Hedging Cryptos with Bitcoin Futures

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Abstract

The introduction of derivatives on Bitcoin, in particular the launch of futures contracts on CME in December 2017 and introduction of cryptocurrency indices like the CRIX or the Bloomberg Galaxy Crypto Index enables investors to hedge risk exposures of Bitcoin by futures or contingent claims on indices. We investigate methods of finding the optimal hedge ratio h^* under different dependence structures modeled by copulae and optimality definition described by a range of risk measures. Because of volatility swings and jumps in Bitcoin prices, the traditional variance-based approach to obtain the hedge ratios is infeasible. The approach is therefore generalised to various risk measures, such as Value-at-Risk, Expected Shortfall and a variety of Spectral Risk Measures. In addition, we deploy different copulae for capturing the dependency between spot and future returns, such as the Gaussian, Student-t, NIG and Archimedean copulae. Various measures of hedge effectiveness in out-of-sample tests give insights in the practice of hedging Bitcoin. We find that across copulae and risk measures, the hedge effectiveness are very similar with the exception of the Frank copula, Expected Shortfall 99% and Value-at-Risk 99%. Our findings are based on an analysis for the time span from 15/12/2017 to 04/02/2021. The results allows investors to construct a stable portfolio

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1 Introduction

Cryptocurrencies (CCs) are a growing asset class. Many more CCs are now available on the market since the first cryptocurrency Bitcoin (BTC) surfaced (Nakamoto, 2009). In response to the rapid development of the CC market, the CME group launched a BTC future contract in Dec 2017. Trading volume in BTC futures surpassed $2 trillion in 2020 (CryptoCompare, 2020). While more and more investors (individuals and institutions) are adding CCs and their derivatives into their portfolios, we see the need to understand the downside risk and find a suitable way to hedge against extreme risks. The price of bitcoin even surged to USD 20,000 at beginning of 2021. This paper analyse modern techniques for the choice of the hedge ratio of the CC portfolios with various copulae and risk measures.

The optimal hedge ratio is the appropriate size of futures contracts which should be held such that the movements in future price cancel that of BTC. The task of determining an optimal hedge ratio is not easy. It relies on the dependence between the BTC price and future price. Copulae provide the flexibility to model multivariate random variable separately by its margins and dependence structure. The concept of copulae was originally developed (but not under this name) by Wassily Hoeffding (Hoeffding, 1940a) , later popularised by the work of Abe Sklar (Sklar, 1959). Different risk measures account for investors' risk attitude. They serve as loss functions in the searching process of optimal hedge ratio. Vast literature discussed the relationship between risk measures and investor's risk attitude, we refer readers to Artzner et al. (1999) for an axiomatic, economic reasoning approach of risk measure construction; Embrechts et al. (2002) for reasoning of using Expected Shortfall (ES) and Spectral Risk Measure (SRM) in addition to VaR; Acerbi (2002) for direct linkage between risk measures and investor's risk attitude using the concept of "risk aversion function".

Financial asset return is known to be non Gaussian (Fama, 1963). In particular, Gaussian models cannot produce so-called fat tails and asymmetry of observed probability densities, which leads to underestimate financial risks. Therefore, one cannot solely rely on 2^{nd} order moment calculations in order to minimize downside risk. Variance as a risk measure doesn't consider the variety of investors' utility functions. However, the investors are tail-risk averse. Bollerslev et al. (2015) find that the jump tail risk is more closely associated with changes in risk-aversion. It is important to link the investor utility's functions as hedging the tail risk. Significant tail risks lead to the need to investigate even static hedge with more refined methods than minimum-variance based (Ederington and Salas, 2008).

In order to capture the risk preferences of investors, in addition to variance, we include other risk measures. We consider also Value-at-Risk (VaR), Expected Shortfall (ES), and Spectral Risk Measure (SRM). VaR is widely used by the industry and easy to understand. ES and SRM are chosen because of their coherence property, in particular, they encourage diversification. SRM is also directly related to individual's utility function. Popular examples are the exponential SRM and power SRM introduced by Dowd et al. (2008).

This paper considers hedging BTC using its future. i.e. to find an optimal hedge ratio h^* such that the risk of a hedged portfolio $r^h = r^S - h^* r^F$ has minimal risk. Here r^S as the log return of BTC spot price, r^F the log return of BTC future. The leptokurtic properties mentioned above leads us to deploy a comprehensive way of modelling dependency namely copulae together with various risk measures as loss function to find optimal hedge ratio. We first calibrate the log returns of BTC and CME future by copulae, then find the optimal quantity of assets in the hedged portfolio according to a range of risk measures. Barbi and Romagnoli (2014) use the C-convolution operator introduced by Cherubini et al. (2011) to derive the distribution of linear combination of margins with copula as their dependence structure. We slightly amend their lemma and come up with a formula for the linear combination of random variables for our purpose.

This paper is organized as follows. Section 2 introduces the notion of optimal hedge ratio; section 3 decribe the method of estimation of copulae; section 4 provides the empirical result; section 5 concludes.All calculations in this work can be reproduced. The codes are available on www.quantlet.com.

2 Optimal hedge ratio

We form a portfolio with two assets, a spot asset and a future contract, for example Bitcoin spot and CME Bitcoin future. Our objective is to minimize the risk of the exposure in the spot. To keep a simple portfolio setting, we long one unit of the spot and short h unit of the future with $h \in [0, \infty)$. Let r^S and r^F be the log returns of the spot and future price, the log return of the portfolio is

$$r^h = r^S - hr^F.$$

We call this portfolio a hedged portfolio: the price movement of spot is hedged by the price movement of future.

Risk is measured by risk measures. Assume the payoff r^h of a portfolio lives in a probability space, $r^h \in L(\Omega, \mathcal{F}, \mathbb{P})$, and there is a risk measure on r^h $\rho : r^h \mapsto \mathbb{R}$. We are looking for an optimal hedge ratio h^* which minimizes risk measure

$$h^* = \underset{h}{\operatorname{argmin}}\, \rho(r^h).$$

Most risk measures are defined as functionals of the portfolio loss distribution F_{r^h}, i.e. $\rho : F_{r^h} \mapsto \mathbb{R}$. For example, Value-at-Risk (VaR) is simply the quantile of r^h multiply with negative one $\mathrm{VaR}_{1-\alpha} = -F_{r^h}^{(-1)}(1-\alpha) = -\inf\{x \in \mathbb{R} : 1 - \alpha \leq F_{r^h}(x)\}$, where α is a parameter chosen by investor. We need the knowledge of F_{r^h} in order to measure risk. By convolution of random variables (Härdle and Simar, 2019), $f_{r^h}(z) = \int_{-\infty}^{\infty} f_{r^S,-hr^F}(x, z-x)dx$, where $f_{r^S,-hr^F}$ is the joint pdf of r^S and $-hr^F$. Obviously the cdf of r^h and risk measure depend on the joint distribution of r^S and $-hr^F$.

Optimising h according to $f_{r^S,-hr^F}$ is unfavorable in a sense that one needs to calibrate a new joint pdf $f_{r^S,-hr^F}$ when updating h. This is too time consuming and unnecessary. Another problem of using joint pdf is that one lacks of flexibility to model the margins. A joint pdf completely determine the form of its marginals, for example, margins of a bivariate t-distribution are univariate t-distributions.

To overcome the problems, we use copulae. The benefit of using copulae is two folded. First, copulae allow us to model the margins and dependence structure separately, see Sklar's Theorem. Second, copulae are invariance under strictly monotone increasing function (Schweizer et al., 1981), see lemma below.

Theorem 1 (Hoeffding Sklar Theorem) *Let F be a joint distribution function with margins F_X, F_Y. Then, there exists a copula $C : [0,1]^2 \mapsto [0,1]$ such that, for all $x, y \in \mathbb{R}$*

$$F(x, y) = C\{F_X(x), F_Y(y)\}. \tag{1}$$

If the margins are continuous, then C is unique; otherwise C is unique on the range of the margins.

Conversely, if C is a copula and F_X, F_Y are univariate distribution functions, then the function F defined by (1) is a joint distribution function with margins F_X, F_Y.

Indeed, many basic results about copulae can be traced back to early works of Wassily Hoeffding (Hoeffding, 1940b, 1941). The works aimed to derive a measure of relationship of variables which is invariant under change of scale. Readers can refer to Fisher and Sen (2012) for English translations of the works.

Lemma 1

$$C_{X,hY}\{F_X(s), F_{hY}(t)\} = C_{X,Y}\{F_X(s), F_Y(t/h)\}. \tag{2}$$

Leveraging the two features of copulae, Barbi and Romagnoli (2014) introduces the distribution of linear combination of random variables using copulae. We slightly edit the Corollary 2.1 of their work and yield the following correct expression of the distribution.

Proposition 2 *Let X and Y be two real-valued continuous random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with absolutely continuous copula $C_{X,Y}$ and marginal distribution functions F_X and F_Y. Then, the distribution function of of Z is given by*

$$F_Z(z) = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \tag{3}$$

Here, $F^{(-1)}$ denotes the inverse of F, i.e., the quantile function.

Here $D_1 C(u, v) = \frac{\partial}{\partial u} C(u, v)$ see e.g. Equation (5.15) of (McNeil et al., 2005):

$$D_1 C_{X,Y}(F_X(x), F_Y(y)) = \mathbf{P}(Y \leq y | X = x). \tag{4}$$

Proof. Using the identity (4) gives

$$\begin{aligned} F_Z(z) &= \mathbf{P}(X - hY \leq z) = \mathsf{E}\left\{\mathbf{P}\left(Y \geq \frac{X - z}{h}\Big| X\right)\right\} \\ &= 1 - \mathsf{E}\left\{\mathbf{P}\left(Y \leq \frac{X - z}{h}\Big| X\right)\right\} = 1 - \int_0^1 D_1 C_{X,Y}\left[u, F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}\right] du. \end{aligned}$$

■

Corollary 1 *Given the formulation of the above portfolio, the pdf of Z can be written as*

$$f_Z(z) = \left|\frac{1}{h}\right| \int_0^1 c_{X,Y}\left[F_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\}, u\right] \cdot f_Y\left\{\frac{F_X^{(-1)}(u) - z}{h}\right\} du \tag{5}$$

, or

$$f_Z(z) = \int_0^1 c_{X,Y}\left[F_X\left\{z + hF_Y^{(-1)}(u)\right\}, u\right] \cdot f_X\left\{z + hF_Y^{(-1)}(u)\right\} du. \tag{6}$$

The two expression are equivalent. Notice that the pdf of Z in the above proposition is readily accessible as long as we have the copula density and the marginal densities. The proof and a generic expression can be found in the appendix.

2.1 Risk Measures

We consider a variety of risk measures: variance, Value-at-Risk (VaR), Expected Shortfall (ES), and Exponential Risk Measure (ERM). A summary of risk measures being used in portfolio selection problem can be found in Härdle et al. (2008).

Let Z be a random variable of distribution F_Z.

1. Variance is $\mathrm{Var}(F_Z)$
2. VaR of a given confidence level α is $\mathrm{VaR}(F_Z) = -F_Z^{(-1)}(1-\alpha)$
3. ES with parameter α is $\mathrm{ES}(F_Z) = -\frac{1}{1-\alpha}\int_0^{1-\alpha} F_Z^{(-1)}(p)dp$
4. ERM with Arrow-Pratt coeficient of absolute risk aversion k is $\mathrm{ERM}_k(F_Z) = \int_0^{1-\alpha} \phi(p)F_Z^{(-1)}(p)dp$ where ϕ is a weight function described in (8) below.

VaR, ES, and ERM fall into the class of Spectral Risk Measure (SRM). SRM has the from (Acerbi, 2002)

$$\rho_\phi(r^h) = -\int_0^1 \phi(p)F_Z^{(-1)}(p)dp, \tag{7}$$

where p is the loss quantile and $\phi(p)$ is a user-defined weighting function defined over $[0,1]$. We consider only admissible risk spectra $\phi(p)$ (named by Acerbi (2002))

i ϕ is positive

ii ϕ is decreasing

iii integrates to one.

The VaR's $\phi(p)$ gives all its weight on the $1-\alpha$ quantile of Z and zero elsewhere, i.e. the weighting function is a Dirac delta function, hence violates the ii property of admissible risk spectra. The ES' $\phi(p)$ gives all tail quantiles the same weight of $\frac{1}{1-\alpha}$ and non-tail quantiles zero weight. ERM assumes investor's risk preference is in a form of exponential utility function $U(x) = -e^{kx}$, its risk spectrum is defined as

$$\phi(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}}, \tag{8}$$

where k is the Arrow-Pratt coefficient of absolute risk aversion.

k has an economic interpretation of being the ratio between the second derivative and first derivative of investor's utility function on an risky asset

$$k = -\frac{U''(x)}{U'(x)}, \tag{9}$$

for x in all possible outcomes.

2.2 Copulae

As we saw from the last section, risk measures we considered are all functionals of the joint distribution of r^S and r^F. We test different copulae: Gaussian-, t-, Frank-, Gumbel-, Clayton-, Plackett-, mixture, and factor copula. This hedging exercise concerns only portfolios with two assets, we only present the bivariate version of copulae and some important features of a copula, they include Kendall's τ, Spearman's ρ, upper tail dependence $\lambda_U \stackrel{\text{def}}{=} \lim_{q\to 1^-} \mathbf{P}\{X > F_X^{(-1)}(q)|Y > F_Y^{(-1)}(q)\}$ and lower tail dependence $\lambda_L \stackrel{\text{def}}{=} \lim_{q\to 0^+} \mathbf{P}\{X \leq F_X^{(-1)}(q)|Y \leq F_Y^{(-1)}(q)\}$. Furthermore, we denote the Fréchet-Hoeffding lower bound as $\boldsymbol{W}$, product copula as $\boldsymbol{\Pi}$, and the Fréchet-Hoeffding upper bound as $\boldsymbol{M}$, they represent cases of perfect negative dependence, independence, and perfect positive dependence respectively. For further detail, we refer readers to Joe (1997) and Nelsen (1999). See also Härdle and Okhrin (2010).

2.2.1 Elliptical Copulae

Elliptical copulae are dependence structure associated with elliptical distributions. The bivariate Gaussian copula is:

$$\begin{aligned}\boldsymbol{C}(u,v) &= \Phi_{2,\rho}\{\Phi^{-1}(u),\Phi^{-1}(v)\} \\ &= \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right\}dsdt \end{aligned} \tag{10}$$

where $\Phi_{2,\rho}$ is the cdf of bivariate Normal distribution with zero mean, unit variance, and correlation ρ. , and Φ^{-1} is quantile function univariate standard normal distribution. Please note that we use rho to represent the correlation parameter in Gaussian copula only for traditional purposes. In other sections, $rho(\cdot)$ is a risk measure. The Gaussian copula density is

$$\begin{aligned}\boldsymbol{c}_\rho(u,v) &= \frac{\boldsymbol{\varphi}_{2,\rho}\{\Phi^{-1}(u),\Phi^{-1}(v)\}}{\varphi\{\Phi^{-1}(u)\}\cdot\varphi\{\Phi^{-1}(v)\}} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right\}, \end{aligned} \tag{11}$$

where $\boldsymbol{\varphi}_{2,\rho}(\cdot)$ is the pdf of $\Phi_{2,\rho}$, and $\varphi(\cdot)$ the standard normal distribution pdf.

The Kendall's τ_K and Spearman's ρ_S of a bivariate Gaussian Copula are

$$\tau_K(\rho) = \frac{2}{\pi}\arcsin\rho \tag{12}$$

$$\rho_S(\rho) = \frac{6}{\pi}\arcsin\frac{\rho}{2} \tag{13}$$

The t-Copula has a form

$$\boldsymbol{C}(u,v) = \boldsymbol{T}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\}$$
$$= \int_{-\infty}^{T_\nu^{-1}(u)} \int_{-\infty}^{T_\nu^{-1}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{1-\rho^2}} \tag{14}$$
$$\left(1 + \frac{s^2 - 2st\rho + t^2}{\nu}\right)^{-\frac{\nu+2}{2}} dsdt, \tag{15}$$

where $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$ denotes the cdf of bivariate t distribution with scale parameter ρ and degree of freedom ν, $T_\nu^{-1}(\cdot)$ is the quantile function of a standard t distribution with degree of freedom ρ.

The copula density is

$$\boldsymbol{c}(u,v) = \frac{\boldsymbol{t}_{2,\rho,\nu}\{T_\nu^{-1}(u), T_\nu^{-1}(v)\}}{t_\nu\{T_\nu^{-1}(u)\} \cdot t_\nu\{T_\nu^{-1}(v)\}}, \tag{16}$$

where $\boldsymbol{t}_{2,\rho,\nu}$ is the pdf of $\boldsymbol{T}_{2,\rho,\nu}(\cdot,\cdot)$, and t_ν the density of standard t distribution.

Like all the other elliptical copulae, t copula's Kendall's τ is identical to that of Gaussian copula (Demarta and reference therein).

2.2.2 Archimedean Copulae

The Archimedean copulae forms a large class of copulae with many convenient features. In general, they take a form

$$\boldsymbol{C}(u,v) = \psi^{-1}\{\psi(u), \psi(v)\}, \tag{17}$$

where $\psi : [0,1] \to [0,\infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1) = 0$ for any permissible dependence parameter θ. ψ is also called generator. ψ^{-1} is the inverse the generator.

The Frank copula (B3 in Joe (1997)) is a radial symmetric copula and cannot produce any tail dependence. It takes the form

$$\boldsymbol{C}_\theta(u,v) = \frac{1}{\theta}\log\left\{1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right\} \tag{18}$$

where $\theta \in [0,\infty]$ is the dependency parameter. $\boldsymbol{C}_{-\infty} = \boldsymbol{M}$, $\boldsymbol{C}_1 = \boldsymbol{\Pi}$, and $\boldsymbol{C}_\infty = \boldsymbol{W}$.

The Copula density is

$$\boldsymbol{c}_\theta(u,v) = \frac{\theta e^{\theta(u+v)(e^\theta - 1)}}{\left\{e^\theta - e^{\theta u} - e^{\theta v} + e^{\theta(u+v)}\right\}^2} \tag{19}$$

Frank copula has Kendall's τ and Spearman's ρ as follow:

$$\tau_K(\theta) = 1 - 4\frac{D_1\{-\log(\theta)\}}{\log(\theta)}, \tag{20}$$

and

$$\rho_S(\theta) = 1 - 12\frac{D_2\{-\log(\theta)\} - D_1\{\log(\theta)\}}{\log(\theta)}, \tag{21}$$

where D_1 and D_2 are the Debye function of order 1 and 2. Debye function is $D_n = \frac{n}{x^n}\int_0^x \frac{t^n}{e^t-1}dt$.

Gumbel copula (B6 in Joe (1997)) has upper tail dependence with the dependence parameter $\lambda^U = 2 - 2^{\frac{1}{\theta}}$ and displays no lower tail dependence.

$$\boldsymbol{C}_\theta(u,v) = \exp -\{(-\log(u))^\theta + (-\log(v))^\theta\}^{\frac{1}{\theta}}, \tag{22}$$

where $\theta \in [1,\infty)$ is the dependence parameter.While Gumbel copula cannot model perfect counter dependence (ref), $\boldsymbol{C}_1 = \boldsymbol{\Pi}$ models the independence, and $\lim\limits_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$ models the perfect dependence.

$$\tau_K(\theta) = \frac{\theta - 1}{\theta} \tag{23}$$

The Clayton copula, by contrast to Gumbel copula, generates lower tail dependence in a form $\lambda^L = 2^{-\frac{1}{\theta}}$, but cannot generate upper tail dependence.

The Clayton copula takes the form

$$\boldsymbol{C}_\theta(u,v) = \left\{\max(u^{-\theta} + v^{-\theta} - 1, 0)\right\}^{-\frac{1}{\theta}}, \tag{24}$$

where $\theta \in (-\infty,\infty)$ is the dependency parameter. $\lim\limits_{\theta\to-\infty} \boldsymbol{C}_\theta = \boldsymbol{M}$, $\boldsymbol{C}_0 = \boldsymbol{\Pi}$, and $\lim\limits_{\theta\to\infty} \boldsymbol{C}_\theta = \boldsymbol{W}$.

Kendall's τ to this copula dependency is

$$\tau_K(\theta) = \frac{\theta}{\theta+2}. \tag{25}$$

2.2.3 Mixture Copula

Mixture copula is a linear combination of copulae. For a 2-dimensional random variable $\boldsymbol{X} = (X_1, X_2)^\top$, its distribution can be written as linear combination K copulae

$$\mathbf{P}(X_1 \le x_1, X_2 \le x_2) = \sum_{k=1}^{K} p^k \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{(k)}(x_1; \boldsymbol{\gamma}_1^{(k)}), F_{X_2}^{(k)}(x_2; \boldsymbol{\gamma}_2^{(k)}); \boldsymbol{\theta^{(k)}}\} \tag{26}$$

where $p^{(k)} \in [0,1]$ is the weight of each component, $\boldsymbol{\gamma}^{(k)}$ is the parameter of the marginal distribution in the k^{th} component, and $\boldsymbol{\theta^{(k)}}$ is the dependence parameter with the copula of the k^{th} component. The weights add up to one $\sum_{k=1}^K p^{(k)} = 1$.

We deploy a simplified version of the above representation by assuming the margins of $\boldsymbol{X}$ are not mixture. By Sklar's theorem one may write

$$\boldsymbol{C}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \tag{27}$$

The copula density is again a linear combination of copula density

$$\boldsymbol{c}(u,v) = \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{c}^{(k)}\{F_{X_1}^{-1}(u), F_{X_2}^{-1}(v); \boldsymbol{\theta^{(k)}}\}. \tag{28}$$

While Kendall's τ of mixture copula is not known in close form, the Spearman's ρ is

Proposition 3 *Let $\rho_S^{(k)}$ be the Spearman's ρ of the k^{th} component and $\sum_{k=1}^{K} p^{(k)} = 1$ holds, the Spearman's ρ of a mixture copula is*

$$\rho_S = \sum_{k=1}^{K} p^{(k)} \cdot \rho_S^{(k)} \tag{29}$$

Proof. Spearman's ρ is defined as (Nelsen, 1999)

$$\rho_S = 12 \int_{\mathbb{I}^2} \boldsymbol{C}(s,t) ds dt - 3. \tag{30}$$

Rewrite the mixture copula into sumation of components

$$\rho_S = 12 \int_{\mathbb{I}^2} \sum_{k=1}^{K} p^{(k)} \cdot \boldsymbol{C}^{(k)}(s,t) ds dt - 3. \tag{31}$$

■

Example 4 *The Fréchet class can be seen as an example of mixture copula. It is a convex combinations of $\boldsymbol{W}$, $\boldsymbol{\Pi}$, and $\boldsymbol{M}$ (Nelsen, 1999)*

$$\boldsymbol{C}_{\alpha,\beta}(u,v) = \alpha \boldsymbol{M}(u,v) + (1-\alpha-\beta)\boldsymbol{\Pi}(u,v) + \beta \boldsymbol{W}(u,v), \tag{32}$$

where α and β are the dependence parameters, with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. Its Kendall's τ and Spearman's ρ are

$$\tau_K(\alpha,\beta) = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3} \tag{33}$$

, and

$$\rho_S(\alpha,\beta) = \alpha - \beta \tag{34}$$

We use a mixture of Gaussian and independent copula in our analysis. We write the copula

$$\boldsymbol{C}(u,v) = p \cdot \boldsymbol{C}^{\text{Gaussian}}(u,v) + (1-p)(uv). \tag{35}$$

The corresponding copula density is

$$\boldsymbol{c}(u,v) = p \cdot \boldsymbol{c}^{\text{Gaussian}}(u,v) + (1-p). \tag{36}$$

This mixture allows us to model how much "random noise" appear in the dependency structure. In this hedging exercise, the structure of the "random noise" is not of our concern nor we can hedge the noise by a two-asset portfolio. However, the proportion of the "random noise" does affect the distribution of r^h, so as the optimal hedging ratio h^*. One can consider the mixture copula as a handful tool for stress testing. Similar to this Gaussian mix Independent copula, t copula is also a two parameter copula allow us to model the noise, but its interpretation of parameters is not as intuitive as that of a mixture. The mixing variable p is the proportion of a manageable (hedgable) Gaussian copula, while the remaining proportion $1-p$ cannot be managed.

2.3 Other Copula

The Plackett copula has an expression

$$\boldsymbol{C}_\theta(u,v) = \frac{1+(\theta-1)(u+v)}{2(\theta-1)} - \frac{\sqrt{\{1+(\theta-1)(u+v)\}^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \tag{37}$$

$$\rho_S(\theta) = \frac{\theta+1}{\theta-1} - \frac{2\theta\log\theta}{(\theta-1)^2} \tag{38}$$

We include Placket copula in our analysis as it possesses a special property, the cross-product ratio is equal to the dependence parameter

$$\begin{aligned} &\frac{\mathbf{P}(U\le u, V\le v)\cdot\mathbf{P}(U>u,V>v)}{\mathbf{P}(U\le u, V>v)\cdot\mathbf{P}(U>u,V\le v)} \\ &= \frac{\boldsymbol{C}_\theta(u,v)\{1-u-v+\boldsymbol{C}_\theta(u,v)\}}{\{u-\boldsymbol{C}_\theta(u,v)\}\{v-\boldsymbol{C}_\theta(u,v)}\\ &= \theta. \end{aligned} \tag{39}$$

That is, the dependence parameter is equal to the ratio between number of concordence pairs and number of discordence pairs of a bivariate random variable.

3 Estimation

3.1 Simulated Method of Moments

This method is suggested by Oh and Patton (2013). In our setting, rank correlation e.g. Spearman's ρ or Kendall's τ, and quantile dependence measures at different levels λ_q are calibrated against their empirical counterparts.

Spearman's rho, Kendall's tau, and quantile dependence of a pair (X,Y) with copula C are defined as

$$\rho_S = 12\int\int_{I^2} C_{\boldsymbol{\theta}}(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3 \tag{40}$$

$$\tau_K = 4\,\mathsf{E}[C_{\boldsymbol{\theta}}\{F_X(x),F_Y(y)\}] - 1, \tag{41}$$

$$\lambda_q = \begin{cases} \mathbf{P}(F_X(X)\le q|F_Y(Y)\le q) = \dfrac{C_{\boldsymbol{\theta}}(q,q)}{q}, & \text{if } q\in(0,0.5],\\ \mathbf{P}(F_X(X)> q|F_Y(Y)> q) = \dfrac{1-2q+C_{\boldsymbol{\theta}}(q,q)}{1-q}, & \text{if } q\in(0.5,1). \end{cases} \tag{42}$$

The empirical counterparts are

$$\hat{\rho}_S = \frac{12}{n}\sum_{k=1}^{n} \hat{F}_X(x_k)\hat{F}_Y(y_k) - 3,$$

$$\hat{\tau}_K = \frac{4}{n}\sum_{k=1}^{n} \hat{C}\{\hat{F}_X(x_i), \hat{F}_X(y_i)\} - 1,$$

$$\hat{\lambda}_q = \begin{cases} \frac{1}{n}\sum_{k=1}^{n} \frac{\mathbf{1}_{\{\hat{F}_X(x_k)\le q, \hat{F}_Y(y_k)\le q\}}}{q}, & \text{if } q \in (0, 0.5], \\ \frac{1}{n}\sum_{k=1}^{n} \frac{\mathbf{1}_{\{\hat{F}_X(x_k)> q, \hat{F}_Y(y_k)> q\}}}{1-q}, & \text{if } q \in (0.5, 1). \end{cases},$$

where $\hat{F}(x) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^{n} 1_{\{x_i \le x\}}$ and $\hat{C}(u,v) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=1}^{n} 1_{\{u_i \le u, v_i \le v\}}$.

We denote $\tilde{\boldsymbol{m}}(\boldsymbol{\theta})$ be a m-dimensional vector of dependence measures according the the dependence parameters $\boldsymbol{\theta}$,and $\hat{\boldsymbol{m}}$ be the corresponding empirical counterpart. The difference between dependence measures and their counterpart is denoted by

$$\boldsymbol{g}(\boldsymbol{\theta}) = \hat{\boldsymbol{m}} - \tilde{\boldsymbol{m}}(\boldsymbol{\theta}).$$

The SMM estimator is

$$\hat{\boldsymbol{\theta}} = \operatorname*{argmin}_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \boldsymbol{g}(\boldsymbol{\theta})^\top \hat{\boldsymbol{W}} \boldsymbol{g}(\boldsymbol{\theta}),$$

where $\hat{W}$ is some positive definite weigh matrix.

In this work, we use $\tilde{\boldsymbol{m}}(\boldsymbol{\theta}) = (\rho_S, \lambda_{0.05}, \lambda_{0.1}, \lambda_{0.9}, \lambda_{0.95})^\top$ for calibration of Bitcoin price and CME Bitcoin future.

3.2 Maximum Likelihood Estimation

By Sklar's theorem, the joint density of a d-dimensional random variable $\boldsymbol{X}$ with sample size n can be written as

$$\boldsymbol{f}_{\boldsymbol{X}}(x_1, ..., x_d) = \boldsymbol{c}\{F_{X_1}(x_1), ..., F_{X_d}(x_d)\} \prod_{j=1}^{d} f_{X_i}(x_i). \tag{43}$$

We follow the treatment of MLE documented in section 10.1 of Joe (1997), namely the inference functions for margins or IFM method. The log-likelihood $\sum_{i=1}^{n} \boldsymbol{f}_{\boldsymbol{X}}(X_{i,1}, ..., X_{i,d})$ can be decomposed into dependence part and marginal part,

$$L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \boldsymbol{c}\{F_{X_1}(x_{i,1}; \boldsymbol{\delta}_1), ..., F_{X_d}(x_{i,d}; \boldsymbol{\delta}_d); \boldsymbol{\gamma}\} + \sum_{i=1}^{n}\sum_{j=1}^{d} f_{X_j}(x_{i,j}; \boldsymbol{\delta}_j) \tag{44}$$

$$= L_C(\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma}) + \sum_{j=1}^{d} L_j(\boldsymbol{\delta}_j) \tag{45}$$

where $\boldsymbol{\delta}_j$ is the parameter of the j-th margin, $\boldsymbol{\gamma}$ is the parameter of the parametric copula, and $\boldsymbol{\theta} = (\boldsymbol{\delta}_1, ..., \boldsymbol{\delta}_d, \boldsymbol{\gamma})$.

Instead of searching the $\boldsymbol{\theta}$ is a high dimensional space, Joe (1997) suggests to search for $\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d$ that maximize $L_1(\boldsymbol{\delta}_1), ..., L_d(\boldsymbol{\delta}_d)$, then search for $\hat{\boldsymbol{\gamma}}$ that maximize $L_C(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \boldsymbol{\gamma})$.

That is, under regularity conditions, $(\hat{\boldsymbol{\delta}}_1, ..., \hat{\boldsymbol{\delta}}_d, \hat{\boldsymbol{\gamma}})$ is the solution of

$$\left(\frac{\partial L_1}{\partial \boldsymbol{\delta}_1}, ..., \frac{\partial L_d}{\partial \boldsymbol{\delta}_d}, \frac{\partial L_C}{\partial \boldsymbol{\gamma}} \right) = \mathbf{0}. \tag{46}$$

However, the IFM requires making assumption to the distribution of of the margins. Genest et al. (1995) suggests to replace the estimation of marginals parameters estimation by non-parameteric estimation. Given non-parametric estimator $\hat{F}_i$ of the margins F_i, the estimator of the dependence parameters $\boldsymbol{\gamma}$ is

$$\hat{\boldsymbol{\gamma}} = \underset{\boldsymbol{\gamma}}{\operatorname{argmax}} \sum_{i=1}^{n} \boldsymbol{c}\{\hat{F}_{X_1}(x_{i,1}), ..., \hat{F}_{X_d}(x_{i,d}); \boldsymbol{\gamma}\}. \tag{47}$$

3.3 Comparison

Both the simulated method of moments and the maximum likelihood estimation are unbiased. The problem remain is which procedure is more suitable for hedging.

Figure 1 shows the empirical quantile dependence of Bitcoin and CME future and the copula implied quantile dependence from MLE and MM calibration procedures. Although the MLE is a better fit to a range of quantile dependence in the middle, it fails to address the situation in the tails. Our data empirically has weaker quantile dependence in the ends, and those points generate PnL to the hedged portfolio. MM is preferred visually as it produces a better fit to the dependence structure in the two extremes.

4 Results

In this section, we provide the result of hedging BTC with BTC future using different copulae and risk measures. The results is drawn using the data from 15/12/2017 to 04/02/2021, where 15/12/2021 is the first trading day of the CME BTC future. BTC price is Bitcoin Reference Rate generated by CME; BTC future price is quoted by CME. Both prices are retrieved as daily closing price from Bloomberg Terminal indexed in Berlin time.

Before estimating the optimal hedge ratio, we pre-process the data as follow. First, we inner joint the daily BTC price and BTC future by date, i.e. match the prices with date and discard the unmatched. Then, we compute the log returns by $r_t = \log \frac{\text{Price}_t}{\text{Price}_{t-1}}$ We use this log return to estimate h and evaluate h's performances.

After preprocessing the data, we search h^* for each copula-risk-measure pairs as follow. First, we compute the the marginal distributions by empirical CDF (ECDF) for each training dataset. Each training dataset consists of 300 returns. The ECDF is $\hat{F}(x) = \frac{1}{300} \sum_{n=1}^{300} r \cdot 1(r \leq x)$. Then, we calibrate copulae with the marginal ECDF by the method of moments mentioned above. Next, we draw samples from copula and, with the samples, numerically search for the h^* which minimise the risk measure. We draw one million samples from copula for each search. The h^* is then applied to the out-of-sample data. The out-of-sample is the data of the consecutive 5 trading day to the training

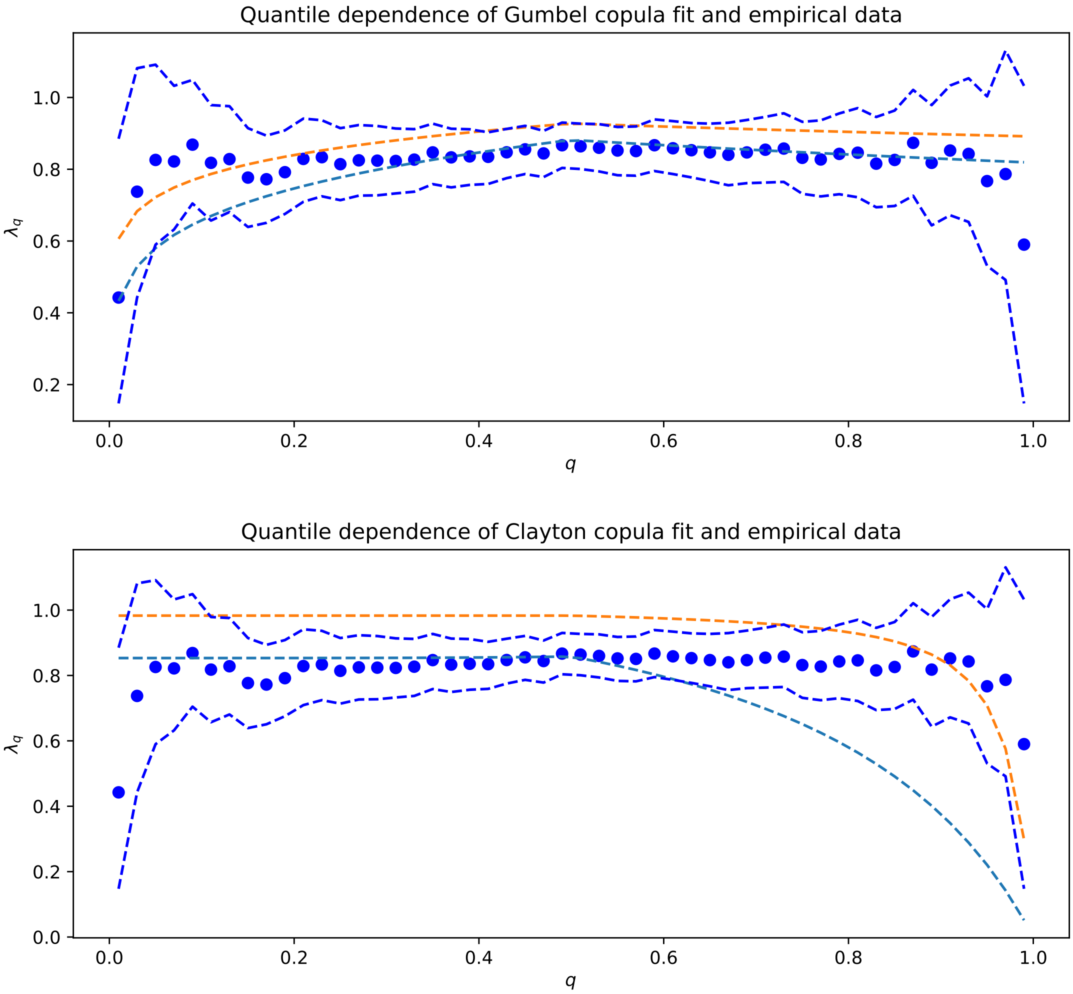

Figure 1: Quantile dependences of Gumbel, and Clayton Copula. The blue circle dots are the quantile dependence estimate of Bitcoin and CME future, blue dotted lines are the estimates' 90% confidnece interval. Orange dotted line is the copula implied quantile dependence by MM estimation. Light blue dotted line is the copula implied quantile dependence by MM estimation.

data (window size of 5). The h^* is indexed with the first date of the out-of-sample data. We then shift the data with 5 trading days and repeat the procedure (step size of 5).

4.1 Profit and Loss

We illustrate the profit and loss (PnL) in in section.

Figure 2 shows the time series of out-of-sample r^h using the NIG factor copula with the objective of reducing variance. The red dots are the 30 most extreme negative returns in Bitcoin. In the figure, we can see the downside risk of Bitcoin is well managed by the hedging procedure with NIG factor copula. Most of the extreme losses of Bitcoin are greatly reduced by introducing the CME future in the hedged portfolio. Two exceptions are found in 25/09/2019 and 26/09/2019, where the CME future failed to follow the large drop in Bitcoin. One of the possible reason is that traders was performing rollover activities on 25-26/09/2019, which 27/09/2019 is the expiry day of the September future. Another reason for the NIG factor copula fail of capturing the loss is dependence break. The Kendall's tau in the training data is 0.2 higher than that of the testing data. Other copulas suffer from the break as well.

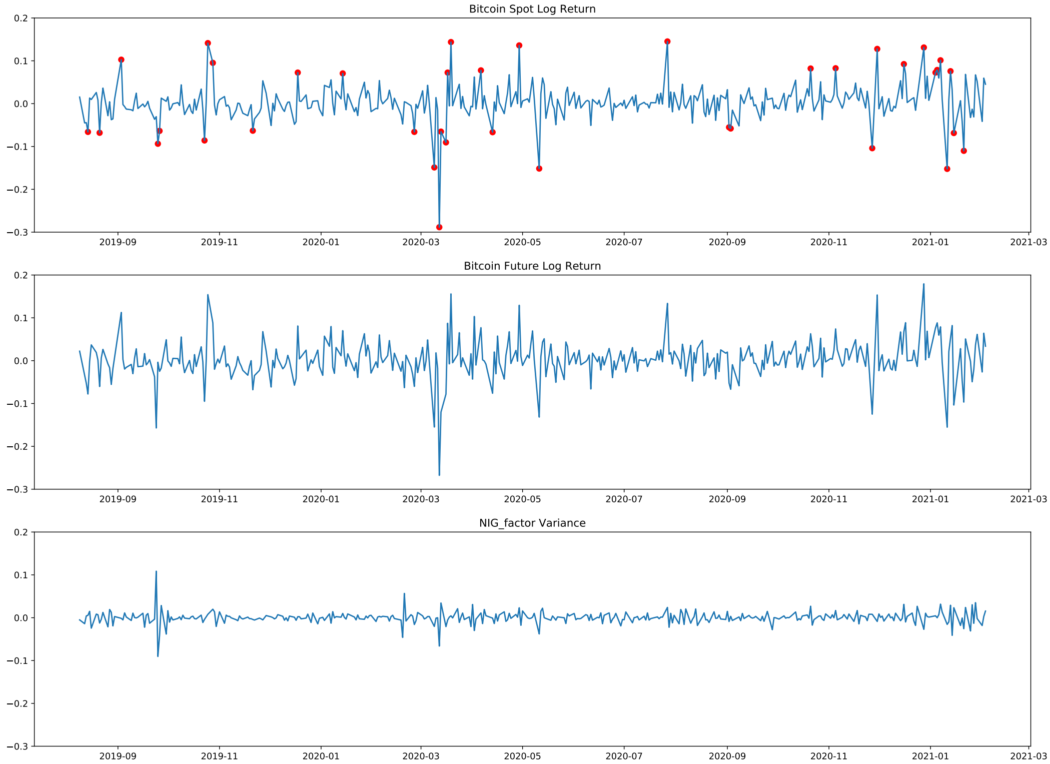

Figure 2: First Panel: Out of Sample Log-return of Bitcoin; Second Panel: Out of Sample Log-return of Future; Third Panel: Out of Sample Log-return of Hedged Portfolio by NIG factor copula with the aim of variance reduction. The red dots indicate the lowest 10% return of Bitcoin, i.e. large negative moments of price.

Results of the other pairs are very similar, they are documented in figure 7 in the appendix. The figure tabulates the time series of out-of-sample returns of hedged portfolio under various copulas and risk reduction objectives.

4.2 Evaluation

In addition to the PnL generated by different copula-risk-measure pairs, we evaluate their performance by hedging effectiveness (HE), root mean squared error (RMSE), and downside semivariance (SV). The purpose of evaluating performances of copula-risk-measure pairs beyond PnL is to assist the judgement of which pairs can estimate the optimal hedge ratio which are suitable to hedge BTC with its future. , in particular, HE measures the reduction of risk, the RMSE and SV relate investors' preference (via utility function) while judging which random outcomes are better than the other.

4.2.1 Hedging Effectiveness

The hedging effectiveness (HE) measures the reduction of portfolio risk. This notion of evaluating of hedging performance was proposed by Ederington (1979) in the context of hedging the newly introduced organized futures market.

HE is defined as

$$1 - \frac{\rho(r^h)}{\rho(r^S)}, \tag{48}$$

where ρ is a risk measure.

We measure the HE of copula-risk-measure pairs according to the risk measure, for example we measure the HE of Gaussian-ES99% pair by

$$1 - \frac{\mathrm{ES}_{99\%}(r^h)}{\mathrm{ES}_{99\%}(r^S)}, \tag{49}$$

where r^h is out-of-sample return generated by Gaussian-ES99%, and r^S is the out-of-sample log return of BTC.

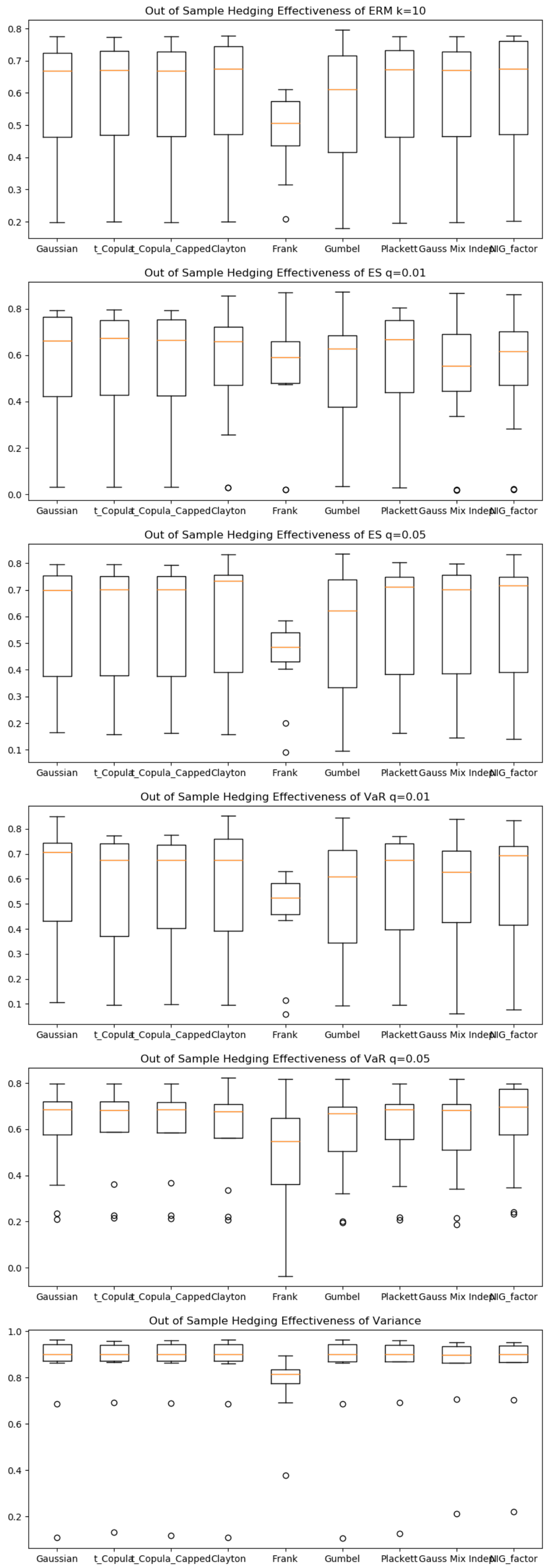

Figure 3: Out of Sample Hedging Effectiveness Box-plot. The HEs are obtained from a set of out-of-sample data, each set consists 30 days log returns of Bitcoin and CME future.

The box-plots in figure 3 show the out-of-sample hedging effectiveness of different copulas under various risk reduction objectives across testing datasets. Observe that most of the copulae perform well. The average HE of copulas and risk reduction objectives is higher than 60% except for Frank-copula. However, the HEs vary a lot in different testing data. In some instances, the HE can be as low as 10%. This reflects the highly violate nature of cryptocurrencies: the optimal hedge ratio in the training data deviates from that of testing data. There is a large literature about structural break points and time changing dependence, to name a few Hafner and Manner (2012), Patton (2006), Creal et al. (2008), Engle (2002), Giacomini et al. (2009), and also Manner and Reznikova (2012).

4.2.2 Root Mean Squared Error and Semivariance

Root mean squared error (RMSE) and semivariance (SV) are special cases of the Bernel Stone's generalized risk measure (Stone, 1973). The purpose of applying a generalized risk measure is to measure the hedge performance of copula-risk-measure pairs in a common ground.

The Bernel Stone's generalized risk measure is

$$\rho(F) = \int_{-\infty}^{\gamma(F)} |x - \eta(F)|^{\alpha} \, dF(x),$$

where F is the distribution of the uncertain return, parameters α is chosen to represent preferences of investors, $\eta(F)$ is a reference level of wealth from which deviations are measured, and $\gamma(F)$ is a range parameter that specifies the range of deviations to be included.

Fishburn (1977) justifies the usage of generalized risk measure (with his α-t model) by connecting the measure with Von Neumann-Morgenstern utility theorem, see also Bawa (1975, 1978) and Morgenstern and Von Neumann (1953). We argue that evaluation of hedging performance of a crypto portfolio does not differ from this classical framework: crypto investors maximise expected utility with given utility functions. Vast body of recent literature remain in this classical framework, to name a few, Sebastião and Godinho (2020); Deng et al. (2020); Cui and Feng (2020); Oglend and Straume (2020). See also Chen et al. (2003) for a review of hedging performance evaluation.

For root mean squared error, we choose $\gamma(F) = \infty$ and $\eta(F) = 0$, i.e. we consider a full range, from $-\infty$ to ∞, of deviations to our target of zero PnL. For semivariance (SV), we choose $\gamma(F) = 0$ and $\eta(F) = \mathbf{E}(r|r \leq 0)$. The setting of semivariance represents our focus on the downside risk. Sometimes, SV is called lower partial moment.

Therefore, for each pair of copula-risk-measure, we calculate

$$\text{RMSE} = \mathsf{E}\{(r - 0)^2\}^{1/2},$$

and

$$\text{SV} = \mathsf{E}\left[\{r - \mathsf{E}(r|r \leq 0)\}^2 | r \leq 0\right].$$

The result is shown in figure 4. The RMSE of the copula-risk-measure pairs is ranging from 0.014 to 0.023. The smallest RMSE is generated by the pair Clayton-Variance, while a number of other pairs generate very similar results. In particular, RMSEs of variance are relatively low while comparing with other risk measures across different copulae. This is a natural result. The Frank copula's RMSEs are

Figure 4: Left panel: RMSE·1000 of different copula-risk-measure pairs; Right panel: $\mathrm{SV}^{0.5} \cdot 1000$ of different copula-risk-measure pairs. Frank copula is inferior to other copulae in terms of RMSE and SV. ES99% and VaR99% have slightly higher RMSE and SV.

high no matter the risk measures. This suggests Frank should not be used to model the dependency structure of BTC and BTC future.

The SV of the copula-risk-measure pairs is ranging from 0.011 to 0.020. The best performing pair in terms of SV is Gumbel-VaR95%. Gumbel copula also has the lowest or second lowest SV while comparing with other copulae across different risk measures. It is not surprising that Gumbel copula is superior than the other copula: the dependency of positive jumps in BTC and BTC future is captured by Gumbel copula, while in our dataset, positive jumps in BTC is frequent. Furthermore, VaR95% has the lowest or second lowest SV while comparing with other risk measure across different copulae. Similar to the result in RMSE, various the copula-risk-measure pairs' SV performance are similar, except for Frank copula, VaR99%, and VaR99%.

5 Conclusion and Discussion

In this paper, we model the dependency between Bitcoin (BTC) and its future via various copulae and search for the optimal hedge ratios h^* minimising different risk measures. We conclude that various copulae, except the Frank copula, are appropriate to model the dependency structure between Bitcoin and its future when we want to minimise risk. In addition, one should avoid using Value-at-Risk 99% or Expected Shortfall 99% as loss function while searching for the optimal h^*. Other risk measures are ready to be deployed according one's objective. The hedging effectiveness (HE) of various copula-risk-measure pairs are close to 65%, i.e. the risk of a portfolio of Bitcoin, measured by a particular risk measure, is reduced by 65% by including an optimal amount h^* of Bitcoin future. Again, the Frank copula is inferior to other copulae in terms of HE. We also compare the root mean squared error (RMSE) and semivariance (SV). Unsurprisingly, we can rule out the Frank copula, Value-at-Risk 99% and Expected Shortfall 99% for hedging Bitcoin and its future.

5.1 About Frank Copula

Frank-copula, in general, is not a good choice to model financial data. We can see from figure 5 that the Frank copula is not fitting the Bitcoin and its future visually, no matter which optimization procedure is being deployed. The samples of Frank diffuse like a strip with parallel edge when the parameter θ decrease (samples are being less dependent). This makes Frank copula not a a good fit to the data.

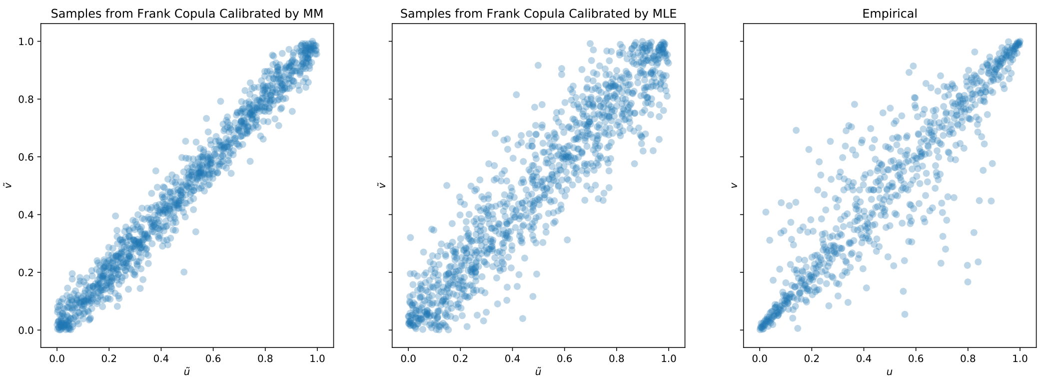

Figure 5: Comparison of Frank Copula Samples and Pseudo Observations of Bitcoin and CME Future Returns.

Aside from the Frank-copula, the HEs of various combination of copula and risk reduction objective are very similar. This is an expected result as the portfolio consists only two assets.

5.2 Robustness

The study of robustness concerns the stability of statistical estimation with respect to violation in assumptions. In our context, the robustness is with respect to outliers (or jumps). It is natural to do we want the optimal hedge ratio react to extreme market changes? In practice, outliers of returns can come from anywhere, for example, a tweet from Elon Musk, a sudden large order from institutional investor, or an incident of system failure in cryptocurrency exchanges. Rapid and drastic changes in portfolio weight causes problem of slippage and transaction cost. Investors should be aware of the cost brought by the sensitivity of the optimal hedge ratio procedure.

The discussion of sensitivity or robustness dates back to Huber and Ronchetti (1981)'s work on robust statistics. Hampel et al. (2011) suggest an infinitesimal approach to investigate sensitivity of statistical procedures. There are three central concepts in this approach, qualitative robustness, influence function, and break-down point. They are loosely related to the concept of continuity, first derivative of functional, and the distance of a functional to its nearest pole (singularity). While the first concept is a qualitative feature of a functional, the second the third concepts are practical tools to measure sensitivity quantitatively. We deploy a finite sample version of the second and third concepts. Details of robustness of risk measures can be found in Cont et al. (2010).

The influence function of $\hat{h}_\rho$ with finite sample size n is

$$\mathrm{IF}(\boldsymbol{z}; \hat{h}_\rho) = \frac{\hat{h}_\rho(\boldsymbol{X}_1, ..., \boldsymbol{X}_n, \boldsymbol{z}) - \hat{h}_\rho(\boldsymbol{X}_1, ..., \boldsymbol{X}_n)}{\frac{1}{n+1}}. \tag{50}$$

The equation describes the effect of a single contamination at point $\boldsymbol{z}$ on the estimate of OHR, standardised by the mass of the contamination.

Figure 6 shows the influence function of $\hat{h}_\rho$ of using t copula estimated by MLE with 300 data points of Bitcoin and CME future returns from 14/12/2018 to 25/02/2020. Contamination are in a set $\{-0.3, -0.27, ..., 0.3\} \times \{-0.3, -0.27, ..., 0.3\}$, in total 900 pairs of contamination. The product is Cartesian product of two sets.

We can see from the plots that Expected Shortfall with $\alpha = 99\%$ is very sensitive the negative return in spot (lower right plot). The h^* obtained this way increases with a single contamination of negative jump in spot price. VaR at 99% is also sensitive to negative jump in spot price but with a lower level (lower left plot). This is a natural result that reflects investor's strong preference on risk avoidance: investor increases her future's short position to compensate a large drop in spot price she saw in her data. The result of ES being more sensitive to VaR as risk measure agrees with the conclusion of Cont et al. (2010).

Other risk measures are relatively less sensitive. Interestingly, although ERM places heavy weights to negative returns, its IF is similar to that of variance, where variance does not exhibit risk preference.

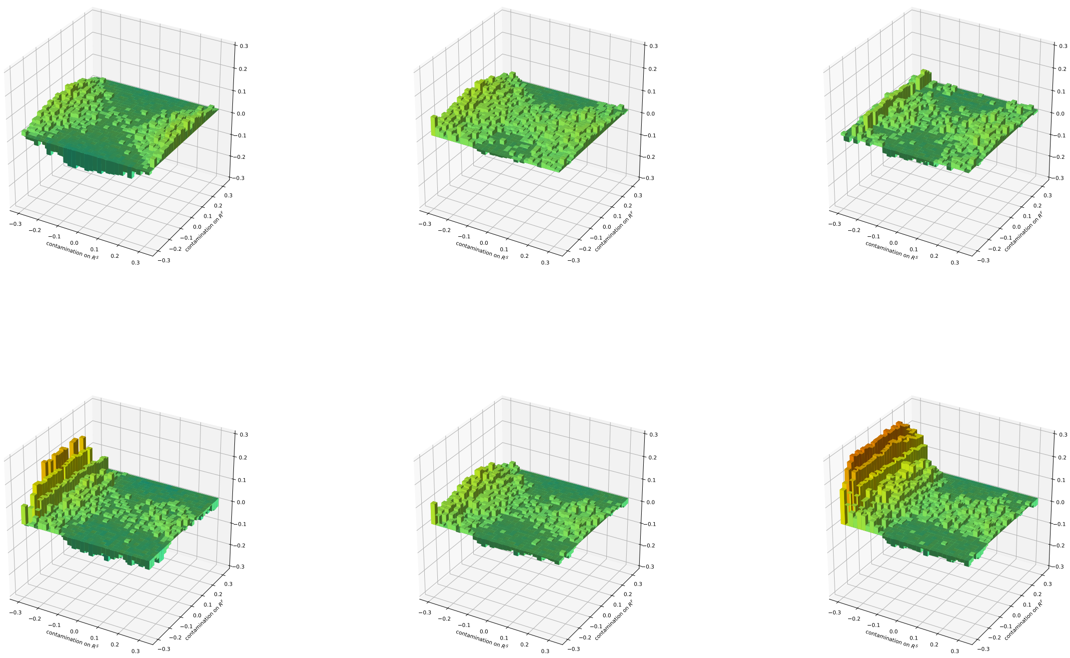

Figure 6: Influence functions (IF) of h^* using t copula copula estimated by MLE. From left to right, top to bottem, the plots are IF of using Var, ERM_{10}, $\mathrm{VaR}_{0.95}$, $\mathrm{VaR}_{0.99}$, $\mathrm{ES}_{0.95}$, and $\mathrm{ES}_{0.99}$ respectively.

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6 Appendix

6.1 Density of linear combination of random variables

Proposition 5 *Let $\boldsymbol{X} = (X_1, ..., X_d)^\top$ be real-valued random variables with corresponding copula density $\boldsymbol{c}_{X_1,...,X_d}$, and continuous marginals $F_{X_1}, ..., F_{X_d}$. Then, pdf of the linear combination of marginals $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ is*

$$f_Z(z) = \left|n_1^{-1}\right| \int_{[0,1]^{d-1}} \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1} \circ S(z) du_2...du_d \tag{51}$$

$$S(z) = \frac{1}{n_1} \cdot z - \frac{n_2}{n_1} \cdot F_{X_2}^{(-1)}(u_2) - ... - \frac{n_d}{n_1} \cdot F_{X_d}^{(-1)}(u_d) \tag{52}$$

Proof. Rewrite $Z = n_1 \cdot X_1 + ... + n_d \cdot X_d$ in matrix form

$$\begin{bmatrix} Z \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & \cdots & n_d \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix}. \tag{53}$$

By transformation variables

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \boldsymbol{f}_{X_1,...,X_d}\left(\boldsymbol{A}^{-1} \begin{bmatrix} z \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \right) \cdot |\det \boldsymbol{A}^{-1}| \tag{54}$$

$$= \left|n_1^{-1}\right| \boldsymbol{f}_{X_1,...,X_d}\{S(z), x_2, ..., x_d\} \tag{55}$$

Let $u_i = F_{X_i}(x_i)$ and use the relationship

$$\boldsymbol{c}_{X_1,...,X_d}(u_1, ..., u_d) = \frac{\boldsymbol{f}_{X_1,...,X_d}(x_1, ..., x_d)}{\prod_{i=1}^d f_{X_i}(x_i)}, \tag{56}$$

we have

$$\boldsymbol{f}_{Z,X_2,...,X_d}(z, x_2, ..., x_d) = \tag{57}$$

$$\left|n_1^{-1}\right| \cdot \boldsymbol{c}_{X_1,...,X_d}\{F_{X_1} \circ S(z), u_2, ..., u_d\} \cdot f_{X_1}\{S(z)\} \cdot \prod_{i=2}^{d} f_{X_i}(x_i) \tag{58}$$

The claim 51 is obtained by integrating out $x_2, ...x_d$ by substituting $dx_i = \frac{1}{f_{X_i}(x_i)} du_i$. ■

6.2 PnL and optimal hedge ratio of all copula-risk-measure pairs

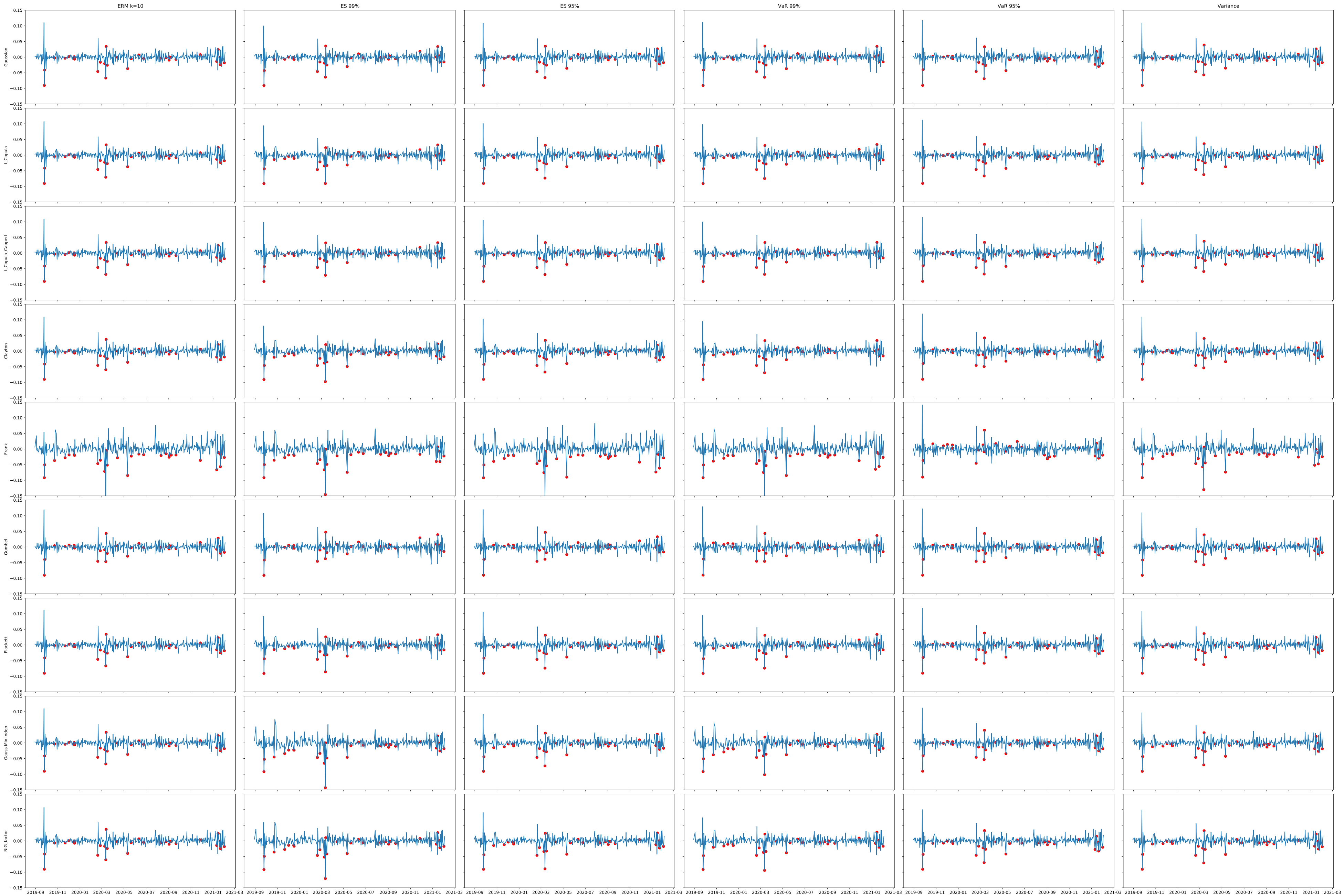

Figure 7: Out-of-Sample Returns of Hedged Portfolio of Copulas and Risk Reduction Objectives.

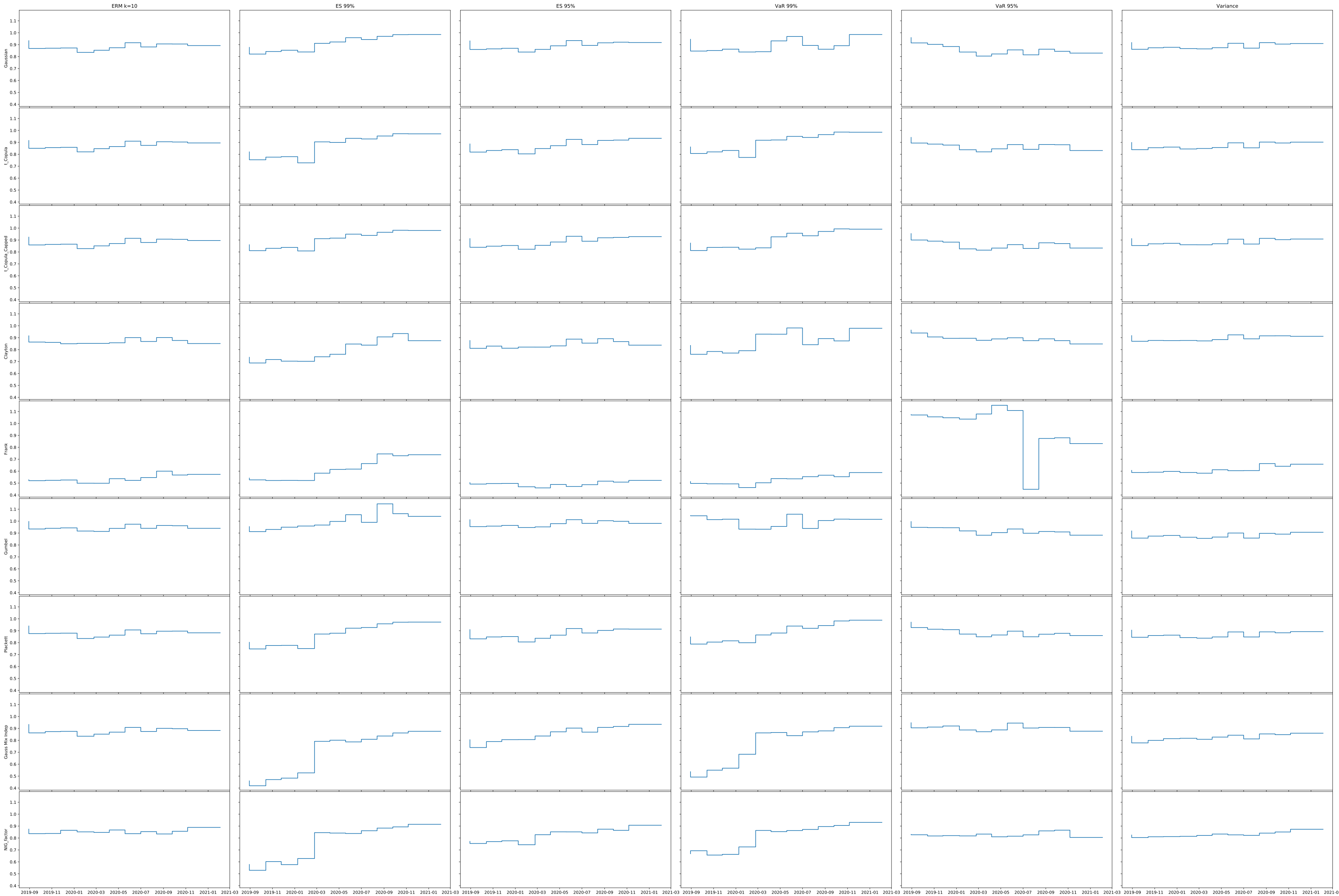

Figure 8: Optimal Hedge Ratio Obtained from Combinations of Copula and Risk Reduction Objective.