

Hedging Cryptos with Bitcoin Futures

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Abstract

The introduction of derivatives on Bitcoin enables investors to hedge risk exposures in cryptocurrencies. Because of volatility swings and jumps in cryptocurrency prices, the traditional variance-based approach to obtaining hedge ratios is infeasible. As a consequence, we consider two extensions of the traditional approach: first, different dependence structures are modelled by different copulae, such as the Gaussian, Student-t, Normal Inverse Gaussian and Archimedean copulae; second, different risk measures, such as value-at-risk, expected shortfall and spectral risk measures, are employed to find the optimal hedge ratio. Extensive out-of-sample tests in the period December 2017 until May 2021 give insights into the practice of hedging various cryptos and crypto indices, including Bitcoin, Ethereum, Cardano, the CRIX index and several crypto-portfolios. Evidence shows that BTC futures can effectively hedge BTC and BTC-involved indices. This promising result is consistent across different risk measures and copulae except for Frank. On the other hand, we observe complex and diverse dependence structures between non-BTC-related crypto assets and the BTC futures. As a consequence, the hedge performance of non-BTC-related crypto assets are mixed and even infeasible for some assets.

JEL classification: G11, G13

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1 Introduction

Cryptocurrency (CC) is a fast-growing asset class, many more CCs now available on the market since the first CC Bitcoin (BTC) surfaced in 2009 (Nakamoto, 2009). Due to the rapid development of the CC market, more and more institutional investors recognise the importance of the market and start introducing CCs to their portfolios. For example, [some examples of reasonable size investment firm that enter the market](#). In response to the rapid development of the CC market, the Chicago Mercantile Exchange (CME) Group launched exchange-traded BTC futures contracts in December 2017. At the time of writing, the CME BTC futures is the only regulated cryptos futures in the market, making CME BTC futures an attractive derivative for institutional investors to participate in the crypto market; the average daily volume and open interest of the CME BTC futures are \$2,518 M and \$2,836 M respectively.

The CC market is highly volatile ever since the introduction of BTC; sharp and sudden price swings happen all the time. For example, in January 2021, Dogecoin's (DOGE) price received a generous boost within 24 hours from around \$0.008 to about \$0.08. However, the price plummeted by approximately 72% the next day to \$0.022. Even though DOGE's price climbed higher and higher in the following months, not all CC made the same recovery. [will pick another example coin that appears in our hedge](#)

The high volatility of CCs' price creates a need for investors to hedge their position, crypto futures products become a natural choice of hedging instrument. This paper analyses static hedges of crypto portfolios with the CME Bitcoin futures while taking the characteristics of CC market and investors' risk measure preferences into account. Based on Cherubini et al. (2011), Barbi and Romagnoli (2014) conducts a similar study for equity and FX portfolios. The authors show that the hedging performance in value-at-risk (VaR) and expected shortfall (ES) can be improved by modelling dependency via copulae.

CC market is proven to be a more volatile market than that of equity and FX (ref); there is no guarantee that the results from the aforementioned works can be carried over to the CC market. In addition to the volatility, the dependence between CC's price and CC futures is, to the best of authors' knowledge, not studied systematically. This motivates us to examine the hedging capability of the CME BTC futures to hedge various CCs. We also slightly extend Cherubini et al. (2011)'s results and develop a density functional for linear combinations of random variables whose multivariate distribution is modelled by copulae.


The main task of establishing a hedge with futures is to determine the appropriate amount of futures contracts to be held short, the amount is known as the optimal hedge ratio (OHR) in many literature. Determining OHR depends on 1. the dependence between the assets and futures prices, 2. choice of risk measure that defines optimal. We analyse the performance of a range of copulae on a set of popular risk measures, ensuring us the flexibility to model multivariate random variables and applicability of these copulae for different risk measure preferences.

The flexibility provided by copulae is critical to our analysis as marginal distribution of CCs' price are very different (ref). In fact, financial asset returns have long been known to be non-Gaussian, see e.g. (Fama, 1963; Cont, 2001). Specifically, Gaussian models cannot produce the heavy tails and the asymmetry observed in asset returns, which in turn implies a consistent underestimation of financial risks. Copulae bring great advantage to cater the non-Gaussianity that one can model the dependence of two random variables via copulae without enforcing limitations in modelling marginal distributions. Wassily Hoeffding shows that one can separately model the margins and the dependence structure

(Hoeffding, 1940a). This concept is later popularised by the work of Abe Sklar (Sklar, 1959), showing the uniqueness of copulae under certain assumptions.

The great flexibility of copulae provides good basis to deploy range of risk measures that represent for investors' different risk attitudes. Risk measures serve as measurement of risk, but in this work, they are also the loss functions in the OHR searching algorithm. In order to capture a variety of risk preferences, especially to cater investors tail-risk aversiveness (Menezes et al., 1980), we include the risk measures value-at-risk (VaR), expected shortfall (ES), and spectral risk measures (SRM) in addition to the commonly used variance. VaR is widely used by the finance industry in reporting and monitoring. ES and SRM are chosen because of their coherence property; i.e. they recognise diversification benefits. In addition, SRM is related to an individual's utility function. Examples are the exponential SRM and power SRM introduced by Dowd et al. (2008). Of the vast literature discussing the relationship between risk measures and investor's risk attitude, we refer readers to Artzner et al. (1999) for an axiomatic, economic reasoning approach of risk measure construction; Embrechts et al. (2002) for reasoning of using expected shortfall (ES) and spectral risk measures (SRM), in addition to value-at-risk (VaR); Acerbi (2002) for direct linkage between risk measures and investor's risk attitude using the concept of a "risk aversion function".

Through a systematic and extensive back-test,¹ we study the hedge effectiveness of the CME BTC futures to various cryptos and crypto indices under different risk preferences. We find the ability of the BTC futures to hedge BTC and BTC-related indices promising regardless of the choices of the copula (with the exception of the Frank copula) and risk measures. On the other hand, the results of BTC futures to hedge other cryptos and indices are inconclusive.

The paper is organised as follows. Section 2 introduces the notion of optimal hedge ratio; section 3 describes the method of estimation of copulae; section 4 provides the empirical result; section 5 concludes. All calculations in this work can be reproduced with the data and codes available at www.quantlet.com .

2 Optimal hedge ratio

2.1 Distribution of hedge portfolio

We form a portfolio with two assets, a spot asset and a futures contract, for example Bitcoin spot and a CME Bitcoin futures contract. Our objective is to minimize the risk of the exposure in the spot. To keep a simple portfolio setting, we go long one unit of the spot and short h units of the future, $h \geq 0$. Letting R^S and R^F be the (discrete) returns of the spot and futures price. The (discrete) return of the portfolio is²

$$R^h = R^S - hR^F.$$

If the portfolio reduces the risk of the spot position, then we call this a hedge portfolio. To measure risk, we define a risk measures ρ to be a mapping from a financial position, such as R^h , to a real number, which is often interpreted as the amount of money to make the position acceptable (e.g. to a regulator), see e.g. (Föllmer and Schied, 2002).

An optimal hedge ratio (OHR) h^* is a parameter that minimizes the risk of the aforementioned

¹We thank the data provider Tiingo (<https://www.tiingo.com/>) for providing the crypto price data.

²This is equivalent to stating that if both the spot price S_{t-1} and the futures price F_{t-1} are normalised to 1, then h units of the future will hedge the value change $\Delta V = \Delta S - h\Delta F$, where $\Delta S = S_t - S_{t-1}$, etc.

portfolio

$$h^* = \underset{h}{\operatorname{argmin}} \rho(R^h).$$

For example, Value-at-Risk (VaR) at the confidence level α is the absolute value of the $1 - \alpha$ -quantile of R^h , i.e., $\operatorname{VaR}_{1-\alpha} = -F_{R^h}^{(-1)}(1 - \alpha) = -\inf\{x \in \mathbb{R} : 1 - \alpha \leq F_{R^h}(x)\}$, where F_{R^h} is the distribution function of R^h .

Obviously the cdf of R^h and the risk measure depend on the joint distribution of R^S and $-hR^F$.

Optimising h according to $f_{R^S, -hR^F}$ is unfavorable in the sense that one would need to calibrate the joint pdf $f_{R^S, -hR^F}$ whenever updating h . Another problem of using the joint pdf is that one lacks the flexibility to model the margins separately from the dependence structure. To overcome both of these problems, we use copulae.

The benefit of using copulae is two fold. First, copulae are invariant under strictly monotone increasing function (Schweizer et al., 1981), see the Lemma below. Second, copulae allow us to model the margins and dependence structure separately, see Sklar's Theorem (Sklar, 1959). See also (Nelsen, 1999; Joe, 1997; McNeil et al., 2005) for Sklar's Theorem.

Theorem 1 (Hoeffding Sklar Theorem) *Let F be a joint distribution function with margins F_X, F_Y . Then, there exists a copula $C : [0, 1]^2 \mapsto [0, 1]$ such that, for all $x, y \in \mathbb{R}$*

$$F(x, y) = C\{F_X(x), F_Y(y)\}. \quad (1)$$

If the margins are continuous, then C is unique; otherwise C is unique on the range of the margins.

Conversely, if C is a copula and F_X, F_Y are univariate distribution functions, then the function F defined by (1) is a joint distribution function with margins F_X, F_Y .

Indeed, many basic results about copulae can be traced back to early works of Wassily Hoeffding (Hoeffding, 1940b, 1941). The works aimed to derive a measure of relationship of variables, which is invariant under change of scale. See also Fisher and Sen (2012) for English translations of the original papers written in German.

Lemma 1

$$C_{X, hY}(F_X(s), F_{hY}(t)) = C_{X, Y}(F_X(s), F_Y(t/h)). \quad (2)$$

Proof. Since copulae are invariant under strictly monotone increasing function (Schweizer et al., 1981, Theorem 3 (i)),

$$C_{X, hY}(F_X(s), F_{hY}(t)) = C_{X, Y}(F_X(s), F_{hY}(t)).$$

We rewrite second argument of the copula

$$\begin{aligned} F_{hY}(t) &= \mathbb{P}(hY \leq t) \\ &= \mathbb{P}(Y \leq t/h) \\ &= F_Y(t/h), \end{aligned}$$

and finish the proof. ■

Leveraging these two features of copulae, Barbi and Romagnoli (2014) introduce the distribution of linear combinations of random variables using copulae. We slightly edit the Corollary 2.1 of their work and yield the following expression of the distribution.

Proposition 2 *Let X and Y be two real-valued continuous random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with absolutely continuous copula $C_{X,Y}$ and marginal distribution functions F_X and F_Y . Then, the distribution function of Z is given by*

$$F_Z(z) = 1 - \int_0^1 D_1 C_{X,Y} \left[u, F_Y \left\{ \frac{F_X^{(-1)}(u) - z}{h} \right\} \right] du. \quad (3)$$

Here, $F^{(-1)}$ denotes the inverse of F , i.e., the quantile function.

Here $D_1 C(u, v) = \frac{\partial}{\partial u} C(u, v)$ and, see e.g. Equation (5.15) of (McNeil et al., 2005),

$$D_1 C_{X,Y} \{F_X(x), F_Y(y)\} = \mathbf{P}(Y \leq y | X = x). \quad (4)$$

Proof. Using the identity (4) gives

$$\begin{aligned} F_Z(z) &= \mathbf{P}(X - hY \leq z) = \mathbb{E} \left\{ \mathbf{P} \left(Y \geq \frac{X - z}{h} \middle| X \right) \right\} \\ &= 1 - \mathbb{E} \left\{ \mathbf{P} \left(Y \leq \frac{X - z}{h} \middle| X \right) \right\} = 1 - \int_0^1 D_1 C_{X,Y} \left[u, F_Y \left\{ \frac{F_X^{(-1)}(u) - z}{h} \right\} \right] du. \end{aligned}$$

■

Corollary 1 *Given the formulation of random variables, the pdf of Z can be written as*

$$f_Z(z) = \left| \frac{1}{h} \right| \int_0^1 c_{X,Y} \left[F_Y \left\{ \frac{F_X^{(-1)}(u) - z}{h} \right\}, u \right] \cdot f_Y \left\{ \frac{F_X^{(-1)}(u) - z}{h} \right\} du, \quad (5)$$

or

$$f_Z(z) = \int_0^1 c_{X,Y} \left[F_X \left\{ z + hF_Y^{(-1)}(u) \right\}, u \right] \cdot f_X \left\{ z + hF_Y^{(-1)}(u) \right\} du. \quad (6)$$

The two expressions are equivalent. Note that the pdf of Z in the above proposition can be assessed via numerical integration as long as we have the copula density and the marginal densities. A generic expression and proof can be found in the appendix.

2.2 Procedure to determine optimal hedge ratio

We introduce the empirical procedure to obtain the optimal hedge ratio (OHR) being used in this work. First, we split the time series of spot and futures into sets of training and testing data. The training data makes up the first 300 observations and its corresponding testing data consists of the consecutive 5 observations. We then roll 5 observations forward (step size of 5) to obtain the next training and test data sets and repeat this until the end of the time series. Note that the testing data are non-overlapping since the step size and equal to test size.

Next, we obtain the OHR as follows:

1. **Construct Univariate Kernel Density Function (KDE).** From the training data we calibrate the spot and futures' univariate kernel density functions using the Gaussian kernel with bandwidth determined by the refined plug-in method (Härdle et al., 2004, section 3.3.3).
2. **Calibrate Copulae.** We then calibrate the copulae outlined in section 3.1 via the method of moments described in section 3.2.1.
3. **Select Copula.** We compute the Akaike Information Criterion. The copula with the best (i.e., lowest) AIC is used for the next step. A discussion of this step is found in 3.2.3.
4. **Determine OHR.** We determine the OHRs numerically using different risk measures as the loss function by drawing samples from the selected copula and KDEs. The risk measures used as risk reduction objectives are outlined in 3.3
5. **Obtain testing log-return of hedged portfolio.** Finally, we apply the OHRs to the test data $R_h = R_s - h^* R_f$.

3 Copulae and risk measures

3.1 Copulae

To capture different aspects of the dependence structure, we consider a number of different copulas, which are layed out in details below Gaussian-, t -, Frank-, Gumbel-, Clayton-, mixture, NIG factor, and Plackett-copula.

As this hedging exercise concerns only portfolios with two assets, we focus on the bivariate version of copulae and some important features of a copula, including Kendall's τ_K and Spearman's ρ_S .

Kendall's τ and Spearman's ρ are measures of association in terms of concordance, see Kruskal (1958) Let (x_i, y_i) and (x_j, y_j) denote two observations from a vector (X, Y) of continuous random variables. A pair of observations is concordant if $x_i < x_j$ and $y_i < y_j$, discordant if $x_i > x_j$ and $y_i < y_j$ or if $x_i < x_j$ and $y_i > y_j$. For a bivariate random variable of n observations, there are $\binom{n}{2}$ distinct pairs.

Let c denote the number of concordant pairs, and d the number of discordant pairs, Kendall's tau is defined as follows (Nelsen, 1999)

$$\begin{aligned}\tau_K &\stackrel{\text{def}}{=} \frac{c - d}{c + d} \\ &= \frac{c - d}{\binom{n}{2}};\end{aligned}$$

Let F_X and F_Y be cdfs of X and Y respectively, Spearman's ρ is

$$\rho_S \stackrel{\text{def}}{=} 12\mathbb{E}(F_X(X)F_Y(Y)) - 3;$$

Upper tail dependence is defined as

$$\lambda_U \stackrel{\text{def}}{=} \lim_{q \rightarrow 1^-} \mathbf{P}\{X > F_X^{(-1)}(q) | Y > F_Y^{(-1)}(q)\};$$

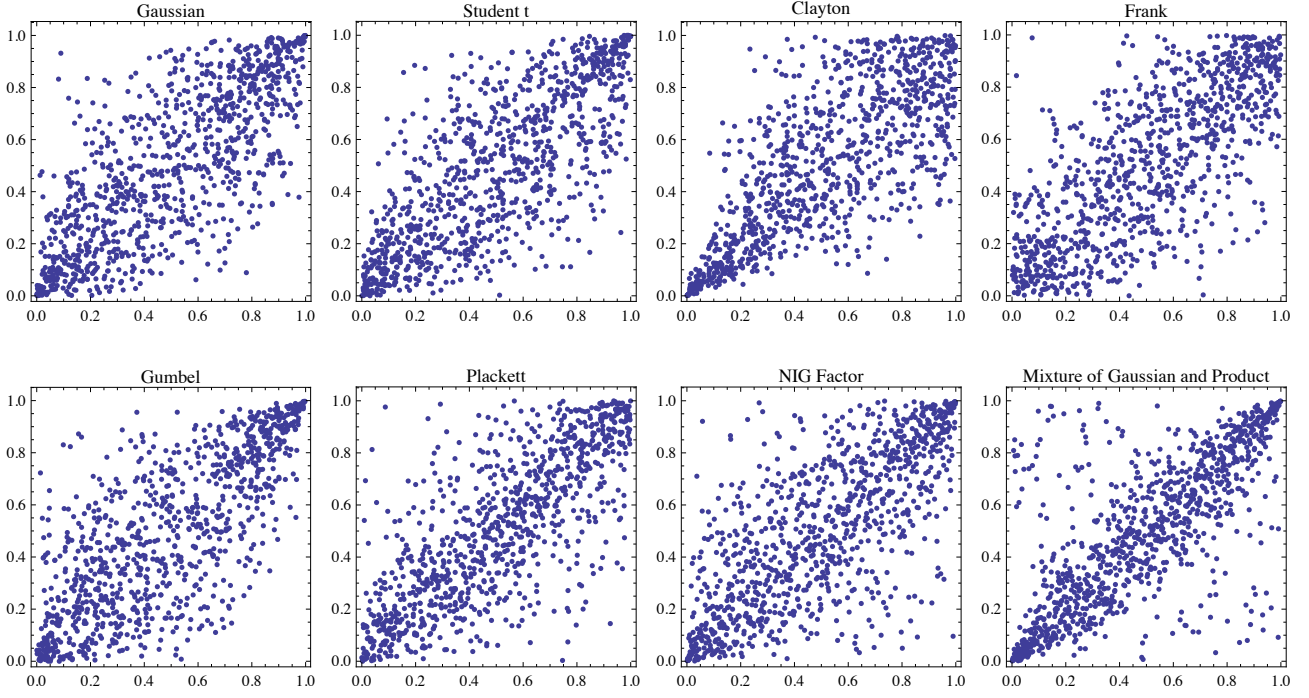


Figure 1: Scatterplots of samples drawn from copulae. All copulae are calibrated to Spearman's ρ of 0.75 before sampling.

Lower tail dependence is defined as

$$\lambda_L \stackrel{\text{def}}{=} \lim_{q \rightarrow 0^+} \mathbf{P}\{X \leq F_X^{(-1)}(q) | Y \leq F_Y^{(-1)}(q)\}.$$

Furthermore, we denote the Fréchet-Hoeffding lower bound by \mathbf{W} , the product copula by $\mathbf{\Pi}$, and the Fréchet-Hoeffding upper bound by \mathbf{M} . They represent cases of perfect negative dependence, independence, and perfect positive dependence, respectively. For further details, we refer readers to Joe (1997) and Nelsen (1999); see also Härdle and Okhrin (2010).

The symmetry property of copulae is also important for modelling financial data. In particular, we are interested in the radially symmetric among other symmetry definitions, see Nelsen (1999).

Definition 3 *Let (U_1, \dots, U_d) be random variables. The random variables is radially symmetric if the joint cdf of (U_1, \dots, U_d) is same as the joint cdf of $(1 - U_1, \dots, 1 - U_d)$*

To illustrate the difference among copulae, we plot random samples drawn from copulae listed below in figure 1.

3.1.1 Gaussian and t Copulae

Gaussian and t copulae are derived from Gaussian and t distributions. Since Gaussian and t distributions are elliptical distributions, Gaussian and t copulae are called elliptical copulae.

Gaussian copula (bivariate) is defined as

$$\begin{aligned} C(u, v) &= \Phi_{2,\rho}\{\Phi^{(-1)}(u), \Phi^{(-1)}(v)\} \\ &= \int_{-\infty}^{\Phi^{(-1)}(u)} \int_{-\infty}^{\Phi^{(-1)}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right\} ds dt, \end{aligned}$$

where $\Phi_{2,\rho}$ is the cdf of bivariate Normal distribution with zero mean, unit variance, and correlation coefficient ρ , and $\Phi^{(-1)}$ is the quantile function univariate standard normal distribution.

Note that we use ρ to denote the correlation parameter as well as a $\rho(\cdot)$ to denote a risk measure.

The Gaussian copula is fully specified by the correlation parameter ρ . Like all elliptical copulas, it is symmetric. It has no tail dependence, which, in a finance context, implies that it often underestimates tail risk.

The Gaussian copula density is

$$\begin{aligned} c_\rho(u, v) &= \frac{\varphi_{2,\rho}\{\Phi^{(-1)}(u), \Phi^{(-1)}(v)\}}{\varphi\{\Phi^{(-1)}(u)\} \cdot \varphi\{\Phi^{(-1)}(v)\}} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right\}, \end{aligned}$$

where $\varphi_{2,\rho}(\cdot)$ is the pdf corresponding to $\Phi_{2,\rho}$, and $\varphi(\cdot)$ the standard normal distribution pdf.

Kendall's τ_K and Spearman's ρ_S of a bivariate Gaussian copula are

$$\tau_K(\rho) = \frac{2}{\pi} \arcsin \rho$$

$$\rho_S(\rho) = \frac{6}{\pi} \arcsin \frac{\rho}{2}.$$

The t -copula has the form

$$\begin{aligned} C(u, v) &= \mathbf{T}_{2,\rho,\nu}\{T_\nu^{(-1)}(u), T_\nu^{(-1)}(v)\} \\ &= \int_{-\infty}^{T_\nu^{(-1)}(u)} \int_{-\infty}^{T_\nu^{(-1)}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \pi \nu \sqrt{1-\rho^2}} \\ &\quad \left(1 + \frac{s^2 - 2st\rho + t^2}{\nu}\right)^{-\frac{\nu+2}{2}} ds dt, \end{aligned}$$

where $\mathbf{T}_{2,\rho,\nu}$ denotes the cdf of bivariate t distribution with dependence parameter ρ and degrees of freedom parameter ν , and where $T_\nu^{(-1)}(\cdot)$ is the quantile function of a standard t distribution with degree of freedom ν .

Contrary to the Gaussian copula, the t -copula has a non-zero tail dependence coefficient, which makes it more appropriate for dependence modelling in finance. The Gaussian copula arises as $\nu \rightarrow \infty$.

The copula density is

$$c(u, v) = \frac{\mathbf{t}_{2,\rho,\nu}\{T_\nu^{(-1)}(u), T_\nu^{(-1)}(v)\}}{t_\nu\{T_\nu^{(-1)}(u)\} \cdot t_\nu\{T_\nu^{(-1)}(v)\}},$$

where $\mathbf{t}_{2,\rho,\nu}$ is the pdf of $\mathbf{T}_{2,\rho,\nu}$ and t_ν the density of standard t distribution.

Like all the other elliptical copulae, the t -copula's Kendall's τ is identical to that of the Gaussian copula (see Demarta and McNeil, 2005, and references therein).

3.1.2 Archimedean copulae

The family of Archimedean copulae forms a large class of copulae with many convenient features. Contrary to elliptical copulas, which are derived from elliptical distributions. Archimedean copulas are determined via a simple parametric form of the dependence structure. A prominent feature is the ability to model asymmetric dependence structures.

In general, they take a form

$$\mathbf{C}(u, v) = \psi^{(-1)}\{\psi(u), \psi(v)\},$$

where $\psi : [0, 1] \rightarrow [0, \infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1) = 0$ for any permissible dependence parameter θ . ψ is also called generator. $\psi^{(-1)}$ is the inverse of the generator.

The Frank copula (B3 in Joe (1997)) is a radial symmetric copula and cannot produce any tail dependence. It takes the form

$$\mathbf{C}_\theta(u, v) = \frac{1}{\theta} \log \left\{ 1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right\}$$

where $\theta \in [0, \infty]$ is the dependency parameter. $\mathbf{C}_{-\infty} = \mathbf{M}$, $\mathbf{C}_1 = \mathbf{\Pi}$, and $\mathbf{C}_\infty = \mathbf{W}$.

The Copula density is

$$c_\theta(u, v) = \frac{\theta e^{\theta(u+v)(e^\theta - 1)}}{\{e^\theta - e^{\theta u} - e^{\theta v} + e^{\theta(u+v)}\}^2}$$

Frank copula has Kendall's τ and Spearman's ρ as follow:

$$\tau_K(\theta) = 1 - 4 \frac{D_1\{-\log(\theta)\}}{\log(\theta)},$$

and

$$\rho_S(\theta) = 1 - 12 \frac{D_2\{-\log(\theta)\} - D_1\{\log(\theta)\}}{\log(\theta)},$$

where D_1 and D_2 are the Debye function of order 1 and 2. Debye function is $D_n = \frac{n}{x^n} \int_0^x \frac{t^n}{e^t - 1} dt$.

The Gumbel copula (B6 in Joe (1997)) has upper tail dependence with the dependence parameter $\lambda^U = 2 - 2^{\frac{1}{\theta}}$ and displays no lower tail dependence.

$$\mathbf{C}_\theta(u, v) = \exp - \{(-\log(u))^\theta + (-\log(v))^\theta\}^{\frac{1}{\theta}},$$

where $\theta \in [1, \infty)$ is the dependence parameter.

While the Gumbel copula cannot model perfect counter-dependence (Nelsen, 2002), $\mathbf{C}_1 = \mathbf{\Pi}$ models the independence, and $\lim_{\theta \rightarrow \infty} \mathbf{C}_\theta = \mathbf{W}$ models the perfect dependence. The copula density takes the form

$$\tau_K(\theta) = \frac{\theta - 1}{\theta}.$$

The Clayton copula, by contrast to Gumbel copula, generates lower tail dependence of the form

$\lambda^L = 2^{-\frac{1}{\theta}}$, but cannot generate upper tail dependence. The Clayton copula takes the form

$$\mathbf{C}_\theta(u, v) = \left\{ \max(u^{-\theta} + v^{-\theta} - 1, 0) \right\}^{-\frac{1}{\theta}},$$

where $\theta \in (-\infty, \infty)$ is the dependence parameter. Moreover, $\lim_{\theta \rightarrow -\infty} \mathbf{C}_\theta = \mathbf{M}$, $\mathbf{C}_0 = \mathbf{\Pi}$, and $\lim_{\theta \rightarrow \infty} \mathbf{C}_\theta = \mathbf{W}$. Kendall's τ of the Clayton copula is given by

$$\tau_K(\theta) = \frac{\theta}{\theta + 2}.$$

3.1.3 Mixture Copula

The mixture copula is a linear combination of copulae. For a 2-dimensional random variable $\mathbf{X} = (X_1, X_2)^\top$, its distribution can be written as linear combination of K copulae

$$\mathbf{C}(u, v) = \sum_{k=1}^K p^{(k)} \cdot \mathbf{C}^{(k)}\{F_{X_1}^{(-1)}(u), F_{X_2}^{(-1)}(v); \boldsymbol{\theta}^{(k)}\}.$$

The copula density is again a linear combination of copula densities

$$\mathbf{c}(u, v) = \sum_{k=1}^K p^{(k)} \cdot \mathbf{c}^{(k)}\{F_{X_1}^{(-1)}(u), F_{X_2}^{(-1)}(v); \boldsymbol{\theta}^{(k)}\}.$$

While Kendall's τ of mixture copula is not known in closed form, Spearman's ρ is specified by the following statement.

Proposition 4 *Let $\rho_S^{(k)}$ be Spearman's ρ of the k -th component Spearman's ρ of the mixture copula is given by*

$$\rho_S = \sum_{k=1}^K p^{(k)} \cdot \rho_S^{(k)}$$

Proof. Since Spearman's ρ is defined as (Nelsen, 1999)

$$\rho_S = 12 \int_{\mathbb{I}^2} \mathbf{C}(s, t) ds dt - 3,$$

Spearman's ρ of the mixture copula is given by summation of the components

$$\rho_S = 12 \int_{\mathbb{I}^2} \sum_{k=1}^K p^{(k)} \cdot \mathbf{C}^{(k)}(s, t) ds dt - 3.$$

■

An example of a mixture copula is the Fréchet class of copulas, which are given by convex combinations of \mathbf{W} , $\mathbf{\Pi}$, and \mathbf{M} (Nelsen, 1999).

We use a mixture of Gaussian and independent copulas in our analysis, i.e.,

$$\mathbf{C}(u, v) = p \cdot \mathbf{C}^{\text{Gaussian}}(u, v) + (1 - p)(uv),$$

with corresponding density is

$$\mathbf{c}(u, v) = p \cdot \mathbf{c}^{\text{Gaussian}}(u, v) + (1 - p).$$

This mixture models the amount of “random noise” that appears in the dependence structure. In the hedging exercise, the “random noise” adds an unhedgable component to the two-asset portfolio, whose weight $(1 - p)$ is calibrated from market data.

3.1.4 NIG factor copula

The *normal inverse Gaussian* (NIG) distribution, introduced by (Barndorff-Nielsen, 1997), has density function

$$g(x; \alpha, \beta, \mu, \delta) = \frac{\alpha}{\pi} \mathbf{e}^{\delta \sqrt{\alpha^2 - \beta^2} - \beta \mu} \frac{1}{q((x - \mu)/\delta)} K_1 \left[\delta \alpha q \left(\frac{x - \mu}{\delta} \right) \right] \mathbf{e}^{\beta x}, \quad x > 0,$$

where $q(x) = \sqrt{1 + x^2}$ and where K_1 is the modified Bessel function of third order and index 1. The parameters satisfy $0 \leq |\beta| \leq \alpha$, $\mu \in \mathbb{R}$ and $\delta > 0$. The parameters have the following interpretation: μ and δ are location and scale parameters, respectively, α determines the heaviness of the tails and β determines the degree of asymmetry. If $\beta = 0$, then the distribution is symmetric around μ .

The moment-generating function of the NIG distribution is given by

$$M(u; \alpha, \beta, \mu, \delta) = \exp \left(\delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right) + \mu u \right).$$

As a direct consequence, moments are easily calculated with the expectation and variance of the NIG distribution being

$$\begin{aligned} \mathbb{E}X &= \mu + \frac{\delta \beta}{\sqrt{\alpha^2 - \beta^2}} \\ \text{Var}(X) &= \frac{\alpha^2 \delta}{(\alpha^2 - \beta^2)^{3/2}}. \end{aligned} \tag{7}$$

The $\text{NIG}(\alpha, \beta, \mu, \delta)$ distribution belongs to the class of so-called *normal variance-mean mixture*, (see Section 3.2 of (McNeil et al., 2005)): X follows an $\text{NIG}(\alpha, \beta, \mu, \delta)$ distribution if X conditional on W follows a normal distribution with mean $\mu + \beta W$ and variance W , i.e.,

$$X|W \stackrel{\mathcal{L}}{\sim} \text{N}(\mu + \beta W, W),$$

where W follows an *inverse Gaussian distribution*, denoted by $\text{IG}(\delta, \sqrt{\alpha^2 - \beta^2})$.

It is easily seen from the moment-generating function that the NIG distribution is preserved under linear combinations, provided the variables share the parameters α and β . For this and other reasons, the NIG distribution is popular in many areas of financial modelling; for example, it gives rise to the normal inverse Gaussian Lévy process, which may be represented as a Brownian motion with a time change.

In the setting here, we consider the *NIG factor copula*. This is not directly derived from the multivariate NIG distribution, but determined through a factor structure instead. The factor structure, which was applied e.g. in (Kalemanova et al., 2007) for calibrating CDO's, gives additionally flexibility

as it does not force the components to have a mixing variable W . The following proposition introduces the NIG factor model and some of its properties.

Proposition 5 *Let $Z \sim NIG(\alpha, \beta, \mu, \delta)$ and $Z_i \sim NIG(\alpha, \beta, \mu_i, \delta_i)$, $i = 1, \dots, n$ be independent NIG-distributed random variables. Then:*

1. $X_i = Z + Z_i \sim NIG(\alpha, \beta, \mu + \mu_i, \delta + \delta_i)$,
2. and

$$\begin{aligned} \text{Cov}(X_i, X_j) &= \text{Var}(Z), \\ \text{Corr}(X_i, X_j) &= \frac{\delta}{\sqrt{(\delta + \delta_i)(\delta + \delta_j)}}. \end{aligned} \tag{8}$$

Proof.

1. This follows directly from the moment-generating function.
2. For the covariance,

$$\begin{aligned} \text{Cov}(X_i, X_j) &= \mathbb{E}[(Z + Z_i)(Z + Z_j)] - \mathbb{E}[Z + Z_i]\mathbb{E}[Z + Z_j] \\ &= \mathbb{E}[Z^2] - (\mathbb{E}Z)^2. \end{aligned}$$

The correlation is determined directly from 7. ■

The NIG factor copula is obtained by transforming the margins to uniforms (see Sklar's Theorem), giving (e.g. (Krupskii and Joe, 2013)):

$$C_{r^S, r^F}(F_{r^S}(r^S), F_{r^F}(r^F)) = \int_{\mathbb{R}} F_{Z_1}(F_{X_1}^{(-1)} \circ F_{r^S}(r^S) - z) \cdot F_{Z_2}(F_{X_2}^{(-1)} \circ F_{r^F}(r^F) - z) \cdot f_Z(z) dz$$

If the margins are continuous, then Spearman's rho of NIG factor copula is

$$\rho_S = 12 \int \int \int_{\mathbb{R}^3} F_{X_1}(x_1) \cdot F_{X_2}(x_2) \cdot f_{Z_1}(x_1 - z) \cdot f_{Z_2}(x_2 - z) \cdot f_Z(z) dx_1 dx_2 dz - \frac{1}{48}.$$

3.1.5 Plackett copula

The Plackett copula has an expression

$$C_\theta(u, v) = \frac{1 + (\theta - 1)(u + v)}{2(\theta - 1)} - \frac{\sqrt{\{1 + (\theta - 1)(u + v)\}^2 - 4uv\theta(\theta - 1)}}{2(\theta - 1)}$$

$$\rho_S(\theta) = \frac{\theta + 1}{\theta - 1} - \frac{2\theta \log \theta}{(\theta - 1)^2}$$

We include Plackett copula in our analysis as it possesses a special property, the cross-product ratio

is equal to the dependence parameter

$$\begin{aligned}
& \frac{\mathbf{P}(U \leq u, V \leq v) \cdot \mathbf{P}(U > u, V > v)}{\mathbf{P}(U \leq u, V > v) \cdot \mathbf{P}(U > u, V \leq v)} \\
&= \frac{C_\theta(u, v)\{1 - u - v + C_\theta(u, v)\}}{\{u - C_\theta(u, v)\}\{v - C_\theta(u, v)\}} \\
&= \theta.
\end{aligned} \tag{9}$$

That is, the dependence parameter is equal to the ratio between number of concordance pairs and number of discordance pairs of a bivariate random variable.

3.2 Calibration and selection of copulae

We introduce the method to calibrate copulae to our data in this section. In general, there are two types of calibration procedures to calibrate copulae: maximum likelihood (MLE) and method of moments (MM). We decide to deploy the latter since it calibrates according to the moments desired.

In the following subsection, we present the configuration of the method of moments procedures in this work. In the subsection after, we argue that MM is more suitable to this work by comparing MM with MLE.

3.2.1 Method of moments

We trace back the usage of MM to calibrate copulae to Genest (1987); Genest and Rivest (1993). The moments mainly refer to Kendall's τ or Spearman's ρ . We extend MM to quantile dependence measures denoted by λ_q for quantile level q .

Spearman's ρ , Kendall's τ , and quantile dependence of the copula C are defined as

$$\begin{aligned}
\rho_S &= 12 \int \int_{I^2} C_\theta(u, v) \, du \, dv - 3 \\
\tau_K &= 4\mathbb{E}[C_\theta\{F_X(x), F_Y(y)\}] - 1, \\
\lambda_q &= \begin{cases} \mathbf{P}(F_X(X) \leq q | F_Y(Y) \leq q) = \frac{C_\theta(q, q)}{q}, & \text{if } q \in (0, 0.5], \\ \mathbf{P}(F_X(X) > q | F_Y(Y) > q) = \frac{1 - 2q + C_\theta(q, q)}{1 - q}, & \text{if } q \in (0.5, 1). \end{cases}
\end{aligned}$$

The empirical counterparts are

$$\begin{aligned}
\hat{\rho}_S &= \frac{12}{n} \sum_{k=1}^n \hat{F}_X(x_k) \hat{F}_Y(y_k) - 3, \\
\hat{\tau}_K &= \frac{4}{n} \sum_{k=1}^n \hat{C}\{\hat{F}_X(x_k), \hat{F}_Y(y_k)\} - 1, \\
\hat{\lambda}_q &= \begin{cases} \frac{1}{n} \sum_{k=1}^n \frac{\mathbf{1}_{\{\hat{F}_X(x_k) \leq q, \hat{F}_Y(y_k) \leq q\}}}{q}, & \text{if } q \in (0, 0.5], \\ \frac{1}{n} \sum_{k=1}^n \frac{\mathbf{1}_{\{\hat{F}_X(x_k) > q, \hat{F}_Y(y_k) > q\}}}{1 - q}, & \text{if } q \in (0.5, 1), \end{cases}
\end{aligned}$$

where $\hat{F}(x) = \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{x_i \leq x\}}$ and $\hat{C}(u, v) = \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{u_i \leq u, v_i \leq v\}}$.

Denote by $\mathbf{m}(\boldsymbol{\theta})$ the m -dimensional vector of dependence measures according the dependence parameters $\boldsymbol{\theta}$, and let $\hat{\mathbf{m}}$ be the corresponding empirical counterpart. The difference between dependence measures and their counterpart is denoted by

$$\mathbf{g}(\boldsymbol{\theta}) = \hat{\mathbf{m}} - \mathbf{m}(\boldsymbol{\theta}).$$

The MM estimator is

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \Theta}{\operatorname{argmin}} \mathbf{g}(\boldsymbol{\theta})^\top \hat{\mathbf{W}} \mathbf{g}(\boldsymbol{\theta}),$$

where $\hat{\mathbf{W}}$ is some positive definite weight matrix. In this work, we use $\mathbf{m}(\boldsymbol{\theta}) = (\rho_S, \lambda_{0.05}, \lambda_{0.1}, \lambda_{0.9}, \lambda_{0.95})^\top$ for calibration. $\hat{\mathbf{W}}$ is set to identity matrix.

3.2.2 Comparison between method of moments and maximum likelihood

By the Hoeffding-Sklar theorem, the joint density of a d -dimensional random variable \mathbf{X} with sample size n can be written as

$$\mathbf{f}_{\mathbf{X}}(x_1, \dots, x_d) = \mathbf{c}\{F_{X_1}(x_1), \dots, F_{X_d}(x_d)\} \prod_{j=1}^d f_{X_j}(x_j).$$

We follow the treatment of MLE documented in section 10.1 of Joe (1997), namely the *inference functions for margins (IFM) method*. The log-likelihood $\sum_{i=1}^n \mathbf{f}_{\mathbf{X}}(X_{i,1}, \dots, X_{i,d})$ can be decomposed into a dependence part and a marginal part,

$$\begin{aligned} L(\boldsymbol{\theta}) &= \sum_{i=1}^n \mathbf{c}\{F_{X_1}(x_{i,1}; \boldsymbol{\delta}_1), \dots, F_{X_d}(x_{i,d}; \boldsymbol{\delta}_d); \boldsymbol{\gamma}\} + \sum_{i=1}^n \sum_{j=1}^d f_{X_j}(x_{i,j}; \boldsymbol{\delta}_j) \\ &= L_C(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d, \boldsymbol{\gamma}) + \sum_{j=1}^d L_j(\boldsymbol{\delta}_j) \end{aligned}$$

where $\boldsymbol{\delta}_j$ are the parameters of the j -th margin, $\boldsymbol{\gamma}$ is the parameter of the parametric copula, and $\boldsymbol{\theta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d, \boldsymbol{\gamma})$. Instead of searching the $\boldsymbol{\theta}$ in a high dimensional space, Joe (1997) suggests to search for $\hat{\boldsymbol{\delta}}_1, \dots, \hat{\boldsymbol{\delta}}_d$ that maximize $L_1(\boldsymbol{\delta}_1), \dots, L_d(\boldsymbol{\delta}_d)$, then search for $\hat{\boldsymbol{\gamma}}$ that maximize $L_C(\hat{\boldsymbol{\delta}}_1, \dots, \hat{\boldsymbol{\delta}}_d, \boldsymbol{\gamma})$.

We follow Genest et al. (1995) who suggest to replace the estimation of marginal parameters estimation by non-parametric estimation. Given non-parametric estimator \hat{F}_i of the margins F_i , the estimator of the dependence parameters $\boldsymbol{\gamma}$ is

$$\hat{\boldsymbol{\gamma}} = \underset{\boldsymbol{\gamma}}{\operatorname{argmax}} \sum_{i=1}^n \mathbf{c}\{\hat{F}_{X_1}(x_{i,1}), \dots, \hat{F}_{X_d}(x_{i,d}); \boldsymbol{\gamma}\}.$$

Both the simulated method of moments and the maximum likelihood estimation are unbiased. The question though which procedure is more suitable for hedging.

Figure 2 shows the empirical quantile dependence of Bitcoin and CME future and the copula implied quantile dependence of the MLE and MM calibration procedures. Although the MLE is a better fit to a range of quantile dependence in the middle, it fails to address the situation in the tails. We find that our data empirically has low quantile dependence in the lower ends ($q < 10\%$). We argue that MM is preferred as it produces a better fit to the dependence structure in the tail behaviour,

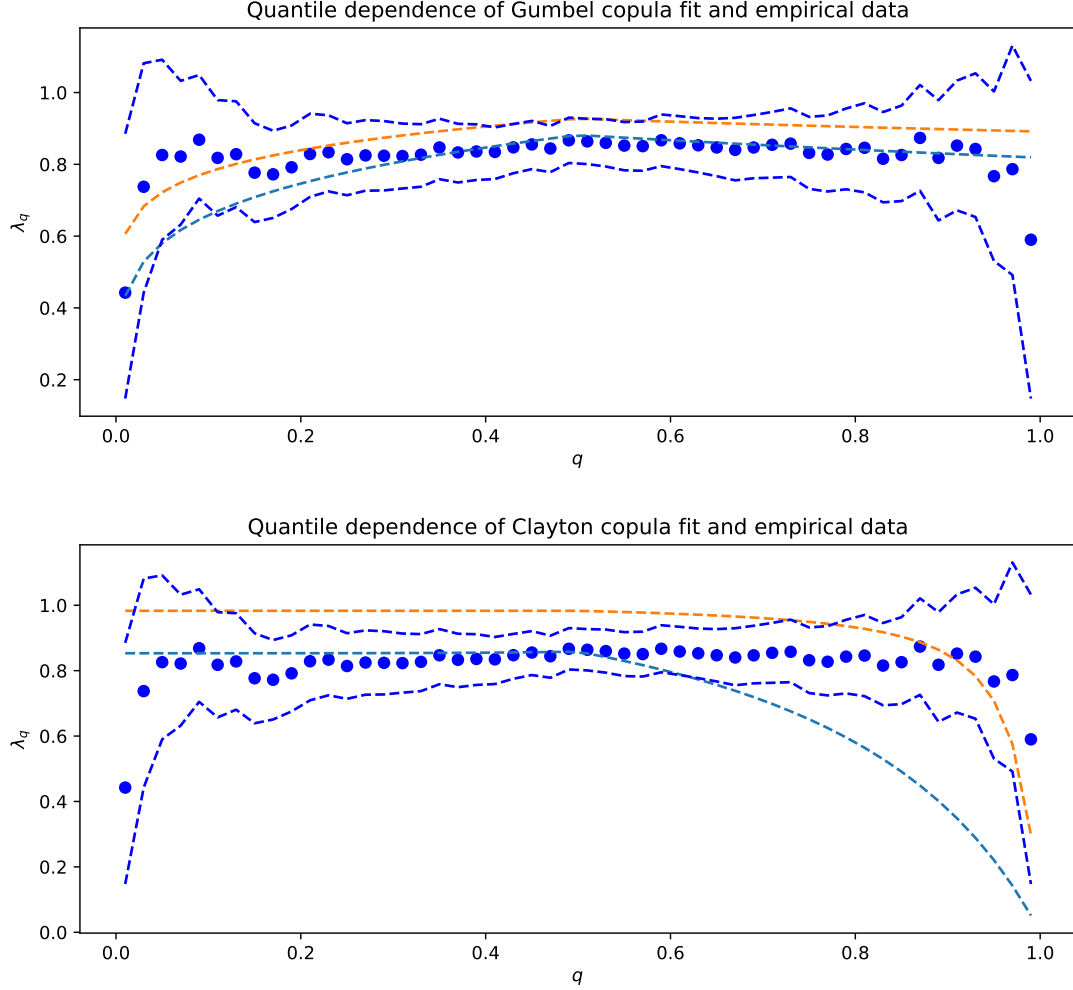


Figure 2: Quantile dependences of Gumbel, and Clayton Copula. The blue circle dots are the quantile dependence estimates of Bitcoin and CME future, blue dotted lines are the estimates' 90% confidence interval. Orange dotted line is the copula implied quantile dependence by MM estimation. Light blue dotted line is the copula implied quantile dependence by MLE estimation.

contrary to MLE.

Therefore, we deploy the method of moments throughout the analysis. We choose the 5th-, 10th-, 90th-, 95th-quantile, and Spearman's ρ as the moments.

3.2.3 Copula selection

The dependence structure of price data changes across time, in which both the dependency parameters and the type of dependence. For this reason, we allow for a flexible choice of the best-fitting copula, by re-calibrating periodically and re-evaluating performance of the various copulas. We select the best-fitting copula, characterised by the lowest *Akaike Information Criterion (AIC)*,

$$AIC = 2k - 2\log(L),$$

where k is the number of estimated parameters and L is the likelihood (Akaike, 1973).

Other model selection criteria, such as the TIC (Takeuchi, 1976) or likelihood ratio test could be used instead. For a survey of model selection and inference, see Anderson et al. (1998). Among various copula selection procedures, AIC is a popular choice for its applicability, see e.g. Breymann

et al. (2003). In our case, the AICs are calculated only with dependency likelihood since the marginals are modelled via kernel density estimators. The selected copula will then be enter the calculation of the optimal hedge ratio.

3.3 Risk measures

The optimal hedge ratio is determined for the following variety of risk measures: variance, Value-at-Risk (VaR), Expected Shortfall (ES), and Exponential Risk Measure (ERM). A summary of risk measures being used in portfolio selection problem can be found in Härdle et al. (2008). The risk measures here serve as risk minimization objectives, i.e. loss functions for searching the optimal hedge ratio.

The risk measures are defined as follows. Let Z be a random variable with distribution function F_Z .

1. Variance: $\text{Var}(Z) = \mathbb{E}[(Z - \mathbb{E}Z)^2]$.
2. VaR at confidence level α : $\text{VaR}_\alpha(Z) = -F_Z^{(-1)}(1 - \alpha)$
3. ES at confidence level α : $\text{ES}(F_Z) = -\frac{1}{1-\alpha} \int_0^{1-\alpha} F_Z^{(-1)}(p) dp$
4. ERM with Arrow-Pratt coefficient of absolute risk aversion k :

$$\text{ERM}_k(F_Z) = \int_0^{1-\alpha} \phi(p) F_Z^{(-1)}(p) dp,$$

where ϕ is a weight function described in (3.3) below.

VaR, ES, and ERM fall into the class of spectral risk measures (SRM), which have the from (Acerbi, 2002)

$$\rho_\phi(r^h) = - \int_0^1 \phi(p) F_Z^{(-1)}(p) dp,$$

where p is the loss quantile and $\phi(p)$ is a user-defined weighting function defined on $[0, 1]$. We consider only so-called admissible risk spectra $\phi(p)$, i.e., fulfilling

- (i) ϕ is positive,
- (ii) ϕ is decreasing,
- (iii) and $\int \phi = 1$.

The VaR's $\phi(p)$ gives all its weight on the $1 - \alpha$ quantile of Z and zero elsewhere, i.e. the weighting function is a Dirac delta function, and hence it violates the (ii) property of admissible risk spectra. The ES' $\phi(p)$ gives all tail quantiles the same weight of $\frac{1}{1-\alpha}$ and non-tail quantiles zero weight. The ERM assumes investors' risk preference are in the form of an exponential utility function $U(x) = 1 - e^{kx}$, so its corresponding risk spectrum is defined as

$$\phi(p) = \frac{ke^{-k(1-p)}}{1 - e^{-k}},$$

where k is the Arrow-Pratt coefficient of absolute risk aversion. The parameter k has an economic interpretation as being the ratio between the second derivative and first derivative of investor's utility

function on an risky asset,

$$k = -\frac{U''(x)}{U'(x)},$$

for x in all possible outcomes. In case of the exponential utility, k is the the constant absolute risk aversion (CARA).

References

- ACERBI, C. (2002): “Spectral measures of risk: A coherent representation of subjective risk aversion,” *Journal of Banking & Finance*, 26, 1505–1518.
- AKAIKE, H. (1973): “Information theory and an extension of the maximum likelihood principle,” in *Second International Symposium on Information Theory*, ed. by B. N. Petrov and F. Csaki, Budapest: Akadémiai Kiado, 267–281.
- ANDERSON, D., K. BURNHAM, AND G. WHITE (1998): “Comparison of Akaike information criterion and consistent Akaike information criterion for model selection and statistical inference from capture-recapture studies,” *Journal of Applied Statistics*, 25, 263–282.
- ARTZNER, P., F. DELBAEN, J.-M. EBER, AND D. HEATH (1999): “Coherent measures of risk,” *Mathematical Finance*, 9, 203–228.
- BARBI, M. AND S. ROMAGNOLI (2014): “A Copula-Based Quantile Risk Measure Approach to Estimate the Optimal Hedge Ratio,” *Journal of Futures Markets*, 34, 658–675.
- BARNDORFF-NIELSEN, O. E. (1997): “Normal inverse Gaussian distributions and stochastic volatility modelling,” *Scandinavian Journal of statistics*, 24, 1–13.
- BREYMAN, W., A. DIAS, AND P. EMBRECHTS (2003): “Dependence structures for multivariate high-frequency data in finance,” .
- CHERUBINI, U., S. MULINACCI, AND S. ROMAGNOLI (2011): “A copula-based model of speculative price dynamics in discrete time,” *Journal of Multivariate Analysis*, 102, 1047–1063.
- CONT, R. (2001): “Empirical properties of asset returns: stylized facts and statistical issues,” *Quantitative Finance*, 1, 223–236.
- DEMARTA, S. AND A. J. MCNEIL (2005): “The t copula and related copulas,” *International statistical review*, 73, 111–129.
- DOWD, K., J. COTTER, AND G. SORWAR (2008): “Spectral risk measures: properties and limitations,” *Journal of Financial Services Research*, 34, 61–75.
- EMBRECHTS, P., A. MCNEIL, AND D. STRAUMANN (2002): “Correlation and dependence in risk management: properties and pitfalls,” *Risk management: value at risk and beyond*, 1, 176–223.
- FAMA, E. F. (1963): “Mandelbrot and the stable Paretian hypothesis,” *The Journal of Business*, 36, 420–429.
- FISHER, N. I. AND P. K. SEN (2012): *The collected works of Wassily Hoeffding*, Springer Science & Business Media.
- FÖLLMER, H. AND A. SCHIED (2002): *Stochastic Finance. An Introduction in Discrete Time*, de Gruyter.
- GENEST, C. (1987): “Frank’s family of bivariate distributions,” *Biometrika*, 74, 549–555.
- GENEST, C., K. GHOUDI, AND L.-P. RIVEST (1995): “A semiparametric estimation procedure of dependence parameters in multivariate families of distributions,” *Biometrika*, 82, 543–552.

- GENEST, C. AND L.-P. RIVEST (1993): “Statistical inference procedures for bivariate Archimedean copulas,” *Journal of the American statistical Association*, 88, 1034–1043.
- HÄRDLE, W. K., N. HAUTSCH, AND L. OVERBECK (2008): *Applied Quantitative Finance*, Springer Science & Business Media.
- HÄRDLE, W. K., M. MÜLLER, S. SPERLICH, AND A. WERWATZ (2004): *Nonparametric and Semiparametric Models*, Springer Science & Business Media.
- HÄRDLE, W. K. AND O. OKHRIN (2010): “De copulis non est disputandum,” *AStA Advances in Statistical Analysis*, 94, 1–31.
- HOEFFDING, W. (1940a): “Masstabinvariante Korrelationstheorie,” *Schriften des Mathematischen Instituts und Instituts für Angewandte Mathematik der Universität Berlin*, 5, 181–233.
- (1940b): “Scale-invariant correlation theory (English translation),” 5, 181–233.
- (1941): “Scale-invariant correlations for discontinuous distributions (English translation),” 7, 49–70.
- JOE, H. (1997): *Multivariate models and multivariate dependence concepts*, CRC Press.
- KALEMANOVA, A., B. SCHMID, AND R. WERNER (2007): “The normal inverse Gaussian distribution for synthetic CDO pricing,” *The Journal of Derivatives*, 14, 80–94.
- KRUPSKII, P. AND H. JOE (2013): “Factor copula models for multivariate data,” *Journal of Multivariate Analysis*, 120, 85–101.
- KRUSKAL, W. H. (1958): “Ordinal measures of association,” *Journal of the American Statistical Association*, 53, 814–861.
- MCNEIL, A., R. FREY, AND P. EMBRECHTS (2005): *Quantitative Risk Management*, Princeton, NJ: Princeton University Press.
- MENEZES, C., C. GEISS, AND J. TRESSLER (1980): “Increasing downside risk,” *The American Economic Review*, 70, 921–932.
- NAKAMOTO, S. (2009): “Bitcoin: A Peer-to-Peer Electronic Cash System,” .
- NELSEN, R. (2002): “Concordance and copulas: A survey,” in *Distributions with Given Marginals and Statistical Modelling*, Kluwer Academic Publishers, 169–178.
- NELSEN, R. B. (1999): *An Introduction to Copulas*, Springer.
- SCHWEIZER, B., E. F. WOLFF, ET AL. (1981): “On nonparametric measures of dependence for random variables,” *Annals of Statistics*, 9, 879–885.
- SKLAR, A. (1959): “Fonctions de répartition à n dimensions et leurs marges,” *Publications de l’Institut de Statistique de l’Université de Paris*, 8, 229–231.
- TAKEUCHI, K. (1976): “Distribution of informational statistics and a criterion of model fitting. Suri-Kagaku (Mathematical Sciences) 153 12-18,” .