Cemef

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Self-similarity, boundary layer & heat transfer in fluid mechanics

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1. Purposes of the lecture

1. Scaling analysis;
2. Self-similarity;
3. Boundary layer theory;
4. Heat transfer in fluid;
5. Learn about the natural convection.

2. Self-similar solution of partial differential equation

- Fluid mechanics obeys to non-linear partial differential equations.
- Exact solutions are very scarce.
- Fortunately, particular solutions exist invariant by groups of affinity transformation.
- Affinity transformations linked to dimensional changes are remarkable in physics & in particular in fluid mechanics.
- Invariant solutions are called “self-similar” solutions.

Figure 1: Barnsley fern created by affinity geometric transformation.

2. Self-similar solution of partial differential equation

2.1 Rayleigh problem

Let $u(y, t)$ x-component of the velocity obeying to

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

$$u(y, 0) = 0, \text{ for } t = 0, \tag{2}$$

$$u(0, t) = U, \text{ for } t > 0. \tag{3}$$

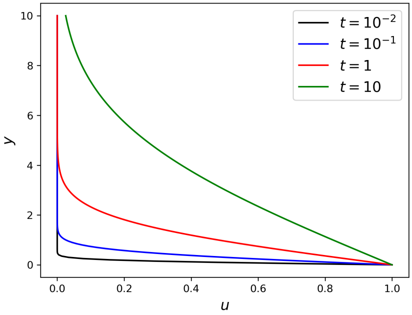

Figure 2: u vs. y at four increasing times.

Normalisation of the equation:

$$\bar{u} = \frac{u}{U}, \; \bar{y} = \frac{y}{\ell}, \; \bar{t} = \frac{t}{\tau}, \tag{4}$$

which gives

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\nu\tau}{\ell^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = 0, \tag{5}$$

$$\bar{u}(\bar{y}, 0) = 0, \; \text{for } \bar{t} = 0, \tag{6}$$

$$\bar{u}(0, \bar{t}) = 1, \; \text{for } \bar{t} > 0. \tag{7}$$

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Takes $\ell = \sqrt{\nu\tau}$.

Theorem 1

If $\bar{u}(\bar{y},\bar{t})$ is solution of (5), then $\bar{u}(\sqrt{a}\bar{y}, a\bar{t})$ is also solution of (5) for all constant a.

An homogeneous solution of $\bar{u}(\bar{y},\bar{t})$ can be taken as $f(\bar{y}/\sqrt{\bar{t}})$ with $\eta = \bar{y}/\sqrt{\bar{t}}$ is the "self-similar" variable. Here, for reason of simplification, η is written as

$$\eta = \frac{\bar{y}}{2\sqrt{\bar{t}}} = \frac{y}{2\sqrt{\nu t}}. \tag{8}$$

$f(\eta)$ is then solution of

$$f'' + 2\eta f' = 0, \text{ with } f' = \frac{df}{d\eta}, \tag{9}$$

$$f(0) = 1, \tag{10}$$

$$\lim_{\eta\to\infty} f(\eta) = 0. \tag{11}$$

The exact solution is

$$f(\eta) = \mathrm{erfc}(\eta). \tag{12}$$

3. Singular perturbation method

- In fluid mechanics, there are various approximations depending on “small” or “large” parameters[1]:
  - Stokes flows: $\mathrm{Re} \ll 1$;
  - Boundary layer: $\mathrm{Re} \gg 1$;
  - Boundary heat & mass layer: $\mathrm{Pe} \gg 1$;
  - Quasi-steady state regime: $\mathrm{St} \ll 1$,
  - ...
- The perturbation method is a useful technique to find approximative solution.

Definition 2

Let’s ϵ a small parameter, if an approximated solution stays valide when $\epsilon \to 0$, the approximation is said regular. Conversely, if the solution is non uniform when $\epsilon \to 0$, the approximation is said singular.

[1] M. Van Dyke: Perturbation methods in fluid mechanics, Stanford, California 1975.

3.1 Example of singular ODE

Consider the ordinary differential equation[2]

$$\epsilon y'' + y' + y = 0, \forall x \in]0, 1[, \tag{13}$$

$$y(0) = 0, \; y(1) = 1, \tag{14}$$

$$\epsilon < 1/4. \tag{15}$$

The exact solution is given by

$$y = \frac{e^{r_1 x} - e^{r_2 x}}{e^{r_1} - e^{r_2}}, \tag{16}$$

$$r_1 = -\frac{1 - \sqrt{1 - 4\epsilon}}{2\epsilon}, \; r_2 = -\frac{1 + \sqrt{1 - 4\epsilon}}{2\epsilon}. \tag{17}$$

The solution is singular since r_2 diverges when $\epsilon \to 0$.

[2] E. J. Hinch: Perturbation Methods, 1991.

3. Singular perturbation method

○ 3.1 Example of singular ODE

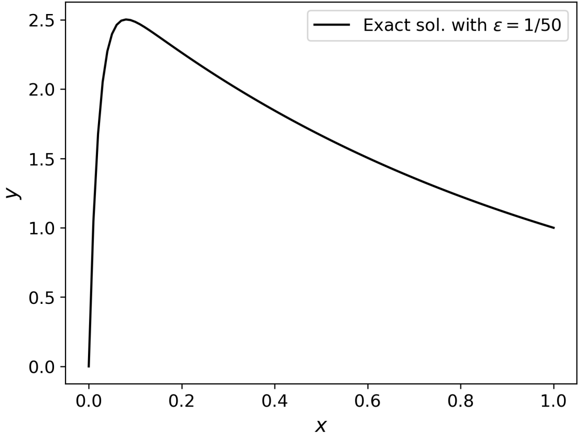

Figure 3: y vs. x for $\epsilon = 1/50$.

To have an approximative solution, y can be expanded as ϵ^n as

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots . \tag{18}$$

After introduction in the ODE, we have

Zeroth order:

$$y_0' + y_0 = 0, \tag{19}$$
$$y_0(0) = 0, \ y_0(1) = 1. \tag{20}$$

First order:

$$y_1' + y_1 = -y_0'', \tag{21}$$
$$y_1(0) = 0, \ y_1(1) = 0. \tag{22}$$

Second order:

$$y_2' + y_2 = -y_1'', \tag{23}$$
$$y_2(0) = 0, \ y_2(1) = 0. \tag{24}$$

- The order of ODE is reduced by one order ➻ signature of the singular equation.
- Impossible to satisfy the two boundary conditions.
- Solution valid only for $x \gg 0$ ➻ **outer** solution.

- To find the adequate approximation close to $x = 0$, **inner** (**stretched**) coordinate has to be introduced:

$$\tilde{x} = \frac{x}{\mu(\epsilon)}, \text{ with } \mu(\epsilon) \ll 1. \tag{25}$$

- By written $\tilde{y} = y(\tilde{x})$, the ODE becomes

$$\frac{\epsilon}{\mu^2}\tilde{y}'' + \frac{\tilde{y}'}{\mu} + \tilde{y} = 0. \tag{26}$$

Definition 3

The "**principle of least degeneracy**" involves that a significant degeneracy of an equation must keep a maximum of terms of the equation[3].

[3] Van Dyke: Perturbation methods in fluid mechanics (see n. 1).

3. Singular perturbation method

○ 3.1 Example of singular ODE

- If the balance is done between in the first and the third terms:

$$\mu = \sqrt{\epsilon}. \tag{27}$$

- The ODE becomes

$$\sqrt{\epsilon}\tilde{y}'' + \tilde{y}' + \sqrt{\epsilon}\tilde{y} = 0. \tag{28}$$

- If $\epsilon \to 0$, the ODE is simply $\tilde{y}' = 0$.
- If the balance is done between in the first and the second terms:

$$\mu = \epsilon. \tag{29}$$

- The ODE becomes

$$\tilde{y}'' + \tilde{y}' + \epsilon\tilde{y} = 0. \tag{30}$$

- If $\epsilon \to 0$, the ODE is simply $\tilde{y}'' + \tilde{y}' = 0$.
- The principle of least degeneracy involves that the significant approximation is the second case.

3. Singular perturbation method

○ 3.1 Example of singular ODE

- The outer solution at the zeroth order is
$$y_0(x) = e^{1-x}. \tag{31}$$
- The inner solution is
$$\tilde{y}_0(\tilde{x}) = A_0 \left(1 - e^{-\tilde{x}}\right). \tag{32}$$
- A_0 is unknown. To find it, a matching between the inner and outer solutions have to do.
- Introduce a new coordinate:
$$\eta = x/\epsilon^\alpha, \text{ with: } 0 < \alpha < 1, \tag{33}$$
$$x = \epsilon^\alpha \eta, \ \tilde{x} = \frac{\eta}{\epsilon^{1-\alpha}}. \tag{34}$$
- For a finite value of η, $x \to 0$ and $\tilde{x} \to \infty$ when $\epsilon \to 0$.
- The matching between the two solutions gives: $A_0 = e$.

3.1 Example of singular ODE

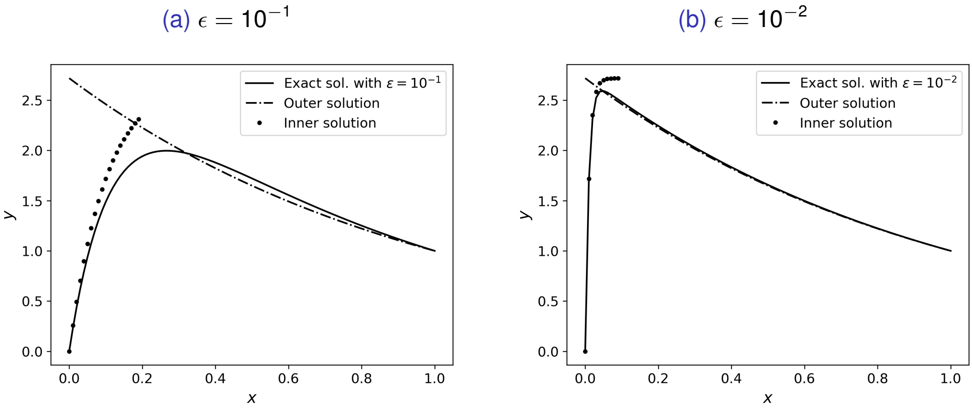

Figure 4: y vs. x with exact, inner and outer solutions.

3.2 Laminar boundary layer

By using $\mathrm{Re} = UL/\nu$, the 2D Navier-Stokes equations for incompressible fluid are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{35}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{36}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{37}$$

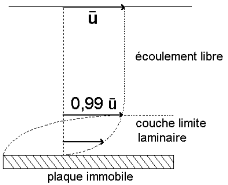

Figure 5: Sketch of the boundary layer close to horizontal wall.

3. Singular perturbation method

3.2 Laminar boundary layer

- A CIMLIB_CFD case of this problem is available on gitlab:franck.pigeonneau/laminarboundlayer.git.

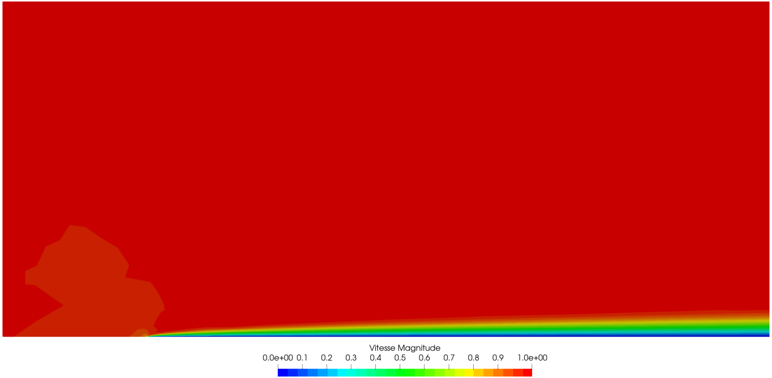

Figure 6: $||\boldsymbol{u}||$ for $\mathrm{Re} = 10^3$.

First, we introduce the “small” parameter

$$\epsilon = \frac{1}{\mathrm{Re}}. \tag{38}$$

The N-S equations become

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, && (39)\\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial P}{\partial x} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), && (40)\\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial P}{\partial y} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). && (41)
\end{aligned}
$$

- Study the boundary layer leads to study the behaviour of the equations for small ϵ.
- As shown above, we use the perturbation method. The solution is developed as a power of ϵ.
- In the first approximation, remove all the terms proportional to ϵ:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, && (42)\\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial P}{\partial x}, && (43)\\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial P}{\partial y}. && (44)
\end{aligned}$$

- Too abusive: **impossible to apply the boundary conditions both on the wall and at the infinity**.

- The spatial scales are different along x and y axis. Let introduce a new scale δ to normalize the y axis:

$$\tilde{y} = \frac{y}{\delta}, \; \tilde{v} = \frac{v}{\delta} \tag{45}$$

- The change of the velocity is required to conserve the continuity equation. Now the N-S equations becomes

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{46}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = -\frac{\partial P}{\partial x} + \frac{\epsilon}{\delta^2}\left(\delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \tilde{y}^2}\right), \tag{47}$$

$$\delta^2\left[u\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}\right] = -\frac{\partial P}{\partial \tilde{y}} + \epsilon\left(\delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial \tilde{y}^2}\right). \tag{48}$$

3.2 Laminar boundary layer

- Following the principle of least degeneracy, the simplification of the equations must keep the maximum of terms.
- In the case of the boundary layer, we must have

$$\delta^2 = \epsilon,\ \delta = \frac{1}{\sqrt{\mathrm{Re}}}. \tag{49}$$

- The Prandtl boundary layer equations[4]:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0, & (50)\\ u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} &= -\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial \tilde{y}^2}, & (51)\\ \frac{\partial P}{\partial \tilde{y}} &= 0. & (52) \end{aligned}$$

[4] L. Prandtl: Zur Berechnung der Grenzschichten, in: Zeitschrift für Angewandte Mathematik und Mechanik 18.1 (1938), pp. 77–82.

- Since P is independent on $\tilde{y}$, P is a function of x.
- P can be matched with the outer (Euler) solution.
- Using the Bernoulli relation:

$$\frac{dP}{dx} = -U\frac{dU}{dx}, \tag{53}$$

  with U the outer solution.
- In the particular case of the horizontal wall, $U = 1$ ➻ the pressure gradient vanishes.

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{54}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = \frac{\partial^2 u}{\partial \tilde{y}^2}. \tag{55}$$

○ 3.2 Laminar boundary layer

- According to Blasius[5], the boundary layer equation can be solved using the stream function:

$$u = \frac{\partial\psi}{\partial\tilde{y}}, \; \tilde{v} = -\frac{\partial\psi}{\partial x}. \tag{56}$$

Then, eq. (55) becomes:

$$\frac{\partial\psi}{\partial\tilde{y}}\frac{\partial^2\psi}{\partial x\partial\tilde{y}} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial\tilde{y}^2} = \frac{\partial^3\psi}{\partial\tilde{y}^3}. \tag{57}$$

Theorem 4

If $\psi(x, \tilde{y})$ is solution of (57), then $\sqrt{a}\psi(ax, \sqrt{a}\tilde{y})$ is also solution of (57) for all constant a.

- Taking $ax = 1$, the self-similar solution is then defined as follows

$$\psi = \sqrt{x} f(\eta), \text{ with } \eta = \frac{\tilde{y}}{\sqrt{x}}. \tag{58}$$

[5] H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung, in: Zeitschrift für Mathematik und Physik 56 (1908), pp. 1–37.

- f is solution of

$$2f''' + ff'' = 0, \tag{59}$$

- The boundary conditions are:

$$f(0) = f'(0) = 0, \tag{60}$$

$$\lim_{\eta \to 0} f'(\eta) = 1. \tag{61}$$

- To solve this equation, we transform in a system of Cauchy problem and a shooting method is used to impose the boundary far away the wall.

○ 3.2 Laminar boundary layer

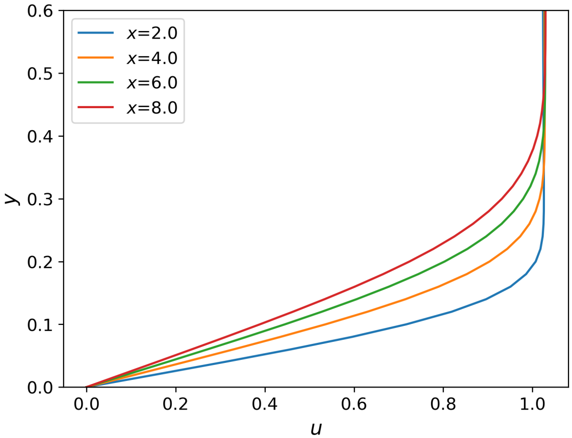

Figure 7: y vs. u obtained with `CIMLIB_CFD` for $\mathrm{Re} = 10^3$ in x=2, 4, 6 & 8.

3. Singular perturbation method

○ 3.2 Laminar boundary layer

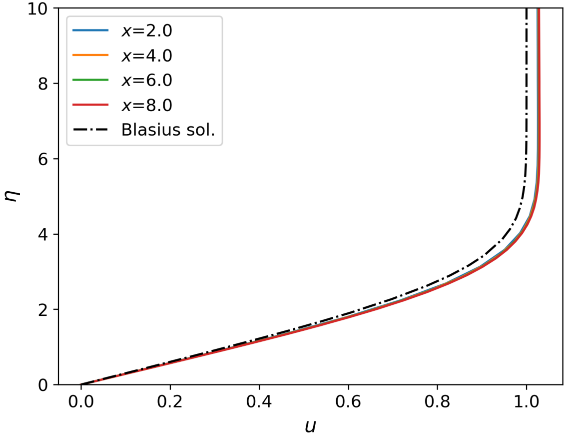

Figure 8: η vs. u obtained with CIMLIB_CFD for $\mathrm{Re} = 10^3$ in x=2, 4, 6 & 8. Comparison to the Blasius solution.

4. Balance equations of heat transfer

○ 4.1 General formulation

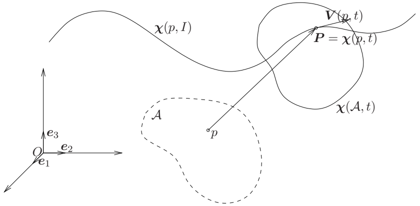

Figure 9: Motion of χ in $\mathcal{R}$.

$$M = \int_{\Omega(t)} \rho dV, \tag{62}$$

$$\boldsymbol{P} = \int_{\Omega(t)} \rho \boldsymbol{u} dV, \tag{63}$$

$$E = \int_{\Omega(t)} \rho e_t dV; \;\; e_t = e + \frac{1}{2}\boldsymbol{u}^2 + \Phi. \tag{64}$$

4.1 General formulation

$$\frac{DM}{Dt} = 0, \tag{65}$$

$$\frac{D\boldsymbol{P}}{Dt} = \int_{\Omega(t)} \rho \boldsymbol{f} dV + \int_{\partial\Omega(t)} \boldsymbol{\sigma} \cdot \boldsymbol{n} dS; \; \boldsymbol{\sigma} = -P\boldsymbol{I} + \boldsymbol{\tau}, \tag{66}$$

$$\frac{DE}{Dt} = \int_{\partial\Omega(t)} \left[-\boldsymbol{q} \cdot \boldsymbol{n} + (\boldsymbol{u} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{n}\right] dS. \tag{67}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{68}$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \rho \boldsymbol{f} - \boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \tag{69}$$

$$\frac{\partial \rho e_t}{\partial t} + \boldsymbol{\nabla} \cdot (\rho e_t \boldsymbol{u}) = \boldsymbol{\nabla} \cdot (-\boldsymbol{q} + \boldsymbol{u} \cdot \boldsymbol{\sigma}). \tag{70}$$

○ 4.1 General formulation

$$
\begin{aligned}
\boldsymbol{q} &= -\lambda \boldsymbol{\nabla} T, \text{ Fourier law,} && (71)\\
\boldsymbol{\tau} &= 2\mu \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{I}\left(\zeta - \frac{2\mu}{3}\right) \boldsymbol{\nabla} \cdot \boldsymbol{u}, \text{ Newtonian fluid,} && (72)\\
\boldsymbol{D}(\boldsymbol{u}) &= \frac{1}{2}\left(\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla}^t \boldsymbol{u}\right). && (73)
\end{aligned}
$$

- λ: thermal conductivity.
- μ: shear viscosity or dynamical viscosity.
- ζ: dilatational viscosity.
- Mostly in nature, ζ is equal to zero (Skokesian behaviour). The trace of the viscous stress is equal to zero.

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$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{74}$$

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$$\rho \boldsymbol{u} \cdot \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \Phi + \boldsymbol{u} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{75}$$

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$$\rho \frac{D}{Dt} \left(\frac{\boldsymbol{u}^2}{2} + \Phi \right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \boldsymbol{D}, \tag{76}$$

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$$\rho \frac{D}{Dt} \left(e + \frac{\boldsymbol{u}^2}{2} + \Phi \right) = \nabla \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \nabla \cdot \boldsymbol{q}, \tag{77}$$

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$$\rho \frac{De}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} - P \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{79}$$

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Equivalent to the first principle of the thermodynamics: $dU = \delta Q - PdV$.

4. Balance equations of heat transfer

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We can use the enthalpy

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From the thermodynamics, we have

$$\rho \frac{Dh}{Dt} = \rho C_p \frac{DT}{Dt} + \left[1 + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P\right] \frac{DP}{Dt}. \tag{82}$$

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$$\rho \frac{Dh}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{81}$$

From the thermodynamics, we have

$$\rho \frac{Dh}{Dt} = \rho C_p \frac{DT}{Dt} + \left[1 + \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_P \right] \frac{DP}{Dt}. \tag{82}$$

$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_P \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{83}$$

○ 4.1 General formulation

For perfect gas,

$$\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P = -1, \tag{84}$$

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot (\lambda \nabla T) + \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{85}$$

○ 4.1 General formulation

For perfect gas,

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$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) + \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{85}$$

For liquid,

$$\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P \approx 0, \tag{86}$$

$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) + \boldsymbol{\tau} : \boldsymbol{D}. \tag{87}$$

In summary, the balance equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{88}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot (2\mu \, \mathring{\boldsymbol{D}}) + \rho \boldsymbol{f}, \tag{89}$$

$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) + \boldsymbol{\tau} : \boldsymbol{D}, \tag{90}$$

$$\mathring{\boldsymbol{D}} = \boldsymbol{D} - \frac{\boldsymbol{\nabla} \cdot \boldsymbol{u}}{3} \boldsymbol{I}. \tag{91}$$

To close the system of equations, an equation of state, $\rho = f(T, P)$ is needed.

- The major of fluids dilate when T increases:

$$\rho = \rho_0 \left[1 - \beta(T - T_0)\right], \tag{92}$$

$$\rho_0\text{: density at } T_0, \tag{93}$$

$$\beta\text{: thermal volumetric coefficient.} \tag{94}$$

- $\beta > 0$ in general.
- Water is an noticeable exception: $\beta < 0$ for $T \in [0, 4]$°C.

- The Boussinesq approximation[6] is used:
  - Fluid is considered as incompressible;
  - The effect of the thermal dilatation is only taken as a source term in the momentum equation.

J. Boussinesq, French mathematician (1842-1929) (my hero!).

[6] J. Boussinesq: Théorie analytique de la chaleur, vol. II, 1903.

○ 4.2 Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{95}$$

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla P + \nabla \cdot [2\mu \boldsymbol{D}(\boldsymbol{u})] - \rho_0 \beta (T - T_0)\boldsymbol{g}, \tag{96}$$

$$\rho_0 C_p \frac{DT}{Dt} = \lambda \nabla^2 T + 2\mu \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u}). \tag{97}$$

- P takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.

4. Balance equations of heat transfer

4.2 Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{95}$$

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla P + \nabla \cdot [2\mu \boldsymbol{D}(\boldsymbol{u})] - \rho_0 \beta (T - T_0)\boldsymbol{g}, \tag{96}$$

$$\rho_0 C_p \frac{DT}{Dt} = \lambda \nabla^2 T. \tag{97}$$

- P takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.
- The viscous dissipation is generally neglected.

4.2 Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{95}$$

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$$\rho_0 C_p \frac{DT}{Dt} = \lambda \nabla^2 T. \tag{97}$$

- P takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.
- The viscous dissipation is generally neglected.
- Fluid motion and the temperature field are fully coupled.

○ 4.2 Boussinesq approximation

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0, && (95)\\
\rho_0 \frac{D\boldsymbol{u}}{Dt} &= -\boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot [2\mu \boldsymbol{D}(\boldsymbol{u})] - \rho_0 \beta (T - T_0)\boldsymbol{g}, && (96)\\
\rho_0 C_p \frac{DT}{Dt} &= \lambda \boldsymbol{\nabla}^2 T. && (97)
\end{aligned}
$$

- P takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.
- The viscous dissipation is generally neglected.
- Fluid motion and the temperature field are fully coupled.
- P loses his “thermodynamic” behaviour: P becomes just a **Lagrangian multiplier**.

5.1 Vertical heated wall

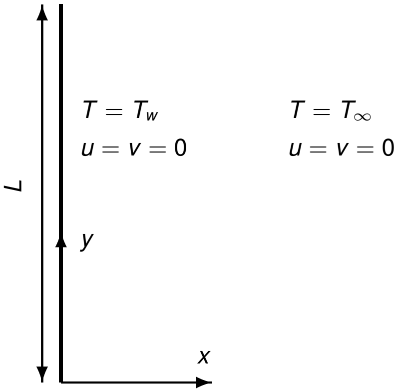

Figure 10: Vertical heated wall geometry with the thermal and kinematic boundary conditions.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{98}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{99}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g\beta(T - T_\infty), \tag{100}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{101}$$

- In natural convection, the order of magnitude of velocity is *a priori* unknown.
- To find an order of magnitude of velocity, a scaling analysis is needed.
- Need to write the balance equations under dimensionless form.
- x & x normalized by L;
- u & v normalized by the unknown characteristic velocity u_0;
- P normalized by δP (unknown);
- T normalized by $T_\infty + \Delta T \theta$ with $\Delta T = T_w - T_\infty$.

5.1 Vertical heated wall

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, && (102)\\
\frac{\rho u_0^2}{L}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= -\frac{\delta P}{L}\frac{\partial P}{\partial x} + \frac{\eta u_0}{L^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), && (103)\\
\frac{\rho u_0^2}{L}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) &= -\frac{\delta P}{L}\frac{\partial P}{\partial y} + \frac{\eta u_0}{L^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g \beta \Delta T \theta, && (104)\\
u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} &= \frac{\kappa}{u_0 L}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right). && (105)
\end{aligned}$$

- The buoyancy source term is the driven force to keep.
- Due to the incompressibility, the gradient of P must be kept.
- From the y-component momentum equation and by balancing inertia with the buoyancy term:

$$u_0 = \sqrt{g\beta\Delta T L}. \quad (106)$$

- The Reynolds number is then

$$\mathrm{Re} = \frac{u_0 L}{\nu} = \sqrt{\mathrm{Gr}}, \text{ with }, \quad (107)$$

$$\mathrm{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \text{ Grashof number.} \quad (108)$$

- The Péclet number defined by

$$\mathrm{Pe} = \frac{u_0 L}{\kappa}, \quad (109)$$

is equal to

$$\mathrm{Pe} = \sqrt{\mathrm{Gr}}\,Pr, \text{ with }, \quad (110)$$

$$Pr = \frac{\nu}{\kappa}, \text{ Prandtl number.} \quad (111)$$

5. Natural convection in open and closed domains IV

5.1 Vertical heated wall

- The problem statement becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{112}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\sqrt{\mathrm{Gr}}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{113}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\sqrt{\mathrm{Gr}}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \theta, \tag{114}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\sqrt{\mathrm{Gr}}\,\mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right). \tag{115}$$

- Solved numerically with `CIMLIB_CFD`, see the case on gitlab:franck.pigeonneau/verticalheatedplate.git.

○ 5.1 Vertical heated wall

Figure 11: θ field obtained numerically for $\mathrm{Gr}=10^4$ and $\mathrm{Pr}=1$.

○ 5.1 Vertical heated wall

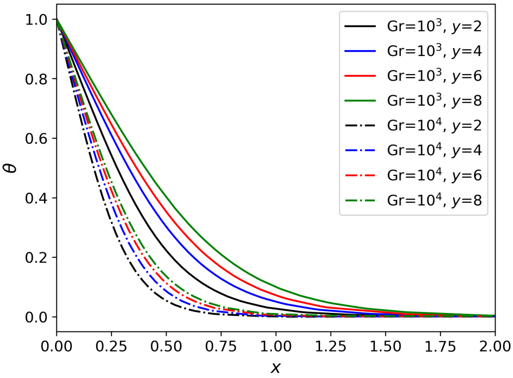

Figure 12: θ vs x for 4 locations in y and for $\mathrm{Gr}=10^3$ & 10^4 and $\mathrm{Pr}=1$.

○ 5.1 Vertical heated wall

- When $\mathrm{Gr} \gg 1$, a boundary layer appears close to the vertical wall on x=0.
- The small parameter is $\epsilon = 1/\sqrt{\mathrm{Gr}}$.
- To find the boundary layer equations, the x-coordinate is stretched:

$$\tilde{x} = \frac{x}{\delta}. \tag{116}$$

- Due to the continuity equation:

$$\tilde{u} = \frac{u}{\delta}. \tag{117}$$

- The balance equations become

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial v}{\partial y} = 0, \tag{118}$$

$$\delta^2 \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial \tilde{x}} + \epsilon \left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \delta^2 \frac{\partial^2 \tilde{u}}{\partial y^2} \right), \tag{119}$$

$$\tilde{u} \frac{\partial v}{\partial \tilde{x}} + v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\epsilon}{\delta^2} \left(\frac{\partial^2 v}{\partial \tilde{x}^2} + \delta^2 \frac{\partial^2 v}{\partial y^2} \right) + \theta, \tag{120}$$

$$\tilde{u} \frac{\partial \theta}{\partial \tilde{x}} + v \frac{\partial \theta}{\partial y} = \frac{\epsilon}{\delta^2 \, \mathrm{Pr}} \left(\frac{\partial^2 \theta}{\partial \tilde{x}^2} + \delta^2 \frac{\partial^2 \theta}{\partial y^2} \right). \tag{121}$$

○ 5.1 Vertical heated wall

- From the principle of least degeneracy:

$$\delta = \sqrt{\epsilon} = \frac{1}{\sqrt[4]{\mathrm{Gr}}}. \tag{122}$$

- Assuming a uniform pressure in the outer area, the boundary layer equations are

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial v}{\partial y} = 0, \tag{123}$$

$$\tilde{u}\frac{\partial v}{\partial \tilde{x}} + v\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial \tilde{x}^2} + \theta, \tag{124}$$

$$\tilde{u}\frac{\partial \theta}{\partial \tilde{x}} + v\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \tilde{x}^2}, \tag{125}$$

with the boundary conditions:

$$\tilde{u} = v = 0,\ \theta = 1,\ \text{in } \tilde{x} = 0, \tag{126}$$

$$\lim_{\tilde{x}\to\infty} \tilde{u} = \lim_{\tilde{x}\to\infty} v = \lim_{\tilde{x}\to\infty} \theta = 0. \tag{127}$$

- Once again, the stream function can be used with

$$\tilde{u} = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial \tilde{x}}. \tag{128}$$

- The two equations to solve are

$$-\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial \tilde{x}^2} + \frac{\partial \psi}{\partial \tilde{x}}\frac{\partial^2 \psi}{\partial y \partial \tilde{x}} = -\frac{\partial^3 \psi}{\partial \tilde{x}^3} + \theta, \tag{129}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial \tilde{x}} - \frac{\partial \psi}{\partial \tilde{x}}\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \tilde{x}^2}. \tag{130}$$

Theorem 5

If $\psi(\tilde{x}, y)$ *and* $\theta(\tilde{x}, y)$ *are solution of* (129) *and* (130) *respectively, then* $a^{3/4}\psi(\sqrt[4]{a}\tilde{x}, ay)$ *and* $\theta(\sqrt[4]{a}\tilde{x}, ay)$ *are also* (129) *and* (130). *This set of solutions belongs to a self-similar group of scale change.*

5.1 Vertical heated wall

- A self-similar solution can be found using

$$\psi = y^{3/4} f(\eta),\ \theta = g(\eta),\ \text{with } \eta = \frac{\tilde{x}}{\sqrt[4]{y}}. \tag{131}$$

- f and g are solution of

$$f''' + \frac{3ff''}{4} - \frac{f'^2}{2} + g = 0, \tag{132}$$

$$g'' - \frac{3\,\mathrm{Pr}\, fg'}{4} = 0. \tag{133}$$

- The boundary conditions are:

$$f(0) = f'(0) = 0,\ g(0) = 1, \tag{134}$$

$$\lim_{\eta\to\infty} f'(\eta) = \lim_{\eta\to\infty} g'(\eta) = 0. \tag{135}$$

- These two equations numerically solved like a Cauchy problem of ODE, see program `python`.

○ 5.1 Vertical heated wall

(a) θ vs. η (b) $v/\sqrt{y}$ vs. η

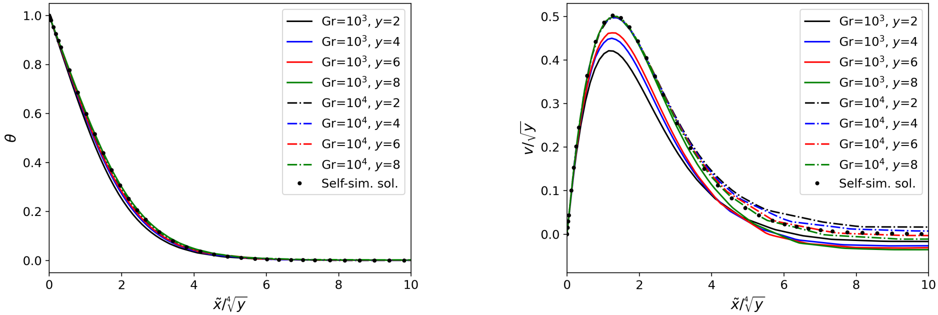

Figure 13: Comparison of the numerical and self-similar solution for Gr=10^3 & 10^4 and Pr=1.

- The heat flux on the wall can be determined as follows

$$\varphi(y) = \lambda \left.\frac{\partial T}{\partial x}\right|_{x=0}. \tag{136}$$

- Using the self-similar solution, $\varphi(y)$ becomes (: here, y is in SI unit.)

$$\varphi(y) = \lambda \Delta T \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \frac{\sqrt[4]{\mathrm{Gr}}}{L^{3/4}\sqrt[4]{y}}. \tag{137}$$

- The heat transfer coefficient is defined by

$$h = \lambda \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \frac{\sqrt[4]{\mathrm{Gr}}}{L^{3/4}\sqrt[4]{y}}. \tag{138}$$

- The local Nusselt number is

$$\mathrm{Nu}(y) = \frac{hL}{\lambda} = \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \sqrt[4]{\mathrm{Gr}}\sqrt[4]{\frac{L}{y}}. \tag{139}$$

- The average Nusselt number over the height of the wall is

$$\langle \mathrm{Nu} \rangle = \frac{4}{3} \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} \sqrt[4]{\mathrm{Gr}}, \tag{140}$$

  or

$$\langle \mathrm{Nu} \rangle = C \sqrt[4]{\mathrm{Gr}}. \tag{141}$$

- C obtained from the self-similar solution when Pr=1 is equal to 0.535 in perfect agreement with the solution given by Whitaker[7].

[7] S. Whitaker: Fundamental Principles of Heat Transfer, 1977, chap. 5.

○ 5.2 Differentially heated square cavity

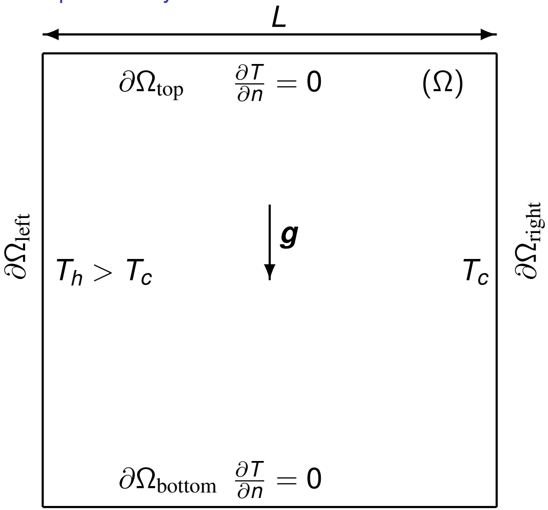

Figure 14: Thermal boundaries on the square cavity. No-slip conditions are taken on $\partial\Omega$.

Theorem 6

Whatever the temperature difference, the fluid in the cavity is in motion.

To proof it, the vorticity is defined by

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{u} = \frac{1}{2}\mathbf{rot}\boldsymbol{u}. \tag{142}$$

The momentum equation can be written as follows

$$\rho_0 \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\boldsymbol{\nabla} P + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} - \rho_0 \beta (T - T_0) \boldsymbol{g}. \tag{143}$$

In tensorial analysis[8], we know that

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \mathbf{rot}\boldsymbol{u} \times \boldsymbol{u} + \boldsymbol{\nabla} \left(\frac{\boldsymbol{u}^2}{2} \right). \tag{144}$$

So, the momentum equation becomes

$$\rho_0 \left[\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{rot}\boldsymbol{u} \times \boldsymbol{u} + \boldsymbol{\nabla} \left(\frac{\boldsymbol{u}^2}{2} \right) \right] = -\boldsymbol{\nabla} P + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} - \rho_0 \beta (T - T_0) \boldsymbol{g}. \tag{145}$$

Taking the rotational of this equation, we obtain:

$$\rho_0 \left[\frac{\partial \mathbf{rot}\boldsymbol{u}}{\partial t} + \mathbf{rot}(\mathbf{rot}\boldsymbol{u} \times \boldsymbol{u}) \right] = \mu \boldsymbol{\nabla}^2 \mathbf{rot}\boldsymbol{u} - \rho_0 \beta \mathbf{rot}[(T - T_0)\boldsymbol{g}]. \tag{146}$$

Using the identity,

$$\mathbf{rot}(\boldsymbol{a} \times \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{a} + \boldsymbol{a} \boldsymbol{\nabla} \cdot \boldsymbol{u} - \boldsymbol{\nabla} \cdot \boldsymbol{a}\boldsymbol{u} - \boldsymbol{\nabla} \boldsymbol{u} \cdot \boldsymbol{a}, \tag{147}$$

and applied for $\boldsymbol{a} = \mathbf{rot}\boldsymbol{u}$ with free-divergence velocity field:

$$\mathbf{rot}(\mathbf{rot}\boldsymbol{u} \times \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{\nabla}(\mathbf{rot}\boldsymbol{u}) - \boldsymbol{\nabla} \boldsymbol{u} \cdot \mathbf{rot}\boldsymbol{u}. \tag{148}$$

Moreover, we use also the identity

$$\mathbf{rot}[(T - T_0)\boldsymbol{g}] = \boldsymbol{\nabla} T \times \boldsymbol{g}, \tag{149}$$

We obtain

$$\rho_0 \left[\frac{D\mathbf{rot}\boldsymbol{u}}{Dt} - \boldsymbol{\nabla}\boldsymbol{u} \cdot \mathbf{rot}\boldsymbol{u} \right] = \mu \boldsymbol{\nabla}^2 \mathbf{rot}\boldsymbol{u} - \rho_0 \beta \boldsymbol{\nabla} T \times \boldsymbol{g}, \tag{150}$$

or with the definition of the vorticity

$$\rho_0 \left[\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\nabla}\boldsymbol{u} \cdot \boldsymbol{\omega} \right] = \mu \boldsymbol{\nabla}^2 \boldsymbol{\omega} - \rho_0 \beta \boldsymbol{\nabla} T \times \boldsymbol{g}, \tag{151}$$

If the thermal gradient is not collinear to the gravity vector, $\boldsymbol{\omega}$ is obviously not equal to zero. The fluid motion always occurs.
The first derivation of this result has been done by Euler in 1764!

[8] R. Aris: Vectors, Tensors and the basic equation of fluid mechanics, New York 1962.

Theorem 7

In the steady-state regime, the thermal flux on the vertical walls defined as follows

$$\Phi_{T,\partial\Omega_{\mathrm{left}}} = \int_{\Omega_{\mathrm{left}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl, \tag{152}$$

$$\Phi_{T,\partial\Omega_{\mathrm{right}}} = \int_{\Omega_{\mathrm{right}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl, \tag{153}$$

verify the relation:

$$\Phi_{T,\partial\Omega_{\mathrm{left}}} + \Phi_{T,\partial\Omega_{\mathrm{right}}} = 0. \tag{154}$$

To proof it, in steady-state regime, we have

$$\frac{\partial T}{\partial t} = 0. \tag{155}$$

So, the energy balance becomes

$$\rho_0 C_p \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T). \tag{156}$$

or for free-divergence velocity

$$\rho_0 C_p \boldsymbol{\nabla} \cdot (\boldsymbol{u} T) = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T). \tag{157}$$

By integration over the whole domain, we obtain:

$$\int_\Omega \rho_0 C_p \boldsymbol{\nabla} \cdot (\boldsymbol{u} T) dS = \int_\Omega \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) dS. \tag{158}$$

Using the theorem of the divergence (Green theorem), we have

$$\int_{\partial\Omega} \rho_0 C_p \boldsymbol{u} \cdot \boldsymbol{n} T dl = \int_{\partial\Omega} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl. \tag{159}$$

Since the fluid velocity vanishes on the wall and due to the adiabatic condition on the two horizontal walls, we obtain

$$\int_{\Omega_{\text{left}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl + \int_{\Omega_{\text{right}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl = 0. \tag{160}$$

This criterion is often used in the numerical computation to control the time convergence.

○ 5.2 Differentially heated square cavity

- Important to set the velocity scaling in the enclosure. Once again, dimensionless form is useful.

$$\nabla \cdot \boldsymbol{u} = 0, \quad (161)$$

$$\frac{\rho_0 u_0^2}{L}\frac{D\boldsymbol{u}}{Dt} = -\frac{\delta P}{L}\nabla P + \frac{\mu u_0}{L^2}\nabla \cdot [2\boldsymbol{D}(\boldsymbol{u})] - \rho_0 g(T_h - T_c)\beta\theta\boldsymbol{e}_g, \quad (162)$$

$$\frac{D\theta}{Dt} = \frac{\kappa}{u_0 L}\nabla^2\theta, \text{ with } \kappa = \frac{\lambda}{\rho_0 C_p}. \quad (163)$$

○ 5.2 Differentially heated square cavity

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c,\ \nu = \mu/\rho_0. \tag{164}$$

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c,\ \nu = \mu/\rho_0. \tag{164}$$

$$\mathrm{Re} = \frac{u_0 L}{\nu} = \mathrm{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \textbf{ Grashof} \text{ number.} \tag{165}$$

○ 5.2 Differentially heated square cavity

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c,\ \nu = \mu/\rho_0. \tag{164}$$

$$\mathrm{Re} = \frac{u_0 L}{\nu} = \mathrm{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \textbf{ Grashof} \text{ number.} \tag{165}$$

2. Low viscous, balance between inertia and buoyancy forces:

$$u_0 = \sqrt{g\beta\Delta T L}. \tag{166}$$

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c, \ \nu = \mu/\rho_0. \tag{164}$$

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$$\mathrm{Re} = \sqrt{\mathrm{Gr}}. \tag{167}$$

5.2 Differentially heated square cavity

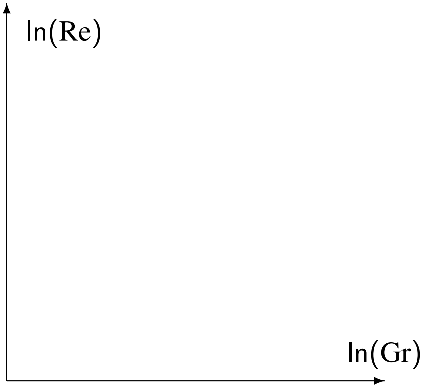

Figure 15: Expected behavior of $\ln(\mathrm{Re})$ vs. $\ln(\mathrm{Gr})$.

5.2 Differentially heated square cavity

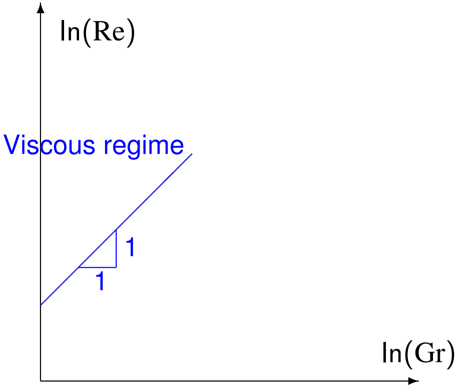

Figure 15: Expected behavior of $\ln(\mathrm{Re})$ vs. $\ln(\mathrm{Gr})$.

○ 5.2 Differentially heated square cavity

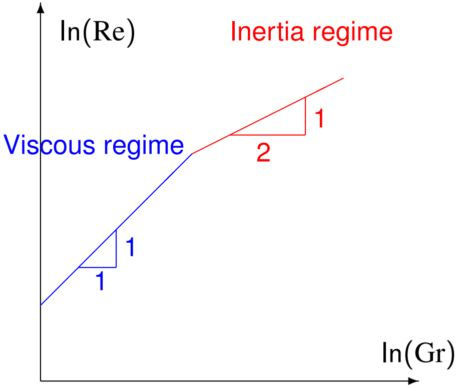

Figure 15: Expected behavior of $\ln(\mathrm{Re})$ vs. $\ln(\mathrm{Gr})$.

5.2 Differentially heated square cavity

- We solved the flow using the scaling of the inertial regime.
- The dimensionless equations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{168}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} P + \frac{1}{\sqrt{\mathrm{Gr}}} \boldsymbol{\nabla}^2 \boldsymbol{u} + \theta \boldsymbol{e}_y, \tag{169}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \frac{1}{\sqrt{\mathrm{GrPr}}} \boldsymbol{\nabla}^2 \theta. \tag{170}$$

- Two dimensionless numbers:

$$\mathrm{Gr} = \frac{g\beta(T_h - T_c)L^3}{\nu^2}, \tag{171}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \tag{172}$$

$$\nu = \frac{\mu}{\rho}, \ \kappa = \frac{\lambda}{\rho_0 C_p}. \tag{173}$$

- `CIMLIB_CFD` case is available on: gitlab:franck.pigeonneau/diffheatingcavity.git.

○ 5.2 Differentially heated square cavity

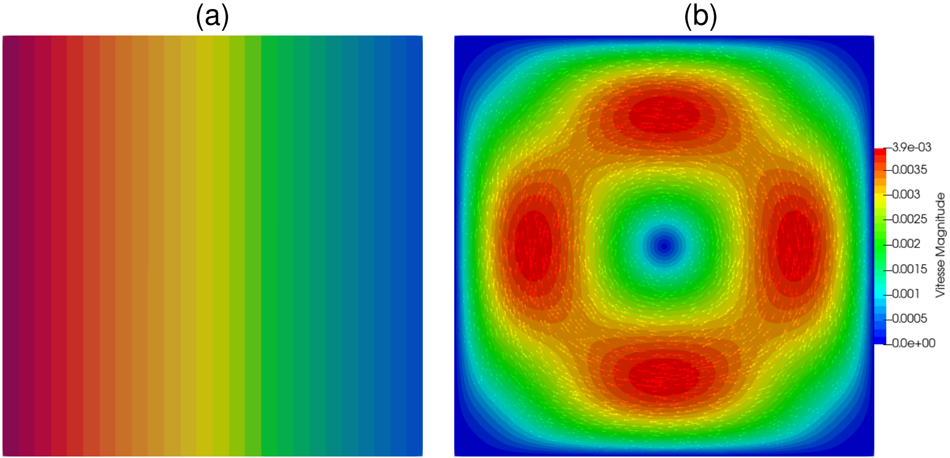

Figure 16: (a) temperature (b) velocity fields for $\mathrm{Gr} = 1$, $\mathrm{Pr} = 1$.

5. Natural convection in open and closed domains

○ 5.2 Differentially heated square cavity

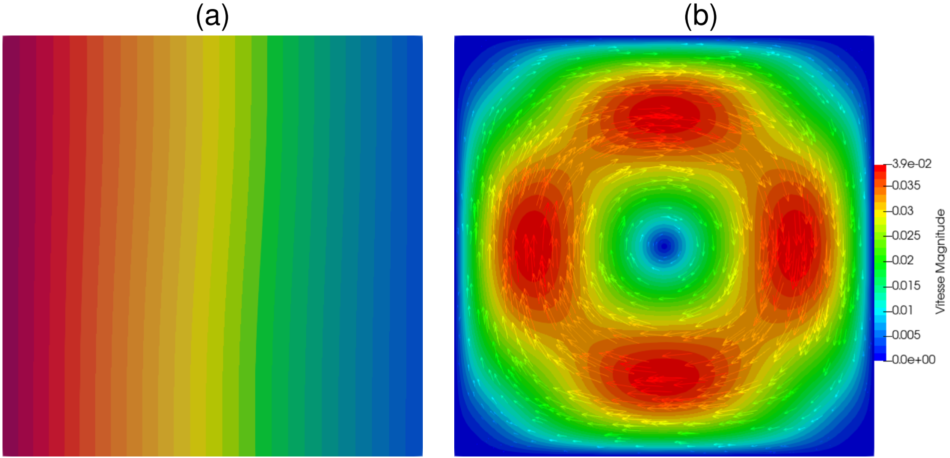

Figure 17: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^2$, $\mathrm{Pr} = 1$.

5. Natural convection in open and closed domains

○ 5.2 Differentially heated square cavity

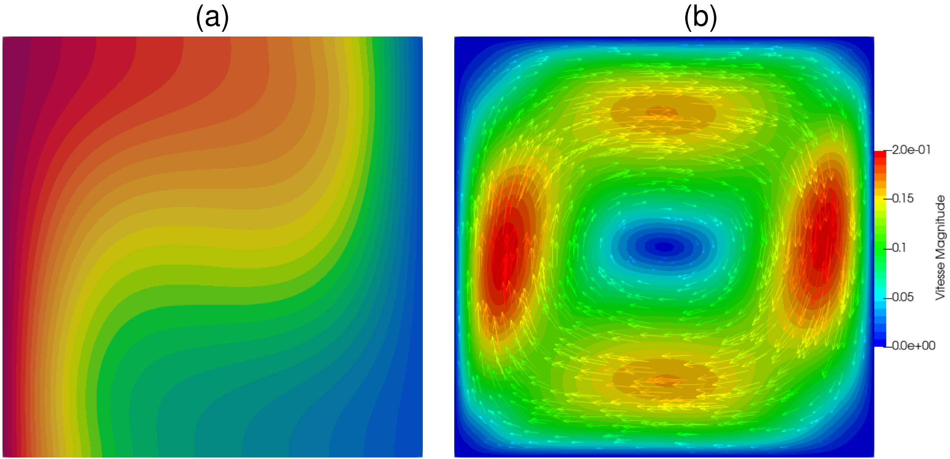

Figure 18: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^4$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity

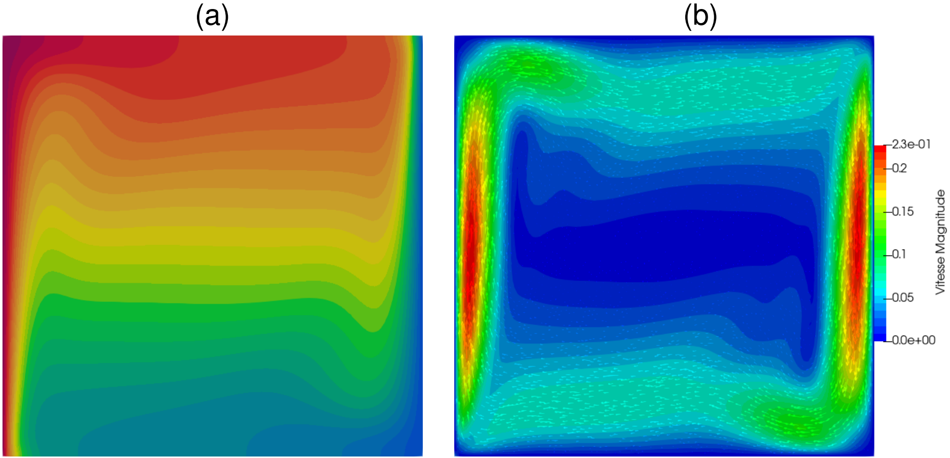

Figure 19: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^6$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity

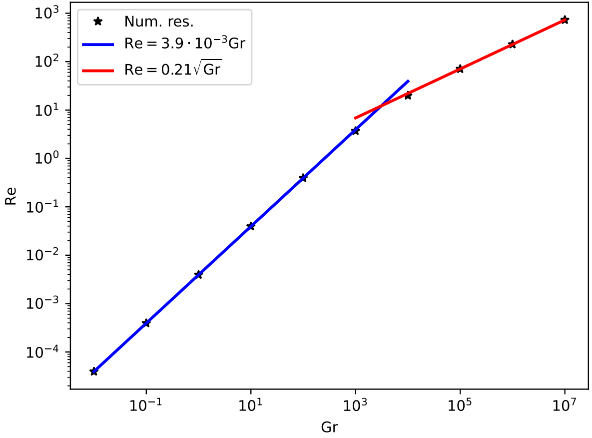

Figure 20: Re vs. Gr.

5. Natural convection in open and closed domains

○ 5.2 Differentially heated square cavity

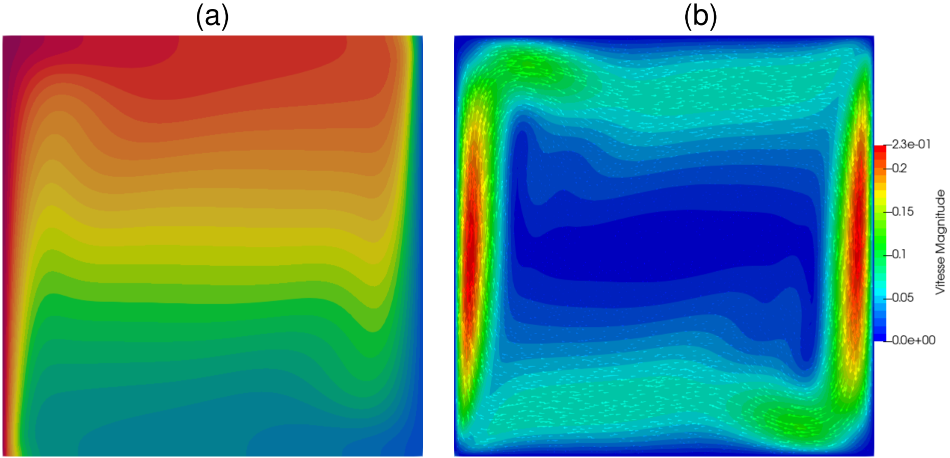

Figure 21: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^6$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity

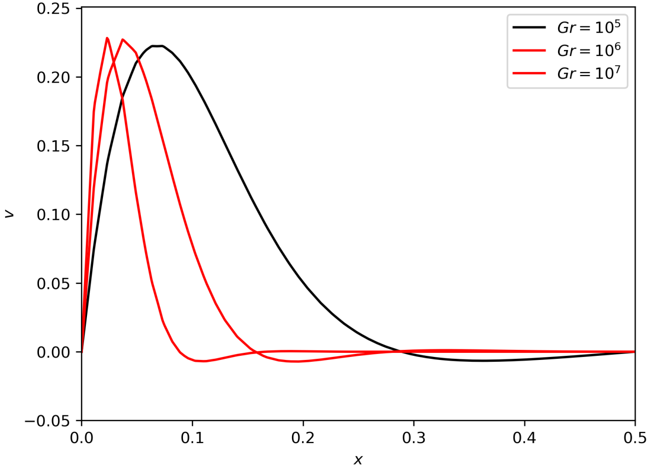

Figure 22: v vs. x on $\partial\Omega_{\text{left}}$ for $\mathrm{Gr} = 10^5$, 10^6 and 10^7.

○ 5.2 Differentially heated square cavity

- Boundary layer appears close to the vertical walls similar to the previous study in open system.
- The boundary layer scales as $1/\sqrt[4]{\mathrm{Gr}}$.

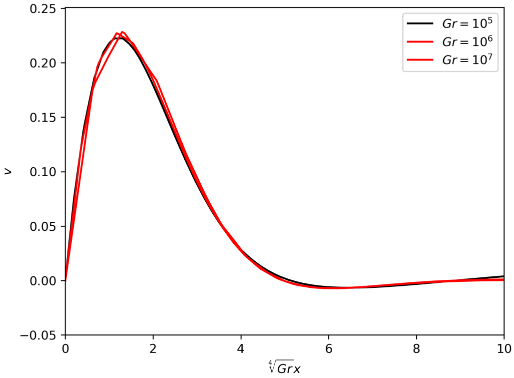

Figure 23: v vs. $x\sqrt[4]{\mathrm{Gr}}$ on $\partial\Omega_{\text{left}}$ for $\mathrm{Gr} = 10^5$, 10^6 and 10^7.

The thermal flux at a boundary is given by

$$\begin{aligned} \varphi &= \lambda \frac{\partial T}{\partial n}, && (174) \\ &= \frac{\lambda}{L} \frac{\partial \theta}{\partial n} \Delta T. && (175) \end{aligned}$$

The thermal heat coefficient is

$$\alpha = \frac{\lambda}{L} \frac{\partial \theta}{\partial n}. \qquad (176)$$

We define the Nusselt number

$$\mathrm{Nu} = \frac{\alpha L}{\lambda} = f(\mathrm{Ra}, \mathrm{Pr}). \qquad (177)$$

○ 5.2 Differentially heated square cavity

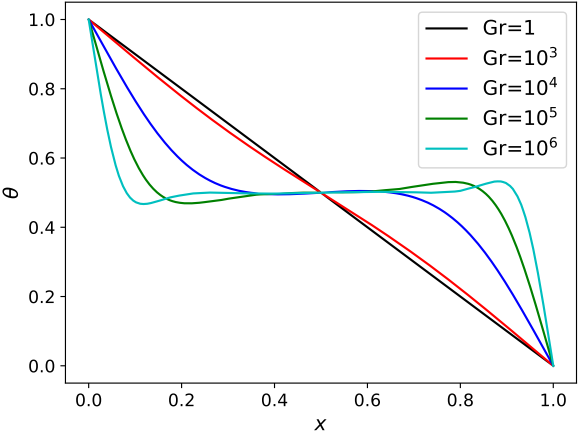

Figure 24: θ vs. x for various Gr.

○ 5.2 Differentially heated square cavity

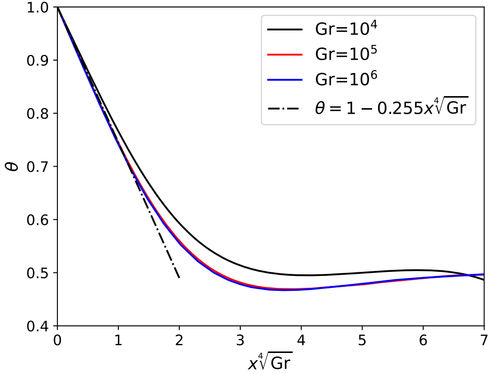

Figure 25: θ vs. $x\sqrt[4]{\mathrm{Gr}}$ for various $\mathrm{Gr} = 10^4$, 10^5 and 10^6.

$$|\mathrm{Nu}| = 0.255\sqrt[4]{\mathrm{Gr}}. \tag{178}$$

6. Synthesis

- Before to investigate a numerical or experimental problem, the scaling analysis must be done:
  - To determine the relevant control parameters (dimensionless numbers);
  - To define scaling laws;
  - To do the best choice of the numerical solvers.
- Crucial to have approximated or exact solutions:
  - To control the accuracy of the numerical solutions.
  - Very useful to write articles coupling numerical solution vs. theory.
- Scaling analysis is a powerful tool to study the turbulent flows.

6. Synthesis

Other beautiful problems not addressed here

(a)

(b)

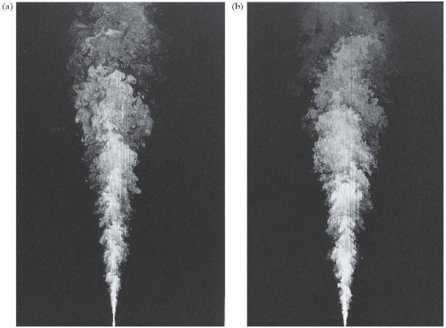

Figure 26: free shear turbulent flows: (a) Kelvin-Helmholtz instability, (b) Turbulent jets, Re=5000 and 20000[9].

[9] S. B. Pope: Turbulent flows, 2000.

6. Synthesis

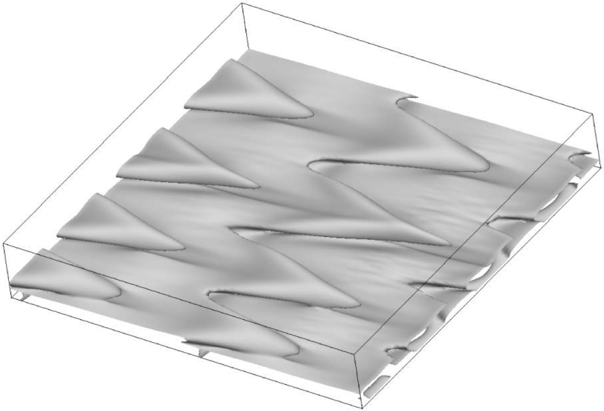

Figure 27: Vorticity of DNS solution of periodic channel flow at high Reynolds number[10].

[10] M. Lesieur: Turbulence in fluids, 4th, 2008.