01NAEX - Lecture 01
Introduction to Design of Experiments (DOE)

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What DOE is and why it matters

Experiment: A test or a series of tests where purposeful changes are made to input variables to observe and identify reasons for changes in the output response.

DOE: The process of planning experiments so that data can be analyzed by statistical methods, resulting in valid, objective, and meaningful conclusions.

All experiments are designed experiments, some are poorly designed, some are well-designed.

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All experiments are designed experiments, some are poorly designed, some are well-designed.

Q: Name a poorly designed experiment you have run or seen.

DOE

- is a methodology for systematically applying statistics to experimentation.
- lets experimenters develop a mathematical model that predicts how input variables interact to create output variables or responses in a process or system.
- can be used for a wide range of experiments for various purposes including nearly all fields of engineering and even in business marketing.

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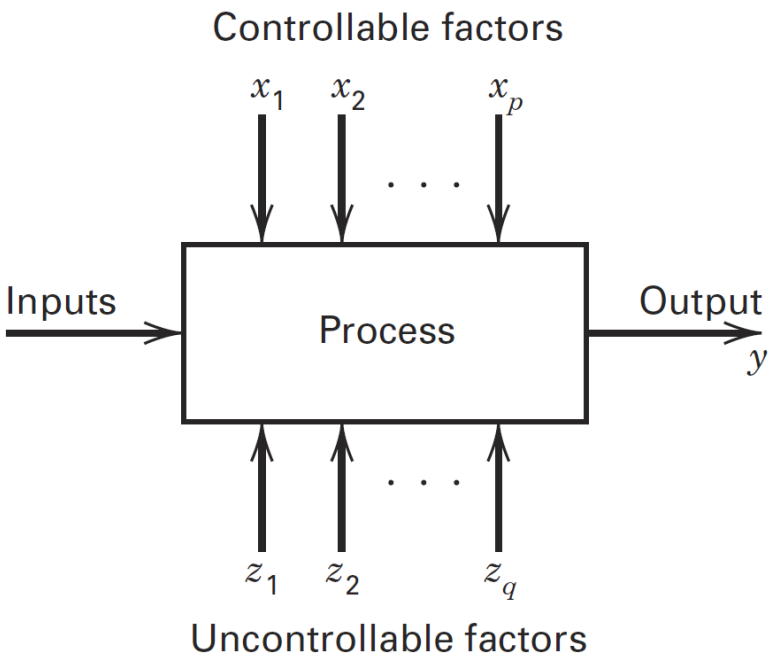
Q: Where would DOE help you or save you the most time this year?

General model of a process or system

The process (combination of operations, machines, methods, people, etc.) or system can be represented by the model that transforms some input into an output that has one or more observable response variables.



The objectives of the experiment:

- Determining which variables are most influential on the response y.
- Determining where to set the influential x's so that y is almost always close to the desired nominal value.
- Determining where to set the influential x's so that variability in y is small.
- Determining where to set the influential x's so that the effects of the noise variables z's are minimized.

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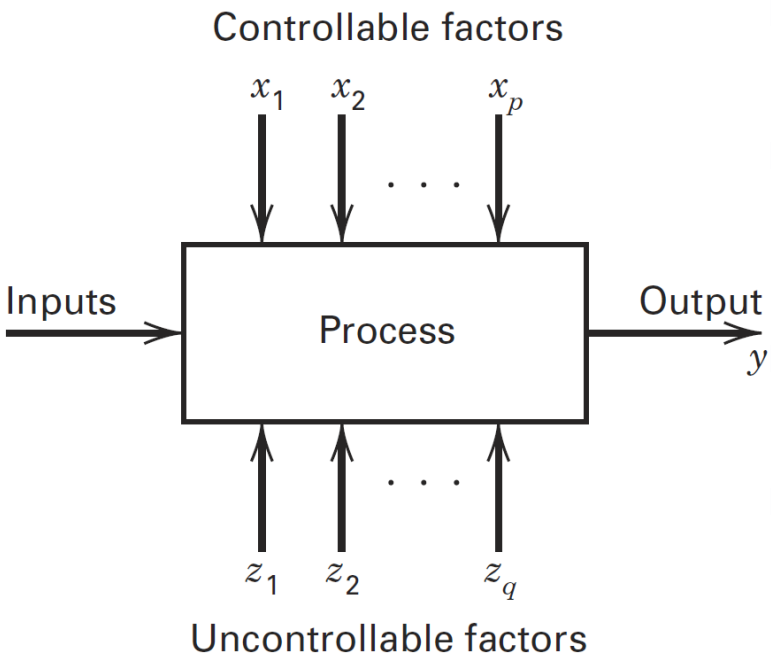

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N: My y is happiness; x is coffee; z is time, what process is yours?

Short motivation example

Imagine you're trying to weigh your luggage before a flight with moody scale to avoid extra fees.

Weighting problem: we want to weight 3 different items A, B, and C.

Design 1: Standard weighing procedure each measurement for one parameter

No. measurement	Notation	Meaning	Model for responses
1	(1)	No item	$(1) = \mu + \varepsilon_1$
2	a	item A	$a = \mu + A + \varepsilon_2$
3	b	item B	$b = \mu + B + \varepsilon_3$
4	c	item C	$c = \mu + C + \varepsilon_4$

Where μ is offset of weighting device, A, B, C are weights of items A,B,C, and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are measurement errors with standard deviation σ_ε.

The estimate of A is given by $\hat{A}^{(1)} = a - (1) = A + \varepsilon_2 - \varepsilon_1$ and the corresponding for B and C.

Short motivation example

Weighting problem: we want to weight 3 different items A, B, and C.

Design 2: Alternative weighing procedure uses all measurements for all parameters

No. measurement	Notation	Meaning	Model for responses
1	(1)	No item	$(1) = \mu + \varepsilon_5$
2	ac	item A and C	$ac = \mu + A + C + \varepsilon_6$
3	bc	item B and C	$bc = \mu + B + C + \varepsilon_7$
4	ab	item A and B	$ab = \mu + A + B + \varepsilon_8$

The estimate of A is now given by $\hat{A}^{(2)} = \frac{-(1)+ac-bc+ab}{2} = A + \frac{-\varepsilon_5+\varepsilon_6-\varepsilon_7+\varepsilon_8}{2}$ and the corresponding for B and C.

$$
\begin{aligned}
Var\left[\hat{A}^{(1)}\right] &= 2\sigma_\varepsilon^2 \\
Var\left[\hat{A}^{(2)}\right] &= \sigma_\varepsilon^2
\end{aligned}
$$

The second design is more precise than the first one, since in the first design not all measurements are used to estimate all parameters, which is the case in the second design.

How does beer and brewery relate to DOE?

Did you know that the origins of modern DOE are linked to beer?

- Back in the early 1900s, William Sealy Gosset, an employee at the Guinness Brewery in Dublin, was trying to improve the quality of stout, things like consistency of bitterness, alcohol content, and raw material selection. Because industrial trials were expensive and limited, he needed methods that still worked with small samples.
- Due to limited resources, he developed small sample statistical methods and his p-values were always stout.
- What was the name of the test that he used and why?

Short history of DOE

- **The agricultural origins, 1920s - 1940s:**
  - Sir R. A. Fisher & his co-workers.
  - Lady tasting tea, The Design of Experiments book (1935)
  - Factorial designs, ANOVA.
- **The first industrial era, 1950s - 1970s:**
  - Box & Wilson, Response surface methodology.
  - Applications in chemical and process industries, finance, etc.
  - Total quality control, Statistical process control.
- **The second industrial era, 1970s - 1990s:**
  - Quality control, Japanese style of TQC.
  - Taguchi and robust parameter design, process robustness.
  - Total Quality Management, Continuous Quality Improvement.
- **The modern era, 1990s - till today:**
  - Wide use of computer technology and experiments in DOE.
  - Expanded use of DOE in Six-Sigma and in business.
  - All sectors of the economy are more competitive and use simulation-based experiments, hyperparameter tuning, and ML/AI workflows.

The basic principles of design of experiments: Randomization

Think of randomization like shuffling a deck of cards before dealing. If you don't shuffle well, someone might end up with all the aces, and that's no fun. Unless you're the one getting the aces.

Randomization:

- Cornerstone underlying the use of statistical methods.
- Running trials in random order to balance out the effects of hidden variables.
- The allocation of the experimental material is random.
- Reduce bias and systematic errors.

The basic principles of design of experiments: Replication

Replication is like trying a new flavor of ice cream multiple times. You know, just to be sure it's really your new favorite!

Replication:

- An independent repeat of each factor combination.
- It allows us to estimate experimental error and improves the precision of our results.
- More data (sample size) gives a better estimate, reducing the impact of any one "odd" result.

The basic principles of design of experiments: Blocking

Blocking is like organizing your wardrobe by seasons. You can't control the weather, but at least you won't end up wearing a swimsuit in a snowstorm!

Blocking:

- It is a design technique used to improve the precision.
- Reduce or eliminate the variability of nuisance factors.
- Increases the precision of the experiment by accounting for known sources of variability.

Strategy of Experimentation

- **Best-guess approach:**
  - Wide-spread, trial and error experiments.
  - More successful than you might suspect.
  - **Disadvantage:**
    - cannot guarantee best solution has been found.
- **One-factor-at-a-time (OFAT) approach:**
  - Sometimes associated with the scientific or engineering method.
  - **Disadvantage:**
    - risks *missing interactions* $\Rightarrow$ slow or wrong direction.
    - very inefficient, requires many test runs.
- **Statistically designed approach:**
  - Based on Fisher's factorial concept $\Rightarrow$ main effects & interactions in *one shot*.
  - Modern and most efficient approach.
  - Can determine how factors interact.
  - Best-guess $+$ iteration $\neq$ factorial.

Factorial Designs

Treatment x Nuisance factors

- Treatment factor - factor of primary interest to us.
- Nuisance factor - factor that is not our primary focus, but we have to deal with them.

Experimental x Classification factors

- Experimental factor - we can specify them and set the levels.
- Classification factor - we can not change or randomly assign them.

Quantitative x Qualitative factors

- Quantitative factor - we can assign any specified level of them.
- Qualitative factor - have categories which are different types.

N: Treatment=additive, Nuisance=operator; Experimental=temp; Classification=machine; Quantitative=pressure; Qualitative=brand.

Guidelines for Designing Experiments

1. **Recognition and statement of problem:**
   - Obvious, but in practice not so simple point.
   - It is necessary to develop all the ideas about the objectives of the experiment, to obtain input from all involved parties, and to use a team approach (PI, operator, analyst, QA).
2. **Choice of factors, levels, and ranges:**
   - The potential design factors are those factors that we wish to vary in the experiment .
   - Classify factors as: design factor, held-constant factor, allow-to-vary factors, nuisance factors.
   - Nuisance factors may have large effects that must be accounted for (controllable, uncontrollable, noise).
   - Need to use engineering judgment or prior test results.
3. **Selection of the response variable(s):**
   - Be sure that the variable really provides useful information about the process under study.
   - Multiple responses are not unusual.

Guidelines for Designing Experiments

4. **Choice of experimental design:**
   - Involves consideration of sample size (no. of replicates), selection of run order, whether blocking or randomization restriction, software to use etc
5. **Performing the experiment:**
   - Vital to monitor the process carefully
   - Up/for planning is crucial to success
   - Easy to underestimate logistical and planning aspects of running and design experiment in development environment
6. **Statistical analysis of the data:**
   - Should be used to analyze the data so that results and conclusions are objective rather than judgmental in nature
   - Use simple graphics whenever possible
7. **Drawing conclusions and recommendations:**
   - Follow-up test runs and confirmation testing to validate the conclusions from the experiment

Useful advices in Planning, Conducting & Analyzing an Experiment

Use your **non-statistical knowledge** of the problem, physical laws, background knowledge, etc. It is crucial to success.

KISS principle (Keep it simple - Stupid!, Keep it simple and straightforward, Keep it short and simple). Don't use complex, sophisticated statistical techniques. If design is good, analysis is relatively straightforward. If design is bad - even the most complex and elegant statistics cannot save the situation.

Think and experiment **sequentially**, pre-experimental planning is vital.

Recognize the difference between **practical and statistical significance**.

Q: Give me an example of practical vs statistical significance.

CHAPTER 2

Short introduction to:

- probability and statistics
- plots of distributions
- CLT and Low of LN
- tests and probability plots
- QQ plots, checking assumptions.
- ...

General definitions from probability - probability distribution

Assume, probability triplet $(\Omega, \mathcal{F}, p)$ and a random variable, say y.

Probability distribution: If y is discrete, we call the $p(y)$ the probability mass function of y. If y is continuous, we call $f(y)$ the probability density function for y.

Mean, expected value:

$$\mu = E(y) \begin{cases} \int_{-\infty}^{+\infty} y f(y) dy & y \text{ is continuous random variable} \\ \sum_{\text{all } y} y p(y) & y \text{ is discrete random variable} \end{cases}$$

Variance, standard deviation:

$$\sigma^2 = Var(y) \begin{cases} \int_{-\infty}^{+\infty} (y-\mu)^2 f(y) dy & y \text{ is continuous random variable} \\ \sum_{\text{all } y} (y-\mu)^2 p(y) & y \text{ is discrete random variable} \end{cases}$$

General definitions: notation that we will use

If the population contains N elements and a sample of n of them is to be selected, and if each of the possible samples has an equal probability of being chosen, then the procedure employed is called random sampling.

Suppose that $y_1, y_2, \ldots y_n$ represents a sample, we define two statistics:
Sample mean:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

The sample mean $\bar{y}$ is an unbiased point estimator of the population mean μ
Sample variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

The sample variance S^2 is a unbiased point estimator of the population variance σ^2. Sometimes $S = \sqrt{S^2}$ is called the **sample standard deviation**. The expression

$$SS = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

is called the corrected **sum of squares** of the observations y_i.

General definitions from probability - Degrees of Freedom

Number of degrees of freedom refers to number of independent elements in sum of squares SS:

$$E(SS) = E\left[\sum_{i=1}^{n}(y_i - \bar{y})^2\right] = (n-1)\sigma^2.$$

The degrees of freedom associated with a sum-of-squares is the degrees-of-freedom of the corresponding component vectors.

Only $(n-1)$ elements from the set $(y_1 - \bar{y}, y_2 - \bar{y}, \ldots, y_n - \bar{y})$ are independent.

General definitions from probability - Normal distribution, χ^2 distribution

Let y be a random variable with normal distribution then the probability density function of y is

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}$$

where $\mu \in R$ is the mean of the distribution, $\sigma^2 > 0$) is the variance.

If $x_1, x_2, \ldots, x_k$ are *iid* random variables with standard normal distribution then $y = x_1^2 + x_2^2 + \ldots + x_k^2$ is a random variable with χ^2 distribution with k degrees of freedom, denoted χ_k^2 and the probability density function of y is

$$f(y) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} y^{\frac{k}{2}-1} e^{-\frac{y}{2}}$$

The distribution is skewed with mean $\mu = k$ and variance $\sigma^2 = 2k$.

General definitions from probability - t distribution

If z and χ^2_k are independent standard normal and chi-squared r.v. then

$$t_k = \frac{z}{\sqrt{\frac{\chi^2_k}{k}}}$$

is r.v. with t distribution with k degrees of freedom and the probability density function of t_k is

$$f(y) = \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})} \frac{1}{(\frac{t^2}{k}+1)^{\frac{k+1}{2}}}$$

The distribution is symmetric with mean $\mu = 0$ and variance $\sigma^2 = \frac{k}{k-2}$ for $k > 2$
If $k = \infty$, the t distribution becomes the standard normal distribution.

General definitions from probability - F distribution

If χ^2_k and χ^2_l are two independent chi-squared r.v. with k and l degrees of freedom then the ratio

$$F_{k,l} = \frac{\frac{\chi^2_k}{k}}{\frac{\chi^2_l}{l}}$$

follows the $F_{k,l}$ distribution.

F distribution is very important for us, suppose we have two independent normal populations with common variance σ.
If $y_{11}, y_{12}, \ldots, y_{1n_1}$ is a random sample of n_1 observations from first population with sample variance S_1^2 and $y_{21}, y_{22}, \ldots, y_{2n_2}$ is a random sample of n_2 observations from second population with sample variance S_2^2 then

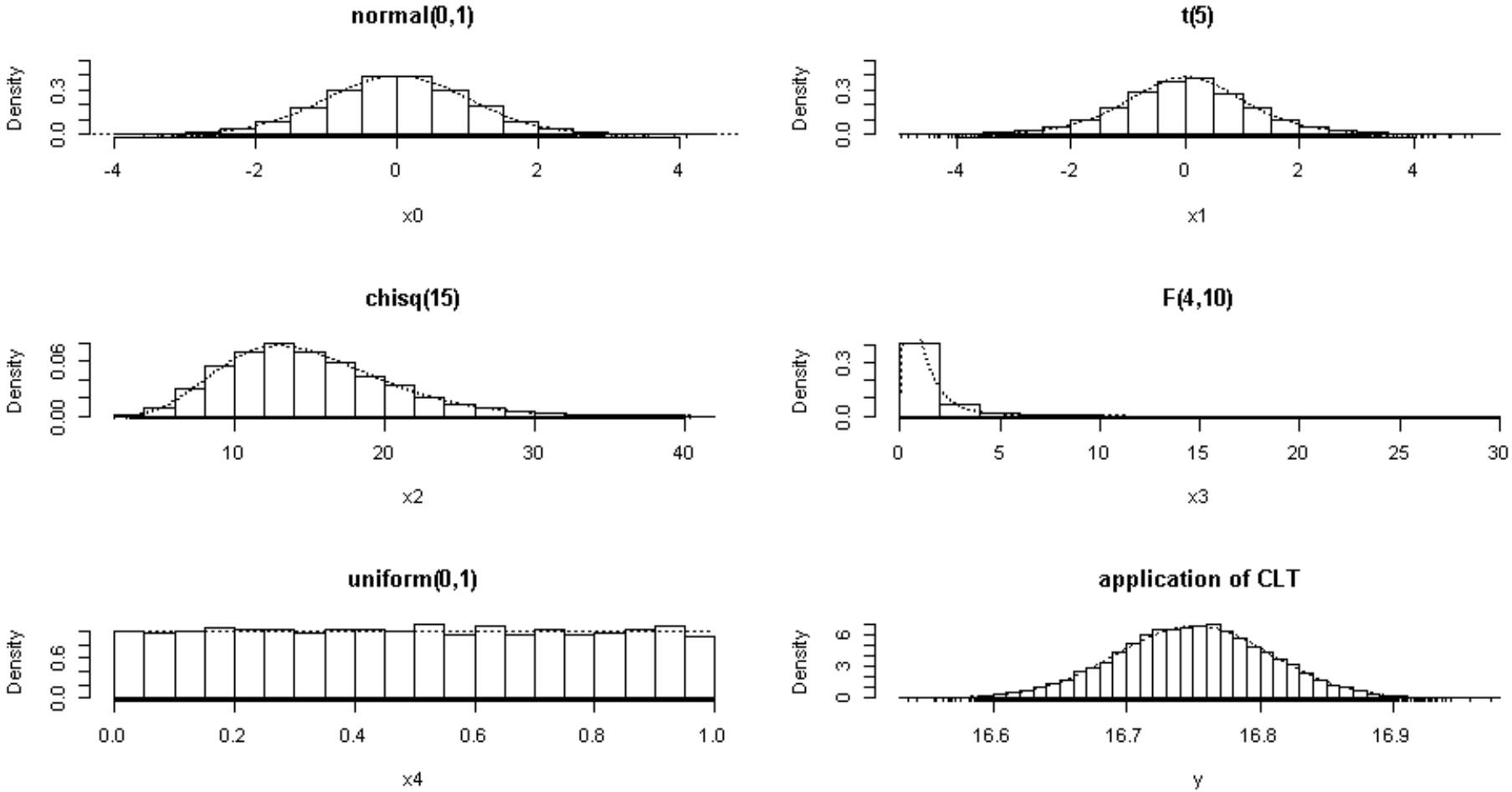
$$\frac{S_1^2}{S_2^2} \sim F_{n_1-1,n_2-1}.$$

The result comes from following

$$\frac{SS}{\sigma^2} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sigma^2} \sim \chi^2_{n-1}.$$

N: Caution: F-test is sensitive to non-normality.

General definitions from probability - CLT



N: One sentence story: sums/averages of i.i.d. variables look Normal quickly.

Q: What breaks CLT?

General definitions from probability - Hypothesis testing

A statistical hypothesis is a statement about the parameters of one or more populations.
Suppose that we partition the parameter space Θ into two disjoint sets Θ_0 and Θ_1 and that we wish to test:

The null hypothesis	H_0 :	$\theta \in \Theta_0$	*versus*
The alternative hypothesis	H_1 :	$\theta \in \Theta_1$	

To test a hypothesis, we devise a procedure for taking a random sample, computing an appropriate test statistics, and then rejecting or failing to reject the null hypothesis H_0

Decisions in Hypothesis Testing

Decision	H_0 Is True	H_0 Is False
Retain H_0	no error	type II error
Reject H_0	type I error	no error

Significance level: $\alpha = P(\text{type I error})$
Power: $1 - \beta = 1 - P(\text{type II error}) = P(\text{reject } H_0 \mid H_0 \text{ is false})$

Q: Does p-value $= 0.03$ means $P(H_0 \text{ is true}) = 0.03$

General definitions from probability - p-value

The p-value is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

- The p-value is the most commonly reported statistic resulting from hypothesis testing.
- The significance level α is a threshold for deciding whether to reject H_0. If the test is rejected at the significance level α, it will be also rejected at the level $\alpha' > \alpha$.
- The p-value is a measure of the evidence against H_0 (the smaller the p-value, the stronger evidence against H_0).
- The p-value in not the probability that H_0 is true (The p-value is calculated under the assumption that H_0 is true.)

$$\text{p-value} \neq P(H_0 \mid \text{Data})$$

$$\text{p-value} = P(\text{Data as extreme or more extreme than observed} \mid H_0)$$

- 1 - (p-value) is not the probability of the alternative hypothesis being true.
- The significance level of the test is not determined by the p-value.

Q: If p-value $= 0.049$ vs p-value $= 0.051$, should management act differently?

General definitions from probability - Test on means with variance known

Tests on Means with Variance Known

Hypothesis	Test Statistic	Fixed Significance Level Criteria for Rejection	*P*-Value
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	$Z_0 = \dfrac{\bar{y} - \mu_0}{\sigma/\sqrt{n}}$	$\lvert Z_0 \rvert > Z_{\alpha/2}$	$P = 2[1 - \Phi(\lvert Z_0 \rvert)]$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$		$Z_0 < -Z_\alpha$	$P = \Phi(Z_0)$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$Z_0 > Z_\alpha$	$P = 1 - \Phi(Z_0)$
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	$Z_0 = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$	$\lvert Z_0 \rvert > Z_{\alpha/2}$	$P = 2[1 - \Phi(\lvert Z_0 \rvert)]$
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 < \mu_2$		$Z_0 < -Z_\alpha$	$P = \Phi(Z_0)$
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 > \mu_2$		$Z_0 > Z_\alpha$	$P = 1 - \Phi(Z_0)$

General definitions from probability - Test on means with variance unknown

Tests on Means of Normal Distributions, Variance Unknown

Hypothesis	Test Statistic	Fixed Significance Level Criteria for Rejection	*P*-Value
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	$t_0 = \dfrac{\bar{y} - \mu_0}{S/\sqrt{n}}$	$\lvert t_0 \rvert > t_{\alpha/2,n-1}$	sum of the probability above t_0 and below $-t_0$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$		$t_0 < -t_{\alpha,n-1}$	probability below t_0
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$t_0 > t_{\alpha,n-1}$	probability above t_0
	if $\sigma_1^2 = \sigma_2^2$		
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	$t_0 = \dfrac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ $v = n_1 + n_2 - 2$	$\lvert t_0 \rvert > t_{\alpha/2,v}$	sum of the probability above t_0 and below $-t_0$
	if $\sigma_1^2 \neq \sigma_2^2$		
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 < \mu_2$	$t_0 = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$	$t_0 < -t_{\alpha,v}$	probability below t_0
$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 > \mu_2$	$v = \dfrac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{(S_1^2/n_1)^2}{n_1 - 1} + \dfrac{(S_2^2/n_2)^2}{n_2 - 1}}$	$t_0 > t_{\alpha,v}$	probability above t_0

General definitions from probability - Test on Variance of Normal Distributions

Tests on Variances of Normal Distributions

Hypothesis	Test Statistic	Fixed Significance Level Criteria for Rejection
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 \neq \sigma_0^2$	$\chi_0^2 = \frac{(n-1)S^2}{\sigma_0^2}$	$\chi_0^2 > \chi^2_{\alpha/2,n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 < \sigma_0^2$		$\chi_0^2 < \chi^2_{1-\alpha,n-1}$
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 > \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha,n-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	$F_0 = \frac{S_1^2}{S_2^2}$	$F_0 > F_{\alpha/2,n_1-1,n_2-1}$ or $F_0 < F_{1-\alpha/2,n_1-1,n_2-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 < \sigma_2^2$	$F_0 = \frac{S_2^2}{S_1^2}$	$F_0 > F_{\alpha,n_2-1,n_1-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 > \sigma_2^2$	$F_0 = \frac{S_1^2}{S_2^2}$	$F_0 > F_{\alpha,n_1-1,n_2-1}$

Q: Do you know any robust alternatives?

Simple Comparative Experiment

We consider experiments to compare two conditions (also called treatments, levels, factors, etc).

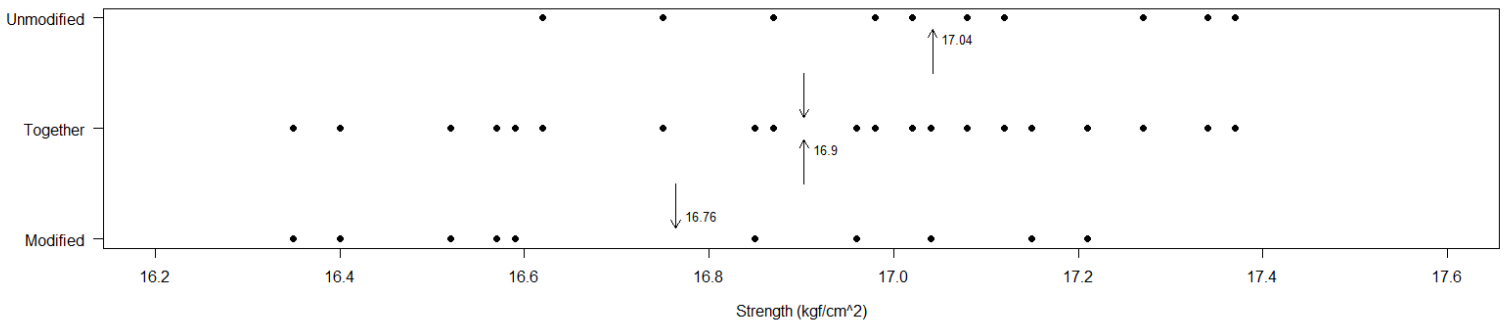
In the simple comparative experiment we use previous mentioned statistical concepts, such as random samples, sampling distributions, hypothesis testing etc.

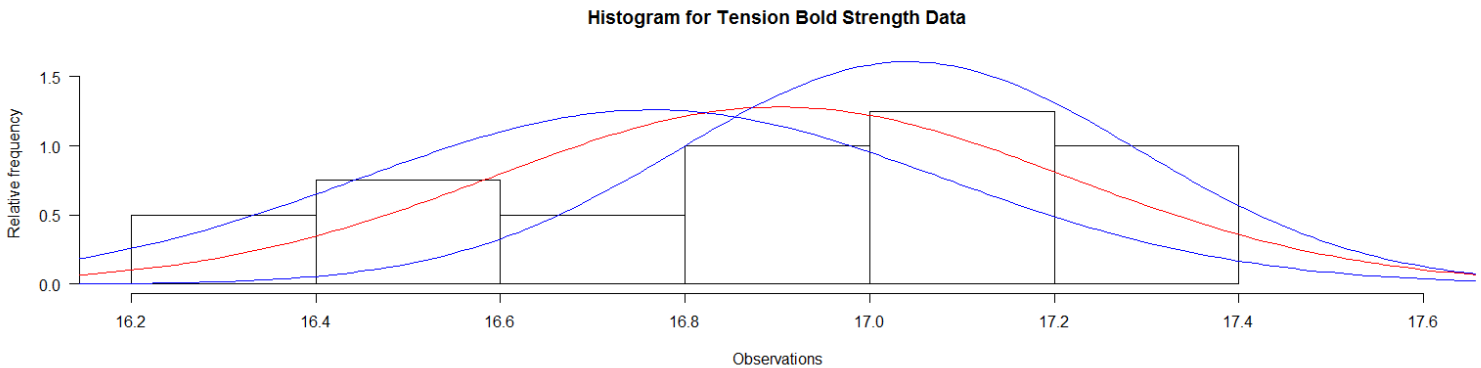
Simple Comparative Experiment - modified mortar with polymer additive

Tension Bond Strength Data
Portland Cement Data

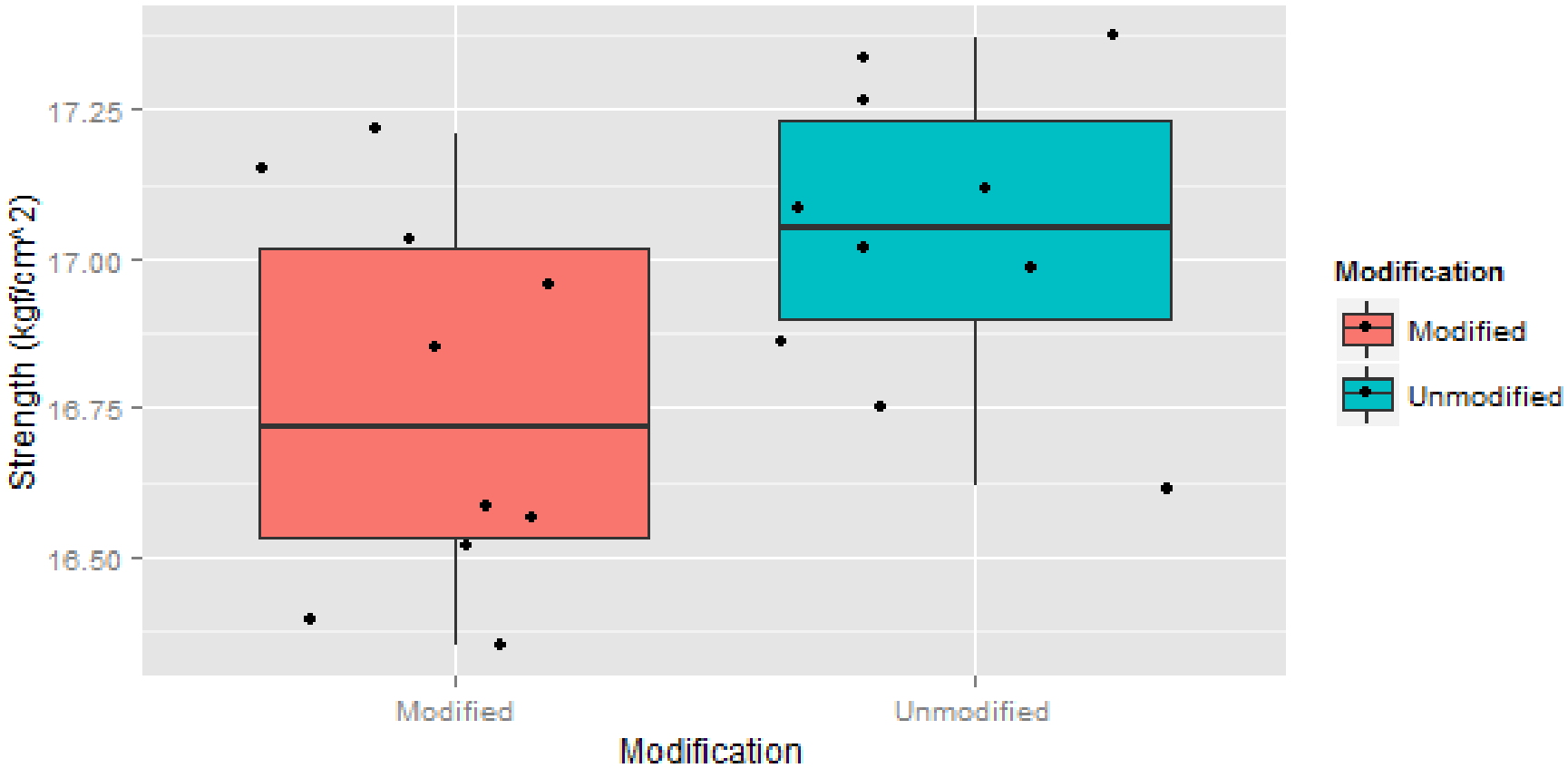
Observation j	Modified Mortar y_{1j}	Unmodified Mortar y_{2j}
1	16.57	18.15
2	16.59	17.96
3	17.15	17.90
4	16.96	18.22
5	17.04	17.75
6	16.52	17.86
7	16.35	18.00
8	17.21	18.25
9	16.40	17.63
10	16.85	17.50

Simple Comparative Experiment - data plot





Simple Comparative Experiment - boxplot



Simple Comparative Experiment - F-test

If we do not know the variances of the data sets, we have to test, if they are identical or not.

```
var.test(y1, y2,ratio = 1,alternative="two.sided",
conf.level = 0.95)
```

```
F test to compare two variances

data:  y1 and y2
F = 1.6293, num df = 9, denom df = 9, p-value = 0.4785
alternative hypothesis:
true ratio of variances is not equal to 1

90 percent confidence interval:
0.5125235 5.1792350
95 percent confidence interval:
0.4046845 6.5593806
99 percent confidence interval:
0.2490804 10.6571184

sample estimates:
ratio of variances 1.629257
```

Simple Comparative Experiment - The Two-Sample *t-test*

If we could assume that the variances of tension bond strengths are identical, or the sample *sd* are reasonably similar, we can use simple two-sample t-test to compare means.

The test statistic:

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = -2.186876,$$

where S_p is an estimate of common variance $\sigma^2 = \sigma_1^2 = \sigma_2^2$

$$S_p = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = 0.2842534.$$

Because $t_0 = -2.186876 < -t_{0.025,18} = -2.100922$, we would reject the H_0 hypothesis at the significance level $\alpha = 0.05$.

Simple Comparative Experiment - The Two-Sample *t-test*

Confidence interval on the true difference in means $\mu_1 - \mu_2$ for the Portland cement problem.

The statistic:

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

is distributed as $t_{n_1+n_2-2}$ and thus,

$$P(\bar{y}_1 - \bar{y}_2 - t_{\frac{\alpha}{2}, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{y}_1 - \bar{y}_2 + t_{\frac{\alpha}{2}, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}})$$

is $100(1-\alpha)$ percent confidence interval for $\mu_1 - \mu_2$.
For $\alpha = 0.05$ we obtain $-0.5450734 \leq \mu_1 - \mu_2 \leq -0.01092661$.

Simple Comparative Experiment - The Two-Sample *t-test*

If we could assume that the variances of tension bond strengths are identical, or the sample *sd* are reasonably similar, we can use simple two-sample t-test to compare means.

```
> t.test(y1,y2,alternative="two.sided",mu=0,
paired=F,var.equal=T,conf.level=0.95)

Two Sample t-test

data:  y1 and y2
t = -2.1869, df = 18, p-value = 0.0422
alternative hypothesis:
true difference in means is not equal to 0

95 percent confidence interval:
-0.54507339 -0.01092661
99 percent confidence interval:
-0.64391308 0.08791308

sample estimates:
mean of x:  16.764, mean of y:  17.042
```

Simple Comparative Experiment - t-test

If the variances are not identical, the classical test statistic is not distributed exactly as t and some approximation of degrees of freedom must be used.

```
> t.test(y1,y2,alternative="two.sided",mu=0,
paired=FALSE,var.equal=FALSE,conf.level=0.95)

Welch Two Sample t-test

data:  y1 and y2
t = -2.1869, df = 17.025, p-value = 0.043
alternative hypothesis:
true difference in means is not equal to 0

95 percent confidence interval:
-0.546174139 -0.009825861
99 percent confidence interval:
-0.64636235 0.09036235

sample estimates:
mean of x:  16.764, mean of y:  17.042
```

Simple Comparative Experiment - checking assumptions

In using classical t-test we make following assumptions:

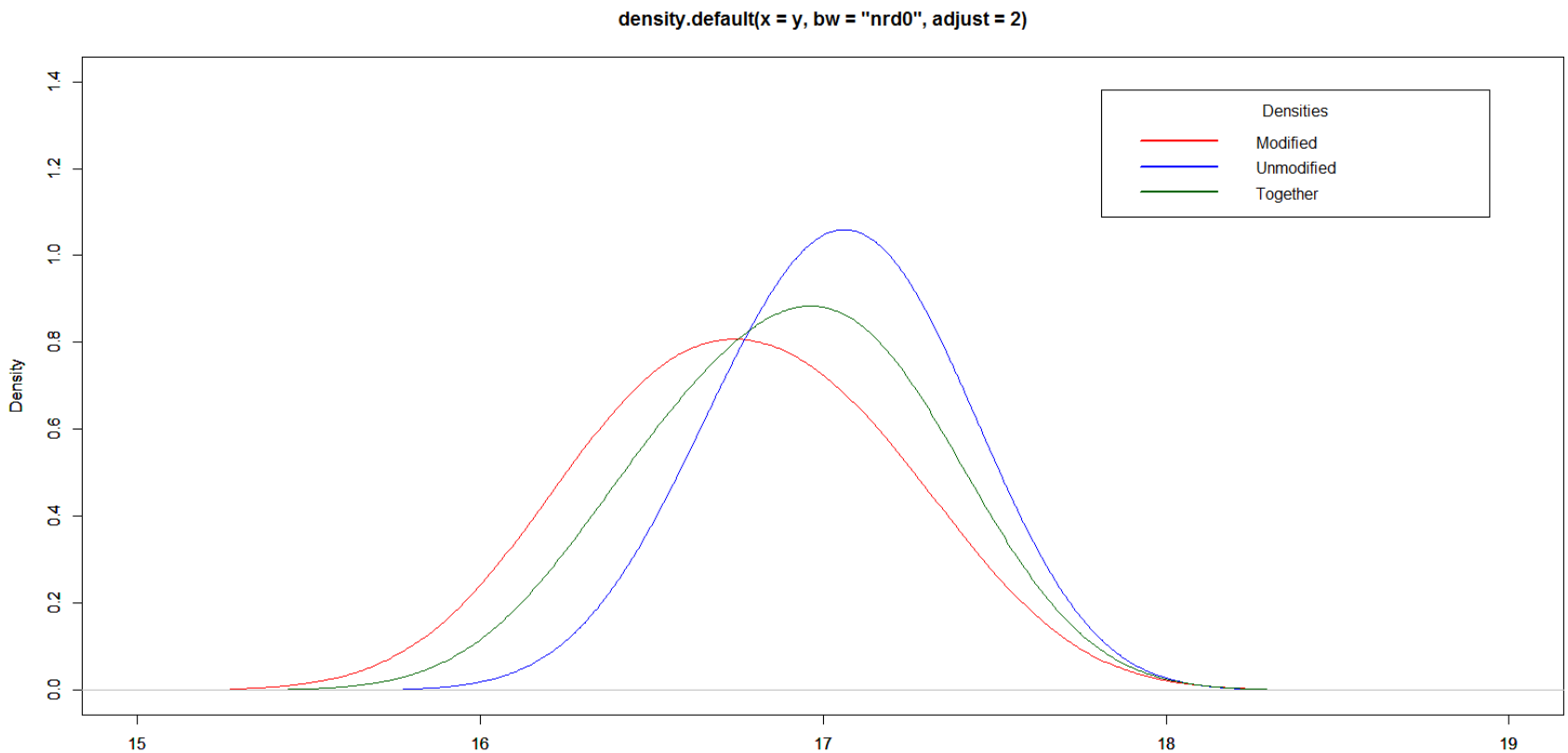
- observations are independent random variables.
- both samples are drawn from independent populations that are **normal distributed**.
- variances of both populations are equal.

Strategy for Checking Model Assumptions:

- check the form of the model.
- check for outliers.
- check for independence.
- check for constant variance.
- check for normality.

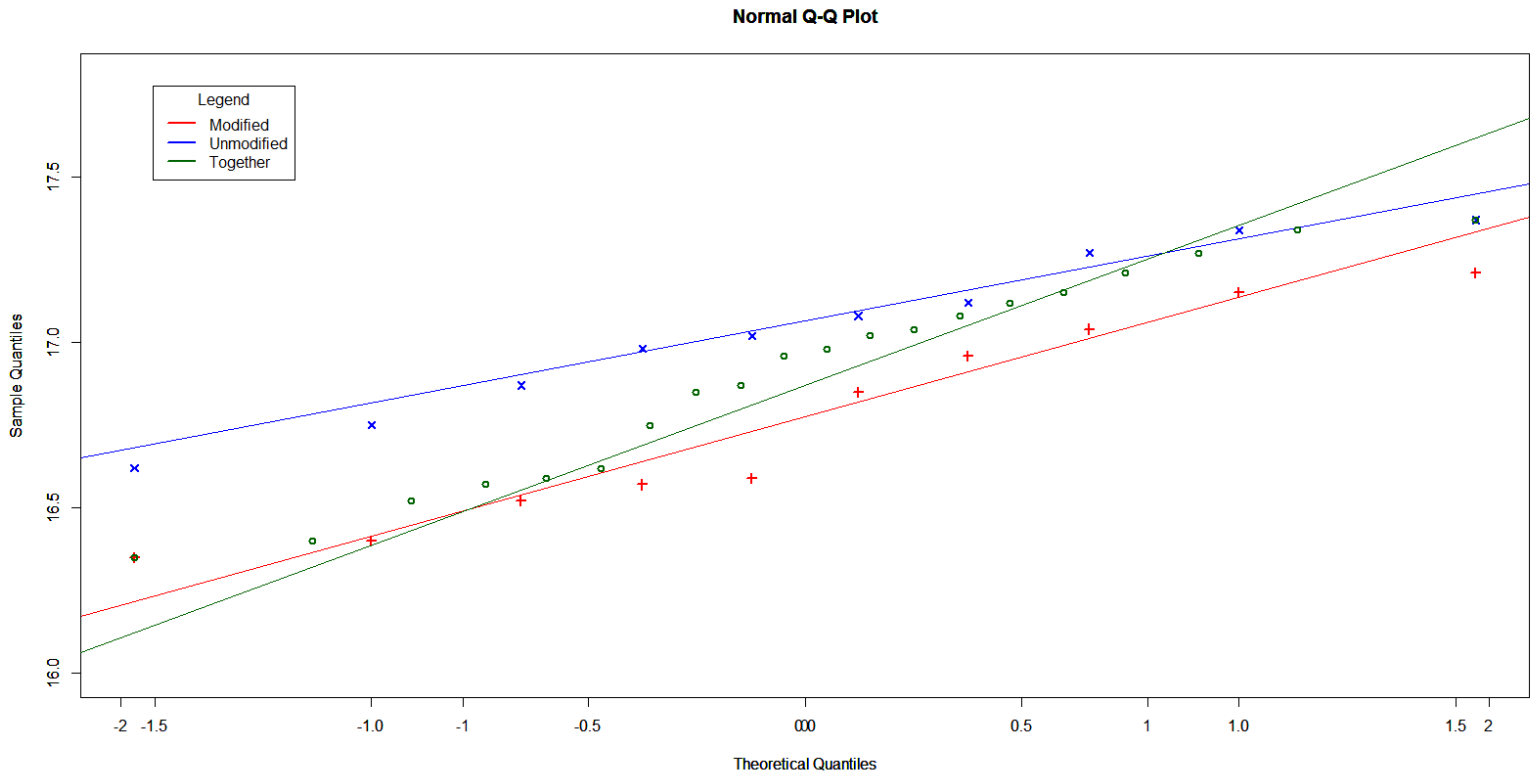
Simple Comparative Experiment - checking assumptions

Kernel density estimations of data sets



Simple Comparative Experiment - checking assumptions

The equal variance and normality assumptions are easy to check using **normal probability plot** (QQ-plot).



Test for the hypothesis of normality.

One sample Kolmogorov - Smirnov test tests the null hypothesis that a probability distribution of given sample is the same as a reference probability distribution.

Shapiro - Wilk normality test and **Anderson - Darling normality test** test the null hypothesis that a given sample came from a normally distributed population.

Hypothesis testing of normality

Data	Shapiro-Wilk test		Kolmogorov-Smirnov test	
	Statistics W	p-value	Statistics D	p-value
Modified	0.9186	0.3457	0.2088	0.7027
Unmodified	0.9626	0.8153	0.1211	0.9943
Together	0.9540	0.4326	0.1226	0.8891

Simple Comparative Experiment -Non-parametric alternative

For two-sample comparisons, two non-parametric tests can be used, depending on the way data are collected. If both sample are independent, we use Mann-Whitney-Wilcoxon rank sum test, while for paired sample the corresponding test is called Wilcoxon signed rank test.

Both are called using R function:

`wilcox.test`

and the option

`paired={TRUE|FALSE}.`

Choice of sample size

In hypothesis testing we eliminate the probability of type I error by setting the significance level α, but how to decrease the probability of type II error β?

In t-test, the Power depends on the significance level α, on the true difference in means δ and the sample size n.

Two-sample t test power calculation

Sample size calculation	Power calculation
n = 28.17	n = 10
delta = 0.28	delta = 0.28
sd = 0.28	sd = 0.28
sig.level = 0.05	sig.level = 0.05
power = 0.95	power = 0.54
alternative = two.sided	alternative = two.sided

```
#calculation of requiared sample size
power.t.test(power = .95,    delta = diff.t, sd = sigma.t,
sig.level = 0.05, type = "two.sample", alternative = "two.sided")
#calculation of the power of the test
power.t.test(n = 10, delta = diff.t, sd = sigma.t,
sig.level = 0.05, type = "two.sample", alternative = "two.sided")
```

Choice of sample size

Calculation of Power for equal but unknown populations SD's:

```
C     = qt(0.975, n1 + n2 - 2)
sd    = sigma*sqrt(1/n1 + 1/n2)
power = 1-pt(C, n1+n2-2, ncp=delta/se)+pt(-C, n1+n2-2, ncp=delta/sd)
```

We have four connected quantities:

- sample size;
- difference between means;
- significance level of the test;
- power of the test.

We can plot the graph of β (the probability of type II error) versus δ - the true difference in means. This curve is called Operating Characteristic curve (OC curve).

Choice of sample size

Operating characteristic curve (OC curve) for two-sided t-test, where $d = \frac{|\delta|}{2\sigma}$.

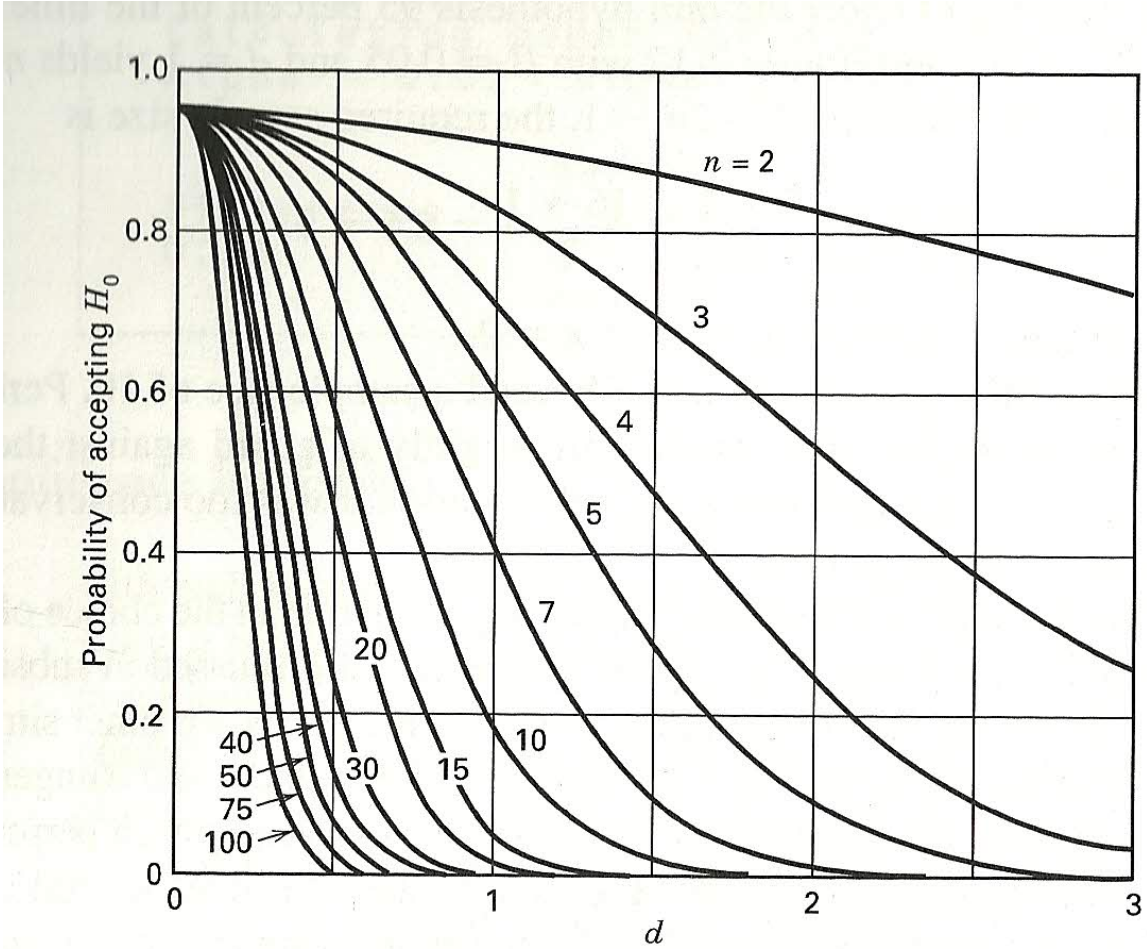
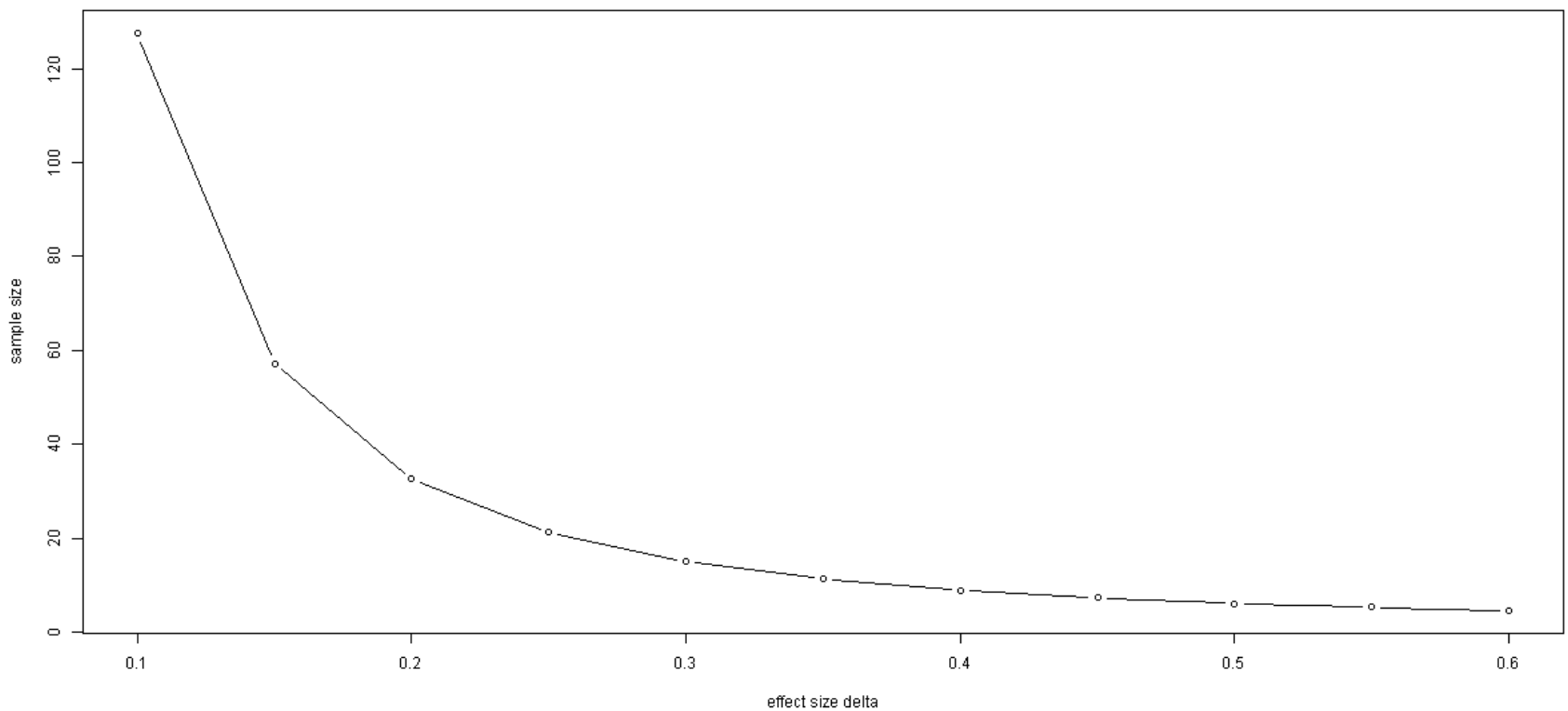


Figure 2-12 Operating characteristic curves for the two-sided t-test with $\alpha = 0.05$. (Reproduced with permission from "Operating Characteristics for the Common Statistical Tests of Significance," C. L. Ferris, F. E. Grubbs, and C. L. Weaver, *Annals of Mathematical Statistics*, June 1946.)

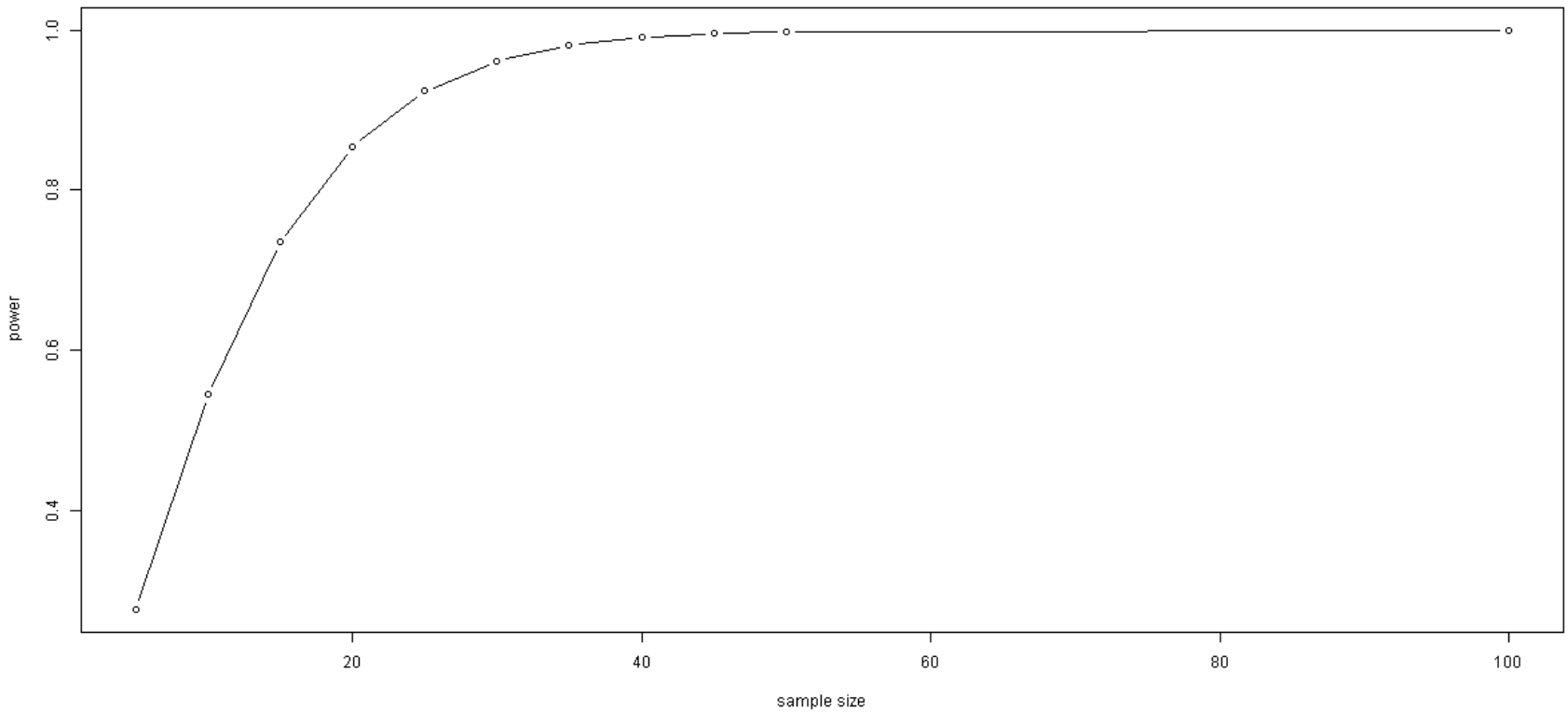
Choice of sample size - Sample curve

```
for (i in c(.1,.15,.2,.25,.3,.35,.4,.45,.5,0.55,0.6))
{
     pwrt<-power.t.test(sd=0.284,delta=i,power=.8,sig.level=.05,
     type="two.sample",alternative="two.sided")
     ptab<-rbind(ptab, cbind(pwrt$d, pwrt$n))
}
plot(ptab[,1],ptab[,2],type="b",xlab="effect size",ylab="sample size")
```



Choice of sample size - Power curve

```
pwrt<-power.t.test(delta=0.278, sd=0.284,
                   n=c(5,10,15,20,25,30,35,40,45,50,100),
                   sig.level=.05,
                   type="two.sample",
                   alternative="two.sided")
```



Next Lecture and Today Exercise

Next Lecture: Comparing several treatment means, linear regression

- Review basic statistical concepts and analysis of variance (ANOVA).
- Design and analyze single factor experiments using ANOVA.
- Analysis of the fixed effects model.
- Multiple linear regression model and submodels.

Today Exercise:

- Run and familiarize with Python (R) and Jupyter NB.
- Solve problems 2.20, 2.26, 2.30 from following slides.
  (originally from Montgomery - Design and Analysis of Experiments).

Exercise 2.20

The shelf life of a carbonated beverage is of interest. Ten bottles are randomly selected and tested, and the following results are obtained:

Days	
108	138
124	163
124	159
106	134
115	139

1. We would like to demonstrate that the mean shelf life exceeds 120 days. Set up appropriate hypotheses for investigating this claim.
2. Test these hypotheses using significant level $\alpha = 0.01$. Find the P-value for the test. What are your conclusions?
3. Construct a 99 percent confidence interval on the mean shelf life.

Exercise 2.26

The following are the burning times (in minutes) of chemical flares of two different formulations. The design engineers are interested in both the mean and variance of the burning times.

Type 1		Type 2	
65	82	64	56
81	67	71	69
57	59	83	74
66	75	59	82
82	70	65	79

1. Test the hypothesis that the two variances are equal. Use $\alpha = 0.05$.
2. Using the results of part 1), test the hypothesis that the mean burning times are equal. Use $\alpha = 0.05$. What is the P-value for this test?
3. Discuss the role of the normality assumption in this problem. Check the assumption of normality for both types of flares.
4. If the mean burning times of the two flares differ by as much as 2 minute, find the power of the test. What sample size would be required to detect an actual difference in mean burning time of 1 minute with a power of at least 0.9?

Exercise 2.30

Front housings for cell phones are manufactured in an injection molding process. The time the part is allowed to cool in the mold before removal is thought to influence the occurrence of a particularly troublesome cosmetic defect, flow lines, in the finished housing. After manufacturing, the housings are inspected visually and assigned a score between 1 and 10 based on their appearance, with 10 corresponding to a perfect part and 1 corresponding to a completely defective part. An experiment was conducted using two cool-down times, 10 and 20 seconds, and 20 housings were evaluated at each level of cool-down time. All 40 observations in this experiment were run in random order.

10 seconds				**20 seconds**			
1	3	2	6	7	6	8	9
1	5	3	3	5	5	9	7
5	2	1	1	5	4	8	6
5	6	2	8	6	8	4	5
3	2	5	3	6	8	7	7

1. Is there evidence to support the claim that the longer cool-down time results in fewer appearance defects? Use $\alpha = 0.05$.
2. What is the P-value for the test conducted in the previous part?
3. Find a 95 percent confidence interval on the difference in means. Provide a practical interpretation of this interval.
4. Check the assumption of normality for the data from this experiment.
5. Compute the power of the test.