Solved selected problems of Classical Electrodynamics - Hans Ohanian

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Chapter 1 - Vector Calculus

Problems

Solution. 8. Knowing that

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi \\
\hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi \\
\hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}}
\end{aligned}$$

We can compute the following

$$\begin{aligned}
\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = \hat{\boldsymbol{z}}\cos^2\phi - (-\hat{\boldsymbol{z}}\sin^2\phi) = \hat{\boldsymbol{z}} \\
\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi = \hat{\boldsymbol{\rho}} \\
\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{z}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{\boldsymbol{x}}\sin\phi - \hat{\boldsymbol{y}}\cos\phi = -\hat{\boldsymbol{\phi}}
\end{aligned}$$

$\square$

Solution. 9. Knowing that

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta \\
\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{x}}\cos\theta\cos\phi + \hat{\boldsymbol{y}}\cos\theta\sin\phi - \hat{\boldsymbol{z}}\sin\theta \\
\hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi
\end{aligned}$$

We can compute the following

$$\begin{aligned}
\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix} \\
&= -\hat{\boldsymbol{x}}\sin^2\theta\sin\phi + \hat{\boldsymbol{y}}\cos^2\theta\cos\phi + \hat{\boldsymbol{z}}\sin\theta\cos\phi\cos\theta\sin\phi \\
&\quad - \hat{\boldsymbol{z}}\cos\theta\cos\phi\sin\theta\sin\phi - \hat{\boldsymbol{x}}\cos^2\theta\sin\phi + \hat{\boldsymbol{y}}\sin^2\theta\cos\phi \\
&= -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi \\
&= \hat{\boldsymbol{\phi}} \\
\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\
&= \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta\cos^2\phi + \hat{\boldsymbol{z}}\cos\theta\sin^2\phi + \hat{\boldsymbol{x}}\sin\theta\cos\phi \\
&= \hat{\boldsymbol{r}} \\
\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\phi}} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\
&= -\hat{\boldsymbol{y}}\cos\theta\sin\phi + \hat{\boldsymbol{z}}\sin\theta\cos^2\phi + \hat{\boldsymbol{z}}\sin\theta\sin^2\phi - \hat{\boldsymbol{x}}\cos\theta\cos\phi \\
&= -\hat{\boldsymbol{\theta}}
\end{aligned}$$

$\square$

Solution. 10. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be vectors then

(a) We know that $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$ can be written as $\varepsilon^{klm} a^l (\varepsilon^{mrs} b^r c^s)$ then

$$\begin{aligned}
\varepsilon^{klm} a^l (\varepsilon^{mrs} b^r c^s) &= \varepsilon^{klm} \varepsilon^{mrs} a^l b^r c^s \\
&= \varepsilon^{klm} \varepsilon^{rsm} a^l b^r c^s \\
&= (\delta^{kr} \delta^{ls} - \delta^{ks} \delta^{lr}) a^l b^r c^s \\
&= \delta^{kr} \delta^{ls} a^l b^r c^s - \delta^{ks} \delta^{lr} a^l b^r c^s \\
&= b^k a^s c^s - c^k a^r b^r
\end{aligned}$$

This implies, written in vector format that

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

(b) We know that $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ can be written as $a^k (\varepsilon^{klm} b^l c^m)$ then since $\varepsilon^{klm} = \varepsilon^{mkl}$ we have that

$$a^k (\varepsilon^{klm} b^l c^m) = (\varepsilon^{mkl} a^k b^l) c^m$$

This implies, written in vector format that

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}$$

□

Solution. 11. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be three arbitrary vectors then expanding the product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ gives us

$$\begin{aligned}
\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) &= \varepsilon^{klm} A^k B^l C^m \\
&= \varepsilon^{123} A^1 B^2 C^3 + \varepsilon^{231} A^2 B^3 C^1 + \varepsilon^{312} A^3 B^1 C^2 \\
&\quad + \varepsilon^{213} A^2 B^1 C^3 + \varepsilon^{132} A^1 B^3 C^2 + \varepsilon^{321} A^3 B^2 C^1 \\
&= A^1 B^2 C^3 + A^2 B^3 C^1 + A^3 B^1 C^2 \\
&\quad - A^2 B^1 C^3 - A^1 B^3 C^2 - A^3 B^2 C^1
\end{aligned}$$

But also we know that

$$\begin{aligned}
\begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix} &= A^1 B^2 C^3 + A^2 B^3 C^1 + A^3 B^1 C^2 \\
&\quad - A^2 B^1 C^3 - A^1 B^3 C^2 - A^3 B^2 C^1
\end{aligned}$$

Hence

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \varepsilon^{klm} A^k B^l C^m = \begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix}$$

On the other hand, we know that the volume of a parallelepiped with edges $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ is the area of the parallelogram formed by the vectors $\boldsymbol{B}$ and $\boldsymbol{C}$ times the "height" given by $|\boldsymbol{A}| \cos \alpha$ where α is the angle formed by $\boldsymbol{A}$ and the vertical, normal to the parallelogram area. The area of the parallelogram is given by $|\boldsymbol{B}||\boldsymbol{C}| \sin \beta$ where β is the angle between $\boldsymbol{B}$ and $\boldsymbol{C}$. So joining these reasonings we get that

$$V = |\boldsymbol{A}|(|\boldsymbol{B}||\boldsymbol{C}| \sin \beta) \cos \alpha$$

Which is the same as saying that

$$V = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$$

$\square$

Solution. 12. Let Q^{kl} be a matrix, then the determinant of Q^{kl} is given by

$$\begin{vmatrix} Q^{11} & Q^{12} & Q^{13} \\ Q^{21} & Q^{22} & Q^{23} \\ Q^{31} & Q^{32} & Q^{33} \end{vmatrix} = Q^{11}Q^{22}Q^{33} + Q^{12}Q^{23}Q^{31} + Q^{13}Q^{21}Q^{32} \\ - Q^{31}Q^{22}Q^{13} - Q^{11}Q^{23}Q^{32} - Q^{12}Q^{21}Q^{33}$$

Now, let us compute the explicit expression of $\varepsilon^{klm}Q^{k1}Q^{l2}Q^{m3}$ as follows

$$\begin{aligned} \varepsilon^{klm}Q^{k1}Q^{l2}Q^{m3} &= \varepsilon^{123}Q^{11}Q^{22}Q^{33} + \varepsilon^{231}Q^{21}Q^{32}Q^{13} + \varepsilon^{312}Q^{31}Q^{12}Q^{23} \\ &\quad + \varepsilon^{213}Q^{21}Q^{12}Q^{33} + \varepsilon^{132}Q^{11}Q^{32}Q^{23} + \varepsilon^{321}Q^{31}Q^{22}Q^{13} \\ &= Q^{11}Q^{22}Q^{33} + Q^{21}Q^{32}Q^{13} + Q^{31}Q^{12}Q^{23} \\ &\quad - Q^{21}Q^{12}Q^{33} - Q^{11}Q^{32}Q^{23} - Q^{31}Q^{22}Q^{13} \end{aligned}$$

Therefore we see that $\varepsilon^{klm}Q^{k1}Q^{l2}Q^{m3} = \det\{Q^{kl}\}$. □

Solution. 15. Let T^{kl} and Q^{lm} be tensors, we want to prove that $T^{kl}Q^{lm}$ is also a tensor. Let us compute $T^{kl}Q^{lm}$ knowing that T^{kl} and Q^{lm} transform as tensors, then

$$\begin{aligned} T'^{kl}Q'^{lm} &= a^{kn}a^{lr}T^{nr}a^{ls}a^{md}Q^{sd} \\ &= a^{kn}a^{md}T^{nr}(a^T)^{rl}a^{ls}Q^{sd} \\ &= a^{kn}a^{md}T^{nr}\delta^{rs}Q^{sd} \\ &= a^{kn}a^{md}T^{nr}Q^{rd} \end{aligned}$$

So we see that $T^{kl}Q^{lm}$ transform like a tensor and therefore $T^{kl}Q^{lm}$ is a tensor. □

Solution. 17.

(a) We know that

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi$$
$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi$$

Then by multiplying the first equation by $\cos\phi$ and the second one by $\sin\phi$ we have that

$$\hat{\boldsymbol{\rho}} \cos\phi = \hat{\boldsymbol{x}} \cos^2\phi + \hat{\boldsymbol{y}} \sin\phi\cos\phi$$
$$\hat{\boldsymbol{\phi}} \sin\phi = -\hat{\boldsymbol{x}} \sin^2\phi + \hat{\boldsymbol{y}} \sin\phi\cos\phi$$

By subtracting the second equation from the first one we get that

$$\begin{aligned} \hat{\boldsymbol{x}} \cos^2\phi + \hat{\boldsymbol{y}} \sin\phi\cos\phi + \hat{\boldsymbol{x}} \sin^2\phi - \hat{\boldsymbol{y}} \sin\phi\cos\phi &= \hat{\boldsymbol{\rho}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi \\ \hat{\boldsymbol{x}} &= \hat{\boldsymbol{\rho}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi \end{aligned}$$

Now by multiplying the equation for $\hat{\boldsymbol{\rho}}$ by $\sin\phi$ and the equation for $\hat{\boldsymbol{\phi}}$ by $\cos\phi$ we get that

$$\hat{\boldsymbol{\rho}} \sin\phi = \hat{\boldsymbol{x}} \cos\phi\sin\phi + \hat{\boldsymbol{y}} \sin^2\phi$$
$$\hat{\boldsymbol{\phi}} \cos\phi = -\hat{\boldsymbol{x}} \sin\phi\cos\phi + \hat{\boldsymbol{y}} \cos^2\phi$$

So by adding both equations, we get that

$$\begin{aligned} \hat{\boldsymbol{x}} \cos\phi\sin\phi + \hat{\boldsymbol{y}} \sin^2\phi - \hat{\boldsymbol{x}} \sin\phi\cos\phi + \hat{\boldsymbol{y}} \cos^2\phi &= \hat{\boldsymbol{\rho}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \\ \hat{\boldsymbol{y}} &= \hat{\boldsymbol{\rho}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \end{aligned}$$

(b) We know that

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}} \sin\theta\cos\phi + \hat{\boldsymbol{y}} \sin\theta\sin\phi + \hat{\boldsymbol{z}} \cos\theta$$
$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}} \cos\theta\cos\phi + \hat{\boldsymbol{y}} \cos\theta\sin\phi - \hat{\boldsymbol{z}} \sin\theta$$
$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi$$

Let us multiply the equation for $\hat{\boldsymbol{r}}$ by $\sin\theta$ and the equation for $\hat{\boldsymbol{\theta}}$ by $\cos\theta$ to get

$$\hat{\boldsymbol{r}} \sin\theta = \sin^2\theta(\hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi) + \hat{\boldsymbol{z}} \cos\theta\sin\theta$$
$$\hat{\boldsymbol{\theta}} \cos\theta = \cos^2\theta(\hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi) - \hat{\boldsymbol{z}} \sin\theta\cos\theta$$

Then by adding both equations, we get that

$$\begin{aligned} \hat{\boldsymbol{r}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta &= \hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi(\sin^2\theta + \cos^2\theta) \\ &= \hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi \end{aligned} \tag{1}$$

Now let us multiply again equation (1) by $\sin\phi$ and the equation for $\hat{\boldsymbol{\phi}}$ by $\cos\phi$ to get

$$\begin{aligned}\hat{\boldsymbol{r}}\sin\theta\sin\phi+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi &= \hat{\boldsymbol{x}}\cos\phi\sin\phi+\hat{\boldsymbol{y}}\sin^2\phi\\ \hat{\boldsymbol{\phi}}\cos\phi &= -\hat{\boldsymbol{x}}\sin\phi\cos\phi+\hat{\boldsymbol{y}}\cos^2\phi\end{aligned}$$

Then by adding both equations, we get that

$$\begin{aligned}\hat{\boldsymbol{y}}\sin^2\phi+\hat{\boldsymbol{y}}\cos^2\phi &= \hat{\boldsymbol{r}}\sin\theta\sin\phi+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi+\hat{\boldsymbol{\phi}}\cos\phi\\ \hat{\boldsymbol{y}} &= \hat{\boldsymbol{r}}\sin\theta\sin\phi+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi+\hat{\boldsymbol{\phi}}\cos\phi\end{aligned}$$

Instead, if we multiply equation (1) by $\cos\phi$ and the equation for $\hat{\boldsymbol{\phi}}$ by $\sin\phi$ we get that

$$\begin{aligned}\hat{\boldsymbol{r}}\sin\theta\cos\phi+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi &= \hat{\boldsymbol{x}}\cos^2\phi+\hat{\boldsymbol{y}}\sin\phi\cos\phi\\ \hat{\boldsymbol{\phi}}\sin\phi &= -\hat{\boldsymbol{x}}\sin^2\phi+\hat{\boldsymbol{y}}\cos\phi\sin\phi\end{aligned}$$

So if we subtract them we get that

$$\begin{aligned}\hat{\boldsymbol{x}}\cos^2\phi+\hat{\boldsymbol{x}}\sin^2\phi &= \hat{\boldsymbol{r}}\sin\theta\cos\phi+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi+\hat{\boldsymbol{\phi}}\sin\phi\\ \hat{\boldsymbol{x}} &= \hat{\boldsymbol{r}}\sin\theta\cos\phi+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi+\hat{\boldsymbol{\phi}}\sin\phi\end{aligned}$$

Finally, let us multiply the equation for $\hat{\boldsymbol{r}}$ by $\cos\theta$ and the equation for $\hat{\boldsymbol{\theta}}$ by $\sin\theta$ to get

$$\begin{aligned}\hat{\boldsymbol{r}}\cos\theta &= \hat{\boldsymbol{x}}\sin\theta\cos\phi\cos\theta+\hat{\boldsymbol{y}}\sin\theta\sin\phi\cos\theta+\hat{\boldsymbol{z}}\cos^2\theta\\ \hat{\boldsymbol{\theta}}\sin\theta &= \hat{\boldsymbol{x}}\cos\theta\cos\phi\sin\theta+\hat{\boldsymbol{y}}\cos\theta\sin\phi\sin\theta-\hat{\boldsymbol{z}}\sin^2\theta\end{aligned}$$

Then by subtracting both equations, we get that

$$\hat{\boldsymbol{z}}=\hat{\boldsymbol{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta$$

$\square$

Solution. 18. Let $\mathbf{x} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{x} : (x_1, x_2, x_3) \to (x_1, x_2, x_3)$ then we have that

(a) We want to show that $\boldsymbol{\nabla} \cdot \mathbf{x} = 3$ hence

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{x} &= \frac{\partial x^k}{\partial x^k} \\
&= \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}
$$

(b) In this case, we want to show that $\boldsymbol{\nabla} 1/|\mathbf{x}| = -\mathbf{x}/|\mathbf{x}|^3$ where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ so we have that

$$
\begin{aligned}
\boldsymbol{\nabla} \frac{1}{|\mathbf{x}|} &= \frac{\partial}{\partial x^k} \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\
&= \left(-\frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, -\frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, -\frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \right) \\
&= \left(-\frac{x_1}{|\mathbf{x}|^3}, -\frac{x_2}{|\mathbf{x}|^3}, -\frac{x_3}{|\mathbf{x}|^3} \right) \\
&= -\frac{\mathbf{x}}{|\mathbf{x}|^3}
\end{aligned}
$$

(c) Now, we want to show that $\boldsymbol{\nabla}^2 1/|\mathbf{x}| = 0$ hence

$$
\begin{aligned}
\boldsymbol{\nabla}^2 \frac{1}{|\mathbf{x}|} &= \frac{\partial}{\partial x^k} \left(\frac{\partial}{\partial x^k} \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right) \\
&= \frac{\partial}{\partial x^k} \left(-\frac{x^k}{|\mathbf{x}|^3} \right) \\
&= \frac{\partial}{\partial x_1} \left(-\frac{x_1}{|\mathbf{x}|^3} \right) + \frac{\partial}{\partial x_2} \left(-\frac{x_2}{|\mathbf{x}|^3} \right) + \frac{\partial}{\partial x_3} \left(-\frac{x_3}{|\mathbf{x}|^3} \right) \\
&= \frac{2x_1^2 - x_2^2 - x_3^2}{|\mathbf{x}|^5} + \frac{2x_2^2 - x_1^2 - x_3^2}{|\mathbf{x}|^5} + \frac{2x_3^2 - x_2^2 - x_1^2}{|\mathbf{x}|^5} \\
&= \frac{1}{|\mathbf{x}|^5} (2x_1^2 - x_2^2 - x_3^2 + 2x_2^2 - x_1^2 - x_3^2 + 2x_3^2 - x_2^2 - x_1^2) \\
&= 0
\end{aligned}
$$

(c) Finally, we want to show that $(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{x} = \mathbf{B}$ so we see that

$$
\begin{aligned}
(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{x} &= B^k \frac{\partial}{\partial x^k}\mathbf{x} \\
&= \left(B_{x_1}\frac{\partial}{\partial x_1} + B_{x_2}\frac{\partial}{\partial x_2} + B_{x_3}\frac{\partial}{\partial x_3} \right)\mathbf{x} \\
&= \Bigg(\left(B_{x_1}\frac{\partial}{\partial x_1}x_1 + B_{x_2}\frac{\partial}{\partial x_2}x_1 + B_{x_3}\frac{\partial}{\partial x_3}x_1 \right), \\
&\qquad \left(B_{x_1}\frac{\partial}{\partial x_1}x_2 + B_{x_2}\frac{\partial}{\partial x_2}x_2 + B_{x_3}\frac{\partial}{\partial x_3}x_2 \right), \\
&\qquad \left(B_{x_1}\frac{\partial}{\partial x_1}x_3 + B_{x_2}\frac{\partial}{\partial x_2}x_3 + B_{x_3}\frac{\partial}{\partial x_3}x_3 \right) \Bigg) \\
&= (B_{x_1}, B_{x_2}, B_{x_3}) \\
&= \mathbf{B}
\end{aligned}
$$

Where we used that $\frac{\partial}{\partial x_i}x_i = 1$.

□

Solution. 19. Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be scalar fields. We prove below a few identities.

(a)

$$
\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\phi\psi}) &= \hat{\boldsymbol{x}}\frac{\partial(\boldsymbol{\phi\psi})}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial(\boldsymbol{\phi\psi})}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial(\boldsymbol{\phi\psi})}{\partial z} \\
&= \hat{\boldsymbol{x}}\left(\boldsymbol{\psi}\frac{\partial\boldsymbol{\phi}}{\partial x} + \boldsymbol{\phi}\frac{\partial\boldsymbol{\psi}}{\partial x}\right) + \hat{\boldsymbol{y}}\left(\boldsymbol{\psi}\frac{\partial\boldsymbol{\phi}}{\partial y} + \boldsymbol{\phi}\frac{\partial\boldsymbol{\psi}}{\partial y}\right) + \hat{\boldsymbol{z}}\left(\boldsymbol{\psi}\frac{\partial\boldsymbol{\phi}}{\partial z} + \boldsymbol{\phi}\frac{\partial\boldsymbol{\psi}}{\partial z}\right) \\
&= \boldsymbol{\psi}\left(\hat{\boldsymbol{x}}\frac{\partial\boldsymbol{\phi}}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial\boldsymbol{\phi}}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial\boldsymbol{\phi}}{\partial z}\right) + \boldsymbol{\phi}\left(\hat{\boldsymbol{x}}\frac{\partial\boldsymbol{\psi}}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial\boldsymbol{\psi}}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial\boldsymbol{\psi}}{\partial z}\right) \\
&= \boldsymbol{\psi}\boldsymbol{\nabla}\boldsymbol{\phi} + \boldsymbol{\phi}\boldsymbol{\nabla}\boldsymbol{\psi}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\phi}\boldsymbol{A}) &= \frac{\partial(\boldsymbol{\phi}A^k)}{\partial x^k} \\
&= \boldsymbol{\phi}\frac{\partial A^k}{\partial x^k} + A^k\frac{\partial\boldsymbol{\phi}}{\partial x^k} \\
&= \boldsymbol{\phi}\boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}\boldsymbol{\phi}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\phi A}) &= \varepsilon^{klm} \frac{\partial}{\partial x^l} \boldsymbol{\phi} A^m \\
&= \varepsilon^{klm} (\boldsymbol{\phi} \frac{\partial}{\partial x^l} A^m + A^m \frac{\partial \boldsymbol{\phi}}{\partial x^l}) \\
&= \boldsymbol{\phi} \varepsilon^{klm} \frac{\partial}{\partial x^l} A^m - \varepsilon^{kml} A^m \frac{\partial \boldsymbol{\phi}}{\partial x^l} \\
&= \boldsymbol{\phi} \boldsymbol{\nabla} \times \boldsymbol{A} - \boldsymbol{A} \times \boldsymbol{\nabla} \boldsymbol{\phi}
\end{aligned}
$$

□

Solution. 20. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be vectors. We will prove a few identities.

(a)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \frac{\partial}{\partial x^k}\varepsilon^{klm}A^l B^m \\
&= \varepsilon^{klm}\frac{\partial}{\partial x^k}A^l B^m \\
&= \varepsilon^{klm}\left(A^l\frac{\partial}{\partial x^k}B^m + B^m\frac{\partial}{\partial x^k}A^l\right) \\
&= -A^l\varepsilon^{lkm}\frac{\partial}{\partial x^k}B^m + B^m\varepsilon^{mkl}\frac{\partial}{\partial x^k}A^l \\
&= \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) &= \varepsilon^{klm}\frac{\partial}{\partial x^l}\ \left(\varepsilon^{mrs}A^r B^s\right) \\
&= \varepsilon^{klm}\varepsilon^{rsm}\left(B^s\frac{\partial}{\partial x^l}A^r + A^r\frac{\partial}{\partial x^l}B^s\right) \\
&= (\delta^{kr}\delta^{ls}-\delta^{ks}\delta^{lr})\left(B^s\frac{\partial}{\partial x^l}A^r + A^r\frac{\partial}{\partial x^l}B^s\right) \\
&= \delta^{kr}\delta^{ls}B^s\frac{\partial}{\partial x^l}A^r + \delta^{kr}\delta^{ls}A^r\frac{\partial}{\partial x^l}B^s \\
&\quad - \delta^{ks}\delta^{lr}B^s\frac{\partial}{\partial x^l}A^r - \delta^{ks}\delta^{lr}A^r\frac{\partial}{\partial x^l}B^s \\
&= B^l\frac{\partial}{\partial x^l}A^k + A^k\frac{\partial}{\partial x^s}B^s - B^k\frac{\partial}{\partial x^r}A^r - A^l\frac{\partial}{\partial x^l}B^k \\
&= (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}
\end{aligned}$$

(c)

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) &= \varepsilon^{klm}\frac{\partial}{\partial x^l}\ \left(\varepsilon^{mrs}\frac{\partial}{\partial x^r}A^s\right) \\
&= \varepsilon^{klm}\varepsilon^{mrs}\frac{\partial}{\partial x^l}\ \left(\frac{\partial}{\partial x^r}A^s\right) \\
&= (\delta^{kr}\delta^{ls}-\delta^{ks}\delta^{lr})\frac{\partial}{\partial x^l}\frac{\partial}{\partial x^r}A^s \\
&= \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}A^l - \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^l}A^k \\
&= \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}A^l - \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^l}A^k \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}
\end{aligned}$$

$\square$

Solution. 23.

(a) Let $\boldsymbol{A}(\mathbf{x}) = 4xy\hat{\boldsymbol{x}} + 2x^2\hat{\boldsymbol{y}} + 3z^2\hat{\boldsymbol{z}}$ then if we define $\boldsymbol{\Phi}(\mathbf{x}) = 2x^2y + z^3 + C$ which we can get by integration we see that $\boldsymbol{A} = \boldsymbol{\nabla}\boldsymbol{\Phi}$ where C is a constant.

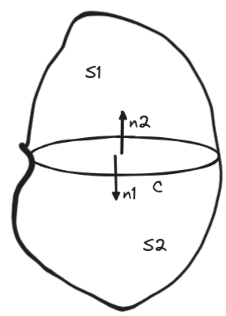
(b) Let $\boldsymbol{A}(\mathbf{x}) = 4xy\hat{\boldsymbol{x}} + 3x^2\hat{\boldsymbol{y}} + 3z^2\hat{\boldsymbol{z}}$ then if $\boldsymbol{\Phi}$ exists such that $\boldsymbol{A} = \boldsymbol{\nabla}\boldsymbol{\Phi}$ then the double derivatives of $\boldsymbol{\Phi}$ commutes, but we see that

$$\frac{\partial}{\partial y}\frac{\partial \boldsymbol{\Phi}}{\partial x} = \frac{\partial(\boldsymbol{A}(\mathbf{x}))_x}{\partial y} = 4x \neq 6x = \frac{\partial(\boldsymbol{A}(\mathbf{x}))_y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \boldsymbol{\Phi}}{\partial y}$$

Therefore there is no scalar field $\boldsymbol{\Phi}$ such that $\boldsymbol{A} = \boldsymbol{\nabla}\boldsymbol{\Phi}$.

□

Solution. 26. Let S be a closed surface and let C be an arbitrary closed path on S. So we can say that we have "separated" the surface S onto two open surfaces (separated by the area enclosed by C), let us call them S_1 and S_2, we can see this as follows



Then we can apply Stokes Theorem for both surfaces as

$$\int_{S_1} \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S} = -\int_C \boldsymbol{A} \cdot d\boldsymbol{l}$$
$$\int_{S_2} \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S} = \int_C \boldsymbol{A} \cdot d\boldsymbol{l}$$

Where we added a minus sign to the first equation because the normal unit vector $\hat{\boldsymbol{n}}_{\mathbf{1}}$ has the opposite direction to $\hat{\boldsymbol{n}}_{\mathbf{2}}$. Therefore we see that

$$\begin{aligned}\int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S} &= \int_{S_1} \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S} + \int_{S_2} \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S} \\ &= -\int_C \boldsymbol{A} \cdot d\boldsymbol{l} + \int_C \boldsymbol{A} \cdot d\boldsymbol{l} \\ &= 0\end{aligned}$$

□

Solution. 27. Let S be a closed surface and V the volume contained in it. Also, let $\boldsymbol{\Phi}$ be any scalar field and let $\boldsymbol{C}$ be a constant vector, then we see because of Problem 19 (b) that

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\Phi}\boldsymbol{C}) &= \boldsymbol{\Phi}\boldsymbol{\nabla}\cdot\boldsymbol{C}+\boldsymbol{C}\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}\\ &= \boldsymbol{C}\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}\end{aligned}$$

Where we used that $\boldsymbol{\nabla}\cdot\boldsymbol{C}=0$ since $\boldsymbol{C}$ is a constant vector. Then applying Gauss Theorem to our vector field $\boldsymbol{\Phi}\boldsymbol{C}$ we see that

$$\begin{aligned}\int_V \boldsymbol{\nabla}\cdot(\boldsymbol{\Phi}\boldsymbol{C})d\boldsymbol{V} &= \int_S(\boldsymbol{\Phi}\boldsymbol{C})\cdot d\boldsymbol{S}\\ \int_V \boldsymbol{C}\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}d\boldsymbol{V} &= \int_S(\boldsymbol{\Phi}\boldsymbol{C})\cdot d\boldsymbol{S}\\ \boldsymbol{C}\cdot\int_V \boldsymbol{\nabla}\boldsymbol{\Phi}d\boldsymbol{V} &= \boldsymbol{C}\cdot\int_S \boldsymbol{\Phi}d\boldsymbol{S}\\ \boldsymbol{C}\cdot\left(\int_V \boldsymbol{\nabla}\boldsymbol{\Phi}d\boldsymbol{V}-\int_S \boldsymbol{\Phi}d\boldsymbol{S}\right) &= 0\end{aligned}$$

Finally, since this must hold for any constant vector $\boldsymbol{C}$ then it could happen that the vector $\boldsymbol{C}$ is not perpendicular to $\int_V \boldsymbol{\nabla}\boldsymbol{\Phi}d\boldsymbol{V}-\int_S \boldsymbol{\Phi}d\boldsymbol{S}$ so it must happen that

$$\int_V \boldsymbol{\nabla}\boldsymbol{\Phi}d\boldsymbol{V}=\int_S \boldsymbol{\Phi}d\boldsymbol{S}$$

□

Solution. 28. Let S be an open surface and C its boundary. Also, let $\boldsymbol{\Phi}$ be any scalar field and let $\boldsymbol{C}$ be a constant vector, then we see because of Problem 19 (c) that

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\Phi C}) &= \boldsymbol{\Phi \nabla} \times \boldsymbol{C} - \boldsymbol{C} \times \boldsymbol{\nabla \Phi} \\ &= -\boldsymbol{C} \times \boldsymbol{\nabla \Phi}\end{aligned}$$

Where we used that $\boldsymbol{\nabla} \times \boldsymbol{C} = 0$ since $\boldsymbol{C}$ is a constant vector. Then applying Stoke's Theorem to our vector field $\boldsymbol{\Phi C}$ we see that

$$\begin{aligned}\int_S (\boldsymbol{\nabla} \times (\boldsymbol{\Phi C})) \cdot d\boldsymbol{S} &= \int_C (\boldsymbol{\Phi C}) \cdot d\boldsymbol{l} \\ \int_S (-\boldsymbol{C} \times \boldsymbol{\nabla \Phi}) \cdot d\boldsymbol{S} &= \boldsymbol{C} \cdot \int_C \boldsymbol{\Phi} d\boldsymbol{l} \\ \int_S -\boldsymbol{C} \cdot (\boldsymbol{\nabla \Phi} \times d\boldsymbol{S}) &= \boldsymbol{C} \cdot \int_C \boldsymbol{\Phi} d\boldsymbol{l} \\ \boldsymbol{C} \cdot \int_S -(\boldsymbol{\nabla \Phi} \times d\boldsymbol{S}) &= \boldsymbol{C} \cdot \int_C \boldsymbol{\Phi} d\boldsymbol{l} \\ \boldsymbol{C} \cdot \left(- \int_S \boldsymbol{\nabla \Phi} \times d\boldsymbol{S} - \int_C \boldsymbol{\Phi} d\boldsymbol{l} \right) &= 0\end{aligned}$$

Where we used the triple product property that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$.

Finally, since this must hold for any constant vector $\boldsymbol{C}$ then it could happen that the vector $\boldsymbol{C}$ is not perpendicular to $-\int_S \boldsymbol{\nabla \Phi} \times d\boldsymbol{S} - \int_C \boldsymbol{\Phi} d\boldsymbol{l}$ so it must happen that

$$-\int_S \boldsymbol{\nabla \Phi} \times d\boldsymbol{S} = \int_C \boldsymbol{\Phi} d\boldsymbol{l}$$

$\square$

Solution. 29. We know that $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ so we can write the left-hand side of the equation as

$$\int_V \nabla^2 \frac{1}{|\boldsymbol{x}|} d\boldsymbol{V} = \int_V \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \frac{1}{|\boldsymbol{x}|} \right) d\boldsymbol{V}$$

By applying Gauss' Theorem we get that

$$\int_V \nabla^2 \frac{1}{|\boldsymbol{x}|} d\boldsymbol{V} = \int_V \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \frac{1}{|\boldsymbol{x}|} \right) d\boldsymbol{V} = \int_S \boldsymbol{\nabla} \frac{1}{|\boldsymbol{x}|} d\boldsymbol{S}$$

Now considering a sphere centered on the origin and working with spherical coordinates, we can write the above equation as

$$\int_V \nabla^2 \frac{1}{|\boldsymbol{x}|} d\boldsymbol{V} = \int_S -\frac{1}{r^2} d\boldsymbol{S}$$

Here we used that the distance from the origin to a point in the sphere surface in spherical coordinates is given by $|\boldsymbol{x}| = r$ and hence $\boldsymbol{\nabla} 1/r = -1/r^2$.

On the other hand, the surface element in spherical coordinates is given by $d\boldsymbol{S} = r^2 \sin\theta d\theta d\phi$ so integrating along the entire sphere surface we get that

$$\begin{aligned}
\int_V \nabla^2 \frac{1}{|\boldsymbol{x}|} d\boldsymbol{V} &= -\int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{r^2} \right) r^2 \sin\theta \; d\theta d\phi \\
&= -\int_0^{2\pi} \int_0^{\pi} \sin\theta \; d\theta d\phi \\
&= -\int_0^{2\pi} \big[-\cos\theta \big]_0^{\pi} \; d\phi \\
&= -\int_0^{2\pi} 2 \; d\phi \\
&= -4\pi
\end{aligned}$$

□

Solution. **30.** Let S be a closed surface and V the volume contained in it. Also, let $\boldsymbol{A}$ be any vector field and let $\boldsymbol{B}$ be a constant vector, then we see because of Problem 20 (a) that

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\boldsymbol{A}) &= \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})-\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\\ &= -\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\end{aligned}$$

Where we used that $\boldsymbol{\nabla}\times\boldsymbol{B}=0$ since $\boldsymbol{B}$ is a constant vector. Then applying Gauss's Theorem to our vector field $\boldsymbol{B}\times\boldsymbol{A}$ we see that

$$\begin{aligned}\int_V \boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\boldsymbol{A})\,d\boldsymbol{V} &= \int_S(\boldsymbol{B}\times\boldsymbol{A})\cdot d\boldsymbol{S}\\ \int_V(-\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}))\,d\boldsymbol{V} &= \int_S \boldsymbol{B}\cdot(\boldsymbol{A}\times d\boldsymbol{S})\\ \boldsymbol{B}\cdot\int_V -(\boldsymbol{\nabla}\times\boldsymbol{A})\,d\boldsymbol{V} &= \boldsymbol{B}\cdot\int_S \boldsymbol{A}\times d\boldsymbol{S}\\ \boldsymbol{B}\cdot\left(-\int_V \boldsymbol{\nabla}\times\boldsymbol{A}\,d\boldsymbol{V}-\int_S\boldsymbol{A}\times d\boldsymbol{S}\right) &= 0\end{aligned}$$

Finally, since this must hold for any constant vector $\boldsymbol{B}$ then it could happen that the vector $\boldsymbol{B}$ is not perpendicular to $-\int_V \boldsymbol{\nabla}\times\boldsymbol{A}\,d\boldsymbol{V}-\int_S\boldsymbol{A}\times d\boldsymbol{S}$ so it must happen that

$$\int_V \boldsymbol{\nabla}\times\boldsymbol{A}\,d\boldsymbol{V} = -\int_S \boldsymbol{A}\times d\boldsymbol{S}$$

$\square$

Solution. 32. Let $\boldsymbol{j} = C(xr\hat{\boldsymbol{x}} + yr\hat{\boldsymbol{y}})$ or in cartesian coordinates

$$\boldsymbol{j} = C(x\sqrt{x^2+y^2+z^2}\ \hat{\boldsymbol{x}} + y\sqrt{x^2+y^2+z^2}\ \hat{\boldsymbol{y}})$$

So $\boldsymbol{\nabla}\cdot\boldsymbol{j}$ is given by

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{j} &= \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} \\
&= C\frac{2x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} + C\frac{x^2+2y^2+z^2}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{C(3(x^2+y^2)+2z^2)}{\sqrt{x^2+y^2+z^2}}
\end{aligned}$$

Or in spherical coordinates

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{j} &= \frac{C(3(r^2-z^2)+2z^2)}{r} \\
&= \frac{C(3r^2-z^2)}{r} \\
&= \frac{C(3r^2-r^2\cos^2\theta)}{r} \\
&= Cr(3-\cos^2\theta)
\end{aligned}$$

On the other hand, from the charge conservation proof we see that

$$\int \boldsymbol{\nabla}\cdot\boldsymbol{j}\ d\boldsymbol{V} = (\text{rate of outflow of charge from V})$$

So we compute the rate of change of the electric charge in the spherical region bounded by $r = R$ as follows

$$\begin{aligned}
\int \boldsymbol{\nabla}\cdot\boldsymbol{j}\ d\boldsymbol{V} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(Cr(3-\cos^2\theta))r^2\sin\theta\ drd\theta d\phi \\
&= C\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^3(3-\cos^2\theta)\sin\theta\ drd\theta d\phi \\
&= C\int_0^{2\pi}\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^R(3-\cos^2\theta)\sin\theta\ d\theta d\phi \\
&= \frac{CR^4}{4}\int_0^{2\pi}\int_0^{\pi}(2+\sin^2\theta)\sin\theta\ d\theta d\phi \\
&= \frac{CR^4}{4}\int_0^{2\pi}\int_0^{\pi} 2\sin\theta+\sin^3\theta\ d\theta d\phi
\end{aligned}$$

Where we used that $3 - \cos^2\theta = 2 + \sin^2\theta$ hence

$$\begin{aligned}
\int \boldsymbol{\nabla} \cdot \boldsymbol{j} \; d\boldsymbol{V} &= \frac{CR^4}{4} \int_0^{2\pi} \int_0^{\pi} 2\sin\theta + \sin^3\theta \; d\theta d\phi \\
&= \frac{CR^4}{4} \int_0^{2\pi} \left[\frac{1}{12} (\cos(3\theta) - 33\cos\theta) \right]_0^{\pi} \; d\phi \\
&= \frac{CR^4}{4} \int_0^{2\pi} \frac{1}{12} [(-1 + 33) - (1 - 33)] \; d\phi \\
&= \frac{CR^4}{4} \int_0^{2\pi} \frac{16}{3} \; d\phi \\
&= \frac{4CR^4}{3} \int_0^{2\pi} \; d\phi \\
&= \frac{8\pi}{3} CR^4
\end{aligned}$$

$\square$