01NAEX - Lecture 08
The Fractional Factorial Design

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The Fractional Factorial Design

Learning objectives for this lesson:

- Understanding the application of Fractional Factorial designs.
- Becoming familiar with the terms design generator, alias structure and design resolution.
- Knowing how to analyze fractional factorial designs in which there are not normally enough degrees of freedom for error.

The Fractional Factorial Design - introduction

If the number k of factors of interest is very large, the size of the 2^k factorial design grows very quickly.

- There may be many variables (often because we don't know much about the system).
- The main interest is in the identification of the factors with large effects.
- Almost always run as unreplicated factorials, but often with center points.

In the last lesson we considered one replicate of a full factorial design and run it in blocks. The treatment combinations in each block of a full factorial can be thought of as a fraction of the full factorial. A fractional factorial design is useful when we can not afford even one full replicate of the full factorial design.

The Fractional Factorial Design - introduction

The Fractional Factorial Design is based on three ideas:

- **The sparsity of effects principle**:
  There may be lots of factors, but few are important. System is dominated by main effects, low order interactions.
- **The projection property**:
  Every fractional factorial contains full factorials in fewer factors.
- **Sequential experimentation**:
  Can add runs to a fractional factorial to resolve difficulties in interpretation.

The One-Half Fraction of the 2^k Factorial Design

The one-half fraction of the 2^k has $\frac{2^k}{2}$ runs and is denoted as 2^{k-1} design.

Let us consider a simple example of 2^3 factorial design.

Treatment Combination	Factorial Effect							
	I	A	B	C	AB	AC	BC	ABC
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

Notice that the 2^{3-1} design is formed by selecting only those treatment combinations that have a plus in the ABC column. Thus, $ABC = I$ is called the generator of this particular fraction. Alternative fraction of the full design has generator $ABC = -I$.

The One-Half Fraction of the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	A	B	C	AB	AC	BC	ABC
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+

In the 2^{3-1} design we have 4 observations, therefore we have 3 degrees of freedom that we may use to estimate the effects.

$$
\begin{aligned}
A = BC &= \frac{1}{2}(a - b - c + abc) \\
B = AC &= \frac{1}{2}(b - a - c + abc) \\
C = AB &= \frac{1}{2}(c - a - b + abc)
\end{aligned}
$$

It is impossible to differentiate between effects of A and BC, B and AC, and C and AB. In fact we do not estimate A, B, and C, but we are really estimating $A + BC$, $B + AC$, and $C + AB$.

Two or more effects that have previous property are called **aliases**.

The One-Half Fraction of the 2^3 Factorial Design

In 2^{3-1} design ABC was the generator, which is equal to the Identity, ($I = ABC$ or $I = -ABC$). This defines the generator of the design and from this we can determine which effects are confounded or aliased with which other effects.

By using the defining relation $I = ABC$ and by multiplying any column (or effect) by the defining relation yields the aliases for that column. Since the square of any column is just the identity I we obtain following relations:

$$\begin{aligned} A \cdot I &= A \cdot ABC = A^2BC = BC \\ B \cdot I &= B \cdot ABC = AB^2C = AC \\ C \cdot I &= C \cdot ABC = ABC^2 = AB \end{aligned}$$

Suppose that after running the principal fraction, the alternate fraction was also run. The two groups of runs can be combined to form a full factorial, i.e. an example of sequential experimentation.

Design Resolution

Definition: A design is of resolution R if no p-factor effect is aliased with another effect containing less than $R - p$ factors.

More simple definition: the resolution is called Resolution R Design, if the generator $I = ABCD \cdots$ has R letters. For example in 2^{3-1} design with $I = ABC$ generator we have Resolution III Design and the main effects are confounded with 2-way interactions.

1. **Resolution III designs:** are designs in which no main effects are aliased with another main effects, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other.
2. **Resolution IV designs:** are designs in which no main effect is aliased with other main effect or with any two-factor interaction, but two-factor interactions are aliased with each other.
3. **Resolution V designs:** are designs in which no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions.

Generally, we want the highest resolution possible, and construct fractional factorial with highest order interaction.

Construction and Analysis of the One-Half Fraction

One possibility, how to construct a one-half fraction of the 2^k design with the highest resolution, is by writing down a basic full 2^{k-1} design and then adding the kth factor by identifying its plus and minus levels with the plus and minus signs of the highest-order interaction $ABC \cdots (K-1)$.
Example of construction of one-half fractions of the 2^3 design:

	Full 2^2 Factorial (Basic Design)		2^{3-1}_{III}, $I = ABC$			2^{3-1}_{III}, $I = -ABC$		
Run	A	B	A	B	$C = AB$	A	B	$C = -AB$
1	−	−	−	−	+	−	−	−
2	+	−	+	−	−	+	−	+
3	−	+	−	+	−	−	+	+
4	+	+	+	+	+	+	+	−

Another way is to partition complete design runs into two blocks with the highest-order interaction $ABC \cdots K$ confounded. Each block is 2^{k-1} fractional factorial design of the highest resolution.

The maximum possible resolution of a one-half fraction of the 2^k design is $R = k$.

2_{IV}^{4-1} design with the defining relation $I = ABCD$

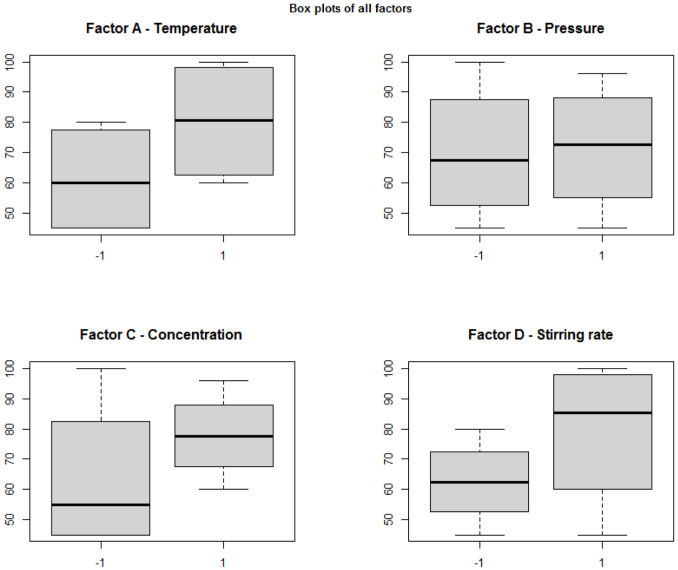
Consider a filtration rate experiment from the last lesson, where we found that the main effects A, C, D, and interactions AC and AD are significant. Now we will simulate what would have happened if 2_{IV}^{4-1} design had been run.

	Basic Design					
Run	A	B	C	$D = ABC$	**Treatment Combination**	**Filtration Rate**
1	$-$	$-$	$-$	$-$	(1)	45
2	$+$	$-$	$-$	$+$	*ad*	100
3	$-$	$+$	$-$	$+$	*bd*	45
4	$+$	$+$	$-$	$-$	*ab*	65
5	$-$	$-$	$+$	$+$	*cd*	75
6	$+$	$-$	$+$	$-$	*ac*	60
7	$-$	$+$	$+$	$-$	*bc*	80
8	$+$	$+$	$+$	$+$	*abcd*	96

Find the resolution and aliases of this design.

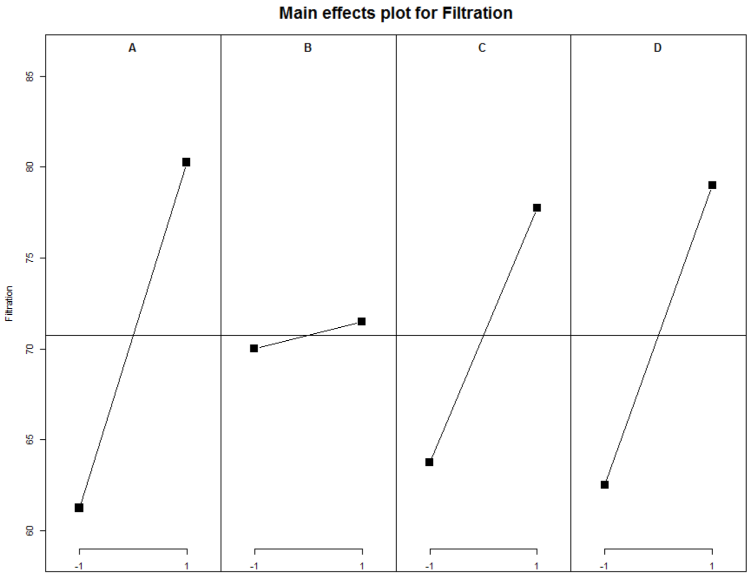
One-half 2^4 filtration rate experiment

Plot of effects:



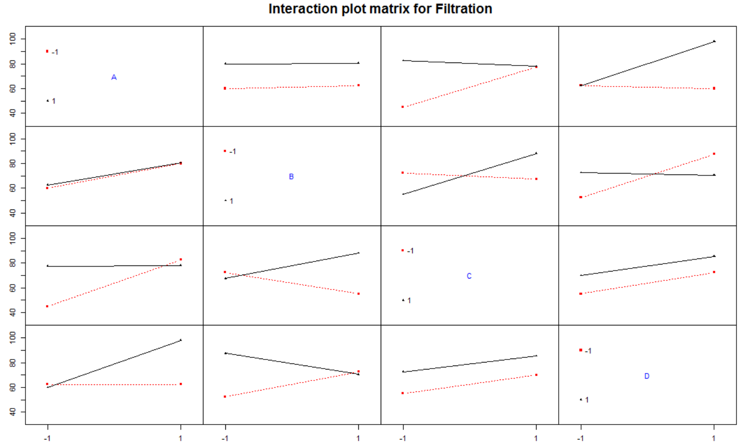
One-half 2^4 filtration rate experiment

Plot of effects:



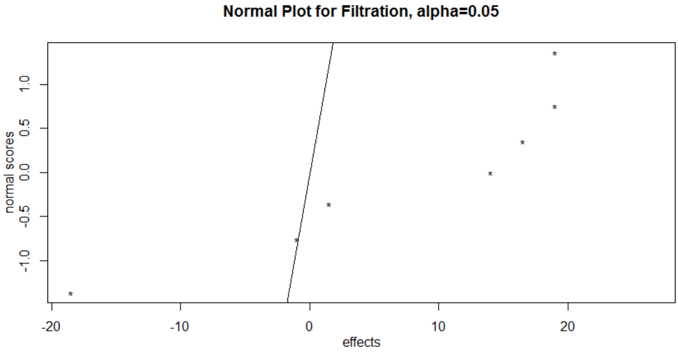
One-half 2^4 filtration rate experiment

Plot of effects:



One-half 2^4 filtration rate experiment

Plot of effects:



2^{4-1}_{IV} design with the defining relation $I = ABCD$

```
> k = 4
> design8_1    <-  FrF2(2^(k-1), k, replications = 1,
                   randomize = FALSE,factor.names = c("A", "B", "C", "D"))
> Filtration   <-  c(45,100,45,65,75,60,80,96)
> design8_1    <-  add.response(design8_1, Filtration)
> summary(design8_1)
Experimental design of type  FrF2 - 8  runs
Responses:        [1] Filtration
generators:       [1] D=ABC
Alias structure: [1] AB=CD AC=BD AD=BC
The design itself:
   A  B  C  D Filtration
1 -1 -1 -1 -1         45
2  1 -1 -1  1        100
3 -1  1 -1  1         45
4  1  1 -1 -1         65
5 -1 -1  1  1         75
6  1 -1  1 -1         60
7 -1  1  1 -1         80
8  1  1  1  1         96
class=design, type= FrF2
```

2^{4-1}_{IV} design with the defining relation $I = ABCD$

One-half 2^4 filtration rate experiment design with generator $I = ABCD$

- A(ABCD) = BCD $[A] \rightarrow A + BCD$
- B(ABCD) = ACD $[B] \rightarrow B + ACD$
- C(ABCD) = ABD $[C] \rightarrow C + ABD$
- D(ABCD) = ABC $[D] \rightarrow D + ABC$
- AB(ABCD) = CD $[AB] \rightarrow AB + CD$
- AC(ABCD) = BD $[AC] \rightarrow AC + BD$
- AD(ABCD) = BC $[AD] \rightarrow AD + BC$

We have seven degree of freedom and the alias structure is a four letter word, therefore this is a Resolution *IV* design, *A*, *B*, *C* and *D* are each aliased with a 3-way interaction, (so we can't estimate them any longer), and the two way interactions are aliased with each other.

2^{4-1}_{IV} filtration rate experiment design

ANOVA analysis results
- model without all third and fourth order interactions:

```
> summary(aov(Filtration ~ A*C*D + B  - A:C:D,
                  data = design8_1 ))
            Df  Sum Sq Mean Sq
A            1   722.0   722.0
C            1   392.0   392.0
D            1   544.5   544.5
B            1     4.5     4.5
A:C          1   684.5   684.5
A:D          1   722.0   722.0
C:D          1     2.0     2.0
```

Still, not enough free degrees of freedom.

2^{4-1}_{IV} filtration rate experiment design

ANOVA analysis results
- model without all third order interactions and without AB=CD interaction :

```
> summary(aov(Filtration ~ A*C*D + B - C:D - A:C:D,
              data = design8_1 ))
             Df Sum Sq Mean Sq F value Pr(>F)
A             1  722.0   722.0  361.00 0.0335 *
C             1  392.0   392.0  196.00 0.0454 *
D             1  544.5   544.5  272.25 0.0385 *
B             1    4.5     4.5    2.25 0.3743
A:C           1  684.5   684.5  342.25 0.0344 *
A:D           1  722.0   722.0  361.00 0.0335 *
Residuals     1    2.0     2.0
```

Because factor *B* is not significant, we can dropped it.

2^{4-1}_{IV} filtration rate experiment design

Analysis results:

```
summary(design8_1_numeric.lm)
Call:
lm.default(formula = Filtration ~ -1+A+B+C+D+
           A:B+A:C+A:D,data = design8_1)
Coefficients:
        Estimate Std. Error t value Pr(>|t|)
A        9.50      70.75    0.134    0.915
B        0.75      70.75    0.011    0.993
C        7.00      70.75    0.099    0.937
D        8.25      70.75    0.117    0.926
A:B     -0.50      70.75   -0.007    0.996
A:C     -9.25      70.75   -0.131    0.917
A:D      9.50      70.75    0.134    0.915
```

If A, C, and D are important main effects, it is logical to conclude that the two interactions alias chains $AC + BD$ and $AD + BC$ have large effects because the AC and AD interactions are also significant.

2_{IV}^{4-1} filtration rate experiment design

Final regression model:

```
summary(design8_1.lm2 )
lm.default(formula = Filtration ~ A + C + D +
          A:C + A:D, data = design8_1)
Residuals:
    1     2     3     4     5     6     7     8
-1.25 -0.25  1.25  0.25 -1.25 -0.25  1.25  0.25
Coefficients:
     Estimate Std.Error  t value Pr(>|t|)
1     70.7500    0.6374  111.00 8.11e-05 ***
A      9.5000    0.6374   14.90  0.00447 **
C      7.0000    0.6374   10.98  0.00819 **
D      8.2500    0.6374   12.94  0.00592 **
A:C   -9.2500    0.6374  -14.51  0.00471 **
A:D    9.5000    0.6374   14.90  0.00447 **

Residual standard error: 1.803 on 2 degrees of freedom
Multiple R-squared: 0.9979,    Adjusted R-squared: 0.9926
F-statistic: 188.6 on 5 and 2 DF,  p-value: 0.005282
```

Compare this model with results that we obtained in previous lectures.

2^{4-1}_{IV} filtration rate experiment design

Because factor B is not significant, we dropped it and projected the 2^{4-1} design into a single replicate of the 2^3 design.

Final regression model is:

$$\hat{y} = 70.75 + 9.5x_1 + 7x_3 + 8.25x_4 - 9.25x_1x_3 + 9.5x_1x_4,$$

where x_1, x_3, x_4 are coded variables $(-1 \leq x_i \leq +1)$ that represents A, C, D. Remember that the intercept $\hat{\beta}_0$ is the average of all responses at the eight runs in the design.

If we can run the alternate fraction to complete the 2^4 design we will block the two fractions and confounding $ABCD$ with blocks.

2^{5-1} design used for Process Improvement Example

Five factors in a manufacturing process for an integrated circuit were investigated in a 2^{5-1} design with the objective of improving the process yield.

Run	Basic Design A	B	C	D	$E = ABCD$	Treatment Combination	Yield
1	−	−	−	−	+	e	8
2	+	−	−	−	−	a	9
3	−	+	−	−	−	b	34
4	+	+	−	−	+	abe	52
5	−	−	+	−	−	c	16
6	+	−	+	−	+	ace	22
7	−	+	+	−	+	bce	45
8	+	+	+	−	−	abc	60
9	−	−	−	+	−	d	6
10	+	−	−	+	+	ade	10
11	−	+	−	+	+	bde	30
12	+	+	−	+	−	abd	50
13	−	−	+	+	+	cde	15
14	+	−	+	+	−	acd	21
15	−	+	+	+	−	bcd	44
16	+	+	+	+	+	$abcde$	63

$I = ABCDE$ is the generator. Consequently, every main effect is aliased with a four-factor interaction and every two-factor interaction is aliased with a three-factor interaction.

The design is of resolution V and we have 15 degree of freedom.

2^{5-1} design used for Process Improvement Example

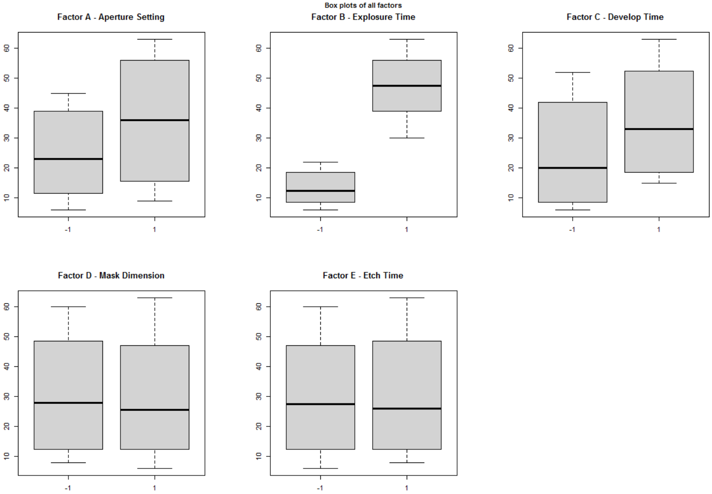
```
k = 5
summary(design8_2)
FrF2(2^(k - 1), k, replications = 1, randomize = FALSE,
factor.names = c("A", "B", "C", "D", "E"))

Experimental design of type  FrF2
16  runs
Generators: E=ABCD
Alias structure:
no aliasing among main effects and 2fis

The design itself:
   A  B  C  D  E yield     A  B  C  D  E yield
1 -1 -1 -1 -1  1   8   9  -1 -1 -1  1 -1   6
2  1 -1 -1 -1 -1   9   10  1 -1 -1  1  1  10
3 -1  1 -1 -1 -1  34   11 -1  1 -1  1  1  30
4  1  1 -1 -1  1  52   12  1  1 -1  1 -1  50
5 -1 -1  1 -1 -1  16   13 -1 -1  1  1  1  15
6  1 -1  1 -1  1  22   14  1 -1  1  1 -1  21
7 -1  1  1 -1  1  45   15 -1  1  1  1 -1  44
8  1  1  1 -1 -1  60   16  1  1  1  1  1  63
class=design, type= FrF2
```

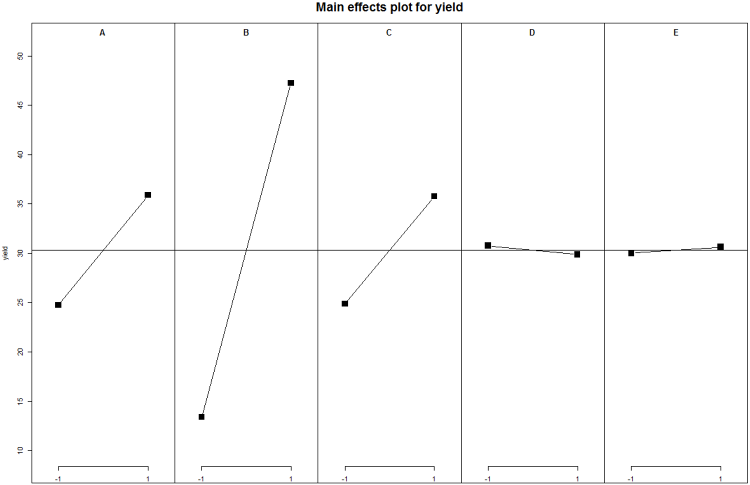
2^{5-1} design used for Process Improvement Example

Plot of effects:



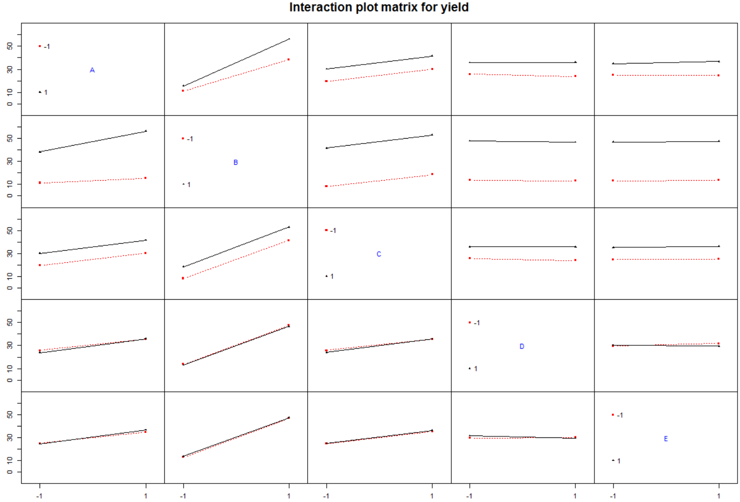
2^{5-1} design used for Process Improvement Example

Plot of effects:



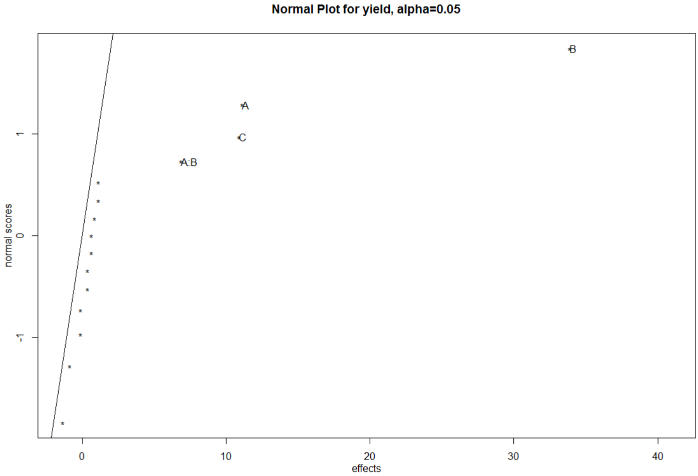
2^{5-1} design used for Process Improvement Example

Plot of effects:



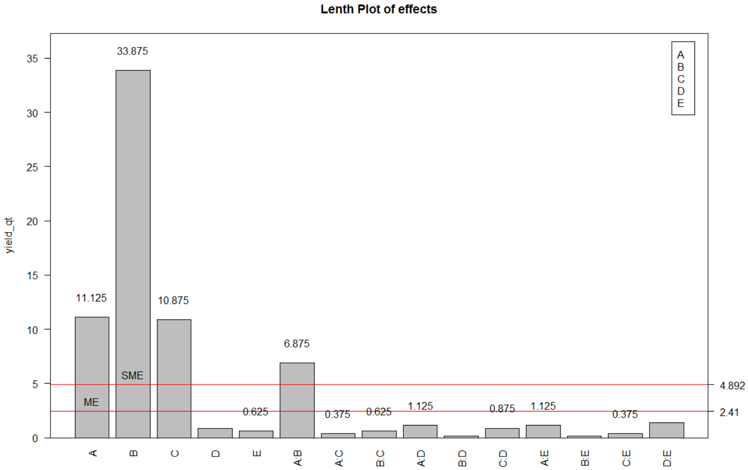
2^{5-1} design used for Process Improvement Example

Plot of effects:



2^{5-1} design used for Process Improvement Example

Plot of effects:



2^{5-1} design used for Process Improvement Example

Effects, regression coefficients and sum of squares:

Variable	Name	−1 Level	+1 Level
A	Aperture	Small	Large
B	Exposure time	−20%	+20%
C	Develop time	30 s	40 s
D	Mask dimension	Small	Large
E	Etch time	14.5 min	15.5 min

Variable	Regression Coefficient	Estimated Effect	Sum of Squares
Overall Average	30.3125		
A	5.5625	11.1250	495.062
B	16.9375	33.8750	4590.062
C	5.4375	10.8750	473.062
D	−0.4375	−0.8750	3.063
E	0.3125	0.6250	1.563
AB	3.4375	6.8750	189.063
AC	0.1875	0.3750	0.563
AD	0.5625	1.1250	5.063
AE	0.5625	1.1250	5.063
BC	0.3125	0.6250	1.563
BD	−0.0625	−0.1250	0.063
BE	−0.0625	−0.1250	0.063
CD	0.4375	0.8750	3.063
CE	0.1875	0.3750	0.563
DE	−0.6875	−1.3750	7.563

2^{5-1} design used for Process Improvement Example

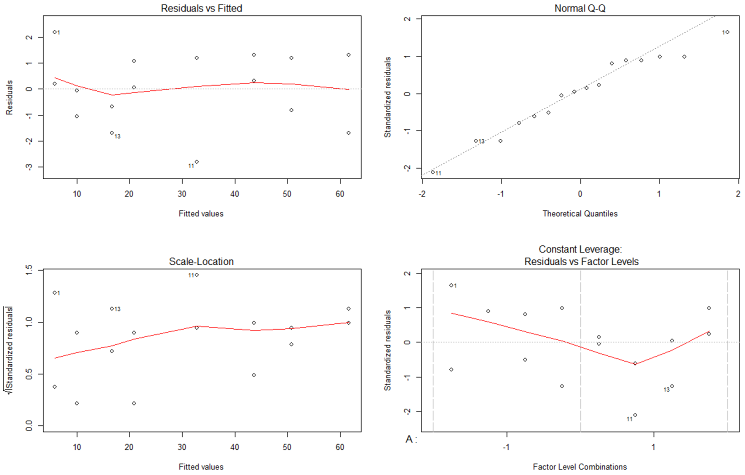
R - results

```
> summary(aov(yield~A*B +C, data = design8_2) )
             Df Sum Sq Mean Sq F value   Pr(>F)    
A             1    495     495  193.19 2.53e-08 ***
B             1   4590    4590 1791.24 1.56e-13 ***
C             1    473     473  184.61 3.21e-08 ***
A:B           1    189     189   73.78 3.30e-06 ***
Residuals    11     28       3
```

The 2^{5-1} design has collapsed into two replicates of a 2^3 design, because factors *D* and *E* have little effect on average process yield.

2^{5-1} design used for Process Improvement Example

Model adequacy checking:



The one-quarter fraction of the 2^k design

The one-quarter fraction of the 2^k design contains 2^{k-2} runs and is usually called a 2^{k-2} **fractional factorial.**

One-quarter fraction of the 2^k design has two generators. If P and Q represents the generators chosen, then $I = P = Q = PQ$ are called the generating relations. We call the elements P, Q, and PQ the defining relation **words**. The aliases of any effect are produced by the multiplication of the column for that effect by each word in the defining relation.

Each effect has three aliases.

Be careful in choosing the generators so that potentially important effects are not aliased each other.

Example of one-quarter fraction - the 2^{6-2}_{IV} design

Construction of the 2^{6-2}_{IV} design with the generators $I = ABCE$ and $I = BCDF$.

Run	Basic Design A	B	C	D	$E = ABC$	$F = BCD$
1	−	−	−	−	−	−
2	+	−	−	−	+	−
3	−	+	−	−	+	+
4	+	+	−	−	−	+
5	−	−	+	−	+	+
6	+	−	+	−	−	+
7	−	+	+	−	−	−
8	+	+	+	−	+	−
9	−	−	−	+	−	+
10	+	−	−	+	+	+
11	−	+	−	+	+	−
12	+	+	−	+	−	−
13	−	−	+	+	+	−
14	+	−	+	+	−	−
15	−	+	+	+	−	+
16	+	+	+	+	+	+

Example of one-quarter fraction - the 2^{6-2}_{IV} design

Alias structure for the 2^{6-2}_{IV} design with the generators $I = ABCE = BCDF = ADEF$.

$A = BCE = DEF = ABCDF$
$B = ACE = CDF = ABDEF$
$C = ABE = BDF = ACDEF$
$D = BCF = AEF = ABCDE$
$E = ABC = ADF = BCDEF$
$F = BCD = ADE = ABCEF$

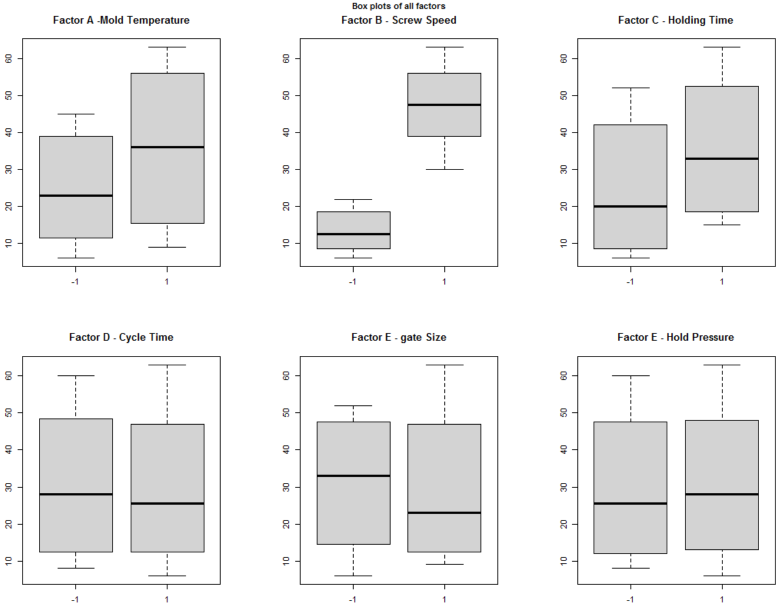
$ABD = CDE = ACF = BEF$
$ACD = BDE = ABF = CEF$

$AB = CE = ACDF = BDEF$
$AC = BE = ABDF = CDEF$
$AD = EF = BCDE = ABCF$
$AE = BC = DF = ABCDEF$
$AF = DE = BCEF = ABCD$
$BD = CF = ACDE = ABEF$
$BF = CD = ACEF = ABDE$

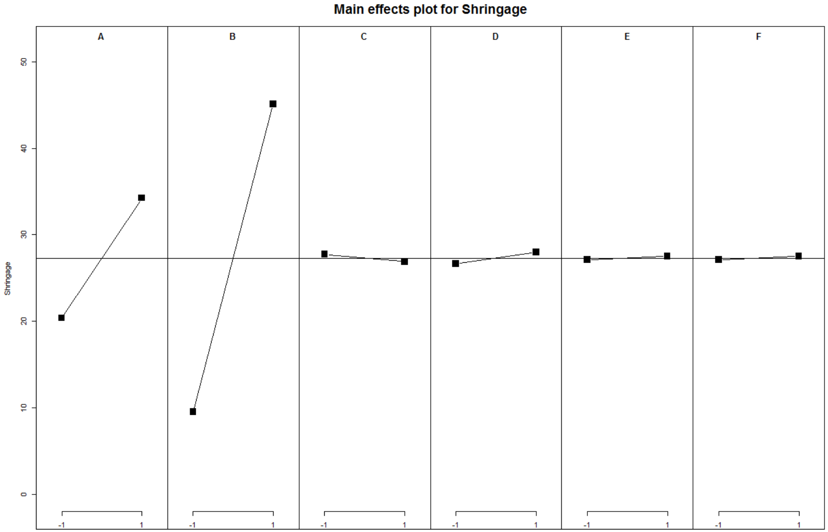
Example of the 2^{6-2}_{IV} design - Injection Molding Experiment

```
design8_4           <-  FrF2(2^(6-2), 6, replications = 1,
                randomize = FALSE,generators = c("ABC","BCD") ,
             factor.names = c("A", "B", "C", "D", "E", "F"))
Shringage           <-  c(6,10,32,60,4,15,26,60,8,12,34,60,16,5,37,52)
design8_4           <-  add.response(design8_4, Shringage)
summary(design8_4)
FrF2(2^(k - 2), k, replications = 1, randomize = FALSE,generators =
    c("ABC","BCD"), factor.names = c("A", "B", "C", "D", "E", "F"))
Generators:
E=ABC F=BCD
Alias structure:
AB=CE, AC=BE, AD=EF, AE=BC=DF, AF=DE, BD=CF, BF=CD
The design itself:
    A  B  C  D  E  F Shringage      A  B  C  D  E  F Shringage
1  -1 -1 -1 -1 -1 -1         6   9 -1 -1 -1  1 -1  1         8
2   1 -1 -1 -1  1 -1        10  10  1 -1 -1  1  1  1        12
3  -1  1 -1 -1  1  1        32  11 -1  1 -1  1  1 -1        34
4   1  1 -1 -1 -1  1        60  12  1  1 -1  1 -1 -1        60
5  -1 -1  1 -1  1  1         4  13 -1 -1  1  1  1 -1        16
6   1 -1  1 -1 -1  1        15  14  1 -1  1  1 -1 -1         5
7  -1  1  1 -1 -1 -1        26  15 -1  1  1  1 -1  1        37
8   1  1  1 -1  1 -1        60  16  1  1  1  1  1  1        52
```

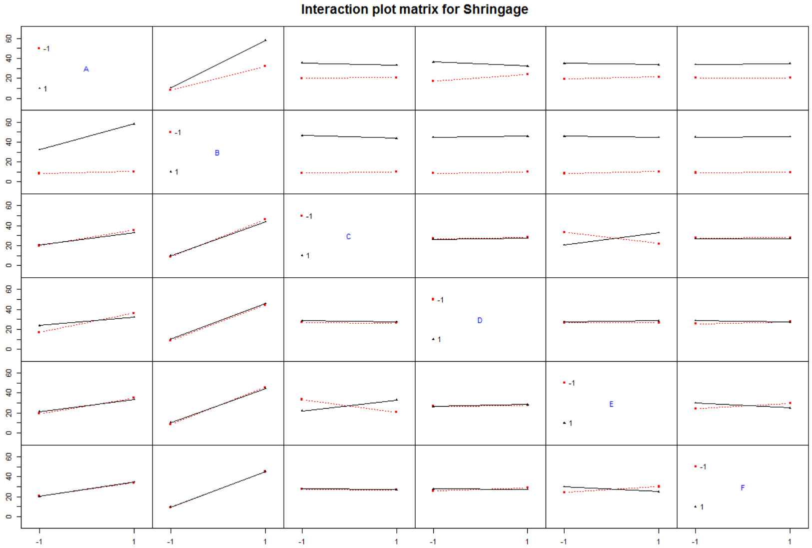
Example of the 2_{IV}^{6-2} design - Injection Molding Experiment



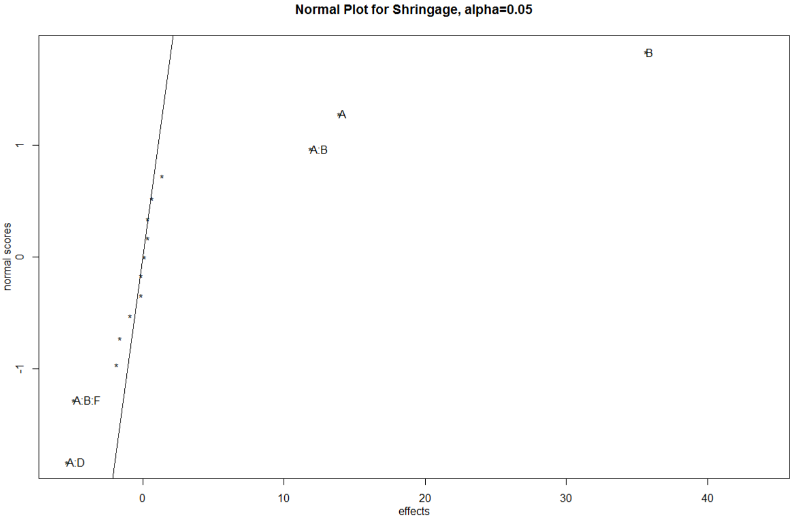
Example of the 2^{6-2}_{IV} design - Injection Molding Experiment



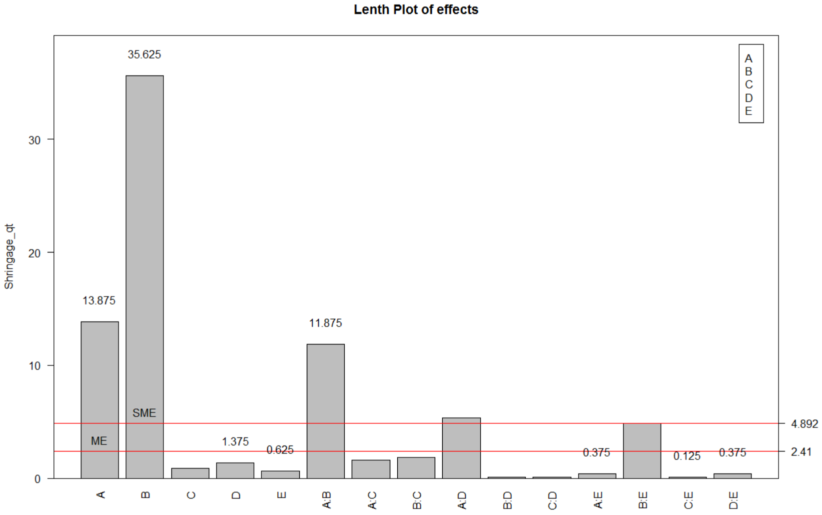
Example of the 2_{IV}^{6-2} design - Injection Molding Experiment



Example of the 2^{6-2}_{IV} design - Injection Molding Experiment



Example of the 2_{IV}^{6-2} design - Injection Molding Experiment



Example of the 2_{IV}^{6-2} design - Injection Molding Experiment

Anova for final model for Injection Molding Experiment:

```
> summary(aov(yield~A*B, data = design8_4) )
            Df Sum Sq Mean Sq F value   Pr(>F)
A            1    495     495  11.852  0.00487 **
B            1   4590    4590 109.887 2.15e-07 ***
A:B          1    189     189   4.526  0.05480 .
Residuals   12    501      42
```

Example of the 2_{IV}^{6-2} design - Injection Molding Experiment

Final linear regression model for Injection Molding Experiment:

```
Call: lm.default(formula =
       Shringage ~ A.num * B.num, data = design8_4)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   27.312      1.138  23.996 1.65e-11 ***
A.num          6.938      1.138   6.095 5.38e-05 ***
B.num         17.812      1.138  15.649 2.39e-09 ***
A.num:B.num    5.938      1.138   5.216 0.000216 ***
---
Residual standard error: 4.553 on 12 degrees of freedom
Multiple R-squared: 0.9626,     Adjusted R-squared: 0.9533
F-statistic: 103.1 on 3 and 12 DF,  p-value: 7.837e-09
```

Final regression model is:

$$\hat{y} = 27.312 + 6.938x_1 + 17.812x_2 + 5.938x_1x_2,$$

where x_1, x_2 are coded variables ($-1 \leq x_i \leq +1$) that represents A, B.

The general 2^{k-p} fractional factorial design

Highest possible resolution and minimum aberration design:
it is important to select the p generators for 2^{k-p} fractional design in such a way that we obtain the best possible alias relationships.

1) we want to obtain the highest possible resolution.

Design A Generators: $F = ABC$, $G = BCD$ $I = ABCF = BCDG = ADFG$	Design B Generators: $F = ABC$, $G = ADE$ $I = ABCF = ADEG = BCDEFG$	Design C Generators: $F = ABCD$, $G = ABDE$ $I = ABCDF = ABDEG = CEFG$
Aliases (Two-Factor Interactions)	Aliases (Two-Factor Interactions)	Aliases (Two-Factor Interactions)
$AB = CF$	$AB = CF$	$CE = FG$
$AC = BF$	$AC = BF$	$CF = EG$
$AD = FG$	$AD = EG$	$CG = EF$
$AG = DF$	$AE = DG$	
$BD = CG$	$AF = BC$	
$BG = CD$	$AG = DE$	
$AF = BC = DG$		

2) we want to obtain minimum number of words in the defining relation that are of minimum length. We call such a design a **minimum aberration design**.

The general 2^{k-p} fractional factorial design

Number of Factors, k	Fraction	Number of Runs	Design Generators
3	2^{3-1}_{III}	4	$C = \pm AB$
4	2^{4-1}_{IV}	8	$D = \pm ABC$
5	2^{5-1}_{V}	16	$E = \pm ABCD$
	2^{5-2}_{III}	8	$D = \pm AB$
			$E = \pm AC$
6	2^{6-1}_{VI}	32	$F = \pm ABCDE$
	2^{6-2}_{IV}	16	$E = \pm ABC$
			$F = \pm BCD$
	2^{6-3}_{III}	8	$D = \pm AB$
			$E = \pm AC$
			$F = \pm BC$
7	2^{7-1}_{VII}	64	$G = \pm ABCDEF$
	2^{7-2}_{IV}	32	$F = \pm ABCD$
			$G = \pm ABDE$
	2^{7-3}_{IV}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
	2^{7-4}_{III}	8	$D = \pm AB$
			$E = \pm AC$
			$F = \pm BC$
			$G = \pm ABC$
8	2^{8-2}_{V}	64	$G = \pm ABCD$
			$H = \pm ABEF$
	2^{8-3}_{IV}	32	$F = \pm ABC$
			$G = \pm ABD$
			$H = \pm BCDE$
	2^{8-4}_{IV}	16	$E = \pm BCD$
			$F = \pm ACD$
			$G = \pm ABC$
			$H = \pm ABD$
9	2^{9-2}_{VI}	128	$H = \pm ACDFG$
			$J = \pm BCEFG$
	2^{9-3}_{IV}	64	$G = \pm ABCD$
			$H = \pm ACEF$
			$J = \pm CDEF$
	2^{9-4}_{IV}	32	$F = \pm BCDE$
			$G = \pm ACDE$
			$H = \pm ABDE$
			$J = \pm ABCE$

Number of Factors, k	Fraction	Number of Runs	Design Generators
	2^{9-5}_{III}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
			$H = \pm ABD$
			$J = \pm ABCD$
10	2^{10-3}_{V}	128	$H = \pm ABCG$
			$J = \pm ACDE$
			$K = \pm ACDF$
	2^{10-4}_{IV}	64	$G = \pm BCDF$
			$H = \pm ACDF$
			$J = \pm ABDE$
			$K = \pm ABCE$
	2^{10-5}_{IV}	32	$F = \pm ABCD$
			$G = \pm ABCE$
			$H = \pm ABDE$
			$J = \pm ACDE$
			$K = \pm BCDE$
	2^{10-6}_{III}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
			$H = \pm ABD$
			$J = \pm ABCD$
			$K = \pm AB$
11	2^{11-5}_{IV}	64	$G = \pm CDE$
			$H = \pm ABCD$
			$J = \pm ABF$
			$K = \pm BDEF$
			$L = \pm ADEF$
	2^{11-6}_{IV}	32	$F = \pm ABC$
			$G = \pm BCD$
			$H = \pm CDE$
			$J = \pm ACD$
			$K = \pm ADE$
			$L = \pm BDE$
	2^{11-7}_{III}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$
			$H = \pm ABD$
			$J = \pm ABCD$
			$K = \pm AB$

Number of Factors, k	Fraction	Number of Runs	Design Generators
			$L = \pm AC$
12	2^{12-8}_{III}	16	$E = \pm ABC$
			$F = \pm ABD$
			$G = \pm ACD$
			$H = \pm BCD$
			$J = \pm ABCD$
			$K = \pm AB$
			$L = \pm AC$
			$M = \pm AD$
13	2^{13-9}_{III}	16	$E = \pm ABC$
			$F = \pm ABD$
			$G = \pm ACD$
			$H = \pm BCD$
			$J = \pm ABCD$
			$K = \pm AB$
			$L = \pm AC$
			$M = \pm AD$
			$N = \pm BC$
14	2^{14-10}_{III}	16	$E = \pm ABC$
			$F = \pm ABD$
			$G = \pm ACD$
			$H = \pm BCD$
			$J = \pm ABCD$
			$K = \pm AB$
			$L = \pm AC$
			$M = \pm AD$
			$N = \pm BC$
			$O = \pm BD$
15	2^{15-11}_{III}	16	$E = \pm ABC$
			$F = \pm ABD$
			$G = \pm ACD$
			$H = \pm BCD$
			$J = \pm ABCD$
			$K = \pm AB$
			$L = \pm AC$
			$M = \pm AD$
			$N = \pm BC$
			$O = \pm BD$
			$P = \pm CD$

Blocking Fractional Factorials

In general choose some aliases (preferably with longest words) and generate block by the help of (+) and (-) signs.

Example of the 2_{IV}^{6-2} design - Two blocks with ABD confounded:

- **Block 1:** (1), abf, cef, abce,abef,bde,acd,bcdf
- **Block 2:** ae,acf,bef,bc,df,abd,cde,abcdef

Remember: the generator of blocks ABD is aliased

$$ABD = CDE = ACF = BEF.$$

Plackett-Burman Design

Plackett-Burman Design is two-level fractional factorial design for studying $k = N - 1$ variables in N runs, where N is a multiple of 4. Sometimes it is called nongeometric design.

- Developed by Plackett & Burmann in 1946.
- These designs can not be represented as cubes.
- There are no design generators and the alias structure is complicated.
- N is a multiple of 4 (i.e. $N \in \{4, 8, 12, 16, 20, 24, 28, 32, \dots\}$).
- PB designs, where N is not a power of 2, are called non-geometric (i.e. $N \in \{12, 20, 24, \dots\}$).
- PB designs are non-regular designs. Regular design is one in witch all effects can be estimated independently of the other effects.

Plackett-Burman Design

The designs for $N = 12$, 20, 24, and 36 are obtained by writing the appropriate signs as a column (or row). A second column (or row) is then generated from the first one by moving the element of the column (or row) down (or right) one position and placing the last element in the first position. The process is continued until k columns (rows) are generated. A row of minus signs is then added.

Plus and Minus Signs for the Plackett–Burman Designs

$k = 11, N = 12$	+ + − + + + − − − + −
$k = 19, N = 20$	+ + − − + + + + − + − + − − − − + + −
$k = 23, N = 24$	+ + + + + − + − + + − − + + − − + − + − − − −
$k = 35, N = 36$	− + − + + + − − − + + + + + − + + + − − + − − − − + − + − + + − − + −

$k = 27, N = 28$

+ − + + + + − − −	− + − − − + − − +	+ + − + − + + − +
+ + − + + + − − −	− − + + − − + − −	− + + + + − + + −
− + + + + + − − −	+ − − − + − − + −	+ − + − + + − + +
− − − + − + + + +	− − + − + − − − +	+ − + + + − + − +
− − − + + − + + +	+ − − − − + + − −	+ + − − + + + + −
− − − − + + + + +	− + − + − − − + −	− + + + − + − + +
+ + + − − − + − +	− − + − − + − + −	+ − + + − + + + −
+ + + − − − + + −	+ − − + − − − − +	+ + − + + − − + +
+ + + − − − − + +	− + − − + − + − −	− + + − + + + − +

For $N = 28$, the three blocks are written as: X,Y,Z - first row, Z, X, Y - second row, Y, Z, X - third row and a rows of minus signs is added to these 27 rows.

Plackett-Burman Design

An example of the Plackett-Burman Design for $N = 12,\ k = 11$.

Run	A	B	C	D	E	F	G	H	I	J	K
1	+	−	+	−	−	−	+	+	+	−	+
2	+	+	−	+	−	−	−	+	+	+	−
3	−	+	+	−	+	−	−	−	+	+	+
4	+	−	+	+	−	+	−	−	−	+	+
5	+	+	−	+	+	−	+	−	−	−	+
6	+	+	+	−	+	+	−	+	−	−	−
7	−	+	+	+	−	+	+	−	+	−	−
8	−	−	+	+	+	−	+	+	−	+	−
9	−	−	−	+	+	+	−	+	+	−	+
10	+	−	−	−	+	+	+	−	+	+	−
11	−	+	−	−	−	+	+	+	−	+	+
12	−	−	−	−	−	−	−	−	−	−	−

For example: AB interaction is aliased with the nine main effects and each main effect is partially aliased with 45 two-factor interactions.

$$[A] = A - \frac{1}{3}BC - \frac{1}{3}BD - \frac{1}{3}BE + \frac{1}{3}BF + \cdots - \frac{1}{3}KL$$

PB- design is part of FrF2 package in R.

Homework 02

Design and analyze **Problem from 01NAEX_HW2_xxx**.

Form groups (prefere 4 students in each), measure data according to instructions in the file 01NAEX_HW2_xxx.pdf

Submit solutions with a written report in pdf format and used code with data file till November the 28th 2022.