1 Green's First Identity

$$\int \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi \, dV = \oint \psi \nabla \phi \cdot d\vec{S}$$

2 Zonal channel Boundary Conditions

The zonal channel is infinite in the x direction, and has a finite length, L, in the y direction. Volume and surface integrals must been taken in the limit.

Since the channel is infinite in the x direction, this imples that there is no x-independence in the stream function. Mathematically,

$$\phi = \phi(y,t), \psi = \psi(y,t)$$

For any functions $u(y,t), v(y,t)$, the surface integral evaluates to

$$\begin{aligned}
&\oint u \vec{\nabla} v \cdot d\vec{S} \\
&= \lim_{X \to \infty} \frac{1}{2X} \oint_S u(y,t) v_y(y,t) \, dy + u(y,t) v_x(y,t) \, dx \\
&= \lim_{X \to \infty} \frac{1}{2X} \oint_S u(y,t) v_y(y,t) \, dy \\
&= \lim_{X \to \infty} \frac{1}{2X} \left[\int_0^L u(y,t) u_y(y,t) \, dy + \int_L^0 u(y,t) v_y(y,t) \, dy \right] \\
&= 0
\end{aligned}$$

If we assign $u = \psi_i$ and either $v = \dot{\psi}_i$ or $v = \psi_i$, we get 0. This result is used in the derivation of the Hamiltonian.

3 Doubly Periodic Boundary Conditions

The boundary is a rectangle with side lengths L_x and L_y, where the top edge is equivalent with the bottom edge, and the left is equivalent to the right. Mathematically, this is

$$\begin{aligned}
f(0,y,t) &= f(L_x,y,t) \\
f(x,0,t) &= f(x,L_y,t) \\
f_t(0,y,t) &= f_t(L_x,y,t) \\
f_t(x,0,t) &= f_t(x,L_y,t) \\
\vec{\nabla} f(0,y,t) &= \vec{\nabla} f(L_x,y,t) \\
\vec{\nabla} f(x,0,t) &= \vec{\nabla} f(x,L_y,t)
\end{aligned}$$

For any functions $u(y,t), v(y,t)$ that are doubly periodic, the surface integral evaluates to

$$\begin{aligned}
&\oint u\vec{\nabla} v \cdot d\vec{S} \\
&= \int_0^{L_x} u(x,0,t)v_x(x,0,t)\,dx + \int_0^{L_y} u(L_x,y,t)v_y(L_x,y,t)\,dy \\
&\quad + \int_{L_x}^0 u(x,L_y,t)v_x(x,L_y,t)\,dx + \int_{L_x}^0 u(0,y,t)v_y(0,y,t)\,dy \\
&= \int_0^{L_x} u(x,0,t)v_x(x,0,t)\,dx + \int_0^{L_y} u(0,y,t)v_y(0,y,t)\,dy \\
&\quad + \int_{L_x}^0 u(x,0,t)v_x(x,0,t)\,dx + \int_{L_x}^0 u(0,y,t)v_y(0,y,t)\,dy \\
&= 0
\end{aligned}$$