FSF3847 Convex optimization with engineering applications
Spring 2023
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Homework Assignment 2
Due Tuesday April 18 2023

Exercise 2.1. Let $x^* = (0\ \ 1\ -1)^T$. Determine if x^* is optimal to the optimization problem

$$\begin{array}{ll} \underset{x \in I\!R^3}{\text{minimize}} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & -1 \leq x_j \leq 1, \quad j = 1, 2, 3, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}.$$

Exercise 2.2. Consider a linear program on the form

$$(LP) \qquad \begin{array}{ll} \underset{u \in I\!R^m}{\text{minimize}} & -b^T u \\ \text{subject to} & A^T u \leq c. \end{array}$$

We would normally associate (LP) with a dual problem. The difference is that the maximization of $b^T u$ has been replaced by a minimization of $-b^T u$. This is an equivalent problem, the only difference is the change of sign in the objective function.
Your task is to show that the dual of the dual is the primal, i.e., that the dual of (LP) is equivalent to a primal linear program on standard form. Do this in two different ways.

a) Formulate a dual problem associated with (LP) by rewriting (LP) to a linear program on standard form and use the duality result for

$$(PLP) \quad \begin{array}{ll} \text{minimize} & \widetilde{c}^T x \\ \text{subject to} & \widetilde{A} x = \widetilde{b}, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (DLP) \quad \begin{array}{ll} \max & \widetilde{b}^T y \\ \text{subject to} & \widetilde{A}^T y + s = \widetilde{c}, \\ & s \geq 0. \end{array}$$

You are meant to introduce suitable variables and/or constraints to define $\widetilde{A}$, $\widetilde{b}$ and $\widetilde{c}$. Finally express your dual problem (DLP) in terms of A, b and c.

Hint: An inequality constraint $A^T u \leq c$ may equivalently be expressed as $A^T u + v = c$, $v \geq 0$. A free variable u may equivalently be expressed as $u_+ - u_-$, with $u_+ \geq 0$ and $u_- \geq 0$.

b) Derive the Lagrange dual problem associated with (LP).

If your problems do not have exactly the form that you expect, comment on the differences.

Exercise 2.3. Assume that a line $y = kx + l$ is to be fit to a set of given points (x_i, y_i), $i = 1, \ldots, m$. Consider two ways of fitting the line:

- Choose k and l so that the maximum deviation in the y-direction is minimized, i.e., let k and l solve $\min_{k,l}\{\max_i |kx_i + l - y_i|\}$.
- Choose k and l so that the sum of the deviations in the y-direction is minimized, i.e., let k and l solve $\min_{k,l} \sum_i |kx_i + l - y_i|$.

a) Formulate these two problems as linear programs.

b) Formulate the dual problems associated with these linear programs.

c) Comment on the problem dimensions of these problems.

d) Use `cvx`, or some other linear programming solver, to solve the linear programs that you derived above, and evaluate their performance on a set of random data. Generate the data as follows

```
m=200; k0=4;l0=1;
x=randn(m,1);
y=k0*x+l0+randn(m,1);
y(end)=3*y(end);
```

(the final line is used to add an "outlier" to the data). Compare the optimized parameters k and l with the values used to generate the data and comment on your observations.

Exercise 2.4. Solve the optimal activity level problem described in Exercise 4.17 in *Convex Optimization*, p. 195, for the instance with problem data

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 1 & 0 & 3 & 2 \end{pmatrix}, \quad c^{\text{max}} = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \end{pmatrix}, \quad p = \begin{pmatrix} 3 \\ 2 \\ 7 \\ 6 \end{pmatrix}, \quad p^{\text{disc}} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \end{pmatrix}, \quad q = \begin{pmatrix} 4 \\ 10 \\ 5 \\ 10 \end{pmatrix}$$

You can do this by first deriving a linear programming formulation, and then solving it using (for example) `cvx`. Give the optimal activity levels, the revenue generated by each one, and the total revenue generated by the optimal solution. Also give the average price per unit for each activity level, *i.e.*, the ratio of revenue associated with each activity to the associated activity.

Good luck!