FSF3847
Convex Optimization
with Engineering Applications
Assignment 2

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Exercise 2.1

Problem

Let $x^* = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$. Determine if x^* is optimal to the optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^3}{\text{minimize}} \quad & \frac{1}{2} x^T H x + c^T x \\ \text{subject to} \quad & -1 \leq x_j \leq 1, \; j = 1, 2, 3 \end{aligned}$$

where

$$H = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \; c = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

Solution

First, the constraints can be written as standard inequality constraints:

$$-1 \leq x_j \leq 1 \iff \begin{cases} x_j + 1 \geq 0 \\ -x_j + 1 \geq 0 \end{cases}, \; j = 1, 2, 3$$

To simplify the notation, the constraints can be written in the matrix form $Ax - b \succeq 0$ with:

$$\begin{aligned} A &:= \begin{bmatrix} \mathbb{I}_{3\times 3} \\ -\mathbb{I}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{6\times 3} \\ b &:= -\mathbf{1} \in \mathbb{R}^6 \end{aligned}$$

Thus, the optimization problem can be written as:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^3}{\text{minimize}} & \frac{1}{2} x^T H x + c^T x \\ \text{subject to} & Ax - b \succeq 0 \end{array}$$

The objective function, denoted f, can be written as:

$$f(x) \coloneqq \underbrace{\frac{1}{2} x^T H x}_{f_1 \coloneqq} + \underbrace{c^T x}_{f_2 \coloneqq}$$

where f_2 is linear in x, so convex. Since H is positive semidefinite[1], the Hessian matrix of f_1 is positive semidefinite:

$$\begin{aligned} \nabla f_1(x) &= \frac{1}{2}(H + H^T)x \\ \nabla^2 f_1(x) &= \frac{1}{2}(H + H^T) \succeq 0 \end{aligned}$$

which implies that f_1 is convex. Therefore, the objective function f is convex, as it is the sum of convex functions.

Since the optimization problem has convex objective function and linear constraints, it is a *linearly constrained convex problem.* Thus, it is possible to check if x^* is optimal by using the *KKT optimality conditions.* In particular, x^* is optimal if and only if there exists a $\lambda^* \in \mathbb{R}^6$ such that[2]:

$$Ax^* - b \succeq 0 \tag{1}$$

$$\nabla f(x^*) - A^T \lambda^* = 0 \tag{2}$$

$$\lambda^* \succeq 0 \tag{3}$$

$$\lambda^* \odot (Ax^* - b) = 0 \tag{4}$$

Equation (1) is verified:

$$Ax^* - b = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 2 \end{bmatrix}^T \succeq 0 \tag{5}$$

Equation (4) constraints the Lagrangian multipliers corresponding to the non-zero entries of $Ax^* - b$ (i.e., inactive constraints) to be 0. By checking the non-zero entries in Eq. (5), we have:

$$\begin{cases} \lambda_1^* = 0 \\ \lambda_2^* = 0 \\ \lambda_4^* = 0 \\ \lambda_6^* = 0 \end{cases} \tag{6}$$

[1] I checked the eigenvalues in MATLAB.

[2] $\odot$ denotes element-wise multiplication.

Equation (2) gives:

$$\begin{aligned} \nabla f(x^*) - A^T\lambda^* &= 0 && \iff \\ \frac{1}{2}(H + H^T)x^* + c - A^T\lambda^* &= 0 && \iff \\ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_1^* - \lambda_4^* \\ \lambda_2^* - \lambda_5^* \\ \lambda_3^* - \lambda_6^* \end{bmatrix} &= 0 && \end{aligned} \tag{7}$$

Plugging Eq. (6) into Eq. (7) clearly proves that the KKT conditions cannot be satisfied at x^* by any λ. Therefore, x^* *is not optimal*.

Exercise 1.2

Problem

Consider a linear program on the form:

$$\begin{aligned} \underset{u \in \mathbb{R}^m}{\text{minimize}} \quad & -b^T u \\ \text{subject to} \quad & A^T u \preceq c \end{aligned} \tag{LP}$$

We would normally associate problem (LP) with a dual problem. The difference is that the maximization of $b^T u$ has been replaced by a minimization of $-b^T u$. This is an equivalent problem, the only difference is the change of sign in the objective function.

Your task is to show that the dual of the dual is the primal, i.e., that the dual of problem (LP) is equivalent to a primal linear program on standard form. Do this in two different ways.

a) Formulate a dual problem associated with (LP) by rewriting (LP) to a linear program on standard form and use the duality result for

$$\begin{aligned} \min \quad & \tilde{c}^T x \\ \text{s.t.} \quad & \tilde{A}x = \tilde{b}, \\ & x \succeq 0 \end{aligned} \quad \text{(PLP)} \qquad \begin{aligned} \max \quad & \tilde{b}^T y \\ \text{s.t.} \quad & \tilde{A}^T y + s = \tilde{c}, \\ & s \succeq 0 \end{aligned} \quad \text{(DLP)}$$

You are meant to introduce suitable variables and/or constraints to define $\tilde{A}$, $\tilde{b}$, $\tilde{c}$. Finally express your dual problem (DLP) in terms of A, b, c.

Hint: An inequality constraint $A^T u \preceq c$ may equivalently be expressed as $A^T u + v = c$, $v \succeq 0$. A free variable u may equivalently be expressed as $u_+ - u_-$, with $u_+ \succeq 0$ and $u_- \succeq 0$.

b) Derive the Langrange dual problem associated with (LP).

If your problems do not have exactly the form that you expect, comment on the differences.

Solution

Part (a)

Starting from (LP), introduce a slack variable v:

$$\begin{aligned} \text{minimize} \quad & -b^T u \\ \text{subject to} \quad & A^T u + v = c, \\ & v \succeq 0 \end{aligned}$$

Express $u = u_+ - u_-$ as the difference of two nonnegative variables $u_+ \succeq 0$ and $u_- \succeq 0$:

$$\begin{aligned} \text{minimize} \quad & -b^T u_+ + b^T u_- \\ \text{subject to} \quad & A^T u_+ - A^T u_- + v = c, \\ & u_+ \succeq 0, \\ & u_- \succeq 0, \\ & v \succeq 0 \end{aligned}$$

Introduce suitable variables to express it in the standard form:

$$\begin{aligned} \tilde{c} &:= \begin{bmatrix} -b^T & b^T & 0 \end{bmatrix}^T \\ x &:= \begin{bmatrix} u_+^T & u_-^T & v^T \end{bmatrix}^T \\ \tilde{A} &:= \begin{bmatrix} A^T & -A^T & \mathbb{I} \end{bmatrix} \\ \tilde{b} &:= c \end{aligned}$$

Using these variables, (LP) is in the standard primal form:

$$\begin{aligned} \text{minimize} \quad & \tilde{c}^T x \\ \text{subject to} \quad & \tilde{A} x = \tilde{b}, \\ & x \succeq 0 \end{aligned}$$

and the duality result can be used:

$$\begin{aligned} \text{maximize} \quad & \tilde{b}^T y \\ \text{subject to} \quad & \tilde{A}^T y + s = \tilde{c}, \\ & s \succeq 0 \end{aligned} \tag{DLP}$$

Change the sign of the objective function:

$$\begin{aligned} \text{minimize} \quad & -\tilde{b}^T y \\ \text{subject to} \quad & \tilde{A}^T y + s = \tilde{c}, \\ & s \succeq 0 \end{aligned}$$

Remove the slack variable s:

$$\begin{aligned} &\text{minimize} && -\tilde{b}^T y \\ &\text{subject to} && \tilde{A}^T y \preceq \tilde{c} \end{aligned}$$

Replace $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ with their definitions:

$$\begin{aligned} &\text{minimize} && -c^T y \\ &\text{subject to} && \begin{bmatrix} A \\ -A \\ \mathbb{I} \end{bmatrix} y \preceq \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} \end{aligned}$$

Expand the constraints:

$$\begin{aligned} &\text{minimize} && -c^T y \\ &\text{subject to} && Ay \preceq -b, \\ &&& Ay \succeq -b, \\ &&& y \preceq 0 \end{aligned}$$

The first two inequality constraints are equivalent to an equality constraint:

$$\begin{cases} Ay \preceq -b \\ Ay \succeq -b \end{cases} \implies Ay = -b \iff -Ay = b$$

Therefore, (DLP) can be written as:

$$\begin{aligned} &\text{minimize} && -c^T y \\ &\text{subject to} && -Ay = b, \\ &&& y \preceq 0 \end{aligned}$$

Change the variable $z = -y$ in order to remove the negative signs:

$$\begin{aligned} &\text{minimize} && c^T z \\ &\text{subject to} && Az = b, \\ &&& z \succeq 0 \end{aligned} \tag{PLP}$$

which is in the standard primal form. We have proven that *the dual problem (DLP) of a dual linear program (LP) is equivalent to a primal linear program in standard form (PLP).*

Part (b)

Starting from (LP), the Lagrangian is:

$$\begin{aligned} L(u, \lambda) &= -b^T u + \lambda^T (A^T u - c) \\ &= (A\lambda - b)^T u - c^T \lambda \end{aligned}$$

with $\lambda \succeq 0$. The Lagrangian dual function is:

$$\begin{aligned} g(\lambda) &= \inf_u L(u,\lambda) \\ &= \inf_u (A\lambda - b)^T u - c^T\lambda \\ &= -c^T\lambda + \inf_u \underbrace{(A\lambda - b)^T u}_{g_2(u,\lambda):=} \end{aligned}$$

Since $g_2(u,\lambda)$ is linear in u, it is unbounded below except for the special case when it is identically 0:

$$g(\lambda) = \begin{cases} -c^T\lambda \text{ if } A\lambda - b = 0 \\ -\infty \text{ otherwise} \end{cases}$$

Therefore, the Lagrangian dual problem is:

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & -c^T\lambda \\ \text{subject to} \quad & A\lambda = b, \\ & \lambda \succeq 0 \end{aligned} \tag{DLP}$$

Change the sign of the objective function:

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & c^T\lambda \\ \text{subject to} \quad & A\lambda = b, \\ & \lambda \succeq 0 \end{aligned} \tag{PLP}$$

which is in the standard primal form. We have proven that *the dual problem (DLP) of a dual linear program (LP) is equivalent to a primal linear program in standard form (PLP).*

Exercise 1.3

Problem

Assume that a line $y = kx + l$ is to be fit to a set of given points $(x_i, y_i),\ i = 1, ..., m$. Consider two ways of fitting the line:

- Choose k and l so that the maximum deviation in the y-direction is minimized, i.e., let k and l solve $\min_{k,l}\{\max_i |kx_i + l - y_i|\}$.
- Choose k and l so that the sum of the deviations in the y-direction is minimized, i.e., let k and l solve $\min_{k,l} \sum_i |kx_i + l - y_i|$

a) Formulate these two problems as linear programs.

b) Formulate the dual problems associated with these linear programs.

c) Comment on the problem dimensions of these problems.

d) Use *cvx*, or some other linear programming solver, to solve the linear programs that you derived above, and evaluate their performance on a set of random data. Generate the data as follows

```
m=200; k0=4; l0=1;
x=randn(m,1);
y=k0*x+l0+randn(m,1);
y(end)=3*y(end);
```

(the final line is used to add an "outlier" to the data). Compare the optimized parameters k and l with the values used to generate the data and comment on your observations.

Solution

Recall that $\inf_{x,y} f(x,y) = \inf_x \inf_y f(x,y)$. That is, minimizing first over some variables and then over the others is the same as minimizing jointly over all the variables [1]. This observation is useful to convert the given norm approximation problems to linear programs.

Part a - Problem 1

Let x and y be the vectors with the set of given points, i.e., $[x]_i = x_i$ and $[y]_i = y_i$. Problem 1 is an unconstrained convex optimization problem with the l_∞ norm as objective function:

$$\underset{k,l \in \mathbb{R}}{\text{minimize}} \quad \|kx + l\mathbf{1} - y\|_\infty \tag{P1}$$

The epigraph form of (P1) is [1]:

$$\begin{aligned} \underset{t,k,l \in \mathbb{R}}{\text{minimize}} \quad & t \\ \text{subject to} \quad & t \geq \|kx + l\mathbf{1} - y\|_\infty \end{aligned}$$

The equivalence can be understood as follows. Assume we optimize first over t and then over k and l. While optimizing over t, k and l are considered fixed. Since $t \geq \|kx + l\mathbf{1} - y\|_\infty$ and the objective is t, the optimal point is $t = \|kx + l\mathbf{1} - y\|_\infty$ and the optimal value is $p^*(k,l) = \|kx + l\mathbf{1} - y\|_\infty$. Thus, when we optimize over k and l afterwards, the problem is again (P1).

The objective function t is linear, but the constraints are still not linear. Observe that t is greater than the maximum of a set of elements if and only if t is greater than each element:

$$\begin{aligned} & t \geq \|kx + l\mathbf{1} - y\|_\infty && \iff \\ & t \geq \max_i |kx_i + l - y_i| && \iff \\ & t \geq |kx_i + l - y_i|, \ i = 1, ..., m && \iff \end{aligned}$$

For each $i = 1, ..., m$, expand the absolute value:

$$t \geq |kx_i + l - y_i| \iff \begin{cases} t + kx_i + l - y_i \geq 0 \\ t - kx_i - l + y_i \geq 0 \end{cases} \tag{8}$$

Thus, (P1) is equivalent to:

$$\begin{aligned} \underset{t,k,l \in \mathbb{R}}{\text{minimize}} \quad & t \\ \text{subject to} \quad & t\mathbf{1} + kx + l\mathbf{1} - y \succeq 0, \\ & t\mathbf{1} - kx - l\mathbf{1} + y \succeq 0 \end{aligned} \tag{PLP1}$$

which is a linear program.

Part a - Problem 2

Let x and y be the vectors with the set of given points, i.e., $[x]_i = x_i$ and $[y]_i = y_i$. Problem 2 is an unconstrained convex optimization problem with the l_1 norm as objective function:

$$\underset{k,l \in \mathbb{R}}{\text{minimize}} \quad \|kx + l\mathbf{1} - y\|_1 \tag{P2}$$

Let $t \in \mathbb{R}^m$ be a vector with elements $[t]_i = t_i$. (P2) is equivalent to:

$$\begin{aligned} \underset{t \in \mathbb{R}^m, k,l \in \mathbb{R}}{\text{minimize}} \quad & \mathbf{1}^T t \\ \text{subject to} \quad & t_i \geq |kx_i + l - y_i| \end{aligned}$$

The equivalence can be understood as follows. Assume we optimize first over t and then over k and l. While optimizing over t, k and l are considered fixed. The objective function $\mathbf{1}^T t = \sum_i t_i$ is a sum of independent terms t_i, so it is minimized when each term t_i is minimized. Since $t_i \geq |kx_i + l - y_i|$, the optimal term t_i is $t_i = |kx_i + l - y_i|$ and the optimal value is $p^*(k,l) = \|kx + l\mathbf{1} - y\|_1$. Thus, when we optimize over k and l afterwards, the problem is again (P2).

After expanding the absolute values as done in Eq. (8), (P2) becomes:

$$\begin{aligned} \underset{t \in \mathbb{R}^m, k,l \in \mathbb{R}}{\text{minimize}} \quad & \mathbf{1}^T t \\ \text{subject to} \quad & t + kx + l\mathbf{1} - y \succeq 0, \\ & t - kx - l\mathbf{1} + y \succeq 0 \end{aligned} \tag{PLP2}$$

which is a linear program.

Part b - Problem 1

Starting from (PLP1), the Lagrangian is:

$$L(t, k, l, \lambda, \mu) = t - \lambda^T(t\mathbf{1} + kx + l\mathbf{1} - y) - \mu^T(t\mathbf{1} - kx - l\mathbf{1} + y)$$

with $\lambda \succeq 0$ and $\mu \succeq 0$. The Lagrangian dual function is:

$$
\begin{aligned}
g(\lambda, \mu) &= \inf_{t,k,l} L(t, k, l, \lambda, \mu) \\
&= y^T(\lambda - \mu) + \inf_{t,k,l} t(1 - \mathbf{1}^T(\lambda + \mu)) + kx^T(\lambda - \mu) + l\mathbf{1}^T(-\lambda + \mu)
\end{aligned}
$$

Therefore, the dual problem is:

$$
\begin{array}{ll}
\underset{\lambda, \mu \in \mathbb{R}^m}{\text{maximize}} & y^T(\lambda - \mu) \\
\text{subject to} & \mathbf{1}^T(\lambda + \mu) = 1, \\
& x^T(\lambda - \mu) = 0, \\
& \mathbf{1}^T(\lambda - \mu) = 0
\end{array}
\tag{DLP1}
$$

Part b - Problem 2

Starting from (PLP2), the Lagrangian is:

$$
L(t, k, l, \lambda, \mu) = \mathbf{1}^T t - \lambda^T(t + kx + l\mathbf{1} - y) - \mu^T(t - kx - l\mathbf{1} + y)
$$

with $\lambda \succeq 0$ and $\mu \succeq 0$. The Lagrangian dual function is:

$$
\begin{aligned}
g(\lambda, \mu) &= \inf_{t,k,l} L(t, k, l, \lambda, \mu) \\
&= y^T(\lambda - \mu) + \inf_{t,k,l} t^T(\mathbf{1} - \lambda - \mu) + kx^T(\lambda - \mu) + l\mathbf{1}^T(-\lambda + \mu)
\end{aligned}
$$

Therefore, the dual problem is:

$$
\begin{array}{ll}
\underset{\lambda, \mu \in \mathbb{R}^m}{\text{maximize}} & y^T(\lambda - \mu) \\
\text{subject to} & \lambda + \mu - \mathbf{1} = 0, \\
& x^T(\lambda - \mu) = 0, \\
& \mathbf{1}^T(\lambda - \mu) = 0
\end{array}
\tag{DLP2}
$$

Part c

- (P1) has 2 variables ($k, l \in \mathbb{R}$).
- (P2) has 2 variables ($k, l \in \mathbb{R}$).
- (PLP1) has 3 variables ($t, k, l \in \mathbb{R}$).
- (PLP2) has $m + 2$ variables ($t \in \mathbb{R}^m, k, l \in \mathbb{R}$).
- (DLP1) has $2m$ variables ($\lambda, \mu \in \mathbb{R}^m$).
- (DLP2) has $2m$ variables ($\lambda, \mu \in \mathbb{R}^m$).

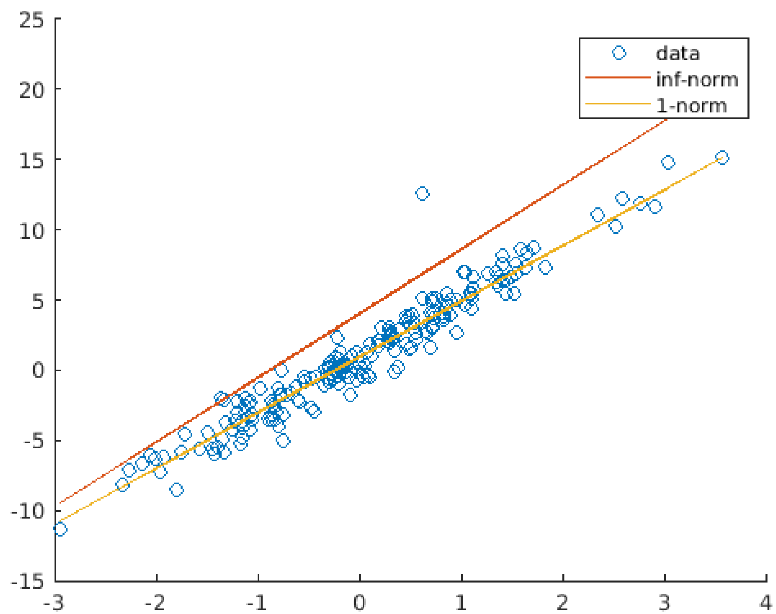

Figure 1: Line fitting using l_∞ and l_1 norms. The dataset contains one outlier.

Table 1: Difference between optimized (learned) parameters and true parameters.

	l_∞ norm	l_1 norm
$\lvert k - k_0 \rvert$	0.579375	0.031039
$\lvert l - l_0 \rvert$	3.018506	0.060093

The original problems (P1) and (P2) have 2 variables (k and l). The transformation to linear programs has little impact on the first problem ((PLP1) has 3 variables instead of 2) but high impact on the second problem ((PLP2) has a dimensionality linear in m). By definition, the dual problems have one variable for each constraint (i.e., for each given point in the dataset), so they have a dimensionality linear in m. In other words, the original problems (P1) and (P2) are parametric (i.e., they use a fixed number of parameters to fit the data), while the dual problems (DLP1) and (DLP2) are non-parametric (i.e., they have one parameter for each sample in the dataset).

Part d

The results are reported in Fig. 1 and Table 1. Clearly, the l_1-norm approximation is much more robust to outliers than the l_{inf} one. Intuitively, the l_1

norm considers all data points, while the l_{inf} norm considers the point which is worst approximated. Thus, for the l_{inf} norm, the outlier has a very high impact "pulls" the line until another point becomes the worst.

Exercise 1.4

Problem

Optimal activity levels. We consider the selection of n nonnegative activity levels, denoted $x1, ..., x_n$. These activities consume m resources, which are limited. Activity j consumes $A_{ij}x_j$ of resource i, where A_{ij} are given. The total resource consumption is additive, so the total resource i consumed is $c_i = \sum_{j=1}^{n} A_{ij}x_j$. (Ordinarily we have $A_{ij} \geq 0$, i.e., activity j consumes resource i. But we allow the possibility that $A_{ij} < 0$, which means that activity j actually generates resource i as a by-product.) Each resource consumption is limited: we must have $c_i \leq c_i^{max}$, where c_i^{max} are given. Each activity generates revenue, which is a piecewise-linear concave function of the activity level:

$$r_j(x_j) = \begin{cases} p_j x_j & 0 \leq x_j \leq q_j \\ p_j q_j + p_j^{disc}(x_j - q_j) & x_j \geq q_j \end{cases}$$

Here $p_j > 0$ is the basic price, $q_j > 0$ is the quantity discount level, and p_j^{disc} is the quantity discount price, for (the product of) activity j. (We have $0 < p_j^{disc} < p_j$.) The total revenue is the sum of the revenues associated with each activity, i.e., $\sum_{j=1}^{n} r_j(x_j)$. The goal is to choose activity levels that maximize the total revenue while respecting the resource limits.

Solve the optimal activity level problem for the instance with problem data

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 1 & 0 & 3 & 2 \end{bmatrix}, \quad c^{max} = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}, \quad p = \begin{bmatrix} 3 \\ 2 \\ 7 \\ 6 \end{bmatrix}, \quad p^{disc} = \begin{bmatrix} 3 \\ 2 \\ 7 \\ 6 \end{bmatrix}, \quad q = \begin{bmatrix} 4 \\ 10 \\ 5 \\ 10 \end{bmatrix}$$

You can do this by first deriving a linear programming formulation, and then solving it using (for example) *cvx*. Give the optimal activity levels, the revenue generated by each one, and the total revenue generated by the optimal solution. Also give the average price per unit for each activity level, i.e., the ratio of revenue associated with each activity to the associated activity.

Solution

The problem can be formulated as the following optimization problem:

$$
\begin{array}{ll}
\underset{x \in \mathbb{R}^n}{\text{maximize}} & \displaystyle\sum_{j=1}^{n} r_j(x_j) \\
\text{subject to} & x \succeq 0, \\
& Ax \preceq c^{max}
\end{array} \tag{P}
$$

which is not a linear program, as the functions $r_j(x_j)$ are piecewise. To identify similarities with exercise 2.3, change the sign of the objective function:

$$
\begin{array}{ll}
\underset{x \in \mathbb{R}^n}{\text{minimize}} & \displaystyle\sum_{j=1}^{n} (-r_j(x_j)) \\
\text{subject to} & x \succeq 0, \\
& Ax \preceq c^{max}
\end{array}
$$

The objective function is a sum of independent terms, as happened in exercise 2.3 for the l_1-norm approximation (P2). Thus, using the same considerations made in exercise 2.3, the problem is equivalent to:

$$
\begin{array}{ll}
\underset{t, x \in \mathbb{R}^n}{\text{minimize}} & \mathbf{1}^T t \\
\text{subject to} & x \succeq 0, \\
& Ax \preceq c^{max}, \\
& t_j \geq -r_j(x_j)
\end{array}
$$

The functions $r_j(x_j)$ can be written as:

$$
\begin{aligned}
r_j(x_j) &= \min\{p_j x_j, p_j q_j + p_j^{disc}(x_j - q_j)\} \\
-r_j(x_j) &= \max\{-p_j x_j, -p_j q_j - p_j^{disc}(x_j - q_j)\}
\end{aligned}
$$

as can be understood graphically in Fig. 2.

As done for the l_∞-norm approximation problem (P1) in exercise 2.3, observe that t is greater than the maximum of a set of elements if and only if t is greater than each element:

$$
\begin{aligned}
& t_j \geq -r_j(x_j) && \Longleftrightarrow \\
& t_j \geq \max\{-p_j x_j, -p_j q_j - p_j^{disc}(x_j - q_j) && \Longleftrightarrow \\
& \quad \begin{cases} t_j \geq -p_j x_j \\ t_j \geq -p_j q_j - p_j^{disc}(x_j - q_j) \end{cases}
\end{aligned}
$$

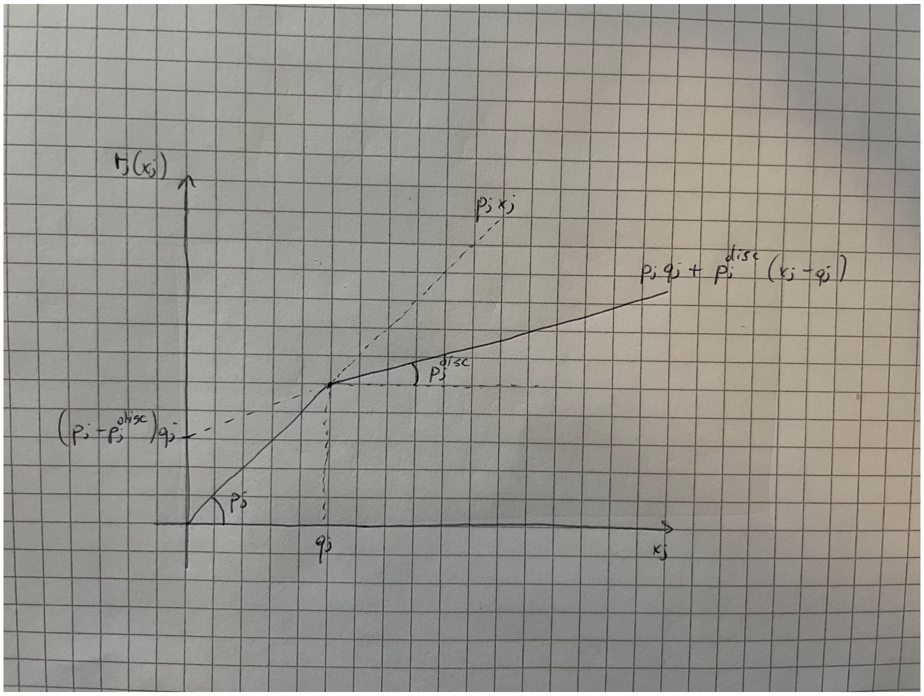

Figure 2: Graphical interpretation of the functions $r_j(x_j)$. Clearly, $r_j(x_j)$ is the minimum of its pieces.

Table 2: Optimal solution to the provided instance of the problem.

	Activity 1	Activity 2	Activity 3	Activity 4	Total
Activity level	4	22.5	31	1.5	-
Revenue	12	32.5	139	9	192.5
Avg. price per unit	3	1.4444	4.4839	6	-

Therefore, the problem is equivalent to:

$$
\begin{array}{ll}
\underset{t, x \in \mathbb{R}^n}{\text{minimize}} & \mathbf{1}^T t \\
\text{subject to} & x \succeq 0, \\
& Ax \preceq c^{max}, \\
& t + p \odot x \succeq 0, \\
& t + p \odot q + p^{disc} \odot (x - q) \succeq 0
\end{array}
$$

The solution to the provided instance of the problem, obtained using *cvx*, is reported in Table 2.

References

[1] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization.* Cambridge university press, 2004.