Solved selected problems of Analytical Mechanics by Nivaldo Lemos

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Solution. **1.1** Let

$$(x^2 + y^2)dx + xzdz = 0 \qquad (x^2 + y^2)dy + yzdz = 0$$

be constraints.

(a) We have two constraints that can be expressed as 1-forms $\omega^1 = 0$ and $\omega^2 = 0$ where

$$\omega^1 = (x^2 + y^2)dx + xzdz \qquad \omega^2 = (x^2 + y^2)dy + yzdz$$

Also, $d\omega^1$ and $d\omega^2$ are

$$d\omega^1 = 2ydy \wedge dx + zdx \wedge dz \quad \text{and} \quad d\omega^2 = 2xdx \wedge dy + zdy \wedge dz$$

So, let us consider them separately, i.e. we want to compute $d\omega^1 \wedge \omega^1$ and $d\omega^2 \wedge \omega^2$ as follows

$$\begin{aligned} d\omega^1 \wedge \omega^1 &= (2ydy \wedge dx + zdx \wedge dz) \wedge ((x^2 + y^2)dx + xzdz) \\ &= 2xyz\ dy \wedge dx \wedge dz \end{aligned}$$

And

$$\begin{aligned} d\omega^2 \wedge \omega^2 &= (2xdx \wedge dy + zdy \wedge dz) \wedge ((x^2 + y^2)dy + yzdz) \\ &= 2xyz\ dx \wedge dy \wedge dz \end{aligned}$$

We see that $d\omega^1 \wedge \omega^1 \neq 0$ unless $x = y = z = 0$ and the same happens for $d\omega^2 \wedge \omega^2$.

Therefore, they are not integrable when considered separately.

(b) The 2-form Ω defined by $\Omega = \omega^1 \wedge \omega^2$ is given by

$$\begin{aligned} \Omega &= (x^2 + y^2)dx + xzdz \wedge (x^2 + y^2)dy + yzdz \\ &= (x^2 + y^2)dx \wedge (x^2 + y^2)dy + (x^2 + y^2)dx \wedge yzdz \\ &\quad + xzdz \wedge (x^2 + y^2)dy \\ &= (x^2 + y^2)^2 dx \wedge dy + yz(x^2 + y^2)dx \wedge dz + xz(x^2 + y^2)dz \wedge dy \end{aligned}$$

It follows that

$$\begin{aligned}d\omega^1 \wedge \Omega &= 2y(x^2+y^2)^2 dy \wedge dx \wedge dx \wedge dy + 2y^2 z(x^2+y^2) dy \wedge dx \wedge dx \wedge dz \\ &\quad + 2yxz(x^2+y^2) dy \wedge dx \wedge dz \wedge dy + z(x^2+y^2)^2 dx \wedge dz \wedge dy \wedge dx \\ &\quad + yz^2(x^2+y^2) dx \wedge dz \wedge dx \wedge dz + xz^2(x^2+y^2) dx \wedge dz \wedge dz \wedge dy \\ &= 0\end{aligned}$$

And in the same way $d\omega^2 \wedge \Omega = 0$.

Therefore, the constraints are integrable when considered together.

On the other hand, let $x, y, z \neq 0$ then we see that

$$d \ln \frac{y}{x} = \frac{1}{y} dy - \frac{1}{x} dx$$

Also, let us consider the following

$$\begin{aligned}y\omega^1 - x\omega^2 &= y(x^2+y^2)dx + xyzdz - x(x^2+y^2)dy - xyzdz \\ &= (x^2+y^2)(ydx - xdy)\end{aligned}$$

But since $\omega^1 = \omega^2 = 0$ then must be that

$$\begin{aligned}ydx - xdy &= 0 \\ \frac{1}{y}dy - \frac{1}{x}dx &= 0\end{aligned}$$

Then the constraints are equivalent to $d \ln(y/x) = 0$.

Now, considering the combination

$$\begin{aligned}x\omega^1 + y\omega^2 &= x(x^2+y^2)dx + x^2 zdz + y(x^2+y^2)dy + y^2 zdz \\ &= (x^2+y^2)(xdx + ydy + zdz)\end{aligned}$$

Must be that $xdx + ydy + zdz = 0$ but we see that

$$d(x^2+y^2+z^2) = xdx + ydy + zdz$$

Therefore, the constraints are equiavalent also to $xd(x^2+y^2+z^2) = 0$.

(c) Finally, integrating the equations

$$xd(x^2 + y^2 + z^2) = 0 \quad \text{and} \quad d \ln \frac{y}{x} = 0$$

We get that

$$\begin{aligned} \int d(x^2 + y^2 + z^2) &= 0 \\ x^2 + y^2 + z^2 + C_1' &= 0 \\ x^2 + y^2 + z^2 &= C_1 \end{aligned}$$

And

$$\begin{aligned} \int d \ln \frac{y}{x} &= 0 \\ \ln \frac{y}{x} + C_2' &= 0 \\ \frac{y}{x} &= 1 - e^{C_2'} \\ y &= C_2 x \end{aligned}$$

Where we renamed $C_1 = -C_1'$ and $C_2 = 1 - e^{C_2'}$.

□

Solution. **1.2** Let us consider the system from Example 1.18 in the case where $m_1 = m_2 = m_3 = m$ then the equations of motion become

$$\begin{aligned} m\ddot{x}_2 - \frac{k}{2}(x_3 - x_2 - l) &= 0 \\ m\ddot{x}_3 - mg + k(x_3 - x_2 - l) &= 0 \end{aligned}$$

Let us define $y = x_3 - x_2 - l$ then

$$\ddot{y} = \ddot{x}_3 - \ddot{x}_2$$

Then, subtracting the first equation from the second one we get that

$$\begin{aligned} m\ddot{x}_3 + k(x_3 - x_2 - l) - m\ddot{x}_2 + \frac{k}{2}(x_3 - x_2 - l) &= mg \\ (\ddot{x}_3 - \ddot{x}_2) + \frac{3k}{2m}(x_3 - x_2 - l) &= g \\ \ddot{y} + \frac{3k}{2m}y &= g \end{aligned}$$

We know that in terms of y the solution to this equation is

$$y(t) = C_1 \sin\left(\sqrt{\frac{3k}{2m}}t\right) + C_2 \cos\left(\sqrt{\frac{3k}{2m}}t\right) + \frac{2gm}{3k}$$

or letting $\omega = \sqrt{\frac{3k}{2m}}$ we get that

$$y(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t) + \frac{2mg}{3k}$$

Using the initial conditions $x_2 = 0$ and $x_3 = l$ we can determine C_2. We see that $y(t = 0) = l - 0 - l = 0$ then $C_2 = -\frac{2mg}{3k}$.
On the other hand,

$$\dot{y}(t) = C_1\omega \cos(\omega t) - C_2\omega \sin(\omega t)$$

And since the system starts from rest i.e. $\dot{y} = 0$ we get that $C_1 = 0$. Therefore the equation becomes

$$y(t) = \frac{2mg}{3k}(1 - \cos(\omega t)) = x_3 - x_2 - l$$

So, replacing in the first original differential equation we get that

$$\ddot{x}_2 + \frac{g}{3}(\cos(\omega t) - 1) = 0$$

Integrating with respect to t gives us

$$\dot{x}_2 = -\frac{g}{3}\int (\cos(\omega t) - 1)\ dt = \frac{g}{3\omega}\Big[\omega t - \sin(\omega t)\Big] + C$$

Where $C = 0$ since $\dot{x}_2 = 0$ at $t = 0$. Integrating again

$$
\begin{aligned}
x_2 &= \frac{g}{3} \int \left[-\frac{1}{\omega} \sin(\omega t) + t \right] dt \\
&= \frac{g}{3} \frac{\cos(\omega t)}{\omega^2} + \frac{g}{6} t^2 + C \\
&= \frac{2mg}{9k} \cos(\omega t) + \frac{g}{6} t^2 + C
\end{aligned}
$$

Where we replaced the value of ω. Now since $x_2 = 0$ when $t = 0$ we have that $C = -2mg/9k$, so, finally

$$
x_2 = \frac{2mg}{9k}(\cos(\omega t) - 1) + \frac{g}{6} t^2
$$

From the equation we got for $\dot{x}_2$, let $t = \pi/2\omega$, at this moment $\sin(\omega t) = 1$ which is as negative as $-\sin(\omega t)$ can get. Hence

$$
\omega t - \sin(\omega t) = \frac{\pi}{2} - 1
$$

But since $\pi/2 > 1$, then, $\dot{x}_2 > 0$ at this moment. For any point later, t will grow even more and hence $\omega t - \sin(\omega t)$ will be positive.
Therefore if $t > 0$ then $\dot{x}_2 > 0$. $\square$

Solution. **1.5** Let us consider Atwood's machine shown in Fig. 1.10. The string connecting M and m must be inextensible, so, let x be the vertical coordinate which determines the position of M, then must be that $x+c+r = l$ where l is the constant length of the string and c is the length between pulleys.
The kinetic energy experienced by M is then $T_M = M\dot{x}^2/2 = M\dot{r}^2/2$.
So the kinetic energy of the system is

$$T = \frac{M}{2}\dot{r}^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{m+M}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2$$

On the other hand, for the potential energy since the string is inextensible then any change in the r coordinate will imply a change in potential energy for the M mass so we can write that

$$V = Mgr - mgr\cos\theta = gr(M - m\cos\theta)$$

Therefore, the Lagrangian is

$$L = T - V = \frac{m+M}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - gr(M - m\cos\theta)$$

Now, we want to write Lagrange's equations for the system. We compute first the derivatives

$$\frac{\partial L}{\partial \dot{r}} = (m+M)\dot{r} \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - g(M - m\cos\theta)$$
$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

Then the Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = (m+M)\ddot{r} - mr\dot{\theta}^2 + g(M - m\cos\theta) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = mr(2\dot{\theta}\dot{r} + r\ddot{\theta}) + mgr\sin\theta = 0$$

□

Solution. **1.6** Let us consider Huygens' cycloidal pendulum with the following parametric equations

$$x = R(\theta - \sin\theta) \qquad y = R(1 - \cos\theta)$$

where the vertical y-axis points downward.
Then the kinetic energy of the system is given by

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \\ &= \frac{m}{2}R^2[(\dot{\theta} - \dot{\theta}\cos\theta)^2 + \dot{\theta}^2\sin^2\theta] \\ &= \frac{m}{2}R^2[\dot{\theta}^2 - 2\dot{\theta}^2\cos\theta + \dot{\theta}^2\cos^2\theta + \dot{\theta}^2\sin^2\theta] \\ &= \frac{m}{2}R^2[1 - 2\cos\theta + 1]\dot{\theta}^2 \\ &= 2mR^2\left[\frac{1-\cos\theta}{2}\right]\dot{\theta}^2 \\ &= 2mR^2\sin^2\left(\frac{\theta}{2}\right)\dot{\theta}^2 \end{aligned}$$

Where we used the trigonometric identity $\sin^2(\theta/2) = (1-\cos\theta)/2$.
On the other hand, the potential energy only depends on y and since the y-axis points downward we have that

$$V = -mgy = -mgR(1-\cos\theta)$$

Therefore, the Lagrangian of the system is

$$L = T - V = 2mR^2\sin^2\left(\frac{\theta}{2}\right)\dot{\theta}^2 + mgR(1-\cos\theta)$$

Let now, $u = \cos(\theta/2)$ then $\theta = 2\arccos(u)$, so the Lagrangian in terms of u is given by

$$\begin{aligned} L &= 2mR^2\sin^2(\arccos(u))\left(-\frac{2\dot{u}}{\sqrt{1-u^2}}\right)^2 + mgR(1-\cos(2\arccos(u))) \\ L &= 2mR^2(1-u^2)\left(\frac{4\dot{u}^2}{1-u^2}\right) + mgR(1-2\cos^2(\arccos(u))+1) \\ L &= 8mR^2\dot{u}^2 + 2mgR(1-u^2) \end{aligned}$$

Then the Lagrange equation in terms of u is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = 16mR^2\ddot{u} + 4mgRu = 0$$

Finally, we can write this equation as

$$\begin{aligned} 4R^2\ddot{u} + gRu &= 0 \\ \ddot{u} &= -\frac{g}{4R}u \end{aligned}$$

This is the equation of a simple harmonic oscillator where $\omega^2 = g/4R$, or in terms of the period of oscillation $\omega = 2\pi/T$ then we get that

$$\begin{aligned} \frac{4\pi^2}{T^2} &= \frac{g}{4R} \\ T^2 &= \frac{16\pi^2 R}{g} \\ T &= 4\pi\sqrt{\frac{R}{g}} \end{aligned}$$

$\square$

Solution. **1.7** Let a projectile fired near the surface of the earth. Taking the Cartesian coordinates as the generalized coordinates we have that the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2)$$

And the potential energy is

$$V = mgy$$

Therefore the lagrangian is given by

$$L = T - V = \frac{1}{2}m(v_x^2 + v_y^2) - mgy$$

We solve then Lagrange's equation to get the equations of motion as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial v_x} &= m\dot{v}_x + \lambda v_x = 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_y}\right) - \frac{\partial L}{\partial y} + \frac{\partial \mathcal{F}}{\partial v_y} &= m\dot{v}_y + mg + \lambda v_y = 0
\end{aligned}$$

Then the equations of motion are

$$\ddot{x} + \frac{\lambda}{m}\dot{x} = 0 \qquad \ddot{y} + \frac{\lambda}{m}\dot{y} = -g$$

□

Solution. **1.8**

(a) Let us define the following lagrangian

$$L = e^{\lambda t/m}\left[\frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy\right]$$

Lagrange's equation for the x coordinate is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t/m} m\dot{x}\right) &= 0 \\
me^{\lambda t/m}\ddot{x} + \lambda e^{\lambda t/m}\dot{x} &= 0 \\
\ddot{x} + \frac{\lambda}{m}\dot{x} &= 0
\end{aligned}$$

And Lagrange's equation for the y coordinate is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t/m} m\dot{y}\right) + mge^{\lambda t/m} &= 0 \\
me^{\lambda t/m}\ddot{y} + \lambda e^{\lambda t/m}\dot{y} + mge^{\lambda t/m} &= 0 \\
\ddot{y} + \frac{\lambda}{m}\dot{y} &= -g
\end{aligned}$$

(b) Let us solve the equations of motion now assuming the projectile is fired from the origin with a velocity of magnitude v_0 making an angle θ_0 with the horizontal.

We see that the equation for x can be solved by integration, first let us rename $v = \dot{x}$ then the equation becomes

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{\lambda}{m}v \\
\int \frac{dv}{v} &= -\frac{\lambda}{m}\int dt \\
\log(v) &= -\frac{\lambda}{m}t + C \\
v &= c_1 e^{-\lambda t/m}
\end{aligned}$$

Now, replacing back again v and integrating again we get that

$$\begin{aligned}
\int dx &= c_1 \int e^{-\lambda t/m} dt \\
x &= -c_1\frac{m}{\lambda}e^{-\lambda t/m} + c_2
\end{aligned}$$

We can determine the values of c_1 and c_2 by plugging in the initial conditions. We know that $x(t=0) = 0$ hence $0 = c_2 - c_1 m/\lambda$.

Also, we know that $\dot{x}(t=0) = v_0 \cos\theta_0$ so since $\dot{x} = c_1 e^{-\lambda t/m}$ we get that $c_1 = v_0 \cos\theta_0$, and using this result we get that $c_2 = v_0 m \cos\theta_0/\lambda$. Then the equation for x becomes

$$x = \frac{v_0 \cos\theta_0 m}{\lambda}(1 - e^{-\lambda t/m})$$

On the other hand, to solve the equation for y first we solve the homogeneous equation $\ddot{y} + \lambda\dot{y}/m = 0$ and then we add a particular solution to get the general solution. The homogeneous equation has the same form as the one we solved for x so the solution is

$$y = c_2 - c_1 \frac{m}{\lambda} e^{-\lambda t/m}$$

Now, let us try the particular solution $y = -(mg/\lambda)t$ then we see that $\dot{y} = -mg/\lambda$ and $\ddot{y} = 0$ so we get that

$$0 + \frac{\lambda}{m}\left(- \frac{mg}{\lambda}\right) = -g$$

So $y = -(mg/\lambda)t$ is a particular solution. Then the general solution is

$$y = c_2 - c_1 \frac{m}{\lambda} e^{-\lambda t/m} - \frac{mg}{\lambda} t$$

When $t = 0$ we get that $y(t=0) = 0$ then $c_2 = c_1 m/\lambda$. Also, we have that $\dot{y} = c_1 e^{-\lambda t/m} - mg/\lambda$ and when $t = 0$ we know that $\dot{y} = v_0 \sin\theta_0$ so $c_1 = v_0 \sin\theta_0 + mg/\lambda$ and hence $c_2 = v_0 \sin\theta_0 m/\lambda + m^2 g/\lambda^2$. Therefore the general solution becomes

$$y = \left(v_0 \sin\theta_0 \frac{m}{\lambda} + \frac{m^2 g}{\lambda^2}\right)(1 - e^{-\lambda t/m}) - \frac{mg}{\lambda} t$$

(c) From the equation for x we can get the time t as follows

$$\begin{aligned} 1 - e^{-\lambda t/m} &= \frac{\lambda x}{m v_0 \cos\theta_0} \\ -\frac{\lambda t}{m} &= \log\left(1 - \frac{\lambda x}{m v_0 \cos\theta_0}\right) \\ t &= -\frac{m}{\lambda}\log\left(1 - \frac{\lambda x}{m v_0 \cos\theta_0}\right) \end{aligned}$$

Now, replacing t in the equation for y we get the equation for the trajectory of the projectile

$$\begin{aligned} y &= \left(v_0 \sin\theta_0 \frac{m}{\lambda} + \frac{m^2 g}{\lambda^2}\right) \frac{\lambda x}{m v_0 \cos\theta_0} + \frac{m^2 g}{\lambda^2} \log\left(1 - \frac{\lambda x}{m v_0 \cos\theta_0}\right) \\ y &= \left(\tan\theta_0 + \frac{mg}{\lambda v_0 \cos\theta_0}\right) x + \frac{m^2 g}{\lambda^2} \log\left(1 - \frac{\lambda x}{m v_0 \cos\theta_0}\right) \end{aligned}$$

□

***Solution.* 1.9**

(a) Taking the generalized coordinate x from the middle point when both springs are at its natural length and θ measured from vertical then the coordinates of the mass m are

$$\begin{aligned} x' &= x + l\sin\theta \\ y' &= l\cos\theta \end{aligned}$$

So the velocities of m are

$$\begin{aligned} \dot{x'} &= \dot{x} + \dot{\theta} l\cos\theta \\ \dot{y'} &= -\dot{\theta} l\sin\theta \end{aligned}$$

Then the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m((\dot{x} + \dot{\theta} l\cos\theta)^2 + (-\dot{\theta} l\sin\theta)^2) \\ &= \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta} l\cos\theta + \dot{\theta}^2 l^2\cos^2\theta + \dot{\theta}^2 l^2\sin^2\theta) \\ &= \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta} l\cos\theta + \dot{\theta}^2 l^2) \end{aligned}$$

And the potential energy is

$$V = -mgl\cos\theta + \frac{1}{2}k(x^2 + x^2) = -mgl\cos\theta + kx^2$$

Where we assumed that the natural length of the springs is 0 for simplicity.
Then the Lagrangian is given by

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta} l\cos\theta + \dot{\theta}^2 l^2) + mgl\cos\theta - kx^2$$

Now, we solve the Lagrange's equation as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\Big(m(\dot{x} + \dot{\theta} l\cos\theta)\Big) + 2kx &= 0 \\ m(\ddot{x} + \ddot{\theta} l\cos\theta - \dot{\theta}^2 l\sin\theta) + 2kx &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\Big(m(\dot{x} l\cos\theta + \dot{\theta} l^2)\Big) + m\dot{x}\dot{\theta} l\sin\theta + mgl\sin\theta &= 0 \\ m(\ddot{x} l\cos\theta - \dot{x}\dot{\theta} l\sin\theta + \ddot{\theta} l^2) + m\dot{x}\dot{\theta} l\sin\theta + mgl\sin\theta &= 0 \\ m(\ddot{x} l\cos\theta + \ddot{\theta} l^2) + mgl\sin\theta &= 0 \end{aligned}$$

(b) In the case of small angular oscillations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ and terms such as θ^n and $\dot{\theta}^n$ can be neglected. Then the equations of motion become

$$m(\ddot{x} + \ddot{\theta}l) + 2kx = 0$$
$$m(\ddot{x} + \ddot{\theta}l) + mg\theta = 0$$

Then subtracting both equations we see that

$$2kx - mg\theta = 0$$
$$x = \frac{mg}{2k}\theta$$

So x can be written as $x = \alpha\theta$ where $\alpha = mg/2k$.

Now, computing $\ddot{x} = \alpha\ddot{\theta}$ and replacing it in the second approximated equation of motion gives us

$$m(\alpha\ddot{\theta} + \ddot{\theta}l) + mg\theta = 0$$
$$\ddot{\theta}(l + \alpha) + g\theta = 0$$
$$\ddot{\theta} + \frac{g}{l'}\theta = 0$$

We see that the last equation is the differential equation which governs a small oscillation pendulum of length l' where l' is

$$l' = l + \frac{mg}{2k}$$

□

Solution. **1.10** Let the force between two charged particles in motion be

$$F = \frac{ee'}{r^2}\left[1 + \frac{r\ddot{r}}{c^2} - \frac{\dot{r}^2}{2c^2}\right] = ee'\left[\frac{1}{r^2} + \frac{\ddot{r}}{rc^2} - \frac{\dot{r}^2}{2r^2c^2}\right]$$

Let us define the generalised potential as follows

$$U = ee'\left[\frac{1}{r} + \frac{\dot{r}^2}{2rc^2}\right]$$

Then we see that F can be written as

$$\begin{aligned} F &= -\frac{\partial U}{\partial r} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{r}}\right) \\ &= ee'\left[\left(\frac{1}{r^2} + \frac{\dot{r}^2}{2r^2c^2}\right) + \left(\frac{\ddot{r}}{rc^2} - \frac{\dot{r}^2}{r^2c^2}\right)\right] \\ &= ee'\left[\frac{1}{r^2} + \frac{\ddot{r}}{rc^2} - \frac{\dot{r}^2}{2r^2c^2}\right] \end{aligned}$$

So U is the generalised potential associated to F.
Now, if a charge is in the presence of another charge held fixed at the origin, then the force F is the force between them, but r is the polar radial coordinate of the free charge. Then the kinetic energy of the system is

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

Where m is the mass of the charge. Then the Lagrangian of the system is

$$L = T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - ee'\left[\frac{1}{r} + \frac{\dot{r}^2}{2rc^2}\right]$$

Therefore the Lagrange's equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r} - \frac{ee'\dot{r}}{rc^2}\right) - \left[mr\dot{\theta}^2 + ee'\left[\frac{1}{r^2} + \frac{\dot{r}^2}{2r^2c^2}\right]\right] &= 0 \\ m\ddot{r} - ee'\left[\frac{\ddot{r}}{rc^2} - \frac{\dot{r}^2}{r^2c^2}\right] - \left[mr\dot{\theta}^2 + ee'\left[\frac{1}{r^2} + \frac{\dot{r}^2}{2r^2c^2}\right]\right] &= 0 \\ m(\ddot{r} - r\dot{\theta}^2) - ee'\left[\frac{1}{r^2} + \frac{\ddot{r}}{rc^2} - \frac{\dot{r}^2}{2r^2c^2}\right] &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ mr^2\dot{\theta} &= 0 \end{aligned}$$

But, we see from the last equation that $\dot{\theta} = 0$ then the first equation of motion becomes

$$m\ddot{r} = ee'\left[\frac{1}{r^2} + \frac{\ddot{r}}{rc^2} - \frac{\dot{r}^2}{2r^2c^2}\right]$$

□

Solution. **1.16** Let Bateman's Lagrangian

$$L = e^{\lambda t}\left(\frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2\right)$$

Then Lagrange's equation for this Lagrangian is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t} m\dot{x}\right) - e^{\lambda t} m\omega^2 x &= 0 \\
\lambda e^{\lambda t} m\dot{x} + e^{\lambda t} m\ddot{x} - e^{\lambda t} m\omega^2 x &= 0 \\
\ddot{x} + \lambda\dot{x} - \omega^2 x &= 0
\end{aligned}$$

We see that this is the equation of motion of a damped harmonic oscillator. Let us define $q = e^{\lambda t/2}x$ then we see that $q^2 = e^{\lambda t}x^2$, also, from the derivative of q with respect to time we see that

$$\begin{aligned}
\dot{q} &= \frac{\lambda}{2}e^{\lambda t/2}x + e^{\lambda t/2}\dot{x} \\
e^{\lambda t/2}\dot{x} &= \dot{q} - \frac{\lambda}{2}e^{\lambda t/2}x \\
e^{\lambda t/2}\dot{x} &= \dot{q} - \frac{\lambda}{2}q \\
e^{\lambda t}\dot{x}^2 &= \left(\dot{q} - \frac{\lambda}{2}q\right)^2
\end{aligned}$$

Then the Lagrangian L becomes

$$L = e^{\lambda t}\frac{m}{2}\dot{x}^2 - e^{\lambda t}\frac{m\omega^2}{2}x^2 = \frac{m}{2}\left(\dot{q} - \frac{\lambda}{2}q\right)^2 - \frac{m\omega^2}{2}q^2$$

Hence the Lagrangian doesn't have an explicit time dependence. $\square$

Solution. **1.18**

(a) Let us consider Atwood's machine in terms of $x = x_1$ with a massive pulley and a massive string with uniform linear mass density λ.
The kinetic energy of the system is then given by

$$T = \frac{1}{2}(m_1 + m_2 + m_s)\dot{x}^2 + \frac{1}{2}I\omega^2$$

Where the first term takes into account the masses m_1, m_2 and the mass of the string m_s. The second term is the rotational kinetic energy because of the massive pulley, but we know that $\omega = \dot{x}/R$ where R is the radius of the pulley, then

$$T = \frac{1}{2}(m_1 + m_2 + m_s + I/R^2)\dot{x}^2$$

On the other hand, the potential energy because of the masses m_1 and m_2 is $-m_1gx + m_2gx$ since when one mass goes up the other goes down by the same amount.
Now, for the string, let l be the length of the hanging part of the string, then the mass of the string on the left side is $\lambda x_1 = \lambda x$ and the mass on the right side is then $\lambda(l - x)$. Taking into account the displacements of the centers of mass of each side we get that the total potential energy is

$$\begin{aligned} V &= -m_1gx + m_2gx - \lambda xg\frac{x}{2} - \lambda(l - x)g\frac{(l - x)}{2} \\ &= (m_2 - m_1)gx - \frac{\lambda g}{2}x^2 - \frac{\lambda g}{2}(l - x)^2 \end{aligned}$$

Therefore the Lagrangian of the system is

$$L = \frac{1}{2}(m_1 + m_2 + m_s + I/R^2)\dot{x}^2 + (m_1 - m_2)gx + \frac{\lambda g}{2}x^2 + \frac{\lambda g}{2}(l - x)^2$$

(b) Lagrange's equation for x is then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\big((m_1 + m_2 + m_s + I/R^2)\dot{x}\big) - (m_1 - m_2)g - \lambda gx + \lambda g(l - x) &= 0 \\ (m_1 + m_2 + m_s + I/R^2)\ddot{x} - (m_1 - m_2)g - \lambda gx + \lambda g(l - x) &= 0 \end{aligned}$$

(c) Let $m_1 = m_2 = 0$ then the equation of motion becomes

$$\begin{aligned} (m_s + I/R^2)\ddot{x} - \lambda gx + \lambda g(l - x) &= 0 \\ (m_s + I/R^2)\ddot{x} - 2\lambda gx + \lambda gl &= 0 \\ \ddot{x} - b^2x + \frac{b^2 l}{2} &= 0 \end{aligned}$$

Where $b = \sqrt{2\lambda g/(m_s + I/R^2)}$. We know that the solution to this differential equation is

$$x(t) = c_1 e^{bt} + c_2 e^{-bt} + \frac{l}{2}$$

From the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ we can determine c_1 and c_2, so we get that

$$c_1 + c_2 = x_0 - \frac{l}{2}$$

And from $\dot{x}(t) = c_1 b e^{bt} - c_2 b e^{-bt}$ we get that $0 = c_1 - c_2$ so

$$c_1 = c_2 = \frac{1}{2}\left(x_0 - \frac{l}{2}\right)$$

Therefore

$$\begin{aligned} x(t) &= \frac{1}{2}\left(x_0 - \frac{l}{2}\right)(e^{bt} + e^{-bt}) + \frac{l}{2} \\ &= \left(x_0 - \frac{l}{2}\right)\frac{e^{bt} + e^{-bt}}{2} + \frac{l}{2} \\ &= \left(x_0 - \frac{l}{2}\right)\cosh bt + \frac{l}{2} \end{aligned}$$

If $x_0 = l/2$ we get that $x(t) = l/2$ which is the right physically expected solution since each half of the string is hanging statically on each side of the pulley.

(d) The exponential behaviour of the speed of the string for t larger than a few time constants is physically correct since we are considering the case $m_1 = m_2 = 0$, so the string would slip and fall off the pulley increasing its velocity infinitely.

□

Solution. **1.19** Let the following lagrangian

$$L = \frac{1}{2}\frac{1}{1+\lambda x^2}\left(\dot{x}^2 - \omega_0^2 x^2\right) \quad \text{with} \quad \lambda > 0$$

(a) Then Lagrange's equation for x is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{1+\lambda x^2}\right) - \frac{1}{2}\left(-\frac{2\lambda x}{(1+\lambda x^2)^2}(\dot{x}^2 - \omega_0^2 x^2) - \frac{2\omega_0^2 x}{1+\lambda x^2}\right) &= 0 \\
\left(\frac{\ddot{x}}{1+\lambda x^2} - \frac{2\lambda x\dot{x}^2}{(1+\lambda x^2)^2}\right) + \left(\frac{\lambda x}{(1+\lambda x^2)^2}(\dot{x}^2 - \omega_0^2 x^2) + \frac{\omega_0^2 x}{1+\lambda x^2}\right) &= 0 \\
\left(\frac{\ddot{x}(1+\lambda x^2) - 2\lambda x\dot{x}^2}{(1+\lambda x^2)^2}\right) + \frac{\lambda x\dot{x}^2 - \omega_0^2\lambda x x^2 + \omega_0^2 x(1+\lambda x^2)}{(1+\lambda x^2)^2} &= 0 \\
\ddot{x}(1+\lambda x^2) - 2\lambda x\dot{x}^2 + \lambda x\dot{x}^2 - \omega_0^2\lambda x x^2 + \omega_0^2 x(1+\lambda x^2) &= 0 \\
\ddot{x}(1+\lambda x^2) - \lambda x\dot{x}^2 + \omega_0^2 x &= 0
\end{aligned}$$

(b) Let

$$x = A\sin(\omega t + \delta)$$

Then

$$\dot{x} = A\omega\cos(\omega t + \delta) \quad \text{and} \quad \ddot{x} = -A\omega^2\sin(\omega t + \delta)$$

So replacing in the equation of motion we got from part (a) we see that

$$\begin{aligned}
&\ddot{x}(1+\lambda x^2) - \lambda x\dot{x}^2 + \omega_0^2 x = 0 \\
&-A\omega^2\sin(\omega t+\delta)(1+\lambda A^2\sin^2(\omega t+\delta)) - \lambda A\sin(\omega t+\delta)A^2\omega^2\cos^2(\omega t+\delta) \\
&\quad + \omega_0^2 A\sin(\omega t+\delta) = 0 \\
&-A\omega^2\sin(\omega t+\delta) - \lambda A^3\omega^2\sin^3(\omega t+\delta) - \lambda A^3\omega^2\sin(\omega t+\delta)\cos^2(\omega t+\delta) \\
&\quad + \omega_0^2 A\sin(\omega t+\delta) = 0
\end{aligned}$$

We see that if $t = n\pi/\omega - \delta$ for $n = 0, 1, ...$ the equation is satisfied, so let us assume $t \neq n\pi/\omega - \delta$ and hence $\sin(\omega t + \delta) \neq 0$ then dividing by $\sin(\omega t + \delta)$ we get that

$$\begin{aligned}
-A\omega^2 - \lambda A^3\omega^2\sin^2(\omega t+\delta) - \lambda A^3\omega^2\cos^2(\omega t+\delta) + \omega_0^2 A &= 0 \\
-A\omega^2 - \lambda A^3\omega^2(\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)) + \omega_0^2 A &= 0 \\
\omega^2 + \lambda A^2\omega^2 - \omega_0^2 &= 0
\end{aligned}$$

So we see that this equation is satisfied if $\omega^2 + \lambda A^2\omega^2 = \omega_0^2$. Therefore ω^2 must have the following value

$$\omega^2 = \frac{\omega_0^2}{1+\lambda A^2}$$

□

Solution. **1.20** From equation (1.162) and taking $V(r) = -\kappa/r$ we get that

$$E = \frac{m}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{l^2}{2mr^2} - \frac{\kappa}{r}$$

So

$$\begin{aligned}\frac{m}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 &= E + \frac{\kappa}{r} - \frac{l^2}{2mr^2} \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \sqrt{\frac{2}{m}}\sqrt{E + \frac{\kappa}{r} - \frac{l^2}{2mr^2}}\end{aligned}$$

Now, we integrate from $t = 0$ where $r = r_{min}$ (since the planet is at the perihelion) to t and r

$$\int_0^t dt = \sqrt{\frac{m}{2}}\int_{r_{min}}^{r} \frac{dr}{\sqrt{E + \frac{\kappa}{r} - \frac{l^2}{2mr^2}}}$$

Also, we see that

$$\begin{aligned}t &= \sqrt{\frac{m}{2}}\int_{r_{min}}^{r} \frac{dr}{\sqrt{\frac{\kappa}{r^2}\left(\frac{Er^2}{\kappa} + r - \frac{l^2}{2m\kappa}\right)}} \\ &= \sqrt{\frac{m}{2\kappa}}\int_{r_{min}}^{r} \frac{rdr}{\sqrt{\frac{Er^2}{\kappa} + r - \frac{l^2}{2m\kappa}}} \\ &= \sqrt{\frac{m}{2\kappa}}\int_{r_{min}}^{r} \frac{rdr}{\sqrt{\frac{Er^2}{\kappa} + r - \frac{p}{2}}}\end{aligned}$$

Where $p = l^2/m\kappa$. We know as well that $e = \sqrt{1 + 2El^2/m\kappa^2}$ so

$$-\frac{m}{2l^2}(1 - e^2) = \frac{E}{\kappa^2}$$

But also since $a = p(1 - e^2)^{-1}$ we get that

$$\frac{E}{\kappa} = -\frac{m\kappa}{2l^2}(1 - e^2) = -\frac{1}{2p}(1 - e^2) = -\frac{1}{2a}$$

then

$$t = \sqrt{\frac{m}{2\kappa}}\int_{r_{min}}^{r} \frac{rdr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-e^2)}{2}}}$$

Where we also used that $p = a(1 - e^2)$.
Now, let us define ψ as $r = a(1 - e\cos\psi)$ so $dr = ea\sin\psi d\psi$. Then, replacing

in the integral we get that

$$\begin{aligned}
t &= \sqrt{\frac{m}{2\kappa}} \int_0^{\psi} \frac{ea^2(1-e\cos\psi)\sin\psi d\psi}{\sqrt{a(1-e\cos\psi) - \frac{a(1-e\cos\psi)^2}{2} - \frac{a(1-e^2)}{2}}} \\
&= \sqrt{\frac{m}{2\kappa}} \int_0^{\psi} \frac{e\sqrt{a^3}(1-e\cos\psi)\sin\psi d\psi}{\sqrt{1-e\cos\psi - \frac{1-2e\cos\psi+e^2\cos^2\psi}{2} - \frac{1-e^2}{2}}} \\
&= \sqrt{\frac{ma^3}{2\kappa}} \int_0^{\psi} \frac{e(1-e\cos\psi)\sin\psi d\psi}{\sqrt{1-e\cos\psi - \frac{1}{2} + e\cos\psi - \frac{e^2\cos^2\psi}{2} - \frac{1}{2} + \frac{e^2}{2}}} \\
&= \sqrt{\frac{ma^3}{2\kappa}} \int_0^{\psi} \frac{e(1-e\cos\psi)\sin\psi d\psi}{\sqrt{\frac{e^2}{2} - \frac{e^2\cos^2\psi}{2}}} \\
&= \sqrt{\frac{ma^3}{2\kappa}} \int_0^{\psi} \frac{\sqrt{2}(1-e\cos\psi)\sin\psi d\psi}{\sqrt{(1-\cos^2\psi)}} \\
&= \sqrt{\frac{ma^3}{\kappa}} \int_0^{\psi} \frac{(1-e\cos\psi)\sin\psi d\psi}{\sin\psi} \\
&= \sqrt{\frac{ma^3}{\kappa}} \int_0^{\psi} (1-e\cos\psi) d\psi \\
&= \sqrt{\frac{ma^3}{\kappa}} (\psi - e\sin\psi)
\end{aligned}$$

Therefore naming $\omega = \sqrt{\kappa/ma^3}$ we get that

$$\omega t = \psi - e\sin\psi$$

□

Solution. **Exercise 1.7.1** Let a circular orbit such that $r = r_0$ then energy of the system is

$$E_0 = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$$

But since r is constant then we have that $\dot{r} = 0$ and hence

$$E_0 = V_{\text{eff}}(r_0)$$

Then taking the time derivative we get that $V'_{\text{eff}}(r_0) = 0$ since E_0 is constant. But also we know that $V_{\text{eff}}(r) = l^2/2mr^2 + V(r)$ hence taking the derivative and making it equal to 0 we get that

$$-\frac{l^2}{mr^3} + V'(r) = 0$$

or replacing $r = r_0$ we get that $V'(r_0) = l^2/mr_0^3$.
Finally, taking the second derivative of $V_{\text{eff}}(r)$ we get that

$$V''_{\text{eff}}(r_0) = \frac{3l^2}{mr_0^4} + V''(r_0) = \frac{3}{r_0}\frac{l^2}{mr_0^3} + V''(r_0) = \frac{3}{r_0}V'(r_0) + V''(r_0)$$

$\square$

Solution. **Exercise 1.7.2**

(1) Let $0 < e < 1$, the equation for r is

$$r = \frac{l^2}{m\kappa} \frac{1}{(1 + e\cos\phi)}$$

Then the maximum and minimum of r happen for $\phi = \pi$ and $\phi = 0$ respectively. For $\phi = \pi$ we have that $\cos\pi = -1$ and hence

$$r_{\text{max}} = \frac{l^2}{m\kappa} \frac{1}{1-e} = p(1-e)^{-1}$$

And for $\phi = 0$ we get that $\cos 0 = 1$ so

$$r_{\text{min}} = \frac{l^2}{m\kappa} \frac{1}{1+e} = p(1+e)^{-1}$$

Where $p = l^2/m\kappa$. Also, since the semi-major axis of an ellipse is given by $a = (r_{\text{max}} + r_{\text{min}})/2$ we get that

$$a = \frac{p(1-e)^{-1} + p(1+e)^{-1}}{2} = p\frac{1+e+1-e}{2(1-e)(1+e)} = p(1-e^2)^{-1}$$

(2) We know that $dA = \frac{1}{2}r^2 d\phi$ then dividing by dt

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

But we know that $l = mr^2\dot{\phi}$ then

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2m}mr^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{l}{2m}$$

Which is Kepler's second law.

(3) From Kepler's second law we know that $dA = (l/2m)dt$ so by integration we get that

$$\frac{l}{2m}\int_0^\tau dt = \int_0^A dA$$
$$\tau = \frac{2m}{l}A$$

Replacing $A = \pi ab$ and $b = a\sqrt{1-e^2}$ we get that

$$\tau = \frac{2m}{l}\pi ab = \frac{2\pi ma^2}{l}\sqrt{1-e^2}$$

Also from $a = p(1-e^2)^{-1}$ we know that $1 - e^2 = p/a$ hence

$$\tau = \frac{2\pi ma^2}{l}\sqrt{\frac{p}{a}} = \frac{2\pi ma^2}{l}\sqrt{\frac{l^2}{m\kappa a}} = \frac{2\pi a^2}{\sqrt{a}}\sqrt{\frac{m}{\kappa}} = 2\pi a^{3/2}\sqrt{\frac{m}{\kappa}}$$

□