Hamiltonian fluid dynamics and hydrodynamic stability

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August 14, 2015

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Newtonian Dynamics

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- $\mathbf{F} = -\nabla\Pi$
- Second order ODE

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Hamiltonian Dynamics

- $\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$
- $\frac{d\mathbf{q}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}$
- System of first order ODEs

Fluids

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Navier-Stokes Equations

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- $\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \sigma + \mathbf{F}$
- $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u} = 0$
- System of coupled, nonlinear, PDEs

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Hamiltonian

- $\frac{d\mathbf{p}}{dt} = -\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$
- $\frac{d\mathbf{q}}{dt} = \frac{\delta \mathcal{H}}{\delta \mathbf{p}}$
- More analytical tools to work with

Hamiltonian Fluid Dynamics

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System of PDEs

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$$\mathbf{q}_t = \mathbf{J}\frac{\delta\mathcal{H}}{\delta\mathbf{q}}$$

Variational Derivatives

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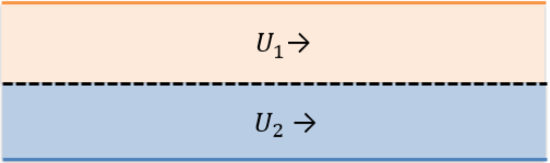
$$\lim_{\epsilon\to 0} \frac{\partial}{\partial \epsilon}\mathcal{F}(\mathbf{q} + \epsilon\delta\mathbf{q}) \equiv \left\langle \frac{\delta\mathcal{F}}{\delta\mathbf{q}}, \delta\mathbf{q} \right\rangle \equiv \int_{\Omega} \frac{\delta\mathcal{F}}{\delta\mathbf{q}}\delta\mathbf{q}\, d\mathbf{x}$$

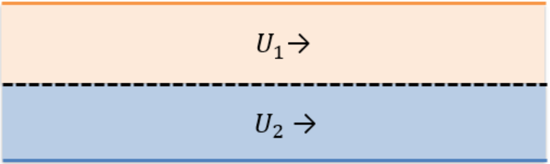
Poisson Bracket

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t} &= \int_\Omega \frac{\partial F}{\partial t}\, d\mathbf{x} \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^\top \mathbf{q}_t\, d\mathbf{x} \\
&= \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^\top \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}}\, d\mathbf{x} \\
&= \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \right\rangle \\
&\equiv \{\mathcal{F}, \mathcal{H}\}
\end{aligned}
$$

Quasigeostrophic Model

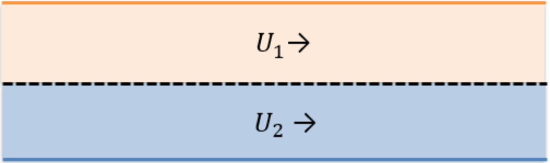


Quasigeostrophic Model



$$\psi_n = -U_n y$$

Quasigeostrophic Model



$$\psi_n = -U_n y$$
$$q_1 = \frac{1}{\alpha_1}\nabla^2\psi_1 + (\psi_2 - \psi_1) = (U_1 - U_2)y$$
$$q_2 = \frac{1}{\alpha_2}\nabla^2\psi_2 + (\psi_1 - \psi_2) = (U_2 - U_1)y$$

Quasigeostrophic Model

$$
\begin{aligned}
0 &= \frac{\partial q_1}{\partial t} + D(\psi_1, q_1) \\
0 &= \frac{\partial q_2}{\partial t} + D(\psi_2, q_2) \\
D(a, b) &= \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}
\end{aligned}
$$

Quasigeostrophic Model

$$0 = \frac{\partial q_1}{\partial t} + D(\psi_1, q_1)$$
$$0 = \frac{\partial q_2}{\partial t} + D(\psi_2, q_2)$$
$$D(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

$$\mathcal{H} = \frac{1}{2}\iint \sum_{n=1}^{2}\left[\frac{1}{\alpha_n}\vec{\nabla}\psi_n \cdot \vec{\nabla}\psi_n + 2C_n(q_n)\right] + (\psi_1 - \psi_2)^2 \, dA$$

Second Variation

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$$\delta^2\mathcal{H} = \iint \sum_n \left[\frac{1}{\alpha_n}\vec{\nabla}\delta\psi_n \cdot \vec{\nabla}\delta\psi_n + \psi_n' \cdot (\delta q_n)^2\right] + (\delta\psi_1 - \delta\psi_2)^2 \, dA$$

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$$U_n \frac{dq_n}{dy} > 0, \quad n = 1, 2$$

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$$U_1(U_1 - U_2) > 0$$

$$U_2(U_2 - U_1) > 0$$

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