Neoclassical Growth Model

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Introduction

- We want now an application to our first set of RL methods.
- One of the fields that is quickly adopting these methods is Macroeconomics.
- In this lecture, we will be studying a simple but powerful model where we can apply DP methods.
- It is nice if you master this model, as extensions of it can lead you to answer some interesting economic questions.
- We will illustrate first what this model is about, and two ways of solving it: the brute force way, and the one in which we apply DP.

Introduction

- This is one of the core models in macro to study growth: parsimonious and nice predictive properties.
- General Equilibrium model with two types of agents:
  1. Large number of infinitely lived identical households (representative household).
  2. Large number of identical firms that produce a single good (representative firm).
- It is a closed economy without government model .

Outline

Ingredients

Equilibrium and Social Planner's Problem

Sequential Solution of the Social Planner's Problem

Recursive Representation

Policy Iteration and Value Function Iteration

Households

- Representative household of size L_t:
  - Preferences: intertemporal utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t/L_t)$$

    u satisfies $u' > 0$ y $u'' < 0$ y

$$\lim_{c \to 0} u'(c) = \infty$$

  - Endowments: Labor and capital (rented to firms).

Budget Constraint

- Households face the following budget constraint:

$$C_t + I_t = w_t L_t + r_t K_t + \Pi_t$$

  Price of consumption good normalized to 1, $\forall t$.

- Capital follows a law of motion:

$$K_{t+1} = (1-\delta)K_t + I_t$$

- Number of workers grow at a rate η:

$$L_{t+1} = (1+\eta)L_t$$

Technology

- Aggregate production function.

$$Y_t = F(K_t, L_t)$$

  Technology satisfies (i) constant returns to scale, (ii) concavity y (iii) Inada conditions.

- Objective function: Profits:

$$\Pi_t = Y_t - w_t L_t - r_t K_t$$

Model in intensive units

- (variables c_t, i_t, K_t, y_t expressed in units of the unique good per worker)

$$u\left(\frac{C_t}{L_t}\right) = u(c_t)$$

$$\frac{K_{t+1}}{L_t} = (1+\eta)k_{t+1}$$

$$y_t = \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

with $f' > 0$ y $f'' > 0$ y

$$\lim_{k\to 0} f'(k) = \infty \quad \lim_{k\to\infty} f'(k) = 0$$

- For now, we will assume that there is no population growth.

Outline

- A competitive equilibrium is a set of sequences for the sequences c_t, i_t, y_t y k_{t+1} and prices $w_t y r_t$ such that:
  - Given $k_0 > 0$. w_t y r_t, the quantities $\{c_t.\ i_t$ y $k_{t+1}\}_{t=0}^{\infty}$ solve the problem of the representative household:

$$max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$s.a.$$
$$c_t + i_t = w_t + r_t k_t \quad \forall t$$
$$(1+\eta)k_{t+1} = (1-\delta)k_t + i_t \quad \forall t$$
$$c_t, i_t \geq 0$$

Competitive Equilibrium

- In each period t, given w_t and r_t, the quantities y_t y k_t solve the problem of the representative firm:

$$max \quad y_t - w_t - r_t k_t$$
$$y_t = f(k_t)$$

  and profits are equal to zero.

$$y_t = w_t + r_t k_t + w_t$$

- In each period, markets clear:

$$y_t = c_t + i_t$$

  Including the input markets!

Social Planner's Problem

- In Economics, under very general conditions, there is always a way to characterize allocations that are 'efficient'.
- This 'efficient' allocations come from solving a problem of a 'benevolent' Social Planner.
- It basically consists of giving this planner full control of all resources, and he takes care of allocating the resources in a way that maximizes social welfare.
- In other words, under the solution of the planner, there is no notion of markets. The planner only takes the technology, endowment, and preferences as given, and seeks to maximize welfare based on that.

Social Planner's Problem

- Given $k_0 > 0$, a benevolent Social Planner solves:

$$
\begin{gathered}
Max \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.a. \\
c_t + i_t = f(k_t) \quad \forall t \\
(1+\eta)k_{t+1} = (1-\delta)k_t + i_t \forall t \\
c_t, i_t \geq 0
\end{gathered}
$$

The sequences c_t, i_t y k_{t+1} that result from this optimization are Pareto efficient.

- Without distortions, such as taxes or externalities:
  1. A competitive equilibrium is Pareto efficient (first welfare theorem): your CE is consistent with solving the Planner's problem.
  2. For each Pareto efficient allocation, there exists a price system such that the allocation and such prices constitute a competitive equilibrium (second welfare theorem): Planner's problem can be implemented as a CE.

  Strategy: Characterize the CE and find the prices such that it is consistent with Planner's problem.

First Order Conditions

- Lagrangean of Social Planner's problem:

$$L = \sum_{t=0}^{\infty} [\beta^t u(c_t) - \lambda_{1.t}(c_t + i_t - f(k_t)) - \lambda_{2,t}((1+\eta)k_{t+1} - (1-\delta)k_t - i_t)]$$

  Why can we omit the non-negativity conditions?

- Maximizing with respect to L, we obtain the FOCs:
- Plus the transversality condition:

$$lim_{t\to\infty}\left(\frac{\lambda_{2,t}}{\lambda_{2,0}}k_{t+1}\right)=0$$

where $\lambda_{2,t}$ represents the shadow price of a unit of capital.

- Using the first-order conditions, we can rewrite the transversality condition as:

$$\lim_{t\to\infty}\beta^t\left(\frac{u'(c_t)}{u'(c_0)}\right)k_{t+1}=0$$

- Doing some algebra:
- **Ecuación de Euler:**

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{f'(k_{t+1}) + (1-\delta)}{1+\eta}$$

- **Transversality condition:**

$$c_t = f(k_t) - (1+\eta)k_{t+1} + (1-\delta)k_t$$

Consumption is equal to the final output minus investment.

- Nonlinear system of two first-order difference equations in c_t and k_t, initial condition k_0 and transversality condition.
- Prices are obtained from the firm's problem:

$$max \quad f(k_t) - w_t - r_t k_t$$

from where:

$$r_t = f'(k_t)$$

$$w_t = f(k_t) - f'(k_t)k_t$$

- The optimal paths for c_t y k_t, together with these prices, constitute a CE.

- To verify that we indeed have a CE, we characterize the solution of the representative household:

$$L = \sum_{t=0}^{\infty} [\beta^t u(c_t) - \lambda_{1.t}(c_t + i_t - w_t - r_t k_t) -$$
$$\lambda_{2,t}((1+\eta)k_{t+1} - (1-\delta)k_t - i_t)]$$

- From the first-order conditions and the transversality condition we have:
- **Euler's equation:**

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{r_{t+1} + (1-\delta)}{1+\eta}$$

and the **Feasibility constraint**:

$$c_t = w_t + [r_t + (1-\delta)]k_t - (1+\eta)k_{t+1}$$

- Using the prices that come from the firm's problem:

$$r_t = f'(k_t) \quad w_t + r_t k_t = f(k_t)$$

  Therefore, the CE and the SP are equivalent.

Steady State

- A steady state is a CE in which all quantities per worker are constant over time:

$$c_{t+1} = c_t = c^*$$
$$k_{t+1} = k_t = k^*$$

  Therefore, the quantities in levels C_t y K_t grow at a rate η

- From the Euler's equation:

$$f'(k^*) = \frac{1+\eta}{\beta} - (1-\delta)$$

  Therefore, there exists a unique level for capital in steady state k^*

Outline

- Solving the deterministic problem of the Social Planner's, we obtain a system of equations in difference of first order:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + (1-\delta)$$
$$c_t = f(k_t) - k_{t+1} + (1-\delta)k_t$$

we can write this in general terms as:

$$\Psi_K(k_t, k_{t+1}, c_t, c_{t+1}) = 0$$
$$\Psi_C(k_t, k_{t+1}, c_t, c_{t+1}) = 0$$

- We can also combine both conditions to obtain:

$$\frac{u'[f(k_t) - k_{t+1} + (1-\delta)k_t]}{\beta u'[f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1}]} = f'(k_{t+1}) + (1-\delta)$$

A equation of differences of second order that we can write as:

$$\Psi(k_t, k_{t+1}, k_{t+2}) = 0$$

Finally, we know that k_t converges monotonically to its steady state value:

$$k^* = (f')^{-1}\left[\frac{1}{\beta} - (1-\delta)\right]$$

Problem: Given the functional forms for u, f, and the value for the parameters β and δ,

1. Find sequences of values for k_t, c_t that solve the system of equations in differences $\Psi_K = 0$, $\Psi_C = 0$ or

2. Find a sequence of values for k_t that solve the equation in differences $\Psi(.) = 0$

... with initial condition $k_0 > 0$ and final $\lim_{t \to \infty} k_t = k^*$

Using directly Newton-Raphson

Assuming that the model reaches the steady state in a finite number of periods (T). The approximated solution must satisfy the system of equations:

$$
\begin{aligned}
\Psi_K(k_0, k_1, c_0, c_1) &= 0 \\
\Psi_C(k_0, k_1, c_0, c_1) &= 0 \\
\Psi_K(k_1, k_2, c_1, c_2) &= 0 \\
\Psi_C(k_1, k_2, c_1, c_2) &= 0
\end{aligned}
$$

$$\ldots\ldots\ldots\ldots$$

$$
\begin{aligned}
\Psi_K(k_{T-1}, k_T, c_{T-1}, c_T) &= 0 \\
\Psi_C(k_{T-1}, k_T, c_{T-1}, c_T) &= 0
\end{aligned}
$$

with 2T equations and $2(T+1)$ unknowns (including k_0,c_0,k_T and c_T)

Using directly Newton-Raphson

- The first missing equation is $k_0 = ...$ (whatever it is its initial value).
- The other missing equation can be k_T^* or $k_T = k_{T-1}$.
- We can the solve the system of equations using the Newton-Raphson method (or the secant method).
- There are so many equations (T is at least 100), but it usually works.
- We need to propose initial sequences for k_0^0, ...k_T^0 and c_0^0,...,c_0^T. For example, a straight line between k_0 and $k_T = k^*$.

Using directly Newton-Raphson

- The method can also be applied to the equation in differences of second order in k:

$$\Psi(k_0, k_1, k_2) = 0$$
$$\Psi(k_1, k_2, k_3) = 0$$
$$.........$$
$$\Psi(k_{T-2}, k_{T-1}, k_T) = 0$$

  This time we have $T-1$ equations and $T+1$ unknowns, the missing equations are $k_0 = ...$ and some terminal condition $k_T = k^*$ or $k_T = k_{T-1}$.

- An algorithm for this problem that does not require to solve so many equations simultaneously is the one of Gauss-Seidel.

Outline

- Have you noticed that the Planner's problem can be written as a MDP?
- Our goal: maximize intertemporal utility.
- Our action: Choose the level of capital for tomorrow.
- Our state variable: Everything can be expressed in terms of our capital stock of today.

- In each period, we can decompose:

$$\underbrace{\sum_{s=0}^{\infty} \beta^s u(c_{t+s})}_{v_t} = u(c_t) + \beta \underbrace{\sum_{s=0}^{\infty} \beta^s u(c_{t+1+s})}_{v_{t+1}}$$

The unique difference between the period t and t+1 is the stock of capital.

- If the problem is recursive, we can write:

$$v(k_t) = u(c(k_t)) + \beta v(k'(k_t))$$

where $c(k_t)$ and $k'(k_t)$ are optimal decision rules for consumption and the capital of tomorrow that depend uniquely on the state of the economy (the stock of capital).

Recursive Representation of the Social Planner

- The Social Planner chooses functions $v(k)$, $c(k)$, $i(k)$, $k'(k)$ that solve the Bellman's equation:

$$v_*(k) = max_{c,i,k'} \{u(c) + \beta v_*(k')\}$$
$$s.t.$$
$$c + i = f(k)$$
$$k' = (1-\delta)k + i$$

for all $k > 0$. This is a functional equation in v.

- The Principle of Optimality guarantees that the solution of this problem is equivalent to the solution of the sequential problem:

$$c_t = c(k_t) \quad k_{t+1} = k'(k_t)$$

and also:

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Departing from k_0, we can build recursively:

$$k_1 = k'(k_0) \quad k_2 = k'(k_1) = k'(k'(k_0))...$$

First Order Conditions

Going back to the SPP in recursive language we have:

$$v(k) = max_{k'} \left\{ u \left[f(k) + (1-\delta)k - k' \right] + \beta v(k') \right\}$$

$$s.t.$$

$$k' \in [0, f(k) + (1-\delta)k]$$

First Order Conditions

- Solving this problem (taking all the assumptions of the Neoclassical Growth Model), the first order conditions for an interior solution are:

$$\frac{\partial v(k,k')}{\delta k'} = -u'\left[f(k) + (1-\delta)k - k'\right] + \beta v'(k') = 0$$

and using Benveniste-Scheinkman:

$$v'(k') = u'\left[f(k') + (1-\delta)k' - k''\right]\left(f'(k') + (1-\delta)\right)$$

Replacing and simplifying we obtain the following Euler Equation:

$$\frac{u'\left[f(k) + (1-\delta)k - k'\right]}{\beta u'\left[f(k') + (1-\delta)k' - k''\right]} = f'(k') + (1-\delta)$$

And note the similarity with the Euler Equation obtained in the sequential representation.

Steady State

- In the recursive language, a steady state is a value for k^* such that:

$$k^* = k'(k^*)$$

  where k' is the optimal policy rule of the social planner.

Using the Euler Equation:

$$k^* = (f')^{-1}\left(\frac{1}{\beta} - (1-\delta)\right)$$

The Inada Conditions over f guarantee that there exists a unique steady state k^*

- To go from the SPP to the recursive equilibrium, we need to distinguish the individual state variable k from the aggregate state variable K.
- The prices depend on the aggregate capital, not the individual (perfect competition).
- The consumers choose the law of motion of the individual capital $k'(k, K)$, taking the law of motion of the aggregate capital $K' = \Gamma(K)$ as given.
- In equilibrium, both laws of motion must be consistent.

A recursive competitive equilibrium is a set of functions $v(k,K)$, $c(k,K)$, $i(k,K)$, prices $w(K)$ and $r(K)$, and a law of motion $\Gamma(K)$ such that:

- For each pair (k,K), given the functions w, r and Γ, the value function $v(k,K')$ solves the Bellman's equation:

$$v(k,K) = max_{c,i,k'}\{u(c) + \beta v(k',K')\}$$
$$s.t.$$
$$c + i = w(K) + r(K)k$$
$$k' = (1-\delta)k + i$$
$$K' = \Gamma(K)$$

and $c(k,K)$, $i(k,K)$, $k'(k,K)$ are optimal decision rules for this problem.

- For each K, prices satisfy the marginal conditions of the firm:

$$r(K) = f'(K)$$
$$w(K) = f(K) - f'(K)K$$

- For each K, the markets clear:

$$f(K) = c(K, K) + i(K, K)$$

- For each K, the law of motion is consistent with the decisions of the agents:

$$\Gamma(K) = k'(K, K)$$

- Once the competitive equilibrium is solved, departing from $k_0 > 0$ given, we can build a sequence for the stock of capital:

$$k_1 = k'(k_0, k_0)$$
$$k_2 = k'(k_1, k_1) = k'\left(k'(k_0, k_0), k'(k_0, k_0)\right)$$
$$\dots\dots\dots\dots\dots\dots$$

 and for the rest of variables:

$$c_t = c(k_t, k_t), \quad i_t = i(k_t, k_t)$$
$$w_t = w(k_t) \quad r_t = r(k_t)$$

- The principle of optimality guarantees that those sequences are the same that one would obtain by solving the sequential competitive equilibrium.

Outline

Ingredients

Equilibrium and Social Planner’s Problem

Sequential Solution of the Social Planner’s Problem

Recursive Representation

Policy Iteration and Value Function Iteration

- We know already what is the problem we want to solve.
- This problem is suitable to be solved with DP methods, because we fully know the dynamics of the model.
- In our DP session, we learned two general algorithms to solve these kind of problems:
  1. Policy Iteration.
  2. Value Iteration.
- In this section, we implement such algorithms.

- Whether we implement one algorithm or other, we first need to set up some ingredients for the numerical implementation.
- First, we need to specify some functional forms for our utility function and our technology:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad f(k) = k^{1-\alpha}$$

- Therefore, we also need to set parameter values for β, σ, α, and δ.
- We also need to set a grid for capital $K_{grid} = \{k_{min}, ..., k_{max}\}$.
- We want to obtain policy functions $k'(k)$ and value functions $v(k)$.

Policy Iteration

- Let's start solving the Planner's problem by policy iteration.
- Remember that to implement policy iteration, we need to perform two big steps:
  1. Policy Evaluation.
  2. Policy Improvement.
- Remember that I told you two lectures ago that your state defines your set of possible actions? You can't consume negative amounts of goods:

$$k' \in \Gamma(k) = [0, f(k) + (1-\delta)k].$$

- Therefore, in our numerical implementation, we have to make sure we are only choosing from our feasible set.
- To initialize the algorithm, we first need a guess of a policy: $\pi_0(k)$, arbitrary and a guess for $v_{\pi 0} = 0$.
- We also need a tolerance criterion for convergence: $tol = 10^{-6}$.

- One way of enforcing that we choose in our feasible set is by making $U(f(k) - k' + (1-\delta)k) = -\infty$ for all choices that are not in our feasibility set.
- Then, departing from our initial policy, we perform the following loop:
- **Policy Evaluation:**

1. Use the Bellman Equation as an operator:

$$v_{\pi 0}^{new}(k) = U(f(k) - \pi_0(k) + (1-\delta)k) + \beta v_{\pi 0}^{old}(k)$$

2. Compute $d = \max |v_{\pi 0}^{new}(k) - v_{\pi 0}^{old}(k)|$. If $d < tol$, store $v_{\pi 0}^{new}(k)$. Otherwise, go to the previous step.

Policy Iteration

- Once we have evaluated our policy, the next step is to improve it. To do this, we generate a policy π_{impro} such that:

$$v_{\pi impro}(k) = \max_{k'} U(f(k) - k' + (1-\delta)k) + \beta v_{\pi 0}^{new}(k)$$

- Compute $d_{pol} = \max|\pi_{impro} - \pi_0|$. If $d_{pol} < tol$, then $v_*(k) = v_{\pi impro}$ and $\pi_*(k) = \pi_{impro}$.

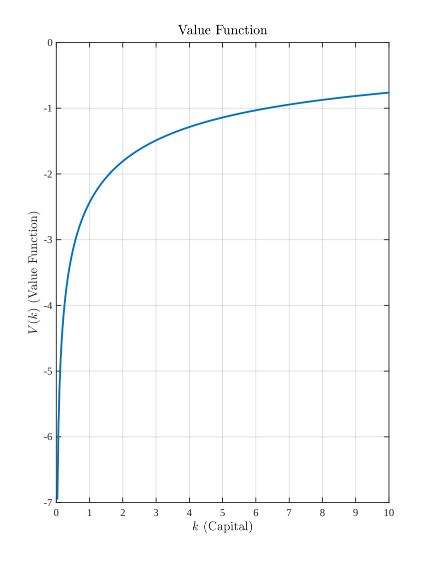
- If not, go back to the evaluation step, setting $\pi_0 = \pi_{impro}$.

- Remember, in value iteration, we do not need to do the full policy evaluation.
- Instead, we guess a value function $v_0(k)$ arbitrary and set $v_{old}(k) = v_0(k)$, and perform the following operation:

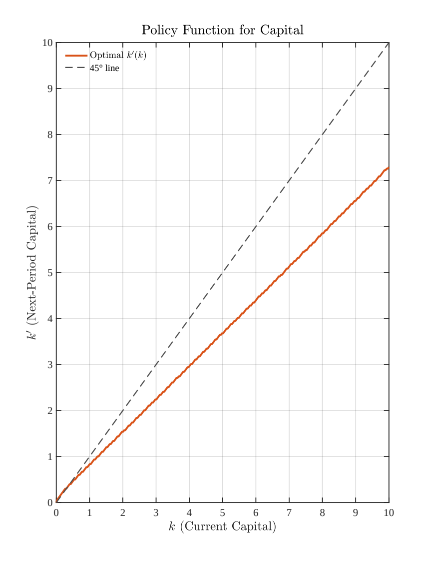
$$v_{new}(k) = \max_{k'} U(f(k) - k' + (1-\delta)k) + \beta v_{old}(k')$$

- Compute $d_{val} = |v_{new}(k) - v_{old}(k)|$. If $d_{val} < tol$, then, $v_*(k) = v_{new}(k)$ and $\pi_*(k)$ is the argument that maximizes the expression above.

Results - Value Function



Results - Policy Function



Results - Consumption