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Distributed Observer Analysis and Design

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Abstract

A Distributed observer design is described for estimating the state of a continuous-time, input free, linear system. This thesis explains how to construct the local estimators, which comprise the observer inputs and outputs, and it is shown which are the requirements to deal with this structure. Every agent senses an output signal from the system and distributes it across a fixed-time network to its neighbors. The information flow increases the capability of each agent to estimate the state of the system and uses collaboration to improve the quality of data.

The proposed solution has several positive features compared to recent results in the literature, which include milder assumptions on the network connectivity and the maximum dimension of the state of each observer does not exceed the order of the plant. The conditions are reduced to certain detectability requirements for each cluster of agents in the network, where a cluster is identified as a subset of agents that satisfy specific properties. Instead, the dimension of each observer is reduced to the number of possible observable states of the system, collected by the agent and by the neighbors.

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Chapter 1

Introduction

The project's purpose is to evaluate and investigate possible distributed control algorithms to be used with a swarm of drones in a research mission. The algorithm has the goal to guide quad-copter drones, in order to find people buried under snow in avalanches, so to minimize time spent on search and to ensure a prompt intervention from rescuers.
The European project airborne https://www.airborne-project.eu/ is involved in the development of mixed ground and aerial robotics platforms, to support humans in search and rescue activities in hostile environments, like alpine rescuing scenario, which is specifically targeted by the project. The distributed estimate is carried out in order to cover a larger research area and to allow a reduction in search times. Moreover, the weight of the device used to estimate the position of the person involved increases with the capacity of providing an accurate measure. Thus, the achievement of a distributed framework decreases the need to have to transport a heavy device and increases the duration and search time for the quad-copters.

Related works

This dissertation deals with the problem of distributed estimation when many agents are present. If agents share information among each other according to some topology, both the questions of optimal control law for a given topology *"What should an agent do?"* and how to design the topology *"Whom should an agent talk to?"* become important. There is a huge literature that seeks to deal with distributed Kalman filters [7] or distributed observers [5, 6], but many results present strictly assumptions [10] or redundant stability techniques [4]. Mostly, a consensus gain is designed in order to deal with the distributed structure, as in the work reported in [6], where the information received by the neighboring agents is weighted to perform linear combinations with the own agent information. Furthermore, there is a numbered of distributed estimation techniques proposed and analyzed for noiseless models. In [10], the distributed estimation problem is tackled as the problem of designing a decentralized stabilizing controller for an LTI system. Differently from the approach presented in this thesis, that work relies on state augmentations. Another interesting approach can be found in [5], where the authors rely on an orthonormal coordinate transformation matrix in order to tackle the design of the distributed observer. However, the distributed design of the observer needs global information about the communication graph and the proposed observer lacks a tuning method to adjust the error convergence speed.

Contributions

The work reported proposes a novel approach to this problem based on the internal reconstruction of the neighbor state and clarifies and expands the results of [10] in several ways. Firstly, jointly observability hypothesis is substituted with collective detectability assumption and the condition of non zero output matrices for all agents is relaxed. Secondly, our solution is able to keep the maximum size of the observers as the size of the plant and in this case, the computational effort is decreased and the efficiency of the overall system is increased. Thirdly, the strong connectivity imposed at the graph is not necessary to ensure the stability properties, instead, the detectability condition is only necessary inside a cluster of agents.

Organization

The thesis is organized as follows. Chapter 2 presents an introduction to network systems and linear system theory, with the notation used. In Section 2.3 the formal problem definition is presented with necessary conditions. In Section 2.4, the state-of-arts of a distributed observer is evaluated in order to be comparable with the final solution and an example is presented in Section 2.4.3. In Chapter 3, the proposed solution at this problem is explained with a steps procedure and a description in pseudo-code. In the last chapter, the model is analyzed and results are presented with short numerical examples.

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Chapter 2

Mathematical Background

In this chapter, preliminaries and background information with a brief literature review of distributed observers. In section 2.1, necessary technical preliminaries from network systems are introduced. In section 2.2, some tools and properties are extracted from the huge field of linear system theory. In section 2.3, a mathematical description of the problem and in section 2.4 the literature on the design of a linear distributed observer is reviewed with an example.

2.1 Introduction to network systems

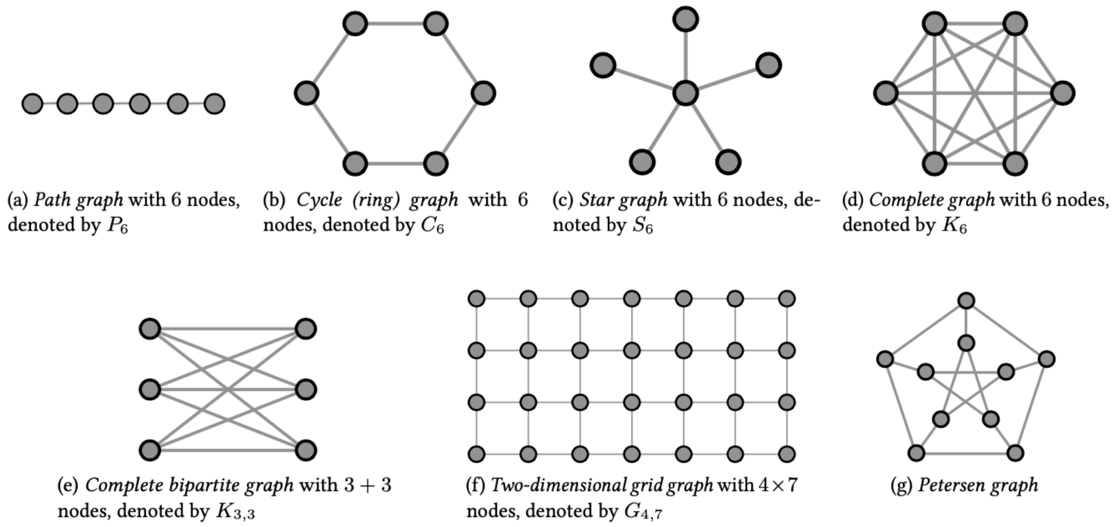
Graph theory provides key concepts to model, analyze and design network systems and distributed algorithms. In mathematics, graph theory is the study of *graphs*, which are structures used to model pairwise relations between objects. In what follows, the most important and useful definitions from [1] are summarized and will be exploited in the next sections in order to define the distributed observer structure and to state the assumptions with mathematical notation.

2.1.1 Graphs and digraphs

A graph in this context is made up of *vertices* (also called nodes) which are connected by *edges*.

[*Edge*] An edge is a pair of nodes, it is important to define the difference between an ordered and an unordered pair. An ordered pair (a, b) is a pair of objects and the order in which the objects appear in the pair is significant: the ordered pair (a, b) is different from the ordered pair (b, a) unless $a = b$. In contrast, the unordered pair $\{a, b\}$ equals the unordered pair $\{b, a\}$.

[*Graphs*] An *undirected graph* (in short, a *graph*) consists of a set $\mathbf{V}$ of *nodes* and of a set $\mathbf{E}_u$ of unordered pairs of nodes, called *edges*. For $u, v \in \mathbf{V}$ and $u \neq v$, the set $\{u, v\}$



(a) *Path graph* with 6 nodes, denoted by P_6

(b) *Cycle (ring) graph* with 6 nodes, denoted by C_6

(c) *Star graph* with 6 nodes, denoted by S_6

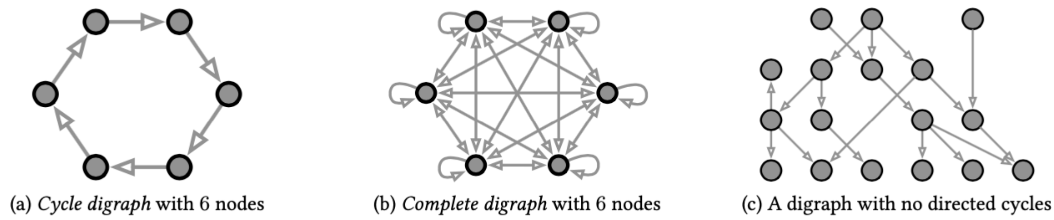
(d) *Complete graph* with 6 nodes, denoted by K_6

(e) *Complete bipartite graph* with $3+3$ nodes, denoted by $K_{3,3}$

(f) *Two-dimensional grid graph* with 4×7 nodes, denoted by $G_{4,7}$

(g) *Petersen graph*

Figure 2.1: Example of graphs



(a) *Cycle digraph* with 6 nodes

(b) *Complete digraph* with 6 nodes

(c) A digraph with no directed cycles

Figure 2.2: Example of digraphs

denotes an unordered edge.
Two nodes u and v of a given graph are neighbors if $\{u, v\}$ is an unordered edge. Given a graph $\mathbb{G}$, we let $\mathcal{N}_G(v)$ denote the set of neighbors of v. Figure 2.1 illustrates some examples of graphs.

[*Digraphs*] A *directed graph* (in short, a *digraph*) of order n is a pair $\mathbb{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V}$ is a set with n elements called nodes and $\mathbf{E}$ is a set of ordered pairs of nodes called *edges*. In other words, $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$. As for graphs, $\mathbf{V}$ and $\mathbf{E}$ are the *node set* and *edge set*, respectively. Figure 2.2 illustrates some examples of digraphs.
For $u, v \in \mathbf{V}$, the ordered pair (u, v) denotes an edge from u to v. A digraph is *undirected* if $(v, u) \in \mathbf{E}$ anytime $(u, v) \in \mathbf{E}$. In a digraph, a *self-loop* is an edge from a node to itself.

[*Subgraphs*] A digraph $(\mathbf{V}', \mathbf{E}')$ is a *subgraph* of a digraph $(\mathbf{V}, \mathbf{E})$ if $\mathbf{V}' \subseteq \mathbf{V}$ and $\mathbf{E}' \subseteq \mathbf{E}$. A digraph $(\mathbf{V}', \mathbf{E}')$ is a *spanning subgraph* of $(\mathbf{V}, \mathbf{E})$ if it is a subgraph and $\mathbf{V}' = \mathbf{V}$.

The subgraph of $(\mathbf{V},\mathbf{E})$ induced by $\mathbf{V}' \subseteq \mathbf{V}$ is the digraph $(\mathbf{V}',\mathbf{E}')$, where $\mathbf{E}'$ contains all edges in $\mathbf{E}$ between two nodes in $\mathbf{V}'$.

[*In- and out-neighbors*] In a digraph $\mathbb{G}$ with an edge $(u,v) \in \mathbf{E}$, u is called an *in-neighbor* of v, and v is called an *out-neighbor* of u. We let $\mathcal{N}^{in}(v)$ (resp., $\mathcal{N}^{out}(v)$) denote the set of in-neighbors, (resp. the set of out-neighbors) of v.
Given a digraph $\mathbb{G} = (\mathbf{V},\mathbf{E})$, an *in-neighbor* of a non empty set of nodes $\mathbf{U}$ is a node $v \in \mathbf{V}\backslash\mathbf{U}$ for which there exists an edge $(v,u) \in \mathbf{E}$ for some $u \in \mathbf{U}$.

[*In- and out-degree*] The *in-degree* $d_{in}(v)$ and *out-degree* $d_{out}(v)$ of v are the number of in-neighbors and out- neighbors of v, respectively. Note that a self-loop at a node v makes v both an in-neighbor as well as an out-neighbor of itself.

2.1.2 Cluster and connectivity

[*Paths*] A *path* in a graph is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is an edge of the graph. A path is *simple* if no node appears more than once in it, except possibly for the case in which the initial node is the same as the final node.

[*Connectivity and connected components*] A graph is *connected* if there exists a path between any two nodes. If a graph is not connected, then it is composed of multiple connected components, that is, multiple connected subgraphs.

[*Directed paths*] A *directed path* in a digraph is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is a directed edge of the digraph. A directed path is *simple* if no node appears more than once in it, except possibly for the initial and final node.

[*Spanning trees*] A subgraph of $\mathbb{G}$ is a *spanning tree*, if it has the same vertex set $\mathbf{V} = \{1,\dots,m\}$, has no cycles, has $m-1$ edges and contains a node from which every other node of $\mathbb{G}$ can be reached by traversing along the directed edges of $\mathbb{G}$ (the root node).

[*Cluster*] A subgraph of $\mathbb{G}$ is a cluster, if it satisfies the following requirements:

1. It contains a spanning tree; and
2. It is a maximal subgraph in the sense that none of its spanning trees can be extended by adding nodes from the set $\mathbf{V}$.

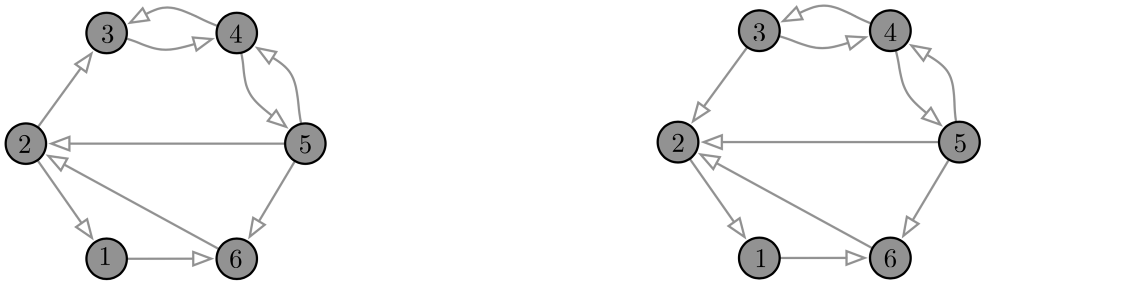
We present four useful connectivity notions for a digraph $\mathbb{G}$:

(i) $\mathbb{G}$ is *strongly connected* if there exists a directed path from any node to any other node;

(ii) $\mathbb{G}$ is *weakly connected* if the undirected version of the digraph is connected;

(iii) $\mathbb{G}$ possesses a *globally reachable node* if one of its nodes can be reached from any other node by traversing a directed path;

(iv) $\mathbb{G}$ possesses a *directed spanning tree* if one of its nodes is the root of directed paths to every other node.



(a) A strongly connected digraph

(b) A weakly connected digraph with a globally reachable node

Figure 2.3: Connectivity examples for digraphs

[*Strongly connected components*] A subgraph $\mathbb{H}$ is a *strongly connected component* of $\mathbb{G}$ if $\mathbb{H}$ is strongly connected and any other subgraph of $\mathbb{G}$ strictly containing $\mathbb{H}$ is not strongly connected.

As important difference, by definition clusters cannot have outgoing edges connecting them to the outside nodes whereas strongly connected components of a digraph can have such outgoing edges.

2.1.3 The Adjacency matrix

Given a digraph $\mathbb{G} = (\mathbf{V}, \mathbf{E}, \{a_{ij}\}_{ij\in\mathbf{E}})$, with $\mathbf{V} = \{1, ..., m\}$, the *binary adjacency matrix* $\mathscr{A} \in \{0, 1\}^{m\times m}$ is defined by

$$a_{ij} = \begin{cases} 1, & if\ (i,j) \in \mathbf{E}, \\ 0, & otherwise. \end{cases}$$

Here, a binary matrix is any matrix taking values in 0, 1.
Then, the *out-degree matrix* $\mathscr{D}_{out}$ and the *in-degree matrix* $\mathscr{D}_{in}$ of a digraph are the diagonal matrices defined by

$$\mathscr{D}_{out} = \text{diag}(\mathscr{A}\mathbf{1}_m) = \begin{bmatrix} d_{out}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{out}(m) \end{bmatrix},$$

$$\mathscr{D}_{in} = \text{diag}(\mathscr{A}^\intercal\mathbf{1}_m)$$

2.1.4 The Laplacian matrix

Given a digraph $\mathbb{G}$ with adjacency matrix $\mathscr{A}$ and out-degree matrix $\mathscr{D}_{out}$, the Laplacian matrix of $\mathbb{G}$ is:

$$\mathscr{L} = \mathscr{D}_{out} - \mathscr{A}$$

In components $\mathscr{L} = (\ell_{ij})_{i,j} \in \{1, ..., m\}$

$$\ell_{ij} = \begin{cases} -1, & if\ (i,j) \text{ is an edge and not self-loop,} \\ d(i), & if\ i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem is one of the results stated in [8], that will be useful to provide detectability conditions for the system presented in the next section.

Theorem 2.1.1 *The multiplicity of the zero eigenvalue of $\mathscr{L}$ is equal to the number of clusters in the graph $\mathbb{G}$*

2.1.5 The Incidence matrix

Let $\mathbb{G}$ be an undirected graph with m nodes and n edges (and no self-loops, as by convention). Assign to each edge of $\mathbb{G}$ a unique identifier $e \in \{1, ..., n\}$ and an arbitrary direction. The (oriented) *incidence matrix* $\mathscr{B} \in \mathbb{R}^{m \times n}$ of the graph $\mathbb{G}$ is defined component-wise by

$$\mathscr{B}_{ie} = \begin{cases} +1, & \text{if node } i \text{ is the source node of edge } e, \\ -1, & \text{if node } i \text{ is the sink node of edge } e, \\ 0, & \text{otherwise.} \end{cases}$$

An example of a ring graph with five nodes and five edges is shown below.

$$\mathscr{B}_{ie} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 Linear System Theory Notions

In what follows, few properties of linear systems with linear algebra tools. These concepts are used during the discussion of the internal structure for the observer.

2.2.1 Observability

In the study of dynamical systems, the key property for the design of observers is the observability of the system. The definition of observability and some conditions that are necessary and sufficient for a widely applied class of continuous-time linear systems are provided. The class of input free systems can be written in general form as follows,

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \tag{2.1}$$

where $x \in \mathbf{R}^n$ is a state vector and $y \in \mathbf{R}^s$ are the outputs. Matrices A and C are of dimensions $n \times n$ and $s \times n$, respectively. The definition of observability is provided from [3],

Definition 2.2.1 *The state equation (2.1) is said to be observable if, for any unknown initial state $x(0)$, there exists a finite time $t_1 > 0$ such that the knowledge of the output y over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable.*

The values of internal states can be inferred by the output y over some time interval. If $(A,\ C)$ is observable, its observability matrix $\mathcal{O}$ has rank n,

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.2}$$

and consequently n linearly independent rows. The observability property is invariant under any equivalence transformation.

2.2.2 Similarity Transformation

A similarity transformation is a linear map whose transformed matrix $\bar{A}$ can be written in the form

$$\bar{A} = TAT^{-1} \tag{2.3}$$

where A and $\bar{A}$ are called similar matrices. Similarity transformations transform objects in space to similar objects.

2.2.3 Canonical Decomposition

Let's suppose that the system in (2.1) is not observable, so $n_o = \text{rank}(\mathcal{O})$, the rank of observability matrix $\mathcal{O}$ is lower than n. In this case, it is possible to apply a similarity transformation to the system (2.1) with matrix T defined as,

$$T = [\ w_{n_o+1},\ \dots w_n, v_1,\ \dots\ v_{n_o}] \tag{2.4}$$

where the last n_o rows are any n_o linearly independent rows of $\mathcal{O}$, and the remaining rows can be chosen arbitrarily as long as T is nonsingular. In the equivalence transformation $z = Tx$ will transform (2.1) into

$$\begin{cases} \dot{z} = TA\,(T)^{-1}\,z = \begin{bmatrix} A_{\bar{o}} & A_p \\ \mathbf{0} & A_o \end{bmatrix} \begin{bmatrix} z_{\bar{o}} \\ z_o \end{bmatrix} \\ y = C(T)^{-1}z = [\mathbf{0} \quad C_o] \begin{bmatrix} z_{\bar{o}} \\ z_o \end{bmatrix} \end{cases} \tag{2.5}$$

where A_o is $n_o \times n_o$ and $A_{\bar{o}}$ is $(n - n_o) \times (n - n_o)$, and the n_o-dimensional subequation of (2.5),

$$\begin{cases} \dot{z}_o = A_o z_o \\ y_o = C_o z_o \end{cases} \tag{2.6}$$

is observable and as the same transfer matrix as (2.1).

2.2.4 Detectability

Let recall the definition of the undetectable subspace of a matrix pair (C, A), $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{s\times n}$. Let $\alpha_A(s)$ denotes the minimal polynomial of A, i.e., the monic polynomial of least degree such that $\alpha_A(A) = 0$, [11], factored as $\alpha_A^-(s)\alpha_A^+(s)$; the zeros of $\alpha_A^-(s)$ and $\alpha_A^+(s)$ are in the open left and closed right half-planes of the complex plane, respectively. Note that Ker $\alpha_A^-(A) \cap$ Ker $\alpha_A^+(A) = \{0\}$, and Ker $\alpha_A^-(A) +$ Ker $\alpha_A^+(A) = \mathbb{R}^n$ [11]. The undetectable subspace of (C, A) is the subspace

$$\mathscr{C} \triangleq \bigcap_{l=1}^{n} \text{Ker } (CA^{l-1}) \cap \text{Ker } \alpha_A^+(A) \tag{2.7}$$

furthermore, let $\mathscr{O}$ denote the unobservable subspace of (C, A)

$$\mathscr{O} \triangleq \bigcap_{l=1}^{n} Ker(CA^{l-1}) \tag{2.8}$$

2.3 Formal Problem Definition

State observers have had a huge impact on the entire field of estimation and control. An observer for a process modeled by a continuous-time, time-invariant linear system:

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \tag{2.9}$$

with state $x \in \mathbb{R}^n$, is a time-invariant linear system:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases} \tag{2.10}$$

which is capable of generating an asymptotically correct estimate of x at a preassigned but arbitrarily fast convergence rate. As is well known, the only requirement on the system (2.9) for such an observer to exist is that the matrix pair (C, A) is observable. The distributed structure, which follows, is one of the classical frameworks present in literature.

2.3.1 Distributed structure

It is considered a fixed network of $m > 0$ autonomous agents labeled $1, 2, \ldots, m$, which are able to receive information from their neighbors. Neighbor relations between distinct pairs of agents are characterized by a directed graph $\mathbb{G} = (\mathbf{V}, \mathbf{E})$, with a set of vertices $\mathbf{V} = \{1, 2, \ldots, m\}$ and a set of arcs $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$ defined so that there is an arc from vertex j to vertex i whenever agent j is a distinct neighbor of agent i; thus $\mathbb{G}$ has no self-arcs, i.e. $(i, i) \notin \mathbf{E}$. $\mathscr{A}$, $\mathscr{D}$ and $\mathscr{L}$ are the adjacency matrix, the in-degree matrix and the Laplacian matrix of the graph $\mathbb{G}$, respectively.
So, each agent i can sense a signal $y_i \in \mathbb{R}^{s_i}$, $i \in \mathbf{m} = \{1, 2, \ldots, m\}$, where the system is:

$$\begin{cases} \dot{x} = Ax \\ y_i = C_i x + \nu_i, \quad i \in \mathbf{m} \end{cases} \tag{2.11}$$

with $x \in \mathbb{R}^n$ and ν_i is output disturbance for agent i.
Agent i estimates x using an n_i-dimensional linear system with state vector $\hat{x}_i$ and it is assumed the information agent i can receive from neighbor $j \in \mathcal{N}_i$ is $\hat{x}_j(t)$, as described in [9].

$$\begin{cases} \dot{\hat{x}}_i = A\hat{x}_i + L_i(y_i - C_i\hat{x}_i) + \displaystyle\sum_{j \in \mathcal{N}_i} H_{ij} \left(\hat{x}_j - T_{ji}\hat{x}_i\right) \\ \\ \hat{y}_i = C_i\hat{x}_i \end{cases} \tag{2.12}$$

Here H_{ij}, $i, j = 1, \ldots, m$, are given matrices. According to (2.12), each node computes its estimate of the plant state x and the neighbours state $T_{ji}\hat{x}_i$, from its local measurements y_i, and the inputs $H_{ij}\hat{x}_j$ received from its neighbors, and also communicates to the neighbors its outputs $\hat{x}_i$. By exploiting all agents equations, it is possible to write,

$$\begin{aligned}
\dot{\hat{x}}_1 &= A\hat{x}_1 + L_1(y_1 - C_1\hat{x}_1) + H_{12}\left(\hat{x}_2 - T_{21}\hat{x}_1\right) + \cdots + H_{1m}\left(\hat{x}_m - T_{m1}\hat{x}_1\right) \\
\dot{\hat{x}}_2 &= A\hat{x}_2 + L_2(y_2 - C_2\hat{x}_2) + H_{21}\left(\hat{x}_1 - T_{12}\hat{x}_2\right) + \cdots + H_{2m}\left(\hat{x}_m - T_{m2}\hat{x}_2\right) \\
\vdots \quad & \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\
\dot{\hat{x}}_m &= A\hat{x}_m + L_m(y_m - C_m\hat{x}_m) + H_{m1}\left(\hat{x}_1 - T_{1m}\hat{x}_m\right) + \cdots + H_{mm-1}\left(\hat{x}_{m-1} - T_{m-1m}\hat{x}_m\right)
\end{aligned}$$

with $H_{ij} = [\mathbf{0}]$ if $j \notin \mathcal{N}_i$. Then in matrix form,

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \vdots \\ \dot{\hat{x}}_m \end{bmatrix} = \begin{bmatrix} A - L_1C_1 - \sum_{j=2}^{m} H_{1j}T_{j1} & H_{12} & \cdots & H_{1m} \\ \vdots & & \ddots & \vdots \\ H_{m1} & & \cdots & A - L_mC_m - \sum_{j=1}^{m-1} H_{mj}T_{jm} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_m \end{bmatrix}$$
$$+ \begin{bmatrix} L_1y_1 & L_2y_2 & \cdots & L_my_m \end{bmatrix}'$$

Now, if it is taken $\hat{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m]'$ and $y = [y_1, y_2, \ldots, y_m]'$,

$$\tilde{A} = \mathbf{I}_m \otimes A = \begin{bmatrix} A & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & A & \cdots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & A \end{bmatrix}_{nm\times nm} \tag{2.13}$$

then $\tilde{C} = \operatorname{diag}[C_1, C_2, \ldots, C_m]$, $L = \operatorname{diag}[L_1, L_2, \ldots, L_m]$ and $\tilde{H} = [H_{ij}]_{i,j=1,\ldots,m}$ where $H_{ij} = -\sum_{j=1,j\neq i}^{m} H_{ij}T_{ji}$ if $j = i$ or $H_{ij} = [\mathbf{0}]$ if $j \notin \mathcal{N}_i$.

$$\dot{\hat{x}} = (\tilde{A} - L\tilde{C} - \tilde{H})\hat{x} + Ly \tag{2.14}$$

This notation, will be useful to state detectability conditions for the overall system. The problem of interest is to construct a suitably defined family of linear systems, in such a way so that no matter what the initializations of (2.12) are, each signal $\hat{x}_i(t)$ is an asymptotically correct estimate of $x(t)$ in the sense that each estimation error $e_i(t) = x(t) - \hat{x}_i(t)$ converges to zero as $t \to \infty$ at a preassigned, but arbitrarily fast convergence rate. Formally, it amounts to the existence of gain matrices $L_i,\ H_i,\ i = 1, \ldots, m$, such that the interconnected system consisting of the error dynamics subsystems

$$\dot{e}_i = (A - L_iC_i)e_i - \sum_{j\in\mathcal{N}_i} H_{ij}\tilde{e}_j \tag{2.15}$$

is globally asymptotically stable. Where $e_i = x - \hat{x}_i$ is the local estimation error at node i and $\tilde{e}_j = \hat{x}_j - T_{ji}\hat{x}_i$ is the estimation error of the state dynamic of agent j from i. Our chief objective is to establish conditions which guarantee detectability for the distributed observer in (2.14). Such a detectability property is necessary for this estimation problem to have a solution. The main result in the following section characterize the detectability property of this large scale system in terms of detectability properties of its components. The necessary condition for the large scale detectability is expressed in terms of the "local" detectability of the plant through individual node measurements, and the observability properties of the node through interconnections. The observer presented in (2.12) has a maximum dimension equal to the plant dimension n, this happen when agent i has fully observability of the plant. In case of partial observable plant, but fully detectable, the dimension of the observer is equal to the dimension of the detectable state.

2.3.2 Detectability conditions

It is important to declare conditions for distributed detectability described in [9] and [8]. The term distributed detectability, refers to the detectability property achieved by the entire network, in contrast to the local detectability which refers to the detectability of the plant from the measurements taken by individual nodes of the network. The undetectable subspace of (C_i, A) is the subspace $\mathscr{C}_i$, from (2.7), and let $\mathscr{O}_{H_i}$ denote the unobservable subspace of (H_i, A), from (2.8). The following theorem shows necessary conditions for collective detectability of the pair $([\tilde{C}', \tilde{H}']', \tilde{A})$, for the special case where $H_i = H$.
From [9]

Theorem 2.3.1 *Suppose the pair* $\left([\tilde{C}', \tilde{H}']', \tilde{A}\right)$ *is detectable. Then, the following statements hold:*

(i)
$$\bigcap_{i=1}^{m} \mathscr{C}_i = \{0\}$$

(ii)
$$\mathscr{O}_H \cap \mathscr{C}_i = \{0\} \quad \forall i \in \{1, \ldots, m\}$$

(iii)
$$rank\ \mathscr{O}_H \geq max_i\ dim\ \mathscr{C}_i$$

Statement (ii) of Theorem 1.2.2 means that every undetectable state of (C_i, A) must necessarily be an observable state of (H_i, A). Also, every unobservable state of (H_i, A) must be a detectable state of one of the pairs (C_i, A).
Detectability of the pair $([\tilde{C}', \tilde{H}']', \tilde{A})$ is necessary but is not sufficient for the existence of the set of observer gains L_i which ensure that the matrix $\tilde{A} - L\tilde{C} - \tilde{H}$ is Hurwitz. Therefore, the collective detectability property is a necessary condition for the observer (2.12) to provide an estimate of the plant (2.11).
By using theorem 1 from [8], which states that the multiplicity of the zero eigenvalue of $\mathscr{L}$ is equal to the number of clusters in the graph $\mathbb{G}$, is possible to rewrite Theorem 2.3.1 by

considering a general case with different matrices H_i. Let's define the observability matrices associated with the matrix pairs (C_i, A), and (H_i, A):

$$\mathcal{O}_{C_i} \triangleq \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{n-1} \end{bmatrix} \quad \mathcal{O}_{H_i} \triangleq \begin{bmatrix} H_i \\ H_i A \\ \vdots \\ H_i A^{n-1} \end{bmatrix}$$

Then from Lemma 2 of [8], The pair $([\tilde{C}', \tilde{H}']', \tilde{A})$ is detectable if and only if

$$\operatorname{Ker}\left(\operatorname{diag}\left[\mathcal{O}_{H_1}, \ldots, \mathcal{O}_{H_m}\right](\mathscr{L} \otimes I_m)\right) \cap \prod_{i=1}^{m} \mathscr{C}_i = \{0\} \tag{2.16}$$

and then the Theorem 2 from [8] states,

Theorem 2.3.2 *Suppose the pair* $([\tilde{C}', \tilde{H}']', \tilde{A})$ *is detectable. Then, for every cluster* $\mathbf{G}(i_1, \ldots, i_s)$ *the following statements hold:*

(i)

$$\bigcap_{i \in \{i_1, \ldots, i_s\}} \mathscr{C}_i = \{0\}$$

(ii) for all $i \in \{i_1, \ldots, i_s\}$,

$$\left(\bigcap_{j \in \mathcal{N}_i} \mathscr{O}_{H_j}\right) \cap \mathscr{O}_{H_i} \cap \mathscr{C}_i = \{0\}$$

The interpretation of claims (i) and (ii) of Theorem 2.3.2 is as follows. Claim (i) states that every state of a collectively detectable plant is necessarily detectable by at least one observer within each cluster of the network. Also, condition (ii) states that communications between the observer nodes in a collectively detectable system must be designed so that each plant x has at least one of the three properties at every node of every cluster: (a) it is detectable by the node from its measurements (i.e., $x \notin \mathscr{C}_i$), or (b) it is observable from the information the node receives from its neighbours (i.e, $x \notin \mathscr{O}_{H_i}$), or (c) it is observable by at least one of the neighbours with whom the node communicates (i.e, there exists j such that $j \in \mathcal{N}_i$ and $x \notin \mathscr{O}_{H_j}$).

2.4 Cutting-edge

In this section, it is reported a simply distributed observer structure defined in [10], in which the layout is similar to the one states in (2.12). In this case, a stabilization technique, recalled from the decentralized theory, is implemented in order to asymptotically estimates the state of the plant. Furthermore, the assumptions and limitations associated with this framework are evaluated and will be relaxed in the next chapter.

2.4.1 Structure

As a first assumption, the system described in (2.11) is considered without disturbances, in order to deal with a simple structure, so

$$\begin{cases} \dot{x} = Ax \\ y_i = C_i x \quad i \in \mathbf{m} \end{cases} \tag{2.17}$$

with $x \in \mathbb{R}^n$, $y_i \in \mathbb{R}^{s_i}$ and $i \in \mathbf{m} = \{1, 2, \dots, m\}$. Then it is assumed throughout that $C_i \neq 0$, $\forall i \in \mathbf{m}$, and the system defined by (2.17) is *jointly observable*, i.e. if $C = [C_1', \dots, C_m']'$ then the system (A,C) is observable. Another important assumption is to ensure that network graph $\mathbb{G}$ is strongly connected. The structure considers the case when the only information transmitted between neighboring agents are estimator states z_i. So, we will focus on observers with the following structure

$$\begin{aligned} \dot{z}_i &= \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_i y_i \quad i \in \mathbf{m} \\ \hat{x}_i &= \sum_{j \in \mathcal{N}_i} M_{ij} z_i \quad i \in \mathbf{m} \end{aligned} \tag{2.18}$$

where the state estimation error $e_i = z_i - x$ is given by

$$\begin{aligned} \hat{x}_i - x &= \sum_{j \in \mathcal{N}_i} M_{ij} e_j \\ \dot{e}_i &= \sum_{j \in \mathcal{N}_i} H_{ij} z_j + (K_i C_i - A)\, x \end{aligned} \tag{2.19}$$

The easiest way to satisfy the observer design equations is to pick

$$I = \sum_{j \in \mathcal{N}_i} M_{ij} \quad i \in \mathbf{m} \tag{2.20}$$

$$A - K_i C_i = \sum_{j \in \mathcal{N}_i} H_{ij} \quad i \in \mathbf{m} \tag{2.21}$$

where $H_{ij} = 0$ if $j \notin \mathcal{N}_i$, it is possible to express these equations in a more explicit form, which takes into account the constraints on the H_{ij} imposed by (2.21). In order to deal with a new structure, three new matrices U_i, C_{ii} and C_{ij} are defined as

$$C_{ij} = b_{ij} \otimes I_{n\times n} \quad \text{with } j \in \mathcal{N}_i$$

$$U_i = u_i \otimes I_{n\times n} = \begin{bmatrix} 0_{n\times n} \\ \vdots \\ I_{n\times n} \\ \vdots \\ 0_{n\times n} \end{bmatrix}_{nm\times n}$$

$$C_{ii} = C_i U_i' = \begin{bmatrix} 0_{s_i\times n} & \cdots & C_i & \cdots & 0_{s_i\times n} \end{bmatrix}_{s_i\times nm}$$

where u_i is the ith unit vector in $\mathbb{R}^n$ and b_{ij} is the row in the transpose of the incidence matrix $\mathscr{B}$ associated to the graph $\mathbb{G}$. It is then possible to express the H in a more compact form:

$$H = \tilde{A} + \sum_{i\in\mathbf{m}} \sum_{j\in\mathcal{N}_i} U_i F_{ij} C_{ij} \tag{2.22}$$

where $F_{ii} = -K_i$ and $F_{ij} = H_{ij}$ with $i \in \mathbf{m}$, $j \in \mathcal{N}_i$ and $j \neq i$. This structure is equal to the one defined in eq. (2.14) for matrix $\tilde{H}$. Note that there are no constraints on the F_{ij} and the problem of constructing a distributed observer of this type, thus, reduces to trying to choose the F_{ij} to at least stabilize H if such matrices exist. This problem is mathematically the same as the classical decentralized stabilization problem for which there is substantial literature.

2.4.2 Decentralized Control

One approach to decentralized stabilization problem was presented by [4] and it consists to try to choose the F_{pq} such that for given $p \in \mathbf{m}$ and $q \in \mathcal{N}_p$, the matrix pairs (H, U_p) and (C_{pq}, H) are controllable and observable, respectively. Having accomplished this, let (H, U_p, C_{pq}) represent a controllable and observable plant, we define:

$$\begin{aligned} H_{dyn} &= \begin{bmatrix} H & 0_{nm\times l} \\ 0_{l\times nm} & 0_{l\times l} \end{bmatrix}_{(nm+l)\times(nm+l)} \\ U_{dyn} &= \begin{bmatrix} U_p & 0_{nm\times l} \\ 0_{l\times n} & I_{l\times l} \end{bmatrix}_{(nm+l)\times(n+l)} \\ C_{dyn} &= \begin{bmatrix} C_{pq} & 0_{s_i\times l} \\ 0_{l\times n} & I_{l\times l} \end{bmatrix}_{(s_p+l)\times(n+l)} \end{aligned} \tag{2.23}$$

represents the observer dynamics plus l additional integrators. Let

$$D(s) = |sI - H| = s^{nm} + \alpha_{nm-1}s^{nm-1} + ... + \alpha_0 \tag{2.24}$$

be the characteristic polynomial of H and ρ_c and ρ_o the smallest integers that

$$\text{rank}[U_p, HU_p, \cdots, H^{\rho_c}U_p] = n$$

and

$$rank[C'_{pq}, C'_{pq}H, \cdots, C'_{pq}H^{\rho_o}] = n$$

For convenience call $\beta = min(\rho_c, \rho_o)$ and define the set $\Lambda_l = \{\lambda_1, .., \lambda_{nm+l}\}$ to be a set of arbitrary complex numbers subject only to the requirement that λ_i with $Im\lambda_i \neq 0$ appears as one of a complex conjugate pair.
Theorem 2 from [4] states that:

Theorem 2.4.1 *Let (H, U_p, C_{pq}) be a controllable observable system and let $H_\beta, U_\beta, C_\beta$ be as defined in (2.23). Given any set Λ_β, there is a matrix K such that the eigenvalues of $H_\beta + U_\beta K C_\beta$ are precisely the elements of the set Λ_β.*

Now assume H is cyclic (its characteristic polynomial is the same as its minimal polynomial) and setting $\beta = m - 1$, because the matrix H is cyclic and the pair (H, C_{pq}) is observable, there are row vectors η' and g' with $g' = \eta' C_{pq}$ such that the pair (H, g') is observable. In order to construct the gain matrix K, that stabilise the system, lets define

$$P(s) = \prod_{i=1}^{nm+l} (s - \lambda_i) \quad \lambda_i \in \Lambda_l$$

as the characteristic polynomial of the closed-loop system. then the problem is to choose K so that

$$\Lambda_l(s) \equiv P(s) \equiv s^{l+nm} + \beta_{l+nm-1} s^{l+nm-1} + \cdots + \beta_0$$

Equation coefficients of s leads to $\Phi\delta = \beta^\star$ defined as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \vert \\ \alpha_{nm-1} & 1 & \cdots & \vdots & \vdots & \vdots & L \\ \vdots & \alpha_{nm-1} & \ddots & \vdots & \vdots & 0 & \vert \\ \alpha_0 & \vdots & \ddots & 0 & 0 & \vert & \vert \\ 0 & \alpha_0 & & 1 & \vert & L & 0 \\ \vdots & 0 & & \alpha_{nm-1} & L & \vert & \vdots \\ \vdots & \vdots & & \vdots & \vert & \vert & \vdots \\ 0 & 0 & \cdots & \alpha_0 & \vert & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1^0 \\ \vdots \\ \delta_l^0 \\ \delta_{l+1}^1 - \delta_{l+1}^0 \\ \vdots \\ \delta_{l+1}^{s_p} - \delta_{l+1}^0 \\ \delta_l^1 - \delta_l^0 \\ \vdots \\ \delta_l^{s_p} - \delta_l^0 \\ \vdots \\ \delta_1^{s_p} - \delta_1^0 \end{bmatrix} = \begin{bmatrix} \beta_{nm+l-1} - \alpha_{nm-1} \\ \beta_{nm+l-2} - \alpha_{nm-2} \\ \vdots \\ \vdots \\ \vdots \\ \beta_l - \alpha_0 \\ \vdots \\ \vdots \\ \beta_0 \end{bmatrix}$$

where L corresponds to

$$L = \begin{bmatrix} g'U_p \\ g'HU_p + \alpha_{nm-1}g'U_p \\ \vdots \\ g'H^{i-1}U_p + \sum_{j=1}^{i-1} \alpha_{nm-j}g'H^{i-j-1}U_p \\ \vdots \\ g'H^{nm-1}U_p + \sum_{j=1}^{nm-1} \alpha_{nm-j}g'H^{nm-j-1}U_p \end{bmatrix} \tag{2.25}$$

To arbitrary obtain pole placement with output feedback it must be shown then that Φ has full rank and that, given vector δ, the elements of the feedback matrix K may be determined as follows:

$$K_0 = \begin{bmatrix} k_{r+1,s_p+1} & \cdots & k_{r+1,s_p+l} \\ \vdots & \ddots & \vdots \\ k_{r+l,s_p+1} & \cdots & k_{r+l,s_p+l} \end{bmatrix} \quad K_j = \left[\begin{array}{c|ccc} k_j & k_{j,s_p+1} & \cdots & k_{j,s_p+l} \\ \hline k_{n+1} & & & \\ \vdots & & K_0 & \\ k_{n+l} & & & \end{array}\right]$$

with $j = 1, 2, \ldots n$ and $\delta_i^j = (-1)^i$ times the sum of all the principal minors of order i of K_j, $i = 1, \ldots, l+1$; $j = 0, 1, \ldots, s_p$ and $\sigma_{l+1}^0 \equiv 0$. For any δ there is a matrix K, of appropriate dimensions, such that $\delta_1^j, \ldots, \delta_{l+1}^j$ are the coefficients of the characteristic polynomial of K_j. For example, one choice is

$$K_0 = \begin{bmatrix} -\delta_1^0 & -\delta_2^0 & \cdots & & -\delta_l^0 \\ 1 & 0 & \cdots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \cdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad K_j = \left[\begin{array}{c|c} \multicolumn{2}{c}{h_i'} \\ \hline 1 & \\ 0 & \\ \vdots & K_0 \\ 0 & \end{array}\right]$$

$$h_j = -\begin{bmatrix} \delta_1^j - \delta_1^0 \\ \vdots \\ \delta_l^j - \delta_l^0 \\ \delta_{l+1}^j - \delta_{l+1}^0 \end{bmatrix} - (\delta_1^0 - \delta_1^j) \begin{bmatrix} 0 \\ \delta_1^0 \\ \vdots \\ \delta_l^0 \end{bmatrix}$$

A step back it is needed in order to deal with the structure of H matrix defined in eq. (2.22), that now is express in terms of H_{dyn}. For simplicity, assume that $p = m$, so we need to build the K such that the system $(H_{dyn} + U_{dyn}KC_{dyn})$ is Hurwitz. As it can be seen in the following formulation, the stabilization part of the system is only pose on the last agent

m,

$$U_{dyn}KC_{dyn} = \left[\begin{array}{ccc|c|ccc} 0_{n\times n} & \cdots & 0_{n\times n} & 0_{n\times n} & & 0_{n\times l} & \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \\ \vdots & \cdots & \vdots & 0_{n\times n} & & 0_{n\times l} & \\ \hline & & & & k_{1,s_p+1} & \cdots & k_{1,s_p+l} \\ 0_{n\times n} & \cdots & 0_{n\times n} & \bar{D}C_m & & & \\ & & & & k_{n,s_p+1} & \cdots & k_{n,s_p+l} \\ \hline 0_{l\times n} & \cdots & 0_{l\times n} & \bar{B}C_m & & K_0 & \end{array}\right]$$

where $\bar{D} = [k_1 k_2 \ldots, k_n]'$ and $\bar{B} = [k_{n+1}, \ldots, k_{n+l}]'$. So, we could rewrite the full system $(H_{dyn} + U_{dyn}KC_{dyn})$, as follows

$$\hat{H} = \begin{bmatrix} H + U_m\bar{D}C_m & U_m\bar{E} \\ \bar{B}C_m & K_0 \end{bmatrix} \tag{2.26}$$

where

$$\bar{E} = \begin{bmatrix} k_{1,s_p+1} & \cdots & k_{1,s_p+l} \\ \vdots & & \vdots \\ k_{n,s_p+1} & \cdots & k_{n,s_p+l} \end{bmatrix}$$

At the end, the stabilization can be achieved by increasing the size of the last observer by $l = m-1$ and by replace the matrices H_{mi}, $i \in \mathbf{m}$, $i \neq m$, and H_{mm} in the eq. (2.22) with

$$\begin{bmatrix} H_{mi} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} H_{mm} + \bar{D}C_m & \bar{E} \\ \bar{B}C_m & K_0 \end{bmatrix} \quad \text{respectively.}$$

theorem 1 from [10] can be state,

Theorem 2.4.2 *Suppose that (2.17) is a jointly observable system and that $C_i \neq 0$, $\forall i \in \mathbf{m}$. If graph $\mathbb{G}$ is strongly connected, then for each symmetric set of $mn + m - 1$ complex numbers Λ, there is a distributed observer (2.18) for which the spectrum of the $(mn + m - 1) \times (mn + m - 1)$ matrix $\hat{H} \triangleq [H_{ij}]$ is Λ.*

This observer, for a process modeled by a continuous-time, time-invariant linear system as (2.17), is capable of generating an asymptotically correct estimate of x exponentially fast at a preassigned convergence rate, no matter what the initializations of (2.17) and (2.18) are.

Figure 2.4: Graph $\mathbb{G}$ of the first distributed observer

2.4.3 Numerical example

An example of a system with two agents is considered in order to demostrate the stability properties previously described. Let's consider the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix} x,$$

with $x \in \mathbb{R}^4$, which is being observed by two agents ($m = 2$) in such a way that

$$\begin{aligned} y_1 &= [1\ 0\ 0\ 0]\, x = C_1 x, \\ y_2 &= [0\ 0\ 1\ 0]\, x = C_2 x \end{aligned}$$

so the jointly observability condition is satisfied with the matrix $C = [C_1', C_2']'$, and the observability matrix $\mathcal{O}$ has $\mathrm{rank}(\mathcal{O}) = 4$. Furthermore, C_1, C_2 are $\neq 0$, the graph $\mathbb{G}$ is shown in fig. 2.4, is strongly connected and the incidence matrix $\mathscr{B}$ associated is described as follows

$$\mathscr{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

used to construct matrix H, with matrices U_1 and U_2,

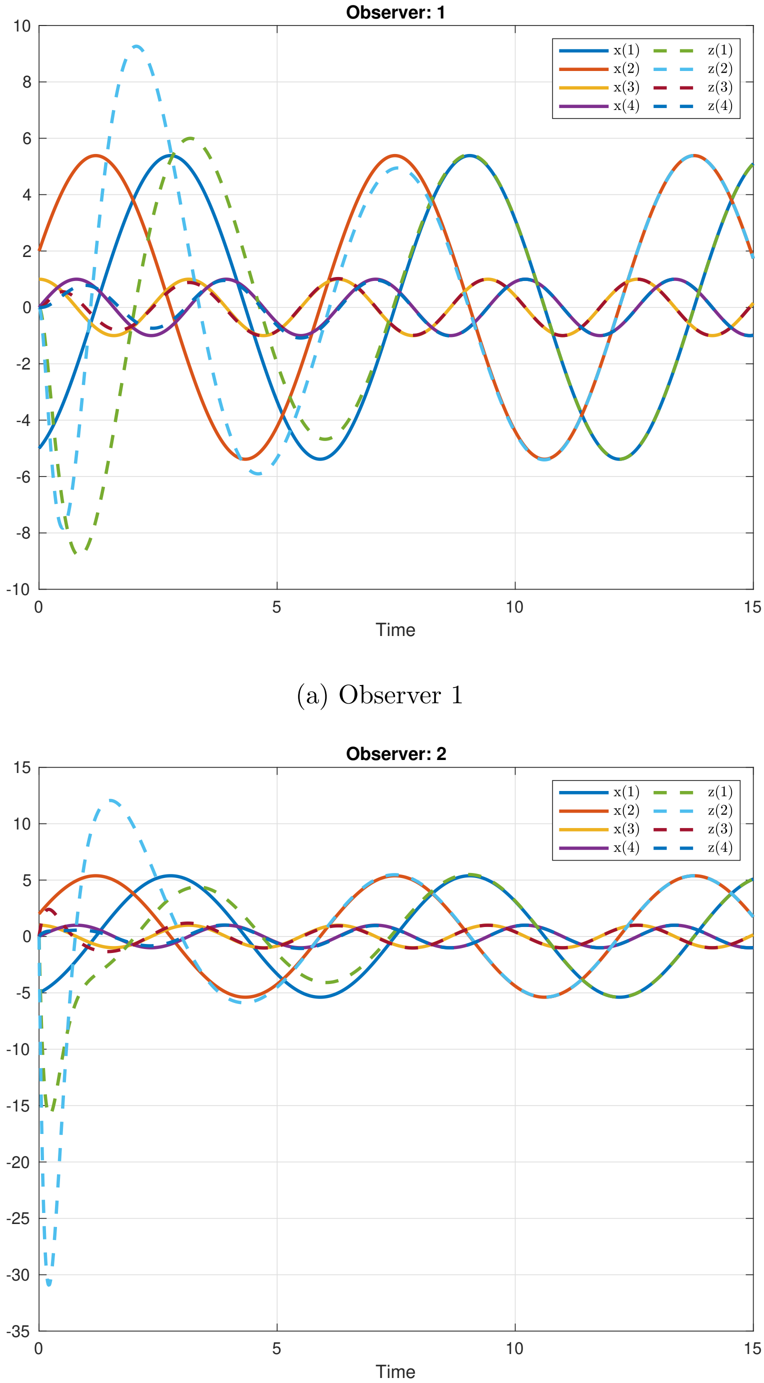
$$U_1 = u_1 \otimes I_4 = \begin{bmatrix} I_4 \\ 0_{4\times 4} \end{bmatrix}, \quad U_2 = u_2 \otimes I_4 = \begin{bmatrix} 0_{4\times 4} \\ I_4 \end{bmatrix},$$

$u_1 = [1\ 0]'$, $u_2 = [0\ 1]'$ and I_4 is the identity matrix. Furthermore, matrices C_{11}, C_{12}, C_{21} and C_{22} are described in the following way,

$$\begin{aligned} C_{11} &= C_1 U_1' = \begin{bmatrix} C_1 & 0_{4\times 4} \end{bmatrix}_{1\times 8} \qquad & C_{12} &= b_1 \otimes I_4 = \begin{bmatrix} I_4 & -I_4 \end{bmatrix}_{8\times 8} \\ C_{21} &= b_2 \otimes I_4 = \begin{bmatrix} -I_4 & I_4 \end{bmatrix}_{8\times 8} \qquad & C_{22} &= C_2 U_2' = \begin{bmatrix} 0_{4\times 4} & C_2 \end{bmatrix}_{1\times 8} \end{aligned}$$

The matrix H can now be states as

$$\begin{aligned} H &= \tilde{A} - U_1 K_1 C_{11} + U_1 F_{12} C_{12} + U_2 F_{21} C_{21} - U_2 K_2 C_{22} \\ &= \begin{bmatrix} A - K_1 C_1 + F_{12} & -F_{12} \\ F_{21} & A - K_2 C_2 + F_{21} \end{bmatrix} \end{aligned}$$



(a) Observer 1

(b) Observer 2

Figure 2.5: State of the plant in solid lines and estimates of agents in dashed lines

in which K_1, K_2, F_{12} and F_{21} are free matrices, used to stabilize the system or at least make the matrix pairs (H, U_2) and (C_2, H), controllable and observable, respectively. In this way, the size of the second agent can be improved, with $l = 1$ and by using the decentralized control presented in section 2.4.2, it is possible to stabilize the full system with the dynamic compensator. The new matrix $\hat{H}$ is described as follows,

$$\hat{H} = \left[\begin{array}{cc|c} A - K_1C_1 + F_{12} & -F_{12} & 0_{4\times1} \\ F_{21} & A - K_2C_2 + F_{21} + \bar{D}C_2 & \bar{E} \\ \hline 0_{1\times4} & \bar{B}C_2 & K_0 \end{array}\right]$$

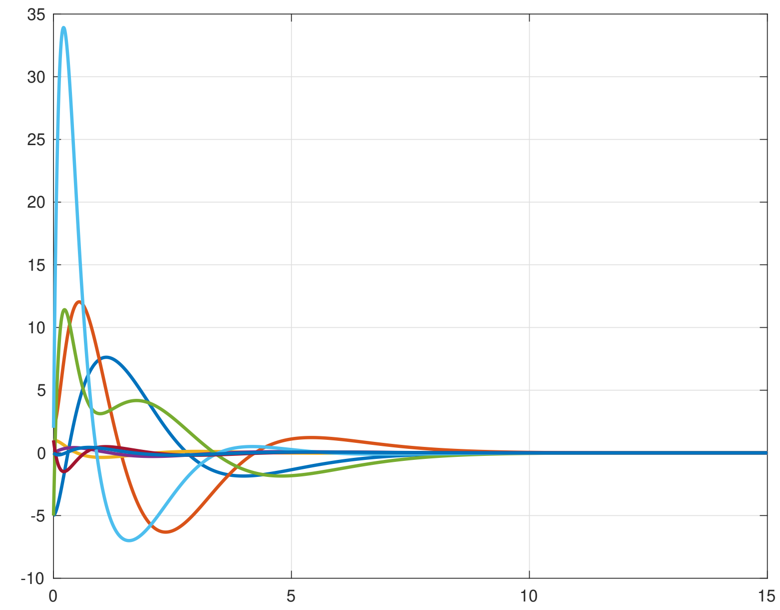
with eigenvectors distribute between 1.5 to 2.5. The full system, composed by agents 1 and 2 is shown in fig. 2.5 and is described below

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ z_{dyn} \end{bmatrix} = \begin{bmatrix} A - K_1C_1 + F_{12} & -F_{12} & 0_{4\times1} \\ F_{21} & A - K_2C_2 + F_{21} + \bar{D}C_2 & \bar{E} \\ 0_{1\times4} & \bar{B}C_2 & K_0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_{dyn} \end{bmatrix} + \begin{bmatrix} K_1 & 0_{4\times1} \\ 0_{4\times1} & K_2 - \bar{D} \\ 0 & \bar{B} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

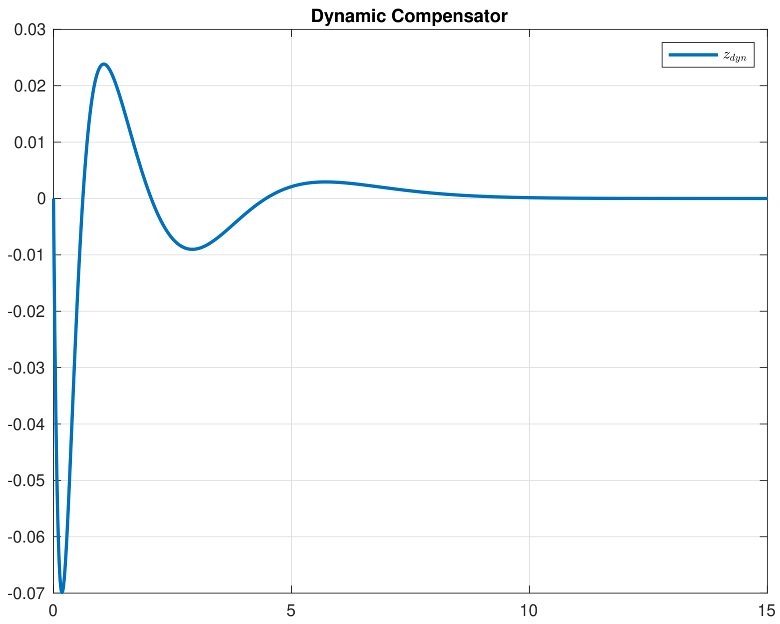
The error is shown in fig. 2.6a and the new state introduced by the dynamic compensator is shown in fig. 2.6b.

2.4.4 Considerations

In this example, the overall dimension of the observer is 9×9, with the first agent having the same plant size, while the last has an extra state introduced by the dynamic compensator. With this approach, the dynamic compensator is forcing the stabilization of the entire system to ensure the converge of the estimation. The strong connectivity of the graph is a necessary condition for implementing the decentralized control technique from the last observer.



(a) Evolution of errors for the two agents



(b) The evolution of the state introduced by the dynamic compensator

Figure 2.6: Errors and dynamic compensator

Chapter 3

A Novel Study on Distributed Observer Design

This chapter presents the main result. In Section 3.1, the design methodology is described step by step. In Section 3.2, one classic control technique is used to stabilize the observer, then in Section 3.3, a review of the stability properties.

3.1 Design procedure

Roughly speaking, the idea behind is to share the transformation matrices, used for the canonical decomposition, between neighboring agents, in order to reconstruct internally the neighbor state. Furthermore, each agent sends his number of observable states, in this way, the construction of the neighbor state shrinks to the observable one. The design is described as a step procedure, in which at each step the knowledge of the system from the agent i increases, if the information provided from neighbors is linearly independent respects to the own one.

3.1.1 Step 1

Consider the state estimation problem for a general system with the same framework presented in (2.11) seen by agent i

$$\begin{cases} \dot{x} = Ax \\ y_i = C_i x + \nu_i, \quad i \in \mathbf{m} \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $y_i \in \mathbb{R}^{s_i}$ and $\nu_i \in \mathbb{R}^{s_i}$.

At the first step, agent i computes the observability matrix $\mathcal{O}_i^{[1]}$, as defined in (2.2), where $n_{o_i}^{[1]} = \text{rank}(\mathcal{O}_i^{[1]}) \leq n$. In order to apply the canonical decomposition, that makes a change of coordinates and splits observable and unobservable states, the transformation matrix $T_i^{[1]}$ is needed. In the new coordinates $z_i^{[1]} = T_i^{[1]}x$, the system has matrices

$$\begin{cases} \dot{z}_i^{[1]} = T_i^{[1]} A \left(T_i^{[1]}\right)^{-1} z_i^{[1]} = \begin{bmatrix} A_{\bar{o}_i}^{[1]} & A_{p_i}^{[1]} \\ \mathbf{0} & A_{o_i}^{[1]} \end{bmatrix} \begin{bmatrix} z_{\bar{o}_i}^{[1]} \\ z_{o_i}^{[1]} \end{bmatrix} \\ \\ y_i^{[1]} = C_i (T_i^{[1]})^{-1} z_i^{[1]} + \nu_i = \begin{bmatrix} \mathbf{0} & C_{o_i}^{[1]} \end{bmatrix} \begin{bmatrix} z_{\bar{o}_i}^{[1]} \\ z_{o_i}^{[1]} \end{bmatrix} + \nu_i \end{cases} \tag{3.2}$$

in which with z_{o_i}, is identified the observable part of system (3.1) and with $z_{\bar{o}_i}$ the unobservable. Equation (3.2) can be rewritten in more compact way as

$$\begin{cases} \dot{z}_i^{[1]} = \bar{A}_i^{[1]} z_i^{[1]} \\ y_i^{[1]} = \bar{C}_i^{[1]} z_i^{[1]} + \ \nu_i \end{cases} \tag{3.3}$$

where $\bar{A}_i^{[1]} = T_i^{[1]} A (T_i^{[1]})^{-1}$ and $\bar{C}_i^{[1]} = C_i (T_i^{[1]})^{-1}$.
Each agent is interested only in the information that knows and is able to share with the network. In this coordinates system, agent i will use only the observable part, and from equations (3.2), instead of (3.3), will follows

$$\begin{cases} \dot{z}_{o_i}^{[1]} = A_{o_i}^{[1]} z_{o_i}^{[1]} \\ y_{o_i}^{[1]} = C_{o_i}^{[1]} z_{o_i}^{[1]} + \ \nu_i \end{cases} \tag{3.4}$$

At the end of step 1, agent i sends matrix $T_i^{[1]}$ and $n_{o_i}^{[1]}$ at every neighbor j with $j \in \mathcal{N}_i$.

3.1.2 Step 2

At the beginning of the second step, agent i has received $T_j^{[1]}$ and $n_{o_j}^{[1]}$ from every neighbor j, with $j \in \mathcal{N}_i$, and it can reconstruct the state z_j of agent j.

$$\begin{cases} z_j^{[1]} = T_j^{[1]} x \\ z_i^{[1]} = T_i^{[1]} x \quad \rightarrow x = (T_i^{[1]})^{-1} z_i^{[1]} \end{cases}$$

$$\rightarrow z_j^{[1]} = T_j^{[1]} (T_i^{[1]})^{-1} z_i^{[1]} = T_{ji}^{[1]} z_i^{[1]}$$

where it is defined $T_{ji}^{[1]} = T_j^{[1]} (T_i^{[1]})^{-1}$, then the state $z_j^{[1]}$ is decomposed into the observable and the unobservable part. $n_{o_j}^{[1]}$ is equal to the number of observable rows of matrix $T_{ji}^{[1]}$ and $(n - n_{o_j}^{[1]})$ are the remain unobservable rows.

$$\begin{bmatrix} z_{\bar{o}_j}^{[1]} \\ \\ z_{o_j}^{[1]} \end{bmatrix} = \begin{array}{r} (n - n_{o_j}^{[1]}) \ \{ \\ \\ n_{o_j}^{[1]} \quad \{ \end{array} \begin{bmatrix} T_{\bar{o}_{ji}}^{[1]} \\ \text{-----} \\ T_{o_{ji}}^{[1]} \end{bmatrix} z_i^{[1]} \tag{3.5}$$

So, the system (3.4) for agent i, now it is composed by

$$\begin{cases} \dot{z}_i^{[1]} = \bar{A}_i^{[1]} z_i^{[1]} \\ y_i^{[1]} = \bar{C}_i^{[1]} z_i^{[1]} + \nu_i \\ z_{o_j}^{[1]} = T_{o_{ji}}^{[1]} z_i^{[1]} \quad \forall j \in \mathcal{N}_i \end{cases} \tag{3.6}$$

The new output matrix $C_i^{[2]}$ at step 2 of agent i takes into account all the matrices $T_{o_{ji}}^{[1]}$, as in the following structure

$$C_i^{[2]} = \begin{bmatrix} C_i \\ T_{o_{wi}}^{[1]} \\ \vdots \\ T_{o_{ji}}^{[1]} \\ \vdots \\ T_{o_{ui}}^{[1]} \end{bmatrix} \quad \text{with } \{w, \dots, j, \dots, u\} \in \mathcal{N}_i \tag{3.7}$$

where w, j and u are neighbors for the agent i. So, the matrix (3.7) is considering only the observable part, for each neighbour. As seen in step 1, the observability matrix $\mathcal{O}_i^{[1]}$, for the second step, is updated by computing it with matrices $C_i^{[2]}$ and A, named $\mathcal{O}_i^{[2]}$. Furthermore, $n_{o_i}^{[2]} = \text{rank}(\mathcal{O}_i^{[2]})$ and transformation matrix $T_i^{[2]}$, defined as in (2.4), are updated according to the new observability matrix $\mathcal{O}_i^{[2]}$. By exploiting the transformation matrix $T_i^{[2]}$

$$z_i^{[2]} = T_i^{[2]} x \tag{3.8}$$

the system (3.3) is transformed into

$$\begin{cases} \dot{z}_i^{[2]} = \bar{A}_i^{[2]} z_i^{[2]} \\ y_i^{[2]} = \bar{C}_i^{[2]} z_i^{[2]} + \ \nu_i \end{cases} \tag{3.9}$$

and the same consideration given in (3.4) is used to rewrite the system (3.6) as

$$\begin{cases} \dot{z}_{o_i}^{[2]} = A_{o_i}^{[2]} z_{o_i}^{[2]} \\ y_{o_i}^{[2]} = C_{o_i}^{[2]} z_{o_i}^{[2]} + \nu_i \\ z_{o_j}^{[2]} = T_{o_{ji}}^{[2]} z_{o_i}^{[2]} \quad \forall j \in \mathcal{N}_i \end{cases} \tag{3.10}$$

where the observable state z_{o_i} is used to reconstruct the neighbour observable state z_{o_j}, so, only the observable part of the plant is used as own injection output for agent i. As before, at the end of step 2, the matrix $T_i^{[2]}$ and the observability number n_{o_i} are sent to neighbours.

3.1.3 Step k

The generalization of the algorithm leads to a more concise description of the step procedure. Let's define as parameters for step k, the message composed by the matrix $T_j^{[k-1]}$ and $n_{o_j}^{[k-1]}$, given from every neighbour $j \in \mathcal{N}_i$ and the own agent matrices A, C_i. So, the initialization consists in the assignment of C_i, to the first rows of $C_i^{[k]}$, as described in Function STEP_K. Then, in the for-loop, for every message received, $T_{ji}^{[k-1]}$ is created with the product between $T_j^{[k-1]}$ and the inverse of $T_i^{[k-1]}$. Only the first n_{o_j} rows are assigned to $C_i^{[k]}$, if $n_{o_j} < n$, as described with the notation $\left(\ell : \ell + n_{o_j},\ :\right)$. ℓ is considered as counter that progressively increases the size of $C_i^{[k]}$, with the observability number n_{o_j} provided by neighbors . At the end, the observability decomposition is computed and the updated transformation $T_i^{[k]}$, with the observability number associated $n_{o_i}^{[k]}$ are sent to neighbours $j \in \mathcal{N}_i$.

Function `STEP_k`$(A,\ C_i)$**:**

  Data: $A \in \mathbb{R}^{n\times n},\ C_i \in \mathbb{R}^{s_i\times n},\ \bigcup_{j\in\mathcal{N}} \texttt{MSG}_j^{[k-1]}$

  Result: $\texttt{MSG}_i^{[k]}$

  begin

    initialization : $C^{[k]}_{i\ (1:s_i,\ :)} \leftarrow C_i$

    forall $\left(T_j^{[k-1]}, n_{o_j}^{[k-1]}\right) \in \bigcup_{j\in\mathcal{N}} \texttt{MSG}_j^{[k-1]}$ **do**

      $T_{ji}^{[k-1]} \leftarrow T_j^{[k-1]} \left(T_i^{[k-1]}\right)^{-1}$

      $C^{[k]}_{i\ \left(\ell:\ell+n_{o_j},\ :\right)} \leftarrow T^{[k-1]}_{o_{ji}\ \left(n-n_{o_j}:n,\ :\right)}$

      $\ell \leftarrow \ell + n_{o_j} + 1$

    end

    $(\bar{A}_i^{[k]},\ \bar{C}_i^{[k]},\ T_i^{[k]}, n_{o_i}^{[k]}) \leftarrow \texttt{OBSF}(A,\ C_i^{[k]})$

    $\texttt{MSG}_i^{[k]} \leftarrow \left(T_i^{[k]},\ n_{o_i}^{[k]}\right) \quad \forall j \in \mathcal{N}_i$

  end

  return $\bar{A}_i^{[k]},\ \bar{C}_i^{[k]},\ T_i^{[k]}, n_{o_i}^{[k]}$

3.2 Control

The end of iterations appears when matrix T_i remains invariant for each step after k, and after a number of steps greater than $m-1$, with $\mathbf{m} = \{1, \ldots, m\}$ equal to the number of agents in the graph $\mathbb{G}$. Furthermore, full observability for agent i is reached when the observability number n_{o_i} is equal to the plant dimension n. Instead if $n_{o_i} < n$, the observer dimension is equal to n_{o_i} and only the detectable dynamics are followed by agent i. By calling the detectability theorem 65 of [2],

Theorem 3.2.1 *Given a system (A, B, C_i, D), with the undetectable space $\mathscr{C}_i$ associated to the couple (A, C_i) as described in (2.7), then for $A \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{s_i\times n}$, there exist $K_i \in \mathbb{R}^{n\times s_i}$ such that*

$$\sigma\left[A + K_iC_i\right] \subset \mathbb{C}^-$$

if and only if the pair (A, C_i) is detectable.

The gain K_i, in our case, is constructed using $\left(A, C_i^{[k]}\right)$, with step k as last step.

3.2.1 Full observability

Firstly, let's consider a full observable system for all the agents $n_{o_j} = n$, for each $j \in \mathbf{m}$, so the observer structure, for agent i, is described as follows

$$\begin{gathered}\dot{\hat{z}}_i = \bar{A}_i\hat{z}_i + K_i\left(\tilde{y}_i - \bar{C}_i^{[k]}\hat{z}_i\right)\\ \text{inputs: } \tilde{y}_i = \begin{bmatrix} \bar{C}_i z_i \\ \hat{z}_w \\ \vdots \\ \hat{z}_j \\ \vdots \\ \hat{z}_u \end{bmatrix} \qquad \text{outputs: } \bar{C}_i^{[k]}\hat{z}_i = \begin{bmatrix} \bar{C}_i \\ T_{wi} \\ \vdots \\ T_{ji} \\ \vdots \\ T_{ui} \end{bmatrix}\hat{z}_i\end{gathered} \tag{3.11}$$

with $\bar{C}_i^{[k]}$ composition of col$\left(T_{ji}^{[k-1]}\right)$, for each $j, w, u \in \mathcal{N}_i$, with the first n rows equal to $\bar{C}_i$, as in (3.7), and the step number is omitted, for a clear explanation. Eq. (3.11) is expanded in order to better show the inputs $\tilde{y}_i$ and outputs for the observer and the matrices associated to neighbours. Furthermore, it is possible to split the gain matrix K_i and associated each gain to each dynamics, in a explicit way,

$$\dot{\hat{z}}_i = \bar{A}_i\hat{z}_i + \left[\begin{array}{c:ccccc} L_i & H_{iw} & \cdots & H_{ij} & \cdots & H_{iu}\end{array}\right]\left(\begin{bmatrix} \bar{C}_i z_i \\ \hdashline \hat{z}_w \\ \vdots \\ \hat{z}_j \\ \vdots \\ \hat{z}_u \end{bmatrix} - \begin{bmatrix} \bar{C}_i \\ \hdashline T_{wi} \\ \vdots \\ T_{ji} \\ \vdots \\ T_{ui} \end{bmatrix}\hat{z}_i\right)$$

where $K_i = [L_i\ H_{iw}\ \cdots\ H_{ij}\ \cdots\ H_{iu}]$, with $\{w, \ldots, j, \ldots, u\} \in \mathcal{N}_i$, The observer i, described by $\left(A, C_i, L_i, \sum_{j\in\mathcal{N}_i} H_j\right)$ as in the structure (2.12), is now shown below

$$\dot{\hat{z}}_i = \bar{A}_i\hat{z}_i + L_i\left(y_i - \bar{C}_i\hat{z}_i\right) + \sum_{j\in\mathcal{N}_i} H_{ij}\left(\hat{z}_j - T_{ji}\hat{z}_i\right) \tag{3.12}$$

Algorithm 1: Distributed observer algorithm

Data: $A \in \mathbb{R}^{n\times n},\ C_i \in \mathbb{R}^{s_i\times n}$

Result: $\hat{z}_i$

begin

  initialization : $T_i^{[1]} \leftarrow$ `OBSF(`$A,\ C_i$`)`

  while $T_i^{[k-1]} \neq T_i^{[k]}$ **or** $k < m$ **do**

    $\bar{A}_i^{[k]},\ \bar{C}_i^{[k]},\ T_i^{[k]}, n_{o_i}^{[k]} \leftarrow$`STEP_k(`$A,\ C_i$`)`

  end

  $K_i \leftarrow K_i : \{\sigma\left[\bar{A}_i + K_i\bar{C}_i^{[k]}\right] \subset \mathbb{C}^-\}$

  $L_i \leftarrow K_{i\ (:,\ 1:n_{o_i})}$

  $\ell \leftarrow n_{o_i} + 1$

  forall $j \in \mathcal{N}_i$ **do**

    $H_{ij} \leftarrow K_{i\ (:,\ \ell:\ell+n_{o_j})}$

    $T_{ji} \leftarrow \bar{C}^{[k]}_{i\ (\ell:\ell+n_{o_j},\ :)}$

    $\ell \leftarrow \ell + n_{o_j} + 1$

  end

  $\dot{\hat{z}} \leftarrow \bar{A}_i\hat{z}_i + L_i\left(y_i - \bar{C}_i\hat{z}_i\right) + \sum_{j\in\mathcal{N}_i} H_{ij}\left(\hat{z}_j - T_{ji}\hat{z}_i\right)$

end

3.2.2 Full detectability

In this case each agent has a number of observable states lower or equal than the dimension of plant $n_{o_i} \leq n$, so the observer structure, for agent i, is described as follows

$$\dot{\hat{z}}_{o_i} = A_{o_i}\hat{z}_{o_i} + K_i\left(\tilde{y}_{o_i} - C_{o_i}^{[k]}\hat{z}_{o_i}\right)$$
$$\text{inputs: } \tilde{y}_{o_i} = \begin{bmatrix} C_{o_i}z_{o_i} \\ \hat{z}_{o_w} \\ \vdots \\ \hat{z}_{o_j} \\ \vdots \\ \hat{z}_{o_u} \end{bmatrix} \qquad \text{outputs: } C_{o_i}^{[k]}\hat{z}_i = \begin{bmatrix} C_{o_i} \\ T_{o_{wi}} \\ \vdots \\ T_{o_{ji}} \\ \vdots \\ T_{o_{ui}} \end{bmatrix}\hat{z}_{o_i} \tag{3.13}$$

with $C_{o_i}^{[k]}$ composition of col $\left(T_{o_{ji}}^{[k-1]}\right)$, for each $j, w, u \in \mathcal{N}_i$, with the first n_{o_i} rows equal to C_{o_i}, as in (3.11), and the step number is omitted, for a clear explanation. In this case the gain matrix K_i is computed in order to ensure the eigenvalues of the system $\left(A_{o_i} - K_iC_{o_i}^{[k]}\right)$ to be with negative real part. The observer i, in this case, is described by the following explicit equation

$$\dot{\hat{z}}_{o_i} = A_{o_i}\hat{z}_{o_i} + L_i\left(y_{o_i} - C_{o_i}\hat{z}_{o_i}\right) + \sum_{j\in\mathcal{N}_i} H_{ij}\left(\hat{z}_{o_j} - T_{o_{ji}}\hat{z}_{o_i}\right) \tag{3.14}$$

with $T_{o_{ji}}\hat{z}_{o_i}$ the projection of the observable state $\hat{z}_{o_i}$ to the observable state $\hat{z}_{o_j}$.

3.3 Stability properties

The asymptotically stability properties are considered in order to demonstrate the robustness of solution. So, the error $e_i = z_i - \hat{z}_i$ and error dynamic are stated as in (2.15),

$$\dot{e}_i = \left(\bar{A}_i - L_i\bar{C}_i\right)e_i - \sum_{j\in\mathcal{N}_i} H_{ij}\left(\hat{z}_j - T_{ji}\hat{z}_i\right) \tag{3.15}$$

and the right hand side is given by the difference between the internal state $\hat{z}_j$ received from agent j and the reconstructed state $T_{ji}\hat{z}_i$ from agent i. By substituting terms $\hat{z}_i = z_i - e_i$ and $\hat{z}_j = z_j - e_j$ inside the sum,

$$\begin{aligned}
\dot{e}_i &= \left(\bar{A}_i - L_i\bar{C}_i\right)e_i - \sum_{j\in\mathcal{N}_i} H_{ij}\left(z_j - e_j - T_{ji}(z_i - e_i)\right) \\
&= \left(\bar{A}_i - L_i\bar{C}_i\right)e_i + \sum_{j\in\mathcal{N}_i} H_{ij}\left(e_j - T_{ji}e_i\right) + \sum_{j\in\mathcal{N}_i} H_{ij}\left(T_{ji}z_i - z_j\right) \\
&= \left(\bar{A}_i - L_i\bar{C}_i\right)e_i
\end{aligned} \tag{3.16}$$

where the terms deleted, are the projection of state z_i, e_i to z_j and e_j, respectively. The structure from (2.14) is recalled with the consideration of all agents, with $e = [e_1, \ldots, e_m]'$ the error dynamic can be written in a compact form as,

$$\dot{e} = \left(\tilde{A} - \tilde{L}\tilde{C}\right)e \tag{3.17}$$

where the matrices $\tilde{A} = \text{blockdiag}(\bar{A}_1, \ldots, \bar{A}_m)$, $\tilde{C} = \text{blockdiag}(\bar{C}_1, \ldots, \bar{C}_m)$ and $\tilde{L} = \text{blockdiag}(L_1, \ldots, L_m$. As discuss in Section 2.3.2, the distributed detectability for the full system in (2.14) is achieved if the statements of Theorem 2.3.2 are satisfied, then the matrix $\left(\tilde{A} - \tilde{L}\tilde{C}\right)$ is Hurwitz and the error decreases exponential fast at a preassigned converge rate.

3.3.1 Considerations

The design introduced has some advantages of respect to the layout given in Section 2.4. Firstly, in the case of a fully observable system, the overall dimension of the observer is equal to $(nm) \times (nm)$, with n as plant dimension and m as the total number of agents. Instead the dimension proposed in the literature layout in (2.26), is $(mn + l) \times (nm + l)$, with $l = m - 1$, the number of states introduced by the dynamic compensator in order to stabilize the system. Secondly, the assumption of joint observability is substituted with the necessary condition of collective detectability, and in case of a fully detectable system, the overall dimension of the observer is even lower than before and equal to $\sum_{i=0}^{m} n_{o_i}$. Thirdly, the hypothesis for matrices $C_i \neq 0$, $\forall i \in \mathbf{m} = \{1, \ldots, m\}$ is relaxed and in the following chapter an example shows this achievement. Fourthly, the strong connectivity of graph $\mathbb{G}$ is not necessary to ensure the stability properties, as seen before, the detectability condition is only necessary inside a cluster of agents.

Chapter 4

Simulation Results

Simulations are presented in order to show the effectiveness of the proposed solution. Firstly a generic system with sinusoidal behavior, in which agents cooperate to estimate the plant dynamics. Then in the second section, the properties for the convergence of the observers are showed in a use case relates to the search and rescue problem. Matlab is the tool used to test the algorithm and to present the results.

4.1 System example

Specifically, it is posted a case in which one of the agents is without output measures, so, the condition from the literature approach in section 2.4 is exceeded. Furthermore, the graph is not strongly connected, in fact, one agent is only able to provide his information to the rest of the network and not to receive. Let's consider the following continuous-time system, as in Section 2.4.3,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix} x,$$

with $x \in \mathbb{R}^4$, which is being observed by four agents ($m = 4$) in such a way that

$$\begin{aligned} y_1 &= [1\ 0\ 0\ 0]\, x = C_1 x, \\ y_2 &= [0\ 0\ 0\ 0]\, x = C_2 x, \\ y_3 &= [0\ 0\ 1\ 0]\, x = C_3 x, \\ y_4 &= [0\ 0\ 0\ 1]\, x = C_4 x. \end{aligned}$$

each agent is able to see only one dynamic of the system, except for agent 2, and by considering the matrix $C = [C_1', C_2', C_3', C_4']'$ the requirement of jointly observability for (A, C) is satisfied. Then the following topology for the graph $\mathbb{G}$ of the agents is showed in

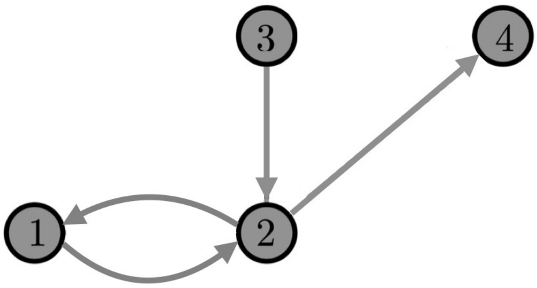


Figure 4.1: Graph $\mathbb{G}$ of simulation

figure 4.1 and the adjacency matrix is described as follows:

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.1}$$

The graph is weakly connected and agent 4 is the globally reachable node for $\mathbb{G}$. After the first step each observability transformation matrix $T_i^{[1]}$ and the number of observable states $n_i^{[1]}$ is computed.

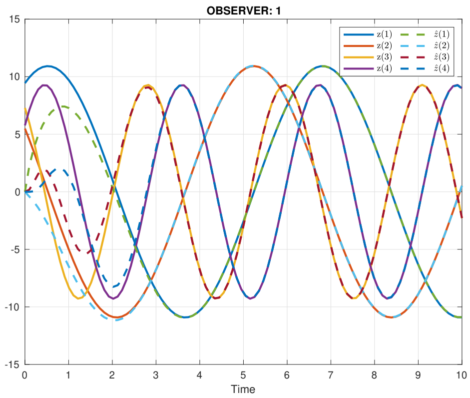
$$T_1^{[1]} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad T_2^{[1]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^{[1]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad T_4^{[1]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

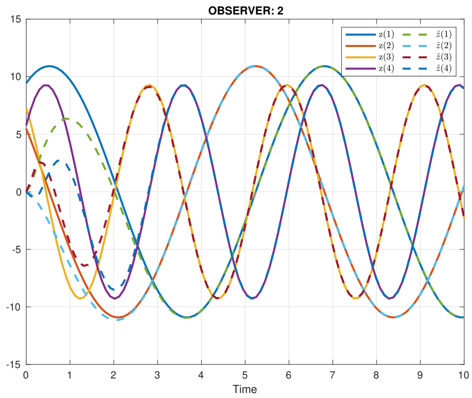
$n_2^{[1]} = 0$, is the number of dynamics seen by agent 2, instead all agents $n_1^{[1]} = n_3^{[1]} = n_4^{[1]} = 2$ have a number of observable states equal than two.

They send the transformation matrix to their neighbours in order to proceed with the second step. According to the second step, the output matrices are updated, and agent 2 has covered the lack of information thanks to agents 1 and 3 and is described as follows:

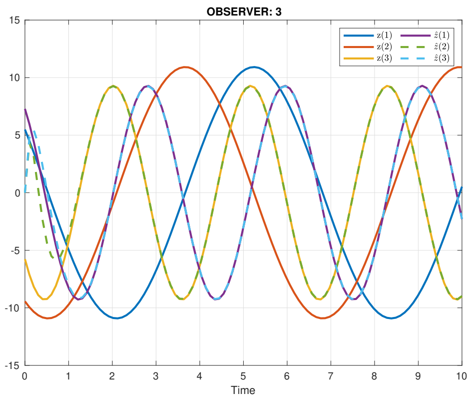
$$C_2^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



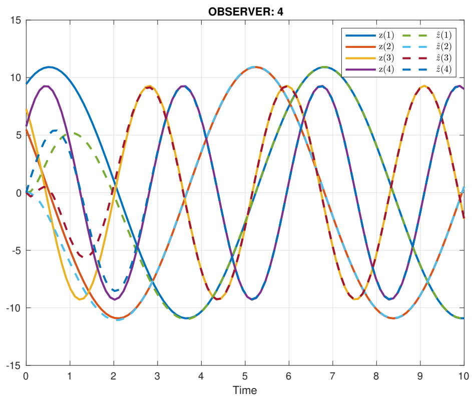
(a) Observer 1



(b) Observer 2



(c) Observer 3



(d) Observer 4

Figure 4.2: State of the plant in solid lines and estimates of agents in dashed lines

with the new output matrix, agent 2 is able to share the full knowledge of the state, $n_2^{[2]} = 4$, to agents 1 and 4. Instead, agents 1 and 4 still have to complete the full observability of the system, in fact, $T_1^{[1]} = T_1^{[2]}$ and $T_4^{[1]} = T_4^{[2]}$, and they have to wait for the end of the second step in order to receive the transformation matrix from agent 2. So, at the third step, the transformation matrices for agents 1 and 4 become identities,

$$T_1^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_4^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

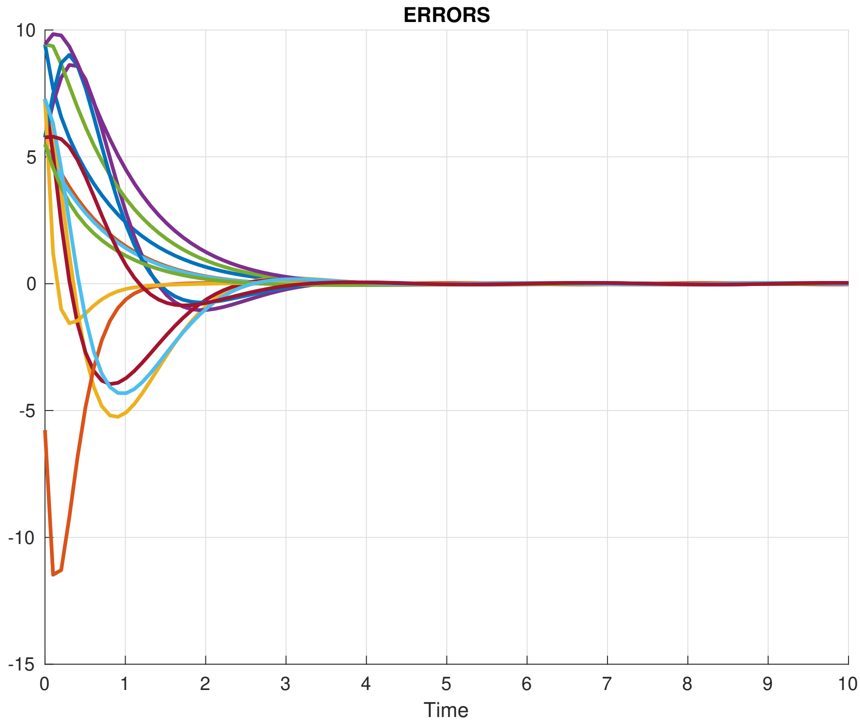
with a number of observable states equal than 4, $n_1^{[3]} = n_4^{[3]} = 4$. Agent 3 is not able to achieve the full observability of the system, the reason is related to the structure of the network, i.e. agent 3 has not in-coming edges. After the third step, the knowledge of the system for each agent remains the same, so it is possible to define the gain matrices for the observers. In this case, the full structure is described as follow:

$$\begin{aligned}
\dot{\hat{z}}_1 &= \left(\bar{A}_1 - L_1\bar{C}_1\right)\hat{z}_1 + L_1 y_1 + H_{12}\left(\hat{z}_2 - T_{21}\bar{z}_1\right) \\
\dot{\hat{z}}_2 &= \left(\bar{A}_2 - L_2\bar{C}_2\right)\hat{z}_2 + H_{21}\left(\hat{z}_1 - T_{12}\bar{z}_2\right) + H_{23}\left(\hat{z}_3 - T_{32}\bar{z}_2\right) \\
\dot{\hat{z}}_{o3} &= \left(A_{o3} - L_3 C_{o3}\right)\hat{z}_{o3} + L_3 y_{o3} \\
\dot{\hat{z}}_4 &= \left(\bar{A}_4 - L_4\bar{C}_4\right)\hat{z}_4 + L_4 y_4 + H_{42}\left(\hat{z}_2 - T_{24}\hat{z}_4\right)
\end{aligned}$$

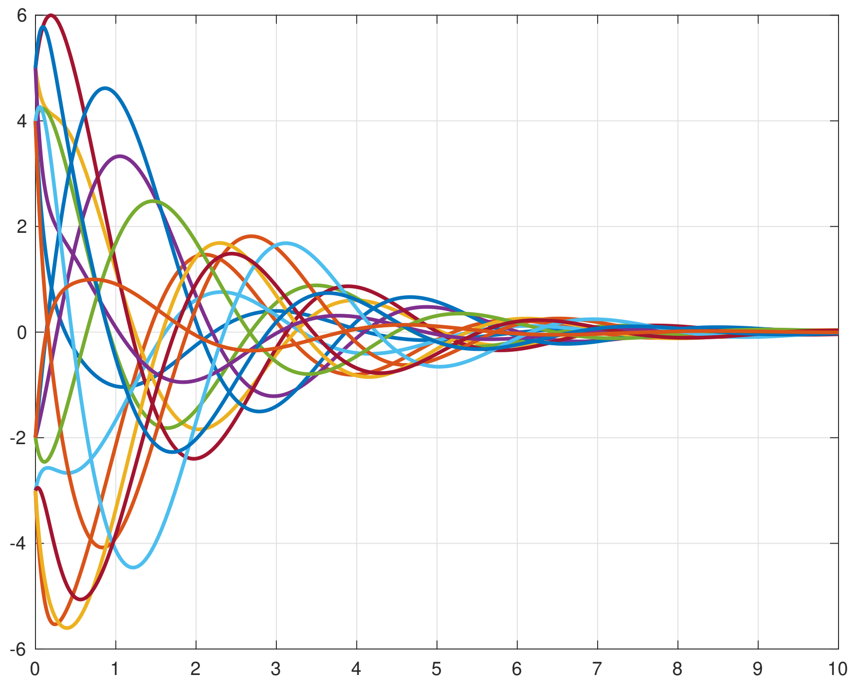
Agent 3 is estimating only the observable part of the system, that consists of the last two states, so the size of the gain L_3 is 2×2.

4.1.1 Comparison

The system previously described was implemented in a Matlab script and simulated. Fig 4.2 shows the evolution of the plant state modes, transformed with matrices $T_i^{[3]}$, in order to be projected into the subspace of the observers, and the estimates for the proposed distributed observer. Note that for agent 3, only two dynamics of the plant are followed, fig 4.2c. Furthermore, in the case of agent 2, the estimation of state variables is given from its neighbors. The design has been carried out placing the poles around -5 and the error dynamics from all the observers are shown in fig. 4.3a. The overall dimension reached by the observer is 14×14, and compare to the implementation with the design in Section 2.4, having an overall dimension of 19×19, with the error dynamics in fig. 4.3b, our design is more stable and efficient. The error dynamics presented in fig. 4.3b are given from a fully connected observer, with $C_2 = [0\ 1\ 0\ 0]$ and poles placed around -5.



(a) Error dynamics from agents with the design proposed



(b) Error dynamics from agents with the literature proposed

Figure 4.3: Evolution of errors for all the agents for the design proposed and for the literature one

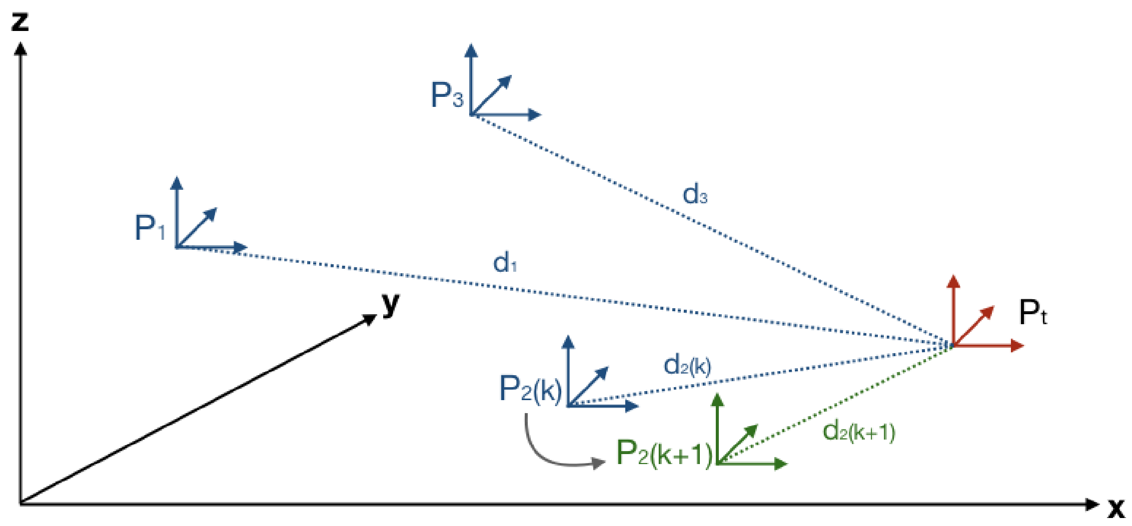


Figure 4.4: Simulated scenario for target position estimation

4.2 Simulated scenario

In this section, one use case is provided to better describe the implementation of the distributed estimation framework previously presented. Regarding three agents, with a strongly connected ring graph, which is able to estimate the 3D position of a target system by only sensing the range distance, named d_i for agent i. The system is states as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = Ax \quad \text{with } x = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} \tag{4.2}$$

where the state x is the 3D position of target t. At time k our agents are situated at positions $P_i(k) = [x_i(k),\ y_i(k),\ z_i(k)]'$ with $i = \{1,\ 2,\ 3\}$, as shown in fig. 4.4. The squared range distance $d_i(k)^2$ at time k sense from agent i is described as follows,

$$d_i^2(k) = (x_i(k) - x_t)^2 + (y_i(k) - y_t)^2 + (z_i(k) - z_t)^2$$

after a new step, named $k+1$, agent i is able to build his output matrix C_i as the difference between two consecutive steps.

$$\begin{aligned} d_i^2(k+1) - d_i^2(k) = {} & (x_i(k+1) - x_t)^2 - (x_i(k) - x_t)^2 + (y_i(k+1) - y_t)^2 - (y_i(k) - y_t)^2 + \\ & + (z_i(k+1) - z_t)^2 - (z_i(k) - z_t)^2 \\ & = x_i^2(k+1) - 2x_t x_i(k+1) + x_i^2(k) - 2x_t x_i(k) + y_i^2(k+1) - 2y_t y_i(k+1) + \\ & + y_i^2(k) - 2y_t y_i(k) + z_i^2(k+1) - 2z_t z_i(k+1) + z_i^2(k) - 2z_t z_i(k) \end{aligned}$$

where $x_i^2(k+1)$, $x_i^2(k)$, $x_i^2(k+1)$, $x_i^2(k)$, $x_i^2(k+1)$ and $x_i^2(k)$ are known quantities, so it is possible to define the new range $\hat{d}_i(k)$ as,

$$\begin{aligned}\hat{d}_i^2(k) &= d_i^2(k) - x_i^2(k) - y_i^2(k) - z_i^2(k) \\ &= -2x_t x_i(k) - 2y_t y_i(k) - 2z_t z_i(k)\end{aligned}$$

and the same for $\hat{d}_i^2(k+1)$, then the output matrix C_i can be states,

$$\begin{aligned}y_i &= \hat{d}_i^2(k+1) - \hat{d}_i^2(k) \\ &= -2\left(x_i(k+1) - x_i(k)\right)x_t - 2\left(y_i(k+1) - y_i(k)\right)y_t - 2\left(z_i(k+1) - z_i(k)\right)z_t \\ &= -2\begin{bmatrix} (x_i(k+1) - x_i(k)) & (y_i(k+1) - y_i(k)) & (z_i(k+1) - z_i(k)) \end{bmatrix}\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} \\ &= C_i x\end{aligned}$$

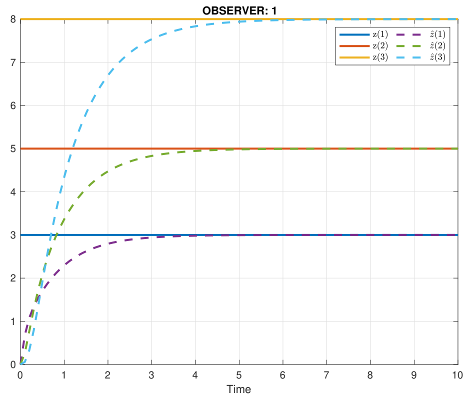
As we can see, the elements of C_i are different than zero if the new position reaches by agent i is not at the same coordinate respect to the previous one. Another important constraint imposes by the construction of matrix C_i is that all agents have to move orthogonally respect to each other, in order to ensure full observability of the system $\dot{x} = Ax$. In figures 4.5, are shown the dynamics of the three agents with the following output matrices,

$$\begin{aligned}y_1 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x = C_1 x \\ y_2 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x = C_2 x \\ y_3 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x = C_3 x\end{aligned}$$

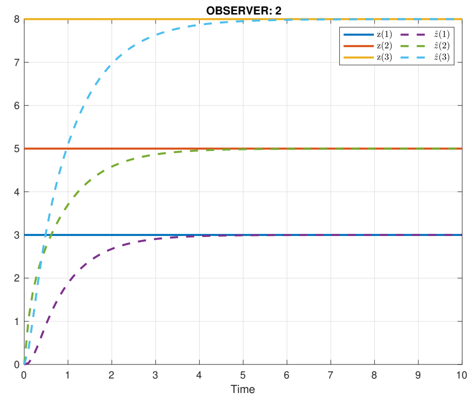
In this case, each agent moves along one coordinate, and the full observability is reached with one step. The target position is estimated by sharing the knowledge between agents and the error dynamics is presented in figure 4.5d.

4.3 Conclusion

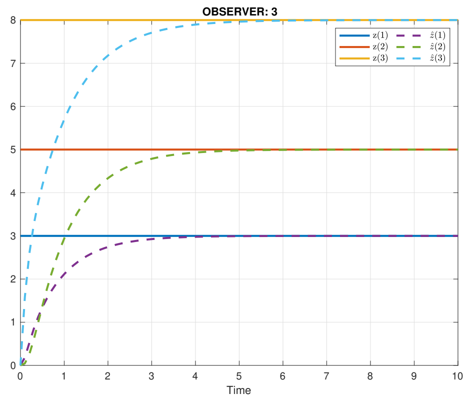
This thesis has introduced and detailed a new technique of the distributed state estimation for a class of linear systems. The design was developed in two main steps. The first step involved the design of an observability transformation for linear systems that arise in the application of a model reduction technique. The second step involved the design of the gain matrices used to reach the dynamic of a plant sense by distributed multiple sensors. Through the exchange of information from each neighboring agent that exploits the availability of multiple measurements, local state estimates can be obtained. The observer from the last step is shown to provide better performance than the decentralized controlled one.



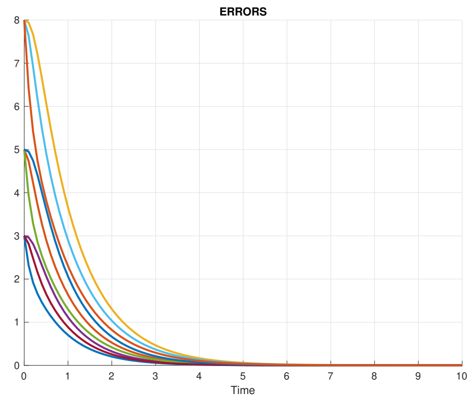
(a) Observer 1 scenario



(b) Observer 2 scenario



(c) Observer 3 scenario



(d) Errors for all agents in scenario

Figure 4.5: State of the plant in solid lines and estimates of agents in dashed lines

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