Optimization Techniques

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Outline

Functions

Function Optimization

Maxima and Minima

Unconstrained Optimization

Constrained Optimization

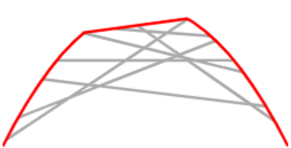
Convex and Concave Functions

- The second derivative of the function depicts how the function is curved, unlike the first derivative which tells us about the slope of the tangent function. A function that has an increasing first derivative bends upwards and is known as a convex function. On the other hand, a function, that has a decreasing first derivative is known as a concave function and bends downwards.

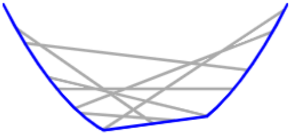
Theorem :

- If the function f and its derivative f' can be differentiated at a, then:
  - The f is convex at a if $f''(a) > 0$
  - The f is concave at a if $f''(a) < 0$

Convex and Concave Functions



A concave function:
no line segment joining
two points on the graph
lies above the graph
at any point



A convex function:
no line segment joining
two points on the graph
lies below the graph
at any point

A function that is neither
concave nor convex:
the line segment shown lies
above the graph at some
points and below it at others

Example

- Identify the curve of the following function and determine whether it is a concave or a convex function:

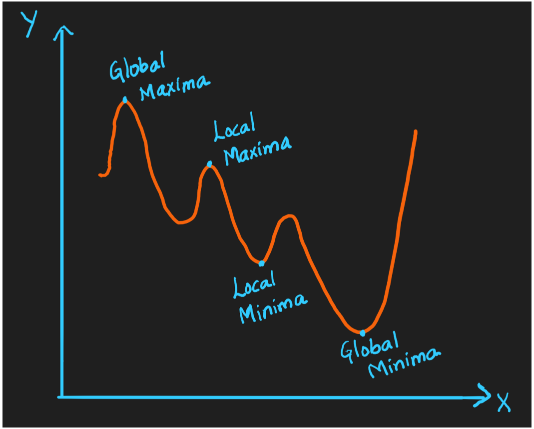
$$f(x) = 3x^2 + 7x - 9$$

Function Optimization

- Function optimization is a widely used tool bag of techniques employed in practically all scientific and engineering disciplines.
- Practically, function optimization describes a class of problems for finding the input to a given function that results in the minimum or maximum output from the function.
- Function Optimization involves three elements: the input to the function (e.g. x), the objective function itself (e.g. f()) and the output from the function (e.g. cost).
  - Input (x): The input to the function to be evaluated, e.g. a candidate solution.
  - Function (f()): The objective function or target function that evaluates inputs.
  - Cost: The result of evaluating a candidate solution with the objective function, minimized or maximized.

Maxima and Minima

- Maxima is the largest and Minima is the smallest value of a function within a given range. We represent them as below:



Maxima and Minima

- Global Maxima and Minima: It is the maximum value and minimum value respectively on the entire domain of the function
- Local Maxima and Minima: It is the maximum value and minimum value respectively of the function within a given range.
- There can be only one global minima and maxima but there can be more than one local minima and maxima.

Example

- Find the maximum of the function $f(x) = x^4 - 8x^2 + 3$ on the interval [-1, 3].

Unconstrained Optimization

- Unconstrained optimization problems consider the problem of minimizing an objective function that depends on real variables with no restrictions on their values.
- Mathematically, let $x \epsilon n$ be a real vector with $n \geq 1$ components and let $f : R^n \rightarrow R$ be a smooth function. Then, the unconstrained optimization problem is :

$$min_x f(x) \tag{1}$$

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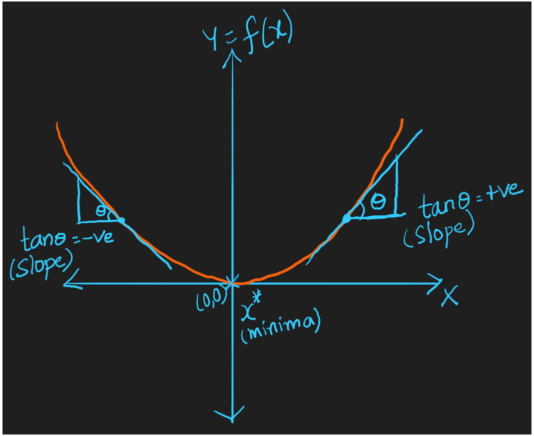
Gradient Descent

- Gradient Descent is an optimization algorithm and it finds out the local minima of a differentiable function. It is a minimization algorithm that minimizes a given function.

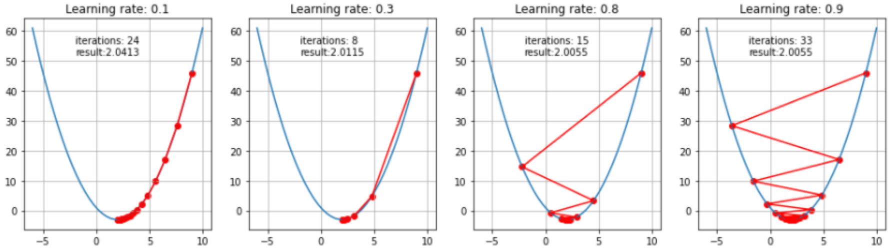


Gradient Descent

- Gradient descent algorithm does not work for all functions. There are two specific requirements. A function has to be:
  - differentiable
  - convex
- Gradient Descent method's steps are:
  1. choose a starting point (initialisation)
  2. calculate gradient at this point
  3. make a scaled step in the opposite direction to the gradient (objective: minimise)
  4. repeat points 2 and 3 until one of the criteria is met:
     - maximum number of iterations reached
     - step size is smaller than the tolerance.

Learning Rate

- If the learning rate is too high, we might OVERSHOOT the minima and keep bouncing, without reaching the minima.
- If the learning rate is too small, the training might turn out to be too long.

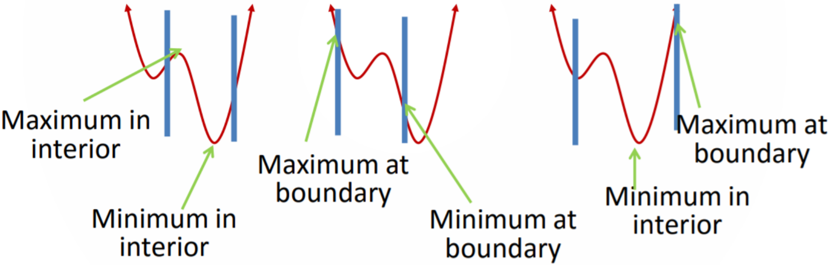


Learning Rate

- Learning rate is optimal, model converges to the minimum
- Learning rate is too small, it takes more time but converges to the minimum
- Learning rate is higher than the optimal value, it overshoots but converges.
- Learning rate is very large, it overshoots and diverges, moves away from the minima, performance decreases on learning

Constrained Optimization

- In the previous slides, most of the functions we examined were unconstrained, meaning they either had no boundaries, or the boundaries were soft.
- A constraint is a hard limit placed on the value of a variable, which prevents us from going forever in certain directions.



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Examples

- If you are attempting to maximize the objective function, typical constraints might involve time, money, and resources. The amounts of these things are limited, and these limits also place limits on the best possible value of the objective function.
- if a single gizmo costs 4, then two gizmos cost 8, five gizmos cost 20, and g gizmos cost 4g. If you buy g gizmos at 4 and s sprockets at 2, then your total cost will be 4g + 2s. If you only have 70 dollars to spend at the gizmo-and sprocket store, then your total cost must be

$$4g + 2s \leq 70 \tag{2}$$

Graphical Method

- An airline offers coach and first-class tickets. For the airline to be profitable, it must sell a minimum of 25 first-class tickets and a minimum of 40 coach tickets. The company makes a profit of 225 dollar for each coach ticket and 200 dollar for each first-class ticket. At most, the plane has a capacity of 150 travelers. How many of each ticket should be sold in order to maximize profits?

Fundamental Theorem of Linear Programming

- If a solution exists to a bounded linear programming problem, then it occurs at one of the corner points.
- If a feasible region is unbounded, then a maximum value for the objective function does not exist.
- If a feasible region is unbounded, and the objective function has only positive coefficients, then a minimum value exist.

Thank You