1 One-Layer Quasi-Geostrophy

The one-layer QG model can be written as,

$$\partial_t q + J(\psi, q) = 0,$$

with the following potential vorticity (PV) relation

$$q = \nabla^2 \psi - \frac{f_0^2}{gH}\psi.$$

The Hamiltonian can be thought of as the energy in the system and takes the form,

$$H = \frac{1}{2}\iint \vec{\nabla}\psi \cdot \vec{\nabla}\psi + \frac{f_0^2}{gH}\psi^2 \, dA$$

The Casimirs are functions of the PV,

$$\mathcal{C} = \iint C(q)\, dA,$$

where $C(q)$ is an arbitrary function. From this we form the constrained Hamiltonian,

$$\mathcal{H} = \iint \frac{1}{2}\vec{\nabla}\psi \cdot \vec{\nabla}\psi + \frac{1}{2}\frac{f_0^2}{gH}\psi^2 + C(q)\, dA \tag{1}$$

1.1 First Variation

To compute the steady states we must first the first variational derivative of the above functional:

$$\begin{aligned}
\delta\mathcal{H} &= \iint \vec{\nabla}\psi \cdot \vec{\nabla}\delta\psi + \frac{f_0^2}{gH}\psi\delta\psi + C'(q)\delta q\, dA, \\
&= \iint -\psi\left[\nabla^2\delta\psi - \frac{f_0^2}{gH}\delta\psi\right] + C'(q)\delta q\, dA, \\
&= \iint \left[-\psi + C'(q)\right]\delta q\, dA.
\end{aligned}$$

From here we find that the steady solutions, which have a zero first variational derivative, must satisfy the following,

$$\psi_s = C'(q_s).$$

Here, and throughout, we use a subscript s to denote steady solution.

1.2 Second Variation

Similarly, we compute the second variational derivative,

$$\delta^2\mathcal{H} = \iint \left[-\psi + C'(q)\right]\delta^2 q - \delta\psi\delta q + C''(q)(\delta q)^2\, dA,$$
$$= \iint \left[-\psi + C'(q)\right]\delta^2 q + \vec{\nabla}\delta\psi\cdot\vec{\nabla}\delta\psi + \frac{f_0^2}{gH}(\delta\psi)^2 + C''(q)(\delta q)^2\, dA,$$

after integrating by parts.

Next, we evaluate the second variation at an arbitrary steady solution,

$$\delta^2\mathcal{H}_s = \iint \vec{\nabla}\delta\psi\cdot\vec{\nabla}\delta\psi + \frac{f_0^2}{gH}(\delta\psi)^2 + C''(q_s)(\delta q)^2\, dA$$
$$= \delta^2 H_s + \iint C''(q_s)(\delta q)^2\, dA.$$

Note that the $\delta^2 H_s$ term is positive definite and the Casimir term is sign definite.

1.3 Linear Stability Theorems

To prove stability it is sufficient to show that the second variation is sign definite. (FJP: explain why?).

1.3.1 First Theorem

The first theorem is the easiest and corresponds to showing where the second variation is positive definite. It is readily seen that this occurs when

$$C''(q_s) > 0.$$

If we assume that the steady solution only depends on latitude, y, then $\Psi = \Psi(y)$, and we can differentiate our equation that defines the steady state to obtain,

$$U_s = C''(q_s)\frac{dq_s}{dy} = C''(q_s)\left[\frac{d^2U_s}{dy^2} - \frac{f_0^2}{gH}U_s\right]$$

where we used the fact that $U_s = -\frac{d\Psi_s}{dy}$.

If we require that $C'' > 0$ then this is equivalent (assuming the PV is not zero) to,

$$\frac{U_s}{\frac{dQ_s}{dy}} = \frac{U_s}{\left[\frac{d^2U_s}{dy^2} - \frac{f_0^2}{gH}U_s\right]} > 0.$$

Since we are only concerned with the signs this can be rewritten as,

$$U_s\frac{dQ_s}{dy} > 0.$$

1.3.2 Second Theorem

This is harder to show and requires first beginning with the Poincaré eigenvalue problem,

$$q + \kappa\psi = 0.$$

We must multiply this by q integrate over the domain and find a bound on the gradient of $\delta\psi$ squared, which appears in the second variation.

FJP: insert details?

1.4 Nonlinear Stability Theorems

To analyze the the nonlinear stability, we introduce the functional

$$\mathcal{N}(\delta q) = \mathcal{H}(q_s + \delta q) - \mathcal{H}(q_s)$$

since, when expanded about q_s, it becomes

$$\begin{aligned}
\mathcal{N}(\delta q) &= -\mathcal{H}(q_s) + \mathcal{H}(q_s + \delta q) \\
&= -\mathcal{H}(q_s) + \left[\mathcal{H}(q_s) + \delta\mathcal{H}(q_s) + \frac{1}{2}\delta^2\mathcal{H}(q_s) + ...\right] \\
&= \frac{1}{2}\delta^2\mathcal{H}(q_s) + \mathcal{O}((\delta q)^3)
\end{aligned}$$

To leading order, arguments about $\mathcal{N}$ will be the same as arguments about $\delta^2\mathcal{H}$. But $\mathcal{N}$ also captures the nonlinearity of the Hamiltonian, so more work will be required.

$\mathcal{N}(\delta q) = \mathcal{H}(q_s + \delta q) - \mathcal{H}(q_s)$

$$
\begin{aligned}
&= \iint \frac{1}{2}\vec{\nabla}(\psi_s+\delta\psi)\cdot\vec{\nabla}(\psi_s+\delta\psi) + \frac{1}{2}\frac{f_0^2}{gH}(\psi_s+\delta\psi)^2 + C(q_s+\delta q)\,dA \\
&\quad - \iint \frac{1}{2}\vec{\nabla}\psi_s\cdot\vec{\nabla}\psi_s + \frac{1}{2}\frac{f_0^2}{gH}\psi_s^2 + C(q_s)\,dA \\
&= \iint \frac{1}{2}\vec{\nabla}\delta\psi\cdot\vec{\nabla}\delta\psi + \frac{1}{2}\frac{f_0^2}{gH}(\delta\psi)^2 + \vec{\nabla}\psi_s\cdot\vec{\nabla}\delta\psi + \frac{1}{2}\frac{f_0^2}{gH}\psi_s\delta\psi + C(q_s+\delta q) - C(q_s)\,dA \\
&= \iint -\psi_s\left[\nabla^2\delta\psi - \frac{1}{2}\frac{f_0^2}{gH}\delta\psi\right] + \frac{1}{2}\vec{\nabla}\delta\psi\cdot\vec{\nabla}\delta\psi + \frac{1}{2}\frac{f_0^2}{gH}(\delta\psi)^2 + C(q_s+\delta q) - C(q_s)\,dA \\
&= \iint \frac{1}{2}\vec{\nabla}\delta\psi\cdot\vec{\nabla}\delta\psi + \frac{1}{2}\frac{f_0^2}{gH}(\delta\psi)^2 + C(q_s+\delta q) - C(q_s) - C'(q_s)\delta q\,dA \\
&= \frac{1}{2}\delta^2 H_s + \iint C(q_s+\delta q) - C(q_s) - C'(q_s)\delta q\,dA
\end{aligned}
$$

where H_s is the unconstrained Hamiltonian evaluated at the steady state, defined at the beginning of this section.

2 Two-Layer Quasi-Geostrophy

The two-layer QG model can be written as,

$$\partial_t q_i + J(\psi_i, q_i) = 0,$$

for $i = 1, 2$ where the PV relations are given by:

$$
\begin{aligned}
q_1 &= \frac{g'H_1}{f_0^2}\nabla^2\psi_1 - (\psi_1 - \psi_2) - \psi_1, \\
q_2 &= \frac{g'H_2}{f_0^2}\nabla^2\psi_2 - (\psi_2 - \psi_1) + \frac{g'}{f_0}h_b.
\end{aligned}
$$

We use the convention that g is the full gravity at the surface and g' is the reduced gravity between the two dynamic layers.

If we want to consider a rigid lid and a flat bottom, then we neglect the last terms in each of these two equations. We make this choice but can return to this later if we like. We get the following system (Holm *et al.* 1985):

$$
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_1}\nabla^2\psi_1 \\ \frac{1}{\alpha_2}\nabla^2\psi_2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \alpha_i = \frac{f_0^2}{g'H_i} \tag{2}
$$

Note that previous papers usually assume that the constants in front are equal. This is not the case. In QG this might not be as important but in SW it will make a big difference.

2.1 Hamiltonian

To derive an equation for the energy we must multiply the evolution equation by ψ_i for each i and then sum up over the different layers,

$$\begin{aligned}
0 &= \iint \psi_1 \partial_t q_1 + \psi_2 \partial_t q_2 + \psi_1 J(\psi_1, q_1) + \psi_2 J(\psi_2, q_2)\, dA \\
&= \iint \psi_1 \partial_t q_1 + \psi_2 \partial_t q_2 + \frac{1}{2} J(\psi_1^2, q_1) + \frac{1}{2} J(\psi_2^2, q_2)\, dA \\
&= \iint \sum_i \left[\partial_t (q_i \psi_i) - q_i \partial_t \psi_i\right] + \frac{1}{2} J(\psi_1^2, q_1) + \frac{1}{2} J(\psi_2^2, q_2)\, dA \\
&= \iint \sum_i \left[\partial_t (q_i \psi_i) - q_i \partial_t \psi_i\right]\, dA \\
&= \iint \sum_i \partial_t \left(\frac{1}{\alpha_i} \psi_i \nabla^2 \psi_i - \psi_i^2 + \psi_i \psi_{j\neq i}\right) \\
&\quad - \left(\frac{1}{\alpha_i} \psi_{it} \nabla^2 \psi_i - \psi_i \psi_{it} + \psi_{j\neq i} \psi_{it}\right)\, dA \\
&= \iint \sum_i \partial_t \left(-\frac{1}{\alpha_i} \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_i - \psi_i^2 + \psi_i \psi_{j\neq i}\right) \\
&\quad - \left(-\frac{1}{\alpha_i} \vec{\nabla} \psi_{it} \cdot \vec{\nabla} \psi_i - \psi_i \psi_{it} + \psi_{j\neq i} \psi_{it}\right)\, dA \\
&= \iint \sum_i \partial_t \left(-\frac{1}{\alpha_i} \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_i - \psi_i^2\right) + \partial_t (\psi_i \psi_{j\neq i}) \\
&\quad + \left(\frac{1}{\alpha_i} \vec{\nabla} \psi_{it} \cdot \vec{\nabla} \psi_i + \psi_i \psi_{it}\right) - \psi_{j\neq i} \psi_{it}\, dA \\
&= \iint \sum_i \frac{1}{2} \partial_t \left[\frac{1}{\alpha_i} \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_i + \psi_i^2\right] - \partial_t \psi_i \psi_{j\neq i} + \psi_{j\neq i} \psi_{it}\, dA \\
&= \iint \sum_i \frac{1}{2} \partial_t \left[\frac{1}{\alpha_i} \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_i + \psi_i^2\right] - \psi_1 \partial_t \psi_2 - \psi_2 \partial_t \psi_1\, dA
\end{aligned}$$

$$= \iint \sum_i \frac{1}{2}\partial_t \left[\frac{1}{\alpha_i}\vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_i + \psi_i^2\right] - \partial_t(\psi_1\psi_2)\, dA$$

$$= \frac{1}{2}\iint \partial_t \left[\sum_{i=1}^{2} \frac{1}{\alpha_i}\vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_i + (\psi_1 - \psi_2)^2\right] dA$$

This yields that the Hamiltonian of the system is the total energy in the system,

$$H = \frac{1}{2}\iint \sum_{i=1}^{2} \frac{1}{\alpha_i}\vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_i + (\psi_1 - \psi_2)^2\, dA$$

The Casimirs are functions of the PV,

$$\mathcal{C}_i = \iint C_i(q_i)\, dA$$

From this we form the constrained Hamiltonian

$$\mathcal{H} = \frac{1}{2}\iint \sum_{i=1}^{2} \left[\frac{1}{\alpha_i}\vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_i + 2C_i(q_i)\right] + (\psi_1 - \psi_2)^2\, dA \tag{3}$$

2.2 First Variation

$$\delta\mathcal{H} = \frac{1}{2}\iint \sum_{i=1}^{2} \left[\frac{1}{\alpha_i}\vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_i + 2C_i(q_i)\right] + (\psi_1 - \psi_2)^2\, dA$$

$$= \iint \sum_{i=1}^{2} \left[\frac{1}{\alpha_i}\vec{\nabla}\psi_i \cdot \vec{\nabla}\delta\psi_i + C_i'(q_i)\delta q_i\right] + (\psi_1 - \psi_2)(\delta\psi_1 - \delta\psi_2)\, dA$$

$$= \iint \sum_{i=1}^{2} \left[-\frac{1}{\alpha_i}\psi_i\nabla^2\delta\psi_i + C_i'(q_i)\delta q_i - \psi_i(\delta\psi_i - \delta\psi_{j\neq i})\right] dA$$

$$= \iint \sum_{i=1}^{2} (-\psi_i + C_i'(q_i))\delta q_i\, dA$$

Since $\delta\mathcal{H} = 0$ for steady solutions, and q_1 and q_2 are independent coordinates, we have that in vector form

$$\vec{\psi}_s = \begin{bmatrix} \psi_{1s} \\ \psi_{2s} \end{bmatrix} = \begin{bmatrix} C_1'(q_{1s}) \\ C_2'(q_{2s}) \end{bmatrix} = \vec{C}'(\vec{q}_s) \tag{4}$$

2.3 Second Variation

Like the single layer model, the second variation for the two-layer model is

$$\begin{aligned}
\delta^2\mathcal{H} &= \iint \sum_i \left[-\psi_i + C_i'(q_i)\right]\delta^2 q_i - \delta\psi_i\delta q_i + C_i''(q_i)(\delta q_i)^2\,dA \\
\delta^2\mathcal{H}_s &= \iint \sum_i \left[-\psi_{is} + C_i'(q_{is})\right]\delta^2 q_i - \delta\psi_i\delta q_i + C_i''(q_{is})(\delta q_i)^2\,dA \\
&= \iint \sum_i -\delta\psi_i\delta q_i + C_i''(q_{is})(\delta q_i)^2\,dA \\
&= \iint \sum_i -\delta\psi_i\left(\frac{1}{\alpha_i}\nabla^2\delta\psi_i - \delta\psi_i + \delta\psi_{j\neq i}\right) + C_i''(q_{is})(\delta q_i)^2\,dA \\
&= \iint \sum_i \left[\frac{1}{\alpha_i}\vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\delta\psi_i + (\delta\psi_i)^2 + C_i''(q_{is})(\delta q_i)^2\right] - 2\delta\psi_1\delta\psi_2\,dA \\
&= \iint \sum_i \left[\frac{1}{\alpha_i}\vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\delta\psi_i + C_i''(q_{is})(\delta q_i)^2\right] + (\delta\psi_1 - \delta\psi_2)^2\,dA \\
&= \delta^2 H_s + \iint \sum_i C_i''(q_{is})(\delta q_i)^2\,dA
\end{aligned}$$

The $\delta^2 H_s$ term is clearly positive definite, while the $C_i''(q_{is})$ term is sign definite. Like in the single-layer case, the definiteness of the second variation depends on the definiteness of the Casimirs.

2.4 Linear Stability Theorems

2.4.1 First Theorem

The first theorem proceeds exactly like it did for the one-layer case. We require the second variation to be positive definite, so

$$C_i''(q_{is}) > 0, \quad i = 1, 2$$

If we assume that the steady solution only depends on latitude, y, then $\Psi_i = \Psi_i(y)$, and we can differentiate our equation that defines the steady state to obtain

$$U_{is} = C_i''(q_{is})\frac{dq_{is}}{dy}$$

or succinctly,

$$U_{is}\frac{dQ_{is}}{dy} > 0, \quad i = 1, 2$$

2.5 Nonlinear Stability Theorems

We proceed as stated in the single-layer case, by defining $\mathcal{N}$.

$$\begin{aligned}
\mathcal{N}(q) &= \mathcal{H}(q_s + \delta q) - \mathcal{H}(q_s) \\
&= \frac{1}{2}\iint \sum_{i=1}^{2}\left(\frac{1}{\alpha_i}\vec{\nabla}(\psi_{is} + \delta\psi_i)\cdot\vec{\nabla}(\psi_{is} + \delta\psi_i) + 2C_i(q_{is} + \delta q_i)\right) \\
&\quad + ((\psi_{1s} + \delta\psi_1) - (\psi_{2s} + \delta\psi_2))^2 \, dA \\
&\quad - \left[\frac{1}{2}\iint \sum_{i=1}^{2}\left(\frac{1}{\alpha_i}\vec{\nabla}\psi_{is}\cdot\vec{\nabla}\psi_{is} + 2C_i(q_{is})\right) + (\psi_{1s} - \psi_{2s})^2 \, dA\right] \\
&= \frac{1}{2}\iint \sum_{i=1}^{2}\frac{1}{\alpha_i}(2\vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\psi_{is} + \vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\delta\psi_i) \\
&\quad + 2\psi_{1s}\delta\psi_1 + (\delta\psi_1)^2 + 2\psi_{2s}\delta\psi_2 + (\delta\psi_2)^2 - 2\psi_{1s}\delta\psi_2 - 2\delta\psi_1\psi_{2s} - 2\delta\psi_1\delta\psi_2 \, dA \\
&\quad + \iint \sum_{i=1}^{2} C_i(q_{is} + \delta q_i) - C_i(q_{is}) \, dA \\
&= \frac{1}{2}\iint \sum_{i=1}^{2}\frac{1}{\alpha_i}(2\vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\psi_{is} + \vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\delta\psi_i) \\
&\quad + (\delta\psi_1 - \delta\psi_2)^2 + 2(\psi_{1s} - \psi_{2s})(\delta\psi_1 - \delta\psi_2) \, dA \\
&\quad + \iint \sum_{i=1}^{2} C_i(q_{is} + \delta q_i) - C_i(q_{is}) \, dA \\
&= \frac{1}{2}\iint \sum_{i=1}^{2}\frac{1}{\alpha_i}\vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\delta\psi_i + (\delta\psi_1 - \delta\psi_2)^2 \\
&\quad + \iint \sum_{i=1}^{2} C_i(q_{is} + \delta q_i) - C_i(q_{is}) + \vec{\nabla}\delta\psi_i\cdot\vec{\nabla}\psi_{is} + \psi_{is}(\delta\psi_i - \delta\psi_{j\neq i}) \, dA \\
&= \frac{1}{2}\delta^2 H_s + \iint \sum_{i=1}^{2} C_i(q_{is} + \delta q_i) - C_i(q_{is}) - \psi_{is}(\nabla^2\delta\psi_i - (\delta\psi_i - \delta\psi_{j\neq i})) \, dA
\end{aligned}$$

$$= \frac{1}{2}\delta^2 H_s + \iint \sum_{i=1}^{2} C_i(q_{is} + \delta q_i) - C_i(q_{is}) - \psi_{is}\delta q_i \, dA$$

$$= \frac{1}{2}\delta^2 H_s + \iint \sum_{i=1}^{2} C_i(q_{is} + \delta q_i) - C_i(q_{is}) - C_i'(q_{is})\delta q_i \, dA$$

3 Shallow Water

The SW model can be written as,

$$\partial_t \vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} + f\hat{k} \times \vec{u} = -g\vec{\nabla} h,$$
$$\partial_t h + \vec{\nabla} \cdot (h\vec{u}) = 0.$$

3.1 Hamiltonian

As can be found in Shepherd (1990), the regular Hamiltonian is

$$H = \frac{1}{2} \iint h|\vec{u}|^2 + gh^2 \, dA.$$

The PV is defined as,

$$q = \frac{\hat{k} \cdot (\vec{\nabla} \times \vec{u}) + f}{h},$$

and is conserved following the motion.

It can be shown (FJP: insert) that the Casimirs are

$$\mathcal{C} = \iint hC(q) \, dA.$$

Therefore, the constrained Hamiltonian is,

$$\mathcal{H} = \iint \frac{1}{2} \left(h(u^2 + v^2) + gh^2 \right) + hC(q) \, dA.$$

3.2 Equations of Motion

Define the Poisson bracket.

Show that we can recover the equations of motion.

3.3 First Variation

The variational derivative is,

$$\delta \mathcal{H} = \iint \left[\frac{1}{2}(u^2 + v^2) + gh + C(q) \right] \delta h + uh\delta u + vh\delta v + hC'(q)\delta q \, dA.$$

But we need to get everything in terms of the variations of height and velocity. Therefore, we need the identity that

$$\begin{aligned}
\iint hC'(q)\delta q \, dA &= \iint hC'(q)\delta \frac{\partial_x v - \partial_y u + f}{h} \, dA, \\
&= \iint -C'(q)\frac{\partial_x v - \partial_y u + f}{h}\delta h + C'(q)\left(\partial_x \delta v - \partial_y \delta u\right) \, dA, \\
&= \iint \left(-C'(q)q\delta h - \partial_x(C'(q))\delta v + \partial_y(C'(q))\delta u\right) \, dA.
\end{aligned}$$

If we substitute this into our above equation for the first variation we get,

$$\begin{aligned}
\delta \mathcal{H} = &\iint \left[\frac{1}{2}(u^2 + v^2) + gh + C(q) - C'(q)q \right] \delta h \, dA, \\
+ &\iint \left[\partial_y(C'(q)) + uh\right] \delta u + \left[vh - \partial_x(C'(q))\right] \delta v \, dA.
\end{aligned}$$

Steady solutions have a zero first variation and therefore are governed by the following equations,

$$\begin{aligned}
\frac{1}{2}(u^2 + v^2) + gh &= -C(q) + C'(q)q, \\
uh &= -\partial_y(C'(q)) = -C''(q)\partial_y q, \\
vh &= \partial_x(C'(q)) = C''(q)\partial_x q.
\end{aligned}$$

If we restrict ourselves to basic states that are invariant with respect to x, then we find that $v = 0$. This can be used in the first two equations to simplify things. I would expect that the first two equations reduce down to geostrohic balance but I don't honestly know. Shepherd 1990 looks at this in some detail. Maybe a good reference?

3.4 Second Variation

Only after we understand the first variation.