Hamiltonian fluid dynamics and hydrodynamic stability

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Newtonian Dynamics

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- $\mathbf{F} = -\nabla\Pi$
- Second order ODE

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Hamiltonian Dynamics

- $\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$

- $\frac{d\mathbf{q}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}$

- System of first order ODEs

Fluids

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Navier-Stokes Equations

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Navier-Stokes Equations

- $\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \sigma + \mathbf{F}$
- $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u} = 0$
- System of coupled, nonlinear, PDEs

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- System of coupled, nonlinear, PDEs

Hamiltonian

- $\frac{d\mathbf{p}}{dt} = -\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$

- $\frac{d\mathbf{q}}{dt} = \frac{\delta \mathcal{H}}{\delta \mathbf{p}}$

- More analytical tools to work with

Hamiltonian Fluid Dynamics

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System of PDEs

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$$\mathbf{q}_t = \mathbf{J}\frac{\delta \mathcal{H}}{\delta \mathbf{q}}$$

Variational Derivatives

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$$\mathcal{F}(\mathbf{q}) = \int_{\Omega} F(\mathbf{q}) \, d\mathbf{x}$$

Variational Derivatives

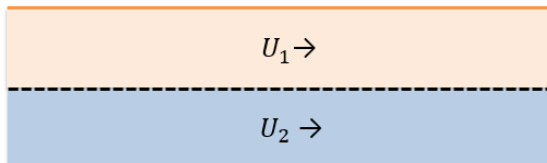$$\mathcal{F}(\mathbf{q}) = \int_\Omega F(\mathbf{q}) \, d\mathbf{x}$$

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \mathcal{F}(\mathbf{q} + \epsilon \delta \mathbf{q}) \equiv \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \delta \mathbf{q} \right\rangle \equiv \int_\Omega \frac{\delta \mathcal{F}}{\delta \mathbf{q}} \delta \mathbf{q} \, d\mathbf{x}$$
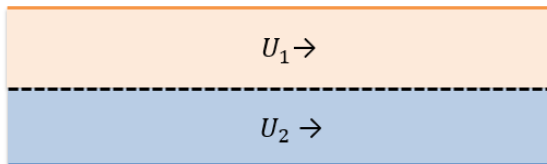
Poisson Bracket

$$\frac{\partial \mathcal{F}}{\partial t} = \int_{\Omega} \frac{\partial F}{\partial t}\, d\mathbf{x}$$

$$= \int_{\Omega} \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^{\top} \mathbf{q}_t \, d\mathbf{x}$$

$$= \int_{\Omega} \frac{\delta \mathcal{F}}{\delta \mathbf{q}}^{\top} \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \, d\mathbf{x}$$

$$= \left\langle \frac{\delta \mathcal{F}}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{q}} \right\rangle$$

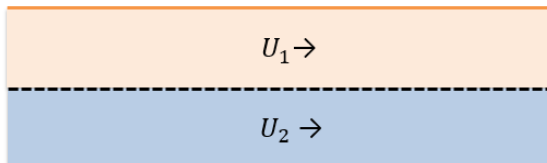$$\equiv \{\mathcal{F}, \mathcal{H}\}$$

Quasigeostrophic Model

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$$\psi_n = -U_n y$$

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$q_1 = \frac{1}{\alpha_1} \nabla^2 \psi_1 + (\psi_2 - \psi_1) = (U_1 - U_2) y$

$q_2 = \frac{1}{\alpha_2} \nabla^2 \psi_2 + (\psi_1 - \psi_2) = (U_2 - U_1) y$

Quasigeostrophic Model

$$0 = \frac{\partial q_1}{\partial t} + D(\psi_1, q_1)$$

$$0 = \frac{\partial q_2}{\partial t} + D(\psi_2, q_2)$$

$$D(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

Quasigeostrophic Model

$$0 = \frac{\partial q_1}{\partial t} + D(\psi_1, q_1)$$

$$0 = \frac{\partial q_2}{\partial t} + D(\psi_2, q_2)$$

$$D(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

$$\mathcal{H} = \frac{1}{2}\iint \sum_{n=1}^{2}\left[\frac{1}{\alpha_n}\vec{\nabla}\psi_n \cdot \vec{\nabla}\psi_n + 2C_n(q_n)\right] + (\psi_1 - \psi_2)^2 \, dA$$

Second Variation

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$$\delta^2 \mathcal{H} = \iint \sum_n \left[\frac{1}{\alpha_n} \vec{\nabla} \delta \psi_n \cdot \vec{\nabla} \delta \psi_n + \psi_n' \cdot (\delta q_n)^2 \right] + (\delta \psi_1 - \delta \psi_2)^2 \, dA$$

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$$U_n \frac{dq_n}{dy} > 0, \quad n = 1, 2$$

Second Variation

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$$U_1(U_1 - U_2) > 0$$

$$U_2(U_2 - U_1) > 0$$

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