Wave Function Collapse as Lie Algebra Contraction: The SU(2) Spin-1/2 Paradigm

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Abstract

We propose a geometric mechanism for the emergence of discrete measurement outcomes in the Stern–Gerlach experiment, based on the Inönü–Wigner contraction of Lie algebras. The inhomogeneous magnetic field of the apparatus explicitly breaks the internal $SU(2)$ symmetry of the spin degree of freedom, inducing a continuous deformation of the spin algebra toward the Euclidean algebra $\mathfrak{e}(2)$. This contraction has three consequences: (i) the spherical coadjoint orbits of $SU(2)$ degenerate into the cylindrical orbits of $E(2)$, geometrically encoding the loss of transverse spin structure; (ii) finite-dimensional spin representations can contract only to one-dimensional characters of $E(2)$, providing a representation-theoretic origin for discrete outcomes; (iii) the Husimi distribution of an arbitrary spin state concentrates onto the poles of the sphere, corresponding to the eigenstates of the spin projection along the field axis.

We derive the contraction rate from the physical parameters of the Stern–Gerlach apparatus and show that typical experimental conditions place the system deep in the contracted regime. The framework explains the discreteness of outcomes and the selection of the preferred measurement basis without invoking environmental decoherence or stochastic collapse models. It does not derive the Born rule, and whether it accounts for single-outcome realization depends on interpretive assumptions that are discussed explicitly. The approach suggests that explicit symmetry breaking may play a more general role in quantum measurement than previously recognized.

Keywords: Quantum measurement Wave function collapse Inönü–Wigner contraction Symmetry breaking Stern–Gerlach experiment.

Contents

1 Introduction

The Stern–Gerlach experiment, performed in Frankfurt in 1922, remains one of the most striking demonstrations of quantum behavior [18]. A beam of silver atoms, each carrying a single valence electron, passes through an inhomogeneous magnetic field and splits into exactly two components, corresponding to the two possible projections of the electron's spin angular momentum along the field axis. The discreteness of this outcome—two spots, not a continuum—was among the first direct evidence for the quantization of angular momentum.

From the perspective of quantum foundations, however, the Stern–Gerlach experiment raises as many questions as it answers. An atom entering the apparatus in a superposition $|\psi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$ emerges in one of two definite states, with probabilities $|\alpha|^2$ and $|\beta|^2$. How does the continuous, deterministic evolution governed by the Schrödinger equation give rise to this discrete, probabilistic outcome? Why does the measurement select the eigenstates of S_z, rather than some other basis? And what physical process, if any, corresponds to the "collapse" of the wave function?

These questions constitute the quantum measurement problem. Standard formulations of quantum mechanics, from von Neumann's projection postulate to the Copenhagen interpretation, treat collapse as a primitive rule rather than a derived phenomenon. Subsequent approaches—environmental decoherence, many-worlds branching, dynamical collapse models—address different aspects of the problem but leave others open. Decoherence explains the suppression of interference without selecting a unique outcome; many-worlds eliminates collapse at the cost of ontological proliferation; dynamical models introduce stochastic terms whose physical origin remains obscure.

1.1 A geometric perspective

This paper proposes a different approach, grounded in the geometry of Lie algebras and their contractions. The starting observation is that quantum spin is not merely described by the algebra $\mathfrak{su}(2)$; it *is* a geometric structure whose state space is the coadjoint orbit $S^2 \subset \mathfrak{su}(2)^*$. Coherent states, which provide the closest quantum analogue to classical spin configurations, are points on this sphere. The full rotational symmetry of the spin degree of freedom is encoded in the non-commutativity of the algebra: $[S_x, S_y] = iS_z$ and cyclic permutations.

A Stern–Gerlach apparatus breaks this symmetry. The inhomogeneous magnetic field selects a preferred axis, and the interaction Hamiltonian $H = -\gamma B_z S_z$ privileges the generator S_z over its transverse counterparts. We argue that this symmetry breaking has a precise algebraic counterpart: the Inönü–Wigner contraction of $\mathfrak{su}(2)$ with respect to the subalgebra generated by S_z.

Under this contraction, the Lie algebra deforms continuously from $\mathfrak{su}(2)$ to the Euclidean algebra $\mathfrak{e}(2)$:

$$[S_z, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = 2S_z \quad \longrightarrow \quad [J, P_\pm] = \pm P_\pm, \qquad [P_+, P_-] = 0.$$

The transverse generators, which encoded the non-commutative structure of spin, become commuting translations. The spherical coadjoint orbits of $SU(2)$ degenerate into the cylindrical orbits of $E(2)$, and coherent states that were spread over the sphere concentrate onto the poles.

This geometric mechanism provides:

1. An explanation of *why* measurement outcomes are discrete: the representation theory of $E(2)$ admits only one-dimensional limits of finite-dimensional $SU(2)$ representations.

2. An explanation of *why* the preferred basis is the S_z eigenbasis: it is selected by the structure of the contraction, not postulated.

3. A quantitative prediction for the *rate* at which transverse coherence is suppressed, derived from the physical parameters of the apparatus.

The framework does not derive the Born rule from geometry alone, nor does it explain why a single outcome is realized in each trial. These limitations are discussed in Sec. 12. What it achieves is a decomposition of the measurement problem: certain features (discreteness, preferred basis, suppression of coherence) receive geometric explanations, while others (probabilistic weights, single-outcome selection) remain as residual interpretive questions.

1.2 Relation to previous work

The Inönü–Wigner contraction was introduced in 1953 as a systematic method for obtaining one Lie algebra as a limiting case of another [6]. The paradigmatic example—the contraction of the Poincaré algebra to the Galilei algebra as $c \to \infty$—shows how relativistic symmetry reduces to non-relativistic symmetry in an appropriate limit. Saletan extended the theory to more general contractions [15], and the geometric aspects were developed through the Kirillov–Kostant orbit method [8, 9].

The application of contraction ideas to quantum measurement is less developed. Geometric quantization provides coherent states on coadjoint orbits [13, 20], and the Berezin–Toeplitz quantization connects these to semiclassical limits [3], but the dynamical process by which measurement induces a contraction has not, to our knowledge, been systematically explored.

The present work fills this gap for the simplest non-trivial case: spin-1/2 in a Stern–Gerlach field. We derive the contraction from the structure of the interaction Hamiltonian, compute the rate of coherence suppression, and show how the limiting behavior of Husimi distributions matches the expected measurement outcomes.

Our approach is complementary to decoherence-based analyses of the Stern–Gerlach experiment [16, 17], which focus on the suppression of interference through environmental entanglement. The contraction mechanism operates at the level of the internal spin algebra and provides a symmetry-based account of basis selection that does not require an external bath. For other geometric perspectives on quantum mechanics and measurement, see [10, 1].

1.3 Outline

Section 2 reviews the $SU(2)$ algebra and its representations. Section 3 introduces coherent states and the Husimi distribution. Section 5 describes the coadjoint orbits of $SU(2)$. Section 6 clarifies the physical identification between internal spin and the magnetic moment. Section 7 presents the Inönü–Wigner contraction and its effect on the internal spin geometry. Section 8 analyzes the representation theory of $E(2)$ and the fate of $SU(2)$ representations under contraction. Section 9 discusses the physical interpretation of the contraction. Section 10 describes the collapse of Husimi distributions. Section 11 derives the contraction rate from the Stern–Gerlach Hamiltonian. Section 12 discusses the status of the Born rule and the single-outcome problem. Section 13 considers extensions, the relationship to hidden-variable theorems, and open questions. Section 14 summarizes our conclusions.

2 SU(2) Algebra, Representations, and Coadjoint Orbits

2.1 The Lie algebra $\mathfrak{su}(2)$

Definition 2.1 (su(2) algebra)**.** *The Lie algebra $\mathfrak{su}(2)$ consists of 2×2 traceless anti-Hermitian matrices:*

$$\mathfrak{su}(2) = \left\{ X \in M_2(\mathbb{C}) : X^\dagger = -X,\ \mathrm{Tr}(X) = 0 \right\}.$$

The Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k.$$

We work with the Hermitian generators

$$S_x = \frac{1}{2}\sigma_1, \qquad S_y = \frac{1}{2}\sigma_2, \qquad S_z = \frac{1}{2}\sigma_3,$$

so that $\{S_x, S_y, S_z\}$ is a basis of $\mathfrak{su}(2)$ with commutation relations

$$[S_x, S_y] = iS_z, \qquad [S_y, S_z] = iS_x, \qquad [S_z, S_x] = iS_y. \tag{1}$$

Remark 2.2. We use units with $\hbar = 1$ throughout. The generators S_i are Hermitian and correspond directly to spin observables.

2.2 Inner product and identification $\mathfrak{su}(2) \cong \mathfrak{su}(2)^*$

We use the Ad-invariant inner product

$$\langle X, Y \rangle := -\frac{1}{2}\mathrm{Tr}(XY), \qquad X, Y \in \mathfrak{su}(2). \tag{2}$$

A direct computation gives

$$\langle S_i, S_j \rangle = \frac{1}{4}\,\delta_{ij}.$$

Thus the orthonormal basis is

$$e_1 = 2S_x, \qquad e_2 = 2S_y, \qquad e_3 = 2S_z,$$

with $\langle e_i, e_j \rangle = \delta_{ij}$.

Via this inner product, we identify the dual space $\mathfrak{su}(2)^*$ with $\mathbb{R}^3$ by the correspondence

$$\xi \in \mathfrak{su}(2)^* \quad \longleftrightarrow \quad \xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \in \mathbb{R}^3.$$

2.3 Coadjoint action and orbits

The coadjoint action of $SU(2)$ on $\mathfrak{su}(2)^*$ is given by

$$\mathrm{Ad}_g^*(\xi) = \xi \circ \mathrm{Ad}_{g^{-1}}, \qquad g \in SU(2),\ \xi \in \mathfrak{su}(2)^*.$$

Under the identification $\mathfrak{su}(2)^* \cong \mathbb{R}^3$ via the Ad-invariant inner product, this becomes the standard rotation action:

$$g \cdot v = R_g v, \qquad R_g \in SO(3).$$

Proposition 2.3 (Coadjoint orbits of $SU(2)$)**.** *The coadjoint orbits of $SU(2)$ are:*

- Generic orbits: *spheres*

$$\mathcal{O}_r = \{\xi \in \mathbb{R}^3 : \|\xi\| = r\} \cong S_r^2, \qquad r > 0;$$

- Singular orbit: *the origin* $\{0\}$.

Each $\mathcal{O}_r$ is a homogeneous space

$$\mathcal{O}_r \cong SU(2)/U(1),$$

where $U(1)$ is the stabilizer of any point on the orbit.

2.4 Kostant–Kirillov–Souriau symplectic form

Each coadjoint orbit carries a natural symplectic structure.

Definition 2.4 (KKS form)**.** *For $\xi \in \mathcal{O}_r$, the Kostant–Kirillov–Souriau (KKS) 2-form is defined by*

$$\omega_r(X_\xi, Y_\xi) = \langle \xi, [X, Y] \rangle, \qquad X, Y \in \mathfrak{su}(2),$$

where X_ξ and Y_ξ are the fundamental vector fields generated by X, Y.

In Cartesian coordinates $\xi = (\xi_1, \xi_2, \xi_3)$ on $\mathbb{R}^3$, one obtains the explicit expression

$$\omega_r = \frac{1}{r} \left(\xi_1 \, d\xi_2 \wedge d\xi_3 + \xi_2 \, d\xi_3 \wedge d\xi_1 + \xi_3 \, d\xi_1 \wedge d\xi_2 \right).$$

This is, up to an overall factor, the standard area form on the sphere S_r^2.

Proposition 2.5 (Symplectic volume)**.** *The symplectic volume of $\mathcal{O}_r$ is*

$$\int_{\mathcal{O}_r} \frac{\omega_r}{2\pi} = 2r.$$

Sketch of proof. The usual area of S_r^2 is $4\pi r^2$. The KKS form integrates to $4\pi r$, so dividing by 2π yields $2r$.

2.5 Integrality and quantization levels

Theorem 2.6 (Kostant–Souriau integrality)**.** *A coadjoint orbit $\mathcal{O}_r$ admits a prequantum line bundle if and only if*

$$\frac{\omega_r}{2\pi} \in H^2(\mathcal{O}_r;\mathbb{Z}) \quad \Longleftrightarrow \quad r = \frac{j}{2}, \quad j \in \mathbb{Z}_{\geq 0}.$$

Thus the allowed radii r are quantized, and each integer $j \geq 0$ corresponds to the spin-$j/2$ representation of $SU(2)$.

For $j = 1$ (spin-1/2) one has

$$r = \frac{1}{2}, \qquad \mathcal{O}_{1/2} = S^2_{1/2},$$

which is quantized by the fundamental representation on

$$\mathbb{C}^2 = \operatorname{span}\{|\uparrow\rangle, |\downarrow\rangle\}.$$

3 Coherent States on SU(2) Coadjoint Orbits

3.1 Definition and properties

Definition 3.1 (SU(2) coherent states)**.** *For spin-j, the coherent state labelled by a unit vector $n = (n_1, n_2, n_3) \in S^2$ is defined as*

$$|n\rangle_j := U_n |j, j\rangle,$$

where:

- *$|j, j\rangle$ is the highest-weight state (eigenstate of iS_z with eigenvalue j),*
- *$U_n \in SU(2)$ is any element such that*

  $$U_n e_3 = n$$

  under the induced $SO(3)$ action.

For spherical coordinates

$$n = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),$$

one may take

$$U_n = e^{i\varphi S_z}\, e^{i\theta S_y}.$$

3.2 Spin-1/2 coherent states

For $j = \frac{1}{2}$, the highest weight state is $|1/2, 1/2\rangle = |\uparrow\rangle$. The coherent state takes the explicit form

$$|n\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle. \tag{3}$$

These states satisfy several key properties:

- **Overcompleteness:**

  $$\int_{S^2} |n\rangle\langle n|\ d\Omega(n) = 2\pi\,\mathbb{1}.$$

- **Non-orthogonality:**

  $$|\langle n|m|n|m\rangle|^2 = \frac{1 + n\cdot m}{2}.$$

- **Expectation values:** For the angular momentum vector operator $S = (S_x, S_y, S_z)$,

  $$\langle n|\ iS\ |n\rangle = \frac{1}{2}\,n.$$

  Thus the coherent state $|n\rangle$ points in the classical direction n.

3.3 Husimi probability measure

Definition 3.2 (Husimi (Q-) function). *For any state $|\psi\rangle \in \mathbb{C}^2$, the Husimi probability density on S^2 is*

$$Q_\psi(n) := |\langle n|\psi|n|\psi\rangle|^2.$$

It satisfies the normalization condition

$$\int_{S^2} Q_\psi(n)\,\frac{d\Omega(n)}{4\pi} = 1.$$

Interpretation. $Q_\psi(n)$ is the probability that a measurement of spin along direction n yields the value $+1/2$.

3.4 Classical limit and Ehrenfest theorem

For spin-j with $j \to \infty$, coherent states become sharply peaked. One has the asymptotic overlap

$$|\langle n_0|n|n_0|n\rangle|^{2j} \approx \exp\bigl(-j\, d^2(n, n_0)\bigr)\,,$$

where $d(\cdot,\cdot)$ is the geodesic distance on S^2.

In the limit $j \to \infty$,

$$Q_{|n_0\rangle}(n) \longrightarrow \delta_{S^2}(n - n_0),$$

recovering a classical deterministic angular momentum vector.

Key observation: This is the same type of concentration that will arise later from Lie algebra contraction, but *without requiring* $j \to \infty$. In our framework, the contraction dynamically induces a similar delta-concentration for spin-1/2.

4 Inönü–Wigner Contraction and Symmetry Breaking of $\mathfrak{su}(2)$

4.1 Motivation: symmetry breaking induced by measurement

In the Stern–Gerlach experiment the inhomogeneous magnetic field selects a preferred spatial direction, usually taken as the z–axis. This external field explicitly breaks the internal SU(2) symmetry of the spin degree of freedom, since the interaction Hamiltonian

$$H_{\text{int}} = -\mu\, B_z(z)\, S_z$$

privileges one generator (S_z) over the transverse ones (S_x, S_y). As the magnetic gradient increases, the dynamics becomes increasingly anisotropic: S_z retains its rôle as the axis of stable precession, while the transverse components S_x and S_y become dynamically suppressed.

This symmetry breaking is naturally and precisely described by the classical Inönü–Wigner contraction

$$\mathfrak{su}(2) \longrightarrow \mathfrak{e}(2),$$

which preserves rotations around z but flattens the x and y directions into translations in the Euclidean plane.

4.2 The Inönü–Wigner contraction

Let $0 < \varepsilon \le 1$ be a deformation parameter. Define the linear map $B(\varepsilon) : \mathfrak{su}(2) \to \mathfrak{su}(2)$ by

$$B(\varepsilon)S_x = \varepsilon S_x, \qquad B(\varepsilon)S_y = \varepsilon S_y, \qquad B(\varepsilon)S_z = S_z.$$

Using this map, introduce a deformed bracket on the *same* vector space $\mathfrak{su}(2)$ via

$$[X, Y]_\varepsilon := B(\varepsilon)^{-1}\big[\,B(\varepsilon)X,\ B(\varepsilon)Y\,\big].$$

Computing the commutators, one finds:

$$\begin{aligned}
[S_x, S_y]_\varepsilon &= B(\varepsilon)^{-1}\big[\varepsilon S_x, \varepsilon S_y\big] = B(\varepsilon)^{-1}(\varepsilon^2 S_z) = \varepsilon^2 S_z, \\
[S_z, S_x]_\varepsilon &= B(\varepsilon)^{-1}\big[S_z, \varepsilon S_x\big] = B(\varepsilon)^{-1}(\varepsilon S_y) = S_y, \\
[S_y, S_z]_\varepsilon &= B(\varepsilon)^{-1}\big[\varepsilon S_y, S_z\big] = B(\varepsilon)^{-1}(\varepsilon S_x) = S_x.
\end{aligned}$$

In the limit $\varepsilon \to 0$, the first bracket collapses:

$$[S_x, S_y]_\varepsilon \longrightarrow 0,$$

while the others converge to

$$[S_z, S_x]_0 = S_y, \qquad [S_y, S_z]_0 = S_x.$$

These are exactly the commutation relations of the two-dimensional Euclidean Lie algebra $\mathfrak{e}(2)$, with $J := S_z$ the generator of rotations in the plane and $P_1 := S_x$, $P_2 := S_y$ the two generators of translations:

$$[P_1, P_2] = 0, \qquad [J, P_1] = P_2, \qquad [J, P_2] = -P_1.$$

4.3 Physical interpretation: measurement-induced contracting symmetry

The parameter ε has a clear physical meaning in the presence of a Stern–Gerlach magnetic field. The gradient of the magnetic field along z induces a strong dynamical anisotropy:

$$\lambda \sim \frac{\mu B'}{\hbar v},$$

where $B' = \partial_z B$ is the field gradient and v is the particle velocity. Over the interaction time t, we may take

$$\varepsilon = e^{-\lambda t}.$$

Thus the limit $\varepsilon \to 0$ corresponds precisely to the regime of "infinitely strong" measurement (i.e. large magnetic gradient or long interaction time), in which the transverse spin components S_x and S_y become dynamically irrelevant.

As a consequence:

- the internal SU(2) symmetry of the spin is *explicitly broken* to its U(1) subgroup generated by S_z;
- the transverse generators S_x, S_y contract to translations, losing their non-commutative character;
- the effective internal symmetry algebra becomes $\mathfrak{e}(2)$, with a distinguished axis determined by the magnetic field.

This identification is *effective*: it describes the suppression of off-diagonal matrix elements in the interaction picture, not a fundamental change in the Lie algebra of the system.

In this way, the Inönü–Wigner contraction captures the essential mechanism by which the measurement apparatus selects a preferred direction and suppresses superpositions involving transverse spin components.

5 Coadjoint orbits of $SU(2)$

The internal spin degree of freedom is naturally described by the $\mathfrak{su}(2)$ Lie algebra acting on a $(2s+1)$–dimensional Hilbert space. Before analyzing the effect of the Stern–Gerlach field and its associated symmetry reduction, we summarize the geometric structure of the $\mathfrak{su}(2)$ coadjoint representation. This provides the reference configuration from which the contracted geometry emerges.

5.1 Coadjoint representation and invariants

Let $\mathfrak{su}(2)$ denote the Lie algebra of traceless skew-Hermitian 2×2 matrices with the standard basis $\{J_x, J_y, J_z\}$ satisfying

$$[J_x, J_y] = iJ_z, \qquad [J_y, J_z] = iJ_x, \qquad [J_z, J_x] = iJ_y. \tag{4}$$

We identify $\mathfrak{su}(2)^*$ with $\mathbb{R}^3$ via the pairing

$$\langle J_i, \xi\rangle = \xi_i, \qquad \xi = (\xi_x, \xi_y, \xi_z). \tag{5}$$

Under this identification, the coadjoint action coincides with the standard action of $SO(3)$ on $\mathbb{R}^3$:

$$\mathrm{Ad}^*_g(\xi) = R_g\,\xi, \qquad g \in SU(2),\ R_g \in SO(3). \tag{6}$$

A complete set of invariants is given by the Euclidean norm $\|\xi\|^2 = \xi_x^2 + \xi_y^2 + \xi_z^2$.

5.2 Orbit classification

Since the coadjoint action is rotational, each orbit is a sphere of radius $r > 0$:

$$\mathcal{O}_r = \{\xi \in \mathfrak{su}(2)^* : \xi_x^2 + \xi_y^2 + \xi_z^2 = r^2\}. \tag{7}$$

This coincides with the classical angular-momentum sphere of magnitude r. For a quantum spin-s representation, the relevant radius is

$$r_s = \sqrt{s(s+1)}, \tag{8}$$

representing the semiclassical limit of the spectrum of S^2. These spheres carry a natural symplectic form (the Kirillov–Kostant–Souriau form), but no nontrivial information about the Stern–Gerlach anisotropy is yet encoded at this level.

Remark 5.1. The radius $r_s = \sqrt{s(s+1)}$ is the semiclassical expectation value of the angular momentum magnitude, distinct from the quantized coadjoint orbit radius $r = s$ arising from the integrality condition. For large s, these coincide to leading order: $\sqrt{s(s+1)} \approx s$.

5.3 Role in the spin description

A spin state with maximal polarization along a direction $\mathbf{n} \in S^2$ corresponds to a point on $\mathcal{O}_{r_s}$ under the usual spin–coherent construction. Thus $\mathcal{O}_{r_s}$ parametrizes all possible internal orientations of the spin prior to the interaction with any external field.

In the Stern–Gerlach experiment, the field gradient breaks the full rotational symmetry and singles out a unique physical axis. The effect of this anisotropy is represented internally not by a deformation of the Hilbert space but by a deformation of the coadjoint geometry. The undeformed orbit $\mathcal{O}_{r_s}$ therefore serves as the reference configuration for the contraction analysis carried out in Sec. 7.

6 Internal Spin Structure and Its Physical Realization

6.1 Motivation

Spin is an internal quantum degree of freedom described by the $SU(2)$ algebra. Its components do not represent spatial rotations of a physical body and do not define a vector in real three-dimensional space. Nevertheless, a Stern–Gerlach apparatus couples to spin through a magnetic field and produces spatially separated output beams. This fact implies that only a specific *physical* quantity associated with spin interacts with the external field: the magnetic dipole moment.

For a particle of spin s, the magnetic moment is given by

$$\boldsymbol{\mu} = \gamma\,\mathbf{S}, \tag{9}$$

where γ is the gyromagnetic ratio and $\mathbf{S}$ is the spin operator. In contrast to $\mathbf{S}$, which acts on an internal Hilbert space, $\boldsymbol{\mu}$ is an observable vector in physical space, and the Stern–Gerlach interaction

$$H_{\text{int}} = -\boldsymbol{\mu} \cdot \mathbf{B} \tag{10}$$

makes the apparatus sensitive to its orientation. Thus the SG device does not "measure spin directions" in an abstract internal sense: it measures the spatial orientation of the magnetic dipole moment.

This observation provides the minimal and physically justified identification between internal spin degrees of freedom and real-space directions. Any deformation of the internal spin structure—in particular, the Inönü–Wigner contraction discussed below—is transferred to the physical space accessed by the magnetic moment through this linear identification. The contraction mechanism therefore leads to an effective restriction of accessible magnetic-moment configurations and supplies a physically meaningful mechanism for axis selection in the Stern–Gerlach experiment.

6.2 Internal structure

For completeness, we summarize the internal objects used later. The spin operators satisfy the $SU(2)$ commutation relations and act on a $(2s+1)$-dimensional Hilbert space. The set of maximally polarized spin states, obtained by rotating a highest-weight state with $SU(2)$, parametrizes the orientation of the internal spin degree of freedom. This parametrization is purely internal and does not yet refer to physical space. Internally, the spin degree of freedom is described by the $SU(2)$ algebra and its coadjoint representation. The orientations of spin in the semiclassical sense correspond to points on coadjoint orbits in $\mathfrak{su}(2)^*$, not to vectors in real space. The magnetic moment identification $\boldsymbol{\mu} = \gamma\mathbf{S}$ is therefore understood as a map from the coadjoint-orbit directions in $\mathfrak{su}(2)^*$ to physical directions in $\mathbb{R}^3$, which is the quantity actually probed by the SG apparatus.

The gyromagnetic identification

$$\boldsymbol{\mu} = \gamma\,\mathbf{S} \tag{11}$$

is the unique experimentally grounded bridge between the internal spin algebra and physical three-dimensional orientations. Through this identification, changes in the effective internal symmetry (such as the transition from $SU(2)$ to $E(2)$ produced by the SG field) manifest as corresponding restrictions on the physical configurations of the magnetic moment observed by the apparatus.

In the following sections, we build on this identification to interpret the Inönü–Wigner contraction as a physically meaningful deformation of the spin degree of freedom under the influence of the Stern–Gerlach magnetic field.

7 Inönü–Wigner contraction and its effect on internal spin geometry

7.1 Algebraic deformation

We recall the standard Inönü–Wigner (IW) contraction of $SU(2)$ with respect to the subalgebra generated by J_z. Let $\{J_x, J_y, J_z\}$ denote the standard basis of $\mathfrak{su}(2)$, satisfying

$$[J_x, J_y] = iJ_z, \qquad [J_y, J_z] = iJ_x, \qquad [J_z, J_x] = iJ_y. \tag{12}$$

For $\varepsilon > 0$ define the linear map

$$D_\varepsilon(J_x) = \varepsilon J_x, \qquad D_\varepsilon(J_y) = \varepsilon J_y, \qquad D_\varepsilon(J_z) = J_z, \tag{13}$$

and introduce the deformed generators

$$P_x := D_\varepsilon(J_x), \qquad P_y := D_\varepsilon(J_y), \qquad J := D_\varepsilon(J_z). \tag{14}$$

The deformed commutation relations are

$$[J, P_x] = iP_y, \qquad [J, P_y] = -iP_x, \qquad [P_x, P_y] = i\varepsilon^2 J. \tag{15}$$

In the limit $\varepsilon \to 0$, these converge to the Lie algebra of the Euclidean group $E(2)$,

$$[J, P_x] = iP_y, \qquad [J, P_y] = -iP_x, \qquad [P_x, P_y] = 0, \tag{16}$$

where J generates rotations around the distinguished axis and (P_x, P_y) generate translations in the transverse plane.[1] This is the algebraic signature of the loss of isotropy induced by a Stern–Gerlach field, which selects a preferred axis. (See [6, 15] for classical references.)

7.2 Dual action and coadjoint orbits

While the contraction acts directly on the algebra, the physically relevant geometry is the coadjoint geometry, i.e. the structure of the orbits in $\mathfrak{su}(2)^*$. The dual map

$$D_\varepsilon^* : \mathfrak{su}(2)^* \longrightarrow \mathfrak{su}(2)^*, \tag{17}$$

is defined by

$$\langle D_\varepsilon(X),\ \xi\rangle = \langle X,\ D_\varepsilon^*(\xi)\rangle. \tag{18}$$

From $D_\varepsilon(J_x) = \varepsilon J_x$, $D_\varepsilon(J_y) = \varepsilon J_y$, $D_\varepsilon(J_z) = J_z$, one immediately finds

$$D_\varepsilon^*(J_x^*) = \varepsilon^{-1} J_x^*, \qquad D_\varepsilon^*(J_y^*) = \varepsilon^{-1} J_y^*, \qquad D_\varepsilon^*(J_z^*) = J_z^*. \tag{19}$$

Thus the adjoint and coadjoint behaviour are opposite: the transverse generators contract in the algebra but blow up in the dual. This is the correct geometric setting for the deformation of coadjoint orbits (first clarified in the representation-theoretic literature; see e.g. [8, 13]).

A point of the original coadjoint orbit $\mathcal{O}_s \subset \mathfrak{su}(2)^*$ may be written as

$$\xi = x\, J_x^* + y\, J_y^* + z\, J_z^*, \qquad x^2 + y^2 + z^2 = r_s^2. \tag{20}$$

Under the deformed dual action,

$$D_\varepsilon^*(\xi) = \varepsilon^{-1} x\, J_x^* + \varepsilon^{-1} y\, J_y^* + z\, J_z^*. \tag{21}$$

[1] These are translations in the internal phase space of the spin degree of freedom, not spatial translations in $\mathbb{R}^3$.

Fixing r_s and letting $\varepsilon \to 0$, the image of the sphere under D_ε^* becomes an ellipsoid of increasing eccentricity in the transverse directions. The limit should be understood at the level of the algebra and its associated orbit structure, not as a pointwise limit of individual orbits. As the algebra contracts from $\mathfrak{su}(2)$ to $\mathfrak{e}(2)$, the relevant geometric structures change: the spherical coadjoint orbits of $SU(2)$ are replaced by the cylindrical coadjoint orbits of $E(2)$,

$$\mathcal{O}_\rho^{E(2)} = \{\xi \in \mathfrak{e}(2)^* : x^2 + y^2 = \rho^2\}, \tag{22}$$

together with the degenerate axis orbit $\mathcal{O}_0 = \{(0, 0, z) : z \in \mathbb{R}\}$.

Geometrically, the contraction "forgets" the compact structure of the sphere while preserving the axial coordinate z. Points on the original sphere with the same value of z but different azimuthal angles become indistinguishable in the contracted geometry (see Fig. 1). This is the internal geometric manifestation of axis selection: the isotropic spin sphere is replaced by a geometry in which only the axial direction carries dynamical significance.

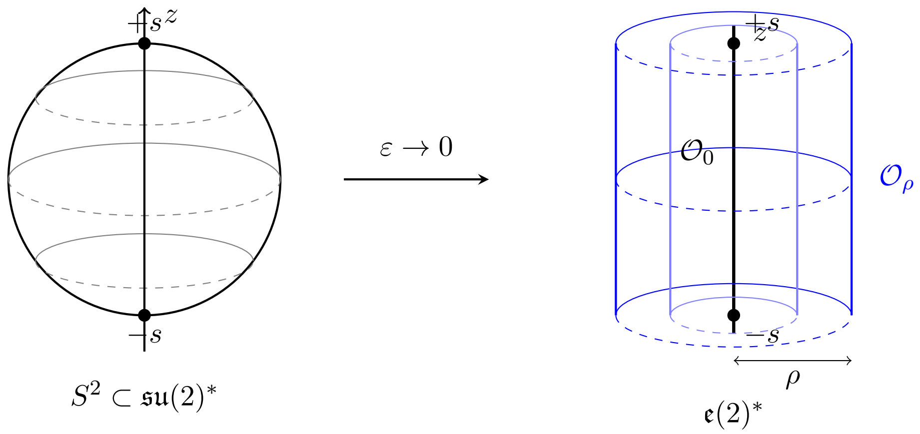

Figure 1: Contraction of coadjoint orbits under the Inönü–Wigner limit $\mathfrak{su}(2) \to \mathfrak{e}(2)$. Left: the spherical orbit S^2 parametrizing internal spin orientations. Right: the contracted geometry, consisting of cylindrical orbits $\mathcal{O}_\rho$ at fixed radius $\rho > 0$ and the degenerate axis orbit $\mathcal{O}_0$. Points at the same latitude on the sphere become indistinguishable in the contracted geometry; only the axial coordinate z (the spin projection) retains physical significance.

7.3 Compatibility with the physical magnetic moment

The physical Stern–Gerlach apparatus does not act on $\mathfrak{su}(2)^*$ directly, but through the magnetic moment

$$\boldsymbol{\mu} = \gamma \mathbf{S}, \tag{23}$$

which must remain finite in the contraction limit. Since the dual deformation D_ε^* dilates the transverse components as ε^{-1}, the physically relevant map is not $\boldsymbol{\mu} \circ D_\varepsilon^*$, but the compensated map

$$\Phi_\varepsilon := \Phi \circ (D_\varepsilon^*)^{-1}, \qquad \Phi(\xi) = \gamma\,(x, y, z). \tag{24}$$

This ensures that the physical dipole moment remains finite and that the contraction acts only on the internal symmetry structure. Concretely: as $\varepsilon \to 0$, the dual map D_ε^* sends transverse components to infinity, but the compensated map Φ_ε rescales them back, so that the image in physical space remains a bounded set of magnetic-moment orientations. Under Φ_ε, the deformed orbit converges to a finite cylindrical set of accessible magnetic-moment directions, with a unique preferred axis. This provides the geometric backbone for the anisotropic "collapse" associated with the Stern–Gerlach interaction, as discussed in the following sections.

8 Irreducible unitary representations of $E(2)$ and their role in the contraction limit

We recall the structure of the unitary dual of the two-dimensional Euclidean group $E(2) = \mathbb{R}^2 \rtimes SO(2)$, following the classical classification due to Itō and Mackey (seminal references in [7, 12]) and the exposition of Sugiura [19, Ch. 5]. Let $p : E(2) \to SO(2)$ denote the projection onto the rotation subgroup. Irreducible unitary representations fall into two disjoint classes.

(i) Principal series. For each $\lambda > 0$ there exists an infinite-dimensional irreducible unitary representation U^λ induced from a character of the translation subgroup, corresponding to the coadjoint orbit

$$\mathcal{O}_\lambda = \{\xi \in \mathfrak{e}(2)^* : x^2 + y^2 = \lambda^2\}, \tag{25}$$

which is a cylinder of radius λ. These representations exhaust the continuous part of the dual.

(ii) Discrete (one-dimensional) series. For each $n \in \mathbb{Z}$, the character

$$\chi_n : SO(2) \to U(1), \qquad \chi_n(R_\theta) = e^{in\theta}, \tag{26}$$

yields, by composition with the projection p, a one-dimensional irreducible representation $\chi_n \circ p$ of $E(2)$. These correspond to the degenerate coadjoint orbit

$$\mathcal{O}_0 = \{\xi \in \mathfrak{e}(2)^* : x = y = 0\}, \tag{27}$$

i.e. the axis generated by J^*. The full unitary dual is therefore

$$\widehat{E(2)} = \{\, U^\lambda \,|\, \lambda > 0 \,\} \;\cup\; \{\, \chi_n \circ p \,|\, n \in \mathbb{Z} \,\}, \tag{28}$$

and no two distinct elements of this set are equivalent [19, Theorem 2.1].

8.1 Contraction limit of finite-dimensional $SU(2)$ representations

A crucial observation for our purposes is that finite-dimensional unitary representations of $SU(2)$ *cannot* contract to principal-series representations of $E(2)$: the latter are necessarily infinite-dimensional and parametrized by a continuous invariant (λ). Under the Inönü–Wigner contraction $SU(2) \to E(2)$, a finite-dimensional representation may converge only to one of the one-dimensional discrete-series representations $\chi_n \circ p$. This follows from the general theory of representation contraction: the limit of a finite-dimensional unitary representation under a Lie algebra contraction can only be a finite-dimensional representation of the contracted group, and the only finite-dimensional unitary representations of $E(2)$ are the one-dimensional characters (see [19], Ch. 5, and [4] for the general theory).

This is the representation-theoretic counterpart of the coadjoint-orbit contraction established in Sec. 7: while the $SU(2)$ coadjoint sphere deforms into a family of cylindrical orbits of $E(2)$, the *representation* itself cannot follow the principal series and collapses onto the degenerate orbit $\mathcal{O}_0$. Hence the only admissible contraction limit of a spin-s representation is

$$\pi^{(s)} \quad \xrightarrow[\varepsilon \to 0]{} \quad \chi_n, \tag{29}$$

for some integer n determined by the surviving eigenvalue of the distinguished generator (in our normalization, the sign of the spin projection).

Physically, this corresponds to the fact that the Stern–Gerlach field selects a unique axis and suppresses all transverse structure: the internal $SU(2)$ degree of freedom collapses to a discrete choice of orientation along the field direction. The appearance of *integer*-labelled one-dimensional representations is not in conflict with the half-integer nature of the original spin:

the contraction removes the compact structure fixing the original spectrum, leaving only the residual $SO(2)$ character that encodes the preferred axis, not the full spin magnitude. In this sense, the IW contraction provides a representation-theoretic mechanism for the discrete "axis-selection" associated with the Stern–Gerlach collapse.

9 Physical interpretation of the $SU(2) \to E(2)$ contraction

The Stern–Gerlach apparatus singles out a unique spatial direction by means of a strong and highly anisotropic magnetic field. Internally, this anisotropy is reflected in the spin degree of freedom by a loss of isotropy: the transverse components of spin cease to play a symmetric dynamical role, while the projection along the field direction remains physically meaningful. The Inönü–Wigner (IW) contraction provides an algebraic mechanism to describe this symmetry reduction.

The internal $SU(2)$ algebra admits a one-parameter family of deformations that preserve the generator associated with the measurement axis and suppress the transverse commutator. As the deformation parameter approaches zero, the compact spin algebra loses its spherical symmetry and approaches the non-compact algebra $E(2)$, whose structure encodes a residual axial rotation and two commuting transverse generators. This algebraic signature matches the operational situation in a Stern–Gerlach device: all orientations that differ by transverse rotations become physically indistinguishable, whereas opposite orientations along the field direction remain dynamically distinct.

At the level of states, the coadjoint sphere of $SU(2)$—which parametrizes the internal spin orientations—does not survive the contraction as a smooth curved surface. Its transverse curvature is flattened, and the orbit degenerates into the union of cylindrical orbits and a single distinguished axis. This geometric collapse captures the experimentally observed reduction of spin orientations to a discrete set of outcomes aligned with the field.

Representation theory provides the complementary structural insight. Finite-dimensional spin representations cannot converge to the principal-series representations of $E(2)$, which are infinite dimensional. The only possible contraction limit is a discrete one-dimensional representation of $E(2)$, corresponding to the degenerate coadjoint orbit. Thus, the contraction mechanism forces the internal symmetry to collapse onto a single rotation character of the residual $SO(2)$ subgroup. In physical terms, the spin degree of freedom is reduced to a binary choice of orientation along the field axis, as observed in the Stern–Gerlach experiment.

In this sense, the IW contraction provides a symmetry-based mechanism for the emergence of discrete outcomes: it describes the loss of internal rotational degrees of freedom induced by the measurement field, leaving only the algebraic and geometric structure needed to support the final, axis-aligned spin value. To be precise: the Lie algebra of spin operators does not change as a mathematical structure. What changes is the *effective* algebra governing the accessible observables: under the SG dynamics, only combinations that commute with S_z retain physical significance, and the effective algebraic relations approach those of $\mathfrak{e}(2)$.

Connection to state-space collapse

The contraction $SU(2) \to E(2)$ describes the symmetry reduction induced by the Stern–Gerlach field at the level of generators and coadjoint orbits. To translate this algebraic deformation into the behaviour of quantum states, one must consider how the contraction acts on coherent-state overlaps and, consequently, on Husimi distributions.

Because coherent states parametrize the internal spin orientations, and because the dual contraction D_ε^* collapses the $SU(2)$ sphere onto the degenerate $E(2)$ axis–plus–cylinders structure, the Husimi distribution of any state becomes anisotropically deformed: the transverse dependence becomes irrelevant, while the longitudinal component is sharpened by the SG evolution.

This geometric contraction is therefore the structural origin of the δ-like Husimi profiles observed after the SG interaction. The next section formalizes this connection.

10 Anisotropic collapse of spin states: Husimi representation and limiting behaviour

Throughout this and the following sections, "collapse" refers to the concentration of the Husimi distribution onto discrete axis-aligned values under the effective dynamics induced by the SG field. This is a geometric and algebraic phenomenon within unitary quantum mechanics, not a non-unitary dynamical collapse in the sense of GRW-type models.

10.1 Husimi distributions for spin states

Let $\mathcal{H}_s$ be the spin-s Hilbert space with basis $\{|s,m\rangle : m = -s,\ldots,s\}$, and let $\{|\mathbf{n}\rangle : \mathbf{n} \in S^2\}$ denote the spin coherent states obtained by rotating the highest-weight vector $|s,s\rangle$ by an element sending the north pole to $\mathbf{n}$. The Husimi distribution of a state $\psi \in \mathcal{H}_s$ is the function

$$Q_\psi(\mathbf{n}) = |\langle \mathbf{n}|\psi\rangle|^2, \tag{30}$$

which is a smooth non-negative function on the sphere S^2. The Husimi representation is particularly natural in the coadjoint-orbit context: coherent states $|\mathbf{n}\rangle$ are in one-to-one correspondence with points on the orbit, and Q_ψ provides a positive (though not pointwise exact) phase-space representation of the quantum state, in contrast to the Wigner function which can take negative values. For coherent states $|\mathbf{n}_0\rangle$, Q_ψ is sharply peaked around $\mathbf{n}_0$ and its width is $O(1/s)$.

In the absence of an external field, the coherent-state manifold S^2 carries the full $SU(2)$ symmetry and the Husimi distribution encodes the rotational content of ψ. A Stern–Gerlach field breaks this symmetry explicitly and induces an anisotropic dynamics of Q_ψ, as we now describe.

10.2 Anisotropic deformation of the Husimi manifold

The internal IW contraction discussed in Sec. 7 has a direct effect on the Husimi representation. A point $\mathbf{n} \in S^2$ parametrizing a coherent state corresponds to a point on the coadjoint orbit of $SU(2)$ under the identification described in Sec. 6. Under the dual deformation D_ε^*, the coadjoint orbit is flattened in the transverse directions; geometrically,

$$S^2 \xrightarrow[\varepsilon\to 0]{\text{IW}} \text{cylindrical orbit} \,\cup\, \text{axis orbit}. \tag{31}$$

In particular, the latitude variable (the z–component) remains distinguished, while the longitudes lose their curvature and become dynamically indistinguishable. Operationally, this corresponds to an anisotropic spreading of the Husimi distribution in the transverse directions and a sharpening along the field axis.

10.3 Dynamics induced by the SG field

The Stern–Gerlach Hamiltonian is

$$H_{\text{SG}} = -\gamma B_z S_z, \tag{32}$$

with B_z strongly inhomogeneous along the beam direction. During the interaction time τ, the unitary evolution $\psi(\tau) = e^{-iH_{\text{SG}}\tau}\psi(0)$ suppresses the relative phases between distinct m-components. As a consequence, the transverse components of the spin operator lose dynamical relevance: only the diagonal contributions in the $|s,m\rangle$ basis survive.

In the Husimi representation, this corresponds to the evolution

$$Q_\psi(\mathbf{n},\tau) = |\langle \mathbf{n}| e^{-iH_{\mathrm{SG}}\tau}\psi(0)\rangle|^2 = \sum_{m,m'} e^{-i\gamma B_z \tau(m-m')}\psi_m \overline{\psi_{m'}}\langle \mathbf{n}|s,m\rangle\langle s,m'|\mathbf{n}\rangle. \tag{33}$$

For sufficiently large field gradients and typical SG times, the oscillatory factors suppress all terms with $m \neq m'$, leaving

$$Q_\psi(\mathbf{n},\tau) \;\approx\; \sum_{m=-s}^{s} |\psi_m|^2\, |\langle \mathbf{n}|s,m\rangle|^2. \tag{34}$$

Thus the Husimi distribution loses its phase information and becomes diagonal in m, which precisely matches the action of the contracted algebra: the transverse structure is suppressed, whereas the projection m along the field axis remains meaningful.

10.4 Limiting profile under contraction

The IW contraction enforces an anisotropic limit on the coherent-state overlaps. As $\varepsilon \to 0$, the overlap $|\langle \mathbf{n}|s,m\rangle|^2$ loses its dependence on the azimuthal angle φ and retains only the dependence on the axial coordinate:

$$|\langle \mathbf{n}|s,m\rangle|^2 \xrightarrow[\varepsilon\to 0]{} f_m(\cos\theta), \tag{35}$$

where f_m depends only on the polar coordinate $\cos\theta$. For the extremal projections $m = \pm s$, the functions $f_{\pm s}$ are sharply concentrated near the poles $\cos\theta = \pm 1$; in the semiclassical limit $s \to \infty$, they converge to delta functions at the poles (see e.g. [13, 11, 14]).

For finite s, the width of these peaks is $O(1/\sqrt{s})$. Combining this with (34), we obtain the limiting behaviour

$$Q_\psi(\mathbf{n},\tau) \xrightarrow[\varepsilon\to 0]{} \sum_{m=-s}^{s} |\psi_m|^2\, \delta(\cos\theta - m/s), \tag{36}$$

where the convergence is understood in the following sense: for any continuous function $g\colon [-1,1] \to \mathbb{R}$ depending only on the axial coordinate,

$$\int_{S^2} Q_\psi(\mathbf{n},\tau)\, g(\cos\theta)\, d\Omega \;\longrightarrow\; \sum_{m=-s}^{s} |\psi_m|^2\, g(m/s) \qquad \text{as } \tau \to \infty. \tag{37}$$

This expresses the concentration of the Husimi distribution onto the discrete set of axial eigenvalues, while the azimuthal dependence becomes uniformly distributed and hence irrelevant. For a spin-1/2 system, this reduces to the binary outcome

$$Q_\psi(\mathbf{n},\tau) \to |\psi_+|^2\, \delta(\cos\theta - 1) + |\psi_-|^2\, \delta(\cos\theta + 1). \tag{38}$$

This is the operational manifestation of the anisotropic collapse: the transverse directions become uniformly irrelevant, while the longitudinal coordinate converges to one of the extremal eigenvalues of S_z.

10.5 Interpretation

The Husimi distribution provides a state-space picture of the collapse induced by the SG field. The IW contraction translates the loss of transverse coherence into a deformation of the internal spin geometry: all orientations differing by a transverse rotation become identified, and the internal degree of freedom collapses onto a discrete axis-aligned character of the contracted symmetry. The limiting delta-like Husimi profiles express this reduction as a distributional concentration on the extremal spin projections, in agreement with the representation- theoretic limit described in Sec. 8.

11 Dynamical implementation of the contraction in the SG apparatus

11.1 Hamiltonian structure

The Stern–Gerlach apparatus subjects the particle to an inhomogeneous magnetic field $\mathbf{B} = (0, 0, B_z(z))$, typically with a strong and monotonic gradient. The spin degree of freedom evolves under the Hamiltonian

$$H_{\mathrm{SG}} = -\gamma\, B_z(z)\, S_z. \tag{39}$$

Since B_z varies along the beam direction, the interaction is effectively time-dependent in the particle's rest frame. For a wave packet with mean velocity v, one may approximate $B_z(z(t)) \approx B_0 + B'vt$ during the interaction time $\tau = L/v$. The evolution operator reads

$$U(\tau) = \exp\left(i\gamma S_z \int_0^\tau B_z(t)\, dt \right). \tag{40}$$

The accumulated phase difference between different spin projections is therefore

$$\Delta\phi_{m,m'} = \gamma(m - m') \int_0^\tau B_z(t)\, dt. \tag{41}$$

When B' is sufficiently large, these phases become rapidly oscillatory and suppress all off-diagonal terms in the density matrix in the S_z basis. This is the dynamical implementation of the anisotropic contraction: transverse spin components cease to contribute to any observable effected by the SG device.

11.2 Effective contraction rate

The characteristic suppression rate is determined by the total phase difference accumulated during the interaction. For a field that varies approximately linearly, $B_z(z) \approx B_0 + B'z$, a particle traversing the apparatus accumulates a phase difference between spin components

$$\Delta\phi_{m,m'} \approx \gamma(m - m')\bar{B}\tau, \tag{42}$$

where $\bar{B}$ is the effective mean field experienced by the particle. The dimensionless suppression parameter is therefore

$$\lambda \;\equiv\; \gamma\bar{B}\tau. \tag{43}$$

It is this parameter that controls the extent to which transverse spin coherences survive the passage through the magnet. For typical SG parameters ($\bar{B} \sim 1$ T, $\tau \sim 10^{-4}$ s, $\gamma_e \approx 1.76 \times 10^{11}$ rad/(s·T)), one finds $\lambda \sim 10^7$, ensuring that the off-diagonal terms are suppressed well before the wave packet reaches the detector.

In the language of Sec. 7, the dynamics generated by H_{SG} implements a flow toward the contracted algebra: transverse generators are rendered dynamically ineffective, while S_z remains the unique physically relevant spin observable during the interaction. Thus the IW contraction is not only a structural descriptor but a dynamical effect.

11.3 Momentum transfer and spatial separation

The force exerted by the inhomogeneous field is

$$F_z = \frac{d}{dz}(\boldsymbol{\mu} \cdot \mathbf{B}) = \gamma\, m\, B', \tag{44}$$

yielding a momentum kick

$$\Delta p_z = \gamma m B' \tau. \tag{45}$$

This depends only on the longitudinal projection m and not on any transverse spin component, consistent with the suppression derived above. As a result, wave packets associated with different m separate spatially along z with no residual transverse structure, thereby manifesting the axis selection predicted by the contraction.

For spin-1/2, this yields the familiar two-beam pattern, whereas higher spin values produce $2s + 1$ packets whose intensities are determined by the initial weights $|\psi_m|^2$.

It is worth emphasizing that the Stern–Gerlach apparatus functions as a *momentum analyzer*: the observable quantity is the deflection of the beam, which is determined by the momentum transfer Δp_z, not by any direct measurement of position. Each atom follows a well-defined classical trajectory through the device, with the quantum discreteness manifesting only in the allowed values of the momentum kick. This point, often obscured in textbook presentations, is discussed in detail in [18].

11.4 Comparison with decoherence

The suppression of transverse components described above is a *unitary* effect generated by H_{SG} and should not be confused with environmental decoherence. No bath degrees of freedom are involved; the loss of transverse coherence arises because the SG Hamiltonian amplifies phase differences between m-components at a rate proportional to the field gradient. The contraction thereby describes an intrinsic, symmetry-driven mechanism rather than an environmentally induced loss of coherence.

This distinction is operationally important: the SG device enforces a preferred axis via the explicit symmetry breaking $SU(2) \to E(2)$, not via coupling to an external bath. Decoherence may occur on longer timescales and, in realistic experimental conditions, typically contributes to the suppression of interference between spatially separated beams. The present analysis addresses the spin degree of freedom in isolation; a complete account of the SG experiment would include the entanglement between spin and motional degrees of freedom and, ultimately, the environment. What the contraction mechanism provides is a symmetry-based explanation of axis selection that complements, rather than replaces, decoherence-based accounts.

11.5 Synthesis

The SG dynamics provides the physical realization of the IW contraction: the inhomogeneous field suppresses the effect of transverse spin generators, enforces S_z-diagonal structure, and produces spatial separation proportional to the representation label m. Combined with the representation-theoretic limit described in Sec. 8, this yields a fully consistent geometric and dynamical picture of spin “collapse” as anisotropic symmetry reduction.

12 The Born rule and the selection problem

The preceding sections have established a geometric mechanism by which the Inönü–Wigner contraction produces discrete, axis-aligned outcomes from an initially isotropic spin state. This section examines more carefully what the framework explains, what it assumes, and what it leaves open regarding the probabilistic structure of quantum measurement.

12.1 What the framework derives

Three features of Stern–Gerlach measurement emerge from the contraction mechanism without additional postulates:

1. *Discreteness of outcomes.* The representation theory of $E(2)$ admits only one-dimensional representations arising as limits of finite-dimensional $SU(2)$ representations (Sec. 8). This

forces the spin degree of freedom into one of finitely many axis-aligned configurations. For spin-1/2, there are exactly two such configurations.

2. *Selection of the preferred basis.* The measurement axis is not imposed by fiat but arises from the structure of the interaction. The Stern–Gerlach field selects the subalgebra with respect to which the contraction is performed, and the eigenstates of S_z are precisely the states that survive as characters of the contracted algebra.

3. *Suppression of transverse coherence.* The off-diagonal elements in the S_z basis correspond to transverse spin components, which lose their algebraic significance under contraction. The geometric content of this suppression is the flattening of the coadjoint sphere into cylindrical orbits where azimuthal distinctions become meaningless.

These three features address the *preferred basis problem*: why does measurement produce eigenstates of S_z rather than arbitrary superpositions? The answer is that the contracted algebra $\mathfrak{e}(2)$ simply does not have room for such superpositions as coherent structures.

12.2 What the framework assumes

Two elements of the standard quantum-mechanical account are not derived but imported into the geometric description:

1. *The Husimi distribution represents probabilities.* The function $Q_\psi(\mathbf{n}) = |\langle \mathbf{n}|\psi\rangle|^2$ is taken to represent a probability distribution over internal configurations. This is consistent with the general principles of geometric quantization, where the symplectic structure on coadjoint orbits induces a natural measure, but the *interpretation* of this measure as encoding measurement probabilities is an assumption inherited from quantum mechanics rather than derived from geometry alone.

2. *The limiting weights become outcome frequencies.* After contraction, the Husimi distribution concentrates at the poles with weights $|\psi_+|^2$ and $|\psi_-|^2$. That these weights correspond to the relative frequencies of measurement outcomes is the content of the Born rule. The framework provides a geometric *representation* of this rule—the weights appear as the masses of the delta functions at the poles—but does not provide a geometric *derivation* of why squared amplitudes yield probabilities.

It is important to be precise about the status of these assumptions. The framework does not contradict the Born rule, nor does it render it mysterious; it simply does not derive it. The Born rule enters as the statement that the Husimi distribution, which has a clear geometric meaning as a function on the coadjoint orbit, also has a probabilistic meaning as a distribution over measurement outcomes.

12.3 The single-outcome problem

Even granting the Born rule, a conceptual gap remains: the framework describes the *structure* of possible outcomes and their *statistical weights*, but does not explain why a single outcome is realized in any given instance.

After contraction, the Husimi distribution has the form

$$Q_\psi \;\longrightarrow\; |\psi_+|^2\,\delta(\cos\theta - 1) \;+\; |\psi_-|^2\,\delta(\cos\theta + 1).$$

This expression describes a distribution over two possibilities, not a definite configuration. The transition from "probability p of spin-up" to "spin-up was observed" is not captured by the contraction mechanism.

This is not a defect peculiar to the present approach. The single-outcome problem—why one result rather than another occurs in an individual trial—is common to all interpretations of quantum mechanics that do not supplement the formalism with additional structure (such as hidden variables or branching worlds). What the contraction framework achieves is a reduction of the measurement problem: it explains why the possible outcomes are discrete and axis-aligned, leaving only the selection among these outcomes as the residual interpretive question.

12.4 The status of internal geometry

The coadjoint orbit $S^2 \subset \mathfrak{su}(2)^*$ has been described as the "internal geometry" of the spin degree of freedom. This language requires clarification.

The sphere S^2 is not a submanifold of physical space $\mathbb{R}^3$. It is a geometric representation of the algebraic structure of spin observables: the non-commutative algebra $\mathfrak{su}(2)$ is encoded in the symplectic geometry of its coadjoint orbits via the Kirillov–Kostant correspondence. A point on this sphere does not represent a position in space but a configuration of the internal degree of freedom.

The coupling $\boldsymbol{\mu} = \gamma \mathbf{S}$ establishes a map from internal geometry to physical space, but this map is only activated at the moment of interaction with the apparatus. Prior to measurement, the internal configuration has no direct spatial manifestation.

This distinction is crucial for the relationship with hidden-variable theorems (see Sec. 13.5). The internal geometry is not a spatially localized property that could serve as a local hidden variable in the sense of Bell's theorem. It is a feature of the symmetry structure of the quantum system.

12.5 Epistemic and ontological readings

The Husimi distribution Q_ψ admits two broad classes of interpretation, corresponding to different views about the completeness of quantum-mechanical description.

Epistemic reading. The quantum state $|\psi\rangle$ is a complete description of the physical situation. The Husimi distribution reflects our knowledge of the internal configuration, and this knowledge is as complete as nature allows. The probabilities $|\psi_\pm|^2$ are fundamental and irreducible; there is no further fact that determines which outcome occurs. On this view, the single-outcome problem is not a gap in our understanding but a reflection of genuine indeterminacy in nature.

Ontological reading. The internal geometry may contain structure beyond what is encoded in $|\psi\rangle$. The Husimi distribution then represents partial knowledge of a more fine-grained configuration. On this view, there is a fact about which pole the system "really" occupies, but this fact is not accessible to us prior to measurement. The contraction mechanism reveals this pre-existing fact by eliminating the geometric structures that previously obscured it.

The present framework is compatible with both readings. It provides a geometric arena—the coadjoint orbit and its contraction—in which either interpretation can be formulated, but it does not adjudicate between them.

12.6 A note on "ontological ignorance"

The phrase "ontological ignorance"—meaning knowledge of what does *not* exist in reality—borders on self-contradiction. If we know that no additional structure exists beyond $|\psi\rangle$, we are not ignorant; we have positive knowledge of completeness. Conversely, if additional structure exists but is inaccessible, our ignorance is epistemic, not ontological.

The point is not merely semantic. Claims about the completeness or incompleteness of quantum-mechanical description are substantive physical assertions. The contraction framework clarifies the geometry of measurement but does not settle these foundational questions. It offers a precise language—coadjoint orbits, symplectic reduction, Husimi distributions—in which the questions can be posed, without prejudging the answers.

12.7 Summary

The Inönü–Wigner contraction provides:

- a geometric mechanism for the discretization of measurement outcomes,
- a dynamical explanation of preferred-basis selection,
- a representation of the Born-rule weights as geometric concentrations on the contracted orbit.

It does not provide:

- a derivation of why squared amplitudes yield probabilities,
- an account of why a single outcome is realized in each trial,
- a resolution of the epistemic/ontological status of the quantum state.

These limitations are not evasions but honest acknowledgments of what remains open. The value of the framework lies in decomposing the measurement problem into distinct components and resolving some of them geometrically, thereby sharpening the questions that remain.

13 Generalizations and Open Questions

The Inönü–Wigner contraction of the spin algebra provides a geometric mechanism for the collapse of the spin state in the presence of an anisotropic measurement interaction (such as the inhomogeneous field of a Stern–Gerlach apparatus). In this section we discuss how this mechanism extends to higher-spin systems, composite systems, and more general contractions, and how it relates to decoherence and foundational questions.

13.1 Higher-spin representations

For $\mathfrak{su}(2)$ the spin-s representation corresponds to the coadjoint orbit S^2_s. Under the IW contraction the orbit degenerates to a cylinder:

$$S^2_s \longrightarrow \mathcal{O}^{\mathfrak{e}(2)}_s = \{(p_1, p_2, j) : p_1^2 + p_2^2 = \rho^2\}.$$

The collapse mechanism remains structurally the same:

- coherence is suppressed in the (S_x, S_y) directions,
- S_z survives as the generator of the residual symmetry,
- the measurement outcomes correspond to $m = -s, -s+1, \ldots, s-1, s$, yielding $2s+1$ discrete values.

For spin-1/2, this gives the familiar two-beam pattern; for spin-1, three beams; and so on. The contraction rate λ depends on the apparatus parameters but not on the spin value s, so the timescale for suppression of transverse coherence is independent of s for fixed experimental conditions.

13.2 Entangled systems

Consider a bipartite system with Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and internal symmetry $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

If a Stern–Gerlach apparatus acts only on the first subsystem, the interaction produces the contraction

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \longrightarrow \mathfrak{e}(2) \oplus \mathfrak{su}(2),$$

that is, the first spin collapses anisotropically while the second remains unaffected.

For example, consider the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\big).$$

After the contraction (on the first subsystem) the Husimi distribution of the first spin collapses onto $j = \pm r$ while the second spin retains its full $\mathfrak{su}(2)$ symmetry.

If both spins undergo measurement along the same axis, the contraction is applied independently to both components:

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \longrightarrow \mathfrak{e}(2) \oplus \mathfrak{e}(2),$$

and the correlations encoded in $|\Phi^+\rangle$ guarantee that the collapse outcomes remain perfectly correlated.

The case of measurements along nonparallel axes raises nontrivial questions, since the two IW contractions privilege different axes. Analyzing the compatibility of such contractions with EPR correlations is an interesting open problem. Any such extension must respect Bell-type no-go theorems: the internal geometry cannot function as a local hidden variable that would allow violation of Bell inequalities to be explained classically. The framework developed here for single-particle spin does not conflict with Bell's theorem (see Sec. 13.5), but its generalization to entangled systems requires careful analysis.

13.3 Other contractions of $\mathfrak{su}(2)$

Besides the IW contraction toward $\mathfrak{e}(2)$, two other contractions of $\mathfrak{su}(2)$ are frequently considered:

1. *Heisenberg-type contraction*, in which the commutator $[S_x, S_y]$ is preserved while other brackets vanish. This leads to a non-compact central extension, but it is not relevant to Stern–Gerlach dynamics.

2. *Isotropic contraction* $B(t) = e^t\mathbf{1}$, which collapses all commutators uniformly and produces the abelian algebra $\mathbb{R}^3_{\mathrm{ab}}$. As discussed earlier, this corresponds to $\hbar_{\mathrm{eff}} \to 0$ and leads to a complete classicalization of the spin degree of freedom. Such an isotropic collapse is too strong to describe SG but may be useful in more foundational contexts.

The IW contraction is therefore the unique contraction compatible with the anisotropic symmetry breaking induced by the SG apparatus.

13.4 Decoherence vs. symmetry breaking

Decoherence suppresses off-diagonal elements in the density matrix by entangling the system with an external environment. However:

- decoherence does not select a unique outcome,

- nor does it determine a preferred basis unless the Hamiltonian already singles out a direction,

- and the resulting state is typically a classical mixture, not a pure eigenstate.

In contrast, the IW contraction affects the *structure* of the observable algebra:

$$\mathfrak{su}(2) \longrightarrow \mathfrak{e}(2),$$

suppressing non-commutativity in two directions and preserving only the axis selected by the apparatus.

The measurement direction (z) becomes an intrinsic feature of the contracted algebra, providing a geometric explanation of the preferred basis and enabling actual collapse rather than mere loss of coherence.

Thus decoherence and contraction are complementary:

- decoherence eliminates interference in the spatial degrees of freedom;
- the IW contraction rigidifies the internal algebra, enforcing one-dimensional outcomes.

13.5 Relation to hidden-variable theorems

The coadjoint sphere $S^2 \subset \mathfrak{su}(2)^*$ should not be confused with a submanifold of physical space $\mathbb{R}^3$. It represents the internal symmetry structure of the spin degree of freedom—a geometric encoding of the non-commutative algebra of spin observables. The "direction" on this sphere is an internal geometric datum, not a local hidden variable in the sense of Bell's theorem [2], which excludes pre-existing properties defined in spatially separated regions.

The coupling $\boldsymbol{\mu} = \gamma \mathbf{S}$ establishes a correspondence between the internal geometry and physical space only at the moment of interaction with the apparatus. Prior to this interaction, the internal configuration is not a spatial property that could be used to construct a local hidden-variable model. Consequently, the present framework does not conflict with Bell-type no-go theorems.

Whether the internal geometry contains structure beyond what is encoded in the quantum state $|\psi\rangle$ is a question the present framework does not settle. The Husimi distribution Q_ψ describes our knowledge of the internal configuration; whether this knowledge is complete or partial remains open. What the framework does provide is a geometric mechanism for the emergence of discrete outcomes and the preferred basis, grounded in the symmetry reduction $\mathfrak{su}(2) \to \mathfrak{e}(2)$.

13.6 Open questions

Several conceptual and mathematical questions remain open:

- *Compatibility with relativistic spin.* Can a similar contraction be defined for the Poincaré algebra?
- *Multiple measurement axes.* How do two IW contractions along different axes interact? This is relevant for entangled systems with separate SG magnets.
- *Foundational meaning of isotropic collapse.* The full abelianization $\mathfrak{su}(2) \to \mathbb{R}^3_{\mathrm{ab}}$ may model $\hbar \to 0$; its physical significance deserves further study.
- *Geometric quantization of* $\mathfrak{e}(2)$. Coherent states adapted to cylindrical orbits require a nontrivial analysis and may relate to plane-wave limits.
- *Extensions to many-body systems.* Understanding how symmetry contraction interacts with entanglement and collective measurements is a natural next step.

14 Summary and outlook

The analysis presented in this work develops a symmetry–based account of the Stern–Gerlach measurement, grounded on the Inönü–Wigner contraction $\mathfrak{su}(2) \to \mathfrak{e}(2)$. The central observation is that the inhomogeneous SG field, by explicitly selecting a preferred spatial axis, induces a reduction of the internal spin symmetry. This reduction is captured, at the algebraic level, by the contraction of the $\mathfrak{su}(2)$ commutation relations toward those of the Euclidean algebra $\mathfrak{e}(2)$; at the geometric level, by the collapse of the $SU(2)$ coadjoint sphere into the degenerate orbit structure of $E(2)$; and at the representation level, by the fact that a finite-dimensional spin representation can contract only to a one-dimensional character of $E(2)$.

Through the gyromagnetic identification $\boldsymbol{\mu} = \gamma \mathbf{S}$, this internal contraction is transferred to the physical space probed by the SG apparatus. The resulting anisotropic suppression of transverse spin components, together with the dynamical amplification of longitudinal phase differences generated by the interaction Hamiltonian $H_{\mathrm{SG}} = -\gamma B_z S_z$, leads to a state-space profile in which the Husimi distribution concentrates on the extremal values of the spin projection along the field direction. This provides a structural and dynamical mechanism for the emergence of discrete outcomes, without invoking environmental decoherence or stochastic collapse models.

The framework developed here offers several directions for further investigation. First, the contraction picture may be extended to higher-spin systems, where the richness of the $E(2)$ representation theory could allow for a finer classification of measurement-induced symmetry reduction. Second, the interplay between the contraction mechanism and entanglement deserves systematic study, particularly in multi-particle SG configurations and in setups involving dynamically rotated magnetic fields. Third, the present analysis suggests that explicit symmetry breaking may play a more general role in quantum measurement; it would be natural to explore analogous contraction-driven mechanisms in other contexts where a preferred basis emerges operationally.

Finally, while the present work focuses on the internal spin algebra, the general strategy—to represent measurement as a deformation of the accessible symmetry structure—could be applied to a broader class of quantum systems. Whether such an approach can shed light on the interface between coherent unitary dynamics and the appearance of definite outcomes remains an open and intriguing question.

A Mathematical background

This appendix collects standard material on coadjoint orbits, coherent states, and Lie algebra contractions. The presentation follows [8, 13, 20]; the reader familiar with geometric quantization may skip to Appendix B.

A.1 Coadjoint orbits and symplectic structure

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *coadjoint action* of G on the dual space $\mathfrak{g}^*$ is defined by

$$\langle \mathrm{Ad}^*_g \xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle,$$

where $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ denotes the natural pairing.

The *coadjoint orbit* through $\xi_0 \in \mathfrak{g}^*$ is the set

$$\mathcal{O}_{\xi_0} = \{ \mathrm{Ad}^*_g \xi_0 : g \in G \}.$$

Each coadjoint orbit carries a canonical symplectic structure, the *Kirillov–Kostant–Souriau form*, defined at $\xi \in \mathcal{O}_{\xi_0}$ by

$$\omega_\xi(X^\#, Y^\#) = \langle \xi, [X, Y] \rangle,$$

where $X^{\#}$ denotes the fundamental vector field generated by $X \in \mathfrak{g}$.

For $G = SU(2)$, the Lie algebra $\mathfrak{su}(2)$ can be identified with $\mathbb{R}^3$ via the basis $\{J_x, J_y, J_z\}$ satisfying

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.$$

The coadjoint orbits are spheres S_r^2 of radius $r > 0$ centered at the origin, together with the trivial orbit $\{0\}$. The symplectic form on S_r^2 coincides (up to normalization) with the standard area form.

A.2 Coherent states and the Husimi representation

For each spin value $s \in \frac{1}{2}\mathbb{Z}_{>0}$, the Hilbert space $\mathcal{H}_s$ carries an irreducible representation of $SU(2)$ with basis $\{|s, m\rangle : m = -s, \ldots, s\}$.

The *spin coherent states* are defined by rotating the highest-weight vector:

$$|\mathbf{n}\rangle = R(\mathbf{n})|s, s\rangle,$$

where $R(\mathbf{n}) \in SU(2)$ is any rotation sending the north pole $\hat{z}$ to $\mathbf{n} \in S^2$. The state $|\mathbf{n}\rangle$ is well-defined up to a phase.

Coherent states satisfy the resolution of identity

$$\frac{2s+1}{4\pi} \int_{S^2} |\mathbf{n}\rangle\langle\mathbf{n}| \, d\Omega = \mathbf{1},$$

and provide an overcomplete basis for $\mathcal{H}_s$.

The *Husimi distribution* (or Q-function) of a state $|\psi\rangle$ is

$$Q_\psi(\mathbf{n}) = |\langle\mathbf{n}|\psi\rangle|^2.$$

This is a smooth, non-negative function on S^2 satisfying

$$\frac{2s+1}{4\pi} \int_{S^2} Q_\psi(\mathbf{n}) \, d\Omega = 1.$$

For a coherent state $|\mathbf{n}_0\rangle$, the Husimi distribution is peaked at $\mathbf{n}_0$ with width $O(1/\sqrt{s})$.

The overlap between coherent states is given by

$$|\langle\mathbf{n}|\mathbf{n}'\rangle|^2 = \left(\frac{1 + \mathbf{n} \cdot \mathbf{n}'}{2}\right)^{2s}, \tag{46}$$

which for large s becomes sharply peaked around $\mathbf{n} = \mathbf{n}'$.

A.3 Inönü–Wigner contraction: general definition

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. Choose a complementary subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

For $\varepsilon > 0$, define the linear map $D_\varepsilon : \mathfrak{g} \to \mathfrak{g}$ by

$$D_\varepsilon(X) = X \quad \text{for } X \in \mathfrak{h}, \qquad D_\varepsilon(Y) = \varepsilon Y \quad \text{for } Y \in \mathfrak{m}.$$

The *deformed bracket* is

$$[X, Y]_\varepsilon = D_\varepsilon^{-1}[D_\varepsilon(X), D_\varepsilon(Y)].$$

The *Inönü–Wigner contraction* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is the Lie algebra

$$\mathfrak{g}_0 = \lim_{\varepsilon \to 0} (\mathfrak{g}, [\cdot, \cdot]_\varepsilon),$$

provided this limit exists.

For the contraction of $\mathfrak{su}(2)$ with respect to $\mathfrak{h} = \mathrm{span}\{J_z\}$:

$$[J_z, J_x]_\varepsilon = iJ_y, \quad [J_z, J_y]_\varepsilon = -iJ_x, \quad [J_x, J_y]_\varepsilon = i\varepsilon^2 J_z.$$

In the limit $\varepsilon \to 0$, this yields the Euclidean algebra $\mathfrak{e}(2)$:

$$[J, P_x] = iP_y, \quad [J, P_y] = -iP_x, \quad [P_x, P_y] = 0,$$

where $J = J_z$ generates rotations and $(P_x, P_y) = (J_x, J_y)$ generate translations.

Note that $\mathfrak{e}(2)$ is *not* abelian: the rotation generator J has nontrivial commutators with the translations. What becomes abelian is the subalgebra generated by the transverse components (P_x, P_y).

B Technical details

B.1 Contraction of coadjoint orbits

The contraction map D_ε on the algebra induces a dual map $D_\varepsilon^* : \mathfrak{g}^* \to \mathfrak{g}^*$ defined by

$$\langle D_\varepsilon^*(\xi), X\rangle = \langle \xi, D_\varepsilon(X)\rangle.$$

From the definition of D_ε, one finds

$$D_\varepsilon^*(J_x^*) = \varepsilon^{-1} J_x^*, \quad D_\varepsilon^*(J_y^*) = \varepsilon^{-1} J_y^*, \quad D_\varepsilon^*(J_z^*) = J_z^*.$$

The dual map dilates transverse components while preserving the axial component.

Under D_ε^*, the sphere $S_r^2 \subset \mathfrak{su}(2)^*$ is mapped to an ellipsoid of increasing eccentricity. The limit $\varepsilon \to 0$ should be understood at the level of algebraic structures: as $\mathfrak{su}(2) \to \mathfrak{e}(2)$, the relevant coadjoint orbits change from spheres to cylinders.

The coadjoint orbits of $E(2)$ are:

- Cylinders $\mathcal{O}_\rho = \{(p_x, p_y, j) : p_x^2 + p_y^2 = \rho^2\}$ for $\rho > 0$,
- The axis $\mathcal{O}_0 = \{(0, 0, j) : j \in \mathbb{R}\}$.

Geometrically, the contraction "forgets" the compact structure of the sphere: points with the same z-coordinate but different azimuthal angles become indistinguishable in the limit.

B.2 Behavior of representations under contraction

The $(2s+1)$-dimensional irreducible representation $\pi^{(s)}$ of $SU(2)$ does not converge to a $(2s+1)$-dimensional representation of $E(2)$. Instead, it decomposes into one-dimensional limits.

To see this, note that the unitary irreducible representations of $E(2)$ are:

1. One-dimensional characters χ_n, $n \in \mathbb{Z}$, where $P_x, P_y \mapsto 0$ and $J \mapsto n$.
2. Infinite-dimensional representations induced from the translation subgroup.

Finite-dimensional representations of $E(2)$ are necessarily built from characters.

Under the contraction, the representation space $\mathcal{H}_s$ splits. In the limit $\varepsilon \to 0$, each weight space $\mathbb{C}|s, m\rangle$ becomes invariant:

$$\pi_\varepsilon^{(s)}(P_x)|s, m\rangle, \ \pi_\varepsilon^{(s)}(P_y)|s, m\rangle \to 0,$$

while $\pi^{(s)}(J)|s, m\rangle = m|s, m\rangle$ is unchanged.

Thus:

$$\pi^{(s)} \xrightarrow{\varepsilon \to 0} \bigoplus_{m=-s}^{s} \chi_m.$$

This is the representation-theoretic mechanism for the emergence of discrete outcomes: the contracted algebra admits only one-dimensional subrepresentations of any finite-dimensional $SU(2)$ module.

B.3 Distributional convergence of Husimi functions

Let $|\psi\rangle = \sum_m \psi_m |s,m\rangle$ be a spin-s state. The Husimi distribution is

$$Q_\psi(\mathbf{n}) = \left| \sum_m \psi_m \langle \mathbf{n}|s,m\rangle \right|^2 .$$

Under the Stern–Gerlach evolution $U(\tau) = e^{i\gamma B_z \tau S_z}$, the evolved distribution becomes

$$Q_\psi(\mathbf{n},\tau) = \sum_{m,m'} e^{i\gamma B_z \tau (m-m')} \psi_m \overline{\psi_{m'}} \langle \mathbf{n}|s,m\rangle\langle s,m'|\mathbf{n}\rangle.$$

For large $\gamma B_z \tau$, the oscillatory factors suppress terms with $m \neq m'$, leaving

$$Q_\psi(\mathbf{n},\tau) \approx \sum_m |\psi_m|^2 |\langle \mathbf{n}|s,m\rangle|^2.$$

The overlap $|\langle \mathbf{n}|s,m\rangle|^2$ depends on $\mathbf{n}$ through the Wigner d-functions:

$$|\langle \mathbf{n}|s,m\rangle|^2 = |d^s_{s,m}(\theta)|^2,$$

where θ is the polar angle of $\mathbf{n}$. For the extremal values $m = \pm s$:

$$|d^s_{s,s}(\theta)|^2 = \cos^{4s}(\theta/2), \quad |d^s_{s,-s}(\theta)|^2 = \sin^{4s}(\theta/2).$$

As $s \to \infty$, these concentrate at $\theta = 0$ and $\theta = \pi$ respectively.

For finite s, the concentration is imperfect but still pronounced. The key observation is that the *azimuthal* dependence is absent: the distributions depend only on $\cos\theta = n_z$.

The limiting behavior is formalized as weak convergence on the sphere. For any continuous test function $g : [-1,1] \to \mathbb{R}$:

$$\int_{S^2} Q_\psi(\mathbf{n},\tau)\, g(\cos\theta)\, d\Omega \xrightarrow{\tau\to\infty} \sum_{m=-s}^{s} |\psi_m|^2\, g(m/s).$$

For spin-1/2, this gives

$$Q_\psi \rightharpoonup |\psi_+|^2 \delta(\cos\theta - 1) + |\psi_-|^2 \delta(\cos\theta + 1),$$

expressing the concentration of the Husimi distribution onto the poles of the sphere.

C Experimental parameters

C.1 Typical Stern–Gerlach parameters

The original Stern–Gerlach experiment [5] and its modern realizations [18] use the following typical parameter ranges:

Parameter	**Symbol**	**Typical value**
Magnetic field gradient	B'	10^2–10^3 T/m
Interaction length	L	10^{-2}–10^{-1} m
Beam velocity (thermal)	v	10^2–10^3 m/s
Interaction time	$\tau = L/v$	10^{-5}–10^{-3} s
Gyromagnetic ratio (electron)	γ_e	1.76×10^{11} rad/(s·T)

For silver atoms (the original Stern–Gerlach system), the magnetic moment is dominated by the single unpaired electron, so $\gamma \approx \gamma_e$.

C.2 Estimate of the contraction rate

The effective contraction parameter is the dimensionless phase accumulated during the interaction:

$$\lambda = \gamma \bar{B} \tau,$$

where $\bar{B}$ is the mean magnetic field experienced by the particle. Substituting typical values ($\bar{B} \sim 1$ T, $\tau \sim 10^{-4}$ s):

$$\lambda \sim (1.76 \times 10^{11}\,\mathrm{rad/(s{\cdot}T)}) \times (1\,\mathrm{T}) \times (10^{-4}\,\mathrm{s}) \sim 10^{7}.$$

This large value ensures that transverse coherences are suppressed by a factor $e^{-\lambda} \approx 0$ well before detection.

The phase difference accumulated between the $m = +1/2$ and $m = -1/2$ components is

$$\Delta\phi = \gamma \Delta m \int_0^\tau B_z(t)\, dt \approx \gamma \bar{B} \tau = \lambda.$$

The spatial separation of the two spin components at the exit of the magnet is

$$\Delta z = \frac{(\Delta p_z)\tau}{2M} = \frac{\mu_B B' \tau^2}{M},$$

where M is the atomic mass and μ_B is the Bohr magneton. For silver ($M \approx 1.8 \times 10^{-25}$ kg) with $B' \sim 10^{3}$ T/m and $\tau \sim 10^{-4}$ s, one finds $\Delta z \sim 10^{-4}$ m, which is easily resolvable.

These estimates confirm that the Stern–Gerlach apparatus operates deep in the contracted regime ($\lambda \gg 1$), where the geometric mechanism described in the main text is fully effective.

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