Solved selected problems of Analytical Mechanics by Nivaldo Lemos

Franco Zacco

Solution. **1.1** Let

$$(x^2+y^2)dx + xzdz = 0 \qquad (x^2+y^2)dy + yzdz = 0$$

be constraints.

(a) We have two constraints that can be expressed as 1-forms $\omega^1 = 0$ and $\omega^2 = 0$ where

$$\omega^1 = (x^2+y^2)dx + xzdz \qquad \omega^2 = (x^2+y^2)dy + yzdz$$

Also, $d\omega^1$ and $d\omega^2$ are

$$d\omega^1 = 2ydy \wedge dx + zdx \wedge dz \quad \text{and} \quad d\omega^2 = 2xdx \wedge dy + zdy \wedge dz$$

So, let us consider them separately, i.e. we want to compute $d\omega^1 \wedge \omega^1$ and $d\omega^2 \wedge \omega^2$ as follows

$$\begin{aligned} d\omega^1 \wedge \omega^1 &= (2ydy \wedge dx + zdx \wedge dz) \wedge ((x^2+y^2)dx + xzdz) \\ &= 2xyz\ dy \wedge dx \wedge dz \end{aligned}$$

And

$$\begin{aligned} d\omega^2 \wedge \omega^2 &= (2xdx \wedge dy + zdy \wedge dz) \wedge ((x^2+y^2)dy + yzdz) \\ &= 2xyz\ dx \wedge dy \wedge dz \end{aligned}$$

We see that $d\omega^1 \wedge \omega^1 \neq 0$ unless $x = y = z = 0$ and the same happens for $d\omega^2 \wedge \omega^2$.

Therefore, they are not integrable when considered separately.

(b) The 2-form Ω defined by $\Omega = \omega^1 \wedge \omega^2$ is given by

$$\begin{aligned} \Omega &= (x^2+y^2)dx + xzdz \wedge (x^2+y^2)dy + yzdz \\ &= (x^2+y^2)dx \wedge (x^2+y^2)dy + (x^2+y^2)dx \wedge yzdz \\ &\quad + xzdz \wedge (x^2+y^2)dy \\ &= (x^2+y^2)^2 dx \wedge dy + yz(x^2+y^2)dx \wedge dz + xz(x^2+y^2)dz \wedge dy \end{aligned}$$

It follows that

$$\begin{aligned}d\omega^1 \wedge \Omega &= 2y(x^2+y^2)^2 dy \wedge dx \wedge dx \wedge dy + 2y^2 z(x^2+y^2) dy \wedge dx \wedge dx \wedge dz \\ &\quad + 2yxz(x^2+y^2) dy \wedge dx \wedge dz \wedge dy + z(x^2+y^2)^2 dx \wedge dz \wedge dy \wedge dx \\ &\quad + yz^2(x^2+y^2) dx \wedge dz \wedge dx \wedge dz + xz^2(x^2+y^2) dx \wedge dz \wedge dz \wedge dy \\ &= 0\end{aligned}$$

And in the same way $d\omega^2 \wedge \Omega = 0$.

Therefore, the constraints are integrable when considered together.

On the other hand, let $x, y, z \neq 0$ then we see that

$$d \ln \frac{y}{x} = \frac{1}{y} dy - \frac{1}{x} dx$$

Also, let us consider the following

$$\begin{aligned}y\omega^1 - x\omega^2 &= y(x^2+y^2)dx + xyzdz - x(x^2+y^2)dy - xyzdz \\ &= (x^2+y^2)(ydx - xdy)\end{aligned}$$

But since $\omega^1 = \omega^2 = 0$ then must be that

$$\begin{aligned}ydx - xdy = 0 \\ \frac{1}{y}dy - \frac{1}{x}dx = 0\end{aligned}$$

Then the constraints are equivalent to $d\ln(y/x) = 0$.

Now, considering the combination

$$\begin{aligned}x\omega^1 + y\omega^2 &= x(x^2+y^2)dx + x^2 zdz + y(x^2+y^2)dy + y^2 zdz \\ &= (x^2+y^2)(xdx + ydy + zdz)\end{aligned}$$

Must be that $xdx + ydy + zdz = 0$ but we see that

$$d(x^2+y^2+z^2) = xdx + ydy + zdz$$

Therefore, the constraints are equiavalent also to $xd(x^2+y^2+z^2) = 0$.

(c) Finally, integrating the equations

$$xd(x^2 + y^2 + z^2) = 0 \quad \text{and} \quad d \ln \frac{y}{x} = 0$$

We get that

$$\begin{aligned} \int d(x^2 + y^2 + z^2) &= 0 \\ x^2 + y^2 + z^2 + C_1' &= 0 \\ x^2 + y^2 + z^2 &= C_1 \end{aligned}$$

And

$$\begin{aligned} \int d \ln \frac{y}{x} &= 0 \\ \ln \frac{y}{x} + C_2' &= 0 \\ \frac{y}{x} &= 1 - e^{C_2'} \\ y &= C_2 x \end{aligned}$$

Where we renamed $C_1 = -C_1'$ and $C_2 = 1 - e^{C_2'}$.

$\square$