

Man or machine? Rational trading without information about fundamentals.*

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Abstract

Systematic trading contingent on past prices by rational agents has long been considered at odds with informationally efficient markets. We show that price-contingent trading is the equilibrium strategy of rational agents in efficient markets in which there is uncertainty about whether a large trader is informed about the fundamental. A large trader who knows his own type will retrieve private information about the fundamental indirectly from market prices even if he does not observe the fundamental directly. Crucially, this large trader will learn more from prices than market participants who still weigh the possibility that he is informed. His trading is then price-contingent and profitable in equilibrium. Our results generalize to a large variety of distributional assumptions. We identify some conditions under which price-contingent trading is positive-feedback or contrarian. One implication is that future order flow is predictable even if markets are semi-strong efficient by construction.

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1 Introduction

Why do agents trade in financial markets? In the traditional paradigm, the answer is unequivocal: rational agents trade only insofar as they have private information about the fundamental value of the asset being traded. Agents who are not informed and only observe market prices do not trade. The reason, as noted by Easley and O'Hara (1991), is that in standard rational expectations models any trading strategy that is contingent on observed prices would be neutralized by any risk-neutral trader who observes the same prices, and thus earn zero or negative profits.

This standard paradigm has been called into question by the unprecedented recent growth of trading by large financial institutions. Many such institutions increasingly develop quantitative trading strategies and implement them by algorithms (e.g., Osler (2003), Hendershott et al. (2011)). Quantitative strategies quickly process a large amount of data to map past prices and other quantifiable public information into orders and trades, often without aiming to trade on superior information about the fundamental value of the assets. The fact that such quantitative strategies are systematically profitable raises the question, are these trades at odds with informationally efficient markets? A large body of literature suggests that systematic uninformed trading contingent on past prices represents a departure from the neoclassical model, stemming instead from behavioral biases, imperfect or bounded rationality, non-standard preferences (e.g., for liquidity), or institutional frictions (see Shleifer (2000) and Barberis and Thaler (2003) for reviews).

We argue that price-contingent trading strategies naturally emerge as optimal and fully rational behavior in a setting with a single departure from an otherwise standard rational expectations framework. We relax the assumption that the types of all traders are public information, and introduce a large trader that may or may not be informed about the fundamental, whereby his type is not known to the market. By itself, such uncertainty about a trader's type generates an information advantage for that same trader who knows his own type, which triggers price-contingent trading. The interpretation of this information advantage is a natural one: a large trader such as a financial institution owns a complex portfolio and faces a set of exposures that the rest of the market does not perfectly observe.

We aim to establish our results on price-contingent trading in a setting with as small a departure as possible from the standard framework. We do not model all real-world nuances of quantitative strategies. We examine a stylized trading model in the tradition of Kyle (1985), with two trading rounds and one risky asset. There is a large risk-neutral trader, type K (same type of agent as in Kyle (1985)), who is informed about the fundamental value of the asset, θ , and noise traders who trade for reasons outside the model and uncorrelated with the fundamental. Our main innovation is to introduce another large risk-neutral trader, type P

(for potentially "price-contingent"), who may be informed or not about θ , and focus on P 's trading incentives.

Strategic traders K and P submit market orders before knowing the execution price and taking into account the expected price impact of their order. As standard in this class of models, we impose that prices are set such that the market is semi-strong efficient, i.e., prices reflect all public information, which includes the total order flow but does not include knowledge of P 's type. Such market efficiency condition is implemented through a hypothetical agent, the Market (often referred to as market maker in the literature). At the same time, the market is not strong-form efficient because traders will profit from private information. Indeed, K always holds private information about θ , and P always knows his own type – informed (I) or uninformed (U) – while the Market does not know P 's type. This is true even when P does not directly observe θ , which is crucial for our results.

In the first trading round P trades only if he knows θ . This is because his only other information is the prior, which is publicly known. Our main result arises in the second period if P is realized to be uninformed. In such case, P knows he has not traded in the previous round, so that the order flow, y_1 , was generated by K and by the noise traders. By contrast, the Market learns from prices and order flow and updates the probability that P is uninformed, but in equilibrium still rationally weighs the possibility that the order flow y_1 reflected an order by informed P . As a result, prices are set to $p_1 = \mathbb{E}[\theta|y_1]$. The uncertainty about P 's type leads different agents to hold different expectations about the fundamental value upon observing date 1 order flow, y_1 . Namely, we show that

$$\mathbb{E}[\theta|y_1, U] \neq \mathbb{E}[\theta|y_1] = p_1,$$

simply because date 1 order flow is in general not independent of the number of informed traders. As a result, uninformed P has incentives to trade at date 2. We start by assuming that noise trading is normally distributed, as it naturally stems from the central limit theorem when applied to a large number of small exogenous orders. The date 2 trading problem is non-trivial, as P 's expected payoff depends on how much the Market will learn about P 's type after observing date 2 order flow, which in turn depends on the unobservable noise trading shock. Still, we prove that there is a unique optimal pure strategy for P . This strategy is characterized by a non-zero trade contingent on past prices and proportional to the standard deviation of noise trading.

We then explore under which conditions price-contingent trading is positive-feedback (i.e., in the same direction of past price movements), or contrarian. As the distribution of the fundamental affects date 1 equilibrium price and expectations, the shape of the distribution turns out to be crucial in determining the direction of optimal price-contingent trading. Specifically,

we focus on two cases based on common distributional assumptions: the normal distribution, which is suitable for modelling such continuous variables as a company's future profits or cash flows, and a two-point distribution, which is more appropriate for discrete events such as the emergence of a takeover bid, the awarding of a patent, the success of a drug trial or the resolution of a legal dispute. We show that these two common assumptions lead to opposite predictions: with a normal prior, price-contingent trading is always positive-feedback; with a two-point distribution, it is always contrarian.

What drives these results? Unlike the normal, the two-point distribution is bounded and has no mass in the center. As a result, the Market will react to the order flow by setting prices too close to the bounds of the distribution, which implies the optimal strategy is contrarian. We examine the effects of boundedness and mass in the center in more detail by studying a symmetric three-point distribution. We find that with a large order flow the optimal strategy is still contrarian, as in the two-point case above. This effect is driven by the bounded support, and is absent in the normal case. On the other hand, if the order flow is small in the three-point case then price-contingent trading becomes positive-feedback, provided there is enough mass in the center of the distribution, i.e., enough probability of "no news". The reason is that, with enough probability mass in the center and a small order flow the Market will tend to underestimate the possibility that the fundamental value is high or low. Therefore, when P is uninformed and knows he did not contribute to the order flow, P is better positioned than the Market to infer that past prices are adjusting in the direction of the fundamental, reflecting only K 's information. As a result, P 's optimal strategy is positive-feedback. In general, the forces that determine the direction of trading can be understood by noting that the order flow signal has a higher mean in state U, which pushes towards positive-feedback trading; but the order flow signal is also more noisy in state U, which pushes towards contrarian trading. Which force dominates depends on the prior.

In addition to solving explicitly the above mentioned cases, we discuss our results in the context of other distributions for the prior and noise trading. Price-contingent trading emerges in general as an equilibrium strategy for uninformed agents, without depending on particular assumptions about the distribution of the fundamental. The shape of this distribution determines only the direction of trading. While assuming normality of noise trading is intuitive due to the central limit theorem, we argue that our results extend beyond the normal case, as long as the noise trading distribution has some desirable and realistic features, most notably, log-concavity. Log-concavity is shown to be necessary for the Market's learning process to be monotonic in the order flow (see also Milgrom (1981)). Log-concavity is desirable since it guarantees that strategic traders K and P face a meaningful trade-off between trading a higher quantity to benefit from their information, as opposed to a lower quantity to prevent their information to

be reflected into prices too quickly. Log-concavity is also realistic, since empirically the market impact is monotonic in the order flow (e.g., Evans and Lyons (2002)).

One notable feature of our equilibrium is that, while the market is semi-strong efficient by construction and therefore future returns are unpredictable as in previous literature, the future order flow is instead predictable from past information. This feature stands in sharp contrast with traditional models of trading in which also the order flow is unpredictable. The reason is that in our setting, while the Market cannot be sure whether P is uninformed, still the Market knows that if P is uninformed he will trade in a predictable, price-contingent direction. Therefore, one robust implication of our model is that in general the order flow is predictable even if returns are not, consistent with available empirical evidence (e.g., Biais, Hillion, and Spatt (1995), Lillo and Farmer (2004)). More generally, our results demonstrate that order flow predictability is entirely consistent with market efficiency.

Section 2 discusses some of the related literature. Section 3 outlines the setup. Section 4 presents the main results assuming normal noise trading; and a normal, a two-point, and a three-point prior using technical Theorems and Lemmas provided in Appendix A. Section 5 discusses the generality of our results, and Section 6 concludes.

2 Literature

The broad literature on asset pricing and learning in micro-founded financial markets is surveyed in Brunnermeier (2001) and Vives (2008), among others. Our work relates to the part of the literature that studies trading in markets with asymmetric information. Our results on the profitability of rational price-contingent trading require that informed traders be large, i.e., that their trades have market impact. We develop our model in a setting that generalizes the Kyle (1985) framework, but similar implications could be obtained in a Glosten and Milgrom (1985) framework in which trades arrive probabilistically and market makers observe individual trades (see also Back and Baruch (2004)).

Our model shows that rational traders with market impact and superior information about their own type can learn from prices better than average market participants. Another strand of the literature studies whether past prices contain useful information for a rational trader (e.g., Grossman and Stiglitz (1980), Brown and Jennings (1989)). However, in these papers there are no profits from uninformed trading in excess of the risk premium.

Our paper also relates to the literature on stock price manipulation, that is, the idea that rational traders may have an incentive to trade against their private information. Provided manipulation is followed by some (exogenously assumed) price-contingent trading, short run losses can be more than offset by long term gains (see Kyle and Viswanathan (2008) for a review).

Somewhat closer to our work, Chakraborty and Yilmaz (2004a, 2004b) study the incentives of an informed trader when there is uncertainty about whether such trader is informed, or is a noise trader instead. If this trader turns out to be informed, he may choose to disregard his information and trade randomly, in order to build a reputation as a noise trader. In their model, uninformed traders are never rational. Therefore, Chakraborty and Yilmaz do not analyze the trading incentives of rational agents when they are uninformed, which is our main focus.

Goldstein and Guembel (2008) show that if stock prices affect real activity then a form of trade-based manipulation such as short-sales by uninformed speculators can be profitable insofar as it causes firms to cancel positive NPV projects, and justifies ex post the "gamble" for a lower firm value. Such manipulation is possible because there is uncertainty about whether speculators are informed. In their setting, both uninformed trading and successful stock price manipulation stem from the feedback effect between stock prices and real activity. By contrast, in our paper there is price-contingent trading but no manipulation. Therefore, our results demonstrate that price-contingent trading does not make uninformed investors the inevitable prey of (potentially informed) speculators.

Finally, our work relates to the literature on rational herding (see Chamley (2004) for a review). Unlike our setting in which traders never disregard their private information, these models characterize conditions under which rational traders ignore their noisy private signal and follow the actions of other traders instead. In particular, in a recent paper Park and Sabourian (2011) show in a framework with a three-point prior how rational herd and contrarian-like behavior can emerge in an efficient market, and identify signal structures that give rise to such behavior. In their setting, all strategic traders observe some relatively imprecise private signal about the fundamental. By contrast, in our setting strategic traders observe either a precise private signal or only quantifiable public information that they interpret better than the market. A common theme of our papers is that rather general information structures are shown to allow for behavior that has traditionally been seen as contradicting trader rationality and informationally efficient markets.

3 Setup

The model is in the spirit of Kyle (1985) and Holden and Subrahmanyam (1992), where a single risky asset with fundamental value θ is traded at date 1 and 2 and the fundamental is realized at date 3. We assume common knowledge of the prior distribution of θ , and we consider a variety of distributional assumptions specified in the relevant sections. We maintain that the

prior probability density (or mass function) is mean zero and symmetric.¹

We focus on two trading rounds to capture traders' short-run incentives in the simplest setting. As in the aforementioned papers, there are large strategic risk-neutral traders and noise traders who submit market orders before knowing the execution price. The equilibrium prices are set by a hypothetical agent, the Market, who observes the total order flow and implements the market efficiency condition.² Namely, he sets period t price,

$$p_t = \mathbb{E} [\theta | \Omega_t^M], \quad (1)$$

where Ω_t^M is the information set available to the Market in $t \in \{1, 2\}$, which includes all publicly available information such as the current and past order flows. We formally specify the information set at the end of this Section.

Our goal is to show how price-contingent trading emerges as optimal in a setting that deviates as little as possible from the standard model; we assume as in Kyle (1985) that there is a large risk-neutral trader K , who learns the value of fundamental θ before date 1 and trades only in date 1. Our main innovation is to introduce another large risk-neutral trader, P , whose type/state, $R \in \{I, U\}$, is not known with certainty to the Market. We denote

$$R = \begin{cases} I & \text{if } P \text{ is "informed (i.e., knows } \theta \text{)"} \\ U & \text{if } P \text{ is "uninformed (i.e., does not know } \theta \text{)"} \end{cases}.$$

The prior probability is $\Pr(I) = \eta$, where $0 < \eta < 1$ ³. When the state is $R = I$, then P is identical to K and only trades at date 1. We are particularly interested in P 's trading incentives when the state is $R = U$. If he is uninformed, he still observes the past information such as order flows and prices, and can trade on both dates. Provided that date 1 price is an invertible function of the order flow, then past order flow and prices have exactly the same information content - so if he trades at date 2, then he can be viewed as a "price-contingent trader". We assume that only uninformed P can trade in date 2 for transparency of the effects. We could interpret that the realization of P 's type may be seen as an outcome of P 's previous

¹If the mean is non-zero, then one can always redefine a new fundamental value with zero mean by subtracting the prior mean from the original random variable. We assume symmetry of all the distributions we consider, to abstract from the additional effects that the skewness of the prior or noise trading distributions can have on trading incentives.

²In models based on Kyle (1985), this agent is frequently referred to as the "market maker". We prefer to call him the Market to emphasize that such agent proxies for the information observed by the whole market, as opposed to any individual broker. The market efficiency condition (1) can also be interpreted as the outcome of Bertrand competition between market makers or as the equilibrium outcome of a large number of small risk-neutral agents who take prices as given.

³We assume that η is not arbitrarily close to one, for technical convenience (see Appendix B), and also because it realistically avoids the case where the Market is too reluctant to update his beliefs about P 's type.

unobservable decision where he decided whether to invest in acquiring fundamental information about θ or to invest in a "machine" that allows him to trade by implementing mechanical rules.

It is natural to assume that P knows his own type. We also assume that K knows P 's type with certainty, but this is not crucial for our results.⁴

Being "large" in this setting means that as in Kyle (1985) strategic traders K and P do not take the (expected) asset price as given, but know that their market orders have a non-negligible impact on prices.

Denote the market order by trader $J \in \{K, P\}$ in state $R \in \{I, U\}$ at date $t \in \{1, 2\}$ as h_t^{RJ} . If both traders are informed, $R = I$, then trader J solves

$$\max_{h_1^{IJ}} \pi_1^{IJ} = \mathbb{E} [h_1^{IJ} (\theta - p_1) | \theta, I], \quad (2)$$

where $J \in \{K, P\}$. If only K is informed about the fundamental, $R = U$, then K solves

$$\max_{h_1^{UK}} \pi_1^{UK} = \mathbb{E} [h_1^{UK} (\theta - p_1) | \theta, U], \quad (3)$$

and P solves⁵

$$\begin{aligned} \max_{h_1^{UP}} \pi_1^{UP} &= \mathbb{E} [h_1^{UP} (\theta - p_1) + h_2^{UP} (\theta - p_2) | U] \\ \max_{h_2^{UP}} \pi_2^{UP} &= \mathbb{E} [h_2^{UP} (\theta - p_2) | y_1, U]. \end{aligned} \quad (4)$$

The total order flow is

$$\begin{aligned} y_1 &= h_1^{RK} + h_1^{RP} + s_1 \text{ for } R = \{I, U\} \\ y_2 &= \begin{cases} s_2 & \text{if } R = I \\ h_2^{UP} + s_2 & \text{if } R = U, \end{cases} \end{aligned} \quad (5)$$

where s_t is date $t \in \{1, 2\}$ demand by noise traders.⁶

In Section 4, we assume that noise traders demand is drawn from a normal distribution with

⁴This assumption makes it ex ante harder for P to develop an information advantage as P never has more information than K . It also allows us to highlight that for price-contingent trading to emerge as optimal, it is important that typical market participants (the Market) do not know large trader's types with certainty, even if other large traders do.

⁵We condition P 's expectation on the order flow (instead of the price or both), because date 1 order flow is always at least as informative as date 1 price. The price is an endogenous function of the order flow to be determined in equilibrium. If price is monotonic in the order flow, then the two have the same information content.

⁶The presence of noise traders is also needed in general for avoiding the Grossman and Stiglitz's (1980) paradox about the impossibility of a fully revealing price in equilibrium.

mean zero and variance σ_s^2 , serially uncorrelated and independent of fundamental and state. Therefore, the probability density function of noise trader's demand for $t \in \{1, 2\}$ is $\varphi_s(s)$, where

$$\varphi_s(s) \equiv \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(-\frac{s^2}{2\sigma_s^2}\right). \quad (6)$$

While being a standard assumption in this literature, there is also a natural economic argument for this distribution choice. We can interpret noise trading as the total demand by a large number of small traders who trade for idiosyncratic reasons unrelated to the fundamental (such as liquidity shocks, private values etc.). In such case, the normality of the distribution of noise trading follows directly from the central limit theorem.

Technically, a useful property of the normal distribution is that it is strictly log-concave, allowing us to use some general properties of log-concave functions.⁷ Log-concavity of noise trading also guarantees some desirable properties of the model, and we discuss generality further in Section 5.⁸

Finally, we state more formally what is contained in the Market's information set. The Market knows all distributions, and observes the total order flow y_t before setting p_t . Crucially, he does not know the realization of P 's type, i.e., the value of R . We denote the Market's information set as $\Omega_1^M = \{y_1\}$ and $\Omega_2^M = \{y_1, y_2\}$. It is worth highlighting that in this setting the order flows provide noisy information about both the fundamental, θ , and P 's type, $R \in \{I, U\}$. This is in contrast to standard settings where all types are known with certainty and the total order flow only reveals information about the fundamental. Using the law of total expectations, we can expand the Market efficiency condition (1) as

$$\begin{aligned} p_1 &= \mathbb{E}[\theta|y_1] = Q_1 \mathbb{E}[\theta|y_1, I] + (1 - Q_1) \mathbb{E}[\theta|y_1, U] \\ p_2 &= \mathbb{E}[\theta|y_1, y_2] = Q_2 \mathbb{E}[\theta|y_1, y_2, I] + (1 - Q_2) \mathbb{E}[\theta|y_1, y_2, U], \end{aligned} \quad (7)$$

where $Q_1 \equiv \Pr(I|y_1)$ and $Q_2 \equiv \Pr(I|y_1, y_2)$ are the probabilities of P being informed conditional on the observed total order flows. We also use notation $p_1(y_1)$, $p_2(y_2)$, $Q_1(y_1)$ and $Q_2(y_2)$ to express these prices and probabilities as functions of contemporaneous order flows.

To summarize the setup, the timing of events is as follows:

- **date 0** - Nature draws $R \in \{I, U\}$ and θ . K and P learn R . If $R = I$, then both K and P learn θ . If $R = U$, only K learns θ .

⁷A function $f(\mathbf{x})$ (where x is a n -component vector) is log-concave if $\ln(f(\mathbf{x}))$ is concave. In the univariate and differentiable case, the following are equivalent: 1) $\partial^2 \ln(f(x))/\partial x^2 < 0$, 2) $f'(x)/f(x)$ is decreasing in x , 3) $f''(x)f(x) - (f'(x))^2 < 0$.

⁸Many other well known distributions are log-concave and symmetric. Notable examples include the beta (with parameters $\alpha = \beta > 1$) and truncated normal. (See Bagnoli and Bergstrom (2005) for an overview and further examples of log-concave densities).

- **date 1** - K , P and noise traders submit market orders before knowing the price. The Market observes total order flow and sets the price p_1 based on the market efficiency condition (1).
- **date 2** - Noise traders submit market orders. If $R = U$, then P also submits a market order before knowing the price. The Market observes total order flow and sets the price p_2 based on the market efficiency condition (1).
- **date 3** - uncertainty resolves and P and K consume profits given the realization of θ .

As standard in the literature we focus on equilibria in pure strategies by K and P .

4 Results

4.1 Importance of uncertainty about types

We start by highlighting why the uncertainty about P 's type is important. For a benchmark, let us assume that P is known to be uninformed with certainty, i.e., $\Pr(I) = \eta = 0$, and consider any symmetric prior (such that the prior mean exists). As we are only interested in P 's incentives, assume that informed trader K 's optimal strategy is given by $g_U(\theta)$ and it holds that $g'_U(\cdot) > 0$ and $g_U(\theta) = -g_U(-\theta)$. This is not restrictive and also holds in all settings we analyze.⁹ We should note that log-concavity of noise trading guarantees that $\mathbb{E}[\theta|\tilde{y}_1, U] - \mathbb{E}[\theta|y_1, U] > 0$ for any $\tilde{y}_1 > y_1$ (See Lemma A.2 in Appendix A) for any prior distribution. This in turn allows us state the following proposition for any symmetric and log-concave density of noise trading (i.e., not just for normal density).

Proposition 1 *If $\eta = 0$: $\mathbb{E}[\theta|y_1, U] = \mathbb{E}[\theta|y_1] = p_1 = p_2$; P can never earn positive expected profits from trading; consequently P does not trade in date 1.*

Proof. By a straightforward application of Bayes' rule, it holds that $\eta = 0 \implies Q_1 = 0$ and $Q_2 = 0$. Also, notice that in this case date 2 order flow is not informative about the fundamental, i.e., it must hold that $\mathbb{E}[\theta|y_1, y_2, U] = \mathbb{E}[\theta|y_1, U]$. From (7), we then find that $p_1 = \mathbb{E}[\theta|y_1] = \mathbb{E}[\theta|y_1, U] = p_2$. From (4), P 's expected date 2 profits are $\mathbb{E}[h_2^{UP}(\theta - p_2)|y_1, U] = h_2^{UP}(\mathbb{E}[\theta|y_1, U] - p_1) = 0$. Therefore, P earns zero profit at date 2 irrespective of the quantity he trades. For date 1, suppose that the Market sets the price under the belief that P trades a known quantity \bar{h}_{1U} at date 1. Given these beliefs, P chooses h_1^{UP} that corresponds to an

⁹Given the symmetry of all distributions assumed, it is natural to expect symmetric strategies. The fact that the informed trader's strategy is increasing in the fundamental follows from the supermodularity of the informed trader's problem (see the discussion in Section 5).

order flow (see (5)) $y_1 = h_1^{UP} + s_1 + g_U(\theta)$. The order flow is uncertain at the time of P 's date 1 trading decision due to the presence of s_1 and θ . Using that P 's profit at date 2 is always zero and $\mathbb{E}[\theta|U] = 0$, (4) can be written as $\pi_1^{UP} = -h_1^{UP}\mathbb{E}[p_1|U]$, where $\mathbb{E}[p_1|U] = \int \int p_1(h_1^{UP} + s_1 + g_U(\theta)) f_s(s_1) f_{\theta U}(\theta) ds_1 d\theta$.¹⁰ The first derivative of profit is $\partial\pi_1^{UP}/\partial h_1^{UP} = -\mathbb{E}[p_1|U] - h_1^{UP}(\partial\mathbb{E}[p_1|U]/\partial h_1^{UP})$. Because $p_1 = \mathbb{E}[\theta|y_1, U]$, by Lemma A.2 in Appendix A, the price is increasing in the order flow, and it holds that $-h_1^{UP}(\partial\mathbb{E}[p_1|U]/\partial h_1^{UP}) < (>) 0$ for any $h_1^{UP} > (<) 0$. Due to the symmetry of distributions and the fact that prices increase in the order flow, it also holds that $-\mathbb{E}[p_1|U] < (>) 0$ for any $h_1^{UP} > (<) 0$.¹¹ Therefore, P 's profit is maximized at $h_1^{UP} = 0$. In equilibrium, the beliefs of the Market must be consistent with P 's optimal strategy, i.e., it holds that $h_1^{UP} = \bar{h}_{1U} = 0$. In such a case, also $\pi_1^{UP} = 0$. ■

Proposition 1 shows that in the special case in which $\eta = 0$, our model supports the Easley and O'Hara (1991) argument against the possibility of uninformed traders profiting from rational price-contingent trading. Indeed, in such a case uninformed P cannot earn positive profits, because prices already reflect all the public information than an uninformed P could have. To be more specific, at date 1 uninformed P 's best guess of the fundamental is the prior mean and any non-zero quantity traded would move the prices such that he obtains an expected loss from trading. Therefore, it is optimal for him not to trade, which yields a zero profit. At date 2, uninformed P does learn new information from the order flow, but the information he obtains is exactly the same as the information that the Market has already obtained, $\mathbb{E}[\theta|y_1, U] = \mathbb{E}[\theta|y_1] = p_1$. Because prices will not change between date 1 and 2, he cannot earn positive profits from trading.¹²

We will demonstrate that in a more general setting where there is uncertainty about P 's type, then it is typically the case that $\mathbb{E}[\theta|y_1, U] \neq \mathbb{E}[\theta|y_1] = p_1$ (and also $Q_1, Q_2 > 0$ and $Q_1 \neq Q_2$), which allows price-contingent trading to be profitable. Essentially, uninformed P will have superior information compared to the Market - not because he knows more about the fundamental directly, but because the order flow reveals different information depending on how many strategic traders are informed. Knowing his own type and past trades (or lack thereof)

¹⁰ A similar argument holds for a discrete distribution of the fundamental and/or of noise trading.

¹¹ From Lemma A.2 in Appendix A it also holds that $p_1(y_1 - \bar{h}_{1U}) = -p_1(-y_1 + \bar{h}_{1U})$. Therefore, $\mathbb{E}[p_1|U] = \int_{\theta \geq 0} \int_{s_1 > 0} P_1(h_1^{UP}, s_1, \theta, \bar{h}_{1U}) f_s(s_1) f_{\theta U}(\theta) ds_1 d\theta$, where $P_1(h_1^{UP}, s_1, \theta, \bar{h}_{1U}) \equiv p_1(h_1^{UP} + s_1 + g_U(\theta) + \bar{h}_{1U} - \bar{h}_{1U}) + p_1(h_1^{UP} + s_1 - g_U(\theta) + \bar{h}_{1U} - \bar{h}_{1U}) - p_1(-h_1^{UP} + s_1 - g_U(\theta) - \bar{h}_{1U} + \bar{h}_{1U}) - p_1(-h_1^{UP} + s_1 + g_U(\theta) - \bar{h}_{1U} + \bar{h}_{1U})$. As prices are increasing in the order flow, it holds that $P_1(h_1^{UP}, s_1, \theta, \bar{h}_{1U})$ is always non-negative and strictly positive for some values of $s_1, \theta > 0$ iff $h_1^{UP} > 0$.

¹² Note that in our setting, the Market knows that date 2 order flow is not informative and thus prices will not change, $p_2 = p_1$. As a result P would earn zero profits from any quantity traded. In this setting $h_2^{UP} = 0$, but also any other constant quantity traded by uninformed P at date 2 can be sustained as an equilibrium. However, the latter only holds because of risk neutrality of P . Clearly, only the equilibrium with $h_2^{UP} = 0$ could be sustained with even a very small degree of risk aversion.

will be enough for P to indirectly retrieve some private information about the fundamental.

4.2 Date 2 problem

Assume $\eta > 0$ and take all date 1 quantities, $\mathbb{E}[\theta|y_1, R]$, p_1 and $Q_1 = \Pr(I|y_1)$ as given. Date 2 problem is only interesting if there is a difference between P 's and the Markets expectations about the fundamental ($\mathbb{E}[\theta|y_1, U] \neq p_1$ or equivalently $\mathbb{E}[\theta|y_1, U] \neq \mathbb{E}[\theta|y_1, I]$) and the Market has not fully learned P 's type ($Q_1 > 0$). For now, we conjecture that this is the case and verify it in next sections for different priors.

As there is no informed trading at date 2, it holds that conditional on a given state $R \in \{I, U\}$ and y_1 , the date 2 order flow only depends on θ through y_1 , which is already incorporated in prices and expectations and therefore $\mathbb{E}[\theta|y_1, y_2, R] = \mathbb{E}[\theta|y_1, R]$. Using (7) we obtain

$$p_2 = p_1 + \frac{(Q_1 - Q_2)}{Q_1} (\mathbb{E}[\theta|y_1, U] - p_1). \quad (8)$$

Clearly prices change between date 1 and 2 only if $Q_2 \neq Q_1$, which implies that they only change if the Market updates his beliefs about P 's type after observing date 2 order flow. Provided that the true state is $R = U$, the Market updates in the "correct" direction if $Q_2 < Q_1$. In such case the prices increase (decrease) if $\mathbb{E}[\theta|y_1, U] > (<) p_1$. Using (8), we can restate P 's problem (4) as

$$\begin{aligned} \max_{h_2^{UP}} \pi_2^{UP} &= h_2^{UP} \mathbb{E}[Q_2|y_1, U] \frac{(\mathbb{E}[\theta|y_1, U] - p_1)}{Q_1} = \\ &= h_2^{UP} \left(\int_{-\infty}^{\infty} Q_2 (h_2^{UP} + s_2) \varphi_s(s_2) s_2 \right) \frac{(\mathbb{E}[\theta|y_1, U] - p_1)}{Q_1}. \end{aligned} \quad (9)$$

We can make some immediate observations. Note that $\mathbb{E}[\theta|y_1, U]$, p_1 and Q_1 only depend on date 1 order flow and are known to P . Suppose that $\mathbb{E}[\theta|y_1, U] > p_1$, i.e., uninformed P expects the fundamental to be higher than date 1 price. On the one hand, P can profit from trading any positive quantity. Ignoring the effect of his trade on $Q_2(\cdot)$ would make him to want to buy an infinitely large quantity of the asset at date 2. On the other hand, the term $\mathbb{E}[Q_2|y_1, U]$ captures the expected updating of P 's type by the Market. Because Q_2 depends on date 2 order flow, P knows that his trade will affect the Markets' beliefs about his type. Since these beliefs directly affect p_2 , one would expect the traditional trade-off between transaction size and information disclosure to be present, but to establish this we need to investigate further the properties of Q_2 .

As we focus on the pure strategies and uninformed P 's trading strategy can only depend on known variables, it must hold that P 's equilibrium quantity is a constant known to the Market

at this stage. So let us suppose that such equilibrium exists and uninformed P trades \bar{h}_2 in equilibrium. Then, from (5) $y_2 = \bar{h}_2 + s_2$ if $R = U$ and $y_2 = s_2$ if $R = I$, we can derive Q_2 by using the Bayes' rule, as

$$Q_2 = \frac{Q_1 f(y_2|y_1, I)}{Q_1 f(y_2|y_1, I) + (1 - Q_1) f(y_2|y_1, U)} = \frac{Q_1}{Q_1 + (1 - Q_1) r(y_2)}, \quad (10)$$

where

$$r(y_2) \equiv \frac{\varphi_s(y_2 - \bar{h}_2)}{\varphi_s(y_2)} \quad (11)$$

is the likelihood ratio and we used the fact that conditional on the state R the date 2 order flow is normal and $\varphi_s(\cdot)$ is given by (6).

Lemma 1 *The following properties hold for $Q_2 = \Pr(I|y_1, y_2)$*

1. Q_2 is decreasing (increasing) in y_2 for any $\bar{h}_2 > (<) 0$.
2. If $\bar{h}_2 > 0$ then $Q_2 > (<) Q_1$ for any $y_2 < (>) \frac{\bar{h}_2}{2}$. If $\bar{h}_2 < 0$ then $Q_2 > (<) Q_1$ for any $y_2 > (<) \frac{\bar{h}_2}{2}$.
3. $Q_2(0) = Q_1 \cdot \left(Q_1 + (1 - Q_1) \frac{\varphi_s(\bar{h}_2)}{\varphi_s(0)} \right)^{-1} = Q_1 \cdot \left(Q_1 + (1 - Q_1) \exp\left(-\frac{(\bar{h}_2)^2}{2\sigma_s^2}\right) \right)^{-1}$,
4. If $\bar{h}_2 > (<) 0$ then $\lim_{y_2 \rightarrow \infty} Q_2(y_2) = 0 (= 1)$ and $\lim_{y_2 \rightarrow -\infty} Q_2(y_2) = 1 (= 0)$.
5. $Q_2(y_2)$ is a log-concave function.

Proof. Part 1: Differentiating and simplifying we obtain $\partial Q_2 / \partial y_2 = -Q_2^2 (1/Q_1 - 1) r'(y_2)$. Because Lemma A.1 in Appendix A shows that log-concavity of φ_s implies the monotone likelihood ratio property (MLRP), i.e., $r'(y_2) > (<) 0$ for any $\bar{h}_2 > (<) 0$. This is because $\varphi_s(\tilde{y}_2 - \bar{h}_2) / \varphi_s(\tilde{y}_2) > (<) \varphi_s(y_2 - \bar{h}_2) / \varphi_s(y_2)$ for any $\tilde{y}_2 > 0$ and $\bar{h}_2 > (<) 0$. Parts 2-4 are straightforward from (6), (10) and (11). Part 5: Taking logs and differentiating, we obtain that $\frac{\partial^2 \ln(Q_2)}{\partial y_2 \partial y_2} = -\frac{(1-Q_1)^2 [(1/Q_1 - 1)r''(y_2) + r''(y_2)r(y_2) - (r'(y_2))^2]}{(Q_1 + (1-Q_1)r(y_2))^2}$. It is sufficient to show that the likelihood ratio (11) is (at least weakly) log-convex.¹³ Indeed from (6) and (11) we find that $\ln(r(y_2))$ is linear in y_2 and therefore weakly convex. ■

Part 1 of Lemma 1 implies that the Market updates his beliefs about P 's type (the state R) in a "sensible" manner. For example, if the Market believes that trader P in state $R = U$ trades a finite and positive quantity, then observing a higher order flow always leads the Market

¹³ $r(y_2)$ is log-convex if $\ln(r(y_2))$ is convex. Equivalently, it must hold that $r''(y_2)r(y_2) - (r'(y_2))^2 \geq 0$. This, together with $r(y_2) > 0$ also implies that $r''(y_2) > 0$.

to assign a lower probability on him being informed. This also confirms that P indeed faces a meaningful trade-off - if he trades more aggressively, then he expects the Market to assign a higher probability on him being uninformed, which reduces his profits.¹⁴ It is worth emphasizing that such realistic property is present only because the likelihood ratio (11) is monotone (for a similar argument, see also Milgrom (1981)). The monotone likelihood ratio property holds for the whole family of log-concave distributions, to which the normal belongs (see Lemma A.1 in Appendix A).

While Bayesian updating itself guarantees that the Market updates his beliefs in the correct direction on average, we can see from part 2 of Lemma 1 that ex post the Market can update the probability that P is informed, Q_2 , in the "correct" or "incorrect" direction. This is because the total order flow includes a random noise trading component. If the realized order flow is relatively small (half of the volume that the Market expects uninformed P to trade) or has an opposite sign to P 's expected trade, then the Market updates in the "correct" direction if the state is $R = I$ and in the "incorrect" direction if the state is $R = U$. It is also immediate from parts 2-4 of Lemma 1 that the Market never learns P 's type perfectly for finite order flows. Therefore, despite of some learning about P 's type, it is clear from (9) that P would always earn positive profits from trading any finite quantity in the correct direction.

Part 4 of Lemma 1 confirms that the Market's learning about P 's type is unbounded. This is necessary to guarantee that P has an incentive to trade a finite amount.¹⁵

While the previous analysis gives some confidence that it may be optimal for uninformed P to trade a finite quantity in equilibrium, it is not yet clear whether P 's problem has a unique (interior) solution. Namely, from (9) P 's expected profit involves an integral over a non-trivial function $Q_2(\cdot)$ that depends on uninformed P 's demand and is always positive for $h_2^{UP} > (<) 0$ provided that $(\mathbb{E}[\theta|y_1, U] - p_1) > (<) 0$.

Lemma 2 *If $(\mathbb{E}[\theta|y_1, U] - p_1) > (<) 0$ then uninformed P 's expected profit (9) is strictly log-concave in $h_2^{UP} > (<) 0$.*

Proof. Assume without loss of generality that $(\mathbb{E}[\theta|y_1, U] - p_1) > 0$ and $h_2^{UP} > 0$. Taking logs of (9), we obtain $\ln(\pi_2^{UP}) = \ln(h_2^{UP}) + \ln(\mathbb{E}[Q_2|y_1, U]) + \ln(\mathbb{E}[\theta|y_1, U] - p_1) - \ln(Q_1)$ and $\partial^2 \ln(\pi_2^{UP}) / \partial h_2^{UP} \partial h_2^{UP} = - (h_2^{UP})^{-2} + \partial^2 \ln(\mathbb{E}[Q_2|y_1, U]) / \partial h_2^{UP} \partial h_2^{UP}$, which is negative if

¹⁴If P trades h_2^{UP} , then the order flow is $y_2 = h_2^{UP} + s_2$ and $\mathbb{E}[Q_2|y_1, U] = \int_{-\infty}^{\infty} Q_2(h_2^{UP} + s_2) \varphi_s(s) ds$. It is clear that $\partial \mathbb{E}[Q_2|y_1, U] / \partial h_2^{UP} = \int_{-\infty}^{\infty} Q_2'(h_2^{UP} + s_2) \varphi_s(s) ds > 0$

¹⁵Suppose instead that learning was bounded (i.e., Q_2 was always larger than some constant $\bar{Q}_2 > 0$) and consider a candidate equilibrium where P trades a finite amount. It is easy to prove that this cannot be an equilibrium as P would earn infinite profits by deviating to trade an infinite quantity. See also Section 5.

$\mathbb{E}[Q_2|y_1, U]$ is log-concave. By change of variables $y_2 = h_2^{UP} + s_2$, we can express

$$\mathbb{E}[Q_2|y_1, U] = \int_{-\infty}^{\infty} Q_2(y_2) \varphi_s(y_2 - h_2^{UP}) dy_2. \quad (12)$$

Using Theorem 6 of Prékopa (1973), restated as Theorem A.3 on Appendix A, a sufficient condition for (12) to be log-concave is that the function $Q_2(y_2) \varphi_s(y_2 - h_2^{UP})$ is log-concave in h_2^{UP} and y_2 . Using (6), $\partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial h_2^{UP} \partial h_2^{UP} = -\sigma_s^{-2}$ and $\partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial h_2^{UP} \partial y_2 = \partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial y_2 \partial h_2^{UP} = \sigma_s^{-2}$. As by part 5 of Lemma 1 $\partial^2 \ln(Q_2(y_2)) / \partial y_2 \partial y_2 < 0$, it is immediate that the Hessian¹⁶ is negative definite, and therefore $\mathbb{E}[Q_2|y_1, U]$ and π_2^{UP} are log-concave. The proof for the case $(\mathbb{E}[\theta|y_1, U] - p_1) < 0$ and $h_2^{UP} < 0$ is similar. ■

Since any univariate log-concave function is also quasi-concave with a unique maximum, we can now state our main result:

Theorem 1 *Uninformed P 's unique equilibrium strategy is to demand a finite amount*

$$h_2^{UP} = \bar{h}_2 = \begin{cases} \sigma_s \kappa & \text{if } \mathbb{E}[\theta|y_1, U] - p_1 > 0 \\ -\sigma_s \kappa & \text{if } \mathbb{E}[\theta|y_1, U] - p_1 < 0 \end{cases}, \quad (13)$$

where $\kappa > 1$ for any $Q_1 \in (0, 1)$ and κ depends on Q_1 only.

Proof. It is immediate from (9) that $h_2^{UP} < (>) 0$ when $\mathbb{E}[\theta|y_1, U] - p_1 > (<) 0$ cannot be optimal as it leads to strictly negative profits. By Lemma 2, the uninformed P 's problem then has a unique maximum at a non-negative h_2^{UP} . Therefore it is sufficient to look at the first order condition only and then impose that in equilibrium beliefs must be consistent with the optimal strategy $h_2^{UP} = \bar{h}_2$. The first order condition, the expression for κ and the proofs of the statements that $\kappa > 0$ and only depends on Q_1 are stated in Appendix B. ■

Because κ only depends on Q_1 , it is most illustrative to present the solution¹⁷ on a graph (see Figure 1). We find that whenever $\mathbb{E}[\theta|y_1, U] \neq p_1 = \mathbb{E}[\theta|y_1]$, it is generally optimal for uninformed P to trade at date 2. The volume traded by him is proportional to the standard deviation of noise trading and is increasing in Q_1 . Both effects are intuitive. When the order flow is more noisy (high σ_s), it is harder for the Market to update his beliefs about the state and it is less costly for uninformed P to trade more aggressively. Because the Market's posterior belief that the state is $R = I$, $Q_2 = \Pr(I|y_1, y_2)$, is increasing in his belief prior to date two,

¹⁶The Hessian is $\begin{vmatrix} -\sigma_s^{-2} & \sigma_s^{-2} \\ \sigma_s^{-2} & \partial^2 \ln(Q_2(y_2)) / \partial y_2 \partial y_2 - \sigma_s^{-2} \end{vmatrix}$

¹⁷It is relatively easy to prove analytically that if $Q_1 = 0.5$, then $\kappa = \sqrt{2}$. The other values are derived using numerical integration.

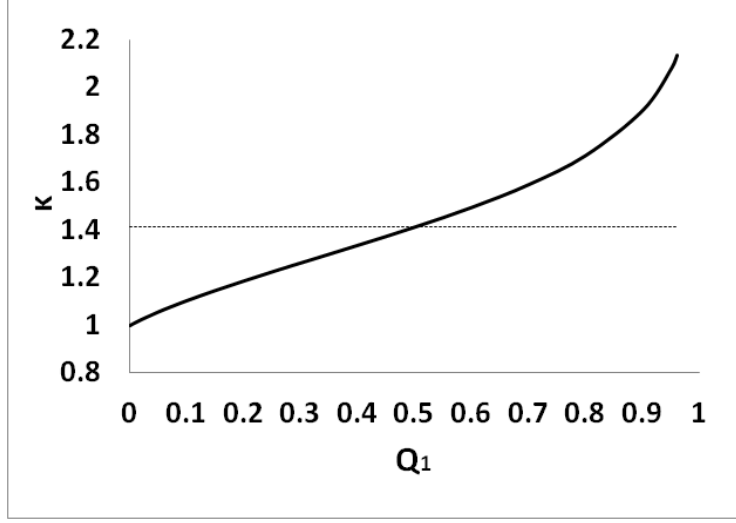


Figure 1: Uninformed P 's trading volume at date 2.

Q_1 , for any order flow, then a higher Q_1 again makes it less costly for an uninformed P to trade more aggressively. Overall P trades a finite quantity as he faces a trade-off between profiting from his superior information and revealing his type too much. This trade-off is fundamentally similar to the one in Kyle (1985), however differently from that setting P 's private information is not about the fundamental directly, but about his impact or lack of impact on date 1 price. Theorem 1 is in stark contrast to Proposition 1, because price-contingent trading is profitable.

Before analyzing the date 1 problem, we can now formally define the two traditional types of price-contingent strategies within the context of our model.

Definition P 's date 2 strategy is called¹⁸

- *positive-feedback (momentum)* for some y_1 if $y_1 > 0$ and $h_2^{UP} > 0$, or $y_1 < 0$ and $h_2^{UP} < 0$
- *negative-feedback (contrarian)* for some y_1 if $y_1 > 0$ and $h_2^{UP} < 0$, or $y_1 < 0$ and $h_2^{UP} > 0$.

Provided the price is monotonic in the order flow (as in all our examples below), it would be equivalent to define P 's strategy through date 1 price.

4.3 Date 1 problem with a normal prior

We did not yet need to specify the joint distribution of fundamental, θ , and the state, R . Here we consider two possibilities. First, we consider a particular dependence structure that preserves linearity of the date 1 equilibrium, which we present in the main text. The advantage

¹⁸The words "momentum" and "contrarian" only refer to P 's strategy. They should not be confused with positive and negative autocorrelation in returns. By the assumption of efficient market (1), there is zero autocorrelation by construction. See also Section 4.5.

of this setting is that it allows us to show how P 's information advantage emerges in a very familiar setting that is directly comparable to previous models such as Kyle (1985). Second, we consider the case in which θ and R are independent. As the results are almost identical to the first case, we relegate this second case to Appendix C.

Assume that

$$\begin{aligned} f_{\theta R}(\theta) &= \Pr(R|\theta) f(\theta) = \Pr(R) f(\theta|R) = \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\eta}{\sigma_I} \exp\left(-\frac{\theta^2}{2\sigma_I^2}\right) \right)^{\tilde{1}_\theta} \left(\frac{1-\eta}{\sigma_U} \exp\left(-\frac{\theta^2}{2\sigma_U^2}\right) \right)^{1-\tilde{1}_\theta}, \end{aligned} \quad (14)$$

where $\tilde{1}_\theta$ is an indicator function that takes values $\tilde{1}_\theta = 1$ if $R = I$ and $\tilde{1}_\theta = 0$ if $R = U$ and where σ_I and σ_U are the standard deviations of the prior distribution in state $R = I$ and $R = U$ respectively. Clearly in (14), the distribution of the fundamental conditional on a state is normal, while unconditionally the density is still symmetric around zero. To preserve linearity of the equilibrium, we first assume that¹⁹

$$\sigma_I = \frac{3}{4}\sigma_U \quad (15)$$

as under this assumption, we will shortly prove that the Market cannot update his beliefs about the state R at date 1.

We solve this problem using the standard technique. We conjecture and later verify that there is a rational expectations equilibrium where P does not trade in date 1 and date 1 price is linear in the order flow, i.e.,

$$p_1 = \lambda_1 y_1, \quad (16)$$

where λ_1 is a constant to be solved for in the equilibrium.²⁰

Proposition 2 *When the prior is given by (14) and (15), then there exists a rational expectations equilibrium where the following holds.*

1. *Informed traders' optimal demand in date 1 is*

$$\begin{aligned} h_1^{UK} &= \frac{\theta}{2\lambda_1}; \quad h_1^{IK} = h_1^{IP} = \frac{\theta}{3\lambda_1} \\ g_U(\theta) &\equiv h_1^{UK} = \frac{\theta}{2\lambda_1} \text{ and } g_I(\theta) \equiv h_1^{IK} + h_1^{IP} = \frac{2\theta}{3\lambda_1} \end{aligned}$$

¹⁹One could argue that the condition (15) captures the possibility that it may be easier/cheaper for P to become informed, if there is less uncertainty about the fundamental.

²⁰The problem can also be solved without immediately imposing this conjecture (see, e.g., Cho and El Karoui, (2000)).

2. *There is no updating about P 's type in the first trading round, i.e.,*

$$Q_1 = \Pr(I|y_1) = \eta.$$

3. *Equilibrium price is given by*

$$p_1 = \lambda_1 y_1,$$

where $\lambda_1 = \frac{\sigma_U}{2\sigma_s} \sqrt{\frac{(2-\eta)}{2}}$ and it holds that

$$\mathbb{E}[\theta|y_1, I] = \frac{3}{4-\eta} \lambda_1 y_1 < p_1 < \frac{4}{4-\eta} \lambda_1 y_1 = \mathbb{E}[\theta|y_1, U]$$

4. *Uninformed P does not trade at date 1, $h_1^{UP} = 0$.*

Proof. Part 1: Given the conjectured price (16) and the total order flow (5), we find that in state $R = U$, the informed trader's expected profit (3) is given by $\mathbb{E}[h_2^{UK}(\theta - p_1)] = h_2^{UK}(\theta - \lambda_1 h_2^{UK})$ and K 's optimal demand is $h_1^{UK} = \frac{\theta}{2\lambda_1}$. From (16) and (5), trader J 's expected profit is $\mathbb{E}[h_2^{IJ}(\theta - p_1)] = h_2^{IJ}(\theta - \lambda_1 h_2^{IJ} - \lambda_1 h_2^{I\bar{J}})$, where $J, \bar{J} \in \{K, P\}$ and $J \neq \bar{J}$ and we find that the optimal demand is the same for K and P , and $h_1^{IK} = h_1^{IP} = \theta/3\lambda_1$. Part 2: The total order flow at date 1 in state $R = U$ is $y_2 = \theta/2\lambda_1 + s_1$. As $\theta|U \sim \mathcal{N}(0, \sigma_U^2)$ it holds that $y_2|U \sim \mathcal{N}(0, \sigma_U^2/4\lambda_1^2 + \sigma_s^2)$. The total order flow in state $R = I$ is $y_2 = \frac{2\theta}{3\lambda_1} + s_1$. Using (15), $\theta|I \sim \mathcal{N}(0, \sigma_I^2) \sim \mathcal{N}(0, 9\sigma_U^2/16)$ and $y_2|I \sim \mathcal{N}(0, 4\sigma_I^2/9\lambda_1^2 + \sigma_s^2) \sim \mathcal{N}(0, \sigma_U^2/4\lambda_1^2 + \sigma_s^2)$. Clearly $f(y_1|U) = f(y_1|I)$, so by Bayes' rule $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + (1-\eta)f(y_1|U)} = \eta$. Part 3: If $R = U$, then the signal that the Market obtains from the order flow is $2\lambda_1 y_2 = \theta + 2\lambda_1 s_1$, where $2\lambda_1 y_2|\theta \sim \mathcal{N}(\theta, 4\lambda_1^2 \sigma_s^2)$. As well known in the case of normally distributed prior and signals, the posterior is a precision-weighted average of the signals, hence we can simplify $\mathbb{E}[\theta|y_1, U] = \frac{\sigma_U^2}{4\lambda_1^2 \sigma_s^2 + \sigma_U^2} 2\lambda_1 y_2$. If $R = I$ then the signal that the Market obtains from the order flow is $\frac{3}{2}\lambda_1 y_2 = \theta + \frac{3}{2}\lambda_1 s_1$, where $\frac{3}{2}\lambda_1 y_2|\theta \sim \mathcal{N}(\theta, \frac{9}{4}\lambda_1^2 \sigma_s^2)$. Using (15) and simplifying, we find $\mathbb{E}[\theta|y_1, I] = \frac{\sigma_U^2}{4\lambda_1^2 \sigma_s^2 + \sigma_U^2} \frac{3}{2}\lambda_1 y_2$. Given this and $Q_1 = \eta$, (7), we obtain that $p_1 = \frac{\sigma_U^2[(3/2)\eta + 2(1-\eta)]}{4\lambda_1^2 \sigma_s^2 + \sigma_U^2} \lambda_1 y_2$. Equating coefficients with those in the conjectured prices (16), we find that $\lambda_1^2 = \sigma_U^2(2-\eta)/8\sigma_s^2$ and the positive solution of this proves the first part of the proposition. We then use the equilibrium value of λ_1 in the expressions of $\mathbb{E}[\theta|y_1, U]$ and $\mathbb{E}[\theta|y_1, I]$. Part 4: See Appendix B. ■

We find from Proposition 2 that while uninformed P optimally does not trade at date 1, the first trading round generates his information advantage for date 2 trading. We can see that for any non-zero order flow, there will be a difference between the expected value conditional on knowing the state, $\mathbb{E}[\theta|y_1, R]$, and date 1 equilibrium price. Furthermore, by the definition of P 's strategy in Section 4.2 and Theorem 1, we find that in this setting P always pursues

a positive feedback trading strategy in date 2, i.e., he buys at date 2 if date 1 order flow is positive (or equivalently, if the price has increased at date 1) and sells when date 1 order flow is negative (or equivalently, if the price has decreased).²¹

Intuitively, P obtains superior information in state $R = U$ because he knows he did not contribute to the date 1 order flow, while the Market sets the price weighting the possibilities that the order flow reflects information of one or two informed traders. For any realization of the fundamental $\theta \neq 0$, the total volume by informed traders is different (i.e., $g_U(\theta) = \frac{\theta}{2\lambda_1}$ versus $g_I(\theta) = \frac{2\theta}{3\lambda_1}$), and therefore the total order flow reveals different information in different states.

Under this prior, uninformed P always perceives the price at date 2 to be too close to the prior mean and therefore pursues a positive-feedback strategy. Indeed, the sensitivity of prices to order flow, λ_1 , in Theorem 1 is too low given that the true state is $R = U$. (If the Market knew the state, he would set $\lambda_1 = \frac{\sigma_U}{2\sigma_s}$ as in Kyle (1985)). This is the outcome of two competing effects. The first effect is related to the value of the fundamental revealed by the order flow alone. The fundamental is the inverse of total informed trader's demand at $y_1 - s_1$ $\theta = g_U^{-1}(y_1 - s_1) = 2\lambda_1(y_1 - s_1)$ and $\theta = g_I^{-1}(y_1 - s_1) = \frac{3}{2}\lambda_1(y_1 - s_1)$. It is clear that, conditional on the order flow *alone*, the expected value of the fundamental is higher in state $R = U$. This is true because the expected value $2\lambda_1 > \frac{3}{2}\lambda_1$. This effect tends to make the Market set the price too low at date 1. The second effect relates to the dispersion of the fundamental conditional on the order flow alone, which is higher in $R = U$ (because the variance $4\lambda_1^2\sigma_s^2 > \frac{9}{4}\lambda_1^2\sigma_s^2$). Because the Market uses Bayes' rule to set the price and puts higher weight on signals that are less noisy, this effect tends to make the Market set the price too high at date 1. With normal prior given by (14) and (15), the first effect always dominates.

In Appendix C we characterize the equilibrium and present the numerical solution of the case of normal prior that is independent of the state R , i.e., $\sigma_I = \sigma_U = \sigma_p$ in (14), assuming that $\eta = 0.5$. In such case the equilibrium price and strategies are non-linear and there is some learning about the state at date 1. However, it turns out that all the results are very similar. In fact, the optimal strategies of informed traders remain very close to linear, there is imperfect learning about the state in general, and very little updating about the state in the case of typical order flows. These findings indicate that our main results for a normal prior in this Section are not driven by the particular dependence structure between θ and R : there is a difference between $\mathbb{E}[\theta|y_1, U]$ and p_1 and the optimal strategy of uninformed P at date 2 is positive-feedback.

²¹When we refer to price changes at date 1, we adopt the convention that date 0 price equals the prior $p_0 = \mathbb{E}[\theta] = 0$, which is consistent with our assumption of market efficiency.

4.4 Date 1 problem with a two- and three-point prior distribution.

In this Section we consider a symmetric three-point prior and assume that the fundamental is independent of the state R .²² Assume

$$\theta = \begin{cases} -\bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \\ 0 & \text{wpr. } \gamma \\ \bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \end{cases}, \quad (17)$$

where $0 \leq \gamma < 1$ and $\bar{\theta} > 0$. Clearly, the case with $\gamma = 0$ corresponds to a two-point distribution and is of special interest, as it is a common assumption in the literature. We first characterize the solution of the date 1 problem and then discuss the direction of uninformed P 's trading at date 2 for different values of γ .

4.4.1 The solution

We first derive the equilibrium price and expectations for different strategies of traders K and P and then derive the optimal equilibrium strategies. As we focus on pure strategies, the demand of K and P must be known to the Market for given realizations of the fundamental $\theta = \{-\bar{\theta}, 0, \bar{\theta}\}$. To shorten the argument, we limit our attention to equilibrium strategies that have some natural properties given the symmetry of the distributions and the fact that P has no superior information at date 1. Namely, we conjecture that in state $R = U$, the informed trader K 's optimal demand is \bar{g}_U if $\theta = \bar{\theta}$, zero if $\theta = 0$ and $h_1^{UK} = -\bar{g}_U$ if $\theta = -\bar{\theta}$; and uninformed trader P does not trade. In state $R = D$, the total demand by informed traders K and P is $\frac{\bar{g}_D}{2}$ if $\theta = \bar{\theta}$, zero if $\theta = 0$ and $-\frac{\bar{g}_D}{2} < 0$ if $\theta = -\bar{\theta}$. This allows to derive the main properties of the equilibrium price and expectations as described by the following lemma.

Lemma 3 *For the equilibrium price and conditional expectations of the fundamental, it holds that*

1. *The price is given by*

$$p_1(y_1) = \bar{\theta} \frac{M_n(y_1) - M_p(y_1)}{M_n(y_1) + M_p(y_1)} \quad (18)$$

where $M_n(y_1) \equiv \frac{1-\gamma}{2} \left(\eta \varphi_s(y_1 - \bar{g}_D) + (1-\eta) \varphi_s(y_1 - \bar{g}_U) + \frac{\gamma}{1-\gamma} \varphi_s(y_1) \right)$
and $M_p(y_1) \equiv \frac{1-\gamma}{2} \left(\eta \varphi_s(y_1 + \bar{g}_D) + (1-\eta) \varphi_s(y_1 + \bar{g}_U) + \frac{\gamma}{1-\gamma} \varphi_s(y_1) \right);$

²²It would be easy to consider, similarly to Section 4.3, a case where there is some dependence of R and θ .

2. The conditional expectation of the fundamental is

$$\mathbb{E}[\theta|y_1, R] = \bar{\theta} \frac{\varphi_s(y_1 - \bar{g}_R) - \varphi_s(y_1 + \bar{g}_R)}{\varphi_s(y_1 - \bar{g}_R) + \varphi_s(y_1 + \bar{g}_R) + \frac{2\gamma}{1-\gamma}\varphi_s(y_1)}; \quad (19)$$

3. The updated probability of state $R = I$ is

$$Q_1(y_1) = \Pr(I|y_1) = \frac{\eta(\varphi_s(y_1 - \bar{g}_I) \frac{1-\gamma}{2} + \varphi_s(y_1) \gamma + \varphi_s(y_1 + \bar{g}_I) \frac{1-\gamma}{2})}{M_n(y_1) + M_p(y_1)}; \quad (20)$$

4. The price is increasing in the order flow, i.e., $p'_1(y_1) > 0$;

5. The price is symmetric around zero, i.e., $p_1(y_1) = -p_1(-y_1)$;

6. It holds that $\lim_{y_1 \rightarrow \infty} p_1(y_1) = \bar{\theta}$ and $\lim_{y_1 \rightarrow -\infty} p_1(y_1) = -\bar{\theta}$;

7. $\bar{\theta} - p_1(y_1) > 0$ for all (finite) y_1 ;

Proof. For parts 1-3 note that (7) implies that, $p_1(y_1) = Q_1 \mathbb{E}[\theta|y_1, I] + (1 - Q_1) \mathbb{E}[\theta|y_1, U]$. By the law of total expectations $\mathbb{E}[\theta|y_1, R] = \bar{\theta} \Pr(\theta = \bar{\theta}|y_1, R) - \bar{\theta} \Pr(\theta = -\bar{\theta}|y_1, R)$ and by Bayes' rule $\Pr(\theta|y_1, R) = \frac{1-\gamma}{2} f(y_1|\theta, R) / f(y_1|R)$, where $f(y_1|\theta = \bar{\theta}, R) = \varphi_s(y_1 - \bar{g}_R)$; $f(y_1|\theta = 0, R) = \varphi_s(y_1)$; $f(y_1|\theta = -\bar{\theta}, R) = \varphi_s(y_1 + \bar{g}_R)$ and $f(y_1|R) = f(y_1|\theta = \bar{\theta}, R) \frac{1-\gamma}{2} + f(y_1|\theta = -\bar{\theta}, R) \frac{1-\gamma}{2} + \gamma f(y_1|\theta = 0, R)$. By Bayes' rule $Q_1(y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + (1-\eta) f(y_1|U)}$. Combining all this proves parts 1-3. For part 4, note that from (6) $\partial \varphi_s(y_1 - c) / \partial y_1 = -\varphi_s(y_1 - c) \frac{y_1 - c}{\sigma_s^2}$ for any constant c , and it holds that $M'_n(y_1) = -\frac{y_1}{\sigma_s^2} M_n(y_1) + \frac{1}{\sigma_s^2} M_{ng}(y_1)$ and $M'_p(y_1) = -\frac{y_1}{\sigma_s^2} M_p(y_1) - \frac{1}{\sigma_s^2} M_{pg}(y_1)$, where $M_{ng}(y_1) \equiv \frac{1-\gamma}{2} (\eta \bar{g}_I \varphi_s(y_1 - \bar{g}_I) + (1 - \eta) \bar{g}_U \varphi_s(y_1 - \bar{g}_U)) > 0$ and $M_{pg}(y_1) \equiv \frac{1-\gamma}{2} (\eta \bar{g}_I \varphi_s(y_1 + \bar{g}_I) + (1 - \eta) \bar{g}_U \varphi_s(y_1 + \bar{g}_U)) > 0$. Using the above and differentiating, $p'_1(y_1) = \frac{2\bar{\theta}}{\sigma_s^2} \frac{M_{ng}(y_1)M_p(y_1) + M_{pg}(y_1)M_n(y_1)}{(M_n(y_1) + M_p(y_1))^2} > 0$. Parts 5-7 are straightforward when using the fact that $\varphi_s(\cdot) > 0$ is an even function, (6) and (18). ■

Lemma 3 confirms some reasonable and desirable properties of date 1 price, e.g., the price is increasing in the order flow, symmetric around zero and always between $-\bar{\theta}$ and $\bar{\theta}$. These properties will be most useful for exploring the informed traders' strategies and eventually the direction of price-contingent strategies. If the state is $R = U$, then the expected profit (3) of K can be written as

$$\pi_2^{UK} = h_1^{UK} \int_{-\infty}^{\infty} (\theta - p(y_1)) \varphi_s(y_1 - h_1^{UK}) dy_1. \quad (21)$$

If the state is $R = I$ then the expected profit (2) of trader $J \in \{K, P\}$ can be written as

$$\pi_2^{IJ} = h_1^{IJ} \int_{-\infty}^{\infty} (\theta - p(y_1)) \varphi_s(y_1 - h_1^{IK} - h_1^{IP}) dy_1. \quad (22)$$

As in Section 4.2, we cannot be immediately sure if the informed trader's profits are quasi-concave in own demand and whether there is a unique maximum. It is clear from (21) and (22) and part 7 in Lemma 3 that whenever $\theta \neq 0$, the informed traders's payoff involves a non-trivial integral and is always positive for orders with the same sign as the fundamental.²³ Furthermore, $\bar{\theta} - p_1(y_1)$ cannot be proved to be log-concave for intermediate values of η . While log-concavity is only a sufficient but not necessary condition to guarantee uniqueness of the solution of the trader's problem and existence of the pure strategy equilibrium, it is usually difficult to establish quasi-concavity of integrals under more general conditions.²⁴ In this setting, however, it turns out that we can still prove quasi-concavity by an alternative approach.

Lemma 4 *The expected profit of an informed trader $J \in \{I, U\}$, in state $R \in \{U, I\}$ is strictly quasi-concave in h_1^{RJ} and there is an interior maximum.*

Proof. See Appendix B. ■

As we know that informed trader's problem has a unique solution, we can now state

Proposition 3 *There is a pure strategy equilibrium at date 1, where the following holds:*

1. *Informed traders' demand is given by*

$$h_1^{UK} = \begin{cases} \bar{g}_U = \sigma_s \mu_U & \text{if } \theta = \bar{\theta} \\ 0 & \text{if } 0 \\ -\bar{g}_U = -\sigma_s \mu_U & \text{if } \theta = -\bar{\theta} \end{cases} \quad \text{and} \quad h_1^{IK} = h_1^{IP} = \begin{cases} \frac{\bar{g}_I}{2} = \sigma_s \frac{\mu_I}{2} & \text{if } \theta = \bar{\theta} \\ 0 & \text{if } 0 \\ -\frac{\bar{g}_I}{2} = -\sigma_s \frac{\mu_I}{2} & \text{if } \theta = -\bar{\theta} \end{cases},$$

where μ_U and μ_I only depend on η and γ .

2. *Total demand by informed traders in the event of news ($\theta = \bar{\theta}$ or $\theta = -\bar{\theta}$) is always higher in absolute value in state $R = I$ compared to state $R = U$, i.e., $g_I > g_U$ (equivalently $\mu_I > \mu_U$).*
3. *In state $R = U$, the uninformed trader P does not trade at date 1, i.e., $h_1^{UP} = 0$.*

²³This can be verified using symmetry and Lemma A.1, in Appendix A,

²⁴See for example Prékopa (1980) for a discussion. Also, note for the objective function to always have a unique maximum, what we need is that the (negative of) first derivative is a single crossing function. Regarding this, Quah and Strulovici (2012) discuss the related complications and propose a more relaxed sufficient conditions for aggregation of single crossing property. However, in our setting, that condition would ultimately require log-concavity of $\bar{\theta} - p_1(y_1)$.

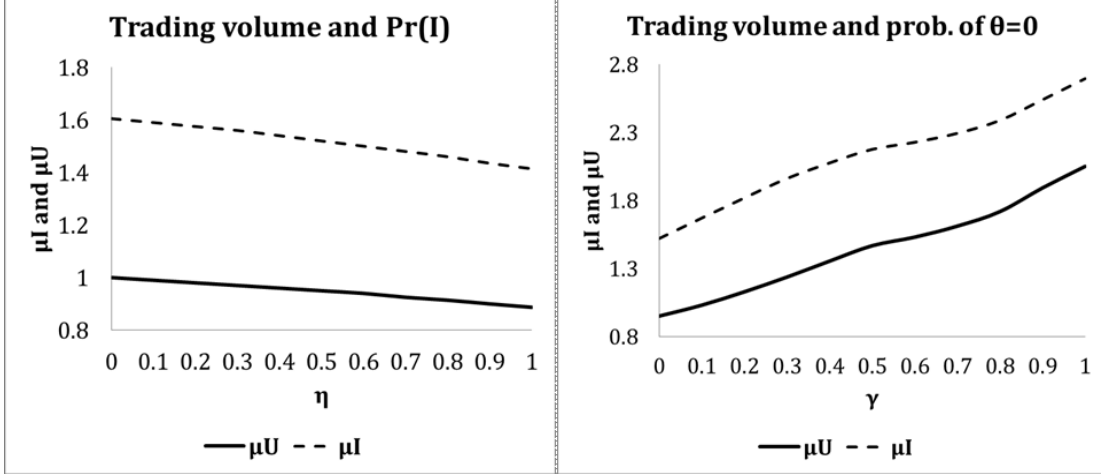


Figure 2: Informed trading volume for different values of η and γ .

Proof. See Appendix B. ■

Proposition 3 confirms that the date 1 equilibrium exists and has some intuitive properties. First, informed traders face the standard trade-off as in Kyle (1985) and Holden and Subrahmanyam (1992). On the one hand, whenever they have private information that indicates $\theta \neq 0$ they earn positive expected profits from trading. On the other hand, they know that due to market impact, trading a higher volume reveals information about the fundamental (and also—less importantly for these traders—about the state R) to the Market. Therefore, they trade a finite amount and the price will not adjust immediately to equal the fundamental value.

As in the previous cases, the trading volume is always proportional to the standard deviation of noise trading. This is because informed traders benefit on average at the expense of noise traders and more noise allows them to hide private information more easily. Because the equilibrium price is proportional to the fundamental (see (18)), the magnitude of the fundamental value does not affect the informed trader's optimal strategy, but clearly profits are higher if $\bar{\theta}$ is higher. While for most values of η and γ we do not have a nice analytical expression for the equilibrium strategies,²⁵ by Lemma 4 we know that the solution is unique and by part 1 of Proposition 3 we know that it depends on only two parameters that are between 0 and 1. Figure 2 illustrates these dependences by plotting on the vertical axis μ_U and μ_I against η (on the left panel, assuming $\gamma = 0$) and against γ (on the right panel, assuming $\eta = 0.5$). These plots are qualitatively similar for different values of η and γ . The effects reported in the figure are intuitive. First, if the prior probability of the state with two informed traders $R = I$ (i.e., $\Pr(I) = \eta$) is higher, then the informed traders are trading less aggressively. This is because

²⁵We can find the analytical solution corresponding to Kyle (1985) and to the two-traders version of Holden and Subrahmanyam (1992) by extending these to a different prior distribution. Namely if $\gamma = 0$ and $\eta = 0$, it holds that $\mu_U = 1$; if $\gamma = 0$ and $\eta = 1$, it holds that $\frac{\mu_I}{2} = \frac{\sqrt{2}}{2}$.

$g_I > g_U$ and the Market expects more informed trading and is updating his beliefs faster. This in turn increases the informed traders' market impact and reduces their willingness to trade aggressively. Second, if the prior probability of "no news" (γ) is higher, the informed traders trade more aggressively whenever they observe $\theta \neq 0$. This is because by Bayes' rule the Market is more reluctant to update his beliefs toward the more extreme realizations of the fundamental. This reluctance reduces the market impact of the informed traders and gives them incentives to trade more.

The most important part of Proposition 3 is the part 3 which states that the total order flow by informed traders is different in the two states. Looking then at the expressions for p_1 , $\mathbb{E}[\theta|y_1, I]$ and $\mathbb{E}[\theta|y_1, U]$ suggest that these are all different. If the state is $R = U$, then P again obtains superior information exactly because he knows that he did not trade and thus we can explore the direction of his trade at date 2.

4.4.2 Direction of price contingent trading

We start by examining the two-point prior, i.e., for now we set $\gamma = 0$.

Proposition 4 *When the prior distribution of the fundamental is a symmetric two-point distribution, it holds that*

$$\mathbb{E}[\theta|y_1, U] < (>) \mathbb{E}[\theta|y_1, I], \text{ for any } y_1 > (<) 0.$$

Whenever $0 < \eta < 1$, the optimal strategy of P in state $R = U$ at date 2 is negative-feedback.

Proof. Assuming $\gamma = 0$, we obtain from (19) that $\text{sgn}(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]) = \text{sgn}\left(\frac{\varphi_s(y_1 - g_U)}{\varphi_s(y_1 + g_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)}\right) = \text{sgn}\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right)$. Because $\varphi_s(\cdot)$ is log-concave and $\bar{g}_I > \bar{g}_U$ by part 2 in Proposition 3, it holds by the property of log-concave distributions in Lemma A.1 in Appendix A that $\text{sgn}\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right) = -1$ if $y_1 > -y_1 \Leftrightarrow y_1 > 0$ and $\text{sgn}\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right) = 1$ if $y_1 < -y_1 \Leftrightarrow y_1 < 0$. By (7) $\text{sgn}(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]) = \text{sgn}(\mathbb{E}[\theta|y_1, U] - p_1)$ for any $0 < Q_1 < 1$, which is true for any $0 < \eta < 1$. Uninformed P 's optimal strategy at date 2, equation (13) in Theorem 1, and the definition of negative-feedback strategy in Section 4.2 complete the proof. ■

With a discrete two-point distribution we find that if the true state is $R = U$, i.e., P is uninformed, then P 's optimal strategy at date 2 is always negative-feedback (contrarian). While there is still optimal price-contingent trading, the direction is exactly the opposite to what we found in Section 4.3 with a normal prior. Recall that in the case of a normal prior, the reason why there was positive-feedback trading was that the higher positive order flow

signalled a higher fundamental in state $R = U$ than in state $R = I$, and this higher signal effect dominated the higher uncertainty of the order flow signal. Clearly with a two-point distribution the higher signal effect is absent, because "news" can either be good or bad, and there is no sense in which they can be 'very' good or 'moderately' good (re. bad).

We can also observe that a two-point prior is bounded and has no mass in the center, while a normal is unbounded and has thin tails and substantial probability mass in the center. To understand the main drivers of the direction of price-contingent trading, it seems then most transparent, while at the same time still tractable, to consider the case of a three-point prior distribution whereby the distribution is still bounded, but we can control the mass in the center by choosing different values of γ (see (17)). It can also be seen as a crude approximation of any continuous prior distribution that is either bimodal at the extremes, uniform or hump-shaped. With a three-point distribution we can establish some general properties about the direction of price-contingent trading.

Proposition 5 *When the prior distribution of the fundamental is a symmetric three-point prior distribution and $R = U$, then for any $0 < \eta < 1$ the following conditions hold for large and very small order flows*

1. *at least for order flows $y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$ and $y_1 \leq -\frac{\bar{g}_I + \bar{g}_U}{2}$, P pursues a negative-feedback strategy at date 2.*
2. *for order flows y_1 in the neighborhood of zero (i.e., for $y_1 \rightarrow 0$), P pursues a positive-feedback strategy at date 2 iff the following condition holds.*

$$\frac{1 + \exp(\mu_I^2) \frac{\gamma}{1-\gamma}}{1 + \exp(\mu_U^2) \frac{\gamma}{1-\gamma}} > \frac{\mu_I}{\mu_U} \quad (23)$$

Proof. To prove part 1 we use (19) to find that $\text{sgn}(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]) = \text{sgn}\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)} + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1 + \bar{g}_U)\varphi_s(y_1 + \bar{g}_I)} B(y_1)\right)$, where $B(y_1) \equiv \varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 + \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) + \varphi_s(y_1 + \bar{g}_I)$. Consider $y_1 > 0$ and let us focus on the sign of $B(y_1)$. Because $\varphi_s(\cdot)$ has a maximum at zero and is decreasing for any positive values, it also holds for any $y_1 > 0$ and $\bar{g}_I > \bar{g}_U$, that $-\varphi_s(y_1 + \bar{g}_U) + \varphi_s(y_1 + \bar{g}_I) < 0$. We can then find a sufficient condition for also $\varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) \leq 0$ to hold. Define $b \equiv y_1 - \frac{\bar{g}_I + \bar{g}_U}{2}$ and $b \geq 0 \iff y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$. By Corollary A.1.1 in Appendix A and by $\bar{g}_I > \bar{g}_U$, it holds for any $b \geq 0$ that $\varphi_s\left(\frac{\bar{g}_I - \bar{g}_U}{2} + b\right) \leq \varphi_s\left(\frac{\bar{g}_I - \bar{g}_U}{2} - b\right) \iff \varphi_s(y_1 - \bar{g}_U) \leq \varphi_s(y_1 - \bar{g}_I)$. Therefore, for any $y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$ it holds that $B(y_1) < 0$. From the proof of Proposition 4 (and Lemma A.1 in Appendix A), we already know that $\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_U)} > \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)}$ for any $y_1 > 0$. Therefore, $\text{sgn}(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]) = \text{sgn}(\mathbb{E}[\theta|y_1, U] - p_1) = -1$ for any $y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2}$ and

$0 < \eta < 1$. The definition of negative-feedback strategy in Section 4.2 completes this part of the proof. The proof for $y_1 \leq -\frac{\bar{g}_I + \bar{g}_U}{2}$ is similar.

To prove part 2, notice from (19), that at $y_1 = 0$, $\mathbb{E}[\theta|y_1, R] = 0$ for $R = \{I, U\}$. For a positive-feedback strategy to be optimal around zero, it is enough to show that the expected value, $\mathbb{E}[\theta|y_1, R]$, is more sensitive to the order flow in state $R = U$ compared to state $R = I$. From (19) and (6) and $\bar{g}_R = \mu_R \sigma_s$, $\partial \mathbb{E}[\theta|y_1, R] / \partial y_1|_{y_1=0} = -\varphi'_s(\bar{g}_R) \left(\varphi_s(\bar{g}_R) + \frac{\gamma}{1-\gamma} \varphi_s(0) \right)^{-1} = \frac{\mu_R}{\sigma_s} \exp(-\mu_R^2) \left(\exp(-\mu_R^2) + \frac{\gamma}{1-\gamma} \right)^{-1}$. Using the definition of positive-feedback trading from Section 4.2, (23) then follows from $\partial \mathbb{E}[\theta|y_1, U] / \partial y_1|_{y_1=0} > \partial \mathbb{E}[\theta|y_1, I] / \partial y_1|_{y_1=0}$. ■

Proposition 5 shows that with a three-point prior both positive-feedback and contrarian strategies are possible at date 2. We also gain insights on which features of the prior distribution drive the direction of price-contingent trading.

Part 1 of Proposition 5 shows that, when the date 1 order flow is large in absolute value, then at date 2 uninformed P always pursues a negative-feedback strategy. This is driven by the bounded support of the prior, and this is why this effect was missing in the case of a normal prior. Intuitively, high order flows are more likely to be driven by high noise trading shocks than by a large draw of the fundamental. For example, if the true state is $R = U$, then any order flow that exceeds \bar{g}_U must mean that there was a positive noise trading shock, while the Market will still consider order flows between \bar{g}_U and \bar{g}_I to be potentially reflecting small or even negative noise trading shocks. This reflects the previously discussed tendency of the Market to underestimate the impact of noise trading in driving the date 1 order flow and the price too high (if the true state is $R = U$).

Part 2 of Proposition 5 shows that if the probability of no-news is large enough, then at least for small order flows uninformed P 's optimal strategy at date 2 is positive-feedback. Namely, there is a threshold level for γ , above which this inequality holds²⁶ and back-of-the-envelope calculations indicate that this threshold is quite low.²⁷ This observation highlights the fact that for positive-feedback trading there should be at least some mass in the center of the distribution. The intuition for small order flows relates to our earlier discussion about the Market tending to underestimate the fundamental conditional on the order flow only. Consider a small positive order flow in this example. If the true state is $R = U$, then the Market is reluctant to believe that it is driven by informed traders who observed $\bar{\theta}$ (as he considers the possibility that 2 informed traders who would trade \bar{g}_I in total, while the actual informed trading could have

²⁶Note that the right hand side of (23) is always bigger than 1 as $\mu_I > \mu_U$ by point 2 in Proposition 3. The right hand side is 1 if $\gamma = 0$, strictly increasing in γ and converges to $\exp(\mu_I^2) / \exp(\mu_U^2)$ when $\gamma \rightarrow 1$. We can also verify that $\exp(\mu_I^2) / \exp(\mu_U^2) > \mu_I / \mu_U$ at the limit. This is because $\exp(\mu_R^2) / \mu_R$ is strictly increasing in μ_R for any $\mu_R > 0.5$. Hence (23) will hold at $\gamma \rightarrow 1$ if $\mu_I > \mu_U > 0.5$. It can also be shown that μ_U is at its lowest when $\eta = 1$ and $\gamma = 0$, and from Figure (2) that in such case μ_U is noticeably higher than 0.5.

²⁷For example, if $\eta = 0.5$ then the threshold is around $\gamma \approx 0.08$.

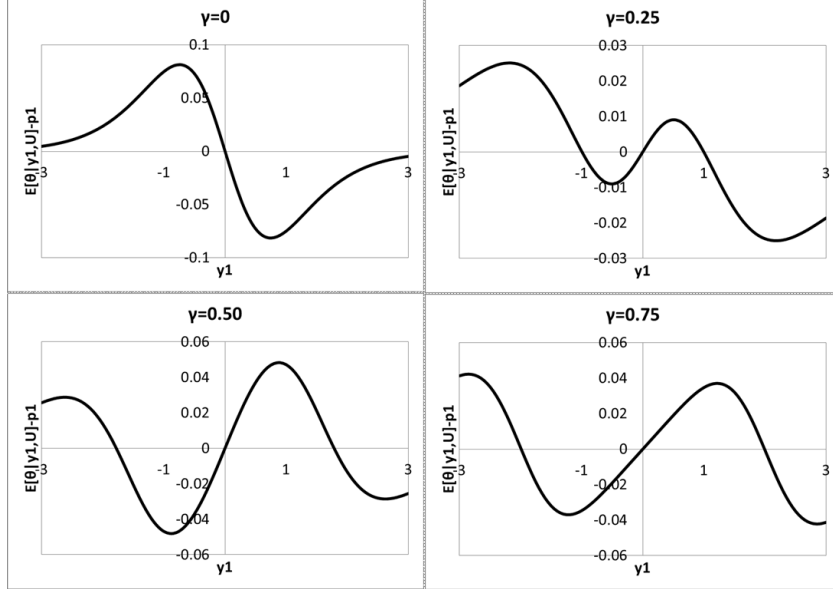


Figure 3: The difference $\mathbb{E}[\theta|y_1, U] - p_1$ for different values of γ .

been at most \bar{g}_U) and sets the price relatively close to zero. Because uninformed P knows at date 2 that his trading did not contribute to date 1 order flow, he benefits from positive-feedback trading on average. Figure (3) illustrates the equilibrium difference, $\mathbb{E}[\theta|y_1, U] - p_1$, (vertical axis) for different values of γ , assuming that $\eta = 0.5$. On the horizontal axis, there is always the date 1 order flow, y_1 . We can see that when γ is high enough, then there is a large set of order flows around zero where $|\mathbb{E}[\theta|y_1, U]| > |p_1|$ and uninformed P 's optimal strategy at date 2 is positive-feedback, while at high order flows in absolute value, it always holds that $|\mathbb{E}[\theta|y_1, U]| < |p_1|$ and uninformed P 's optimal strategy at date 2 is negative-feedback.

The three-point distribution also allows to derive richer empirical implications. We find that price-contingent traders are more likely to act contrarian against large order flows, as these are more likely to be driven by noise trading shocks. In the case of small order flows, they may benefit from momentum strategies, as these are more likely to reflect fundamental information that is not yet incorporated in prices.

4.5 Predictability of order flow and the effect of price-contingent trading on market efficiency.

Here we point out some natural consequences of equilibrium price-contingent trading under either the semi-strong or weak form of market efficiency.

Proposition 6 *While there is no predictability in returns, the order flow is predictable.*

Proof. The lack of predictability in returns is immediate and is due to imposing the efficient

market condition (1). By construction $p_2 = \mathbb{E}[\theta|y_1, y_2]$ and $p_1 = \mathbb{E}[\theta|y_1]$, and by application of the law of iterated expectations, it is clear that $\mathbb{E}[p_2 - p_1|y_1] = \mathbb{E}[\mathbb{E}[\theta|y_1, y_2]|y_1] - p_1 = \mathbb{E}[\theta|y_1] - p_1 = 0$. At the same time by Theorem 1 we know that if the state is $R = U$ then P trades at date 2 a known amount \bar{h}_2 . Therefore, $\mathbb{E}[y_2|y_1] = \Pr(I|y_1)\mathbb{E}[y_2|y_1, I] + \Pr(U|y_1)\mathbb{E}[y_2|y_1, U] = Q_1\mathbb{E}[s_2|y_1, I] + (1 - Q_1)\mathbb{E}[\bar{h}_2 + s_2|y_1, U] = (1 - Q_1)\mathbb{E}[\bar{h}_2|y_1, U] \neq 0$. ■

In Kyle (1985) and Holden and Subrahmanyam (1992) and subsequent models that build on their framework, imposing the market efficiency condition implies *both* the lack of predictability of returns *and* the lack of predictability of the order flow. As discussed in Section 4.1, there is no profitable and predictable price-contingent trading and future order flow can only reflect unpredictable noise trading and informed trading. Matters differ considerably in our more general setting, because the Market cannot be perfectly sure of whether there is a price-contingent trader P or not, but he still knows that if there is one, he will trade in a predictable direction, described in Propositions 3, 5, and 6. For example, if the optimal strategy is positive-feedback (e.g., with a normal prior or with a small order flow in the case of a three-point distribution), the Market expects a positive order flow with some probability; if the actual order flow is zero, the prices fall. While there can be other reasons for predictability of the order flow, our model demonstrates that predictability of the order flow by itself does not imply a failure of market efficiency; and more generally suggests that the order flow can be more predictable than prices, consistent with available empirical evidence (see, e.g., Biais, Hillion, and Spatt (1995), Ellul, Holden, Jain, and Jennings (2007), Lillo and Farmer (2004)).

It should also be noted that the type of price-contingent trading we analyze as emerging in a rational setting without other frictions, on average facilitates price discovery by moving prices closer to the fundamental. In state $R = U$, the best estimate of the fundamental conditional on all the information apart from the fundamental itself is $\mathbb{E}[\theta|y_1, U]$, and not $\mathbb{E}[\theta|y_1]$, so that uninformed P 's price-contingent trading on average pushes date 2 price p_2 closer to $\mathbb{E}[\theta|y_1, U]$. There is also no sense in which contrarian trading is more stabilizing than momentum trading in our setting. It is true that in our setting a rare situation can arise whereby prices change purely because of a large noise trading shock and P 's positive-feedback strategy moves prices further away from the fundamental, but similarly there can be a rare situation whereby P 's optimal strategy is contrarian and due to some extreme draws of noise trading, his contrarian trading delays news about the fundamental from being reflected into prices.

5 Discussion

As discussed in Section 3, we view the noise traders in our model as capturing a large number of traders who trade for idiosyncratic reasons outside the model. Therefore, the main argument

for assuming normally distributed noise trading stems from the central limit theorem. However, technically, many realistic properties of our model rely on the less restrictive assumption of log-concave noise trading. Indeed, a log-concave distribution guarantees that the Market updates at date 2 in the "correct direction" - that is, in state $R = U$, if trader P submits a positive quantity in equilibrium, then higher order flows at date 2 always signal a higher posterior probability that the state is indeed $R = U$. It also guarantees that the expected value $\mathbb{E}[\theta|y_1, R]$ is increasing in date 1 order flow, which for many prior distributions also implies that the date 1 equilibrium price is increasing in the order flow. Both of these properties hold because log-concavity implies the monotone likelihood ratio property (MLRP). These properties are realistic in the context of financial markets and guarantee that sophisticated large traders in our model face a meaningful trade-off in the spirit of Kyle (1985)—namely, an informed trader (either with superior information about the fundamental directly or indirectly due to superior knowledge of his own past actions) benefits from trading a higher volume due to positive expected returns, but trading a higher volume is costly due to market impact as it reveals more about his private information—whether about the fundamental or about his own type.

In our proofs we frequently relied only on log-concavity rather than on the explicit form of the normal density (6). For example, the determinants of the direction of price-contingent trading in the case of two- and three-point prior distribution hold for any (symmetric) log-concave noise trading. Technically, the explicit form of the normal distribution was mostly useful for proving that the trader's problem is quasi-concave in his own action. Because the trader's objective function is an integral of a non-trivial function, quasi-concavity is generally difficult to prove analytically (see also footnote 24). However, at least numerically, it can be verified that similar results hold with other log-concave noise trading distributions. It should also be noted that to avoid a situation whereby P at date 2 has an incentive to trade an infinite amount—which cannot occur in equilibrium—we need at least that the noise trading distribution, $f_s(\cdot)$, is such that the likelihood ratio $r(y_2) = \frac{f_s(y_2 - \bar{h}_2)}{f_s(y_2)}$ is unbounded, which is true for some, but not all log-concave densities.²⁸

While log-concave noise trading is important for generating realistic properties of equilibrium prices and traders' incentives, the choice of the prior distribution is much more flexible. The model can be solved, at least numerically, for many different prior distributions. However, the choice of the prior is clearly crucial for the more qualitative results of our paper. We used the examples of a normal, two-point, and three-point prior. The first two are among the most common assumptions in this literature, as reflected by the fact that there exist direct

²⁸By unbounded, we mean that $\lim_{y_2 \rightarrow \bar{s}} r(y_2) \rightarrow \infty$ for $0 < \bar{h}_2 < \bar{s}$. For example, not just the normal, but also the $Beta(\alpha, \beta)$ distribution with parameters $\alpha, \beta > 1$ is strictly log-concave and has an unbounded likelihood ratio. However, $Gamma(k, \theta)$ with $k > 0$ is strictly log-concave, but the likelihood ratio is not unbounded (Gamma is also not symmetric around zero).

benchmarks for the baseline case with no uncertainty about traders' types—see Kyle (1985) and Holden and Subrahmanyam (1992) for the case with a normal prior and Cho and El Karoui (2000) for the case with a two-point prior. Importantly, we find opposite results for the direction of P 's optimal trading at date 2 between the two-point and the normal prior. We get additional insights using the three-point prior that is a transparent proxy for many distributions in different shapes. In particular, what appears to be crucial is how much probability mass is in the center of the distribution and whether the prior has a bounded support.

It is useful to note that if date 1 price is increasing in the order flow and informed trader's profit at date 1 is quasi-concave (with interior maximum), then there is a pure strategy equilibrium where a set of properties are true, which allows us to argue that the forces that determine the direction of P 's trading remain present for other distributions. For example, assume that the fundamental, θ , is a continuous variable;²⁹ that noise trading is also continuous in the interval $[-\bar{s}, \bar{s}]$; and $f_s(s)$ is log-concave and symmetric.

Proposition 7 *If date 1 price is increasing in the order flow and the informed traders' problem is quasi-concave (with interior maximum), then:*

1. *The total demand of informed traders, $g_R(\theta)$ in state $R \in \{I, U\}$ is strictly increasing in θ .*
2. *It holds that $g_I(\theta) > (<) g_U(\theta)$ for any $\theta > (<) 0$.*

Proof. Part 1 follows from results in the monotone comparative statics literature that we can use to explore informed traders' profits (3) and (2). Denote in state $R = U$ trader K 's expected price when demanding h_1^{UK} as $p_E(h_1^{UK}) \equiv \int_{-\bar{s}}^{\bar{s}} p_1(h_1^{UK} + s_1) f_s(s_1) ds_1$. From Milgrom and Shannon (1994) it is known that $g_U(\theta) = \arg \max_{h_1^{UK}} h_1^{UK} (\theta - p_E(h_1^{UK}))$ is weakly increasing in θ if the trader's problem has increasing differences (which also implies the payoff is supermodular) in h_1^{UK} and θ . This is indeed true as for any $\tilde{\theta} > \theta$ and $\tilde{h}_1^{UK} > h_1^{UK}$, it holds that $\tilde{h}_1^{UK} (\tilde{\theta} - p_E(h_1^{UK})) - \tilde{h}_1^{UK} (\theta - p_E(\tilde{h}_1^{UK})) > h_1^{UK} (\tilde{\theta} - p_E(h_1^{UK})) - h_1^{UK} (\theta - p_E(h_1^{UK})) \iff (\tilde{h}_1^{UK} - h_1^{UK})(\tilde{\theta} - \theta) > 0$. From Edlin and Shannon (1998), it is also known that $g_U(\theta)$ is strictly increasing if additionally the payoff is strictly increasing if the first derivative of the payoff (profit) is strictly increasing in θ , which is also true in our model, as $\partial h_1^{UK} (\theta - p_E(h_1^{UK})) / \partial h_1^{UK} = \theta - p_E(h_1^{UK}) - h_1^{UK} p'_E(h_1^{UK})$ is clearly increasing in θ . The proof is similar for the state $R = I$, where the same monotone comparative statics establish that K 's and P 's individual demand is increasing in θ , and so is the sum of their demands.

For part 2, notice that given the above assumptions, it is enough to only look at the first order conditions to find the unique equilibrium demands by all informed traders. Also, it is easy to

²⁹This is only to simplify notation, as similar arguments hold for a discrete fundamental.

verify that given θ , both K and P demand the same quantity in state $R = I$. We find that the equilibrium total informed demand $g_R(\theta)$ in state R solves

$$\begin{aligned}\theta &= \int_{-\bar{s}}^{\bar{s}} (p_1(s_1 + g_U(\theta)) + g_U(\theta) p'_1(s_1 + g_U(\theta))) f_s(s_1) ds_1 \\ \theta &= \int_{-\bar{s}}^{\bar{s}} \left(p_1(s_1 + g_I(\theta)) + \frac{g_I(\theta)}{2} p'_1(s_1 + g_I(\theta)) \right) f_s(s_1) ds_1\end{aligned}$$

As by part 1 $g_R(\theta)$ is invertible, it must also hold that

$$\begin{aligned}g_U^{-1}(y_\theta) &= \int_{-\bar{s}}^{\bar{s}} (p_1(s_1 + y_\theta) + y_\theta p'_1(s_1 + y_\theta)) f_s(s_1) ds_1 \\ g_I^{-1}(y_\theta) &= \int_{-\bar{s}}^{\bar{s}} \left(p_1(s_1 + y_\theta) + \frac{y_\theta}{2} p'_1(s_1 + y_\theta) \right) f_s(s_1) ds_1,\end{aligned}\tag{24}$$

where an order flow $y_\theta \equiv g_R(\theta)$. It is straightforward to verify that $g_R(\theta) = -g_R(-\theta)$ and clearly $y_\theta > 0$ iff $\theta > 0$. Because date 1 equilibrium price is increasing in the order flow, it holds that $g_U^{-1}(y_\theta) - g_I^{-1}(y_\theta) = \frac{y_\theta}{2} \int_{-\bar{s}}^{\bar{s}} p'_1(s_1 + y_\theta) f_s(s_1) ds_1 > (<) 0$ for any $y_\theta > (<) 0$. Taken $y_\theta = g_U(\theta) > 0$, we find that $g_U^{-1}(y_\theta) > g_I^{-1}(y_\theta) \iff \theta > g_I^{-1}(g_U(\theta)) \iff g_I(\theta) > g_U(\theta)$ for any $\theta > 0$. The case $y_\theta < 0$ is immediate by symmetry. ■

The most important implication of Proposition 7 is that fixing any fundamental, two informed traders trade a higher quantity in equilibrium than one informed trader. This implies that conditional on date 1 order flow y_1 , the state R and the fundamental θ are not independent. In particular, $\Pr(U|\theta, y_1) = \frac{f_s(y_1 - g_U(\theta)) \Pr(U|\theta)}{f_s(y_1 - g_U(\theta)) \Pr(U|\theta) + f_s(y_1 - g_I(\theta)) \Pr(I|\theta)}$ is generally a function of θ . This allows to conclude that in general there will be a difference between uninformed P 's and the markets expectations. Namely, it holds that

$$\mathbb{E}[\theta|y_1, U] - p_1 = \frac{\mathbb{E}[\theta \cdot \Pr(U|\theta, y_1) | y_1]}{\Pr(U|y_1)} - \mathbb{E}[\theta|y_1] = \frac{\text{Cov}(\theta, \Pr(U|\theta, y_1))}{\Pr(U|y_1)},$$

where we used the market efficiency condition (1) and Bayes' rule $f(\theta|y_1, U) = \frac{f(\theta|y_1) \Pr(U|\theta, y_1)}{\Pr(U|y_1)}$. As we argued that $\Pr(U|\theta, y_1)$ is a function of θ , it is clear that θ and $\Pr(U|\theta, y_1)$ are not independent, which means that the covariance will generally not be zero. Therefore, our main result that price-contingent trading is profitable in a setting where there is uncertainty about traders' types is very general.

Also, given Proposition 7, the forces that affect the direction of the difference $\mathbb{E}[\theta|y_1, U] - p_1$ are also present more generally. As an example, assume that both noise trading and the fundamental θ are continuous and have support in $(-\infty, \infty)$ and noise trading has symmetric log-concave density. In such case, by Proposition 7 part 1, we know that $g_R(\cdot)$ is an invert-

ible function. Therefore, in state R we obtain the following signal from the order flow only: given y_1 and unknown noise trading shock s_1^* , the fundamental is $\theta^* = g_R^{-1}(y_1 - s_1^*)$. The cumulative distribution function (c.d.f.) of θ conditional on the order flow only is therefore, $\Pr(\theta^* < \theta) = \Pr(g_R^{-1}(y_1 - s_1^*) < \theta) = 1 - F_s(y_1 - g_R(\theta))$, where F_s is the c.d.f. of noise trading. By part 2 of Proposition 7 and the fact that the c.d.f. is monotonically increasing, $1 - F_s(y_1 - g_U(\theta)) < (>) 1 - F_s(y_1 - g_I(\theta))$ for any $g_I(\theta) > (<) g_U(\theta) \iff \theta > (<) 0$. This confirms that the distribution of θ conditional on the order flow only is more dispersed in state $R = U$ compared to state $R = I$ for any order flow y_1 . The probability density function of the fundamental conditional on order flow only, is $g'_R(\theta) f_s(y_1 - g_R(\theta))$. We find that the expected value conditional on the order flow only is then $\int_{-\infty}^{\infty} \theta g'_R(\theta) f_s(y_1 - g_R(\theta)) d\theta = \int_{-\infty}^{\infty} g_R^{-1}(y_\theta) f_s(y_1 - y_\theta) dy_\theta = \int_0^{\infty} g_R^{-1}(y_\theta) (f_s(y_1 - y_\theta) - f_s(y_1 + y_\theta)) dy_\theta$. It then holds that the difference in expectations is $\int_0^{\infty} (g_U^{-1}(y_\theta) - g_I^{-1}(y_\theta)) (f_s(y_1 - y_\theta) - f_s(y_1 + y_\theta)) dy_\theta > (<) 0$ if $y_1 > (<) 0$. The inequality follows from (24) and Lemma A.1 in Appendix A. Therefore, if $y_1 > 0$, then in state $R = U$ the order flow signal has a higher mean (which pushes toward positive-feedback trading) and a more dispersed distribution (which pushes toward negative-feedback trading) than in state $R = I$. Any Bayesian updating trades off these two effects, and which one dominates depends on the prior. If, additionally, the distributions are bounded then the same effects are present and there are additional effects due to these bounds - most importantly, large order flows in absolute value are always more likely in state $R = I$.³⁰

Apart from the distributions, there are other assumptions in the model that can potentially be relaxed. Changing any of these would not alter our main results and would either be trivial or complicated with limited additional insights. First, it would be trivial to add more type K and type P traders. All the effects would be the same as long as the number of sophisticated traders of type K and P is finite. The reason why K and P trade finite amounts and earn returns on their information is because they have market impact and they are aware of it. If the number of type P traders were infinite, then they would be indistinguishable from the Market; if the number of type K traders were infinite, then there would be strong-form market efficiency, but it would also generate an information acquisition paradox in the spirit of Grossman and Stiglitz (1980). Second, it would also be possible to add more trading rounds where uninformed P can trade. This would complicate the model as P would likely have a Kyle's (1985) type of incentive to split his orders and reveal information more slowly. However, the Market will then be imperfectly and slowly learning about the true state until the price eventually converges to

³⁰If the prior has bounded support $[-\bar{\theta}, \bar{\theta}]$ and noise trading support is noticeably wider (such that any order flows can be generated by a noise trading shock), then conditional on the order flow only in state R the cdf of θ is $\frac{F_s(y_1 + g_R(\bar{\theta})) - F_s(y_1 - g_R(\bar{\theta}))}{F_s(y_1 + g_R(\bar{\theta})) - F_s(y_1 - g_R(\bar{\theta}))}$ and the probability density function is $\frac{g'_R(\bar{\theta}) F_s(y_1 - g_R(\bar{\theta}))}{F_s(y_1 + g_R(\bar{\theta})) - F_s(y_1 - g_R(\bar{\theta}))}$. With some algebra we can then identify the same effects, and find that informed trading at the highest fundamental, $g_R(\bar{\theta})$ in $R = \{I, U\}$ affect both the mean and dispersion.

$\mathbb{E}[\theta|y_1, R]$. Third, to highlight our main effects we have assumed that K only trades at date 1. Our effects would still obtain, however, if we allowed K to trade at date 2 as well. In fact, while K at date 2 may some times want to take an opposite position to P , he would never fully neutralize P 's profit opportunities. The reason is that K still faces the usual trade-off whereby trading too aggressively reveals too much information to the Market. Moreover, often K would want to trade in the same direction as uninformed P , because of his knowledge of θ . As a result, K would be unable to fully neutralize the position of uninformed P . In sum, allowing K to trade at date 2 as well would make the model less transparent and more complicated as there would be more strategic incentives in play, without altering our main insight. Finally, altering the assumption that K knows P 's type or assuming that both traders can be informed or uninformed about the fundamental, would complicate the model but not alter the main result that trading without direct knowledge of the fundamental is profitable in a more general setting that allows for uncertainty about trader's types.

6 Concluding Remarks

We have established that price-contingent trading is the optimal strategy of large rational agents in a setting in which there is uncertainty about whether large traders are informed about the fundamental. We have then provided conditions under which price-contingent trading is positive-feedback or contrarian in equilibrium. A robust implication of our results is that the order flow is predictable from current prices even if the market is semi-strong efficient and future returns cannot be predicted from current prices.

We have started the paper by noting that quantitative strategies by large financial institutions are price-contingent in that they map past prices into orders and trades. We highlight that knowing privately as little as their past trades is sufficient to enable large agents to better learn fundamental information from prices than all market participants who have only imperfect information about these traders' positions. Clark-Joseph (2012) provides preliminary empirical evidence that supports our mechanism. Of course, in the real world quantitative strategies can be a lot more sophisticated than our simple equilibrium momentum and contrarian strategies, and use as input an array of quantifiable public information in addition to prices. Quantifiable information can arise from superior knowledge of market participants' trading styles rather than economic fundamentals as traditionally thought. Extending our model to capture the additional nuances of real-world quantitative strategies is an exciting area for future research.

A Background theorems and lemmas

Lemma A.1 *If $f_s(\cdot)$ is strictly log-concave, then it holds that*

$$\frac{f_s(x_2 - c_2)}{f_s(x_2 - c_1)} > \frac{f_s(x_1 - c_2)}{f_s(x_1 - c_1)} \text{ for any } x_2 > x_1 \text{ and } c_2 > c_1. \quad (25)$$

Proof. By definition of log-concavity it must hold that

$\alpha \ln(f_s(x_1 - c_2)) + (1 - \alpha) \ln(f_s(x_2 - c_1)) < \ln(f_s(\alpha(x_1 - c_2) + (1 - \alpha)(x_2 - c_1)))$ and
 $(1 - \alpha) \ln(f_s(x_1 - c_2)) + \alpha \ln(f_s(x_2 - c_1)) < \ln(f_s((1 - \alpha)(x_1 - c_2) + \alpha(x_2 - c_1)))$ for any
 $0 < \alpha < 1$. Let $\alpha = \frac{x_2 - x_1}{x_2 - x_1 + c_2 - c_1}$. Then $\ln(f_s(\alpha(x_1 - c_2) + (1 - \alpha)(x_2 - c_1))) = \ln(f_s(x_1 - c_1))$
and

$\ln(f_s((1 - \alpha)(x_1 - c_2) + \alpha(x_2 - c_1))) = \ln(f_s(x_2 - c_2))$. Adding up the inequalities, we obtain that $\ln(f_s(x_1 - c_2)) + \ln(f_s(x_2 - c_1)) < \ln(f_s(x_1 - c_1)) + \ln(f_s(x_2 - c_2))$. Exponentiating both sides and rearranging, we obtain (25). ■

Note that, in probability theory, this implies that if we interpret c as a signal about some random variable such that $x = c + s$, where the density $f_s(s)$ is strictly log-concave, then the conditional distribution of $f(x|c) = f_s(x - c)$ satisfies the strict monotone likelihood ratio property (MLRP).

Corollary A.1.1 *If $f_s(\cdot)$ is strictly log-concave and symmetric ($f_s(s) = f_s(-s)$), then for any $x > 0$, it holds that*

$$f_s(x - c) > (<) f_s(x + c) \text{ for any } c > (<) 0$$

Proof. For the case $c > 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_2 > c_1 = 0$. By (25) $\frac{f_s(x-c)}{f_s(x)} > \frac{f_s(-x-c)}{f_s(-x)} = \frac{f_s(x+c)}{f_s(x)} \implies f_s(x - c) > f_s(x + c)$. For the case $c < 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_1 < c_2 = 0$ to obtain that $\frac{f_s(x)}{f_s(x-c)} > \frac{f_s(-x)}{f_s(-x-c)} = \frac{f_s(x)}{f_s(x+c)} \implies f_s(x + c) > f_s(x - c)$. ■

For the next Lemma, assume that the prior distribution in state $R \in \{I, U\}$ is $f(\theta|R) \equiv f_{\theta R}(\theta) = f_{\theta R}(-\theta)$ in between $-\bar{\theta}$ and $\bar{\theta}$ (the case $\bar{\theta} = \infty$, is easy to incorporate in this framework). We consider a continuous prior, but following a similar logic it is easy to prove it for a discrete prior. Assume that informed trader's total demand at date 1 is symmetric around zero and strictly increasing, i.e., $h_1^{UK} = g_U(\theta)$ and $h_1^{IK} + h_1^{IP} = g_I(\theta)$, where $g_R(\theta) = g_R(-\theta)$ and $g'_R(\theta) > 0$ for $R \in \{0, 1\}$. Assume that uninformed P 's demand at date 1 is some constant \bar{h}_{1U} .

Lemma A.2 *If the noise trading distribution is log-concave and symmetric, $f_s(s_1)$, it holds for any prior distribution that $\mathbb{E}[\theta|\tilde{y}_1, R] - \mathbb{E}[\theta|y_1, R] > 0$ for any $\tilde{y}_1 > y_1$. It also holds that $\mathbb{E}[\theta|y_1, I] = \mathbb{E}[\theta|(-y_1), I]$ and $\mathbb{E}[\theta|(y_1 - \bar{h}_{1U}), U] = -\mathbb{E}[\theta|(-y_1 + \bar{h}_{1U}), U]$.*

Proof. The proof uses Milgrom (1981) and log-concavity of $f_s(s)$. Define the observable part of the order flow in state R as follows: $y_{1R} = y_1$ if $R = I$ and $y_{1R} = y_1 - \bar{h}_{1U}$ if $R = U$. It then holds that $y_{1R} = g_R(\theta) + s_1$ for $R = \{I, U\}$. We first show that $\mathbb{E}[\theta|\tilde{y}_1, R] - \mathbb{E}[\theta|y_1, R] > 0$ for any $\tilde{y}_1 > y_1$. It is well known that $\mathbb{E}[\theta|\tilde{y}_1, R] > \mathbb{E}[\theta|y_1, R]$ if the cumulative distribution $F(\theta|\tilde{y}_1, R)$ dominates $F(\theta|y_1, R)$ in the sense of first order stochastic dominance, i.e., $F(\theta|\tilde{y}_1, R) \leq F(\theta|y_1, R)$ for all θ , with strict inequality for some θ . Given that the order flow is $y_{1R} = g_R(\theta) + s_1$, $g_R(\cdot)$ is increasing, we know from Lemma A.1, that log-concavity of $f_s(\cdot)$ implies that $f_s(\tilde{y}_{1R} - g_R(\hat{\theta})) f_s(y_{1R} - g_R(\check{\theta})) > f_s(\tilde{y}_{1R} - g_R(\check{\theta})) f_s(y_{1R} - g_R(\hat{\theta}))$ for every $\hat{\theta} > \check{\theta}$. We can equivalently write this inequality as $f(\tilde{y}_{1R}|\hat{\theta}, R) f(y_{1R}|\check{\theta}, R) > f(\tilde{y}_{1R}|\check{\theta}, R) f(y_{1R}|\hat{\theta}, R)$ for every $\hat{\theta} > \check{\theta}$. The proof that the latter inequality implies first order stochastic dominance for any prior density $f(\theta|R)$ is in Milgrom (1981) Proposition 1. As in state $R = I$, $y_{1R} = y_1$, this immediately proves that $\mathbb{E}[\theta|\tilde{y}_1, I] - \mathbb{E}[\theta|y_1, I] > 0$ for any $\tilde{y}_1 > y_1$. For the state $R = U$, notice that because \bar{h}_{1U} is known, conditioning on y_1 and on $y_{1U} = y_1 - \bar{h}_{1U}$ is equivalent and it must always hold that $\mathbb{E}[\theta|y_1, R] = \mathbb{E}[\theta|y_{1R}, R]$ so we know that $\mathbb{E}[\theta|\tilde{y}_1, U] - \mathbb{E}[\theta|y_1, U] > 0$ for any $\tilde{y}_1 - \bar{h}_{1U} > y_1 - \bar{h}_{1U} \Leftrightarrow \tilde{y}_1 > y_1$. For the second part, note that we can express that $\mathbb{E}[\theta|y_{1R}, R] = \int_{-\bar{\theta}}^{\bar{\theta}} \theta f(\theta|y_{1R}, R) d\theta = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}$. Using the symmetry of $f_s(\cdot)$ and $f_{\theta R}(\theta)$, we then find that $\mathbb{E}[\theta| - y_{1R}, R] = \int_{-\bar{\theta}}^{\bar{\theta}} \theta f(\theta| - y_{1R}, R) d\theta = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(-y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(-y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta} = -\frac{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} \theta f_s(y_{1R} - g_R(\theta)) f_{\theta R}(\theta) d\theta} = -\mathbb{E}[\theta|y_{1R}, R]$. Using then the definition of y_{1R} proves the lemma. ■

Theorem A.3 (Prékopa (1973) Theorem 6) Let $f(\mathbf{x}, \mathbf{y})$ be a function of $n + m$ variables where \mathbf{x} is an n -component and \mathbf{y} is an m -component vector. Suppose that f is logarithmic concave in \mathbb{R}^{n+m} and let A be a convex subset of \mathbb{R}^m . Then the function of the variable \mathbf{x} :

$$\int_A f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

is logarithmic concave in the entire space \mathbb{R}^n .

Theorem A.4 (Chebyshev's integral inequality) Let $\alpha, \beta : [a, b] \rightarrow \mathbb{R}$ and $f(x) : [a, b] \rightarrow \mathbb{R}_+$, where \mathbb{R} is the set of real numbers. We have

1. if α and β are both non-decreasing or both non-increasing, then

$$\int_a^b f(x) dx \times \int_a^b \alpha(x) \beta(x) f(x) dx \geq \int_a^b \alpha(x) f(x) dx \times \int_a^b \beta(x) f(x) dx$$

2. if one is non-increasing and the other is non-decreasing, then the inequality is reversed.

Proof. See for example Proposition 2.1 in Eqozcue, Garcia, and Wong (2009). Also see Mitrinović, Pečarić, and Fink (1993). ■

B Proofs

Proof of the remaining parts of Theorem 1

Assume that $\mathbb{E}[\theta|y_1, U] - p_1 > 0$. It is clear from (9) that the optimal demand h_2^{UP} cannot be negative. Because by Lemma 2 P 's problem at date 2 is log-concave, it is sufficient to only explore the first order condition. Using (6), (9), (10), (11), (12) and noticing that $\frac{\partial \varphi_s(y_2 - h_2^{UP})}{\partial h_2^{UP}} = \frac{y_2 - h_2^{UP}}{\sigma_s^2} \varphi_s(y_2 - h_2^{UP})$, we obtain that

$$\frac{\partial \pi_2^{UP}}{\partial h_2^{UP}} = \int_{-\infty}^{\infty} \frac{Q_1 \varphi_s(y_2)}{Q_1 \varphi_s(y_2) + (1 - Q_1) \varphi_s(y_2 - \bar{h}_2)} \left(1 - \frac{(h_2^{UP})^2}{\sigma_s^2} + \frac{h_2^{UP} y_2}{\sigma_s^2} \right) \varphi_s(y_2 - h_2^{UP}) dy_2 \quad (26)$$

Define $\kappa \equiv \frac{\bar{h}_2}{\sigma_s}$ and $z \equiv \frac{y_2}{\sigma_s}$, where $dy_2 = \sigma_s dz$. From (6) we can notice that then $\varphi_s(y_2 - \bar{h}_2) = \frac{1}{\sigma_s} \phi(z - \kappa)$ and $\varphi_s(y_2) = \frac{1}{\sigma_s} \phi(z)$, where $\phi(\cdot)$ is the p.d.f. of a standard normal. The optimal demand must solve $\frac{\partial \pi_2^{UP}}{\partial h_2^{UP}} = 0$ and it must hold in equilibrium that optimal demand $(h_2^{UP})^* = \bar{h}_2 = \kappa \sigma_s$. Using all this, in (26), we obtain that κ is the positive solution of

$$\int_{-\infty}^{\infty} \frac{Q_1 \phi(z)}{Q_1 \phi(z) + (1 - Q_1) \phi(z - \kappa)} (1 - \kappa^2 + \kappa z) \phi(z - \kappa) dz = 0, \quad (27)$$

which we know to be unique by Lemma 2. Because σ_s does not enter in (27), it also proves that P 's demand is proportional to σ_s and only depends on Q_1 . The proof for the case $\mathbb{E}[\theta|y_1, U] - p_1 < 0$ is similar and in such a case we need the unique negative solution of (27). It is easy to verify that if $\kappa > 0$ solves (27), then also $-\kappa > 0$ solves (27).

Next let us prove that $\kappa > 1$ by contradiction. Suppose instead that $0 < \kappa < 1$ solves (27). From (27), it must then be the case that $\kappa \int_{-\infty}^{\infty} z \frac{Q_1 \phi(z)}{Q_1 \phi(z) + (1 - Q_1) \phi(z - \kappa)} \phi(z - \kappa) dz < 0$. Using that $\phi(\cdot)$ is an even function, we can rewrite this as

$$\kappa \int_0^{\infty} z Q_1 \phi(z) \left(\frac{1}{Q_1 \frac{\phi(z)}{\phi(z - \kappa)} + (1 - Q_1)} - \frac{1}{Q_1 \frac{\phi(z)}{\phi(z + \kappa)} + (1 - Q_1)} \right) dz < 0$$

Because $\phi(\cdot)$ is log-concave, it holds that $\phi(z - \kappa) > \phi(z + \kappa)$ for all $z, \kappa > 0$ by Corollary A.1.1 from Appendix A. This implies that $\frac{1}{Q_1 \frac{\phi(z)}{\phi(z - \kappa)} + (1 - Q_1)} < \frac{1}{Q_1 \frac{\phi(z)}{\phi(z + \kappa)} + (1 - Q_1)}$. So all terms inside the integral are non-negative for all $z \geq 0$ (with strict inequality for $z > 0$), leads to a contradiction and therefore $0 < \kappa < 1$ does not hold.

Proof of the part 4 of Proposition 2

Assume that $R = U$. From parts 2 and 3 of Proposition 2, it holds that $\frac{\mathbb{E}[\theta|y_1, U] - p_1}{Q_1} = \frac{\lambda_1 y_1}{4 - \eta}$. Also, because $Q_1 = \eta$ by part 2 of Proposition 2, we find from (10) that $\mathbb{E}[Q_2|y_1, U]$ is not a function of y_1 , and we can denote $\mathbb{E}[Q_2|y_1, U] = \mathbb{E}[Q_2|U]$. From (13), we then find that

$$\pi_2^{UP} = \begin{cases} \sigma_s \kappa \frac{\lambda_1 \mathbb{E}[Q_2|U]}{4 - \eta} y_1 & \text{if } y_1 > 0 \\ -\sigma_s \kappa \frac{\lambda_1 \mathbb{E}[Q_2|U]}{4 - \eta} y_1 & \text{if } y_1 < 0 \end{cases}$$

Suppose that P deviates and trades some $h_1^{UP} \neq 0$. Then from part 1 and 2 of Proposition 2, we know that in state $R = U$, $y_1|U = h_1^{UP} + \frac{\theta}{2\lambda_1} + s_1 \sim \mathcal{N}\left(h_1^{UP}, \frac{\sigma_U^2}{4\lambda^2} + \sigma_s^2\right) \sim \mathcal{N}(h_1^{UP}, \sigma_{yU}^2)$ and $p_1 = \lambda_1 \cdot \left(\frac{\theta}{2\lambda_1} + h_1^{UP} + s_1\right)$, where $\sigma_{yU} \equiv \sqrt{\frac{4 - \eta}{2 - \eta}} \sigma_s$. Because $\mathbb{E}[\theta|U] = 0$, it holds that $\mathbb{E}[h_1^{UP}(\theta - p_1)|U] = -(h_1^{UP})^2 \lambda_1$. Using the above, we find that expected profit (4) at date 1

$$\pi_1^{UP} = -(h_1^{UP})^2 \lambda_1 + \sigma_s \kappa \frac{\lambda_1 \mathbb{E}[Q_2|U]}{4 - \eta} [\Pr(y_1 > 0|U) \mathbb{E}[y_1|U, y_1 > 0] - \Pr(y_1 < 0|U) \mathbb{E}[y_1|U, y_1 < 0]]$$

Given the moments of truncated normal, $\mathbb{E}[y_1|U, y_1 > 0] = \frac{h_1^{UP}(1 - \Phi(-h_1^{UP}/\sigma_{yU})) + \phi(-h_1^{UP}/\sigma_{yU})\sigma_{yU}}{\Pr(y_1 > 0|U)}$ and $\mathbb{E}[y_1|U, y_1 < 0] = \frac{h_1^{UP}\Phi(-h_1^{UP}/\sigma_{yU}) - \phi(-h_1^{UP}/\sigma_{yU})\sigma_{yU}}{\Pr(y_1 < 0|U)}$, where $\Phi(\cdot)$ and $\phi(\cdot)$ are the c.d.f. and p.d.f. of standard normal, respectively.

Using that $\phi(\cdot)$ is symmetric, we can then express P 's expected profit as

$$\pi_1^{UP} = -(h_1^{UP})^2 \lambda_1 + \sigma_s \kappa \frac{\lambda_1 \mathbb{E}[Q_2|U]}{4 - \eta} \left(h_1^{UP} \left(2\Phi\left(\frac{h_1^{UP}}{\sigma_{yU}}\right) - 1 \right) + 2\phi\left(\frac{h_1^{UP}}{\sigma_{yU}}\right) \sigma_{yU} \right)$$

Differentiating this³¹, we obtain

$$\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}} = -2\lambda_1 h_1^{UP} + \lambda_1 \frac{\sigma_s \kappa \mathbb{E}[Q_2|U]}{4 - \eta} \left(2\Phi\left(\frac{h_1^{UP}}{\sigma_{yU}}\right) - 1 \right)$$

Clearly, $\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}|_{h_1^{UP}=0} = 0$ as $\Phi(0) = 1/2$. The second derivative $\frac{\partial^2 \pi_1^{UP}}{\partial h_1^{UP} \partial h_1^{UP}} = -2\lambda_1 \left(1 - \frac{\sigma_s \kappa \mathbb{E}[Q_2|U] \phi(h_1^{UP}/\sigma_{yU})}{\sigma_{yU}(4 - \eta)} \right)$, and at $h_1^{UP} = 0$, it must hold that $1 - \frac{\sigma_s \kappa \mathbb{E}[Q_2|U]}{\sigma_{yU}(4 - \eta)} \phi(0) > 0$, which can be proved (at least for all values of η that are not too close to one).³² If the inequality holds, then π_1^{UP} is also concave and zero is the global maximum.

³¹Note that $\Phi'(ax) = a\phi(ax)$ and $\phi'(ax) = a^2 x \phi'(x)$

³²Namely, using $\sigma_{yU} \equiv \sqrt{\frac{4 - \eta}{2 - \eta}} \sigma_s$ and $\phi(0) = \frac{1}{\sqrt{2\pi}}$, we can express the inequality as $\kappa < \sqrt{\frac{2\pi(4 - \eta)}{(2 - \eta)}} \frac{(4 - \eta)}{\mathbb{E}[Q_2|U]}$. As the Market updates in the right direction, on average $\mathbb{E}[Q_2|U] < \eta$. So a sufficient condition is $\kappa < \sqrt{\frac{2\pi(4 - \eta)}{(2 - \eta)}} \frac{(4 - \eta)}{\eta}$. Furthermore, it can be shown that $\kappa \leq \sqrt{2}$ for any $\eta \leq 0.5$, where the inequality always holds. It can also be shown that $\kappa < 1/\sqrt{1 - \eta}$ for any $\eta > 0.5$. So we can also be sure that the inequality holds for at least for $\eta < 0.994$.

Proof of Lemma 4

Without loss of generality, assume that $\theta = \bar{\theta}$. We start by the following lemma that benefits from Chebyshev's integral inequality.

Lemma B.1 *For any pair of constants, such that $c_2 > c_1$, it holds that*

$$\frac{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) y_1 \varphi_s(y_1 - c_2) dy_1}{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) \varphi_s(y_1 - c_2) dy_1} \geq \frac{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) y_1 \varphi_s(y_1 - c_1) dy_1}{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) \varphi_s(y_1 - c_1) dy_1}, \quad (28)$$

Proof. Consider a pair of constants $c_2 > c_1$ and the following integral

$$\int_{-\infty}^{\infty} y_1 \left(\frac{\varphi_s(y_1 - c_2)}{\varphi_s(y_1 - c_1)} - 1 \right) (\bar{\theta} - p_1(y_1)) \varphi_s(y_1 - c_1) dy_1$$

where $\varphi_s(\cdot)$ is given by (6). Notice that $\frac{\partial \left(\frac{\varphi_s(y_1 - c_2)}{\varphi_s(y_1 - c_1)} - 1 \right)}{\partial y_1} = \frac{\varphi_s(y_1 - c_2)}{\varphi_s(y_1 - c_1)} \left(\frac{\varphi'_s(y_1 - c_2)}{\varphi_s(y_1 - c_2)} - \frac{\varphi'_s(y_1 - c_1)}{\varphi_s(y_1 - c_1)} \right) > 0$. The inequality is implied by the facts that $\frac{\varphi_s(y_1 - c_2)}{\varphi_s(y_1 - c_1)} > 0$ and $\frac{\varphi'_s(y_1 - c_2)}{\varphi_s(y_1 - c_2)} > \frac{\varphi'_s(y_1 - c_1)}{\varphi_s(y_1 - c_1)}$. The latter is because $y_1 - c_2 < y_1 - c_1$ and for any log-concave function $f_s(\cdot)$ it is true that $(\ln(f_s(x)))' = \frac{f'_s(x)}{f_s(x)}$ is decreasing in x . From (6), we can also explicitly find that $\frac{\varphi'_s(y_1 - c_2)}{\varphi_s(y_1 - c_2)} - \frac{\varphi'_s(y_1 - c_1)}{\varphi_s(y_1 - c_1)} = \frac{c_2 - c_1}{\sigma_s^2} > 0$. Given that y_1 and $\left(\frac{\varphi_s(y_1 - c_2)}{\varphi_s(y_1 - c_1)} - 1 \right)$ are both increasing in y_1 and $(\bar{\theta} - p_1(y_1)) \varphi_s(y - c_1) > 0$ for all finite y_1 (part 7 in Lemma 3), it follows from Chebyshev's inequality³³ (Theorem A.4 in Appendix A) that

$$\frac{\int_{-\infty}^{\infty} y_1 \left(\frac{\varphi_s(y_1 - c_2)}{\varphi_s(y_1 - c_1)} - 1 \right) (\bar{\theta} - p_1(y_1)) \varphi_s(y - c_1) dy_1}{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) (\varphi_s(y - c_2) - \varphi_s(y - c_1)) dy_1} \times \frac{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) y_1 \varphi_s(y_1 - c_1) dy_1}{\int_{-\infty}^{\infty} (\bar{\theta} - p_1(y_1)) \varphi_s(y - c_1) dy_1}$$

Simplifying this gives (28). ■

We can then explore the trader's problem. Consider informed K 's problem in state $R = I$ when $\theta = \bar{\theta}$. K takes the functional form of prices (18) and P 's strategy $h_2^{IP} = \frac{\bar{g}_I}{2}$ as given, and then chooses h_1^{IK} to maximize (22). It is clear that $h_1^{IK} \leq 0$ cannot be optimal as it would lead to negative profits, hence we only consider $h_1^{IK} > 0$. Using that by (6) $\frac{\partial \varphi_s(y_1 - h_1^{IK} - \frac{\bar{g}_I}{2})}{\partial h_1^{IK}} = \frac{y_1 - h_1^{IK} - \frac{\bar{g}_I}{2}}{\sigma_s^2} \varphi_s(y_1 - h_1^{IK} - \frac{\bar{g}_I}{2})$, and simplifying we obtain that

³³Note that Theorem A.4 is stated for a definite integral where x is in the interval $[a, b]$, while here we have an improper integral. This is not a problem, because all functions inside the integrals are integrable and we could have performed a change of variables $y_1 = \frac{x}{1-x^2}$ to consider a definite integral in the interval $[-1, 1]$.

$$\begin{aligned}
-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}} &= - \left(1 - \frac{h_1^{IK} \left(h_1^{IK} + \frac{\bar{g}_I}{2} \right)}{\sigma_s^2} \right) \int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) \varphi_s(y_1 - h_1^{IK} - \frac{\bar{g}_I}{2}) dy_1 \\
&\quad + \frac{h_1^{IK}}{\sigma_s^2} \int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) y_1 \varphi_s(y_1 - h_1^{IK} - \frac{\bar{g}_I}{2}) dy_1
\end{aligned} \tag{29}$$

To prove that $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}$ is a single-crossing function, we need to show that for all $\hat{h} > \check{h} > 0$, $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\check{h}} \geq 0 \implies -\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\hat{h}} \geq 0$ and that $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\check{h}} > 0 \implies -\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\hat{h}} > 0$. So let us assume that $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\check{h}} \geq (>) 0$ holds. Using (29), we can express this as

$$LHS_1 \equiv \left(\frac{\sigma_s^2}{\check{h}} - (\check{h} + h_1^{IP}) \right) \leq (<) \frac{\int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) y_1 \varphi_s(y_1 - \check{h} - h_1^{IP}) dy_1}{\int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) \varphi_s(y_1 - \check{h} - h_1^{IP}) dy_1} \equiv RHS_1$$

Similarly, the inequality $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\hat{h}} \geq (>) 0$ can be written as

$$LHS_2 \equiv \left(\frac{\sigma_s^2}{\hat{h}} - \left(\hat{h} + \frac{\bar{g}_I}{2} \right) \right) \leq (<) \frac{\int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) y_1 \varphi_s(y_1 - \hat{h} - \frac{\bar{g}_I}{2}) dy_1}{\int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) \varphi_s(y_1 - \hat{h} - \frac{\bar{g}_I}{2}) dy_1} \equiv RHS_2$$

Then notice that $\hat{h} > \check{h} > 0 \iff \frac{\sigma_s^2}{\check{h}} - (\check{h} + \frac{\bar{g}_I}{2}) > \frac{\sigma_s^2}{\hat{h}} - \left(\hat{h} + \frac{\bar{g}_I}{2} \right)$, hence $LHS_1 > LHS_2$. We then obtain that $LHS_1 \leq (<) RHS_1$ (because $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\check{h}} \geq (>) 0$) and $RHS_1 \leq RHS_2$ (by property (28)), and indeed $-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}}|_{h_1^{IK}=\hat{h}} > 0$ ($LHS_2 < RHS_2$) because $LHS_2 < LHS_1 \leq (<) RHS_1 \leq RHS_2$. Hence K 's problem is quasi-concave and has a unique maximum. The proof for $\theta = -\bar{\theta}$ is similar.

Then consider the case where $\theta = 0$, then the equilibrium beliefs about the other trader's demand is $h_1^{IP} = 0$. Differentiating and simplifying gives

$-\frac{\partial \pi_2^{IK}}{\partial h_1^{IK}} = \int_{-\infty}^{\infty} p(s_1 + h_1^{IK}) \varphi_s(s_1) ds_1 + h_1^{IK} \int_{-\infty}^{\infty} p'(s_1 + h_1^{IK}) \varphi_s(s_1) ds_1$. It is immediate that this is positive for all $h_1^{IK} < 0$, and negative for $h_1^{IK} > 0$, because by point 3 in Lemma 3.1, the price is increasing in the order flow and K 's demand affects the order linearly. Due to prices being increasing in the order flow and symmetry of the price around zero (part 4 and 5 of Lemma 3), it is also true that the first term

$\int_{-\infty}^{\infty} p(s_1 + h_1^{IK}) \varphi_s(s_1) ds_1 = \int_0^{\infty} (p(s_1 + h_1^{IK}) - p(s_1 - h_1^{IK})) \varphi_s(s_1) ds_1$ is positive iff $h_1^{IK} > 0$. Because informed P 's profit has the same functional form, the proof is exactly the same.

The proof for state $R = U$ is similar. We write down the first derivative of K 's problem (21)

when $\theta = \bar{\theta}$, as it will be useful for the next proof.

$$\begin{aligned} -\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} &= -\left(1 - \frac{h_1^{UK}}{\sigma_s^2}\right) \int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) \varphi_s(y_1 - h_1^{UK}) dy_1 \\ &\quad + \frac{h_1^{UK}}{\sigma_s^2} \int_{-\infty}^{\infty} (\bar{\theta} - p(y_1)) y_1 \varphi_s(y_1 - h_1^{UK}) dy_1 \end{aligned} \quad (30)$$

is a single crossing function, and so are the equivalents for $\theta = 0$ and $\theta = -\bar{\theta}$.

Proof of Proposition 3

From Lemma 4 we know that there is a unique maximum and we can focus on the first order conditions (29) and (30). It is straightforward to prove that K and P must trade the same quantity in equilibrium in state $R = I$ and that the solution is symmetric for $\theta = \bar{\theta}$ and $\theta = -\bar{\theta}$. Overall in the equilibrium the Market's beliefs must be consistent with optimal strategies, i.e., it must hold that $h_1^{IK} = h_1^{IP} = \frac{\bar{g}_I}{2}$ and $h_1^{UK} = \bar{g}_U$. Define $\mu_R \equiv \frac{\bar{g}_R}{\sigma_s}$ for $R \in \{I, U\}$ and $z \equiv \frac{y_2}{\sigma_s}$. From (6) $\varphi_s(y_2 - \bar{g}_R) = \frac{1}{\sigma_s} \phi(z - \mu_R)$, $\varphi_s(y_2) = \frac{1}{\sigma_s} \phi(z)$ and $\varphi_s(y_2 + \bar{g}_R) = \frac{1}{\sigma_s} \phi(z + \mu_R)$, where ϕ is the p.d.f. of a standard normal. Using (18), we then find that

$$\check{p}(z) \equiv p_1(z\sigma_s) = \bar{\theta} \frac{\eta \phi(z - \mu_I) + (1 - \eta) \phi(z - \mu_U) - \eta \phi(z + \mu_I) + (1 - \eta) \phi(z + \mu_U)}{\eta \phi(z - \mu_I) + (1 - \eta) \phi(z - \mu_U) + \eta \phi(z + \mu_I) + (1 - \eta) \phi(z + \mu_U) + \frac{2\gamma}{1-\gamma} \phi(z)}$$

that clearly does not depend on σ_s and it holds that $p_1(z\sigma_s) = -p_1(-z\sigma_s)$.

Using these in (29) and (30) and equating $\frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} = 0$ for $J \in \{K, P\}$; $\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} = 0$, we find that μ_I and μ_U are the positive solutions of

$$\begin{aligned} \frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} \Big|_{h_1^{IJ} = \bar{g}_I} &= -\left(1 - \frac{\mu_I^2}{2}\right) \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) \phi(z - \mu_I) dz + \frac{\mu_I}{2} \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) z \phi(z - \mu_I) dz = 0 \\ \frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} \Big|_{h_1^{UK} = \bar{g}_U} &= -\left(1 - \mu_U^2\right) \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) \phi(z - \mu_U) dz + \mu_U \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) z \phi(z - \mu_U) dz = 0 \end{aligned} \quad (31)$$

We already know from the proof of Lemma 4 that if $\theta = 0$, it is optimal for informed traders to trade zero. This proves part 1 of Proposition 3.

For part 2 notice that from (31), we can express the first order condition of trader $J \in \{K, P\}$ in state $R = I$ as

$$\frac{\partial \pi_2^{IJ}}{\partial h_1^{IJ}} \Big|_{h_1^{IJ} = \bar{g}_I} = -\frac{1}{2} \int_{-\infty}^{\infty} (\bar{\theta} - \check{p}(z)) \phi(z - \mu_I) dz - \frac{1}{2} \frac{\partial \pi_2^{UK}}{\partial h_1^{UK}} \Big|_{h_1^{UK} = \bar{g}_I} = 0$$

By part 6 of Lemma 3 $(\bar{\theta} - p_1(y_1)) > 0$ for all finite y_1 . Therefore, also $(\bar{\theta} - \check{p}(z)) > 0$ for all finite z and $(\bar{\theta} - \check{p}(z)) \phi(z - \mu_I) \geq 0$ with strict inequality for some z . This implies that it

must hold that

$$-\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=\bar{g}_I} > 0.$$

Because $-\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}}$ is a single-crossing function and $\frac{\partial \pi_2^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=\bar{g}_U} = 0$, it then follows that $\bar{g}_I > \bar{g}_U$.

For the uninformed trader's strategy, we need to verify that it is optimal for him to trade zero. We now verify that the first order condition of his problem indeed holds at zero. Define $\Delta(y_1) \equiv \mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]$ and $Q_{1U}(y_1) \equiv \mathbb{E}[Q_2|y_1, U]$. By (19) in Lemma 3 (and also by Lemma A.1 in Appendix A), it holds that $\Delta(y_1) = -\Delta(-y_1)$. Also, it is clear from (20) that it holds that $Q_1(y_1) = Q_1(-y_1)$. Using this in (10) and (12) we confirm that (12) $Q_{1U}(y_1) = Q_{1U}(-y_1)$.

Recalling uninformed P 's optimal trading strategy at date 2 from (13) in Theorem 1 and using (7) and (9), we can then find P 's expected profit at date 2 conditional on y_1 as

$$\pi_2^{UP} = \begin{cases} \sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) & \text{if } \Delta(y_1) > 0 \\ -\sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) & \text{if } \Delta(y_1) < 0 \end{cases}$$

Suppose that at date 1, uninformed P trades h_1^{UP} , then he also knows that the distribution of the total order flow is $f_s(y_1 - h_1^{UP} - \bar{g}_U)$ if $\theta = \bar{\theta}$; $f_s(y_1 - h_1^{UP})$ if $\theta = 0$ and $f_s(y_1 - h_1^{UP} + \bar{g}_U)$ if $\theta = -\bar{\theta}$. Using all this, (17), $\mathbb{E}[\theta|U] = 0$, we can use the law of iterated expectations to express the expected profit of uninformed P before the date 1 trading as

$$\begin{aligned} \pi_1^{UP} = & -h_1^{UP} \int_0^\infty p_1(y_1) (m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP})) dy_1 + \\ & \int_{\Delta(y_1) > 0} \sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) (m(y_1 - h_1^{UP}) + m(y_1 + h_1^{UP})) dy_1, \end{aligned}$$

where $m(x) \equiv \frac{1-\gamma}{2} \varphi_s(x - \bar{g}_U) + \gamma \varphi_s(x) + \frac{1-\gamma}{2} \varphi_s(x + \bar{g}_U)$. Because of symmetry $\varphi_s(\cdot)$ it holds that $m(x) = m(-x)$ and $m'(x) = -m'(-x)$.

The first order condition is

$$\begin{aligned} \frac{\partial \pi_1^{UP}}{\partial h_1^{UP}} = & - \int_0^\infty p_1(y_1) (m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP})) dy_1 \\ & + h_1^{UP} \int_0^\infty p_1(y_1) (m'(y_1 - h_1^{UP}) - m'(y_1 + h_1^{UP})) dy_1 \\ & - \int_{\Delta(y_1) < 0} \sigma_s \kappa Q_{1U}(y_1) \Delta(y_1) (m'(y_1 - h_1^{UP}) - m'(y_1 + h_1^{UP})) dy_1 \end{aligned}$$

Replacing in $h_1^{UP} = 0$, we can now verify that $\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}|_{h_1^{UP}=0} = 0$. Because the zero-trading result of P at date 1 is uncontroversial, we skip here the proof of it being global maximum.

Similarly to the case with a normal prior, it can be verified that zero is a global maximum of P 's problem, at least as long as η is not too close to one.

C Normal prior, independent of the state

We now consider that the prior density of the fundamental is $f_\theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma_p} \exp\left(-\frac{\theta^2}{2\sigma_p^2}\right)$ and the fundamental is independent of the state, i.e., $f(\theta|R) = f_\theta(\theta)$ for $R \in \{I, U\}$. We can no longer conjecture that the price is linear in the order flow, because the Market will learn about the state from the order flow. Instead, we conjecture that the Market believes that total informed trading in state $R \in \{I, U\}$ is $g_R(\theta) = -g_R(-\theta)$ and P does not trade at date one, if the state is $R = U$. Conditional on the state, $\mathbb{E}[\theta|y_1, R] = \int_{-\infty}^{\infty} \theta f(\theta|y_1, R) d\theta$ as we know the distributions and by Bayes' rule it holds that $f(\theta|y_1, R) = \frac{f(y_1|\theta, R)f(\theta|R)}{f(y_1|R)} = \frac{\varphi_s(y_1 - g_R(\theta))f_\theta(\theta)}{f(y_1|R)}$ and $f(y_1|R) = \int_{-\infty}^{\infty} \varphi_s(y_1 - g_R(\theta)) f_\theta(\theta) d\theta$. Also by Bayes's rule $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + f(y_1|U)}$. Using this in (7), we find that date one price is

$$p_1(y_1) = \frac{\int_{-\infty}^{\infty} \theta (\eta \varphi_s(y_1 - g_I(\theta)) + (1 - \eta) \varphi_s(y_1 - g_U(\theta))) f_\theta(\theta) d\theta}{\int_{-\infty}^{\infty} (\eta \varphi_s(y_1 - g_I(\theta)) + (1 - \eta) \varphi_s(y_1 - g_U(\theta))) f_\theta(\theta) d\theta}. \quad (32)$$

It is easy to verify that $p_1(y_1) = -p_1(-y_1)$.

As the aim of this Section is only to verify that our results in Section 4.3 are not driven by the particular dependence structure between θ and R through a numerical exercise, we do not aim to prove analytically that traders' problems are quasi-concave and have unique maximum.³⁴ So to characterize the equilibrium, we only present the first order conditions of the trader's problem. After taking the first order conditions in (3) and (2), it is easy to verify that it must hold that in state $R = I$, both informed traders trade the same optimal quantity $h_1^{IK*} = h_1^{UP*}$. Imposing then that the equilibrium beliefs must be consistent with the actual trades, we find after changing variables and simplifying that $g_U(\theta)$ and $g_I(\theta)$ solve

$$\begin{aligned} \theta &= \int_{-\infty}^{\infty} p_1(y_1) \left(1 - \frac{g_U^2(\theta)}{\sigma_s^2} + \frac{g_U(\theta) y_1}{\sigma_s^2}\right) \varphi_s(y_1 - g_U(\theta)) d\theta \\ \theta &= \int_{-\infty}^{\infty} p_1(y_1) \left(1 - \frac{g_I^2(\theta)}{2\sigma_s^2} + \frac{g_I(\theta) y_1}{2\sigma_s^2}\right) \varphi_s(y_1 - g_I(\theta)) d\theta. \end{aligned} \quad (33)$$

and it is straightforward to verify the symmetry of strategies: $g_R(\theta) = -g_R(-\theta)$. Equations (32) and (33) characterize the functions that determine equilibrium strategies and price. While this problem does not have an analytical solution, the way to solve it is to note that we

³⁴In particular, it will be shown shortly that the price and equilibrium strategies are almost linear, and therefore quasiconcavity of the trader's problem is trivial to verify ex post.

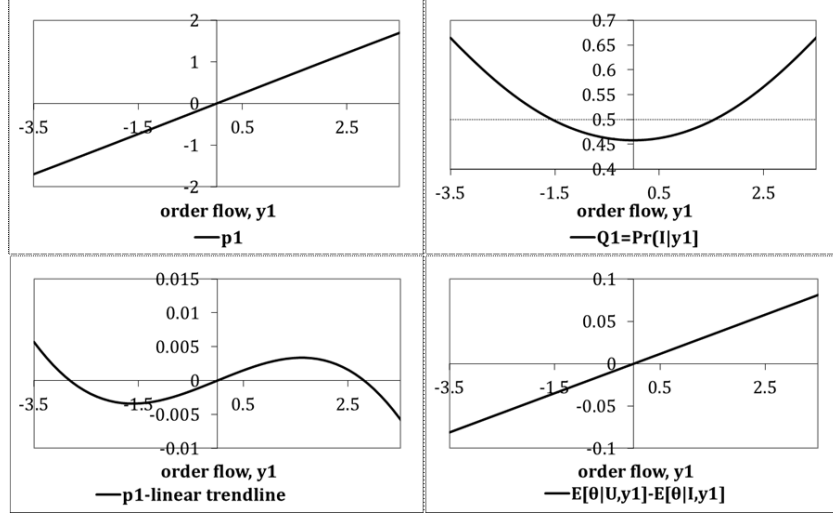


Figure 4: Date 1 price p_1 , updated probability $Q_1 = \Pr(I|y_1)$, and the difference in expectations $\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]$ as a function of the order flow.

can approximate any function with a polynomial. We proceed by assuming that $g_R(\theta)$ is a polynomial, derive the price (32) and change the constants in the polynomial until (33).

For the numerical exercise assume that $\eta = 0.5$, which is the case in which there is most updating about the state R and hence the solution should in principle be most non-linear. Without loss of generality assume $\sigma_s = 1$ and $\sigma_p = 1$ (note that similarly to other settings in this paper, it can be verified that informed trading is proportional to the noise trading variance σ_s). It turns out that informed trader's strategies do not need to be approximated with a high order polynomial, but are already very well approximated by a linear function, namely $g_U(\theta) \approx 1.0284\theta$ and $g_I(\theta) \approx 1.3712\theta$. Figure (4) presents the relevant results. The reason why trader's strategies are close to linear is that the price under linear strategies is "almost linear," i.e., the north-west panel of Figure (4) shows that it is hard to notice nonlinearity of price (the R^2 of the trendline is effectively 1) - only when we zoom and show the difference between the price and a linear trendline (south-west panel), do we see that it is slightly non-linear. Because informed traders care about the expected price that they do not know when they submit their orders, these small nonlinearities have very little effect on their optimal strategies. On the north east panel we see that there is some, but limited updating of trader's types. Because two informed traders jointly trade more than one in absolute value, larger order flows in absolute value tend to signal a higher probability that the state is $R = I$. At small order flows, the Market tends to believe that the state is $R = U$, but even at zero, there is not much learning about P 's type and therefore P 's trading opportunities remain. Finally on the south-east panel, we see that the direction of P 's trading at date 2 is the same as in Section 4.3 - positive feedback. For any $y_1 > (<) 0$, it holds that $\mathbb{E}[\theta|y_1, U] > (<) \mathbb{E}[\theta|y_1, I] \Leftrightarrow \mathbb{E}[\theta|y_1, U] > (<) p_1$.

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