Hamiltonian fluid dynamics and hydrodynamic stability

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August 14, 2015

Contents

Classical Mechanics

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Newtonian Dynamics

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Newtonian Dynamics

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- $\mathbf{F} = -\nabla\Pi$
- $\mathbf{p} = \frac{d}{dt}(m\mathbf{q})$

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Hamiltonian Dynamics

- $\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$
- $\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}$
- $H = \mathbf{p}^2/2m + \Pi(\mathbf{q})$

Fluids

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Navier-Stokes Equations

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- $\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \sigma + \mathbf{F}$

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- System of coupled, nonlinear, partial differential equations

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Hamiltonian

- $\frac{d\mathbf{p}}{dt} = -\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$

- $\frac{d\mathbf{q}}{dt} = \frac{\delta \mathcal{H}}{\delta \mathbf{p}}$

- $\mathcal{H} = \iint \frac{1}{2} \nabla \psi \cdot \nabla \psi \, dA$
  $- \sum_n \lambda_n \oint \mathbf{n} \cdot \nabla \psi \, ds$

Hamiltonian Fluid Dynamics

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System of PDEs

$$0 = \mathbf{F}\left(\mathbf{q}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)$$

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$$\mathbf{q}_t = \mathbf{J}\frac{\delta\mathcal{H}}{\delta\mathbf{q}}$$

Variational Derivatives

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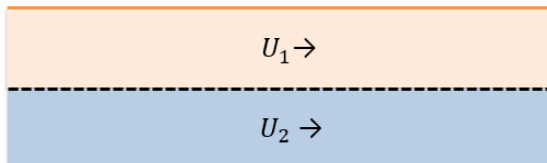$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \mathcal{F}(\mathbf{q} + \epsilon \delta \mathbf{q}) \equiv \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \delta \mathbf{q} \right\rangle \equiv \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}} \delta \mathbf{q} \, d\mathbf{x}$$

Poisson Bracket

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t} &= \int_\Omega \frac{\partial F}{\partial t} \, d\mathbf{x} \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^\top \mathbf{q}_t \, d\mathbf{x} \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^\top \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \, d\mathbf{x} \\
&= \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \right\rangle \\
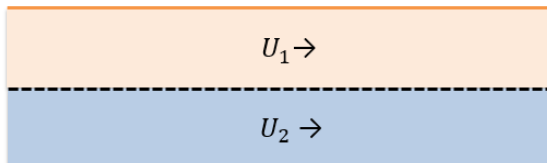&\equiv \{\mathcal{F}, \mathcal{H}\}
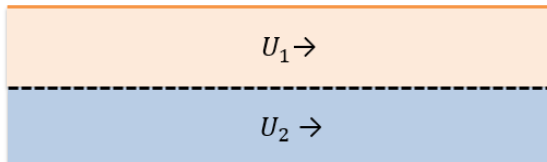\end{aligned}
$$

Quasigeostrophic Model

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$$\psi_n = -U_n y$$

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$\psi_n = -U_n y$

$q_1 = \frac{1}{\alpha_1}\nabla^2\psi_1 + (\psi_2 - \psi_1) = (U_1 - U_2)y$

$q_2 = \frac{1}{\alpha_2}\nabla^2\psi_2 + (\psi_1 - \psi_2) = (U_2 - U_1)y$

Quasigeostrophic Model

$$0 = \frac{\partial q_1}{\partial t} + D(\psi_1, q_1)$$

$$0 = \frac{\partial q_2}{\partial t} + D(\psi_2, q_2)$$

$$D(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

Quasigeostrophic Model

$$0 = \frac{\partial q_1}{\partial t} + D(\psi_1, q_1)$$

$$0 = \frac{\partial q_2}{\partial t} + D(\psi_2, q_2)$$

$$D(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}$$

$$\mathcal{H} = \frac{1}{2} \iint \sum_{n=1}^{2} \left[\frac{1}{\alpha_n} \vec{\nabla} \psi_n \cdot \vec{\nabla} \psi_n + 2C_n(q_n) \right] + (\psi_1 - \psi_2)^2 \, dA$$

Second Variation

Second Variation

$$\delta^2 \mathcal{H} = \iint \sum_n \left[\frac{1}{\alpha_n} \vec{\nabla} \delta \psi_n \cdot \vec{\nabla} \delta \psi_n + \psi_n' \cdot (\delta q_n)^2 \right] + (\delta \psi_1 - \delta \psi_2)^2 \, dA$$

Second Variation

$$\delta^2 \mathcal{H} = \iint \sum_n \left[\frac{1}{\alpha_n} \vec{\nabla} \delta \psi_n \cdot \vec{\nabla} \delta \psi_n + \psi'_n \cdot (\delta q_n)^2 \right] + (\delta \psi_1 - \delta \psi_2)^2 \, dA$$

$$U_1(U_1 - U_2) = 0$$

$$U_2(U_2 - U_1) = 0$$

References

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