Hamiltonian fluid dynamics and stability

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Classical Mechanics

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Newtonian Dynamics

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- $m\frac{d^2\mathbf{q}}{dt} = \mathbf{F}$
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- Second order ODE

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Hamiltonian Dynamics

- $\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$
- $\frac{d\mathbf{q}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}$
- System of first order ODEs

Fluids

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Navier-Stokes Equations

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Navier-Stokes Equations

- $\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \sigma + \mathbf{F}$
- $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u} = 0$
- System of coupled, nonlinear, PDEs

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Hamiltonian

- $\frac{d\mathbf{p}}{dt} = -\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$
- $\frac{d\mathbf{q}}{dt} = \frac{\delta \mathcal{H}}{\delta \mathbf{p}}$
- More analytical tools to work with

Hamiltonian Fluid Dynamics

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System of PDEs

$$\mathbf{0} = \mathbf{F}\left(\mathbf{q}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)$$

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$$\mathbf{q}_t = \mathbf{J}\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$$

Derivatives

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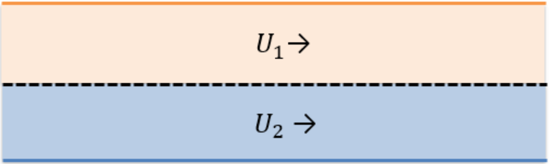
$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \mathcal{F}(\mathbf{q} + \epsilon \delta \mathbf{q}) \equiv \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \delta \mathbf{q} \right\rangle \equiv \int_{\Omega} \frac{\delta \mathcal{F}}{\delta \mathbf{q}} \delta \mathbf{q}\, d\mathbf{x}$$

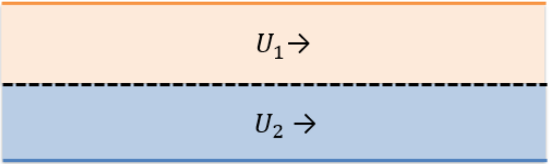
Poisson Bracket

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t} &= \int_\Omega \frac{\partial F}{\partial t}\, d\mathbf{x} \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^\top \mathbf{q}_t \, d\mathbf{x} \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^\top \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \, d\mathbf{x} \\
&= \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \right\rangle \\
&\equiv \{\mathcal{F}, \mathcal{H}\}
\end{aligned}
$$

Quasigeostrophic Model

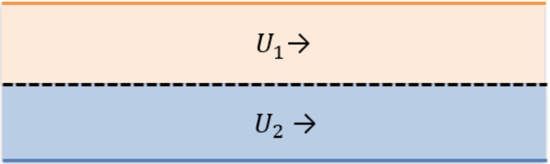


Quasigeostrophic Model



$$\psi_n = -U_n y$$

Quasigeostrophic Model



$$\psi_n = -U_n y$$
$$q_1 = \frac{1}{\alpha_1}\nabla^2\psi_1 + (\psi_2 - \psi_1) = (U_1 - U_2)y$$
$$q_2 = \frac{1}{\alpha_2}\nabla^2\psi_2 + (\psi_1 - \psi_2) = (U_2 - U_1)y$$

Quasigeostrophic Model

$$
\begin{aligned}
0 &= \frac{\partial q_1}{\partial t} + D(\psi_1, q_1) \\
0 &= \frac{\partial q_2}{\partial t} + D(\psi_2, q_2) \\
D(a, b) &= \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}
\end{aligned}
$$

Quasigeostrophic Model

$$0 = \frac{\partial q_1}{\partial t} + D(\psi_1, q_1)$$
$$0 = \frac{\partial q_2}{\partial t} + D(\psi_2, q_2)$$
$$D(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

$$\mathcal{H} = \frac{1}{2}\iint \sum_{n=1}^{2}\left[\frac{1}{\alpha_n}\vec{\nabla}\psi_n \cdot \vec{\nabla}\psi_n + 2C_n(q_n)\right] + (\psi_1 - \psi_2)^2\, dA$$

Second Variation

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$$\delta^2\mathcal{H} = \iint \sum_n \left[\frac{1}{\alpha_n} \vec{\nabla}\delta\psi_n \cdot \vec{\nabla}\delta\psi_n + \psi_n' \cdot (\delta q_n)^2 \right] + (\delta\psi_1 - \delta\psi_2)^2 \, dA$$

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$$U_n \frac{dq_n}{dy} > 0, \quad n = 1, 2$$

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$$U_1(U_1 - U_2) > 0$$

$$U_2(U_2 - U_1) > 0$$

References

1. G. Swaters. "Introduction to Hamiltonian Fluid Dynamics and Stability Theory". Monographs and Surveys in Pure and Applied Mathematics (1999).
2. M. E. McIntyre and T. G. Shepherd. "An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnolds stability theorems". Journal of Fluid Mechanics (1987), vol. 181, pp. 527-565.
3. T. G. Shepherd. "Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics". Advances in Geophysics (1990), vol. 32, pp. 287-338.
4. Jacques Vanneste. "Explosive resonant interaction of baroclinic Rossby waves and stability of multilayer quasi-geostrophic flow". Journal of Fluid Mechanics (1995), vol. 291, pp. 83-107.