Hamiltonian Dynamics of Fluids

James Hawley

University of Waterloo

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Hamiltonian Dynamics

- $\frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial\vec{q}}, \frac{d\vec{q}}{dt} = \frac{\partial\mathcal{H}}{\partial\vec{p}}$
- First order, coupled system

Fluids

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Navier-Stokes Equations

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- $\rho \frac{d\vec{u}}{dt} = -\nabla p + \rho \nabla \Pi + F$
- Coupled, first order, nonlinear system

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Hamiltonian

- $\frac{d\vec{p}}{dt} = -\frac{\delta \mathcal{H}}{\delta \vec{q}}, \frac{d\vec{q}}{dt} = \frac{\delta \mathcal{H}}{\delta \vec{p}}$
- Coupled, first order, linear system

Hamiltonian Fluid Dynamics

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System of PDEs

$$\mathbf{0} = \mathbf{F}\left(\mathbf{q}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)$$

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$$\mathbf{q}_t = \mathbf{J}\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$$

Derivatives

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$$\mathcal{F}(\mathbf{q}) = \int_\Omega F(\mathbf{q})\, d\mathbf{x}$$

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$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \mathcal{F}(\mathbf{q} + \epsilon \delta \mathbf{q}) \equiv \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \delta \mathbf{q} \right\rangle \equiv \int_{\Omega} \frac{\delta \mathcal{F}}{\delta \mathbf{q}} \delta \mathbf{q}\, d\mathbf{x}$$

Poisson Bracket

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$$\frac{\partial \mathcal{F}}{\partial t} = \int_{\Omega} \frac{\partial F}{\partial t} \, d\mathbf{x}$$

Poisson Bracket

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t} &= \int_{\Omega} \frac{\partial F}{\partial t}\, d\mathbf{x} \\
&= \int_{\Omega} \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^{\top} \mathbf{q}_t \, d\mathbf{x}
\end{aligned}
$$

Poisson Bracket

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\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t} &= \int_\Omega \frac{\partial F}{\partial t}\, d\mathbf{x} \\
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&= \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \right\rangle
\end{aligned}$$

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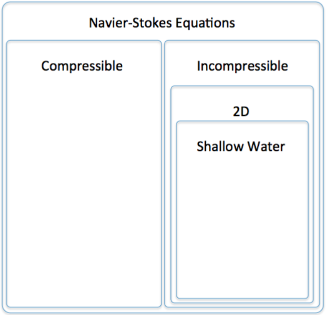
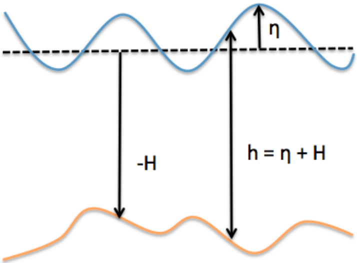
&= \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \right\rangle \\
&\equiv \{\mathcal{F}, \mathcal{H}\}
\end{aligned}
$$

Shallow Water Model



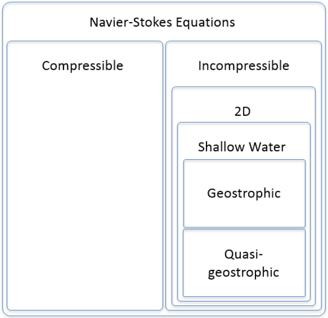
Shallow Water Model



$$0 = \nabla \cdot \vec{u}$$

$$0 = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv)$$

Geostrophic Model



Geostropic Model

$$\frac{d\vec{u}}{dt} + 2\vec{\Omega} \times \vec{u} = -\frac{\nabla p}{\rho} + \nabla \Pi + \frac{F}{\rho}$$

$$\vec{u} = \text{velocity field}$$

$$\vec{\Omega} = \text{rotation vector}$$

$$p = \text{pressure}$$

$$\Pi = \text{scalar potential field}$$

$$F = \text{viscous forces}$$

Quasigeostrophic Model

$$0 = \frac{\partial q}{\partial t} + J(\psi, q)$$

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

References

1. Reference