is equivalent to the Lévy measure ν being of the form

$$\nu(B) = \int_{\mathbf{S}_Q} \sigma(\mathrm{d}\xi) \int_0^\infty \frac{k_\xi(u)}{u} 1_B(u^Q \xi)\,\mathrm{d}u, \quad B \in \mathcal{B}(\mathbf{R}^d \backslash \{0\}), \tag{5.5}$$

where

$$\mathbf{S}_Q = \big\{\xi \in \mathbf{R}^d : |\xi| = 1 \text{ and } |u^Q \xi| > 1 \text{ for all } u > 1\big\},$$

σ is a finite measure on $\mathbf{S}_Q$, $k_\xi(u)$ is measurable in ξ, decreasing in u and nonnegative.

For any signed measure ρ finite on $\mathcal{B}_0(\mathbf{R}^d)$ and for $b > 1$ define $\phi_{Q,b}\rho$ as

$$(\phi_{Q,b}\rho)(B) = \rho(B) - \rho(b^Q B) \tag{5.6}$$

for $B \in \mathcal{B}_0(\mathbf{R}^d)$.

Let $1 \le m < \infty$. From Sato and Yamazato (1985) and Maejima et al. (1999) we find that μ is in $L_m(Q)$ if and only if the following two conditions are satisfied:
(i) for every $z \in \mathbf{R}^d$,

$$\big(-\frac{\mathrm{d}}{\mathrm{d}t}\big)^j \langle \Sigma e^{-tQ^\top} z, e^{-tQ^\top} z\rangle \ge 0, \quad \text{for } j = 1, \dots, m+1\ ;$$

(ii) for every $B \in \mathcal{B}_0(\mathbf{R}^d)$ and $b > 1$, we have

$$(\phi^j_{Q,b}\nu)(B) \ge 0 \ \text{ for } j = 1, \dots, m+1,$$

where $\phi^j_{Q,b}$ is the jth iteration of $\phi_{Q,b}$.

The condition (ii) above is equivalent to saying that, for σ-almost every $\xi \in \mathbf{S}_Q$,

$$(\psi^j_b k_\xi)(u) \ge 0 \ \text{ for } j = 1, \dots, m+1 \ \text{ and } b > 1, \tag{5.7}$$

where ψ^j_b is the jth iteration of the operator ψ_b defined by

$$(\psi_b f)(u) = f(u) - f(bu) \tag{5.8}$$

for any function f on $(0, \infty)$.

5.2 Operator stability

The notion of stability of distributions in the theory of limit distributions was extended by Sharpe (1969) to the case involving operator (i.e. matrix) normalizations.

Let $Q \in M_{(0,\infty)}(d)$. A random vector X or its distribution μ on $\mathbf{R}^d$ is called Q-stable (or operator stable with exponent Q) if, for every positive integer n, there is a $c_n \in \mathbf{R}^d$ such that n independent copies $X_1, \dots, X_n$ of X satisfy

$$X_1 + \ldots + X_n \stackrel{d}{=} n^Q X + c_n. \tag{5.9}$$

It is called strictly Q-stable if c_n can be taken to be 0. If μ is Q-stable, then it is infinitely divisible and the property (5.9) can be strengthened as follows: For every $a > 0$ there is a $c_a \in \mathbf{R}^d$ such that

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{Q^\top} z) e^{\mathrm{i}\langle c_a, z\rangle}. \tag{5.10}$$

If μ is strictly Q-stable, then c_a can be taken to be 0. A Lévy process $\{X(t)\}$ is called Q-stable or strictly Q-stable if $\mathcal{L}(X(1))$ is Q-stable or strictly Q-stable, respectively. We remark that a distribution μ on $\mathbf{R}^d$ is stable with index α if and only if it is operator stable with exponent $\frac{1}{\alpha}I$. While the index of stability is uniquely defined, the exponent of an operator stable distribution is not unique. See e.g. Jurek and Mason (1993).

A probability measure on $\mathbf{R}^d$ is called nondegenerate if its support is not contained in any $(d-1)$-dimensional hyperplane. A stochastic process $\{X(t)\}$ on $\mathbf{R}^d$ is called nondegenerate if, for some t, $\mathcal{L}(X(t))$ is nondegenerate. If $\{X(t)\}$ is a nondegenerate Q-stable Lévy process on $\mathbf{R}^d$, then, for any $t > 0$, $\mathcal{L}(X(t))$ has a C^∞-density $f_t(x)$ on $\mathbf{R}^d$ and we have

$$\int_B f_t(x)\,\mathrm{d}x = \int_{\mathbf{R}^d} f_1(x) 1_B(t^Q x + c_t)\,\mathrm{d}x, \quad B \in \mathcal{B}(\mathbf{R}^d). \tag{5.11}$$

Indeed, the existence of a C^∞-density comes from the estimate $\left|\widehat{\mu}(z)\right| \le c_1 e^{-c_2|z|^\beta}$ with some positive constants c_1, c_2, β for $\mu = \mathcal{L}(X(1))$; the relation (5.11) is equivalent to (5.10).

Sharpe (1969) found the following. Suppose that μ is Q-stable and nondegenerate on $\mathbf{R}^d$. Then $Q \in M_{[\frac{1}{2},\infty)}(d)$ and, moreover, any eigenvalue of Q with real part $\frac{1}{2}$ is a simple root of the minimal polynomial of Q; μ is Gaussian if and only if $Q \in M_{\{\frac{1}{2}\}}(d)$; μ is purely non-Gaussian if and only if $Q \in M_{(\frac{1}{2},\infty)}(d)$.

If μ is Q-stable and nondegenerate on $\mathbf{R}^d$, then there are two Q-invariant subspaces V_1, V_2 of $\mathbf{R}^d$ such that $\mathbf{R}^d = V_1 \oplus V_2$, $\dim V_1 = d_1, \dim V_2 = d_2 = d - d_1$, $[Q]_{V_1} \in M_{\{\frac{1}{2}\}}(d_1)$, $[Q]_{V_2} \in M_{(\frac{1}{2},\infty)}(d_2)$, and $\mu = \mu_1 \times \mu_2$ with μ_1 being a Gaussian $[Q]_{V_1}$-stable distribution on V_1 and μ_2 being a purely non-Gaussian $[Q]_{V_2}$-stable distribution on V_2. Here we allow $d_1 = 0$ or $d_2 = 0$. The square brackets denote the restriction of an operator. This is shown by Sharpe (1969). For necessary and sufficient conditions on the forms of the Gaussian covariance matrix Σ and the Lévy measure ν of a Q-stable distribution, see Sato and Yamazato (1985). Concerning the question when a Q-stable distribution is strictly Q-stable see Sato (1987).

Remark 5.3 Any Q-stable distribution belongs to $L_\infty(Q)$. Moreover the class $L_\infty(Q)$ is generated by cQ-stable distributions in the following sense. Let $S(Q)$ be the class of distributions on $\mathbf{R}^d$ which are cQ-stable for some $c > 0$. Then, $L_\infty(Q)$ is the smallest class containing $S(Q)$ and closed under convolution and weak convergence, as proved in Sato and Yamazato (1985).

5.3 Relation to processes of Ornstein-Uhlenbeck type

The Q-selfdecomposability is closely related to processes of Ornstein-Uhlenbeck type. This fact was found by Wolfe (1982) and Sato and Yamazato (1983,1984) and revealed a deep meaning of Q-selfdecomposability from one aspect. We will state their main results. For another aspect related to selfsimilar processes with independent increments, see Section 16 of Sato (1999).

Let $Q \in M_{(0,\infty)}(d)$ and let $\{X(t)\}$ be a Lévy process on $\mathbf{R}^d$. Consider the linear stochastic equation driven by $\{X(t)\}$

$$\mathrm{d}Z(t) = -QZ(t)\mathrm{d}t + \mathrm{d}X(t), \tag{5.12}$$

where the initial condition $Z(0)$ is independent of $\{X(t)\}$. The solution to this equation is expressed as

$$Z(t) = e^{-tQ}Z(0) + \int_0^t e^{-(t-u)Q}\,\mathrm{d}X(u), \tag{5.13}$$

which is called a *process of Ornstein-Uhlenbeck type* with *background driving process* $\{X(t)\}$.

Recall that a probability measure μ on $\mathbf{R}^d$ is said to be a stationary distribution for (5.12) if $\mathcal{L}(Z(0)) = \mu$ implies $\mathcal{L}(Z(t)) = \mu$ for every t. The equation (5.12) allows at most one stationary distribution. Further, there exists a stationary distribution if and only if

$$\int_{\mathbf{R}^d} \log^+|x|\,\nu_0(\mathrm{d}x) < \infty, \tag{5.14}$$

where ν_0 denotes the Lévy measure of $\{X(t)\}$ and $\log^+|x| = 0 \vee \log|x|$. The condition (5.14) is equivalent to

$$\int_{\mathbf{R}^d} \log^+|x|\,\mu_0(\mathrm{d}x) < \infty, \tag{5.15}$$

where $\mu_0 = \mathcal{L}((X(1))$.

Let μ be a probability measure on $\mathbf{R}^d$. In order that there exist a Lévy process $\{X(t)\}$ on $\mathbf{R}^d$ such that μ is the unique stationary distribution of (5.12) it is necessary and sufficient that μ is Q-selfdecomposable. If μ is Q-selfdecomposable then the law of $\{X(t)\}$ is uniquely determined by μ. Thus the class of Q-selfdecomposable distributions μ on $\mathbf{R}^d$ is in one-to-one correspondence with the class of infinitely divisible distribution μ_0 satisfying (5.15). The relation is expressed as

$$\widehat{\mu}(z) = \exp\Big[\int_0^\infty \log\widehat{\mu}_0\big(e^{-tQ^\top}z\big)\,\mathrm{d}t\Big].$$

The characteristic triplets (Σ, ν, γ) of μ and $(\Sigma_0, \nu_0, \gamma_0)$ of μ_0 are related as follows:

$$
\begin{aligned}
\Sigma &= \int_0^\infty e^{-tQ}\Sigma_0 e^{-tQ^\top}\,\mathrm{d}t,\\
\Sigma_0 &= Q\Sigma + \Sigma Q^\top,\\
\nu(B) &= \int_{\mathbf{R}^d} \nu_0(\mathrm{d}x) \int_0^\infty 1_B(e^{-tQ}x)\,\mathrm{d}t,\\
\nu_0(B) &= -\int_{\mathbf{S}_Q} \sigma(\mathrm{d}\xi) \int_0^\infty 1_B(u^Q\xi)\,\mathrm{d}k_\xi(u),\\
\gamma &= Q^{-1}\gamma_0 + \int_0^\infty \mathrm{d}t \int_{\mathbf{R}^d} e^{-tQ}x\big(1_D(e^{-tQ}x) - 1_D(x)\big)\,\nu_0(\mathrm{d}x),
\end{aligned}
$$

where σ and $k_\xi(u)$ were introduced in (5.5) and $D = \{|x| \le 1\}$.

Remark 5.4 Let μ be Q-selfdecomposable on $\mathbf{R}^d$ corresponding to the law $\mu_0 = \mathcal{L}(X(1))$ of the background driving process. Let $m \in \{1, 2, \ldots, \infty\}$. Then $\mu \in L_m(Q)$ if and only if $\mu_0 \in L_{m-1}(Q)$. Here we read $m - 1 = \infty$ when $m = \infty$. The relation $\widehat{\mu}(z) = \widehat{\mu}_0(z)e^{\mathrm{i}\langle c,z\rangle}$ holds with some $c \in \mathbf{R}^d$ if and only if μ is Q-stable or, equivalently, μ_0 is Q-stable. The relation $\mu = \mu_0$ holds if and only if μ is strictly Q-stable or, equivalently, μ_0 is strictly Q-stable.

6 Role of stability in inheritance of selfdecomposability

In this section we exploit the role of stability of the subordinand in inheritance of selfdecomposability from the subordinator to the subordinated. We also provide a generalization of random variables of type G. Finally two special cases are considered. In Subsection 5.1 we fixed d and defined $L_m(Q)$, for a $d \times d$ matrix Q, as a class of distributions on $\mathbf{R}^d$. But, in the following, we write $L_m(Q; \mathbf{R}^n)$ when Q is $n \times n$ and distributions on $\mathbf{R}^n$ are considered.

6.1 Inheritance of operator selfdecomposability

Let us consider multivariate subordination, as defined in Section 3. That is, $\{X(t)\}$ is an n-dimensional Lévy process with d independent components, $X(t) = \big(X_1(t), \ldots, X_d(t)\big)^\top$, each component $\{X_j(t)\}$ being an n_j-dimensional Lévy process, $n = n_1 + \ldots + n_d$, and $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$

is a d-dimensional subordinator. The processes $\{X(t)\}$ and $\{T(t)\}$ are assumed to be independent and the subordinated process $\{Y(t)\}$ is given in (3.11) by

$$Y(t) = \big(X_1(T_1(t)), \dots, X_d(T_d(t))\big)^\top. \tag{6.1}$$

In the simplest case where $n = d = 1$ and $\{X(t)\}$ is a Brownian motion, Ismail and Kelker (1979) and Halgreen (1979) showed[3] that $\{Y(t)\}$ is selfdecomposable if $\{T(t)\}$ is selfdecomposable. We are interested, in general, under what conditions the selfdecomposability or the class L_m property is inherited from $\{T(t)\}$ to $\{Y(t)\}$. We will show that strict stability of the component processes $\{X_j(t)\}$ is a sufficient condition for this inheritance. When the index α_j of stability of $\{X_j(t)\}$ depends on j we are led to consideration of operator stability and operator selfdecomposability. Therefore it is natural to consider, from the beginning, the case where $\{T(t)\}$ is H-selfdecomposable and each component $\{X_j(t)\}$ is strictly Q_j-stable for some matrices H and Q_j.

Theorem 6.1 *Assume that each $\{X_j(t)\}$ is strictly Q_j-stable for some $Q_j \in M_{[\frac{1}{2},\infty)}(n_j)$. Let $H = \operatorname{diag}(h_1, \dots, h_d)$ with $h_j > 0$ for all j. If $\{T(t)\}$ is H-selfdecomposable, then the subordinated n-dimensional Lévy process $\{Y(t)\}$ is D-selfdecomposable, where $D = \operatorname{diag}(h_1 Q_1, \dots h_d Q_d) \in M_{(0,\infty)}(n)$. More generally, if $\{T(t)\}$ is of class $L_m(H; \mathbf{R}^d)$ with $m \in \{0, 1, \dots, \infty\}$, then $\{Y(t)\}$ is of class $L_m(D; \mathbf{R}^n)$.*

Proof. Recall that $\mu_j = \mathcal{L}(X_j(1))$ and $\mu = \mathcal{L}(X(1))$. Since each μ_j is the weak limit of a sequence of nondegenerate Q_j-stable distributions on $\mathbf{R}^{n_j}$ and since, for any $Q \in M_{(0,\infty)}(n_j)$ and m, the class $L_m(Q; \mathbf{R}^{n_j})$ is closed under weak convergence, we may and do assume that each μ_j is nondegenerate with a density $f_j(x_j)$. Suppose that $\{T(t)\}$ is H-selfdecomposable. As the hardest part in the analysis of the characteristic triplet of $\{Y(t)\}$ is $\nu^{\#}_{(1)}$, let us begin with that. Since $u^H = \operatorname{diag}(u^{h_1}, \dots, u^{h_d})$, equation (5.5) shows that the Lévy measure ρ of $\{T(t)\}$ has the expression

$$\rho(B) = \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_0^\infty \frac{k_\xi(u)}{u} 1_B(u^H \xi)\, \mathrm{d}u, \quad B \in \mathcal{B}(\mathbf{R}^d_+ \backslash \{0\}), \tag{6.2}$$

[3] In the arguments given by these authors it is implicitly used that, in the representation (5.1) of a positive selfdecomposable random variable X, the component Y_b must be positive. This can however be established via the Lévy measure ν of X, which satisfies $\nu(B) \geq \nu(bB)$ for $b > 1$.

where

$$\mathbf{S}_+ = \{\xi \in \mathbf{R}_+^d : |\xi| = 1\},$$

σ is a finite measure on $\mathbf{S}_+$, and $k_\xi(u)$ is measurable in ξ and nonnegative, decreasing in u. Then, by Theorem 3.3, the first term $\nu_{(1)}^{\#}$ of the Lévy measure $\nu^{\#}$ of $\{Y(t)\}$ is expressed for $B \in \mathcal{B}(\mathbf{R}^n \backslash \{0\})$ as

$$\begin{aligned}
\nu_{(1)}^{\#}(B) &= \int_{\mathbf{R}_+^d} \rho(\mathrm{d}\mathbf{s})\, \mu^{\mathbf{s}}(B) \\
&= \int_{\mathbf{R}_+^d} \rho(\mathrm{d}\mathbf{s}) \int_{\mathbf{R}^{n_1}\times\ldots\times\mathbf{R}^{n_d}} 1_B\big(x_1,\ldots,x_d\big)\mu_1^{s_1}(\mathrm{d}x_1)\cdots\mu_d^{s_d}(\mathrm{d}x_d) \\
&= \int_{\mathbf{R}_+^d} \rho(\mathrm{d}\mathbf{s}) \int_{\mathbf{R}^{n_1}\times\ldots\times\mathbf{R}^{n_d}} 1_B\big(s_1^{Q_1}x_1,\ldots,s_d^{Q_d}x_d\big)\prod_{j=1}^d f_j(x_j)\mathrm{d}x_j \\
&= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_0^\infty \frac{k_\xi(u)}{u}\,\mathrm{d}u \\
&\quad \times \int_{\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_d}} 1_B\big((\xi_1 u^{h_1})^{Q_1}x_1,\ldots,(\xi_d u^{h_d})^{Q_d}x_d\big)\prod_{j=1}^d f_j(x_j)\,\mathrm{d}x_j
\end{aligned}$$

by (5.11) and (6.2). Any $x \in \mathbf{R}^n \backslash \{0\}$ can be expressed uniquely as $x = u^D \eta$ with $\eta \in \mathbf{S}_D$ and $u > 0$, where

$$\mathbf{S}_D = \{\eta \in \mathbf{R}^n : |\eta| = 1 \text{ and } \big|u^D\eta\big| > 1 \ \text{ for all } u > 1\}.$$

Correspondingly, the Lebesgue measure on $\mathbf{R}^n$ is disintegrated as

$$\int_A \mathrm{d}x = \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \int_0^\infty 1_A\big(v^D\eta\big)\,\theta_\eta(\mathrm{d}v), \quad A \in \mathcal{B}(\mathbf{R}^n),$$

with some finite measure $\tau(\mathrm{d}\eta)$ on $\mathbf{S}_D$ and σ-finite measures $\theta_\eta(\mathrm{d}v)$ measurable with respect to η. We have $v^D = \mathrm{diag}(v^{h_1Q_1},\ldots,v^{h_dQ_d})$. Hence, if $x = (x_1,\ldots,x_d)^\top = v^D\eta \in \mathbf{R}^{n_1}\times\ldots\times\mathbf{R}^{n_d}$, then $x_j = v^{h_jQ_j}\eta_j$ for $j = 1,\ldots,d$

with $\eta = (\eta_1, \ldots, \eta_d)^\top$ and $\eta_j \in \mathbf{R}^{n_j}$. It follows that

$$\begin{aligned}
\nu^{\#}_{(1)}(B) &= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_0^\infty \frac{k_\xi(u)}{u}\,\mathrm{d}u \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \\
&\quad \times \int_0^\infty 1_B\big((vu)^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, (vu)^{h_d Q_d} \xi_d^{Q_d} \eta_d\big) \theta_\eta(\mathrm{d}v) \\
&\quad \times \prod_{j=1}^d f_j\big(v^{h_j Q_j} \eta_j\big) \\
&= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \int_0^\infty \big(\prod_{j=1}^d f_j(v^{h_j Q_j} \eta_j)\big) \theta_\eta(\mathrm{d}v) \\
&\quad \times \int_0^\infty \frac{k_\xi(wv^{-1})}{w} 1_B\big(w^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, w^{h_d Q_d} \xi_d^{Q_d} \eta_d\big)\,\mathrm{d}w.
\end{aligned}$$

Here we have used change of variable from u to w by $vu = w$ for fixed v. Hence

$$\begin{aligned}
\nu^{\#}_{(1)}(B) &= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \int_0^\infty \frac{k^{\#}_{\xi,\eta}(w)}{w} \\
&\quad \times 1_B\big(w^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, w^{h_d Q_d} \xi_d^{Q_d} \eta_d\big)\,\mathrm{d}w,
\end{aligned} \tag{6.3}$$

where

$$k^{\#}_{\xi,\eta}(w) = \int_0^\infty k_\xi(wv^{-1}) \Big(\prod_{j=1}^d f_j\big(v^{h_j Q_j} \eta_j\big)\Big)\,\theta_\eta(\mathrm{d}v). \tag{6.4}$$

Notice that $k^{\#}_{\xi,\eta}(w)$ is decreasing in w. We claim that

$$(\phi_{D,b} \nu^{\#}_{(1)})(B) \geq 0 \quad \text{for } b > 1 \text{ and } B \in \mathcal{B}_0(\mathbf{R}^n), \tag{6.5}$$

where $\phi_{D,b}$ is defined by (5.6). Since

$$\begin{aligned}
& 1_{b^D B}\big(w^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, w^{h_d Q_d} \xi_d^{Q_d} \eta_d\big) \\
=\;& 1_B\big(b^{-h_1 Q_1} w^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, b^{-h_d Q_d} w^{h_d Q_d} \xi_d^{Q_d} \eta_d\big)
\end{aligned}$$

we have

$$\begin{aligned}
\nu^{\#}_{(1)}(b^D B) &= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \int_0^\infty \frac{k^{\#}_{\xi,\eta}(w)}{w} \\
&\quad \times 1_B\big((b^{-1}w)^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, (b^{-1}w)^{h_d Q_d} \xi_d^{Q_d} \eta_d\big)\,\mathrm{d}w \\
&= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \\
&\quad \times \int_0^\infty \frac{k^{\#}_{\xi,\eta}(bw)}{w}\, 1_B\big(w^{h_1 Q_1} \xi_1^{Q_1} \eta_1, \ldots, w^{h_d Q_d} \xi_d^{Q_d} \eta_d\big)\,\mathrm{d}w.
\end{aligned}$$

This and (6.3) show (6.5), since $k^{\#}_{\xi,\eta}(w) \ge k^{\#}_{\xi,\eta}(bw)$.

Next, consider the case where $\{T(t)\}$ is of class $L_1(H;\mathbf{R}^d)$. This means that

$$(\phi^2_{H,b}\rho)(B) \ge 0 \quad \text{for } b > 1 \text{ and } B \in \mathcal{B}_0(\mathbf{R}^d). \tag{6.6}$$

Equivalently,

$$(\psi^2_b k_\xi)(u) \ge 0 \quad \text{for } b > 1, \tag{6.7}$$

where ψ^2_b is the second iteration of the operator ψ_b defined by (5.8). We have

$$\begin{aligned}(\phi_{D,b}\nu^{\#}_{(1)})(B) &= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \int_0^\infty \frac{(\psi_b k^{\#}_{\xi,\eta})(w)}{w} \\ &\quad \times 1_B\big(w^{h_1Q_1}\xi_1^{Q_1}\eta_1, \ldots, w^{h_dQ_d}\xi_d^{Q_d}\eta_d\big)\,\mathrm{d}w,\end{aligned}$$

and hence

$$\begin{aligned}(\phi_{D,b}\nu^{\#}_{(1)})(b^D B) &= \int_{\mathbf{S}_+} \sigma(\mathrm{d}\xi) \int_{\mathbf{S}_D} \tau(\mathrm{d}\eta) \int_0^\infty \frac{(\psi_b k^{\#}_{\xi,\eta})(bw)}{w} \\ &\quad \times 1_B\big(w^{h_1Q_1}\xi_1^{Q_1}\eta_1, \ldots, w^{h_dQ_d}\xi_d^{Q_d}\eta_d\big)\,\mathrm{d}w.\end{aligned}$$

By (6.4) we have

$$(\psi_b k^{\#}_{\xi,\eta})(w) = \int_0^\infty (\psi_b k_\xi)(wv^{-1})\Big(\prod_{j=1}^d f_j(v^{h_jQ_j}\eta_j)\Big)\,\theta_\eta(\mathrm{d}v). \tag{6.8}$$

Thus we obtain

$$(\phi^2_{D,b}\nu^{\#}_{(1)})(B) \ge 0$$

from (6.7).

If $\{T(t)\}$ is of class $L_m(H;\mathbf{R}^d)$ then we can repeat this procedure m times, arriving at

$$(\phi^j_{D,b}\nu^{\#}_{(1)})(B) \ge 0 \quad \text{for } 1 \le j \le m+1, b > 1, \text{ and } B \in \mathcal{B}_0(\mathbf{R}^n).$$

This finishes the analysis of $\nu^{\#}_{(1)}$.

Next let us consider $\nu^{\#}_{(2)}$, the second term of the Lévy measure of $\{Y(t)\}$. Since

$$b^D B = \{(b^{h_1Q_1}x_1, \ldots, b^{h_dQ_d}x_d) : (x_1, \ldots, x_d)^\top \in B\},$$

it follows from (3.20) that

$$\nu^{\#}_{(2)}(b^D B) = \sum_{j=1}^{d} c_j \nu_j\big(b^{h_j Q_j}(B \cap \mathcal{A}_j)\big), \tag{6.9}$$

where we are making the obvious identification of the set $B \cap \mathcal{A}_j$ with a set in $\mathbf{R}^{n_j}$. Since $\{X_j(t)\}$ is Q_j-stable and since Q_j-stable distributions belong to $L_\infty(Q_j)$ by Remark 5.3, we can see from (6.9) that $\nu^{\#}_{(2)}$ has the property of Lévy measures from the class $L_\infty(D; \mathbf{R}^n)$.

To complete the proof, let us look at the Gaussian covariance matrix $\Sigma^{\#}$ of $\{Y(t)\}$. For $z = (z_1, \ldots, z_d)^\top \in \mathbf{R}^n$ with $z_j \in \mathbf{R}^{n_j}$, we have

$$e^{-tD^\top} z = \big(e^{-th_1 Q_1^\top} z_1, \ldots, e^{-th_d Q_d^\top} z_d\big).$$

Hence, by (3.16),

$$\langle \Sigma^{\#} e^{-tD^\top} z, e^{-tD^\top} z\rangle = \sum_{j=1}^{d} c_j \langle \Sigma_j e^{-th_j Q_j^\top} z_j, e^{-th_j Q_j^\top} z_j\rangle.$$

We have

$$\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\Big)^j \langle \Sigma^{\#} e^{-tD^\top} z, e^{-tD^\top} z\rangle \geq 0 \quad \text{for } j = 1, 2, \ldots, \tag{6.10}$$

since each $\langle \Sigma_j e^{-th_j Q_j^\top} z_j, e^{-th_j Q_j^\top} z_j\rangle$ has the same property, as $\{X_j(t)\}$ is of class $L_\infty(Q_j)$. By (6.10), $\Sigma^{\#}$ has the property of Gaussian covariance matrices from the class $L_\infty(D; \mathbf{R}^n)$. $\square$

Let us give an example showing that a subordinated process is not always selfdecomposable when the subordinator and subordinand are selfdecomposable.

Example 6.2 Let $n = d = 1$, $\{X(t)\}$ and $\{T(t)\}$ be independent gamma processes with parameter 1 and define $Y(t) = X(T(t))$. Then $\{Y(t)\}$ is not selfdecomposable, although $\{X(t)\}$ and $\{T(t)\}$ are selfdecomposable.

Proof. In this case

$$\begin{aligned}
\rho(\mathrm{d}s) &= \frac{e^{-s}}{s}\mathrm{d}s, \\
\nu^{\#}(B) &= \int_0^\infty \rho(\mathrm{d}s) P(X(s) \in B) = \int_0^\infty \frac{e^{-s}}{s}\mathrm{d}s \int_B \frac{1}{\Gamma(s)} x^{s-1} e^{-x} \mathrm{d}x \\
&= \int_B e^{-x}\mathrm{d}x \int_0^\infty \frac{e^{-s}}{s}\frac{x^{s-1}}{\Gamma(s)}\mathrm{d}s.
\end{aligned}$$

Hence,

$$\nu^{\#}(B) = \int_B \frac{k^{\#}(x)}{x} \mathrm{d}x$$

with

$$k^{\#}(x) = e^{-x} \int_0^\infty \frac{e^{-s} x^s}{\Gamma(s+1)} \mathrm{d}s.$$

This is the decomposition of $\nu^{\#}$ in equation (5.5) in one dimension. This shows that $k^{\#}(0+) = 0$. Hence, $k^{\#}(x)$ cannot be a decreasing function, and therefore $\{Y(t)\}$ is not selfdecomposable. $\square$

Notice that in the above example $\{X(t)\}$ is selfdecomposable but not of class L_1. Indeed, the function $h(r)$ defined by $h(r) = k(e^r) = e^{-e^r}$ satisfies $h'(r) = -e^r e^{-e^r}$ and $h''(r) = (e^{2r} - e^r)e^{-e^r} < 0$ for r near $-\infty$, and hence we do not have $(\psi_b^2 k)(u) \geq 0$ for all $b > 1$ and $u > 0$.

Let us complement Theorem 6.1 about inheritance of strict operator stability.

Proposition 6.3 *As in Theorem 6.1, assume that each of the independent components of the subordinand is strictly Q_j-stable for some $Q_j \in M_{[\frac{1}{2},\infty)}(n_j)$ and let $H = \mathrm{diag}(h_1, \ldots, h_d)$ with $h_j > 0$ for all j. If the subordinator $\{T(t)\}$ is strictly H-stable, then the subordinated process $\{Y(t)\}$ is strictly D-stable, where $D = \mathrm{diag}(h_1 Q_1, \ldots, h_d Q_d)$.*

Proof. We only use (3.14). Assume that $\{T(t)\}$ is strictly H-stable. Then the function $\Psi(w)$ now satisfies

$$t\Psi(w) = \Psi(t^H w)$$

for $w = (w_1, \ldots, w_d)^\top \in \mathbf{C}^d$ with $\mathrm{Re} w_j \leq 0$ for all j. Hence, for $z = (z_1, \ldots, z_d)^\top \in \mathbf{R}^n$ with $z_j \in \mathbf{R}^{n_j}$,

$$\begin{aligned} t\Psi(\mathrm{Log}\,\widehat{\mu}(z)) &= \Psi(t^H(\mathrm{Log}\,\widehat{\mu}(z))) \\ &= \Psi(t^{h_1} \log \widehat{\mu}_1(z_1), \ldots, t^{h_d} \log \widehat{\mu}_d(z_d)) \\ &= \Psi\big(\log \widehat{\mu}_1(t^{h_1 Q_1^\top} z_1), \ldots, \log \widehat{\mu}_d(t^{h_d Q_d^\top} z_d)\big). \end{aligned}$$

The last equality is by the strict Q_j-stability of $\{X_j(t)\}$ for each j. Since $t^{D^\top} = \mathrm{diag}(t^{h_1 Q_1^\top}, \ldots, t^{h_d Q_d^\top})$, this shows that

$$t\Psi(\mathrm{Log}\,\widehat{\mu}(z)) = \Psi(\mathrm{Log}\,\widehat{\mu}(t^{D^\top} z)).$$

This, combined with (3.14), proves the strict D-stability of $\{Y(t)\}$. $\square$

6.2 A generalization of random vectors of type G

A random vector Y on $\mathbf{R}^n$ is said to be of type G if there are a standard Gaussian random vector X on $\mathbf{R}^n$ and a nonnegative infinitely divisible random variable T independent of X such that $Y \stackrel{d}{=} T^{1/2}X$. In this subsection we provide a generalization of type G random vectors.

Let Q be an $n \times n$ matrix with $Q = \mathrm{diag}(Q_1, \ldots, Q_d)$, where $Q_j \in M_{[\frac{1}{2},\infty)}(n_j)$, $j = 1, \ldots, d$, and $n = n_1 + \ldots + n_d$. We say that a random vector Y on $\mathbf{R}^n$ is of *type* $G(Q)$ if there are mutually independent strictly Q_j-stable random vectors X_j on $\mathbf{R}^{n_j}$ and an infinitely divisible random vector $T = (T_1, \ldots T_d)^\top$ on $\mathbf{R}^d_+$ independent of $X = (X_1, \ldots, X_d)^\top$, such that

$$Y = (Y_1, \ldots, Y_d)^\top \stackrel{d}{=} (T_1^{Q_1} X_1, \ldots, T_d^{Q_d} X_d)^\top. \tag{6.11}$$

Notice that $T_j^{Q_j}$ is an $n_j \times n_j$ matrix and hence the component Y_j is n_j-dimensional. A special case of this is when $d = 1$ and $Q = \alpha^{-1} I$, namely $X = X_1$ is strictly stable with index α on $\mathbf{R}^n$ and $T = T_1$ is infinitely divisible on $\mathbf{R}_+$. Then $Y \stackrel{d}{=} T^{1/\alpha}X$, and if $\alpha = 2$ we are back to the concept of type G.

The concept of type $G(Q)$ is in essence a special case of multivariate subordination, as the following proposition shows.

Proposition 6.4 *Let Q be as above. A random vector Y on $\mathbf{R}^n$ is of type $G(Q)$ if and only if $Y \stackrel{d}{=} Y(1)$, where $\{Y(t)\}$ is a Lévy process on $\mathbf{R}^n$ obtained by multivariate subordination of a Lévy process $\{X(t)\} = \{(X_1(t), \ldots, X_d(t))^\top\}$ with independent components such that each $\{X_j(t)\}$ is strictly Q_j-stable. As a consequence, Y is infinitely divisible if it is of type $G(Q)$.*

Proof. The 'if' part. Let $\{T(t)\} = \{(T_1(t), \ldots, T_d(t))^\top\}$ be the multivariate subordinator. Then, for every bounded measurable function f,

$$\begin{aligned} E\big[f\big(Y(1)\big)\big] &= E\Big[\Big(E\big[f\big(X_1(t_1), \ldots, X_d(t_d)\big)\big]\Big)_{(t_1,\ldots,t_d)^\top = T(1)}\Big] \\ &= E\Big[\Big(E\big[f\big(t_1^{Q_1} X_1(1), \ldots, t_d^{Q_d} X_d(1)\big)\big]\Big)_{(t_1,\ldots,t_d)^\top = T(1)}\Big] \\ &= E\big[f\big(T_1(1)^{Q_1} X_1(1), \ldots, T_d(1)^{Q_d} X_d(1)\big)\big], \end{aligned}$$

where the second equality is by the strict Q_j-stability of $\{X_j(t)\}$. It follows that

$$Y(1) \stackrel{d}{=} \big(T_1(1)^{Q_1} X_1(1), \ldots, T_d(1)^{Q_d} X_d(1)\big)^\top$$

and Y is of type $G(Q)$.

The 'only if' part. Suppose that Y is of type $G(Q)$ and let X and T be those in the definition. Consider the Lévy process $\{X(t)\}$ and the subordinator $\{T(t)\}$ independent of $\{X(t)\}$ such that $X(1) \stackrel{d}{=} X$ and $T(1) \stackrel{d}{=} T$. Then the proof is similar to the 'if' part. $\square$

Proposition 6.5 *Let $H = \mathrm{diag}(h_1, \ldots, h_d)$, where $h_j > 0$ for all j, and let $Y \stackrel{d}{=} (T_1^{Q_1} X_1, \ldots, T_d^{Q_d} X_d)^\top$ be a random vector of type $G(Q)$, where X and T are as in the definition of type $G(Q)$. If T is H-selfdecomposable, then Y is D-selfdecomposable with $D = \mathrm{diag}(h_1 Q_1, \ldots, h_d Q_d)$. More generally, if T is of class $L_m(H; \mathbf{R}^d)$ for $m \in \{0, 1, \ldots, \infty\}$, then Y is of class $L_m(D; \mathbf{R}^n)$. If T is strictly H-stable, then Y is strictly D-stable.*

This is a consequence of Theorem 6.1 and Propositions 6.3 and 6.4.

6.3 Special case: $n = d$

In the set-up of Subsection 6.1 consider the case where $n = d$, i.e. $n_1 = \ldots = n_d = 1$. Thus $\{X(t)\}$ is a Lévy process on $\mathbf{R}^d$, $X(t) = (X_1(t), \ldots, X_d(t))^\top$, and the coordinates $\{X_j(t)\}$, $j = 1, \ldots, d$, are independent Lévy processes on $\mathbf{R}$. Let $\{T(t)\}$ be a d-dimensional subordinator independent of $\{X(t)\}$. Then the subordinated process is

$$Y(t) = \big(X_1(T_1(t)), \ldots, X_d(T_d(t))\big)^\top.$$

Assume that each $\{X_j(t)\}$ is a strictly stable process with index $\alpha_j \in (0, 2]$. Let $H = \mathrm{diag}(h_1, \ldots, h_d)$. Theorem 6.1 says that, if $\{T(t)\}$ is H-selfdecomposable, then $\{Y(t)\}$ is D-selfdecomposable, where

$$\begin{aligned} D &= HQ = \mathrm{diag}(h_1, \ldots, h_d)\mathrm{diag}(\alpha_1^{-1}, \ldots, \alpha_d^{-1}) \\ &= \mathrm{diag}(h_1 \alpha_1^{-1}, \ldots, h_d \alpha_d^{-1}). \end{aligned}$$

If $\{T(t)\}$ is of class $L_m(H; \mathbf{R}^d)$, then $\{Y(t)\}$ is of class $L_m(HQ; \mathbf{R}^d)$. The simplest case is that $\alpha = \alpha_j$, independent of j, and $H = I$, thus $HQ = \alpha^{-1} I$. Hence, in this case we have the following by Remark 5.1: If $\{T(t)\}$ is of class L_m in the ordinary sense for some $m \in \{0, 1, \ldots, \infty\}$, then $\{Y(t)\}$ is of class L_m.

We also see the following from Propositions 6.4 and 6.5 on random variables of type $G(Q)$ with $Q = \mathrm{diag}(\alpha_1^{-1}, \ldots, \alpha_d^{-1})$.

Proposition 6.6 *Let*

$$Y = (Y_1, \ldots, Y_d)^\top \stackrel{d}{=} (T_1^{1/\alpha_1} X_1, \ldots, T_d^{1/\alpha_d} X_d)^\top,$$

where $X_j, j = 1, \dots, d$, are independent, each X_j is strictly stable on $\mathbf{R}$ with index $\alpha_j \in (0, 2]$, and $T = (T_1, \dots, T_d)^\top$ is infinitely divisible on $\mathbf{R}_+^d$ independent of $X = (X_1, \dots, X_d)^\top$. Then Y is infinitely divisible on $\mathbf{R}^d$. Let $H = \mathrm{diag}(h_1, \dots, h_d)$ with $h_j > 0, j = 1, \dots, d$. If T is of class $L_m(H; \mathbf{R}^d)$ for some $m \in \{0, 1, \dots, \infty\}$, then Y is of class $L_m(HQ; \mathbf{R}^d)$. If T is strictly H-stable, then Y is strictly HQ-stable.

6.4 Special case: $d = 1$

Consider, in the set-up of Subsection 6.1, the case where $d = 1$ and n is arbitrary. Then $X(t) = X_1(t) = \big(X_{11}(t), \dots, X_{1n}(t)\big)^\top$. The process $\{T(t)\}$ is a univariate subordinator independent of $\{X(t)\}$ and we have

$$Y(t) = X_1(T(t)) = \big(X_{11}(T(t)), \dots, X_{1n}(T(t))\big)^\top. \tag{6.12}$$

This is (univariate) subordination as discussed in e.g. Theorem 30.1 in Sato (1999).

In Theorem 6.1 the choice of H is irrelevant when $d = 1$, as is seen from Remark 5.1. Thus we obtain the following.

Proposition 6.7 *If $\{T(t)\}$ is a selfdecomposable subordinator and $\{X(t)\}$ is a strictly Q-stable subordinand on $\mathbf{R}^n$ with some $Q \in M_{[\frac{1}{2},\infty)}(n)$, then the subordinated process $\{Y(t)\}$ is Q-selfdecomposable on $\mathbf{R}^n$. If further $\{T(t)\}$ is of class L_m with $m \in \{1, 2, \dots, \infty\}$, then $\{Y(t)\}$ is of class $L_m(Q; \mathbf{R}^n)$. If $\{T(t)\}$ is strictly stable with index β, then $\{Y(t)\}$ is strictly $\beta^{-1}Q$-stable.*

Remark 6.8 As a special case, assume that $\{X(t)\}$ is strictly stable with index α. Let $\{T(t)\}$ be selfdecomposable. Then $\{Y(t)\}$ is selfdecomposable in the ordinary sense. In fact, the Lévy measure $\nu^\#$ of $\{Y(t)\}$ can be represented by the Lévy measure ρ and the drift c of $\{T(t)\}$ as follows. We have $\rho(\mathrm{d}u) = k(u)u^{-1}\,\mathrm{d}u$ on $(0, \infty)$, with a nonnegative decreasing function k and $\nu^\# = \nu^\#_{(1)} + \nu^\#_{(2)}$. The first term $\nu^\#_{(1)}$ is

$$\nu^\#_{(1)}(B) = \int_{\mathbf{S}} \sigma_0(\mathrm{d}\xi) \int_0^\infty \frac{k^\#_\xi(u)}{u} 1_B(u\xi)\,\mathrm{d}u, \quad B \in \mathcal{B}(\mathbf{R}^n \backslash \{0\})$$

with σ_0 being the surface Lebesgue measure of the unit sphere $\mathbf{S}$ in $\mathbf{R}^n$ and

$$k^\#_\xi(u) = \int_0^\infty k(u^\alpha r^{-\alpha}) f(r\xi) r^{n-1}\,\mathrm{d}r, \tag{6.13}$$

where we assume that $\mu = \mathcal{L}(X(1))$ is nondegenerate with density $f(x)$. A proof of this is given by repeated use of Fubini's theorem and change of variables, since the density $f_t(x)$ of $\mathcal{L}(X(t))$ is represented as $f_t(x) = f(t^{-1/\alpha}x)t^{-n/\alpha}$. The formula (6.13) clearly shows how the properties of k are inherited to $k_\xi^\#$. If $\alpha < 2$ then the second term $\nu_{(2)}^\#$ is just

$$\nu_{(2)}^\#(B) = c \int_{\mathbf{S}} \sigma_X(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)u^{-\alpha-1}\,\mathrm{d}u, \tag{6.14}$$

where $\sigma_X(\mathrm{d}\xi)u^{-\alpha-1}\mathrm{d}u$ is the usual polar expression of the Lévy measure of the stable distribution $\mathcal{L}(X(1))$ with index $\alpha < 2$. If $\alpha = 2$, then $\nu_{(2)}^\# = 0$. The formula (6.13) and the criterion (5.7) make it obvious that, if $\{T(t)\}$ is of class L_m, then $\{Y(t)\}$ is of class L_m.

Finally the extension of type G random vectors given in Subsection 6.2 is reduced to the following when $d = 1$. A random vector Y on $\mathbf{R}^n$ is of type $G(Q)$ for $Q \in M_{[\frac{1}{2},\infty)}(n)$ if

$$Y \stackrel{d}{=} T^Q X, \tag{6.15}$$

where T is infinitely divisible on $\mathbf{R}_+$, X is strictly Q-stable, and T and X are independent. Thus Propositions 6.4 and 6.5 read as follows.

Proposition 6.9 *A random vector Y is of type $G(Q)$ if and only if there exists a subordinated process $\{Y(t)\}$ with a strictly Q-stable subordinand $\{X(t)\}$ such that $Y = Y(1)$.*
Let X be strictly Q-stable and independent of T. If T is of class L_m for some $m \in \{0, 1, \ldots, \infty\}$, then the random vector Y in (6.15) is of class $L_m(Q; \mathbf{R}^n)$.
If T is strictly stable with index β, then Y is strictly $\beta^{-1}Q$-stable.

Example 6.10 Let X and T be independent random variables on $\mathbf{R}$. Assume that X is strictly stable with index α and T is nonnegative. Let $Y = T^{1/\alpha}X$. Feller (1971), p. 176, notes the fact that if T is strictly stable with index β, then Y is strictly stable with index $\alpha\beta$. Bondesson (1992), p. 38, shows that if T has a gamma distribution, then Y is selfdecomposable. These are special cases of our Proposition 6.9. Bondesson's book discusses many class properties related to products of (powers of) independent random variables. $\square$

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