# Hedging Cryptos with Bitcoin Futures

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**Abstract**

The introduction of derivatives on Bitcoin, in particular the launch of futures contracts on CME in December 2017 and introduction of cryptocurrency indices like the CRIX or the Bloomberg Galaxy Crypto Index enables investors to hedge risk exposures of Bitcoin by futures or contingent claims on indices. We investigate methods of finding an optimal hedge ratio $h^*$ under different dependence structures modeled by copulae and employing different optimality definitions based on a range of risk measures. Because of volatility swings and jumps in Bitcoin prices, the traditional variance-based approach to obtain the hedge ratios is infeasible. The techniques are therefore generalised to various risk measures, such as Value-at-Risk, Expected Shortfall and more general, Spectral Risk Measures. In addition, we deploy different copulae for capturing the dependency between spot and future returns, such as the Gaussian, Student-$t$, NIG and Archimedean copulae. Various measures of hedge effectiveness in out-of-sample tests give insights in the practice of hedging Bitcoin. We find that across copulae and risk measures, the hedge effectiveness are very similar with the exception of the Frank copula, Expected Shortfall 99% and Value-at-Risk 99%. Our findings are based on an analysis for the time span from 29/12/2017 to 27/05/2021. The results allow

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investors to construct a stable portfolio with digital assets.

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# 1 Introduction

Cryptocurrencies (CCs) are a growing asset class. Many more CCs are now available on the market since the first cryptocurrency Bitcoin (BTC) surfaced (Nakamoto, 2009). In response to the rapid development of the CC market, the CME group launched a BTC future contract in Dec 2017. Trading volume in BTC futures surpassed $2 trillion in 2020 (CryptoCompare, 2020).

In April 2021, the market value of outstanding coins had risen to 2.3t, more than 6% of the world's narrow money supply and almost 3% of the world GDP. While more and more investors (individuals and institutions) are adding CCs and their derivatives into their portfolios, we see the need to understand the downside risk and find a suitable way to hedge against extreme risks. The price of BTC even surged to USD 60,000 in April 2021. Investment into CCs therefore require proper quantification of their dynamics, but more importantly, exact understanding of their dependency with surfacing contingent claims, e.g. futures and options. This paper analyse modern techniques for the choice of the hedge ratio of the CC portfolios with various copulae and risk measures.

The optimal hedge ratio is the appropriate size of futures contracts which should be held such that the movements in future price cancel that of BTC. The task of determining an optimal hedge ratio is not easy. It relies on the dependence between the BTC price and future price. Copulae provide the flexibility to model multivariate random variable separately by its margins and dependence structure. The concept of copulae was originally developed (but not under this name) by Wassily Hoeffding (Hoeffding, 1940a) , later popularised by the work of Abe Sklar (Sklar, 1959).

Different risk measures account for investors' risk attitude. They serve as loss functions in the searching process of optimal hedge ratio. Vast literature discussed the relationship between risk measures and investor's risk attitude, we refer readers to Artzner et al. (1999) for an axiomatic, economic reasoning approach of risk measure construction; Embrechts et al. (2002) for reasoning of using Expected Shortfall (ES) and Spectral Risk Measure (SRM) in addition to VaR; Acerbi (2002) for direct linkage between risk measures and investor's risk attitude using the concept of "risk aversion function".

Financial asset return is known to be non Gaussian (Fama, 1963). In particular, Gaussian models cannot produce so-called fat tails and asymmetry of observed probability densities, which leads to underestimate financial risks. Therefore, one cannot solely rely on $2^{\text{nd}}$ order moment calculations in order to minimize downside risk. Variance as a risk measure doesn't consider the variety of investors' utility functions. However, the investors are tail-risk averse. Bollerslev et al. (2015) find that the jump tail risk is more closely associated with changes in risk-aversion. It is important to link the investor utility's functions as hedging the tail risk. Significant tail risks lead to the need to investigate even static hedge with more refined methods than minimum-variance based (Ederington and Salas, 2008).

In order to capture the risk preferences of investors, in addition to variance, we include other risk measures. We consider also Value-at-Risk (VaR), Expected Shortfall (ES), and Spectral Risk Measure (SRM). VaR is widely used by the industry and easy to understand. ES and SRM are chosen because of their coherence property, in particular, they encourage diversification. SRM is also directly related to individual's utility function. Popular examples are the exponential SRM and power SRM introduced by Dowd et al. (2008).

This paper considers hedging BTC using its future. i.e. to find an optimal hedge ratio $h^*$ such that the risk of a hedged portfolio $r^h = r^S - h^* r^F$ has minimal risk. Here $r^S$ as the log return of

BTC spot price, $r^F$ the log return of BTC future. The leptokurtic properties mentioned above leads us to deploy a comprehensive way of modelling dependency namely copulae together with various risk measures as loss function to find optimal hedge ratio. We first calibrate the log returns of BTC and CME future by copulae, then find the optimal quantity of assets in the hedged portfolio according to a range of risk measures. Barbi and Romagnoli (2014) use the C-convolution operator introduced by Cherubini et al. (2011) to derive the distribution of linear combination of margins with copula as their dependence structure. We slightly amend their lemma and come up with a formula for the linear combination of random variables for our purpose.

This paper is organized as follows. Section 2 introduces the notion of optimal hedge ratio; section 3 decribe the method of estimation of copulae; section 4 provides the empirical result; section 5 concludes.All calculations in this work can be reproduced. The codes are available on www.quantlet.com.

## 2 Optimal hedge ratio

We form a portfolio with two assets, a spot asset and a future contract, for example Bitcoin spot and CME Bitcoin future. Our objective is to minimize the risk of the exposure in the spot. To keep a simple portfolio setting, we long one unit of the spot and short $h$ unit of the future with $h \in [0, \infty)$. Let $r^S$ and $r^F$ be the log returns of the spot and future price, the log return of the portfolio is

$$r^h = r^S - hr^F.$$

We call this portfolio a hedged portfolio: the price movement of spot is hedged by the price movement of future.

Risk is measured by risk measures. Assume the payoff $r^h$ of a portfolio lives in a probability space, $r^h \in L(\Omega, \mathcal{F}, \mathbb{P})$, and there is a risk measure on $r^h$ $\rho : r^h \mapsto \mathbb{R}$. We are looking for an optimal hedge ratio $h^*$ which minimizes risk measure

$$h^* = \underset{h}{\operatorname{argmin}}\, \rho(r^h).$$

Most risk measures are defined as functionals of the portfolio loss distribution $F_{r^h}$, i.e. $\rho : F_{r^h} \mapsto \mathbb{R}$. For example, Value-at-Risk (VaR) is simply the quantile of $r^h$ multiply with negative one $\mathrm{VaR}_{1-\alpha} = -F^{(-1)}_{r^h}(1-\alpha) = -\inf\{x \in \mathbb{R} : 1-\alpha \leq F_{r^h}(x)\}$, where $\alpha$ is a parameter chosen by investor. We need the knowledge of $F_{r^h}$ in order to measure risk. By convolution of random variables (Härdle and Simar, 2019), $f_{r^h}(z) = \int_{-\infty}^{\infty} f_{r^S,-hr^F}(x, z-x)dx$, where $f_{r^S,-hr^F}$ is the joint pdf of $r^S$ and $-hr^F$. Obviously the cdf of $r^h$ and risk measure depend on the joint distribution of $r^S$ and $-hr^F$.

Optimising $h$ according to $f_{r^S,-hr^F}$ is unfavorable in a sense that one needs to calibrate a new joint pdf $f_{r^S,-hr^F}$ when updating $h$. This is too time consuming and unnecessary. Another problem of using the joint pdf is that one lacks the flexibility to model the margins. A joint pdf completely determines the form of its marginals, for example, margins of a bivariate $t$-distribution are univariate $t$-distributions.

To overcome such a problem, we use copulae. The benefit of using copulae is two folded. First, copulae allow us to model the margins and dependence structure separately, see Sklar's Theorem. Second, copulae are invariance under strictly monotone increasing function (Schweizer et al., 1981), see lemma below.

**Theorem 1 (Hoeffding Sklar Theorem)** *Let $F$ be a joint distribution function with margins $F_X, F_Y$. Then, there exists a copula $C : [0,1]^2 \mapsto [0,1]$ such that, for all $x, y \in \mathbb{R}$*

$$F(x,y) = C\{F_X(x), F_Y(y)\}. \tag{1}$$

*If the margins are continuous, then $C$ is unique; otherwise $C$ is unique on the range of the margins.*

*Conversely, if $C$ is a copula and $F_X, F_Y$ are univariate distribution functions, then the function $F$ defined by (1) is a joint distribution function with margins $F_X, F_Y$.*

Indeed, many basic results about copulae can be traced back to early works of Wassily Hoeffding (Hoeffding, 1940b, 1941). The works aimed to derive a measure of relationship of variables which is invariant under change of scale. See also Fisher and Sen (2012) for English translations of the original papers written in German. The following lemma is not hard to prove.

**Lemma 1**

$$C_{X,hY}\{F_X(s), F_{hY}(t)\} = C_{X,Y}\{F_X(s), F_Y(t/h)\}. \tag{2}$$

Leveraging the two features of copulae, Barbi and Romagnoli (2014) introduces the distribution of linear combination of random variables using copulae. We slightly edit the Corollary 2.1 of their work and yield the following correct expression of the distribution.

**Proposition 2** *Let $X$ and $Y$ be two real-valued continuous random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with absolutely continuous copula $C_{X,Y}$ and marginal distribution functions $F_X$ and $F_Y$. Then, the distribution function of of $Z$ is given by*

$$F_Z(z) = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \tag{3}$$

Here, $F^{(-1)}$ denotes the inverse of $F$, i.e., the quantile function.

Here $D_1 C(u,v) = \dfrac{\partial}{\partial u} C(u,v)$ see e.g. Equation (5.15) of (McNeil et al., 2005):

$$D_1 C_{X,Y}\{F_X(x), F_Y(y)\} = \mathbf{P}(Y \le y | X = x). \tag{4}$$

***Proof.*** Using the identity (4) gives

$$\begin{aligned} F_Z(z) &= \mathbf{P}(X - hY \le z) = \mathsf{E}\left\{\mathbf{P}\left(Y \ge \frac{X-z}{h}\Big| X\right)\right\} \\ &= 1 - \mathsf{E}\left\{\mathbf{P}\left(Y \le \frac{X-z}{h}\Big| X\right)\right\} = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \end{aligned}$$

■

**Corollary 1** *Given the formulation of the above portfolio, the pdf of $Z$ can be written as*

$$f_Z(z) = \left|\frac{1}{h}\right| \int_0^1 c_{X,Y}\left[F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}, u\right] \cdot f_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\} du \tag{5}$$

, *or*

$$f_Z(z) = \int_0^1 c_{X,Y}\left[F_X\left\{z + hF_Y^{(-1)}(u)\right\}, u\right] \cdot f_X\left\{z + hF_Y^{(-1)}(u)\right\} du. \tag{6}$$

The two expression are equivalent. Notice that the pdf of $Z$ in the above proposition is readily accessible as long as we have the copula density and the marginal densities. The proof and a generic expression can be found in the appendix.

### 2.1 Construction of Hedged Portfolio

To evaluate the BTC future hedging effectiveness on various crypto assets and indexes, we obtain the OHR of different risk measures with training data and compute the hedging effectiveness using the testing data.

For each set of training and testing data,

1. **Construct Univariate Kernel Density Function (KDE)**. With the training data, we obtain the spot and future's univariate kernel density function using the Gaussian kernel with bandwidth determined by the refined plug-in method (Härdle et al., 2004, section 3.3.3).
2. **Calibrate Copulae**. We then calibrate copulae outlined in 2.3 via the method of moments described in 3.1.
3. **Select Copula**. We compute the Akaike Information Criterion AIC $= -2l + 2k$, where $l$ is the likelihood of the copula and $k$ is the number of parameter in the copula. The copula with the lowest AIC is used for the next step.
4. **Search for OHR**. We search OHRs of different risk measures numerically by drawing samples from the selected copula and KDEs.
5. **Obtain testing log-return of hedged portfolio**. We apply the OHRs to the testing data $r_h = r_s - h^{*r_f}$.

### 2.2 Risk Measures

We consider a variety of risk measures: variance, Value-at-Risk (VaR), Expected Shortfall (ES), and Exponential Risk Measure (ERM). A summary of risk measures being used in portfolio selection problem can be found in Härdle et al. (2008).

Let $Z$ be a random variable of distribution $F_Z$.

1. Variance is $\mathrm{Var}(F_Z)$
2. VaR of a given confidence level $\alpha$ is $\mathrm{VaR}(F_Z) = -F_Z^{(-1)}(1-\alpha)$
3. ES with parameter $\alpha$ is $\mathrm{ES}(F_Z) = -\frac{1}{1-\alpha}\int_0^{1-\alpha} F_Z^{(-1)}(p)dp$
4. ERM with Arrow-Pratt coeficient of absolute risk aversion $k$ is $\mathrm{ERM}_k(F_Z) = \int_0^{1-\alpha} \phi(p)F_Z^{(-1)}(p)dp$ where $\phi$ is a weight function described in (8) below.

VaR, ES, and ERM fall into the class of Spectral Risk Measure (SRM). SRM has the from (Acerbi, 2002)

$$\rho_\phi(r^h) = -\int_0^1 \phi(p) F_Z^{(-1)}(p) dp, \tag{7}$$

where $p$ is the loss quantile and $\phi(p)$ is a user-defined weighting function defined over $[0,1]$. We consider only admissible risk spectra $\phi(p)$

i $\phi$ is positive

ii $\phi$ is decreasing

iii integrates to one.

The VaR's $\phi(p)$ gives all its weight on the $1-\alpha$ quantile of $Z$ and zero elsewhere, i.e. the weighting function is a Dirac delta function, hence violates the ii property of admissible risk spectra. The ES' $\phi(p)$ gives all tail quantiles the same weight of $\frac{1}{1-\alpha}$ and non-tail quantiles zero weight. ERM assumes investor's risk preference is in a form of exponential utility function $U(x) = 1 - e^{kx}$, its corresponding risk spectrum is defined as

$$\phi(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}}, \tag{8}$$

where $k$ is the Arrow-Pratt coefficient of absolute risk aversion.

The parameter $k$ has an economic interpretation of being the ratio between the second derivative and first derivative of investor's utility function on an risky asset

$$k = -\frac{U''(x)}{U'(x)}, \tag{9}$$

for $x$ in all possible outcomes. In case of the exponential utility, $k$ is the the constant absolute risk aversion (CARA).

### 2.3 Copulae

As we saw from the last section, risk measures we considered are all functionals of the joint distribution of $r^S$ and $r^F$. We test different copulae: Gaussian-, $t$-, Frank-, Gumbel-, Clayton-, Plackett-, mixture, and factor copula. This hedging exercise concerns only portfolios with two assets, we only present the bivariate version of copulae and some important features of a copula, they include Kendall's $\tau$, Spearman's $\rho$, upper tail dependence $\lambda_U \stackrel{\text{def}}{=} \lim_{q\to 1^-} \mathbf{P}\{X > F_X^{(-1)}(q) | Y > F_Y^{(-1)}(q)\}$ and lower tail dependence $\lambda_L \stackrel{\text{def}}{=} \lim_{q\to 0^+} \mathbf{P}\{X \le F_X^{(-1)}(q) | Y \le F_Y^{(-1)}(q)\}$. Furthermore, we denote the Fréchet-Hoeffding lower bound as $\boldsymbol{W}$, product copula as $\mathbf{\Pi}$, and the Fréchet-Hoeffding upper bound as $\boldsymbol{M}$, they represent cases of perfect negative dependence, independence, and perfect positive dependence respectively. For further detail, we refer readers to Joe (1997) and Nelsen (1999). See also Härdle and Okhrin (2010).

### 2.3.1 Elliptical Copulae

Elliptical copulae define dependence structures associated with elliptical distributions. The bivariate Gaussian copula is:

$$\begin{aligned}\boldsymbol{C}(u,v) &= \Phi_{2,\rho}\{\Phi^{-1}(u),\Phi^{-1}(v)\}\\ &= \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right\}dsdt \end{aligned}\tag{10}$$

where $\Phi_{2,\rho}$ is the cdf of bivariate Normal distribution with zero mean, unit variance, and correlation $\rho$. , and $\Phi^{-1}$ is quantile function univariate standard normal distribution. Please note that we use $\rho$ here to represent the correlation parameter in a Gaussian copula only for traditional purposes. In other sections, $\rho(\cdot)$ is a risk measure. The Gaussian copula density is

$$\begin{aligned}\boldsymbol{c}_\rho(u,v) &= \frac{\boldsymbol{\varphi}_{2,\rho}\{\Phi^{-1}(u),\Phi^{-1}(v)\}}{\varphi\{\Phi^{-1}(u)\}\cdot\varphi\{\Phi^{-1}(v)\}}\\ &= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right\},\end{aligned}\tag{11}$$

where $\boldsymbol{\varphi}_{2,\rho}(\cdot)$ is the pdf of $\Phi_{2,\rho}$, and $\varphi(\cdot)$ the standard normal distribution pdf.

The Kendall's $\tau_K$ and Spearman's $\rho_S$ of a bivariate Gaussian Copula are

$$\tau_K(\rho) = \frac{2}{\pi}\arcsin\rho \tag{12}$$

$$\rho_S(\rho) = \frac{6}{\pi}\arcsin\frac{\rho}{2} \tag{13}$$

The $t$-Copula has a form

$$\begin{aligned}\boldsymbol{C}(u,v) &= \boldsymbol{T}_{2,\rho,\nu}\{T_\nu^{-1}(u),T_\nu^{-1}(v)\}\\ &= \int_{-\infty}^{T_\nu^{-1}(u)}\int_{-\infty}^{T_\nu^{-1}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{1-\rho^2}} \end{aligned}\tag{14}$$

$$\left(1+\frac{s^2-2st\rho+t^2}{\nu}\right)^{-\frac{\nu+2}{2}} dsdt, \tag{15}$$

where $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$ denotes the cdf of bivariate $t$ distribution with scale parameter $\rho$ and degree of freedom $\nu$, $T_\nu^{-1}(\cdot)$ is the quantile function of a standard $t$ distribution with degree of freedom $\rho$.

The copula density is

$$\boldsymbol{c}(u,v) = \frac{\boldsymbol{t}_{2,\rho,\nu}\{T_\nu^{-1}(u),T_\nu^{-1}(v)\}}{t_\nu\{T_\nu^{-1}(u)\}\cdot t_\nu\{T_\nu^{-1}(v)\}}, \tag{16}$$

where $\boldsymbol{t}_{2,\rho,\nu}$ is the pdf of $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$, and $t_\nu$ the density of standard $t$ distribution.

Like all the other elliptical copulae, $t$ copula's Kendall's $\tau$ is identical to that of Gaussian copula (see Demarta and McNeil, 2005, and references therein).

### 2.3.2 Archimedean Copulae

The family of Archimedean copulae forms a large class of copulae with many convenient features. In general, they take a form

$$\boldsymbol{C}(u,v)=\psi^{-1}\{\psi(u),\psi(v)\}, \tag{17}$$

where $\psi:[0,1]\rightarrow[0,\infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1)=0$ for any permissible dependence parameter $\theta$. $\psi$ is also called generator. $\psi^{-1}$ is the inverse the generator.

The Frank copula (B3 in Joe (1997)) is a radial symmetric copula and cannot produce any tail dependence. It takes the form

$$\boldsymbol{C}_\theta(u,v)=\frac{1}{\theta}\log\left\{1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right\} \tag{18}$$

where $\theta\in[0,\infty]$ is the dependency parameter. $\boldsymbol{C}_{-\infty}=\boldsymbol{M}$, $\boldsymbol{C}_1=\boldsymbol{\Pi}$, and $\boldsymbol{C}_\infty=\boldsymbol{W}$.

The Copula density is

$$\boldsymbol{c}_\theta(u,v)=\frac{\theta e^{\theta(u+v)(e^\theta-1)}}{\left\{e^\theta-e^{\theta u}-e^{\theta v}+e^{\theta(u+v)}\right\}^2} \tag{19}$$

Frank copula has Kendall's $\tau$ and Spearman's $\rho$ as follow:

$$\tau_K(\theta)=1-4\frac{D_1\{-\log(\theta)\}}{\log(\theta)}, \tag{20}$$

and

$$\rho_S(\theta)=1-12\frac{D_2\{-\log(\theta)\}-D_1\{\log(\theta)\}}{\log(\theta)}, \tag{21}$$

where $D_1$ and $D_2$ are the Debye function of order 1 and 2. Debye function is $D_n=\frac{n}{x^n}\int_0^x\frac{t^n}{e^t-1}dt$.

Gumbel copula (B6 in Joe (1997)) has upper tail dependence with the dependence parameter $\lambda^U=2-2^{\frac{1}{\theta}}$ and displays no lower tail dependence.

$$\boldsymbol{C}_\theta(u,v)=\exp-\{(-\log(u))^\theta+(-\log(v))^\theta\}^{\frac{1}{\theta}}, \tag{22}$$

where $\theta\in[1,\infty)$ is the dependence parameter.While Gumbel copula cannot model perfect counter dependence (Nelsen, 2002), $\boldsymbol{C}_1=\boldsymbol{\Pi}$ models the independence, and $\lim_{\theta\rightarrow\infty}\boldsymbol{C}_\theta=\boldsymbol{W}$ models the perfect dependence.

$$\tau_K(\theta)=\frac{\theta-1}{\theta} \tag{23}$$

The Clayton copula, by contrast to Gumbel copula, generates lower tail dependence in a form $\lambda^L=2^{-\frac{1}{\theta}}$, but cannot generate upper tail dependence.

The Clayton copula takes the form

$$\boldsymbol{C}_\theta(u,v) = \left\{\max(u^{-\theta}+v^{-\theta}-1,0)\right\}^{-\frac{1}{\theta}}, \tag{24}$$

where $\theta \in (-\infty,\infty)$ is the dependency parameter. $\lim_{\theta\to-\infty} \boldsymbol{C}_\theta = \boldsymbol{M}$, $\boldsymbol{C}_0 = \mathbf{\Pi}$, and $\lim_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$.

Kendall's $\tau$ to this copula dependency is

$$\tau_K(\theta) = \frac{\theta}{\theta+2}. \tag{25}$$

#### 2.3.3 Mixture Copula

Mixture copula is a linear combination of copulae. For a 2-dimensional random variable $\boldsymbol{X} = (X_1, X_2)^\top$, its distribution can be written as linear combination $K$ copulae

$$\mathbf{P}(X_1 \le x_1, X_2 \le x_2) = \sum_{k=1}^{K} p^k \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{(k)}(x_1;\boldsymbol{\gamma}_1^{(k)}), F_{X_2}^{(k)}(x_2;\boldsymbol{\gamma}_2^{(k)}); \boldsymbol{\theta}^{(\boldsymbol{k})}\} \tag{26}$$

where $p^{(k)} \in [0,1]$ is the weight of each component, $\boldsymbol{\gamma}^{(k)}$ is the parameter of the marginal distribution in the $k^{\text{th}}$ component, and $\boldsymbol{\theta}^{(\boldsymbol{k})}$ is the dependence parameter with the copula of the $k^{\text{th}}$ component. The weights add up to one $\sum_{k=1}^{K} p^{(k)} = 1$.

We deploy a simplified version of the above representation by assuming the margins of $\boldsymbol{X}$ are not mixture. By Sklar's theorem one may write

$$\boldsymbol{C}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta}^{(\boldsymbol{k})}\}. \tag{27}$$

The copula density is again a linear combination of copula density

$$\boldsymbol{c}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{c}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta}^{(\boldsymbol{k})}\}. \tag{28}$$

While Kendall's $\tau$ of mixture copula is not known in close form, the Spearman's $\rho$ is

**Proposition 3** *Let $\rho_S^{(k)}$ be the Spearman's $\rho$ of the $k^{th}$ component and $\sum_{k=1}^{K} p^{(k)} = 1$ holds, the Spearman's $\rho$ of a mixture copula is*

$$\rho_S = \sum_{k=1}^{K} p^{(k)} \cdot \rho_S^{(k)} \tag{29}$$

***Proof.*** Spearman's $\rho$ is defined as (Nelsen, 1999)

$$\rho_S = 12 \int_{\mathbb{I}^2} \boldsymbol{C}(s,t)\,ds dt - 3. \tag{30}$$

Rewrite the mixture copula into sumation of components

$$\rho_S = 12 \int_{\mathbb{I}^2} \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}(s,t) ds dt - 3. \tag{31}$$

■

**Example 4** *The Fréchet class can be seen as an example of mixture copula. It is a convex combinations of $\boldsymbol{W}$, $\boldsymbol{\Pi}$, and $\boldsymbol{M}$ (Nelsen, 1999)*

$$\boldsymbol{C}_{\alpha,\beta}(u,v) = \alpha \boldsymbol{M}(u,v) + (1-\alpha-\beta)\boldsymbol{\Pi}(u,v) + \beta \boldsymbol{W}(u,v), \tag{32}$$

*where $\alpha$ and $\beta$ are the dependence parameters, with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. Its Kendall's $\tau$ and Spearman's $\rho$ are*

$$\tau_K(\alpha,\beta) = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3} \tag{33}$$

*, and*

$$\rho_S(\alpha,\beta) = \alpha - \beta \tag{34}$$

We use a mixture of Gaussian and independent copula in our analysis. We write the copula

$$\boldsymbol{C}(u,v) = p \cdot \boldsymbol{C}^{\text{Gaussian}}(u,v) + (1-p)(uv). \tag{35}$$

The corresponding copula density is

$$\boldsymbol{c}(u,v) = p \cdot \boldsymbol{c}^{\text{Gaussian}}(u,v) + (1-p). \tag{36}$$

This mixture allows us to model how much "random noise" appear in the dependency structure. In this hedging exercise, the structure of the "random noise" is not of our concern nor we can hedge the noise by a two-asset portfolio. However, the proportion of the "random noise" does affect the distribution of $r^h$, so as the optimal hedging ratio $h^*$. One can consider the mixture copula as a handful tool for stress testing. Similar to this Gaussian mix Independent copula, t copula is also a two parameter copula allow us to model the noise, but its interpretation of parameters is not as intuitive as that of a mixture. The mixing variable $p$ is the proportion of a manageable (hedgable) Gaussian copula, while the remaining proportion $1-p$ cannot be managed.

### 2.4 Other Copula

The Plackett copula has an expression

$$\boldsymbol{C}_\theta(u,v) = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{\{1+(\theta-1)(u+v)\}^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \tag{37}$$

$$\rho_S(\theta) = \frac{\theta+1}{\theta-1} - \frac{2\theta \log\theta}{(\theta-1)^2} \tag{38}$$

We include Placket copula in our analysis as it possesses a special property, the cross-product ratio is equal to the dependence parameter

$$\begin{aligned}
&\frac{\mathbf{P}(U \le u, V \le v) \cdot \mathbf{P}(U > u, V > v)}{\mathbf{P}(U \le u, V > v) \cdot \mathbf{P}(U > u, V \le v)} \\
&= \frac{\boldsymbol{C}_\theta(u,v)\{1 - u - v + \boldsymbol{C}_\theta(u,v)\}}{\{u - \boldsymbol{C}_\theta(u,v)\}\{v - \boldsymbol{C}_\theta(u,v)} \\
&= \theta. \qquad (39)
\end{aligned}$$

That is, the dependence parameter is equal to the ratio between number of concordence pairs and number of discordence pairs of a bivariate random variable.

# 3 Estimation

## 3.1 Simulated Method of Moments

This method is suggested by (Oh and Patton, 2013). In our setting, rank correlation e.g. Spearman's $\rho$ or Kendall's $\tau$, and quantile dependence measures at different levels $\lambda_q$ are calibrated against their empirical counterparts.

Spearman's rho, Kendall's tau, and quantile dependence of a pair $(X, Y)$ with copula $C$ are defined as

$$\rho_S = 12 \int\int_{I^2} C_{\boldsymbol{\theta}}(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3 \qquad (40)$$

$$\tau_K = 4\,\mathsf{E}[C_{\boldsymbol{\theta}}\{F_X(x), F_Y(y)\}] - 1, \qquad (41)$$

$$\lambda_q = \begin{cases} \mathbf{P}(F_X(X) \le q | F_Y(Y) \le q) = \dfrac{C_{\boldsymbol{\theta}}(q,q)}{q}, & \text{if } q \in (0, 0.5], \\ \mathbf{P}(F_X(X) > q | F_Y(Y) > q) = \dfrac{1 - 2q + C_{\boldsymbol{\theta}}(q,q)}{1-q}, & \text{if } q \in (0.5, 1). \end{cases} \qquad (42)$$

The empirical counterparts are

$$\begin{aligned}
\hat{\rho}_S &= \frac{12}{n} \sum_{k=1}^{n} \hat{F}_X(x_k)\hat{F}_Y(y_k) - 3, \\
\hat{\tau}_K &= \frac{4}{n} \sum_{k=1}^{n} \hat{C}\{\hat{F}_X(x_i), \hat{F}_X(y_i)\} - 1, \\
\hat{\lambda}_q &= \begin{cases} \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \dfrac{\mathbf{1}_{\{\hat{F}_X(x_k) \le q, \hat{F}_Y(y_k) \le q\}}}{q}, & \text{if } q \in (0, 0.5], \\ \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \dfrac{\mathbf{1}_{\{\hat{F}_X(x_k) > q, \hat{F}_Y(y_k) > q\}}}{1-q}, & \text{if } q \in (0.5, 1). \end{cases},
\end{aligned}$$

where $\hat{F}(x) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^{n} 1_{\{x_i \le x\}}$ and $\hat{C}(u,v) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^{n} 1_{\{u_i \le u, v_i \le v\}}$.

We denote $\tilde{\boldsymbol{m}}(\boldsymbol{\theta})$ be a $m$-dimensional vector of dependence measures according the the dependence parameters $\boldsymbol{\theta}$,and $\hat{\boldsymbol{m}}$ be the corresponding empirical counterpart. The difference between dependence measures and their counterpart is denoted by

$$\boldsymbol{g}(\boldsymbol{\theta}) = \hat{\boldsymbol{m}} - \tilde{\boldsymbol{m}}(\boldsymbol{\theta}).$$

The SMM estimator is

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmin}}\, \boldsymbol{g}(\boldsymbol{\theta})^\top \hat{\boldsymbol{W}} \boldsymbol{g}(\boldsymbol{\theta}),$$

where $\hat{W}$ is some positive definite weigh matrix.

In this work, we use $\tilde{\boldsymbol{m}}(\boldsymbol{\theta}) = (\rho_S, \lambda_{0.05}, \lambda_{0.1}, \lambda_{0.9}, \lambda_{0.95})^\top$ for calibration of Bitcoin price and CME Bitcoin future.

### 3.2 Maximum Likelihood Estimation

By the Hoeffding-Sklar's theorem, the joint density of a $d$-dimensional random variable $\boldsymbol{X}$ with sample size $n$ can be written as

$$\boldsymbol{f}_{\boldsymbol{X}}(x_1, ..., x_d) = \boldsymbol{c}\{F_{X_1}(x_1), ..., F_{X_d}(x_d)\} \prod_{j=1}^{d} f_{X_i}(x_i). \tag{43}$$

We follow the treatment of MLE documented in section 10.1 of Joe (1997), namely the inference functions for margins or IFM method. The log-likelihood $\sum_{i=1}^{n} f_{\boldsymbol{X}}(X_{i,1}, ..., X_{i,d})$ can be decomposed into dependence part and marginal part,

$$\begin{aligned} L(\boldsymbol{\theta}) &= \sum_{i=1}^{n} \boldsymbol{c}\{F_{X_1}(x_{i,1}; \boldsymbol{\delta}_1), ..., F_{X_d}(x_{i,d}; \boldsymbol{\delta}_d); \boldsymbol{\gamma}\} + \sum_{i=1}^{n} \sum_{j=1}^{d} f_{X_j}(x_{i,j}; \boldsymbol{\delta}_j) & (44) \\ &= L_C(\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma}) + \sum_{j=1}^{d} L_j(\boldsymbol{\delta}_j) & (45) \end{aligned}$$

where $\boldsymbol{\delta}_j$ is the parameter of the $j$-th margin, $\boldsymbol{\gamma}$ is the parameter of the parametric copula, and $\boldsymbol{\theta} = (\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma})$.

Instead of searching the $\boldsymbol{\theta}$ is a high dimensional space, Joe (1997) suggests to search for $\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d$ that maximize $L_1(\boldsymbol{\delta}_1), ..., L_d(\boldsymbol{\delta}_d)$, then search for $\hat{\boldsymbol{\gamma}}$ that maximize $L_C(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \boldsymbol{\gamma})$.

That is, under regularity conditions, $(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \hat{\boldsymbol{\gamma}})$ is the solution of

$$\left( \frac{\partial L_1}{\partial \boldsymbol{\delta}_1}, ..., \frac{\partial L_d}{\partial \boldsymbol{\delta}_d}, \frac{\partial L_C}{\partial \boldsymbol{\gamma}} \right) = \mathbf{0}. \tag{46}$$

However, the IFM requires making assumption to the distribution of of the margins. Genest et al. (1995) suggests to replace the estimation of marginals parameters estimation by non-parameteric estimation. Given non-parametric estimator $\hat{F}_i$ of the margins $F_i$, the estimator of the dependence parameters $\boldsymbol{\gamma}$ is

$$\hat{\boldsymbol{\gamma}} = \underset{\boldsymbol{\gamma}}{\operatorname{argmax}} \sum_{i=1}^{n} \boldsymbol{c}\{\hat{F}_{X_1}(x_{i,1}), ..., \hat{F}_{X_d}(x_{i,d}); \boldsymbol{\gamma}\}. \tag{47}$$

## 3.3 Comparison

Both the simulated method of moments and the maximum likelihood estimation are unbiased. The problem remain is which procedure is more suitable for hedging.

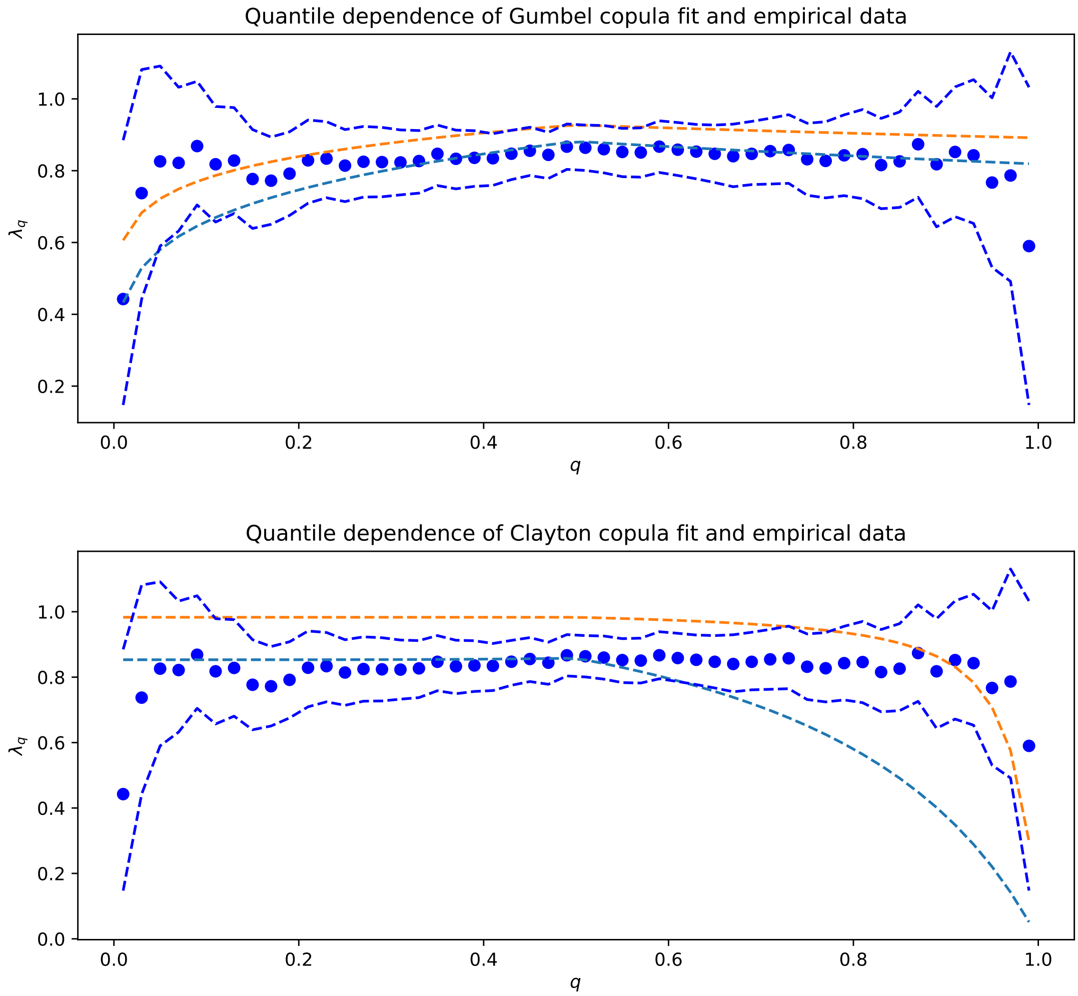


**Figure 1:** Quantile dependences of Gumbel, and Clayton Copula. The blue circle dots are the quantile dependence estimate of Bitcoin and CME future, blue dotted lines are the estimates' 90% confidnece interval. Orange dotted line is the copula implied quantile dependence by MM estimation. Light blue dotted line is the copula implied quantile dependence by MM estimation.

Figure 1 shows the empirical quantile dependence of Bitcoin and CME future and the copula implied quantile dependence from MLE and MM calibration procedures. Although the MLE is a better fit to a range of quantile dependence in the middle, it fails to address the situation in the tails. Our data empirically has weaker quantile dependence in the ends, and those points generate PnL to the hedged portfolio. MM is preferred visually as it produces a better fit to the dependence structure in the two extremes. Therefore, we deploy the method of moments throughout the analysis. We choose the $5^{\text{th}}$-, $10^{\text{th}}$-, $90^{\text{th}}$-, $95^{\text{th}}$-quantile, and Spearman's $\rho$ as the moments.

# 4 Data

In the empirical analysis, we consider the the risk reduction capability of the BTC future on five cryptos , BTC, ETH, ADA, LTC, and XRP, and five crypto indexes, BITX, BITW100, CRIX, BITW20, and BITW70, For each of the 10 hedging portfolios, a crypto or index is considered as the spot and held in a unit size long position, and the BTC future is held in short position of OHR unit in order to reduce the risk of the spot. All the hedging portfolios are cross asset hedging except the BTC-future portfolio. ETH, ADA, LTC, and XRP are popular cryptos tradable in various exchanges and have large market capitalization. BITX, BITW100, and CRIX are market-cap weighted crypto indexes with BTC as constituent. BITX and BITW100 tracks the total return of the 10 and 100 cryptos with largest market-cap respectively. CRIX decides the number of constituents by AIC and track that number of cryptos with largest market-cap. In our case, the number of constituents in CRIX is 5. BITW20 is also a market-cap weighted crypto index but with 20 largest market-cap cryptos outside the constituents of BITX. BITW70 has the same construction as BITW20 but with 70 largest market-cap cryptos outside BITX and BITW20. Therefore, BTC is excluded as constituent in BITW20 and BITW70.

We collect the spots' and BTC future's daily price at 15:00 US Central Time (CT). The reason of choosing this particular time is that the CME group determines the daily settlements for BTC futures based on the trading activities on CME Globex between 14:59 and 15:00 CT. 15:00 CT is also the reporting time of the daily closing price by the Bloomberg Terminal (BBT). Cryptos data are collected from a data provider called Tiingo. Tiingo aggregates crypto OHLC (open, high, low, and close) prices fed by APIs from various exhcanges. Tiingo covers major exchanges, e.g. Binance, Gemini, Poloniex etc., so Tiingo's aggregated OHLC price is a good representation a market tradable price. For each crypto, we match the opening price at 15:00 CT from Tiingo with the daily closing price of BTC future from BBT. Since CRIX is not available at 15:00 CT, we recalculate a hourly CRIX using the monthly constituents weights and the hourly OHLC price data collected from Tiingo. BITX, BITW20, BITW70, and BITW100 are collected from the official website of their publisher Bitwise.com. The daily reporting time of the Bitwise indexes is 15:00 CT.

## 5 Results

In this section, we provide the result of hedging BTC with BTC future using different copulae and risk measures. The results is drawn using the data from 29/12/2017 to 27/05/2021 (15/12/2017 is the first trading day of the CME BTC future). BTC price is obtained from Tiingo, a data provider who aggregate BTC prices of major exhanges in the market; BTC future price is quoted by CME and retrieved as daily closing price from Bloomberg Terminal.

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# 6 Appendix

## 6.1 Density of linear combination of random variables

**Proposition 5** *Let $\boldsymbol{X} = (X_1, ..., X_d)^\top$ be real-valued random variables with corresponding copula density $\boldsymbol{c}_{X_1,...,X_d}$, and continuous marginals $F_{X_1}, ..., F_{X_d}$. Then, pdf of the linear combination of marginals $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ is*

$$f_Z(z) = \left|n_1^{-1}\right| \int_{[0,1]^{d-1}} \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1} \circ S(z) du_2 ... du_d \tag{48}$$

$$S(z) = \frac{1}{n_1} \cdot z - \frac{n_2}{n_1} \cdot F_{X_2}^{(-1)}(u_2) - ... - \frac{n_d}{n_1} \cdot F_{X_d}^{(-1)}(u_d) \tag{49}$$

***Proof.*** Rewrite $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ in matrix form

$$\begin{bmatrix} Z \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & \cdots & n_d \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix}. \tag{50}$$

By transformation variables

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \boldsymbol{f}_{X_1,...,X_d}\left(\boldsymbol{A}^{-1} \begin{bmatrix} z \\ x_2 \\ \vdots \\ x_d \end{bmatrix}\right) \cdot |\det \boldsymbol{A}^{-1}| \tag{51}$$

$$= \left|n_1^{-1}\right| \boldsymbol{f}_{X_1,...,X_d}\{S(z), x_2, ..., x_d\} \tag{52}$$

Let $u_i = F_{X_i}(x_i)$ and use the relationship

$$\boldsymbol{c}_{X_1,...,X_d}(u_1, ..., u_d) = \frac{\boldsymbol{f}_{X_1,...,X_d}(x_1, ..., x_d)}{\prod_{i=1}^d f_{X_i}(x_i)}, \tag{53}$$

we have

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \tag{54}$$

$$\left|n_1^{-1}\right| \cdot \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1}\{S(z)\} \cdot \prod_{i=2}^d f_{X_i}(x_i) \tag{55}$$

The claim 48 is obtained by integrating out $x_2, ...x_d$ by substituting $dx_i = \frac{1}{f_{X_i}(x_i)} du_i$. ■