1 Green's First Identity

Green's first identity:

$$\int \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi \, dV = \oint \psi \nabla \phi \cdot d\vec{S}$$

What I get by integrating the QG equation is

$$0 = \partial_t \left[\frac{1}{2} \iint \nabla \psi \cdot \nabla \psi + F \psi^2 \, dA + \frac{1}{2} \, \psi^2(y,t) \Big|_{y=0}^{L} \right]$$

Am I able to neglect the last term? I know I can set ψ on one of the boundaries to be 0 at a certain time, but can I do it for all time? And if so, can I set ψ on the other boundary to be a constant for all time, too? If I can, then that last term is just a constant and can be neglected, which gives us the exact same Hamiltonian as the doubly periodic BC's, which will give the same stability criteria.

2 Zonal channel Boundary Conditions

The zonal channel is infinite in the x direction, and has a finite length, L, in the y direction. Volume and surface integrals must been taken in the limit. No x-independence implies

$$\phi = \phi(y,t), \psi = \psi(y,t)$$

For any functions $u(y,t), v(y,t)$, the surface integral evaluates to

$$\begin{aligned}
&\oint u\vec{\nabla}v \cdot d\vec{S} \\
&= \lim_{X\to\infty} \frac{1}{2X} \oint_S u(y,t)v_y(y,t)\,dy + u(y,t)v_x(y,t)\,dx \\
&= \lim_{X\to\infty} \frac{1}{2X} \oint_S u(y,t)v_y(y,t)\,dy \\
&= \lim_{X\to\infty} \frac{1}{2X} \left[\int_0^L u(y,t)u_y(y,t)\,dy + \int_0^L u(y,t)v_y(y,t)\,dy\right] \\
&= 0
\end{aligned}$$

If we assign $u = \psi_i$ and either $v = \dot{\psi}_i$ or $v = \psi_i$, we get 0. This result is used in the derivation of the Hamiltonian.

3 Doubly Periodic Boundary Conditions

The boundary is a rectangle with side lengths L_x and L_y, where the top edge is equivalent with the bottom edge, and the left is equivalent to the right. Mathematically, this is

$$
\begin{aligned}
f(0,y,t) &= f(L_x,y,t) \\
f(x,0,t) &= f(x,L_y,t) \\
f_t(0,y,t) &= f_t(L_x,y,t) \\
f_t(x,0,t) &= f_t(x,L_y,t) \\
\vec{\nabla} f(0,y,t) &= \vec{\nabla} f(L_x,y,t) \\
\vec{\nabla} f(x,0,t) &= \vec{\nabla} f(x,L_y,t)
\end{aligned}
$$

For any functions $u(y,t), v(y,t)$ that are doubly periodic, the surface integral evaluates to

$$
\begin{aligned}
&\oint u\vec{\nabla} v \cdot d\vec{S} \\
&= \int_0^{L_x} u(x,0,t)v_x(x,0,t)\,dx + \int_0^{L_y} u(L_x,y,t)v_y(L_x,y,t)\,dy \\
&\quad + \int_{L_x}^0 u(x,L_y,t)v_x(x,L_y,t)\,dx + \int_{L_x}^0 u(0,y,t)v_y(0,y,t)\,dy
\end{aligned}
$$