# A Modified Static Hedging Method for Continuous Barrier Options

**SAN-LIN CHUNG***
**PAI-TA SHIH**
**WEI-CHE TSAI**

This study modifies the static replication approach of Derman, E., Ergener, D., and Kani, I. (1995, DEK) to hedge continuous barrier options under the Black, F. and Scholes, M. (1973) model. In the DEK method, the value of the static replication portfolio, consisting of standard options with varying maturities, matches the *zero value* of the barrier option at *n* evenly spaced time points when the stock price equals the barrier. In contrast, our modified DEK method constructs a portfolio of standard options and binary options with varying maturities to match not only the *zero value* but also *zero theta* on the barrier. Our numerical results indicate that the modified DEK approach improves performance of static hedges

We acknowledge helpful comments by the Editor, Professor Bob Webb, and an anonymous referee. We also thank Keng-Yu Ho, Han-Hsing Lee, Yaw-Huei Wang, and the conference participants at the 2nd NTU Conference on Cross-Strait Banking and Finance for their helpful comments. San-Lin Chung and Pai-Ta Shih gratefully thank the National Science Council of Taiwan for the financial support.

*Correspondence author, Department of Finance, National Taiwan University, Room 707, No. 85, Section 4, Roosevelt Road, Taipei, 106, Taiwan, R.O.C. Tel: 886-2-33661084, Fax: 886-2-23660764, e-mail: chunga@management.ntu.edu.tw.

*Received September 2009; Accepted November 2009*

- *San-Lin Chung is a Professor of Finance at the Department of Finance, National Taiwan University, Taipei 106, Taiwan, R.O.C.*
- *Pai-Ta Shih is an Associate Professor of Finance at the Department of Finance, National Taiwan University, Taipei 106, Taiwan, R.O.C.*
- *Wei-Che Tsai is a Ph.D. Candidate at the Department of Finance, National Taiwan University, Taipei 106, Taiwan, R.O.C.*

The Journal of Futures Markets, Vol. 30, No. 12, 1150–1166 (2010)

View this article online at wileyonlinelibrary.com
DOI: 10.1002/fut.20451

significantly for an up-and-out call option under the BS model even if the bid–ask spreads are considered.  Jrl Fut Mark 30:1150–1166, 2010

## INTRODUCTION

Hedging exotic options is an important task for the issuing banks to manage their risk. The hedging methods are usually classified into two categories: the dynamic hedging and the static hedging approaches. The dynamic hedging method continuously adjusts the weights in a portfolio consisting of risky stock and risk less zero-coupon bonds as time passes or the stock price moves. In contrast, the static hedging method is to form a portfolio with *fixed weights* of standard European options, with varying strikes and maturities, which requires no further adjustment in the future (unless the boundary conditions are met).

In comparison to dynamic hedging, static hedging has at least three advantages discussed in the literature. First, static hedging may be considerably cheaper than dynamic hedging when the transaction cost is large. For example, for options with large gamma (such as barrier options), the dynamic hedge portfolio has to be adjusted often and this will increase the transaction cost.[1] Second, static hedging method has vegas and gammas close to those of the target option being hedged, but dynamic hedging method has zero vega and gamma. Therefore, dynamic hedging method has greater vega and gamma risk than static hedging method under a real world with frequent changes in volatility or price. Finally, several articles have documented that static hedging is less sensitive to the model risk such as volatility misspecification, see for example Thomsen (1998) and Tompkins (2002).

In the literature, the static hedging portfolio is formulated in two different ways. The first approach, proposed by Bowie and Carr (1994), Carr and Chou (1997), and Carr, Ellis, and Gupta (1998), is to construct static positions in a continuum of standard European options of all strikes, with the expiration date $T$ matching that of the exotic option being hedged (e.g. a barrier option). The second approach, developed by Derman, Ergener, and Kani (1995), uses a standard European option to match the boundary at maturity of the exotic option and a continuum of standard European options with different maturities but with the strike price equaling the boundary value before maturity (e.g. the barrier level of a knock-out option) to match boundary before maturity of the exotic option. Chou and Georgiev (1998) show how a static replication

[1]Although the bid–ask spreads involved in option trading are typically an order of magnitude larger than for trading in the underlying, Siven and Poulsen (2008) demonstrate that with realistic frictions, static hedging with options is still the best way to remove the very large hedge errors. Taleb (1996) also discusses in details so-called liquidity holes and other problems associated with dynamic hedging of barrier options. Moreover, the cost of monitoring changes in hedge ratios in addition to the direct transaction fees for rebalancing the portfolio makes dynamic hedging less favorable. We thank the referee for clarifying the discussion here.

portfolio of options with a single maturity and multiple strikes can be converted into a static replication portfolio of options with a single strike and multiple maturities under the Black and Scholes (1973) (BS) model.

Since both the standard options and the barrier option written on the same stock all satisfy the same partial differential equation (PDE), DEK construct a static hedge portfolio of standard European options with different maturities to match the terminal condition and boundary conditions of the barrier option, i.e. the DEK portfolio values are zero on the barrier. When this is done, the values of the barrier option and the static hedge portfolio are equivalent everywhere inside the boundary because their values are uniquely determined by the same terminal condition and boundary conditions.

Although the DEK method is theoretically appealing, the hedging performance may be not as good as one would expect when it is implemented discretely. The main reason for the replication errors is because the value of the DEK portfolio is not zero on the barrier except at some discrete time points. Although one can increase the number of time points, where the DEK portfolio values are zero on the barrier, to enhance the accuracy of static hedging, however, it would require using a lot of standard options with different maturities to form the portfolio.

This study proposes a modified method to form a static hedge portfolio with a smaller replication error than that of the DEK portfolio. We find that the DEK portfolio may have large theta values on the barrier and thus results in large hedging errors. To overcome this problem, the modified DEK method constructs a static portfolio, which has not only *zero value* but also *zero theta* on the barrier at discrete time points. To satisfy these two constraints on the barrier, we add cash-or-nothing binary call options as well as standard options into the static hedge portfolio.[2] In the past, binary options were only available in the over-the-counter market. After regulatory authorization, the Chicago Board Options Exchange (CBOE) offered binary options on the Standard & Poor's 500 Index and the CBOE Volatility Index in 2008, and the American Stock Exchange (AMEX) also listed binary option for several liquid U.S. stocks.[3] Therefore, using binary options as component options of a static hedge portfolio becomes possible nowadays.

When the bid–ask spread is not considered, numerical results show that the portfolio values and thetas on the barrier are very close to zero at all times before maturity (not just at some discrete time points) under the modified DEK method. However, since the bid–ask spread of a binary option is larger than

[2]A cash-or-nothing binary call option is a type of option in which the payoff is either a fixed amount of compensation if the option expires in the money or nothing at all if the option expires out of the money.
[3]The AMEX offers binary options (fixed return options) on ETFs and highly liquid equities, such as Apple, Cisco Systems, Citigroup, DIAMONDS, General Electric, PowerShares QQQ, Wachovia Corp, etc.

that of a standard option,[4] the modified DEK method is not necessarily superior to the DEK method. To further investigate this issue, we incorporate bid–ask spreads and simulate 50,000 stock price paths to compute the profit/loss distributions at the first hitting time or at expiry for the DEK and the modified DEK portfolios. The numerical results (see Table III of this study) indicate that, even when the bid–ask spreads are included, the modified DEK method still outperforms the DEK method under four risk measures suggested by Siven and Poulsen (2009).

The rest of this study is organized as follows. The following section reviews the static hedging approach developed by DEK (1995). The later section shows how the static replication portfolio is formulated under the modified DEK method. The penultimate section presents the numerical results of static replications for the DEK and the modified DEK methods under the BS model. The final section concludes the study.

## OVERVIEW OF DERMAN ET AL. (1995)

In this section, we review the static hedging approach proposed by Derman et al. (1995). The idea behind the DEK method is as follows. Since both the standard options and the barrier option written on the same stock all satisfy the same PDE, it is feasible to formulate a static hedge portfolio of standard European options with different maturities to match the terminal condition and boundary conditions of the barrier option. When this is done, the values of the barrier option and the static hedge portfolio are equivalent everywhere inside the boundary because their values are uniquely determined by the same terminal condition and boundary conditions.

As a demonstration, assume that we want to replicate a "reverse" up-and-out call (UOC) option with strike price $X$, expiration date $T$, and barrier value $B$.[5] If the barrier is not reached before $T$, the terminal condition of the UOC option is exactly the same as the corresponding European call option. Therefore, the static hedge portfolio starts with one unit of the corresponding European call option. For simplicity, we assume that the present time is time zero and suppose that the static hedge portfolio matches the boundary conditions of the UOC option before maturity at $n$ evenly spaced time points, i.e. $t_0 = 0, t_1, \ldots, t_{n-1} = T - \Delta t$, where $\Delta t = T/n$. To match the payoff on the barrier $B$ at time $t_i (i = 0, 1, \ldots, n-1)$, DEK add $W_i$ units of a standard

[4]The bid–ask spread of a binary option is about 2.4 times that of a standard option, see later section for details. We thank the referee for pointing out this issue.

[5]A knock-out call option is a call option that becomes worthless when a pre-specified barrier level is reached. If the barrier level is in-the-money (e.g. $B > X$ for call options), the knock-out option is called a "reverse" or "live-out" option, see Wystup (2007) and Gatheral (2006). We thank the referee for bringing our attention to this point.

European option, maturing at time $t_{i+1}$ and with a strike price equaling $B$, into the static hedge portfolio. DEK then solve $W_i$ using the *value-matching* condition, i.e. the value of the static hedge portfolio is zero on the barrier.

Similar to the binomial models, DEK work backward to determine the number of standard European options for the above $n$-point static hedge portfolio. At time $t_{n-1}$ when stock price equals the barrier price $B$, *value-matching* condition imply that

$$C(B, X, \sigma, r, q, T - t_{n-1}) + W_{n-1}C(B, B, \sigma, r, q, T - t_{n-1}) = 0 \qquad (1)$$

where $C(S, X, \sigma, r, q, \tau)$ is the European call price with initial stock price $S$, strike price $X$, volatility $\sigma$, risk free rate $r$, dividend yield $q$, and time to maturity $\tau$. Thus, $W_{n-1}$ can be easily solved from Equation (1). Using a similar procedure, DEK work backward to determine the number of units of the standard European option, $W_i$, at time $t_i$, $i = n - 2, n - 3, \ldots, 0$. After solving all $W_i$s ($i = 0, 1, \ldots, n - 1$), the value of the $n$-point static hedge portfolio $UOC_n$ at time 0 is obtained as follows:

$$\begin{aligned} UOC_n = {} & C(S_0, X, \sigma, r, q, T) + W_{n-1}C(S_0, B, \sigma, r, q, T) \\ & + W_{n-2}C(S_0, B, \sigma, r, q, t_{n-1}) + \cdots + W_0 C(S_0, B, \sigma, r, q, t_1). \qquad (2) \end{aligned}$$

Note that the values $UOC_1$, $UOC_2$, $UOC_3$, . . ., define a sequence with the limit equaling the UOC value. However, when $n$ is not large, the value of the static hedge portfolio may be deviated from the UOC price significantly. The main reason for the replication errors is because the value of the DEK portfolio is not zero on the barrier except at these $n$ time points. For example, Figure 5 of DEK (1995) indicates that the static replication portfolio value on the barrier may be far from zero especially when time to expiration is small. In other words, the DEK portfolio value on the barrier may be sensitive to the change of time, i.e. the theta values of the DEK portfolio may be significantly different from zero when the stock price equals the barrier price.

To further investigate the above explanation for the replication errors, we derive the asymptotic theta value of the DEK static replication portfolio on the barrier at time $t_{n-1}^{+}$. Since the theta value near expiration of an in-the-money option is small, the theta values of the static portfolio on the barrier at $t_{n-1}^{+}$ can be approximated as the following:

$$\begin{aligned} \left.\frac{\partial UOC_n}{\partial t}\right|_{\substack{S=B \\ t=t_{n-1}^{+}}} &= \left.\frac{\partial C(S, X, \sigma, r, q, T-t)}{\partial t}\right|_{\substack{S=B \\ t=t_{n-1}^{+}}} + W_{n-1}\left.\frac{\partial C(S, B, \sigma, r, q, T-t)}{\partial t}\right|_{\substack{S=B \\ t=t_{n-1}^{+}}} \\ &\approx W_{n-1}\left.\frac{\partial C(S, B, \sigma, r, q, T-t)}{\partial t}\right|_{\substack{S=B \\ t=t_{n-1}^{+}}}. \qquad (3) \end{aligned}$$

Using Equation (1), it is straightforward to show that

$$W_{n-1} = -\frac{C(S, X, \sigma, r, q, T-t)}{C(S, B, \sigma, r, q, T-t)}\bigg|_{\substack{S=B \\ t=t_{n-1}}} \approx -\frac{\sqrt{2\pi}(B-X)}{B\sigma\sqrt{\Delta t}} \quad \text{as} \quad \Delta t \to 0 \tag{4}$$

and

$$\begin{aligned}\frac{\partial C(S, B, \sigma, r, q, T-t)}{\partial t}\bigg|_{\substack{S=B \\ t=t_{n-1}^{+}}} &= -\frac{Bn(d_1)\sigma e^{-q\Delta t}}{2\sqrt{\Delta t}} \\ &\quad - qBN(-d_1)e^{-q\Delta t} + rXe^{-r\Delta t}N(-d_2) \\ &\approx -\frac{B\sigma}{2\sqrt{\Delta t}}\frac{1}{\sqrt{2\pi}} \quad \text{as} \quad \Delta t \to 0 \end{aligned} \tag{5}$$

where $d_1 = (r - q + 0.5\sigma^2)\sqrt{\Delta t}/\sigma$,[6] $d_2 = d_1 - \sigma\sqrt{\Delta t}$, and $n(\cdot)$ and $N(\cdot)$ are the probability density function and cumulative distribution function of the standard normal distribution, respectively. Therefore,

$$\frac{\partial UOC_n}{\partial t}\bigg|_{\substack{S=B \\ t=t_{n-1}^{+}}} \approx \frac{\sqrt{2\pi}(B-X)}{B\sigma\sqrt{\Delta t}}\left[\frac{B\sigma}{2\sqrt{\Delta t}}\frac{1}{\sqrt{2\pi}}\right] = \frac{B-X}{2\Delta t} \quad \text{as} \quad \Delta t \to 0. \tag{6}$$

Equation (6) indicates that the theta value of the DEK static hedging portfolio on the barrier is positive and of order $O(1/\Delta t)$ at $t_{n-1}^{+}$. In other words, the DEK static hedging portfolio would have serious mismatches of theta values on the barrier, especially when the option is near expiration. The mismatch of theta values is also the main reason why the static portfolio values are greater than zero on the barrier except at the $n$ time points.[7]

Figure 1 shows the DEK portfolio values and theta values on the knock-out barrier.[8] The parameters are adopted from Table VI of DEK (1995). It is clear that the DEK method includes standard options to obtain zero portfolio value on the barrier at 6 equally spaced time points. Except at the specified time points, the portfolio values on the barrier are greater than zero. This can be explained by the theta values along the barrier. At $t_i^{+}$, the theta of the static portfolio is greater than zero and as the time increases, the theta becomes negative to

[6]When the stock price equals $B$, the European option with a strike price equaling $B$ becomes an at-the-money option and hence $d_1 = (r - q + 0.5\sigma^2)\sqrt{\Delta t}/\sigma$.

[7]Following Fink (2003), one may construct a static hedge portfolio consisting of European call options with a strike price ($B_H$) greater than the barrier price $B$ to replicate the UOC. However, it can be shown that the theta value of such a portfolio on the barrier is of order $O(1/\Delta t^2)$ at $t_{n-1}^{+}$ and thus results in larger replication mismatches than the DEK portfolio. The detailed results can be obtained from the authors on request.

[8]Since the component options of the static portfolio at $t_i^{-}$ and $t_i^{+}$ are different, the theta values are discontinuous at $t_i$.

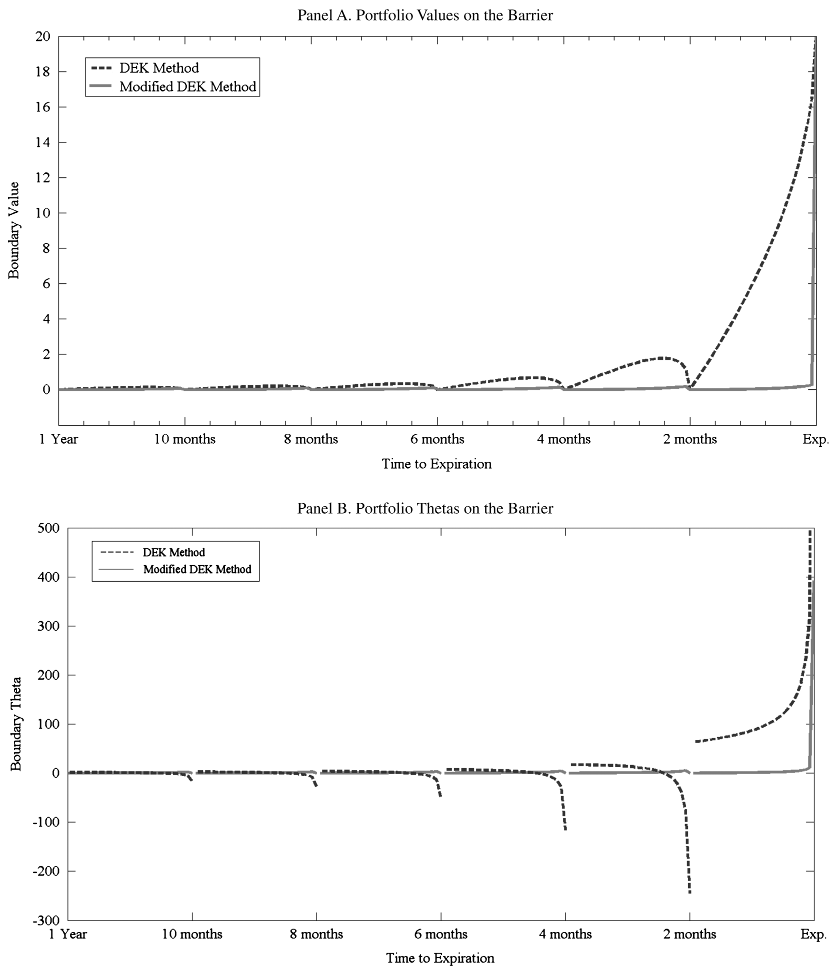


**FIGURE 1**

The Values and Thetas on the Barrier for Two Static Replication Portfolios Panel A. Portfolio Values on the Barrier Panel B. Portfolio Thetas on the Barrier. This figure shows values and thetas on the barrier at all times prior to expiration for the DEK and the modified DEK portfolios. The target option to be hedged is an UOC option with the following parameters: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, and $q = 3\%$. The values of the DEK portfolios match the zero value of the UOC on the barrier every two months (i.e. $n = 6$). The values of the modified DEK portfolios match the zero value and zero theta of the UOC on the barrier every two months.

match the zero value condition on the barrier at time $t_{i+1}$. Overall, the portfolio value on the barrier between $t_i$ and $t_{i+1}$. is positive. Hence, the DEK static portfolio value over-hedges the UOC price.

## IMPLEMENTING THE MODIFIED DEK METHOD

Given the fact that the DEK portfolio may have large theta values on the barrier, this study proposes a modified static hedge portfolio to overcome the problem.

The idea of our modified DEK method is to construct a static portfolio which has not only *zero value* but also *zero theta* on the barrier at $n$ evenly spaced time points. To satisfy these two constraints on the barrier, we add cash-or-nothing binary call options as well as standard options into the static hedge portfolio.[9] Without loss of generality, we define the payoff of a binary option as one dollar if the option is in the money at the expiration date.

The detailed procedures for forming a static hedge portfolio under the modified DEK method are as follows. Suppose that the static hedge portfolio matches the *zero value* and *zero theta* on the barrier for the UOC before maturity at $n$ evenly spaced time points, i.e. $t_0 = 0, t_1, \ldots, t_{n-1} = T - \Delta t$, where $\Delta t = T/n$. To do so, we add $W_i$ units of a standard European option and $\hat{W}_i$ units of a binary option, both maturing at time $t_{i+1}$ and with a strike price equaling $B$, into the static hedge portfolio. We then solve $W_i$ and $\hat{W}_i$ using *value-matching* and *theta-matching* conditions.

As before, we work backward to determine $W_i$ and $\hat{W}_i$ of the above $n$-point static hedge portfolio. At time $t_{n-1}$ when the stock price equals the barrier price $B$, the *value-matching* condition and the *theta-matching* condition imply that

$$C(B, X, \sigma, r, q, T - t_{n-1}) + W_{n-1}C(B, B, \sigma, r, q, T - t_{n-1}) + \hat{W}_{n-1}\,\mathrm{Bin}(B, B, \sigma, r, q, T - t_{n-1}) = 0 \tag{7}$$

$$\frac{\partial C(B, X, \sigma, r, q, T - t_{n-1})}{\partial t} + W_{n-1}\frac{\partial C(B, B, \sigma, r, q, T - t_{n-1})}{\partial t} + \hat{W}_{n-1}\frac{\partial \mathrm{Bin}(B, B, \sigma, r, q, T - t_{n-1})}{\partial t} = 0 \tag{8}$$

where $\mathrm{Bin}(S, X, \sigma, r, q, \tau)$is the European cash-or-nothing binary call price with initial stock price $S$, strike price $X$, volatility $\sigma$, risk-free rate $r$, dividend yield $q$, and time to maturity $\tau$. Thus, $W_{n-1}$ and $\hat{W}_{n-1}$ can be easily solved from Equations (7) and (8). Using a similar procedure, we work backward to determine $W_i$ and $\hat{W}_i$ at time $t_i$, $i = n - 2, n - 3, \ldots, 0$. After solving all $W_i$s and $\hat{W}_i$s ($i = 0, 1, \ldots, n - 1$), the value of the modified $n$-point static hedge portfolio $UOC'_n$ at time 0 is obtained as follows:

[9]One can also choose standard options with strike prices $B$ and $B_H$ ($B_H > B$) to match zero value and zero theta on the barrier. Our numerical experiments show that a static portfolio consisting of standard options performs worse than a static portfolio consisting of standard options and binary options. However, it still reduces significant replication errors of the DEK method. The detailed results are available from the authors on request.

$$UOC'_n = C(S_0, X, \sigma, r, q, T) + W_{n-1}C(S_0, B, \sigma, r, q, T) + \hat{W}_{n-1}Bin(S_0, B, \sigma, r, q, T)$$
$$+ W_{n-2}C(S_0, B, \sigma, r, q, t_{n-1}) + \hat{W}_{n-2}Bin(S_0, B, \sigma, r, q, t_{n-1})$$
$$+ \cdots + W_0C(S_0, B, \sigma, r, q, t_1) + \hat{W}_0Bin(S_0, B, \sigma, r, q, t_1). \quad (9)$$

## NUMERICAL ANALYSIS AND DISCUSSIONS

This section applies the modified DEK method to form a static portfolio for an UOC and compares the hedging performance of the DEK method with that of the modified DEK method.[10] For the purpose of a concise study, the following numerical results represent the hedging performance under the BS model.[11]

We first compare the static hedge portfolio of the DEK method with that of the modified DEK method under the BS model. The parameters are taken from Table VI of DEK (1995) as follows: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, and $q = 3\%$. The closed-form solution of the UOC price is obtained from Reiner and Rubinstein (1991). Table I shows these two static replication portfolios with $n = 6$. There are several points worth noting from Table I. First, the quantity of standard call option changes dramatically when binary options are included in the static hedge portfolio. Second, the replication error of the modified DEK method is only \$0.01, which is only about 0.54% of the UOC price. In contrast, the dollar (percentage) error of the DEK method is \$0.37 (19.45%), which is 37 times of the error of the modified DEK method. Finally, the DEK static hedge portfolio value is negative $(6.76 - 7.28 = -0.52)$ if we only match zero value on the barrier at time $t_{n-1} = T - \Delta t$. However, the modified DEK method still generates positive portfolio value $(6.76 - 4.76 = 2.00)$ even if we only match *zero value* and *zero theta* on the barrier at time $t_{n-1} = T - \Delta t$. These observations suggest the important role of *theta-matching* condition in forming the static hedge portfolio.

The portfolio values and thetas on the barrier are plotted in Figure 1 for the DEK method and the modified DEK method. Panel A of Figure 1 indicates that the portfolio values on the barrier using the modified DEK method are much closer to zero than those using the DEK method. Moreover, Panel B of Figure 1 shows that the theta values on the barrier using the modified DEK method are

[10]Nalholm and Poulsen (2006b) showed that an UOC option is harder to be hedged than a down-and-out call option. Hence, we choose the UOC option as our target option.

[11]We have also implemented numerical experiments under the constant elasticity of variance (CEV) model of Cox (1975). Similar to the case of the Black–Scholes model, numerical results indicate that the modified DEK portfolio can improve the hedging performance of the DEK portfolio significantly. Detailed results under the CEV model are available from the authors on request.

**TABLE I**
Two Static Replication Portfolios

| *Quantity of European Call* | *Strike* | *Expiration (months)* | *Value for $S_0 = 100$* |
|---|---|---|---|
| *Panel A. DEK Method* | | | |
| 0.165720 | 120 | 2 | 0.000553 |
| 0.255330 | 120 | 4 | 0.018793 |
| 0.441691 | 120 | 6 | 0.110170 |
| 0.923678 | 120 | 8 | 0.461913 |
| 2.794490 | 120 | 10 | 2.225826 |
| −6.496245 | 120 | 12 | −7.276219 |
| 1.000000 | 100 | 12 | 6.756088 |
| Net | | | 2.297124 |

| *Quantity of European Call* | *Quantity of Binary Call* | *Strike* | *Expiration (months)* | *Value for $S_0 = 100$* |
|---|---|---|---|---|
| *Panel B. ModifiedDEK Method* | | | | |
| −0.044761 | 0.195096 | 120 | 2 | 0.000155 |
| −0.055474 | 0.241373 | 120 | 4 | 0.000463 |
| −0.070235 | 0.305136 | 120 | 6 | −0.003620 |
| −0.089810 | 0.390398 | 120 | 8 | −0.016672 |
| −0.109261 | 0.479128 | 120 | 10 | −0.040802 |
| −0.135875 | −39.207506 | 120 | 12 | −4.762147 |
| 1.000000 | | 100 | 12 | 6.756088 |
| Net | | | | 1.933466 |

*Note.* This table shows two static replication portfolios for hedging an UOC option under the Black − Scholes model with the following parameters: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, and $q = 3\%$. The values of the DEK replication portfolios match the zero value of the UOC on the barrier every two months. The values of the modified DEK portfolios match the zero value and zero theta of the UOC on the barrier every two months. The benchmark value of the target UOC option is 1.923008 and is computed from the closed-form solution of Reiner and Rubinstein (1991).

also closer to zero than those using the DEK method. These evidences explain why the modified DEK method can improve the hedging performance of the DEK method significantly.

Figure 2 shows the replication mismatches of the DEK portfolio and the modified DEK portfolio over a range of stock prices and times to expiration when $n = 6$. The result indicates that the mismatches of the modified DEK method are far smaller than those of the DEK method. Therefore, the modified DEK portfolio can improve the hedging performance of the DEK portfolio significantly over a range of stock prices and times to expiration.

In order to have a comprehensive comparison of the DEK portfolio and the modified DEK portfolio, we report their values, deltas, and gammas in Table II for $n = 4,6,12,52$. There are three points worth discussing from Table II. First, Table II indicates that the hedging errors of our modified DEK method are far smaller than those of the DEK method. For example, the percentage hedging error of the modified DEK method with $n = 4$ is even lower than that

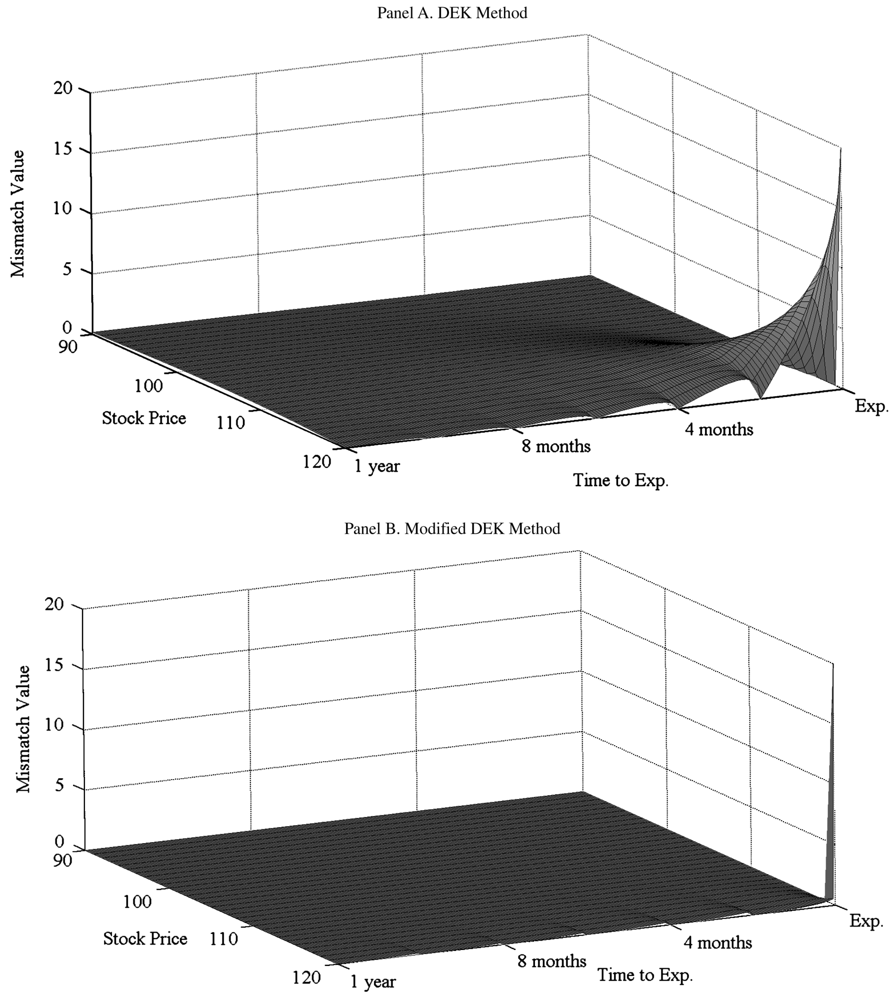


**FIGURE 2**

The Replication Mismatches of Two Static Replication Portfolios Panel A. DEK Method Panel B. Modified DEK Method. This figure presents the mismatch in dollar terms between the benchmark values of the 6-point static hedge portfolio and the target option, over a range of stock prices from 90 to 120 as time passes. The target option to be hedged is an UOC option with the following parameters: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, and $q = 3\%$.

of the DEK method with $n = 52$. In other words, the modified DEK portfolio consisting of nine options (five standard options and four binary options) outperforms the DEK portfolio consisting of 53 standard options for replicating the UOC option. In addition, hedge ratios (such as deltas and gammas) of the modified DEK method are also more accurate than those of the DEK method. Second, the modified DEK method improves the errors of the DEK method gradually when $n$ increases. Define the improvement ratio as the absolute error of the DEK portfolio over that of the modified DEK portfolio. It is clear from

**TABLE II**
Values, Deltas, and Gammas of Two Static Replication Portfolios

| | *DEK Method* | | | *Modified DEK Method* | | | *Improvement Ratio* | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | *Value* | *Delta* | *Gamma* | *Value* | *Delta* | *Gamma* | *Value* | *Delta* | *Gamma* |
| 4 | 2.472396 | 0.047762 | −0.016105 | 1.942729 | 0.024803 | −0.013179 | 27.86 | 15.43 | 107.37 |
| 6 | 2.297124 | 0.038060 | −0.015402 | 1.933466 | 0.024069 | −0.013186 | 35.77 | 17.33 | 109.80 |
| 12 | 2.113646 | 0.029629 | −0.014424 | 1.926626 | 0.023517 | −0.013196 | 52.69 | 21.04 | 121.80 |
| 52 | 1.967738 | 0.024427 | −0.013511 | 1.923399 | 0.023245 | −0.013204 | 114.40 | 36.82 | 152.50 |

*Note*. This table shows values, deltas, and gammas of two static replication portfolios for an UOC option under the Black–Scholes model with the following parameters: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, and $q = 3\%$. The benchmark values of the UOC price, delta, and gamma are 1.923008, 0.023212, and − 0.013206, respectively. The benchmark values are computed from the closed-form solution of Reiner and Rubinstein (1991). The improvement ratio is defined as the absolute error of the DEK portfolio over that of the modified DEK portfolio.

Table II that the improvement ratio is increasing with the increase of $n$. In other words, the modified DEK portfolio not only has a smaller replication error but also converges more quickly to the target option value than the DEK portfolio. Third, hedging performance of the modified DEK portfolio is good for practical implementation. From Table II, we observe that even if we just match the boundary conditions at six discrete time points, the value of our static hedge portfolio is very close to that of the UOC option with an absolute relative error of 0.5%. The result indicates that our static hedge portfolio is a feasible and practical hedging method for the UOC options under the BS model even if the standard European options and binary options only have a few maturities traded in the option exchanges.[12]

In Figure 3, we show the convergence patterns of two static hedging portfolios as $n$ increases. It is evident from Figure 3 that the portfolio value using the modified DEK method converges to the theoretical price of the UOC option more accurately and faster than that using the DEK method.

Since the bid–ask spreads of binary options are larger than those of standard options, it is important to study the impact of this market friction on the static hedging performance. We follow Siven and Poulsen (2008, 2009) to investigate static hedging performance of barrier options in the presence of bid–ask spreads. Specifically, we simulate 50,000 paths of the stock prices, with time discretized to 25,200 time steps per year, to compute the profit and loss distribution, i.e. the difference between the continuous barrier option value and the liquidation value of the static hedge portfolio under each path. As usual, the DEK portfolio and the modified DEK portfolio are still determined by Equations (2), (7), and (8), respectively, and the static hedge portfolios are

[12]One major criticism of the static hedging approach is that it usually needs standard options with many different maturities to achieve an accurate replication. For example, DEK (1995) utilize standard options with 24 maturities to form a static hedge portfolio with a 5% replication error.

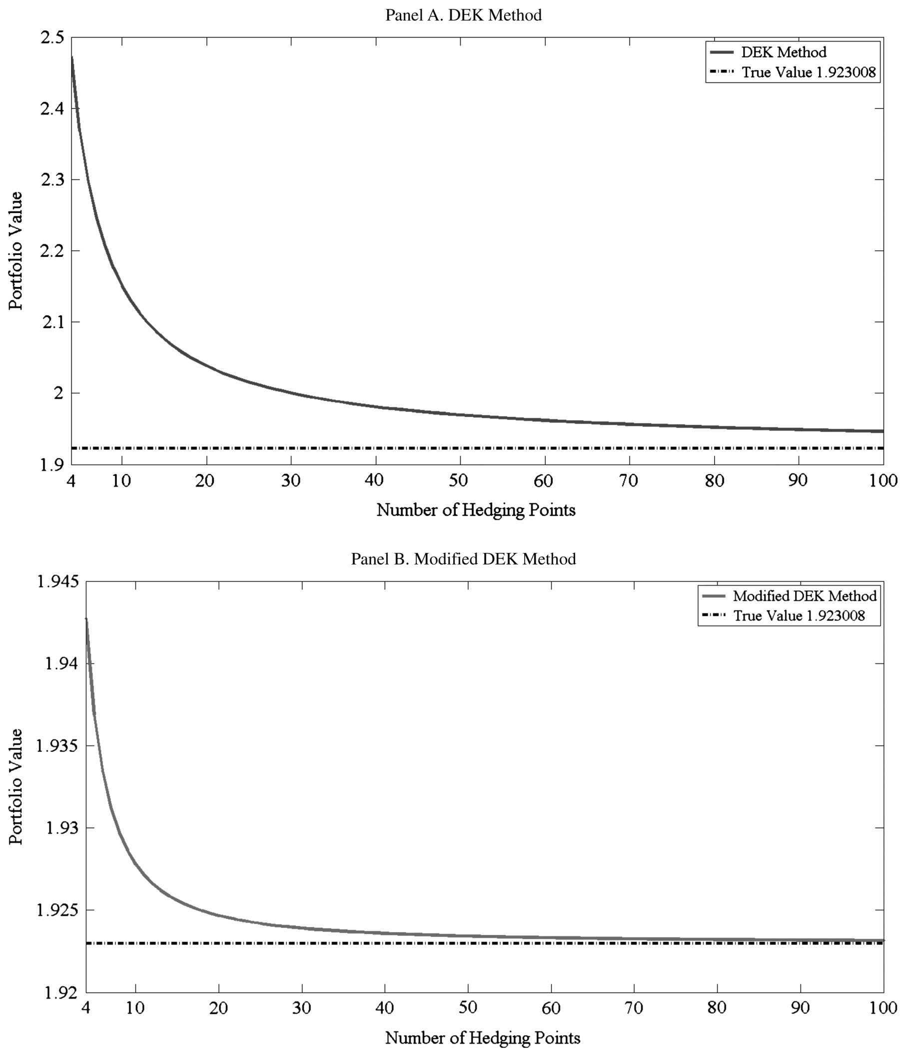


**FIGURE 3**

The Convergence Patterns of Two Static Replication Portfolios Panel A. DEK Method Panel B. Modified DEK Method. This figure shows the convergence patterns of two replicating portfolios with the increase of the number of matching time points ($n$). We use the following parameters in the analysis: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, and $q = 3\%$. The benchmark value of the target UOC option is 1.923008 and is computed from the closed-form solution of Reiner and Rubinstein (1991).

liquidated when the barrier price is breached. Without market frictions, the component options of the static hedge portfolios are sold and bought at the BS model prices. When the bid–ask spreads are considered, the component options are sold (bought) at bid (ask) prices.

Following Chordia, Roll, and Subrahmanyam (2000), we define the proportional quoted spread as $(P_A - P_B)/P_M$, where $P_A$ is the ask price, $P_B$ is the bid

**TABLE III**
Hedging Performance of Two Static Replication Portfolios

| | *DEK Method* | | *Modified DEK Method* | |
|---|---|---|---|---|
| *Risk Measures* | *W/o Bid–Ask Spreads* | *With Bid–Ask Spreads* | *W/o Bid–Ask Spreads* | *With Bid–Ask Spreads* |
| $E[HE(S_\tau)^2]$ | 2.6697 | 2.8055 | 0.0086 | 0.9990 |
| $E[HE(S_\tau)^+]$ | 0.0403 | 1.7591 | 0.0735 | 2.0411 |
| $VAR_{0.05}$ | 0.0 | 2.2716 | 0.0387 | 2.1011 |
| $ES_{0.05}$ | 0.0048 | 2.5110 | 0.1648 | 2.1867 |

*Note.* This table compares the hedging performance of two static replication portfolios, using the DEK and the modified DEK methods, for an UOC option under the Black–Scholes model with the following parameters: $S_0 = 100$, $X = 100$, $B = 120$, $T = 1$ year, $r = 5\%$, $\sigma = 15\%$, $q = 3\%$, and $n = 6$. We simulate 50,000 stock price paths, with time discretized to 25,200 time steps per year, to compute the profit and loss distributions of two static hedge portfolios. The proportional quoted spreads of standard and binary options used in the calculations are 6 and 14.2%, respectively. We adopt four risk measures used by Siven and Poulsen (2009) to evaluate the hedging performance of the two static hedge portfolios. The first risk measure $E[HE(S_\tau)^2]$ represents the quadratic hedging errors, where $HE(S_\tau) = UOC(S_\tau) - UOC_n(S_\tau)$ is the hedge error, $\iota$ is the simulated first hitting time, $S_\tau$ is the underlying asset price at time $\tau$, $UOC(S_\tau)$ and $UOC_n(S_\tau)$ are the continuous barrier option value and the static hedge portfolio value, respectively, at time $\tau$. The second risk measure is the expected loss: $E[HE(S_\tau)^+ = \max(0, HE(S_\tau))]$. The third risk measure, 5% value-at-risk, is defined by $VAR_{0.05} = \inf\{z \in \text{R Pr } (\text{HE}(S_\tau) \geq z) \leq 0.05\}$. The fourth measure, the expected shortfall, is the mean loss beyond value-at-risk: $ES_{0.05} = E[HE(S_\tau)|HE(S_\tau) \geq VAR_{0.05}]$.

price, and $P_M$ is the bid–ask midpoint. According to the option price data downloaded from the CBOE's website (http://www.cboe.com/), the proportional quoted spreads of standard and binary options are 6 and 14.2%, respectively. Thus, we assume that, upon liquidation, the standard (binary) options of the static hedge portfolios are sold at the BS model prices times 0.97 (0.929) and bought at the BS model prices times 1.03 (1.071), respectively.

To compare the hedging performance of the DEK and the modified DEK methods, we adopt four risk measures suggested by Siven and Poulsen (2009) to evaluate the profit and loss distributions. The first risk measure, $E[HE(S_\tau)^2]$, represents the quadratic hedging error, where $HE(S_\tau) = UOC(S_\tau) - UOC_n(S_\tau)$ is the hedge error, $\tau$ is the simulated first hitting time, $S_\tau$ is the underlying asset price at time $\tau$, $UOC(S_\tau)$ and $UOC_n(S_\tau)$ are the continuous barrier option value and the static hedge portfolio value, respectively, at time $\tau$. The second risk measure, the expected loss $E[HE(S_\tau)^+ = \max(0, HE(S_\tau))]$, only concerns the losses of the static hedge portfolios and thus is a one-sided risk measure. The third risk measure, 5% value-at-risk, is defined by $VAR_{0.05} = \inf\{z \in \Re;\ \Pr(HE(S_\tau) \geq z) \leq 0.05\}$. This risk measure is one of the most widely used risk measures in practice probably due to the Basel accords for banking regulations. The fourth risk measure, the expected shortfall (also known as conditional value- at-risk), is the mean loss beyond value-at-risk: $ES_{0.05} = E[HE(S_\tau)|HE(S_\tau) \geq VAR_{0.05}]$.

In Table III, we compare the hedging performance of the DEK and the modified DEK methods with and without bid–ask spreads. When bid–ask spreads are not considered, the DEK method has slightly lower expected loss, value-at-risk, and expected shortfall, than the modified DEK method. This is expected because the DEK method over-hedges the UOC option and thus its static hedge portfolio costs \$0.3637 (18.81%) more than the modified DEK method. On the contrary, the modified DEK method has much smaller quadratic hedging error than the DEK method. Thus, the above results indicate that the modified DEK method has similar or less risks (depending on which risk measures are used) than the DEK method while its hedging cost is much smaller.

When bid–ask spreads are included, Table III shows that the modified DEK method is less risky than the DEK method in terms of all risk measures except for the expected loss measure. Although the expected loss of the modified DEK method is larger than that of the DEK method, their difference (\$0.2820) is still less than the difference between the replication cost of the DEK method and that of the modified DEK method. To sum up, Table III suggests that the modified DEK method can improve the hedging performance of the DEK method with and without bid–ask spreads.

## CONCLUSION

This study proposes a modified method to improve the static hedge portfolio of Derman et al. (1995). We include both the standard options and binary options into our modified static hedge portfolio to match *zero value* and *zero theta* on the barrier at $n$ evenly spaced time points. We do several numerical experiments under the BS model without market frictions to compare the replication performance of the DEK method with that of the modified DEK method. The numerical results indicate that our modified DEK method significantly improves not only the hedging performance but also the speed of convergence to the continuous barrier option price. Thus, the proposed static hedge portfolio offer a feasible and practical hedging method for barrier options even if the standard European options and binary options only have a few maturities traded in the option exchanges. In addition, we further incorporate bid–ask spreads to compute the profit/loss distributions at the first hitting time or at expiry for the DEK and the modified DEK portfolios. The numerical results indicate that, even when the bid–ask spreads are included, the modified DEK method still outperforms the DEK method under four risk measures suggested by Siven and Poulsen (2009).

There are two possible extensions of our method for future research. First, several articles, such as Fink (2003), Nalholm and Poulsen (2006a,b), Chung and Shih (2009), and Maruhn and Sachs (2009), have extended the DEK method to other option pricing models (e.g. stochastic volatility model of Heston, 1993) or other types of options (e.g. American options). It is possible to apply the proposed method to the existing literature on this direction. How effective is our static hedging method under the other option pricing models remains an open question. Second, the hedging performance or model risk of the DEK portfolio has been tested in the literature, see e.g. Toft and Xuan (1998), Tompkins (2002), and Nalholm and Poulsen (2006b). Since the modified DEK method outperforms the DEK method substantially, it is worth trying to reinvestigate this issue using our method.

## BIBLIOGRAPHY


Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of Political Economy, 81, 637–654.

Bowie, J., & Carr, P. (1994). Static simplicity. Risk Magazine, 7, 45–49.

Carr, P., & Chou, A. (1997). Breaking barriers: Static hedging of barrier securities. Risk Magazine, 10, 139–145.

Carr, P., Ellis, K., & Gupta, V. (1998). Static hedging of exotic options. Journal of Finance, 3, 1165–1190.

Chordia, T., Roll, R., & Subrahmanyam, A. (2000). Commonality in liquidity. Journal of Financial Economics, 56, 3–28.

Chou, A., & Georgiev, G. (1998). A uniform approach to static replication. Journal of Risk, 1, 73–87.

Chung, S.-L., & Shih, P.-T. (2009). Static hedging and pricing American options. Journal of Banking and Finance, 33, 2140–2149.

Cox, J. (1975). Notes on option pricing I: Constant elasticity of variance diffusion. Stanford University working paper and reprinted in Journal of Portfolio Management, December Special Issue, 23(1), 15–17, 1996.

Derman, E., Ergener, D., & Kani, I. (1995). Static options replication. Journal of Derivatives, 2, 78–95.

Fink, J. (2003). An examination of the effectiveness of static hedging in the presence of stochastic volatility. Journal of Futures Markets, 23, 859–890.

Gatheral, J. (2006). The volatility surface. New York: Wiley.

Heston, S. L. (1993). A closed form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies, 6, 327–343.

Maruhn, J. H., & Sachs, E. W. (2009). Robust static hedging of barrier options in stochastic volatility models. Mathematical Methods of Operations Research, 70, 405–433.

Nalholm, M., & Poulsen, R. (2006a). Static hedging of barrier options under general asset dynamics: Unification and application. Journal of Derivatives, 13, 46–60.

Nalholm, M., & Poulsen, R. (2006b). Static hedging and model risk for barrier options. Journal of Futures Markets, 26, 149–163.

Reiner, E., & Rubinstein, M. (1991). Breaking down the barriers. Risk Magazine, 4, 75–83.

Siven, J., & Poulsen, R. (2008). The long and short of static hedging with frictions. Wilmott Magazine, 38, 62–67.

Siven, J., & Poulsen, R. (2009). Auto-static for the people: Risk-minimizing hedges of barrier options. Review of Derivatives Research, 12, 193–211.

Taleb, N. (1996). Dynamic hedging: Managing vanilla and exotic options. New York: Wiley.
Thomsen, H. (1998). Barrier options—Evaluation and hedging. Unpublished doctoral dissertation, University of Aarhus, Aarhus, Denmark.
Toft, K. B., & Xuan, C. (1998). How well can barrier options be hedged by a static portfolio of standard options? Journal of Financial Engineering, 7, 147–175.
Tompkins, R. (2002). Static versus dynamic hedging of exotic options: An evaluation of hedge performance via simulation. Journal of Risk Finance, 3, 6–34.
Wystup, U. (2007). FX options and structured products. New York: Wiley.