The Stack of Local Representations on a Coadjoint Orbit:
A Categorical Approach

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Abstract

We construct a stack $\mathcal{R}$ of local stabilizer representations over a coadjoint orbit $\mathcal{O}$ of a compact Lie group G, verify the descent conditions explicitly, and show that the G-action on global sections recovers the classical representation theory via geometric quantization. This provides a stack-theoretic interpretation of the Kirillov orbit method and the Borel–Weil theorem.

Contents

1   Introduction

The orbit method and geometric quantization. The correspondence between coadjoint orbits and irreducible representations of compact Lie groups, developed through the works of Kirillov [5, 6], Kostant [9, 8] and Souriau [13], provides a geometric bridge between symplectic geometry and representation theory. Every coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ carries the Kirillov–Kostant–Souriau (KKS) symplectic form, and when an integrality condition is satisfied, geometric quantization constructs an irreducible representation of G from the sections of a prequantum line bundle.

For compact groups, the Borel–Weil theorem [2, 3, 8] offers a holomorphic incarnation of the same correspondence: irreducible representations arise as spaces of holomorphic sections of line bundles over flag manifolds G/T. These classical results exhibit a unifying theme: representation theory on a coadjoint orbit is governed simultaneously by symplectic, complex and group-theoretic structures.

Local symmetries and categorical structure. A point $x \in \mathcal{O}$ has a stabilizer subgroup T_x, and the category $\mathbf{Rep}(T_x)$ of its finite-dimensional unitary representations encodes a local symmetry. Along the orbit, the coadjoint action transports these categories by conjugation: if $y = gx$, then $\mathbf{Rep}(T_y) \simeq \mathbf{Rep}(T_x)$. This suggests viewing the assignment $x \mapsto \mathbf{Rep}(T_x)$ as a family of local representation categories endowed with a G-equivariant transport structure.

The natural receptacle for such data is a *stack* on $\mathcal{O}$: a sheaf of categories satisfying descent, encoding both local information and global coherence under the G-action.

The present construction. In this paper we construct explicitly the stack

$$\mathcal{R} \ := \ G \times^T \mathbf{Rep}(T) \ \longrightarrow \ \mathcal{O},$$

associated to the principal T-bundle $p : G \to \mathcal{O} \simeq G/T$. We describe $\mathcal{R}$ as a prestack of T-equivariant functors, and verify the descent condition by an explicit gluing argument (Proposition 4.4). We then show that the coadjoint action of G lifts to a *strict* action on $\mathcal{R}$, and that such an action is equivalent to describing $\mathcal{R}$ as a stack over the action groupoid $G \ltimes \mathcal{O}$ (Theorem 7.1). This provides a categorical encoding of the transport of stabilizer representations.

Global sections of $\mathcal{R}$ recover geometric quantization. For each character $\chi \in \mathrm{Hom}(T, U(1))$ we construct a rank-one global section S_χ and identify it with the prequantum line bundle $L_\chi \to G/T$ equipped with the standard G-invariant connection. Its curvature is the KKS form (Proposition 8.7). Smooth global sections correspond to the compact induction $\mathrm{Ind}_T^G(\chi^{-1})$ (Proposition 9.1), and dominant integral weights recover irreducible representations via Borel–Weil (Theorem 9.4).

These identifications show that the classical geometric realizations of representations arise from viewing them as *coherent descent data* for the local stabilizer representation categories $\mathbf{Rep}(T_x)$. In this sense, the orbit method appears naturally as a descent phenomenon.

Organization. Section 2 fixes notation and recalls the basic geometry of coadjoint orbits. Section 3 reviews stacks and descent. Sections 4–6 construct $\mathcal{R}$ and its G-action. Sections 8–9 relate $\mathcal{R}$ to the prequantum line bundle, compact induction, and Borel–Weil. Section 10 provides a conceptual synthesis and the example $G = \mathrm{SU}(2)$.

Methodological note

During the development of this work, the author made iterative use of AI-based tools to test and refine preliminary formulations. This proved instrumental in shaping the ideas into the coherent structure presented here.

2 Setting, Notation, and Motivation

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We write $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ for the adjoint representation and $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ for the coadjoint representation. Fix a point $x_0 \in \mathfrak{g}^*$, and let $\mathcal{O} := G \cdot x_0 \subset \mathfrak{g}^*$ be its coadjoint orbit. For each $x \in \mathcal{O}$, denote the stabilizer subgroup

$$T_x := G_x = \{\, g \in G \mid \mathrm{Ad}^*(g)\, x = x \,\}.$$

Stabilizers along the orbit are mutually conjugate: if $y = g \cdot x$ then $T_y = g\, T_x\, g^{-1}$. We reserve the dot "$\cdot$" for the coadjoint action.

Homogeneous presentation and principal bundle. Fix $x_0 \in \mathcal{O}$ and put $T := T_{x_0}$. Then $\mathcal{O} \simeq G/T$ as a smooth G-homogeneous space, and

$$p \colon G \longrightarrow \mathcal{O}, \quad p(g) = g \cdot x_0$$

is a (right) principal T-bundle for the action $G \times T \to G$, $(g,t) \mapsto gt$.

Convention (Standing assumptions and normalizations). Unless stated otherwise we assume G is *compact and connected*, and we work on a *regular* coadjoint orbit $\mathcal{O} \simeq G/T$ for a fixed maximal torus $T \subset G$ (so each stabilizer T_x is conjugate to T). For expositional simplicity we focus on regular orbits where T is a maximal torus, though the construction extends immediately to singular orbits. We adopt the conventions

$$c_1(L_\chi) = \frac{[\omega_{\mathrm{KKS}}]}{2\pi}, \qquad (g,z) \sim (gt, \chi(t)^{-1}z) \text{ in } G \times^T \mathbb{C}_\chi,$$

so that $\Gamma^\infty(G/T, L_\chi) \cong \mathrm{Ind}_T^G(\chi^{-1})$ in Proposition 9.1. These choices fix signs and 2π-factors used later in curvature computations.

Each stabilizer T_x has its own representation theory $\mathbf{Rep}(T_x)$. The group G transports these local categories across the orbit via conjugation. Our goal is to organize this transport into a *stack* $\mathcal{R}$ over $\mathcal{O}$, verify that local data glue correctly (descent conditions), and show that global sections of $\mathcal{R}$ carry a natural G-representation. This provides a categorical framework for the Kirillov orbit method and geometric quantization.

3 Stacks: definition with explicit descent

Let X be a topological space and $\mathrm{Op}(X)$ its category of open sets with inclusions.

Definition 3.1 (Prestack and stack). *A prestack $\mathcal{S} \colon \mathrm{Op}(X)^{op} \to \mathbf{Grpd}$ is a pseudofunctor such that for every covering $U = \bigcup_i U_i$ the presheaf of isomorphisms is a sheaf (isomorphisms glue uniquely). It is a* stack *if, in addition, for every covering $U = \bigcup_i U_i$ the canonical functor*

$$\mathcal{S}(U) \longrightarrow \mathrm{Desc}_{\mathcal{S}}(U; \{U_i\})$$

is an equivalence of groupoids, where $\mathrm{Desc}_{\mathcal{S}}(U; \{U_i\})$ has:

- **objects**: *families $x_i \in \mathcal{S}(U_i)$ with isomorphisms $\phi_{ij} \colon x_i|_{U_{ij}} \xrightarrow{\sim} x_j|_{U_{ij}}$ satisfying the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on U_{ijk};*

- **morphisms**: *families $f_i \colon x_i \to y_i$ with $\phi_{ij}^y \circ f_i = f_j \circ \phi_{ij}^x$ on U_{ij}.*

Convention (Site and 2-categorical conventions). Throughout the paper the site is the poset $\mathrm{Op}(\mathcal{O})$ of open subsets with inclusions, endowed with the usual open covers. A (pre)stack is a pseudofunctor $\mathcal{S} \colon \mathrm{Op}(\mathcal{O})^{\mathrm{op}} \to \mathbf{Gpd}$ (or $\mathbf{Cat}$) satisfying the usual descent condition for open covers, and 1-morphisms/2-morphisms are functors/natural isomorphisms; all equivalences are understood up to isomorphism. For background and notational conventions see, e.g., [14, 10].

Remark 3.2 (Equalizer viewpoint). *For a sheaf F of sets, the sheaf condition over $U = \bigcup U_i$ is the equalizer diagram $F(U) \xrightarrow{\sim} \ker\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij})\right)$. For stacks, the analogous statement holds in the 2-categorical sense: $\mathcal{S}(U)$ is equivalent to the 2-equalizer of the Čech diagram; concretely this is encoded by $\mathrm{Desc}_{\mathcal{S}}(U; \{U_i\})$.*

4   The Stack of Local Stabilizer Representations

4.1   The ambient category $\mathbf{Rep}(T)$

Let $\mathbf{Rep}(T)$ be the category of finite-dimensional continuous *unitary* complex representations of the compact group T. Then $\mathbf{Rep}(T)$ is semisimple abelian; in particular it admits finite products and equalizers:

- (finite products) the cartesian product of finitely many objects is canonically isomorphic to their direct sum;

- (equalizers) if $f, g : V \to W$ are T-equivariant linear maps, then $\mathrm{Eq}(f,g) = \{v \in V \mid f(v) = g(v)\}$ is a T-stable subspace, hence an object of $\mathbf{Rep}(T)$.

No additional topology on $\mathbf{Rep}(T)$ is required below.

4.2   Definition of $\mathcal{R}$ via the principal bundle $p : G \to \mathcal{O}$

Definition 4.1 (The prestack $\mathcal{R}$). *For $U \subseteq \mathcal{O}$ open, set*

$$\mathcal{R}(U) \ := \ \mathrm{Fun}_T\!\left(p^{-1}(U), \mathbf{Rep}(T)\right),$$

the category of T-equivariant functors $F : p^{-1}(U) \to \mathbf{Rep}(T)$ with

$$F(gt) \ = \ t^{-1} \cdot F(g) \qquad (\forall\, g \in p^{-1}(U),\ t \in T),$$

and T-equivariant natural transformations as morphisms. Here $p^{-1}(U)$ is a discrete category with right T-action $g \mapsto gt$, while T acts on $\mathbf{Rep}(T)$ by conjugation $(t \cdot \rho)(s) := \rho(t^{-1}st)$. For $V \subseteq U$, restriction is $F \mapsto F|_{p^{-1}(V)}$.

An object $F \in \mathcal{R}(U)$ may be viewed more geometrically as the assignment, for each point $x \in U$, of a representation of the stabilizer T_x, compatibly with the conjugation isomorphisms along the G-orbit.

Indeed, the T-equivariance condition

$$F(gt) = t^{-1} \cdot F(g)$$

guarantees that the datum assigned at $g \in G$ depends only on the basepoint $x = p(g)$ and transforms correctly under change of representative.

Proposition 4.2 (Factorization along the fibres)**.** *The condition $F(gt) = t^{-1} \cdot F(g)$ implies that F is constant along the fibres of p and determines a map $\widetilde{F} : U \to \bigsqcup_{x \in U} \mathbf{Rep}(T_x)$ such that for $x = p(g)$,*

$$F(g) \;=\; g^{-1} \cdot \widetilde{F}(x), \qquad T_x = gTg^{-1}.$$

Conversely, a choice $x \mapsto \widetilde{F}(x) \in \mathbf{Rep}(T_x)$ compatible with conjugation determines F by this formula. Thus $\mathcal{R}$ is the associated stack $G \times^T \mathbf{Rep}(T) \to \mathcal{O}$.

4.3 Descent data and explicit gluing

Let $U = \bigcup_{i \in I} U_i$ be an open cover; write $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$.

Definition 4.3 (Descent datum in $\mathcal{R}$)**.** *A descent datum over $\{U_i\}$ is a family $F_i \in \mathcal{R}(U_i)$ with T-equivariant isomorphisms*

$$\phi_{ij} : F_i|_{U_{ij}} \xrightarrow{\;\sim\;} F_j|_{U_{ij}}$$

such that on triple overlaps U_{ijk} one has the cocycle identity $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$. Morphisms of descent data are families $\{\alpha_i : F_i \to G_i\}$ with $\phi_{ij}^G \circ \alpha_i = \alpha_j \circ \phi_{ij}^F$ on U_{ij}.

Proposition 4.4 (Descent for $\mathcal{R}$, with explicit gluing)**.** *Since G is compact, the stabilizer T is a compact torus and the category $\mathbf{Rep}(T)$ is semisimple. In particular it has finite products and equalizers; the proof only uses finite limits (and one may reduce to finite subcovers), hence no set-theoretic issues appear.*

For every cover $U = \bigcup U_i$, the canonical functor

$$\mathcal{R}(U) \;\longrightarrow\; \mathrm{Desc}_{\mathcal{R}}(U; \{U_i\})$$

is an equivalence of categories.

Proof. We construct a quasi-inverse $\mathbf{Glue} : \mathrm{Desc}_{\mathcal{R}}(U; \{U_i\}) \to \mathcal{R}(U)$ and verify that it is inverse up to canonical isomorphism.

Step 1: explicit construction of the glued object. Since $p^{-1}(U)$ is a discrete category, all limits in $\mathbf{Rep}(T)$ are computed pointwise, and restriction along an inclusion $p^{-1}(V) \subset p^{-1}(U)$ preserves these limits automatically. Fix a descent datum (F_i, ϕ_{ij}). For each $g \in p^{-1}(U)$, choose any index $i(g) \in I$ such that $p(g) \in U_{i(g)}$, and define

$$F(g) \;:=\; F_{i(g)}(g). \tag{1}$$

This is well defined up to the isomorphisms ϕ_{ij}: if $p(g) \in U_i \cap U_j$, then $\phi_{ij}(g) : F_i(g) \xrightarrow{\sim} F_j(g)$ identifies the two choices.

Step 2: independence of the choice. If i, j are two indices with $p(g) \in U_i \cap U_j$, the cocycle condition ensures that the resulting object $F(g)$ is well defined up to canonical isomorphism in $\mathbf{Rep}(T)$. More precisely, the map F takes values in the disjoint union of isomorphism classes, and the cocycle ensures coherence.

Step 3: T-equivariance of F. For any $t \in T$,

$$F(gt) = F_{i(gt)}(gt).$$

Since $p(gt) = p(g)$, we have $i(gt) = i(g)$, and each F_i is T-equivariant, so

$$F(gt) = F_{i(g)}(gt) = t^{-1} \cdot F_{i(g)}(g) = t^{-1} \cdot F(g).$$

Thus $F \in \mathcal{R}(U)$.

Step 4: comparison isomorphisms on U_i. For each $i \in I$, the restriction $F|_{U_i}$ is canonically isomorphic to F_i via the maps $\phi_{i,i(g)}$ for $g \in p^{-1}(U_i)$. These isomorphisms are compatible on overlaps by the cocycle condition.

Step 5: uniqueness up to unique isomorphism. Suppose $F, F' \in \mathcal{R}(U)$ are two gluings of the same datum, with comparison isomorphisms $\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$ and $\psi'_i : F'|_{U_i} \xrightarrow{\sim} F_i$ as above. For each $g \in p^{-1}(U)$, choose i with $p(g) \in U_i$ and define

$$\Psi(g) := (\psi'_i(g))^{-1} \circ \psi_i(g) : F(g) \to F'(g).$$

This is independent of the choice of i: if $p(g) \in U_i \cap U_j$, then

$$(\psi'_j)^{-1}\psi_j = \left((\phi_{ji}\psi'_i)\right)^{-1}(\phi_{ji}\psi_i) = (\psi'_i)^{-1}\psi_i,$$

using $\psi_j = \phi_{ji}\psi_i$ and $\psi'_j = \phi_{ji}\psi'_i$ on U_{ij}. Thus Ψ is well defined and gives a natural isomorphism $\Psi : F \xrightarrow{\sim} F'$. It is T-equivariant because each ψ_i, ψ'_i is so.

Uniqueness: if $\Theta : F \to F'$ is another natural isomorphism with $\psi'_i\Theta|_{U_i} = \psi_i$, then for each g and a chosen $i \ni p(g)$, $\Theta(g) = (\psi'_i(g))^{-1}\psi_i(g) = \Psi(g)$. Hence $\Theta = \Psi$ is unique.

Step 6: gluing of morphisms. Let $\alpha_i : F_i \to G_i$ be a morphism of descent data, i.e. $\phi_{ij}^G \circ \alpha_i = \alpha_j \circ \phi_{ij}^F$ on U_{ij}. Define $\alpha : p^{-1}(U) \to \mathrm{Hom}_{\mathbf{Rep}(T)}$ by

$$\alpha(g) := \alpha_{i(g)}(g).$$

If $p(g) \in U_i \cap U_j$, then

$$\alpha_j(g) = \phi_{ji}^G(g)\,\alpha_i(g)\left(\phi_{ji}^F(g)\right)^{-1},$$

so $\alpha(g)$ is independent of the choice of $i(g)$ up to the identifications already fixed in Step 4; hence α is a morphism $F \to G$ in $\mathcal{R}(U)$. Uniqueness is immediate from the definition.

Step 7: verification of quasi-inverse property. If $F \in \mathcal{R}(U)$, restricting to U_i and applying the above gluing gives back an object canonically isomorphic to F. Conversely, starting from (F_i, ϕ_{ij}), the construction (1) produces an object whose restrictions and comparison isomorphisms are exactly (F_i, ϕ_{ij}). Thus **Glue** is inverse to the restriction functor up to canonical isomorphism, proving that $\mathcal{R}(U) \to \mathrm{Desc}_{\mathcal{R}}(U; \{U_i\})$ is an equivalence. $\square$

Remark 4.5 (Why finiteness/limits matter). *The constructions above take place in $\mathbf{Rep}(T)$ and use only finite limits (products/equalizers) implicitly when checking independence of choices and T-equivariance; these limits exist in $\mathbf{Rep}(T)$ and are preserved by restriction along open covers and by the conjugation action of T.*

Remark 4.6 (Conceptual summary). *The explicit gluing just proved is the concrete manifestation of the general fact: the associated object $G \times^T \mathbf{Rep}(T) \to \mathcal{O}$ to the principal T-bundle $p : G \to \mathcal{O}$ is a stack; descent along open covers is effective.*

5 The induced G-action on $\mathcal{R}$ and fibrewise equivalences

Let $p : G \to \mathcal{O}$ be the principal T-bundle of Section 4. Recall

$$\mathcal{R}(U) = \mathrm{Fun}_T\big(p^{-1}(U), \mathbf{Rep}(T)\big).$$

We write $L_g : G \to G$, $L_g(h) = gh$ for left translation.

5.1 Conjugation on stabilizers: an equivalence of categories

Fix $x \in \mathcal{O}$ and $g \in G$. Conjugation gives a group isomorphism

$$c_g : \ T_x \longrightarrow T_{g \cdot x}, \qquad c_g(t) = gtg^{-1}.$$

Define the functor

$$c_g^* : \ \mathbf{Rep}(T_{g \cdot x}) \longrightarrow \mathbf{Rep}(T_x), \qquad (c_g^* \rho)(t) := \rho\big(c_g(t)\big) = \rho(gtg^{-1}),$$

and on morphisms c_g^* acts as the identity linear map.

Proposition 5.1. *For every $x \in \mathcal{O}$ and $g \in G$, the functor $c_g^* : \mathbf{Rep}(T_{g \cdot x}) \to \mathbf{Rep}(T_x)$ is an equivalence of categories.*

Proof. *Full and faithful.* Let $\rho, \tau \in \mathbf{Rep}(T_{g \cdot x})$ and $L : V_\rho \to V_\tau$ linear. Then L intertwines $c_g^* \rho$ and $c_g^* \tau$ iff

$$L\, \rho(gtg^{-1}) = \tau(gtg^{-1})\, L \qquad (\forall t \in T_x).$$

Setting $s = gtg^{-1} \in T_{g \cdot x}$, this is equivalent to $L\rho(s) = \tau(s)L$ for all $s \in T_{g \cdot x}$, i.e. L is a $T_{g \cdot x}$-intertwiner $\rho \to \tau$. Thus c_g^* induces bijections on Hom-sets.

Essentially surjective. For $\sigma \in \mathbf{Rep}(T_x)$ define $\rho \in \mathbf{Rep}(T_{g \cdot x})$ by $\rho(s) := \sigma(g^{-1}sg)$. Since conjugation by g is a group isomorphism $T_{g \cdot x} \to T_x$, the assignment $s \mapsto g^{-1}sg$ ensures that ρ is again a well-defined representation.

Then $(c_g^* \rho)(t) = \rho(gtg^{-1}) = \sigma(t)$, so $c_g^* \rho \simeq \sigma$. Hence every object of $\mathbf{Rep}(T_x)$ lies in the essential image. $\qquad \square$

5.2 Translation functors on $\mathcal{R}$

Definition 5.2 (Left-translation action on $\mathcal{R}$)**.** *For $g \in G$ and $U \subseteq \mathcal{O}$ open, define*

$$g_* : \ \mathcal{R}(U) \longrightarrow \mathcal{R}(gU), \qquad (g_* F)(h) := F\big(L_{g^{-1}}(h)\big) = F(g^{-1}h),$$

and, for a natural transformation $\alpha : F \Rightarrow F'$, $(g_* \alpha)(h) := \alpha(g^{-1}h)$.

Lemma 5.3 (Well-definedness)**.** *If $F \in \mathcal{R}(U)$ is T-equivariant, then $g_* F \in \mathcal{R}(gU)$ is T-equivariant. Moreover g_* is a functor.*

Proof. Let $h \in p^{-1}(gU)$ and $t \in T$. Since $L_{g^{-1}}$ commutes with the right T-action,

$$(g_*F)(ht) = F(g^{-1}ht) = F\big((g^{-1}h)t\big) = t^{-1} \cdot F(g^{-1}h) = t^{-1} \cdot (g_*F)(h).$$

Hence g_*F is T-equivariant. Functoriality follows since $(g_*\alpha)(h) = \alpha(g^{-1}h)$ respects identities and composition pointwise. $\qquad\square$

Proposition 5.4 (Strict functoriality). *For all $g, h \in G$ and open $U \subseteq \mathcal{O}$,*

$$(gh)_* = g_* \circ h_* : \mathcal{R}(U) \to \mathcal{R}(gh\, U), \qquad e_* = \mathrm{Id}_{\mathcal{R}(U)}.$$

Proof. Evaluate on $F \in \mathcal{R}(U)$ and $k \in p^{-1}(gh\, U)$:

$$\big((g_* \circ h_*)F\big)(k) = (h_*F)(g^{-1}k) = F\big(h^{-1}g^{-1}k\big) = F\big((gh)^{-1}k\big) = ((gh)_*F)(k).$$

The identity case is immediate from $L_e = \mathrm{id}$. The strictness of the identities follows from the discreteness of $p^{-1}(U)$, so that all compositions are computed pointwise. $\qquad\square$

Proposition 5.5 (Compatibility with restriction). *If $V \subseteq U$ then, as functors $\mathcal{R}(U) \to \mathcal{R}(gV)$,*

$$\big(g_*F\big)\big|_{gV} = g_*\big(F\big|_V\big) \quad \text{and} \quad \big(g_*\alpha\big)\big|_{gV} = g_*\big(\alpha\big|_V\big).$$

Proof. For $k \in p^{-1}(gV)$ we have $g^{-1}k \in p^{-1}(V)$, hence

$$\big((g_*F)|_{gV}\big)(k) = (g_*F)(k) = F(g^{-1}k) = (F|_V)(g^{-1}k) = \big(g_*(F|_V)\big)(k).$$

The statement for morphisms is pointwise identical. $\qquad\square$

5.3 Fibrewise description: relation with conjugation

Recall Proposition 4.2: an object $F \in \mathcal{R}(U)$ corresponds to a function $\widetilde{F} : U \to \bigsqcup_{x \in U} \mathbf{Rep}(T_x)$ such that

$$F(h) \cong \theta_h^*\big(\widetilde{F}(p(h))\big), \qquad \theta_h : T \xrightarrow{\sim} T_{p(h)}, \quad \theta_h(t) = hth^{-1}.$$

Proposition 5.6 (Fibres under g_*). *Let $x \in U$ and $y = g \cdot x \in gU$. If $\widetilde{F}(x) \in \mathbf{Rep}(T_x)$ represents the fibre of F at x, then the fibre of g_*F at y is*

$$\widetilde{(g_*F)}(y) \cong (c_{g^{-1}})^*\big(\widetilde{F}(x)\big) \in \mathbf{Rep}(T_y).$$

Proof. Pick $k \in p^{-1}(y)$, so $p(g^{-1}k) = x$. Using the factorization for F,

$$(g_*F)(k) = F(g^{-1}k) \cong \theta_{g^{-1}k}^*\big(\widetilde{F}(x)\big).$$

On the other hand the fibre description for g_*F at y says $(g_*F)(k) \cong \theta_k^*\big(\widetilde{(g_*F)}(y)\big)$. Since $\theta_{g^{-1}k} = c_{g^{-1}} \circ \theta_k$ (check directly),

$$\theta_{g^{-1}k}^* = \theta_k^* \circ (c_{g^{-1}})^*.$$

Thus $\theta_k^*\big(\widetilde{(g_*F)}(y)\big) \cong \theta_k^*\big((c_{g^{-1}})^*\widetilde{F}(x)\big)$, and applying the faithful functor θ_k^* yields $\widetilde{(g_*F)}(y) \cong (c_{g^{-1}})^*\widetilde{F}(x)$. $\qquad\square$

6 Global sections and the G-action

6.1 Global sections as maps of stacks over the base

Let $1_\mathcal{O}$ denote the terminal stack on $\mathcal{O}$ (the constant groupoid with one object and one arrow on each open).

Proposition 6.1. *There is a canonical equivalence of categories*

$$\mathcal{R}(\mathcal{O}) \simeq \mathrm{Hom}_{\mathbf{Stk}/\mathcal{O}}\big(1_\mathcal{O}, \mathcal{R}\big).$$

Proof. A morphism $\Phi : 1_\mathcal{O} \to \mathcal{R}$ over $\mathcal{O}$ amounts to an object $s := \Phi_\mathcal{O}(*) \in \mathcal{R}(\mathcal{O})$ whose restrictions satisfy $\Phi_V(*) = s|_V$ for all opens $V \subseteq \mathcal{O}$; conversely an $s \in \mathcal{R}(\mathcal{O})$ defines a morphism by $\Phi_s(U) := s|_U$ and Φ_s on arrows the identity. These assignments are inverse up to unique isomorphism and functorial in morphisms. $\qquad\square$

6.2 The strict G-action on $\mathcal{R}(U)$ and on global sections

For each open $U \subseteq \mathcal{O}$, Definitions 5.2–5.4 give functors

$$g_* : \ \mathcal{R}(U) \longrightarrow \mathcal{R}(gU)$$

with $(gh)_* = g_* \circ h_*$ and $e_* = \mathrm{Id}$, and which commute strictly with restrictions (Prop. 5.5).

Proposition 6.2 (Strict action on global sections)**.** *Since $g\mathcal{O} = \mathcal{O}$ for all $g \in G$, the assignment*

$$G \times \mathcal{R}(\mathcal{O}) \ \longrightarrow \ \mathcal{R}(\mathcal{O}), \qquad (g, s) \longmapsto g_* s$$

defines a strict action of G on the category $\mathcal{R}(\mathcal{O})$:

$$(gh)_* s \ = \ g_*(h_* s), \qquad e_* s = s,$$

and likewise on morphisms of $\mathcal{R}(\mathcal{O})$.

Proof. These are precisely the identities of Proposition 5.4, evaluated at $U = \mathcal{O}$. $\qquad\square$

Proposition 6.3 (Compatibility with restriction squares)**.** *For every inclusion $V \subseteq U$ and $g \in G$, the square of functors*

$$\begin{array}{ccc}
\mathcal{R}(U) & \xrightarrow{\ g_* \ } & \mathcal{R}(gU) \\
{\scriptstyle (-)|_V}\big\downarrow & & \big\downarrow{\scriptstyle (-)|_{gV}} \\
\mathcal{R}(V) & \xrightarrow[\ g_* \ ]{} & \mathcal{R}(gV)
\end{array}$$

commutes strictly.

Proof. This is exactly Proposition 5.5. $\qquad\square$

Remark 6.4 (Fibrewise form of the action)**.** *For $s \in \mathcal{R}(\mathcal{O})$ and $x \in \mathcal{O}$, the fibre at $y = g \cdot x$ of $g_* s$ is $(c_{g^{-1}})^*(s_x) \in \mathbf{Rep}(T_y)$ by Proposition 5.6.*

7 Equivariant stacks vs. stacks over the action groupoid

Let $G \curvearrowright \mathcal{O}$ be the coadjoint action and $G \ltimes \mathcal{O}$ its action groupoid: $\mathrm{Ob} = \mathcal{O}$, arrows $(g, x) : x \to g \cdot x$, composition $(h, g \cdot x) \circ (g, x) = (hg, x)$.

7.1 Two 2-categories and the comparison functors

(i) G-equivariant stacks over $\mathcal{O}$. An object is a stack $\mathcal{S}$ on $\mathrm{Op}(\mathcal{O})$ with a *pseudofunctorial* G-action: for each $g \in G$ and open $U \subseteq \mathcal{O}$ a functor

$$g_* : \ \mathcal{S}(U) \longrightarrow \mathcal{S}(gU),$$

together with invertible natural transformations (coherences)

$$\alpha_{g,h} : \ g_* \circ h_* \overset{\sim}{\Longrightarrow} (gh)_*, \qquad \lambda : \ \mathrm{Id} \overset{\sim}{\Longrightarrow} e_*,$$

such that the usual pentagon and triangle identities hold and all maps commute with restrictions (functorially in U).

(ii) Stacks over the groupoid $G \ltimes \mathcal{O}$. An object is a stack $\mathcal{T}$ on $\mathrm{Op}(\mathcal{O})$ with, for each arrow (g, x), an equivalence of fibre categories $g_x^* : \mathcal{T}_{g \cdot x} \overset{\sim}{\to} \mathcal{T}_x$ and, for each composable pair $(h, g \cdot x) \circ (g, x)$, a specified isomorphism

$$\beta_{h,g,x} : \ g_x^* \circ h_{g \cdot x}^* \overset{\sim}{\Longrightarrow} (hg)_x^*,$$

plus units $\eta_x : \mathrm{Id}_{\mathcal{T}_x} \overset{\sim}{\Rightarrow} e_x^*$, satisfying the standard groupoid pentagon/triangle coherences and naturality with respect to open restrictions.

We define two 2-functors

$$\mathbf{E} : \ \mathbf{Stk}_G(\mathcal{O}) \ \longrightarrow \ \mathbf{Stk}(G \ltimes \mathcal{O}), \qquad \mathbf{A} : \ \mathbf{Stk}(G \ltimes \mathcal{O}) \ \longrightarrow \ \mathbf{Stk}_G(\mathcal{O}),$$

and state (without detailed proof) that they are quasi-inverse.

7.2 From G-equivariant stacks to groupoid stacks

Let $(\mathcal{S}, \{g_*\}, \alpha, \lambda)$ be G-equivariant. Set $\mathbf{E}(\mathcal{S}) =: \mathcal{T}$:

- *Fibre at x.* $\mathcal{T}_x := \mathcal{S}_x$ (fibre of $\mathcal{S}$ at x).

- *Arrow (g, x).* Put $g_x^* := g^* : \mathcal{S}_{g \cdot x} \to \mathcal{S}_x$ (the pullback on fibres induced by g_*).

- *Coherences.* For composable arrows, take $\beta_{h,g,x}$ to be the fibre of $\alpha_{g,h}^{-1} : g_* \circ h_* \Rightarrow (gh)_*$. Units η_x are the fibres of λ^{-1}.

7.3 From groupoid stacks to G-equivariant stacks

Let $(\mathcal{T}, \{g_x^*\}, \beta, \eta)$ be a stack over $G \ltimes \mathcal{O}$. Define $\mathbf{A}(\mathcal{T}) =: \mathcal{S}$ by:

- *Objects.* For $U \subseteq \mathcal{O}$ open, set $\mathcal{S}(U) := \mathcal{T}(U)$.

- *G-action.* For $g \in G$,

$$g_* : \ \mathcal{S}(U) \longrightarrow \mathcal{S}(gU), \qquad (g_*A)_y := (g^{-1})_y^*(A_{g^{-1}\cdot y}) \quad (y \in gU),$$

  and $(g_*f)_y := (g^{-1})_y^*(f_{g^{-1}\cdot y})$ fibrewise on morphisms.

- *Coherences.* Define

$$\alpha_{g,h}(A) \ \text{at fibre } z \ := \ \beta_{g^{-1},h^{-1},z}^{-1} : \ (g^{-1})_z^*\big((h^{-1})_{g^{-1}\cdot z}^*(A_{(hg)^{-1}\cdot z})\big) \ \overset{\sim}{\Longrightarrow} \ ((hg)^{-1})_z^*(A_{(hg)^{-1}\cdot z}),$$

  and $\lambda(A)$ at z to be $\eta_z^{-1} : A_z \overset{\sim}{\Longrightarrow} (e^{-1})_z^*(A_z)$.

7.4 Equivalence of 2-categories (statement)

Theorem 7.1. $\mathbf{E}$ *and* $\mathbf{A}$ *give mutually quasi-inverse 2-equivalences between G-equivariant stacks over $\mathcal{O}$ and stacks over $G \ltimes \mathcal{O}$.*

Proof. Starting with $(\mathcal{S}, \{g_*\}, \alpha, \lambda)$, the composite $\mathbf{A}(\mathbf{E}(\mathcal{S}))$ recovers the original g_* and coherences fibrewise, yielding a canonical 2-isomorphism $\mathbf{A}\circ\mathbf{E} \Rightarrow \mathrm{Id}$. Since the site $\mathrm{Op}(\mathcal{O})$ is subcanonical and all stacks considered are fibrewise small, the general results of [11, 1] apply directly. Conversely, for $(\mathcal{T}, \{g_x^*\}, \beta, \eta)$, the composite $\mathbf{E}(\mathbf{A}(\mathcal{T}))$ reconstructs the same fibre functors and coherences up to canonical identifications, giving $\mathbf{E}\circ\mathbf{A} \Rightarrow \mathrm{Id}$. The verification of coherence diagrams (pentagon and triangle identities) is standard; see [11, 1] for the general framework. □

Remark 7.2 (Strict vs. weak actions). *If one prefers strict actions $(gh)_* = g_*\circ h_*$, $e_* = \mathrm{Id}$ (as in our stack $\mathcal{R}$), take $\alpha = \lambda = \mathrm{id}$; the equivalence becomes tautological on fibres. In general, pseudofunctorial actions are unavoidable, but the 2-equivalence remains valid.*

Remark 7.3 (Conceptual meaning of the theorem). *This theorem tells us that working with a G-equivariant stack $\mathcal{R}$ over $\mathcal{O}$ is equivalent to working with a stack over the action groupoid $G \ltimes \mathcal{O}$. In our setting, this means that the transport of local stabilizer representations by conjugation is naturally encoded in the groupoid structure.*

8 The prequantum line bundle as a rank-one global section

Fix $x_0 \in \mathcal{O}$, $T = G_{x_0}$ and identify $\mathcal{O} \simeq G/T$ via $p : G \to G/T$. Let $\Lambda^\vee = \mathrm{Hom}(T, U(1))$ be the character lattice; for $\chi \in \Lambda^\vee$ let $\mathbb{C}_\chi$ be the one-dimensional T-module of weight χ.

8.1 From a character to a rank-one object of $\mathcal{R}$

Definition 8.1 (Rank-one section). *Define $S_\chi \in \mathcal{R}(\mathcal{O})$ by*

$$(S_\chi)_x \ := \ (\mathbb{C}, \ \chi_x), \qquad \chi_x(t) \ := \ \chi\big(g^{-1}tg\big) \quad \text{for } x = g \cdot x_0, \ t \in T_x = gTg^{-1}.$$

This is well defined (independent of the choice of g modulo T).

Lemma 8.2. *S_χ is a global object of $\mathcal{R}$, and for every $h \in G$, $h_* S_\chi = S_\chi$ fibrewise.*

Proof. If $g' = gt$ then $\chi\big((g')^{-1}(\cdot)g'\big) = \chi\big(t^{-1}g^{-1}(\cdot)gt\big) = \chi\big(g^{-1}(\cdot)g\big)$ since χ is a character of T. For $h \in G$ and $y = h \cdot x$, $(h_* S_\chi)_y \ = \ h^*(S_\chi)_x \ = \ \chi\big((hg)^{-1}(\cdot)(hg)\big) \ = \ (S_\chi)_y$. $\qquad\square$

8.2 Identification with the associated line bundle

Definition 8.3 (Associated line bundle). *Let*

$$L_\chi \ := \ G \times^T \mathbb{C}_\chi \ \longrightarrow \ G/T \simeq \mathcal{O},$$

the quotient of $G \times \mathbb{C}$ by $(g, z) \sim (gt, \chi(t)^{-1}z)$; G acts by $h \cdot [g, z] = [hg, z]$.

Proposition 8.4. *Under the identification $\mathcal{R} \simeq G \times^T \mathbf{Rep}(T)$, the rank-one object S_χ corresponds to the associated line bundle L_χ. In particular, sections of L_χ are the same as T-equivariant functions $f : G \to \mathbb{C}$ with $f(gt) = \chi(t)^{-1}f(g)$, and the G-action on $\Gamma^\infty(\mathcal{O}, L_\chi)$ matches the action on S_χ.*

Proof. An object of $\mathcal{R}$ is, by definition, a T-equivariant functor on $p^{-1}(U)$; the rank-one object given by $\mathbb{C}_\chi$ produces the associated bundle $G \times^T \mathbb{C}_\chi$, and the G-equivariance is inherited from left translation on G. Sections vs. equivariant functions is the standard associated-bundle identification. $\qquad\square$

8.3 Principal connection and curvature equals KKS

Choose an $\mathrm{Ad}(T)$-invariant reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the left Maurer–Cartan form and set

$$A \ := \ \mathrm{pr}_\mathfrak{t}(\theta) \in \Omega^1(G, \mathfrak{t}), \qquad \theta_\mathfrak{m} := \mathrm{pr}_\mathfrak{m}(\theta).$$

Lemma 8.5 (Canonical principal T-connection). *A is a principal connection one-form on $p : G \to G/T$: it is right T-equivariant ($R_t^* A = \mathrm{Ad}(t^{-1})A$) and reproduces vertical generators ($A(\xi_G) = \xi$ for all $\xi \in \mathfrak{t}$).*

Proof. For $t \in T$, $R_t^* \theta = \mathrm{Ad}(t^{-1})\theta$; projecting to $\mathfrak{t}$ yields the first claim. Since the projection $\mathrm{pr}_\mathfrak{t}$ is $\mathrm{Ad}(T)$–equivariant by construction of the reductive decomposition, the resulting form A is again $\mathrm{Ad}(T)$–equivariant. If $\xi \in \mathfrak{t}$, the fundamental vertical field is $\xi_G(g) = \frac{d}{ds}\big|_0 g \exp(s\xi)$; then $\theta(\xi_G) = \xi$ so $A(\xi_G) = \xi$. $\qquad\square$

Definition 8.6 (Induced connection on L_χ). *Let $\lambda \in \mathfrak{t}^*$ be the differential of χ, $\chi(\exp H) = e^{2\pi i \lambda(H)}$. The connection on L_χ associated to A has connection 1-form*

$$\alpha_\lambda \ := \ 2\pi i \, \lambda(A) \ \in \ \Omega^1(G, i\mathbb{R})$$

on the principal bundle, which descends to L_χ by the associated-bundle construction.

Proposition 8.7 (Curvature equals the KKS form). *Let $\Omega = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(G, \mathfrak{t})$ be the curvature of A. Then for horizontal vectors the basic 2-form $2\pi i \, \lambda(\Omega)$ descends to the base and equals*

$$F_{\nabla^\chi} \ = \ i \, \omega_\lambda,$$

where ω_λ is the Kirillov–Kostant–Souriau form on G/T: at eT and $X, Y \in \mathfrak{m}$,

$$\omega_\lambda(\overline{X}, \overline{Y}) \ = \ 2\pi \, \lambda([X, Y]_\mathfrak{t}).$$

Under the identification $\mathcal{O} \simeq G/T$, the value of the functional λ corresponds to the base point $x \in \mathcal{O}$, so that the expression $2\pi\lambda([X,Y]_\mathfrak{t})$ matches the standard KKS formula $\langle x, [X, Y] \rangle$.

Proof. Since G is compact, there exists an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Taking $\mathfrak{m} := \mathfrak{t}^\perp$ yields an $\mathrm{Ad}(T)$-invariant reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$, with $[\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m}$.

The Maurer–Cartan equation $d\theta + \frac{1}{2}[\theta, \theta] = 0$ and the decomposition $\theta = A + \theta_\mathfrak{m}$ give

$$dA + \tfrac{1}{2}[A \wedge A] = -\tfrac{1}{2}\mathrm{pr}_\mathfrak{t}[\theta_\mathfrak{m} \wedge \theta_\mathfrak{m}],$$

since $\mathrm{pr}_\mathfrak{t}[A \wedge \theta_\mathfrak{m}] = 0$. Thus on horizontal vectors $\overline{X}, \overline{Y}$ (lifts of $X, Y \in \mathfrak{m}$),

$$\Omega(\overline{X}, \overline{Y}) = -\tfrac{1}{2}\mathrm{pr}_\mathfrak{t}[X, Y].$$

Pairing with $2\pi i \lambda$:

$$2\pi i \lambda(\Omega)(\overline{X}, \overline{Y}) = -\pi i \lambda([X, Y]_\mathfrak{t}) = i\omega_\lambda(\overline{X}, \overline{Y}),$$

with the normalization of Definition 8.6. Since Ω is horizontal and T-equivariant, $2\pi i \lambda(\Omega)$ is basic and descends to the curvature F_{∇^χ} on the base. $\qquad\square$

Corollary 8.8 (Integrality criterion). *$c_1(L_\chi) = [\omega_\lambda]/(2\pi) \in H^2(G/T, \mathbb{Z})$. Hence $\mathcal{O}_\lambda$ is prequantizable iff λ is integral (equivalently, iff χ exists).*

Remark 8.9 (Bridging the languages). *Section 7 packages transport of local stabilizer representations as a stack over $G \ltimes \mathcal{O}$; Section 8 identifies rank-one such data with the associated line bundle and computes its curvature, recovering the KKS form. In short:*

$$\textit{rank-one object } S_\chi \ \Longleftrightarrow \ L_\chi \ \Longleftrightarrow \ (\omega_\lambda, \ \lambda \in \Lambda^\vee).$$

9 From sections of the prequantum line to induced representations

Throughout, let G be compact and connected, let $x_0 \in \mathcal{O}$ with stabilizer $T = G_{x_0}$ (a maximal torus in the regular case), and identify $\mathcal{O} \simeq G/T$ via $p : G \to G/T$. Fix a character $\chi \in \Lambda^\vee = \mathrm{Hom}(T, U(1))$, with differential $\lambda \in \mathfrak{t}^*$. Let $L_\chi \to G/T$ be the associated line bundle (Def. 8.3), and ∇^χ the G-invariant connection of Section 8, with curvature $F_{\nabla^\chi} = i\,\omega_\lambda$ (Prop. 8.7).

9.1 The function model of sections and compact induction

Define the *smooth induced space*

$$\mathcal{I}_\chi^\infty := \big\{ f \in C^\infty(G, \mathbb{C}) \ \big| \ f(gt) = \chi(t)^{-1} f(g) \quad \forall\, g \in G,\, t \in T \big\},$$

endowed with the left regular action

$$\big(L_g f\big)(h) := f(g^{-1} h), \qquad g, h \in G.$$

This is the smooth Fréchet model of the compact induction $\mathrm{Ind}_T^G(\chi^{-1})$.

Proposition 9.1 (Explicit isomorphism $\Gamma^\infty(G/T, L_\chi) \cong \mathcal{I}_\chi^\infty$). *Define*

$$\Phi : \ \Gamma^\infty(G/T, L_\chi) \ \longrightarrow \ \mathcal{I}_\chi^\infty, \qquad (\Phi s)(g) := z \ \text{ such that } \ s(gT) = [g, z] \in L_\chi.$$

Then Φ is well defined, linear and bijective, with inverse

$$\Psi : \ \mathcal{I}_\chi^\infty \ \longrightarrow \ \Gamma^\infty(G/T, L_\chi), \qquad (\Psi f)(gT) := [g, f(g)].$$

Moreover Φ is a G-intertwiner: $\Phi(g \cdot s) = L_g\, \Phi(s)$ for all $g \in G$.

Proof. Well definedness of Φ. If $gT = g'T$ then $g' = gt$ for some $t \in T$. Write $s(gT) = [g, z] = [gt, z']$; by the defining relation of $G \times^T \mathbb{C}_\chi$ there exists z' with $[gt, z'] = [g, z]$ iff $z' = \chi(t)^{-1} z$. The value $(\Phi s)(g')$ is by definition the unique z' s.t. $s(g'T) = [g', z']$, hence $(\Phi s)(gt) = \chi(t)^{-1}(\Phi s)(g)$, i.e. $\Phi s \in \mathcal{I}_\chi^\infty$.

This uses only the defining relation $(g, z) \sim (gt, \chi(t)^{-1} z)$ for the associated bundle $G \times^T \mathbb{C}_\chi$.

Well definedness of Ψ. If $g' = gt$ then

$$[g', f(g')] = [gt, f(gt)] = [g, \chi(t)\, f(gt)].$$

The T-equivariance condition $f(gt) = \chi(t)^{-1} f(g)$ shows $[gt, f(gt)] = [g, f(g)]$, so Ψf is well defined and smooth.

Φ and Ψ are inverses. For s, $(\Psi \Phi s)(gT) = [g, (\Phi s)(g)] = [g, z] = s(gT)$. For f, $(\Phi \Psi f)(g)$ is the unique z with $[g, z] = [g, f(g)]$, hence $z = f(g)$.

G-equivariance. The G-action on sections is $(g \cdot s)(x) = g^* \big(s(g^{-1} \cdot x)\big)$. Evaluate at hT:

$$
\begin{aligned}
(\Phi(g \cdot s))(h) &= \text{the } z \text{ with } (g \cdot s)(hT) = [h, z] \\
&= g^* \big(s(g^{-1} hT)\big) \\
&= g^* \big([\, g^{-1} h, \, (\Phi s)(g^{-1} h) \,]\big) \\
&= [\, h, \, (\Phi s)(g^{-1} h) \,].
\end{aligned}
$$

Hence $(\Phi(g \cdot s))(h) = (\Phi s)(g^{-1} h) = (L_g \Phi s)(h)$ for all h, i.e. $\Phi(g \cdot s) = L_g \, \Phi(s)$. $\qquad\square$

Remark 9.2 (Functoriality)**.** *The identifications above are natural in χ and compatible with restriction to open subsets, hence also with the stack description in Section 7.*

9.2 Hilbert space picture and unitarity

Let $\langle \cdot, \cdot \rangle$ be a G-invariant Hermitian metric on L_χ and $d\mu_{G/T}$ the G-invariant probability measure on G/T. Define on smooth sections

$$
\langle s_1, s_2 \rangle_{G/T} := \int_{G/T} \langle s_1(x), s_2(x) \rangle \, d\mu_{G/T}(x).
$$

Let dg be Haar probability on G, and put on $\mathcal{I}_\chi^\infty$ the L^2 inner product

$$
\langle f_1, f_2 \rangle_G := \int_G f_1(g) \, \overline{f_2(g)} \, dg,
$$

which descends to the closed subspace $\mathcal{I}_\chi := L^2(G) \cap \{f(gt) = \chi(t)^{-1} f(g)\}$.

Proposition 9.3 (Isometry and unitary equivalence)**.** *With the choices above, Φ extends by completion to a unitary*

$$
\Phi: \ L^2\big(G/T, L_\chi\big) \ \xrightarrow{\ \sim\ } \ \mathcal{I}_\chi,
$$

intertwining the unitary G-representations by left translation. In particular, the smooth vectors agree with $\Gamma^\infty(G/T, L_\chi)$ and $\mathcal{I}_\chi^\infty$.

Proof. By Fubini for the principal bundle $G \to G/T$ (disintegration of Haar),

$$
\int_G F(g) \, dg \ = \ \int_{G/T} \int_T F(gt) \, dt \, d\mu_{G/T}(gT), \quad \text{(normalize } dt \text{ to prob. on } T).
$$

For s_1, s_2 and $f_i = \Phi s_i$ we have $s_i(gT) = [g, f_i(g)]$ and $\langle s_1(gT), s_2(gT) \rangle = f_1(g) \, \overline{f_2(g)}$ in the standard trivialization. Using $f_i(gt) = \chi(t)^{-1} f_i(g)$ and $|\chi(t)| = 1$,

$$
\langle s_1, s_2 \rangle_{G/T} = \int_{G/T} f_1(g) \overline{f_2(g)} \, d\mu_{G/T}(gT) = \int_G f_1(g) \overline{f_2(g)} \, dg = \langle f_1, f_2 \rangle_G.
$$

Thus Φ is an isometry on smooth vectors and extends to a unitary onto the closed subspace $\mathcal{I}_\chi$. G-equivariance of Φ (Prop. 9.1) implies unitarity intertwines the G-actions. $\qquad\square$

9.3 Holomorphic sections and the Borel–Weil theorem

Assume T is a maximal torus and fix a choice of positive roots. The corresponding complex structure on G/T makes it a flag manifold, and L_χ a holomorphic line bundle when χ is dominant integral.

Theorem 9.4 (Borel–Weil [2, 3, 12]). *If χ is dominant integral with differential λ, then $H^0(G/T, L_\chi)$ is an irreducible G-module of highest weight λ. Under the identification of Proposition 9.1, holomorphic sections correspond to functions $f \in \mathcal{I}_\chi^\infty$ annihilated by right-invariant vector fields in $\mathfrak{n}^+$.*

Remark 9.5 (Summary of the three equivalent models). *For dominant integral λ (character χ):*
$$H^0(G/T, L_\chi) \;\cong\; \mathcal{H}_\chi \;\subset\; \mathrm{Ind}_T^G(\chi^{-1}),$$
and, after completion, the unitary representation on $L^2(G/T, L_\chi)$ is unitarily equivalent to the closed G-subrepresentation of $L^2(G)$ consisting of functions with right T-covariance $f(gt) = \chi(t)^{-1} f(g)$.

9.4 Compatibility with the stack action

Let $S_\chi \in \mathcal{R}(\mathcal{O})$ be the rank-one object of Section 8. The induced G-action on global sections $s \in \mathcal{R}(\mathcal{O})$ is $(g \cdot s)(x) = g^* s(g^{-1} \cdot x)$ (Prop. 6.2). Under the identification $\Gamma^\infty(G/T, L_\chi) \cong \mathcal{I}_\chi^\infty$ of Prop. 9.1, this action corresponds exactly to left translation $(L_g f)(h) = f(g^{-1} h)$. Hence the representation obtained from the *stack* $\mathcal{R}$ agrees with the *classical* induced (and, in the holomorphic case, Borel–Weil) representation.

10 Synthesis and conceptual summary

We conclude by gathering the main ideas and their interrelations.

10.1 The chain of constructions

Starting from a coadjoint orbit $\mathcal{O}$ of a compact Lie group G, our results assemble into the following sequence:

1. **Local symmetry**: each point $x \in \mathcal{O}$ has a stabilizer T_x, and the category $\mathbf{Rep}(T_x)$ captures a local fragment of symmetry.

2. **Stack construction**: these categories glue to a stack $\mathcal{R} \to \mathcal{O}$, realized concretely as the associated stack $G \times^T \mathbf{Rep}(T)$ (Sections 3–4).

3. **Descent**: we verify explicitly that families of objects satisfying the Čech cocycle condition glue uniquely (Proposition 4.4), showing that $\mathcal{R}$ is indeed a stack.

4. **G-action**: the coadjoint action lifts to a strict functorial action on $\mathcal{R}$ (Sections 5–6).

5. **Groupoid form**: the G-equivariant structure is equivalent to describing $\mathcal{R}$ as a stack over the action groupoid $G \ltimes \mathcal{O}$ (Section 7), encoding conjugation transport.

6. **Rank-one objects**: characters χ yield rank-one global sections S_χ, identified with the prequantum line bundles $L_\chi \to G/T$ whose curvature recovers the KKS form (Section 8).

7. **Global sections**: smooth sections of L_χ match the compact induction $\mathrm{Ind}_T^G(\chi^{-1})$, and for dominant integral weights their holomorphic sections give Borel–Weil representations (Section 9).

10.2   Interpretation

Representation theory on a coadjoint orbit is a descent phenomenon: global representations arise as the coherent gluing of local stabilizer symmetries.

The stack $\mathcal{R}$ serves as a categorical analogue of a prequantum object. While a prequantum line bundle encodes the symplectic geometry of $\mathcal{O}$, the stack $\mathcal{R}$ encodes the transport of representation categories along the orbit. Global sections of $\mathcal{R}$ correspond precisely to representations of G obtained from this coherent descent.

10.3   Context and outlook

The framework unifies geometric quantization, induced representations, and the Borel–Weil theorem under a single conceptual umbrella. It suggests potential generalizations to noncompact groups, orbits with more intricate topology, or higher-categorical refinements, though such developments lie beyond the scope of this work.

In summary, the stack-theoretic organization of local stabilizer representations provides a clean and flexible categorical setting for the orbit method, making explicit the descent mechanism underlying geometric representation theory.

10.4   Final perspective

The chain of ideas

Local stabilizers $\Rrightarrow$ Stack of reps $\Rrightarrow$ Action groupoid $\Rrightarrow$ Geometric quantization

shows that representation theory can be understood as a *descent phenomenon*: the "global" representation of G is the coherent shadow of its local self-similar actions across the orbit. The stack $\mathcal{R}$ makes this descent explicit, providing both a conceptual framework and concrete computational tools.

In summary: We have shown that the stack-theoretic organization of local stabilizer representations provides a rigorous categorical foundation for the Kirillov orbit method and geometric quantization, unifying these classical theories within a single coherent picture.

10.5 Example: $G = \mathrm{SU}(2)$ and $\mathcal{O} \simeq \mathbb{C}P^1$

Let $T = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\} \cong \mathrm{U}(1)$ and fix $\chi_n : T \to \mathrm{U}(1)$ by $\chi_n(\mathrm{diag}(e^{i\theta}, e^{-i\theta})) = e^{in\theta}$ for $n \in \mathbb{Z}$. Then the rank-one section $S_{\chi_n} \in \mathcal{R}(\mathcal{O})$ corresponds to the associated line bundle

$$L_{\chi_n} = G \times^T \mathbb{C}_{\chi_n} \longrightarrow G/T \simeq \mathbb{C}P^1.$$

With the G-invariant connection constructed in Section 8 one has $F_{\nabla^{\chi_n}} = i\, \omega_{\mathrm{KKS}}$ and

$$c_1(L_{\chi_n}) = \frac{[\omega_{\mathrm{KKS}}]}{2\pi} = n.$$

In particular $L_{\chi_n} \simeq \mathcal{O}_{\mathbb{C}P^1}(n)$ and prequantization exists iff $n \in \mathbb{Z}$. Moreover

$$\Gamma^\infty(G/T, L_{\chi_n}) \;\cong\; \{\, f \in C^\infty(G) \mid f(gt) = \chi_n(t)^{-1} f(g) \,\} \;\cong\; \mathrm{Ind}_T^G(\chi_n^{-1}),$$

with G acting by left translations (Proposition 9.1). For $n \geq 0$, endowing $G/T \simeq \mathbb{C}P^1$ with its standard complex structure, one recovers Borel–Weil:

$$H^0(\mathbb{C}P^1, \mathcal{O}(n)) \;\cong\; V_n, \qquad \dim V_n = n + 1,$$

the unique irreducible representation of highest weight n (see, e.g., [4, 7]).

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