# MINES PARIS | PSL★

## Cemef

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# ■ Self-similarity, boundary layer & heat transfer in fluid mechanics

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1. Scaling analysis;
2. Self-similarity;
3. Boundary layer theory;
4. Heat transfer in fluid;
5. Learn about the natural convection.

# 2. Self-similar solution of partial differential equation

- ► Fluid mechanics obeys to non-linear partial differential equations.
- ► Exact solutions are very scarce.
- ► Fortunately, particular solutions exist invariant by groups of affinity transformation.
- ► Affinity transformations linked to dimensional changes are remarkable in physics & in particular in fluid mechanics.
- ► Invariant solutions are called "self-similar" solutions.



Figure 1: Barnsley fern created by affinity geometric transformation.

○ 2.1 Rayleigh problem

Let $u(y,t)$ $x$-component of the velocity obeying to

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

$$u(y,0) = 0, \text{ for } t = 0, \tag{2}$$

$$u(0,t) = U, \text{ for } t > 0. \tag{3}$$
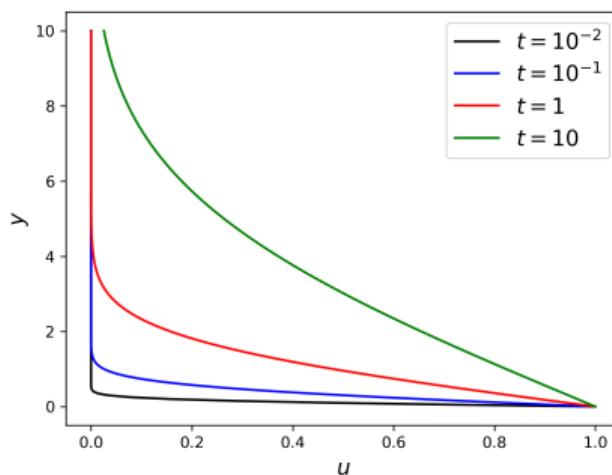


Figure 2: $u$ vs. $y$ at four increasing times.

# 2. Self-similar solution of partial differential equation

○ 2.1 Rayleigh problem

Normalisation of the equation:

$$\bar{u} = \frac{u}{U}, \ \bar{y} = \frac{y}{\ell}, \ \bar{t} = \frac{t}{\tau}, \tag{4}$$

which gives

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\nu\tau}{\ell^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = 0, \tag{5}$$

$$\bar{u}(\bar{y}, 0) = 0, \ \text{for} \ \bar{t} = 0, \tag{6}$$

$$\bar{u}(0, \bar{t}) = 1, \ \text{for} \ \bar{t} > 0. \tag{7}$$

○ 2.1 Rayleigh problem

Normalisation of the equation:

$$\bar{u} = \frac{u}{U}, \ \bar{y} = \frac{y}{\ell}, \ \bar{t} = \frac{t}{\tau}, \tag{4}$$

which gives

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = 0, \tag{5}$$

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$$\bar{u}(0, \bar{t}) = 1, \ \text{for} \ \bar{t} > 0. \tag{7}$$

Takes $\ell = \sqrt{\nu\tau}$.

○ 2.1 Rayleigh problem

## Theorem 1
*If $\bar{u}(\bar{y}, \bar{t})$ is solution of* (5)*, then $\bar{u}(\sqrt{a}\bar{y}, a\bar{t})$ is also solution of* (5) *for all constant a.*

An homogeneous solution of $\bar{u}(\bar{y}, \bar{t})$ can be taken as $f(\bar{y}/\sqrt{\bar{t}})$ with $\eta = \bar{y}/\sqrt{\bar{t}}$ is the "self-similar" variable.
Here, for reason of simplification, $\eta$ is written as

$$\eta = \frac{\bar{y}}{2\sqrt{\bar{t}}} = \frac{y}{2\sqrt{\nu t}}. \tag{8}$$

$f(\eta)$ is then solution of

$$f'' + 2\eta f' = 0, \text{ with } f' = \frac{df}{d\eta}, \tag{9}$$

$$f(0) = 1, \tag{10}$$

$$\lim_{\eta \to \infty} f(\eta) = 0. \tag{11}$$

The exact solution is

$$f(\eta) = \mathrm{erfc}(\eta). \tag{12}$$

# 3. Singular perturbation method

▶ In fluid mechanics, there are various approximations depending on "small" or "large" parameters[1]:

  ▶ Stokes flows: $\text{Re} \ll 1$;
  ▶ Boundary layer: $\text{Re} \gg 1$;
  ▶ Boundary heat & mass layer: $\text{Pe} \gg 1$;
  ▶ Quasi-steady state regime: $\text{St} \ll 1$,
  ▶ ...

▶ The perturbation method is a useful technique to find approximative solution.

## Definition 2
Let's $\epsilon$ a small parameter, if an approximated solution stays valide when $\epsilon \to 0$, the approximation is said regular. Conversely, if the solution is non uniform when $\epsilon \to 0$, the approximation is said singular.

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[1]M. Van Dyke: Perturbation methods in fluid mechanics, Stanford, California 1975.

# 3. Singular perturbation method

○ 3.1 Example of singular ODE

Consider the ordinary differential equation[2]

$$\epsilon y'' + y' + y = 0, \forall x \in ]0, 1[, \tag{13}$$

$$y(0) = 0, \ y(1) = 1, \tag{14}$$

$$\epsilon < 1/4. \tag{15}$$

The exact solution is given by

$$y = \frac{e^{r_1 x} - e^{r_2 x}}{e^{r_1} - e^{r_2}}, \tag{16}$$

$$r_1 = -\frac{1 - \sqrt{1 - 4\epsilon}}{2\epsilon}, \ r_2 = -\frac{1 + \sqrt{1 - 4\epsilon}}{2\epsilon}. \tag{17}$$

The solution is singular since $r_2$ diverges when $\epsilon \to 0$.

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[2]E. J. Hinch: Perturbation Methods, 1991.

# 3. Singular perturbation method

○ 3.1 Example of singular ODE
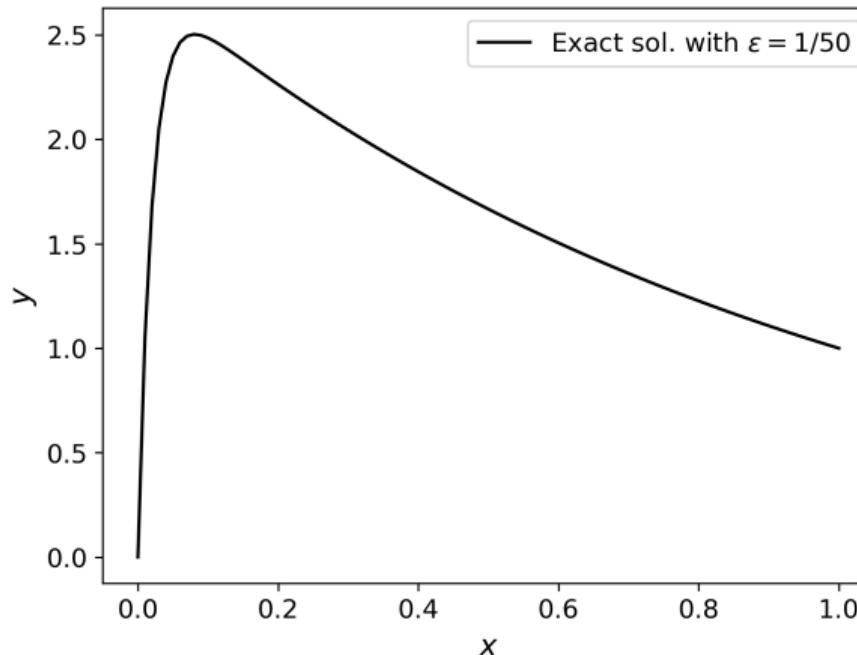


Figure 3: $y$ vs. $x$ for $\epsilon = 1/50$.

# 3. Singular perturbation method

○ 3.1 Example of singular ODE

To have an approximative solution, $y$ can be expanded as $\epsilon^n$ as

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots. \tag{18}$$

After introduction in the ODE, we have

Zeroth order:

$$y_0' + y_0 = 0, \tag{19}$$
$$y_0(0) = 0, \ y_0(1) = 1. \tag{20}$$

First order:

$$y_1' + y_1 = -y_0'', \tag{21}$$
$$y_1(0) = 0, \ y_1(1) = 0. \tag{22}$$

Second order:

$$y_2' + y_2 = -y_1'', \tag{23}$$
$$y_2(0) = 0, \ y_2(1) = 0. \tag{24}$$

▶ The order of ODE is reduced by one order � arrow signature of the singular equation.

▶ Impossible to satisfy the two boundary conditions.

▶ Solution valid only for $x \gg 0$ ➥ **outer** solution.

○ 3.1 Example of singular ODE

▶ To find the adequate approximation close to $x = 0$, **inner** (**stretched**) coordinate has to be introduced:

$$\tilde{x} = \frac{x}{\mu(\epsilon)}, \text{ with } \mu(\epsilon) \ll 1. \tag{25}$$

▶ By written $\tilde{y} = y(\tilde{x})$, the ODE becomes

$$\frac{\epsilon}{\mu^2} \tilde{y}'' + \frac{\tilde{y}'}{\mu} + \tilde{y} = 0. \tag{26}$$

### Definition 3
The "**principle of least degeneracy**" involves that a significant degeneracy of an equation must keep a maximum of terms of the equation[3].

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[3]Van Dyke: Perturbation methods in fluid mechanics (see n. 1).

○ 3.1 Example of singular ODE

▶ If the balance is done between in the first and the third terms:

$$\mu = \sqrt{\epsilon}. \tag{27}$$

▶ The ODE becomes

$$\sqrt{\epsilon}\tilde{y}'' + \tilde{y}' + \sqrt{\epsilon}\tilde{y} = 0. \tag{28}$$

▶ If $\epsilon \to 0$, the ODE is simply $\tilde{y}' = 0$.

▶ If the balance is done between in the first and the second terms:

$$\mu = \epsilon. \tag{29}$$

▶ The ODE becomes

$$\tilde{y}'' + \tilde{y}' + \epsilon\tilde{y} = 0. \tag{30}$$

▶ If $\epsilon \to 0$, the ODE is simply $\tilde{y}'' + \tilde{y}' = 0$.

▶ The principle of least degeneracy involves that the significant approximation is the second case.

# 3. Singular perturbation method

○ 3.1 Example of singular ODE

▶ The outer solution at the zeroth order is

$$y_0(x) = e^{1-x}. \tag{31}$$

▶ The inner solution is

$$\tilde{y}_0(\tilde{x}) = A_0 \left(1 - e^{-\tilde{x}}\right). \tag{32}$$

▶ $A_0$ is unknown. To find it, a matching between the inner and outer solutions have to do.

▶ Introduce a new coordinate:

$$\eta = x/\epsilon^{\alpha}, \text{ with: } 0 < \alpha < 1, \tag{33}$$
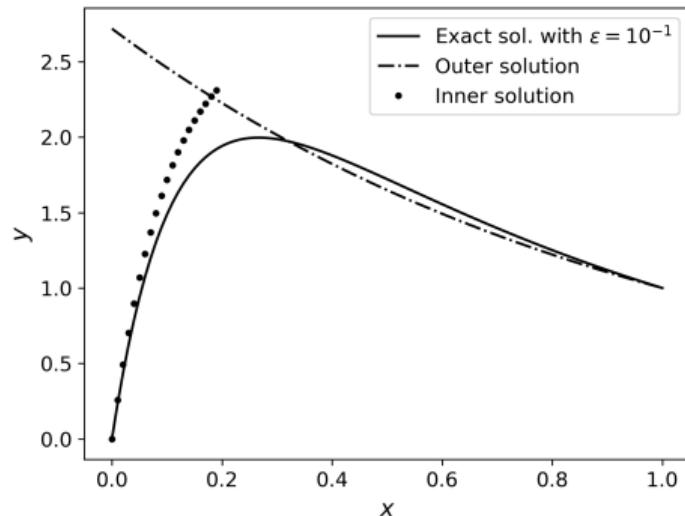
$$x = \epsilon^{\alpha}\eta, \ \tilde{x} = \frac{\eta}{\epsilon^{1-\alpha}}. \tag{34}$$

▶ For a finite value of $\eta$, $x \to 0$ and $\tilde{x} \to \infty$ when $\epsilon \to 0$.

▶ The matching between the two solutions gives: $A_0 = e$.

# 3. Singular perturbation method

○ 3.1 Example of singular ODE



(a) $\epsilon = 10^{-1}$        (b) $\epsilon = 10^{-2}$
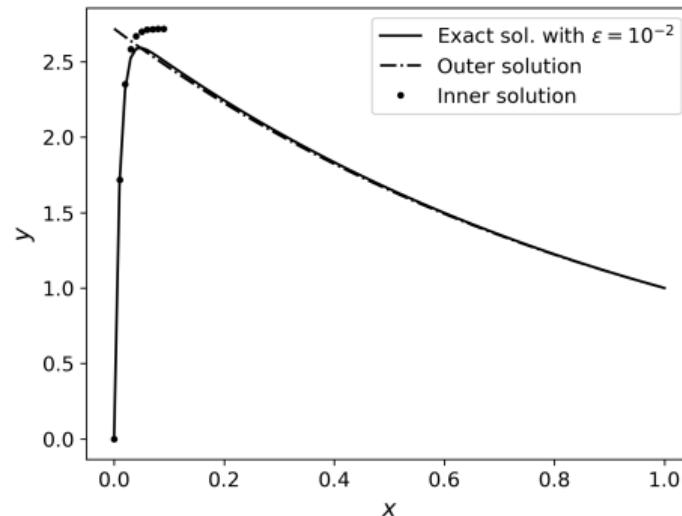
Figure 4: $y$ vs. $x$ with exact, inner and outer solutions.

# 3. Singular perturbation method

○ 3.2 Laminar boundary layer

By using $\mathrm{Re} = UL/\nu$, the 2D Navier-Stokes equations for incompressible fluid are

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{35}
$$

$$
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{36}
$$

$$
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{37}
$$
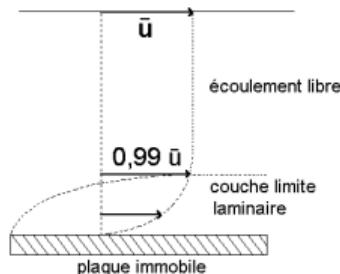


Figure 5: Sketch of the boundary layer close to horizontal wall.

# 3. Singular perturbation method

○ 3.2 Laminar boundary layer

▶ A `CIMLIB_CFD` case of this problem is available on gitlab:franck.pigeonneau/laminarboundlayer.git.



0.0e+00 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0e+00
Vitesse Magnitude
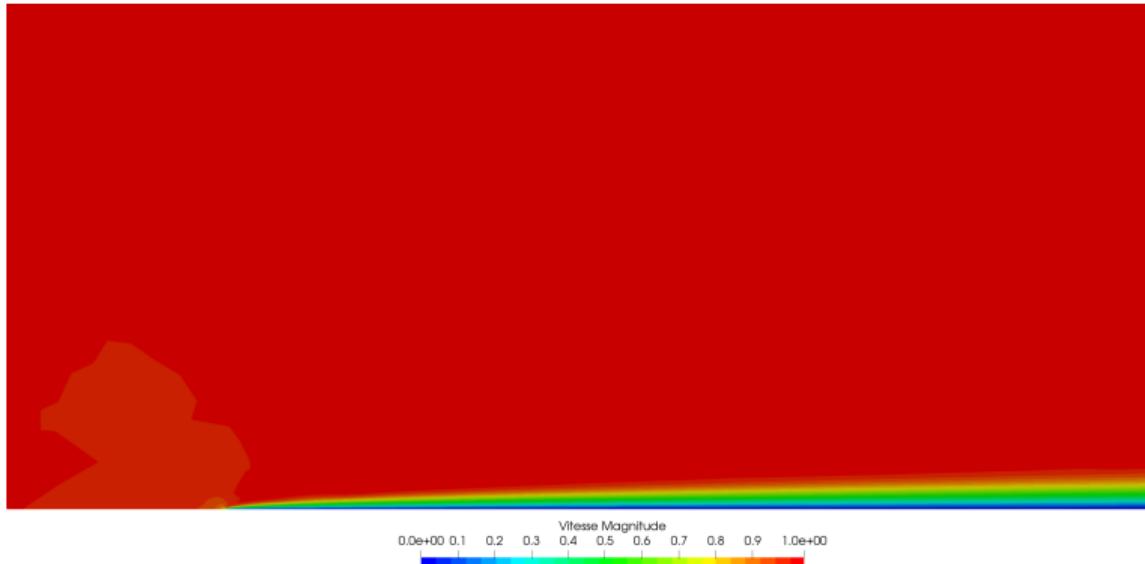
Figure 6: $||\boldsymbol{u}||$ for $\mathrm{Re} = 10^3$.

○ 3.2 Laminar boundary layer

First, we introduce the "small" parameter

$$\epsilon = \frac{1}{\text{Re}}. \tag{38}$$

The N-S equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{39}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{40}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{41}$$

# 3. Singular perturbation method

○ 3.2 Laminar boundary layer

▶ Study the boundary layer leads to study the behaviour of the equations for small $\epsilon$.

▶ As shown above, we use the perturbation method. The solution is developed as a power of $\epsilon$.

▶ In the first approximation, remove all the terms proportional to $\epsilon$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{42}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x}, \tag{43}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y}. \tag{44}$$

▶ Too abusive: **impossible to apply the boundary conditions both on the wall and at the infinity**.

# 3. Singular perturbation method

○ 3.2 Laminar boundary layer

▶ The spatial scales are different along *x* and *y* axis. Let introduce a new scale $\delta$ to normalize the *y* axis:

$$\tilde{y} = \frac{y}{\delta}, \ \tilde{v} = \frac{v}{\delta} \tag{45}$$

▶ The change of the velocity is required to conserve the continuity equation. Now the N-S equations becomes

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{46}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = -\frac{\partial P}{\partial x} + \frac{\epsilon}{\delta^2}\left(\delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \tilde{y}^2}\right), \tag{47}$$

$$\delta^2\left[u\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}\right] = -\frac{\partial P}{\partial \tilde{y}} + \epsilon\left(\delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial \tilde{y}^2}\right). \tag{48}$$

# 3. Singular perturbation method

○ 3.2 Laminar boundary layer

▶ Following the principle of least degeneracy, the simplification of the equations must keep the maximum of terms.

▶ In the case of the boundary layer, we must have

$$\delta^2 = \epsilon, \ \delta = \frac{1}{\sqrt{\text{Re}}}. \tag{49}$$

▶ The Prandtl boundary layer equations[4]:

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{50}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = -\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial \tilde{y}^2}, \tag{51}$$

$$\frac{\partial P}{\partial \tilde{y}} = 0. \tag{52}$$

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[4] L. Prandtl: Zur Berechnung der Grenzschichten, in: Zeitschrift für Angewandte Mathematik und Mechanik 18.1 (1938), pp. 77–82.

# 3. Singular perturbation method

○ 3.2 Laminar boundary layer

▶ Since $P$ is independent on $\tilde{y}$, $P$ is a function of $x$.

▶ $P$ can be matched with the outer (Euler) solution.

▶ Using the Bernoulli relation:

$$\frac{dP}{dx} = -U\frac{dU}{dx}, \tag{53}$$

with $U$ the outer solution.

▶ In the particular case of the horizontal wall, $U = 1$ � the pressure gradient vanishes.

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{54}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = \frac{\partial^2 u}{\partial \tilde{y}^2}. \tag{55}$$

○ 3.2 Laminar boundary layer

▶ According to Blasius[5], the boundary layer equation can be solved using the stream function:

$$u = \frac{\partial \psi}{\partial \tilde{y}}, \ \tilde{v} = -\frac{\partial \psi}{\partial x}. \tag{56}$$

Then, eq. (55) becomes:

$$\frac{\partial \psi}{\partial \tilde{y}} \frac{\partial^2 \psi}{\partial x \partial \tilde{y}} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial \tilde{y}^2} = \frac{\partial^3 \psi}{\partial \tilde{y}^3}. \tag{57}$$

## Theorem 4
*If $\psi(x, \tilde{y})$ is solution of* (57)*, then $\sqrt{a}\psi(ax, \sqrt{a}\tilde{y})$ is also solution of* (57) *for all constant a.*

▶ Taking $ax = 1$, the self-similar solution is then defined as follows

$$\psi = \sqrt{x} f(\eta), \ \text{with} \ \eta = \frac{\tilde{y}}{\sqrt{x}}. \tag{58}$$

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[5]H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung, in: Zeitschrift für Mathematik und Physik 56 (1908), pp. 1–37.

○ 3.2 Laminar boundary layer

► $f$ is solution of

$$2f''' + ff'' = 0, \tag{59}$$

► The boundary conditions are:

$$f(0) = f'(0) = 0, \tag{60}$$

$$\lim_{\eta \to 0} f'(\eta) = 1. \tag{61}$$

► To solve this equation, we transform in a system of Cauchy problem and a shooting method is used to impose the boundary far away the wall.

○ 3.2 Laminar boundary layer
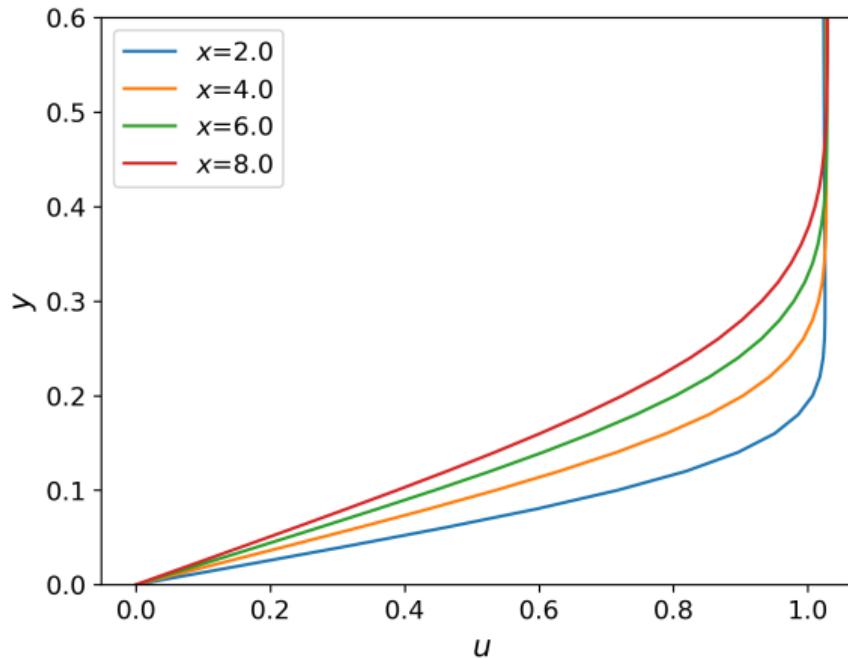


Figure 7: $y$ vs. $u$ obtained with `CIMLIB_CFD` for Re $=10^3$ in $x=2$, 4, 6 & 8.

○ 3.2 Laminar boundary layer
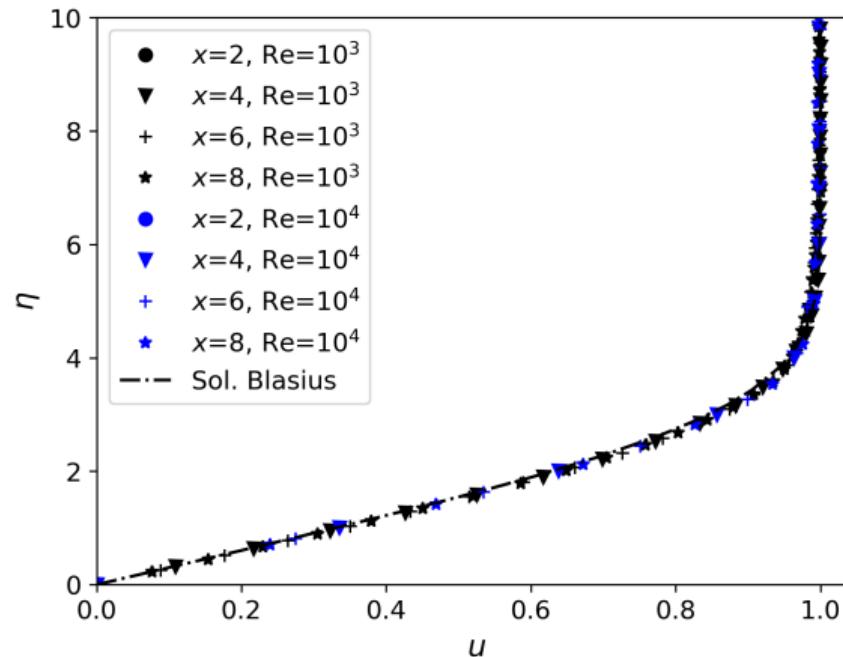


Figure 8: $\eta$ vs. $u$ obtained with `CIMLIB_CFD` for $\mathrm{Re} = 10^3$ in $x$=2, 4, 6 & 8. Comparison to the Blasius solution.

# 4. Balance equations of heat transfer

○ 4.1 General formulation
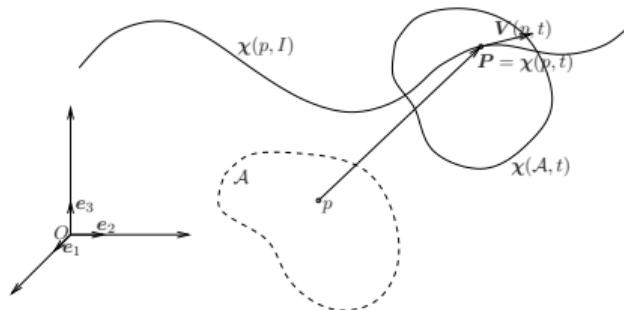


Figure 9: Motion of $\chi$ in $\mathcal{R}$.

$$M = \int_{\Omega(t)} \rho \, dV, \tag{62}$$

$$\boldsymbol{P} = \int_{\Omega(t)} \rho \boldsymbol{u} \, dV, \tag{63}$$

$$E = \int_{\Omega(t)} \rho e_t \, dV; \ e_t = e + \frac{1}{2} \boldsymbol{u}^2 + \Phi. \tag{64}$$

# 4. Balance equations of heat transfer

○ 4.1 General formulation

$$\frac{DM}{Dt} = 0, \tag{65}$$

$$\frac{D\boldsymbol{P}}{Dt} = \int_{\Omega(t)} \rho \boldsymbol{f} dV + \int_{\partial\Omega(t)} \boldsymbol{\sigma} \cdot \boldsymbol{n} dS; \; \boldsymbol{\sigma} = -P\boldsymbol{I} + \boldsymbol{\tau}, \tag{66}$$

$$\frac{DE}{Dt} = \int_{\partial\Omega(t)} [-\boldsymbol{q} \cdot \boldsymbol{n} + (\boldsymbol{u} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{n}] \, dS. \tag{67}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{68}$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \rho \boldsymbol{f} - \boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \tag{69}$$

$$\frac{\partial \rho e_t}{\partial t} + \boldsymbol{\nabla} \cdot (\rho e_t \boldsymbol{u}) = \boldsymbol{\nabla} \cdot (-\boldsymbol{q} + \boldsymbol{u} \cdot \boldsymbol{\sigma}). \tag{70}$$

○ 4.1 General formulation

$$
\begin{aligned}
\boldsymbol{q} &= -\lambda \boldsymbol{\nabla} T, \text{ Fourier law,} & (71) \\
\boldsymbol{\tau} &= 2\mu \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{I} \left( \zeta - \frac{2\mu}{3} \right) \boldsymbol{\nabla} \cdot \boldsymbol{u}, \text{ Newtonian fluid,} & (72) \\
\boldsymbol{D}(\boldsymbol{u}) &= \frac{1}{2} \left( \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla}^t \boldsymbol{u} \right). & (73)
\end{aligned}
$$

▶ $\lambda$: thermal conductivity.

▶ $\mu$: shear viscosity or dynamical viscosity.

▶ $\zeta$: dilatational viscosity.

▶ Mostly in nature, $\zeta$ is equal to zero (Skokesian behaviour). The trace of the viscous stress is equal to zero.

# 4. Balance equations of heat transfer

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○ 4.1 General formulation

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{f} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{74}$$

# 4. Balance equations of heat transfer

○ 4.1 General formulation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, \tag{74}$$

$$\rho \boldsymbol{u}\cdot\frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi + \boldsymbol{u}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, \tag{75}$$

○ 4.1 General formulation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{\nabla} \Phi + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{74}$$

$$\rho \boldsymbol{u} \cdot \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \Phi + \boldsymbol{u} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{75}$$

$$\rho \frac{D}{Dt} \left( \frac{\boldsymbol{u}^2}{2} + \Phi \right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \boldsymbol{D}, \tag{76}$$

○ 4.1 General formulation

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$$\rho \frac{D}{Dt}\left(\frac{\boldsymbol{u}^2}{2} + \Phi\right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \boldsymbol{D}, \tag{76}$$

$$\rho \frac{D}{Dt}\left(e + \frac{\boldsymbol{u}^2}{2} + \Phi\right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\nabla} \cdot \boldsymbol{q}, \tag{77}$$

○ 4.1 General formulation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{\nabla} \Phi + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{74}$$

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$$\rho \frac{D}{Dt}\left(e + \frac{\boldsymbol{u}^2}{2} + \Phi\right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\nabla} \cdot \boldsymbol{q}, \tag{77}$$

$$\rho \frac{De}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \boldsymbol{\sigma} : \boldsymbol{D}, \tag{78}$$

○ 4.1 General formulation

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\rho\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, \tag{74}$$

$$\rho\boldsymbol{u}\cdot\frac{D\boldsymbol{u}}{Dt} = -\rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi + \boldsymbol{u}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, \tag{75}$$

$$\rho\frac{D}{Dt}\left(\frac{\boldsymbol{u}^2}{2} + \Phi\right) = \boldsymbol{\nabla}\cdot(\boldsymbol{u}\cdot\boldsymbol{\sigma}) - \boldsymbol{\sigma}:\boldsymbol{D}, \tag{76}$$

$$\rho\frac{D}{Dt}\left(e + \frac{\boldsymbol{u}^2}{2} + \Phi\right) = \boldsymbol{\nabla}\cdot(\boldsymbol{u}\cdot\boldsymbol{\sigma}) - \boldsymbol{\nabla}\cdot\boldsymbol{q}, \tag{77}$$

$$\rho\frac{De}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{q} + \boldsymbol{\sigma}:\boldsymbol{D}, \tag{78}$$

$$\rho\frac{De}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{q} - P\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\tau}:\boldsymbol{D}. \tag{79}$$

○ 4.1 General formulation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{\nabla} \Phi + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{74}$$

$$\rho \boldsymbol{u} \cdot \frac{D\boldsymbol{u}}{Dt} = -\rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \Phi + \boldsymbol{u} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, \tag{75}$$

$$\rho \frac{D}{Dt} \left( \frac{\boldsymbol{u}^2}{2} + \Phi \right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma} : \boldsymbol{D}, \tag{76}$$

$$\rho \frac{D}{Dt} \left( e + \frac{\boldsymbol{u}^2}{2} + \Phi \right) = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{\nabla} \cdot \boldsymbol{q}, \tag{77}$$

$$\rho \frac{De}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \boldsymbol{\sigma} : \boldsymbol{D}, \tag{78}$$

$$\rho \frac{De}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} - P \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{79}$$

Equivalent to the first principle of the thermodynamics: $dU = \delta Q - P dV$.

○ 4.1 General formulation

We can use the enthalpy

$$h = e + \frac{P}{\rho}. \tag{80}$$

○ 4.1 General formulation

We can use the enthalpy

$$h = e + \frac{P}{\rho}. \tag{80}$$

$$\rho \frac{Dh}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{81}$$

# 4. Balance equations of heat transfer

○ 4.1 General formulation

We can use the enthalpy

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From the thermodynamics, we have

$$\rho \frac{Dh}{Dt} = \rho C_p \frac{DT}{Dt} + \left[ 1 + \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P \right] \frac{DP}{Dt}. \tag{82}$$

# 4. Balance equations of heat transfer

PSL CNTS

○ 4.1 General formulation

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$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) - \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{83}$$

33 / 79

# 4. Balance equations of heat transfer

○ 4.1 General formulation

For perfect gas,

$$\left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P = -1, \tag{84}$$

$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) + \frac{DP}{Dt} + \boldsymbol{\tau} : \boldsymbol{D}. \tag{85}$$

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For liquid,

$$\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P \approx 0, \tag{86}$$

$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) + \boldsymbol{\tau} : \boldsymbol{D}. \tag{87}$$

# 4. Balance equations of heat transfer

○ 4.1 General formulation

In summary, the balance equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{88}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}P + \boldsymbol{\nabla} \cdot (2\mu \, \overset{\circ}{\boldsymbol{D}}) + \rho \boldsymbol{f}, \tag{89}$$

$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) + \boldsymbol{\tau} : \boldsymbol{D}, \tag{90}$$

$$\overset{\circ}{\boldsymbol{D}} = \boldsymbol{D} - \frac{\boldsymbol{\nabla} \cdot \boldsymbol{u}}{3} \boldsymbol{I}. \tag{91}$$

To close the system of equations, an equation of state, $\rho = f(T, P)$ is needed.

○ 4.2 Boussinesq approximation

▶ The major of fluids dilate when $T$ increases:

$$\rho = \rho_0 \left[ 1 - \beta (T - T_0) \right], \tag{92}$$

$$\rho_0: \text{density at } T_0, \tag{93}$$

$$\beta: \text{thermal volumetric coefficient.} \tag{94}$$

▶ $\beta > 0$ in general.
▶ Water is an noticeable exception: $\beta < 0$ for $T \in [0, 4]^\circ C$.

# 4. Balance equations of heat transfer

○ 4.2 Boussinesq approximation

► The Boussinesq approximation[6] is used:
  ► Fluid is considered as incompressible;
  ► The effect of the thermal dilatation is only taken as a source term in the momentum equation.



J. Boussinesq, French mathematician (1842-1929) (my hero!).

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[6]J. Boussinesq: Théorie analytique de la chaleur, vol. II, 1903.

○ 4.2 Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{95}$$

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla P + \nabla \cdot [2\mu \boldsymbol{D}(\boldsymbol{u})] - \rho_0 \beta (T - T_0)\boldsymbol{g}, \tag{96}$$

$$\rho_0 C_p \frac{DT}{Dt} = \lambda \nabla^2 T + 2\mu \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u}). \tag{97}$$

▶ $P$ takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.

○ 4.2 Boussinesq approximation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{95}$$

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot [2\mu \boldsymbol{D}(\boldsymbol{u})] - \rho_0 \beta (T - T_0)\boldsymbol{g}, \tag{96}$$

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▶ The viscous dissipation is generally neglected.

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▶ $P$ takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.
▶ The viscous dissipation is generally neglected.
▶ Fluid motion and the temperature field are fully coupled.

○ 4.2 Boussinesq approximation

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$$\rho_0 C_p \frac{DT}{Dt} = \lambda \boldsymbol{\nabla}^2 T. \tag{97}$$

▶ $P$ takes into account the hydrostatic pressure $-\rho_0 \boldsymbol{g} \cdot \boldsymbol{x}$.

▶ The viscous dissipation is generally neglected.

▶ Fluid motion and the temperature field are fully coupled.

▶ $P$ loses his "thermodynamic" behaviour: $P$ becomes just a **Lagrangian multiplier**.

○ 5.1 Vertical heated wall

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{98}$$

$$\rho \left( u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \eta \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{99}$$

$$\rho \left( u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) = -\frac{\partial P}{\partial y} + \eta \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) +$$
$$\rho g \beta (T - T_\infty), \tag{100}$$

$$\rho C_p \left( u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). \tag{101}$$

$T = T_w$  $\quad T = T_\infty$
$u = v = 0$  $\quad u = v = 0$

$L$

$y$

$x$

Figure 10: Vertical heated wall geometry with the thermal and kinematic boundary conditions.

○ 5.1 Vertical heated wall

▶ In natural convection, the order of magnitude of velocity is *a priori* unknown.

▶ To find an order of magnitude of velocity, a scaling analysis is needed.

▶ Need to write the balance equations under dimensionless form.

▶ $x$ & $x$ normalized by $L$;

▶ $u$ & $v$ normalized by the unknown characteristic velocity $u_0$;

▶ $P$ normalized by $\delta P$ (unknown);

▶ $T$ normalized by $T_\infty + \Delta T \theta$ with $\Delta T = T_w - T_\infty$.

○ 5.1 Vertical heated wall

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{102}$$

$$\frac{\rho u_0^2}{L}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\delta P}{L}\frac{\partial P}{\partial x} + \frac{\eta u_0}{L^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{103}$$

$$\frac{\rho u_0^2}{L}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\delta P}{L}\frac{\partial P}{\partial y} + \frac{\eta u_0}{L^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g\beta\Delta T\theta, \tag{104}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{\kappa}{u_0 L}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right). \tag{105}$$

▶ The buoyancy source term is the driven force to keep.

▶ Due to the incompressibility, the gradient of $P$ must be kept.

▶ From the $y$-component momentum equation and by balancing inertia with the buoyancy term:

$$u_0 = \sqrt{g\beta\Delta T L}. \tag{106}$$

○ 5.1 Vertical heated wall

▶ The Reynolds number is then

$$\mathrm{Re} = \frac{u_0 L}{\nu} = \sqrt{\mathrm{Gr}}, \text{ with },$$ (107)

$$\mathrm{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \text{ Grashof number.}$$ (108)

▶ The Péclet number defined by

$$\mathrm{Pe} = \frac{u_0 L}{\kappa},$$ (109)

is equal to

$$\mathrm{Pe} = \sqrt{\mathrm{Gr}}\,\mathrm{Pr}, \text{ with },$$ (110)

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \text{ Prandtl number.}$$ (111)

○ 5.1 Vertical heated wall

▶ The problem statement becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{112}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\sqrt{\mathrm{Gr}}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{113}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\sqrt{\mathrm{Gr}}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \theta, \tag{114}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\sqrt{\mathrm{Gr}}\,\mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right). \tag{115}$$

▶ Solved numerically with `CIMLIB_CFD`, see the case on
gitlab:franck.pigeonneau/verticalheatedplate.git.

○ 5.1 Vertical heated wall



Figure 11: $\theta$ field obtained numerically for Gr=$10^4$ and Pr=1.

○ 5.1 Vertical heated wall



Figure 12: $\theta$ vs $x$ for 4 locations in $y$ and for Gr=$10^3$ & $10^4$ and Pr=1.

# 5. Natural convection in open and closed domains

○ 5.1 Vertical heated wall

▶ When $\mathrm{Gr} \gg 1$, a boundary layer appears close to the vertical wall on $x$=0.
▶ The small parameter is $\epsilon = 1/\sqrt{\mathrm{Gr}}$.
▶ To find the boundary layer equations, the $x$-coordinate is stretched:

$$\tilde{x} = \frac{x}{\delta}. \tag{116}$$

▶ Due to the continuity equation:

$$\tilde{u} = \frac{u}{\delta}. \tag{117}$$

▶ The balance equations become

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial v}{\partial y} = 0, \tag{118}$$

$$\delta^2 \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial \tilde{x}} + \epsilon \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \delta^2 \frac{\partial^2 \tilde{u}}{\partial y^2} \right), \tag{119}$$

$$\tilde{u} \frac{\partial v}{\partial \tilde{x}} + v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\epsilon}{\delta^2} \left( \frac{\partial^2 v}{\partial \tilde{x}^2} + \delta^2 \frac{\partial^2 v}{\partial y^2} \right) + \theta, \tag{120}$$

$$\tilde{u} \frac{\partial \theta}{\partial \tilde{x}} + v \frac{\partial \theta}{\partial y} = \frac{\epsilon}{\delta^2 \mathrm{Pr}} \left( \frac{\partial^2 \theta}{\partial \tilde{x}^2} + \delta^2 \frac{\partial^2 \theta}{\partial y^2} \right). \tag{121}$$

47 / 79

○ 5.1 Vertical heated wall

▶ From the principle of least degeneracy:

$$\delta = \sqrt{\epsilon} = \frac{1}{\sqrt[4]{\mathrm{Gr}}}. \tag{122}$$

▶ Assuming a uniform pressure in the outer area, the boundary layer equations are

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial v}{\partial y} = 0, \tag{123}$$

$$\tilde{u}\frac{\partial v}{\partial \tilde{x}} + v\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial \tilde{x}^2} + \theta, \tag{124}$$

$$\tilde{u}\frac{\partial \theta}{\partial \tilde{x}} + v\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \tilde{x}^2}, \tag{125}$$

with the boundary conditions:

$$\tilde{u} = v = 0,\ \theta = 1,\ \text{in } \tilde{x} = 0, \tag{126}$$

$$\lim_{\tilde{x}\to\infty} \tilde{u} = \lim_{\tilde{x}\to\infty} v = \lim_{\tilde{x}\to\infty} \theta = 0. \tag{127}$$

○ 5.1 Vertical heated wall

▶ Once again, the stream function can be used with

$$\tilde{u} = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial \tilde{x}}. \tag{128}$$

▶ The two equations to solve are

$$-\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial \tilde{x}^2} + \frac{\partial \psi}{\partial \tilde{x}}\frac{\partial^2 \psi}{\partial y \partial \tilde{x}} = -\frac{\partial^3 \psi}{\partial \tilde{x}^3} + \theta, \tag{129}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial \tilde{x}} - \frac{\partial \psi}{\partial \tilde{x}}\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \tilde{x}^2}. \tag{130}$$

### Theorem 5
*If $\psi(\tilde{x}, y)$ and $\theta(\tilde{x}, y)$ are solution of* (129) *and* (130) *respectively, then $a^{3/4}\psi(\sqrt[4]{a}\tilde{x}, ay)$ and $\theta(\sqrt[4]{a}\tilde{x}, ay)$ are also* (129) *and* (130)*. This set of solutions belongs to a self-similar group of scale change.*

# 5. Natural convection in open and closed domains

○ 5.1 Vertical heated wall

▶ A self-similar solution can be found using

$$\psi = y^{3/4} f(\eta), \; \theta = g(\eta), \text{ with } \eta = \frac{\tilde{x}}{\sqrt[4]{y}}. \tag{131}$$

▶ $f$ and $g$ are solution of

$$f''' + \frac{3ff''}{4} - \frac{f'^2}{2} + g = 0, \tag{132}$$

$$g'' - \frac{3 \Pr fg'}{4} = 0. \tag{133}$$

▶ The boundary conditions are:

$$f(0) = f'(0) = 0, \; g(0) = 1, \tag{134}$$

$$\lim_{\eta \to \infty} f'(\eta) = \lim_{\eta \to \infty} g'(\eta) = 0. \tag{135}$$

▶ These two equations numerically solved like a Cauchy problem of ODE, see program `python`.

○ 5.1 Vertical heated wall

(a) $\theta$ vs. $\eta$

(b) $v/\sqrt{y}$ vs. $\eta$
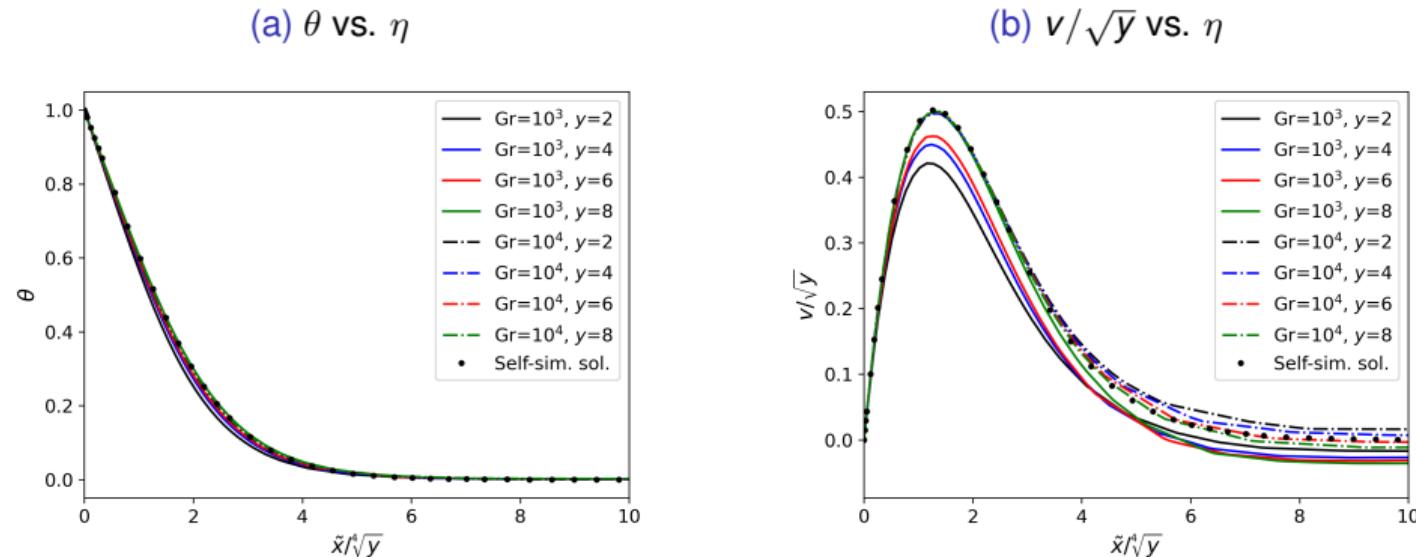


Figure 13: Comparison of the numerical and self-similar solution for $\mathrm{Gr}=10^3$ & $10^4$ and Pr=1.

○ 5.1 Vertical heated wall

▶ The heat flux on the wall can be determined as follows

$$\varphi(y) = \lambda \left. \frac{\partial T}{\partial x} \right|_{x=0}. \tag{136}$$

▶ Using the self-similar solution, $\varphi(y)$ becomes ($\bigotimes$: here, $y$ is in SI unit.)

$$\varphi(y) = \lambda \Delta T \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} \frac{\sqrt[4]{\mathrm{Gr}}}{L^{3/4} \sqrt[4]{y}}. \tag{137}$$

▶ The heat transfer coefficient is defined by

$$h = \lambda \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} \frac{\sqrt[4]{\mathrm{Gr}}}{L^{3/4} \sqrt[4]{y}}. \tag{138}$$

▶ The local Nusselt number is

$$\mathrm{Nu}(y) = \frac{hL}{\lambda} = \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} \sqrt[4]{\mathrm{Gr}} \sqrt[4]{\frac{L}{y}}. \tag{139}$$

○ 5.1 Vertical heated wall

▶ The average Nusselt number over the height of the wall is

$$\langle \mathrm{Nu} \rangle = \frac{4}{3} \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} \sqrt[4]{\mathrm{Gr}}, \tag{140}$$

or

$$\langle \mathrm{Nu} \rangle = C \sqrt[4]{\mathrm{Gr}}. \tag{141}$$

▶ $C$ obtained from the self-similar solution when Pr=1 is equal to 0.535 in perfect agreement with the solution given by Whitaker[7].

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[7]S. Whitaker: Fundamental Principles of Heat Transfer, 1977, chap. 5.

○ 5.2 Differentially heated square cavity
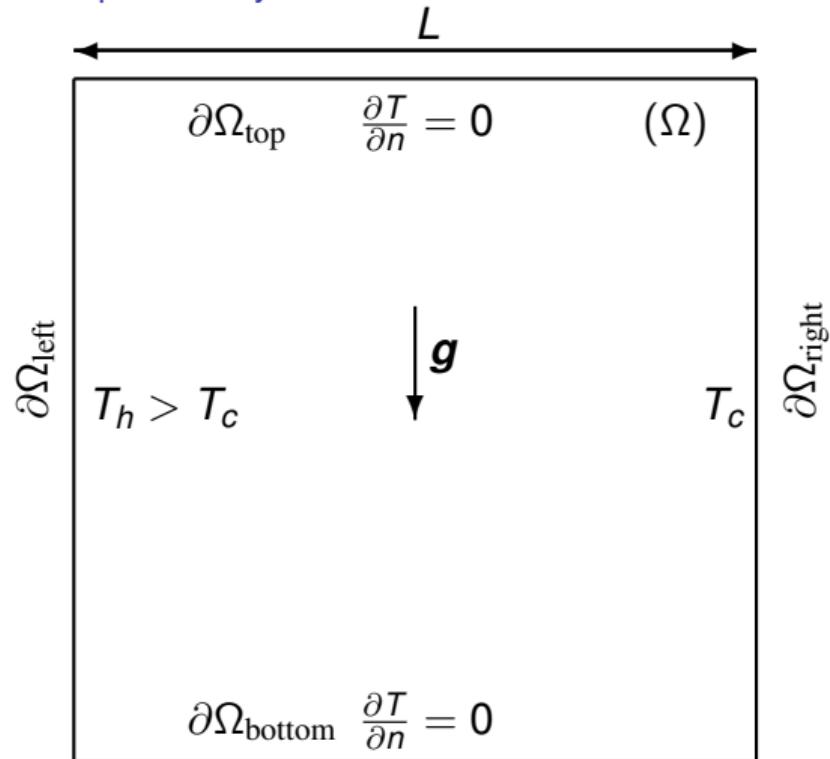


Figure 14: Thermal boundaries on the square cavity. No-slip conditions are taken on $\partial\Omega$.

○ 5.2 Differentially heated square cavity

## Theorem 6

*Whatever the temperature difference, the fluid in the cavity is in motion.*

To proof it, the vorticity is defined by

$$\boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\nabla} \times \boldsymbol{u}. \tag{142}$$

The momentum equation can be written as follows

$$\rho_0 \left( \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\boldsymbol{\nabla} P + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} - \rho_0 \beta (T - T_0) \boldsymbol{g}. \tag{143}$$

In tensorial analysis[8], we know that

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \boldsymbol{\nabla} \left( \frac{\boldsymbol{u}^2}{2} \right). \tag{144}$$

So, the momentum equation becomes

$$\rho_0 \left[ \frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \boldsymbol{\nabla} \left( \frac{\boldsymbol{u}^2}{2} \right) \right] = -\boldsymbol{\nabla} P + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} - \rho_0 \beta (T - T_0) \boldsymbol{g}. \tag{145}$$

○ 5.2 Differentially heated square cavity

Taking the rotational of this equation, we obtain:

$$\rho_0 \left[ \frac{\partial \nabla \times \boldsymbol{u}}{\partial t} + \nabla \times (\nabla \times \boldsymbol{u} \times \boldsymbol{u}) \right] = \mu \boldsymbol{\nabla}^2 \nabla \times \boldsymbol{u} - \rho_0 \beta \nabla \times [(T - T_0)\boldsymbol{g}]. \tag{146}$$

Using the identity,

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{a} + \boldsymbol{a} \boldsymbol{\nabla} \cdot \boldsymbol{u} - \boldsymbol{\nabla} \cdot \boldsymbol{a} \boldsymbol{u} - \boldsymbol{\nabla} \boldsymbol{u} \cdot \boldsymbol{a}, \tag{147}$$

and applied for $\boldsymbol{a} = \nabla \times \boldsymbol{u}$ with free-divergence velocity field:

$$\nabla \times [(\nabla \times \boldsymbol{u}) \times \boldsymbol{u}] = \boldsymbol{u} \cdot \boldsymbol{\nabla}(\nabla \times \boldsymbol{u}) - \boldsymbol{\nabla} \boldsymbol{u} \cdot \nabla \times \boldsymbol{u}. \tag{148}$$

Moreover, we use also the identity

$$\nabla \times [(T - T_0)\boldsymbol{g}] = \boldsymbol{\nabla} T \times \boldsymbol{g}, \tag{149}$$

We obtain

$$\rho_0 \left[ \frac{D\nabla \times \boldsymbol{u}}{Dt} - \boldsymbol{\nabla} \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \right] = \mu \boldsymbol{\nabla}^2 \nabla \times \boldsymbol{u} - \rho_0 \beta \boldsymbol{\nabla} T \times \boldsymbol{g}, \tag{150}$$

○ 5.2 Differentially heated square cavity

or with the definition of the vorticity

$$\rho_0 \left[ \frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\nabla}\boldsymbol{u} \cdot \boldsymbol{\omega} \right] = \mu \boldsymbol{\nabla}^2 \boldsymbol{\omega} - \frac{\rho_0 \beta}{2} \boldsymbol{\nabla} T \times \boldsymbol{g}, \tag{151}$$

If the thermal gradient is not collinear to the gravity vector, $\boldsymbol{\omega}$ is obviously not equal to zero. The fluid motion always occurs.

The first derivation of this result has been done by Euler[9] in 1764!

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[8] R. Aris: Vectors, Tensors and the basic equation of fluid mechanics, New York 1962.

[9] L. Euler: De motu fluidorum a diverso caloris gradu oriundo, in: Novi Commentarii academiae scientiarum Petropolitanae 11 (1767), pp. 232–267.

○ 5.2 Differentially heated square cavity

### Theorem 7

*In the steady-state regime, the thermal flux on the vertical walls defined as follows*

$$\Phi_{T,\partial\Omega_{\text{left}}} = \int_{\Omega_{\text{left}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl, \tag{152}$$

$$\Phi_{T,\partial\Omega_{\text{right}}} = \int_{\Omega_{\text{right}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl, \tag{153}$$

*verify the relation:*

$$\Phi_{T,\partial\Omega_{\text{left}}} + \Phi_{T,\partial\Omega_{\text{right}}} = 0. \tag{154}$$

○ 5.2 Differentially heated square cavity

To proof it, in steady-state regime, we have

$$\frac{\partial T}{\partial t} = 0. \tag{155}$$

So, the energy balance becomes

$$\rho_0 C_p \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T). \tag{156}$$

or for free-divergence velocity

$$\rho_0 C_p \boldsymbol{\nabla} \cdot (\boldsymbol{u} T) = \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T). \tag{157}$$

By integration over the whole domain, we obtain:

$$\int_\Omega \rho_0 C_p \boldsymbol{\nabla} \cdot (\boldsymbol{u} T) dS = \int_\Omega \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} T) dS. \tag{158}$$

○ 5.2 Differentially heated square cavity

Using the theorem of the divergence (Green theorem), we have

$$\int_{\partial\Omega} \rho_0 C_p \boldsymbol{u} \cdot \boldsymbol{n} T dl = \int_{\partial\Omega} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl. \tag{159}$$

Since the fluid velocity vanishes on the wall and due to the adiabatic condition on the two horizontal walls, we obtain

$$\int_{\Omega_{\text{left}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl + \int_{\Omega_{\text{right}}} \lambda \boldsymbol{\nabla} T \cdot \boldsymbol{n} dl = 0. \tag{160}$$

This criterion is often used in the numerical computation to control the time convergence.

○ 5.2 Differentially heated square cavity

▶ Important to set the velocity scaling in the enclosure. Once again, dimensionless form is useful.

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \quad (161)$$

$$\frac{\rho_0 u_0^2}{L} \frac{D\boldsymbol{u}}{Dt} = -\frac{\delta P}{L} \boldsymbol{\nabla} P + \frac{\mu u_0}{L^2} \boldsymbol{\nabla} \cdot [2\boldsymbol{D}(\boldsymbol{u})] - \rho_0 g(T_h - T_c)\beta\theta\boldsymbol{e}_g, \quad (162)$$

$$\frac{D\theta}{Dt} = \frac{\kappa}{u_0 L} \boldsymbol{\nabla}^2 \theta, \text{ with } \kappa = \frac{\lambda}{\rho_0 C_p}. \quad (163)$$

○ 5.2 Differentially heated square cavity

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c,\ \nu = \mu/\rho_0. \tag{164}$$

○ 5.2 Differentially heated square cavity

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c, \ \nu = \mu/\rho_0. \tag{164}$$

$$\mathrm{Re} = \frac{u_0 L}{\nu} = \mathrm{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \textbf{ Grashof } \text{number.} \tag{165}$$

○ 5.2 Differentially heated square cavity

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c, \ \nu = \mu/\rho_0. \tag{164}$$

$$\mathrm{Re} = \frac{u_0 L}{\nu} = \mathrm{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \textbf{ Grashof} \text{ number.} \tag{165}$$

2. Low viscous, balance between inertia and buoyancy forces:

$$u_0 = \sqrt{g\beta\Delta T L}. \tag{166}$$

○ 5.2 Differentially heated square cavity

1. Low inertia, balance between viscous and buoyancy forces:

$$u_0 = \frac{g\beta\Delta T L^2}{\nu}, \text{ with } \Delta T = T_h - T_c, \ \nu = \mu/\rho_0. \tag{164}$$

$$\text{Re} = \frac{u_0 L}{\nu} = \text{Gr} = \frac{g\beta\Delta T L^3}{\nu^2}, \text{ \textbf{Grashof} number.} \tag{165}$$

2. Low viscous, balance between inertia and buoyancy forces:

$$u_0 = \sqrt{g\beta\Delta T L}. \tag{166}$$

$$\text{Re} = \sqrt{\text{Gr}}. \tag{167}$$

○ 5.2 Differentially heated square cavity
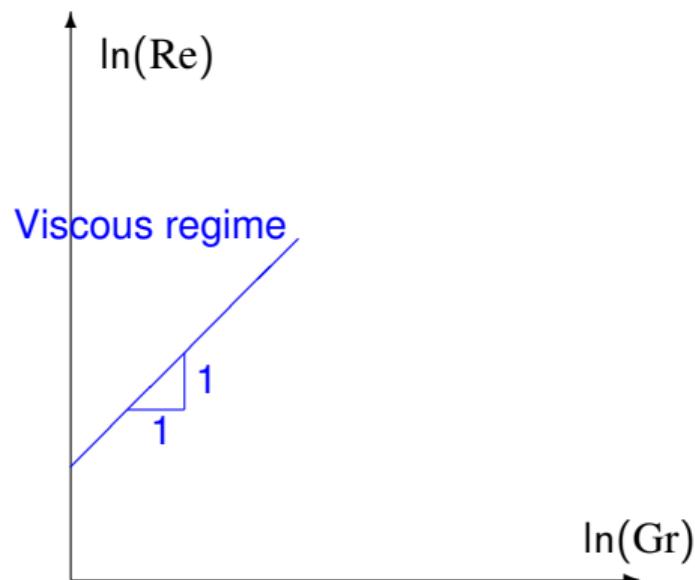


Figure 15: Expected behavior of $\ln(\mathrm{Re})$ vs. $\ln(\mathrm{Gr})$.

○ 5.2 Differentially heated square cavity



Figure 15: Expected behavior of $\ln(\mathrm{Re})$ vs. $\ln(\mathrm{Gr})$.
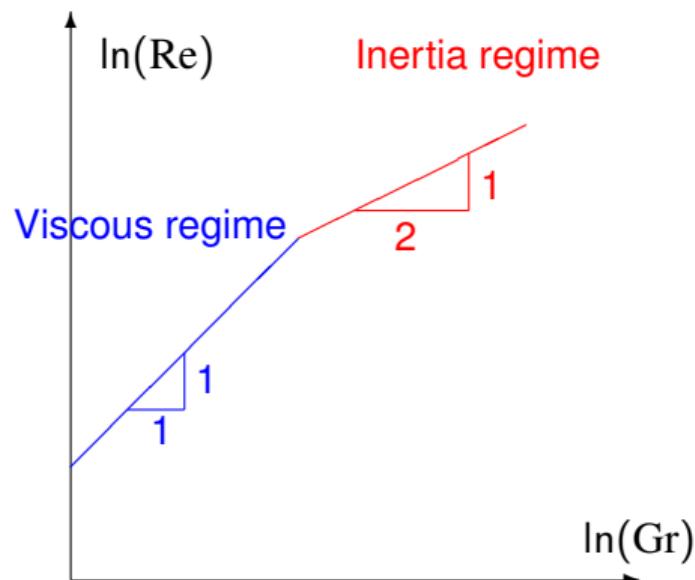
○ 5.2 Differentially heated square cavity



Figure 15: Expected behavior of ln(Re) vs. ln(Gr).

○ 5.2 Differentially heated square cavity

▶ We solved the flow using the scaling of the inertial regime.

▶ The dimensionless equations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{168}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} P + \frac{1}{\sqrt{\mathrm{Gr}}} \boldsymbol{\nabla}^2 \boldsymbol{u} + \theta \boldsymbol{e}_y, \tag{169}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \frac{1}{\sqrt{\mathrm{Gr}}\mathrm{Pr}} \boldsymbol{\nabla}^2 \theta. \tag{170}$$

▶ Two dimensionless numbers:

$$\mathrm{Gr} = \frac{g\beta(T_h - T_c)L^3}{\nu^2}, \tag{171}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \tag{172}$$

$$\nu = \frac{\mu}{\rho}, \ \kappa = \frac{\lambda}{\rho_0 C_p}. \tag{173}$$

▶ `CIMLIB_CFD` case is available on: gitlab:franck.pigeonneau/diffheatingcavity.git.

○ 5.2 Differentially heated square cavity
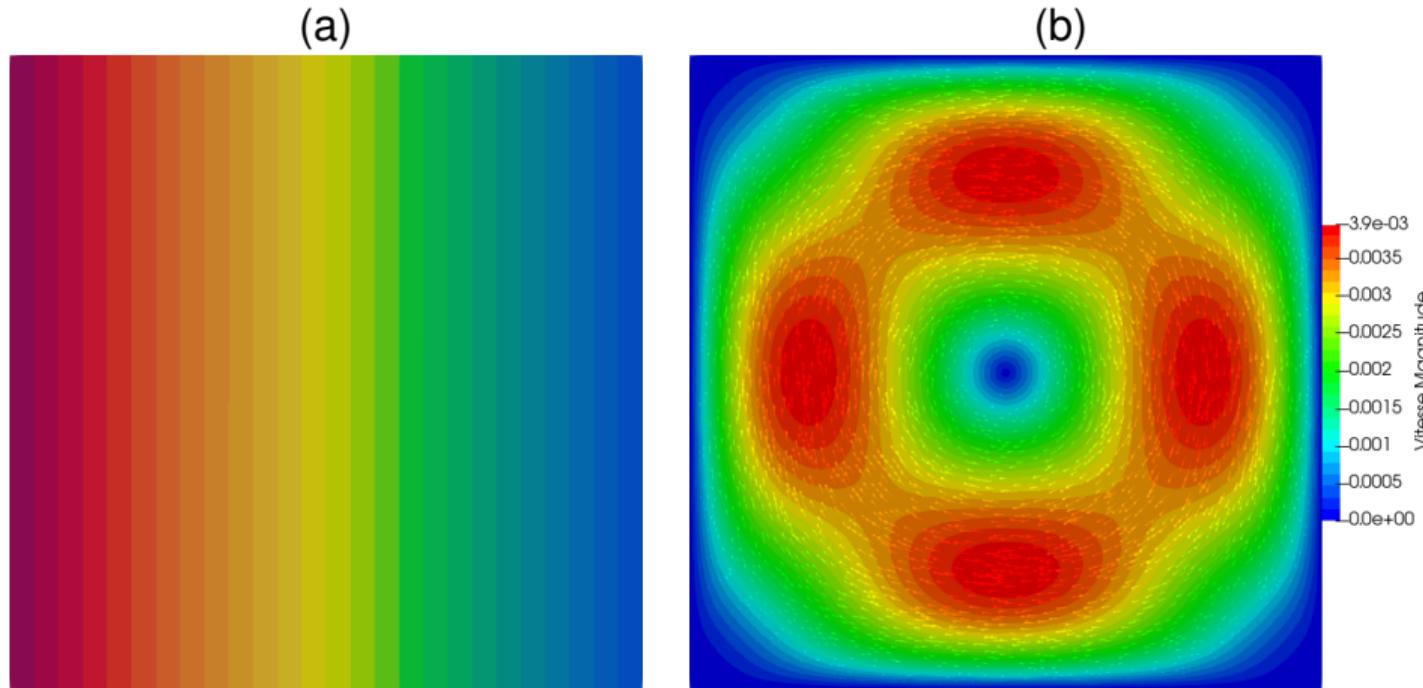


Figure 16: (a) temperature (b) velocity fields for $Gr = 1$, $Pr = 1$.

# 5. Natural convection in open and closed domains

○ 5.2 Differentially heated square cavity
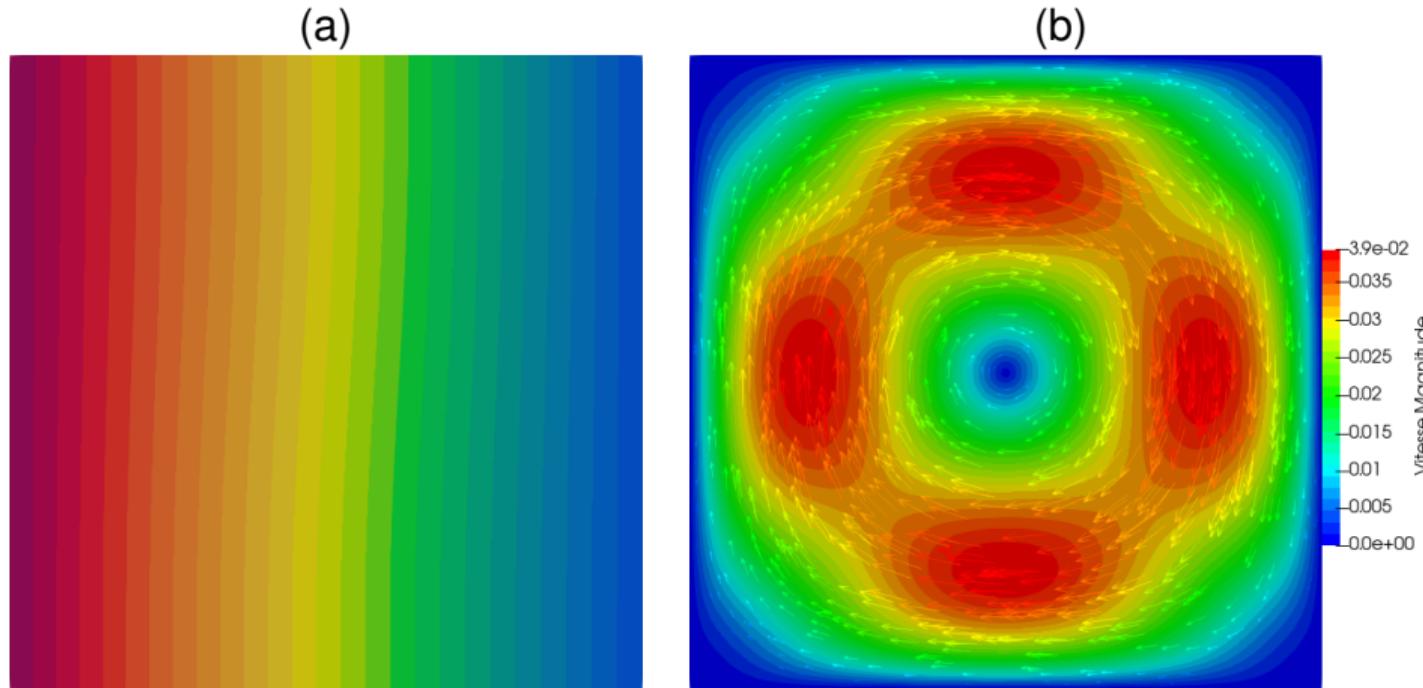


Figure 17: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^2$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity
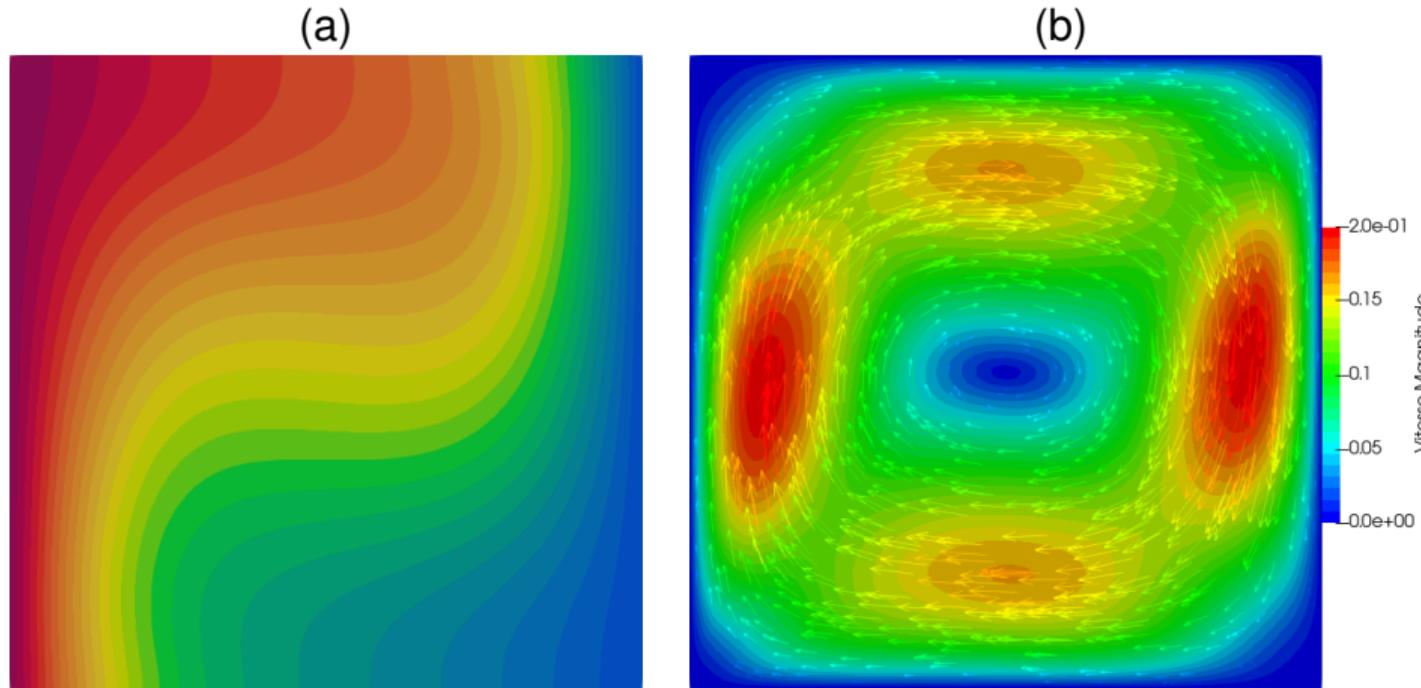


Figure 18: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^4$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity
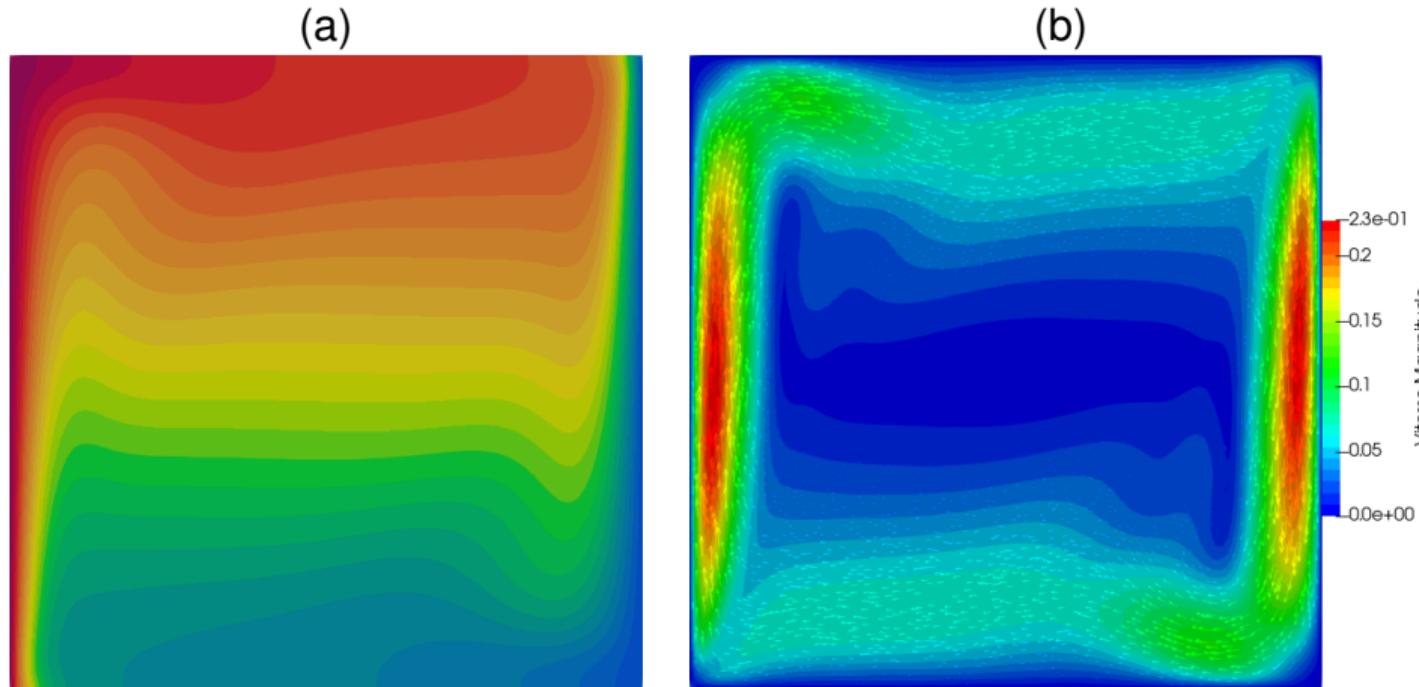


Figure 19: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^6$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity
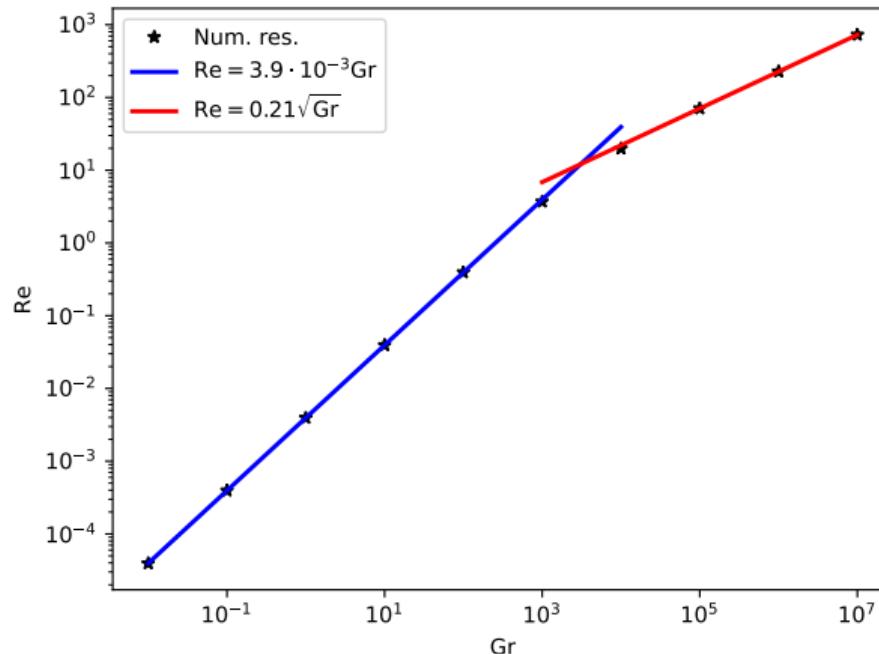


Figure 20: Re vs. Gr.

# 5. Natural convection in open and closed domains

○ 5.2 Differentially heated square cavity
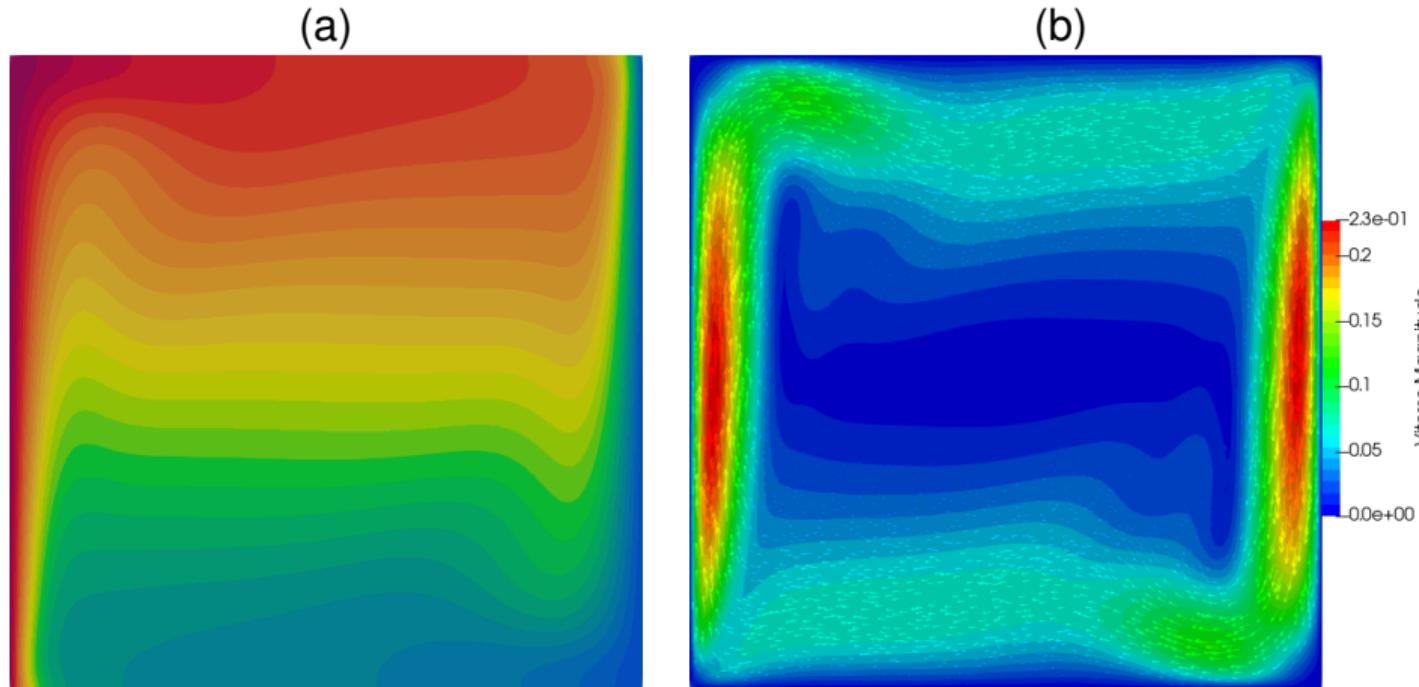


Figure 21: (a) temperature (b) velocity fields for $\mathrm{Gr} = 10^6$, $\mathrm{Pr} = 1$.

○ 5.2 Differentially heated square cavity
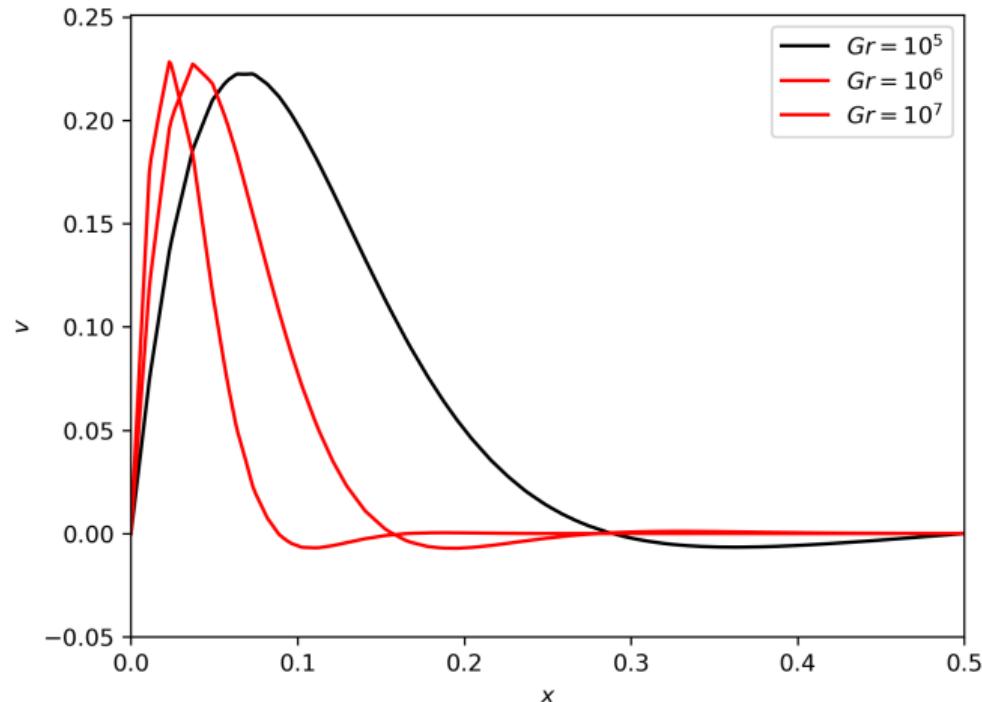


Figure 22: $v$ vs. $x$ on $\partial\Omega_{\text{left}}$ for $\mathrm{Gr} = 10^5$, $10^6$ and $10^7$.

○ 5.2 Differentially heated square cavity

▶ Boundary layer appears close to the vertical walls similar to the previous study in open system.
▶ The boundary layer scales as $1/\sqrt[4]{\text{Gr}}$.
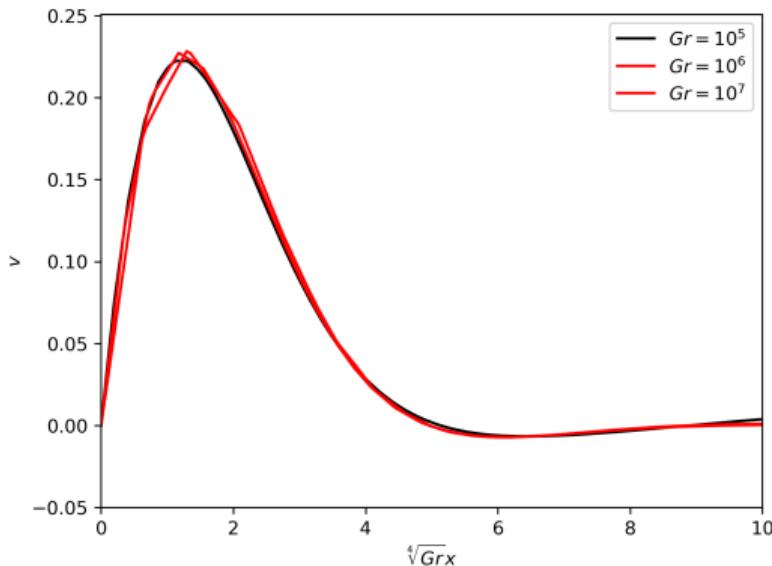


Figure 23: $v$ vs. $x\sqrt[4]{\text{Gr}}$ on $\partial\Omega_{\text{left}}$ for Gr $= 10^5$, $10^6$ and $10^7$.

○ 5.2 Differentially heated square cavity

The thermal flux at a boundary is given by

$$\begin{align}
\varphi &= \lambda \frac{\partial T}{\partial n}, \tag{174} \\
&= \frac{\lambda}{L} \frac{\partial \theta}{\partial n} \Delta T. \tag{175}
\end{align}$$

The thermal heat coefficient is

$$\alpha = \frac{\lambda}{L} \frac{\partial \theta}{\partial n}. \tag{176}$$

We define the Nusselt number

$$\mathrm{Nu} = \frac{\alpha L}{\lambda} = f(\mathrm{Ra}, \mathrm{Pr}). \tag{177}$$

○ 5.2 Differentially heated square cavity
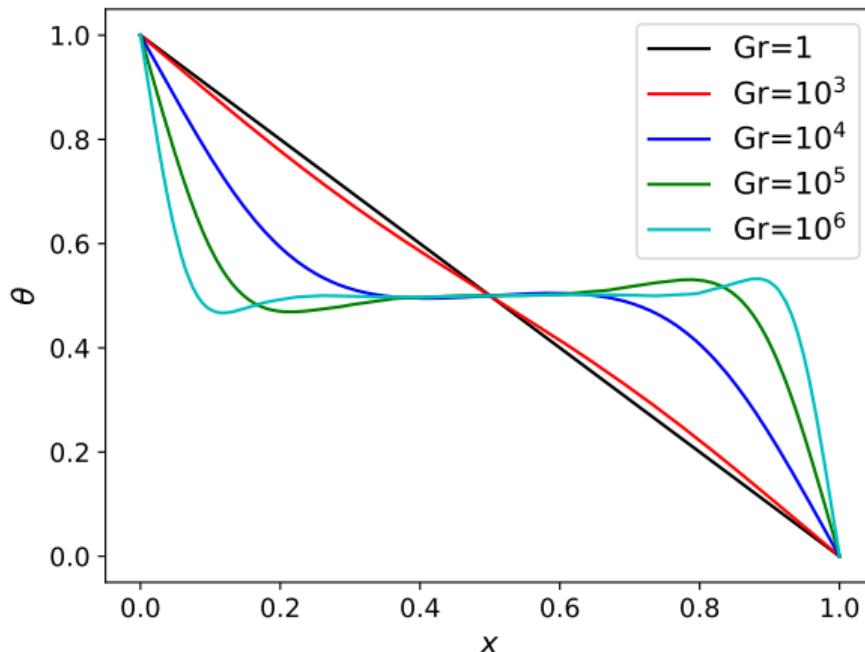


Figure 24: $\theta$ vs. $x$ for various $\mathrm{Gr}$.

# 5. Natural convection in open and closed domains

○ 5.2 Differentially heated square cavity
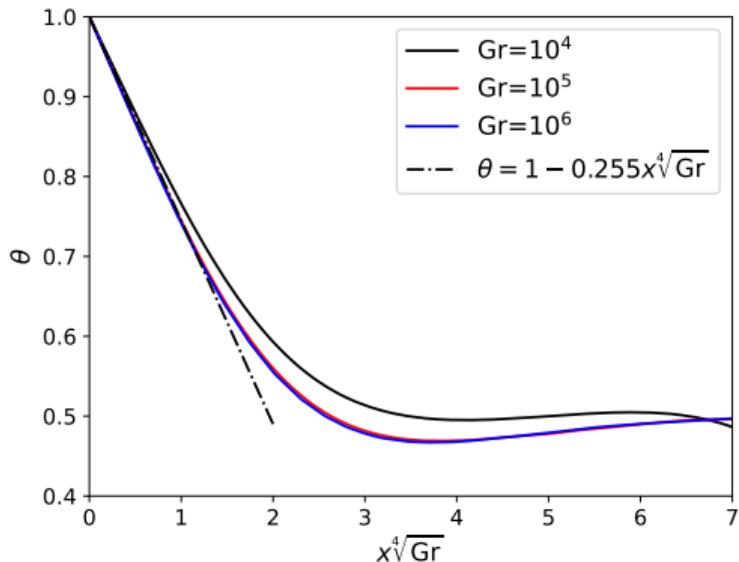


Figure 25: $\theta$ vs. $x\sqrt[4]{\mathrm{Gr}}$ for various $\mathrm{Gr} = 10^4$, $10^5$ and $10^6$.

$$|\mathrm{Nu}| = 0.255\sqrt[4]{\mathrm{Gr}}. \tag{178}$$

- ▶ Before to investigate a numerical or experimental problem, the scaling analysis must be done:
  - ▶ To determine the relevant control parameters (dimensionless numbers);
  - ▶ To define scaling laws;
  - ▶ To do the best choice of the numerical solvers.
- ▶ Crucial to have approximated or exact solutions:
  - ▶ To control the accuracy of the numerical solutions.
  - ▶ Very useful to write articles coupling numerical solution vs. theory.
- ▶ Scaling analysis is a powerful tool to study the turbulent flows.

Other beautiful problems not addressed here

(a)                                                                                      (b)
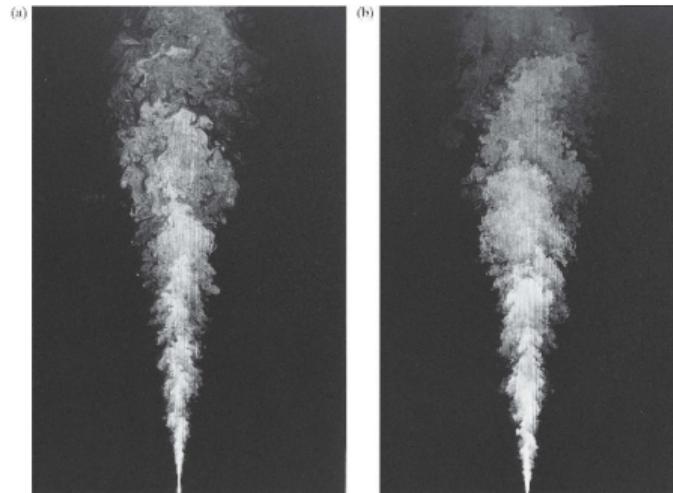


Figure 26: free shear turbulent flows: (a) Kelvin-Helmholtz instability, (b) Turbulent jets, $Re$=5000 and 20000[10].

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[10]S. B. Pope: Turbulent flows, 2000.

# 6. Synthesis
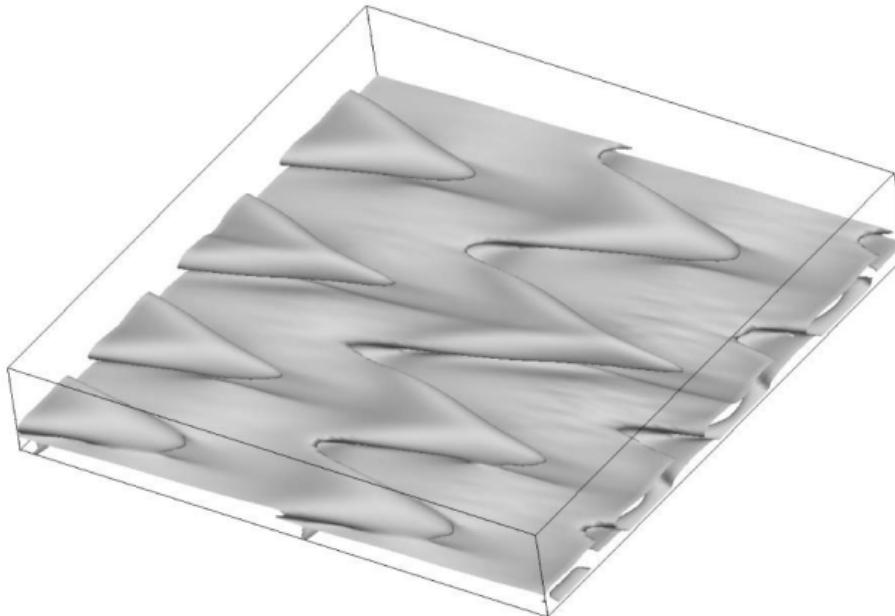
Figure 27: Vorticity of DNS solution of periodic channel flow at high Reynolds number[11].

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[11]M. Lesieur: Turbulence in fluids, 4th, 2008.

# 6. Synthesis

To go further:

▶ G. I. Barenblatt (1996). *Scaling, Self-similarity, and intermediate asymptotics*, Cambridge Univ. Press.

▶ P. Germain (1977). *Méthodes asymptotiques en mécanique des fluides*, in Fluid dynamics, pp. 1-147, Gordon and Breach.

▶ E. J. Hinch (1991). *Perturbation Methods*, Cambridge Univ. Press.

▶ S. B. Pope (2000). *Turbulent flows*, Cambridge Univ. Press.

▶ M. Van Dyke (1975). *Perturbation methods in fluid mechanics*, The parabolic Press.

▶ R. K. Zeytounian (1994). *Modélisation asymptotique en mécanique des fluides newtoniens*, Springer-Verlag.