Solved selected problems of Classical Electrodynamics - Hans Ohanian

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Chapter 2 - Electrostatics

Exercises

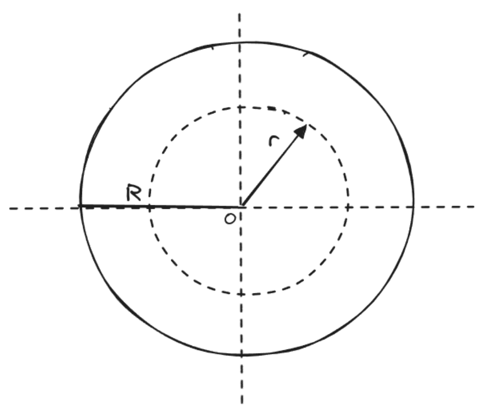
Solution. Exercise 1. Below we show how to express the dyne per esu in centimeters, grams and seconds.

$$\frac{\text{dyn}}{\text{esu}} = \frac{\text{g cm}}{\text{s}^2}\frac{1}{\text{cm}\sqrt{\text{dyn}}} = \frac{\text{g cm}}{\text{s}^2}\frac{\text{s}}{\text{cm}\sqrt{\text{g cm}}} = \frac{1}{\text{s}}\sqrt{\frac{\text{g}}{\text{cm}}}$$

$\square$

Solution. Exercise 2. Let us consider a sphere of radius R with charge Q uniformly distributed over its volume.

Let us consider first the case where $r \leq R$ as shown below



The charge per unit of volume is $3Q/4\pi R^3$ then the charge enclosed in the sphere of radius r is

$$\frac{3Q}{4\pi R^3}\frac{4\pi r^3}{3} = \frac{Qr^3}{R^3}$$

On the other hand, we know that $\boldsymbol{E}(\boldsymbol{x}) = E\hat{\boldsymbol{r}}$ where E is the constant value of the electric field because of the symmetric distribution of charge. Hence

Gauss' law in integral form is given by

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} E\ r^2 \sin\theta d\theta d\phi &= 4\pi\frac{Qr^3}{R^3}\\ 4\pi E r^2 &= 4\pi\frac{Qr^3}{R^3}\\ \boldsymbol{E} &= \frac{Qr}{R^3}\hat{\boldsymbol{r}}\end{aligned}$$

In the case where $r \geq R$ the enclosed charge is just Q and applying again Gauss' Law we get that

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} E\ r^2 \sin\theta d\theta d\phi &= 4\pi Q\\ 4\pi E r^2 &= 4\pi Q\\ \boldsymbol{E} &= \frac{Q}{r^2}\hat{\boldsymbol{r}}\end{aligned}$$

□

Solution. Exercise 3. We know that

$$\Phi(\boldsymbol{x}) = \frac{q'}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

So by applying the gradient to this equation we get that

$$\begin{aligned}
\nabla\Phi(\boldsymbol{x}) &= \frac{\partial\Phi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\Phi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\Phi}{\partial z}\hat{\boldsymbol{z}} \\
&= -\frac{q'(x-x')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\hat{\boldsymbol{x}} \\
&\quad -\frac{q'(y-y')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\hat{\boldsymbol{y}} \\
&\quad -\frac{q'(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\hat{\boldsymbol{z}} \\
&= -\frac{q'((x-x')\hat{\boldsymbol{x}}+(y-y')\hat{\boldsymbol{y}}+(z-z')\hat{\boldsymbol{z}})}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} \\
&= -q'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}
\end{aligned}$$

Therefore

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla\Phi(\boldsymbol{x}) = q'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

On the other hand, for a distributioon of charges with a charge density of $\rho(\boldsymbol{x}')$ we know that

$$\Phi(\boldsymbol{x}) = \int\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}dV' = \int\frac{\rho(\boldsymbol{x}')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dV'$$

Hence the gradient is

$$
\begin{aligned}
\nabla\Phi(\boldsymbol{x}) &= \frac{\partial\Phi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\Phi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\Phi}{\partial z}\hat{\boldsymbol{z}} \\
&= \int \rho(\boldsymbol{x}')\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)dV'\hat{\boldsymbol{x}} \\
&\quad + \int \rho(\boldsymbol{x}')\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)dV'\hat{\boldsymbol{y}} \\
&\quad + \int \rho(\boldsymbol{x}')\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)dV'\hat{\boldsymbol{z}} \\
&= -\int \rho(\boldsymbol{x}')\left(\frac{(x-x')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\hat{\boldsymbol{x}}\right. \\
&\quad + \frac{(y-y')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\hat{\boldsymbol{y}} \\
&\quad \left. + \frac{(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\hat{\boldsymbol{z}}\right)\ dV' \\
&= -\int \rho(\boldsymbol{x}')\left(\frac{(x-x')\hat{\boldsymbol{x}}+(y-y')\hat{\boldsymbol{y}}+(z-z')\hat{\boldsymbol{z}}}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\right)\ dV' \\
&= -\int \rho(\boldsymbol{x}')\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\ dV'
\end{aligned}
$$

Therefore

$$
\boldsymbol{E}(\boldsymbol{x}) = -\nabla\Phi(\boldsymbol{x}) = \int \rho(\boldsymbol{x}')\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\ dV'
$$

□

Solution. Exercise 4. We know the electric field of a sphere of radius R an charge Q uniformly distributed over its volume is given by

$$\boldsymbol{E}(r) = Q\frac{\hat{\boldsymbol{r}}}{r^2} \qquad r \geq R$$
$$\boldsymbol{E}(r) = Q\frac{r\hat{\boldsymbol{r}}}{R^3} \qquad r \leq R$$

Then by using that

$$\Phi(\boldsymbol{x}) = -\int_{x_0}^{x} E(\boldsymbol{x}') \cdot d\boldsymbol{x}'$$

We can compute the electrostatic potential Φ for $r \geq R$ as follows

$$\begin{aligned}
\Phi(r) &= -\int_{\infty}^{r} Q\frac{\hat{\boldsymbol{r}}}{r'^2} \cdot \hat{\boldsymbol{r}} dr' \\
&= -Q\left[-\frac{1}{r'}\right]_{\infty}^{r} \\
&= -Q\left[-\frac{1}{r} + 0\right] \\
&= \frac{Q}{r}
\end{aligned}$$

And for $r \leq R$ we have that

$$\begin{aligned}
\Phi(r) &= -\int_{\infty}^{R} Q\frac{\hat{\boldsymbol{r}}}{r'^2} \cdot \hat{\boldsymbol{r}} dr' - \int_{R}^{r} Q\frac{r'\hat{\boldsymbol{r}}}{R^3} \cdot \hat{\boldsymbol{r}} dr' \\
&= -Q\left[-\frac{1}{r'}\right]_{\infty}^{R} - \frac{Q}{R^3}\left[\frac{r'^2}{2}\right]_{R}^{r} \\
&= Q\left[\frac{1}{R}\right] - \frac{Q}{R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right] \\
&= Q\left[\frac{1}{R} - \frac{r^2}{2R^3} + \frac{1}{2R}\right] \\
&= Q\left[-\frac{r^2}{2R^3} + \frac{3}{2R}\right]
\end{aligned}$$

□

Solution. Exercise 5. Below we show how to express a statvolt in centimeters, grams and seconds.

$$1 \text{ statvolt} = \frac{1 \text{ erg}}{1 \text{ esu}} = \frac{1 \text{ dyn cm}}{1 \text{ cm}\sqrt{1 \text{ dyn}}} = \frac{1 \text{ g cm/s}^2}{1 \text{ (g cm)}^{1/2}\text{/s}} = 1 \; \frac{(\text{g cm})^{1/2}}{\text{s}}$$

□

Solution. Exercise 6. We know that

$$\nabla \times \boldsymbol{E} = \varepsilon^{mrs} \frac{\partial}{\partial x^r} E^s$$

Hence

$$\begin{aligned}
\boldsymbol{p} \times (\nabla \times \boldsymbol{E}) &= \varepsilon^{klm} p^l (\varepsilon^{mrs} \frac{\partial}{\partial x^r} E^s) \\
&= \varepsilon^{klm} \varepsilon^{mrs} p^l (\frac{\partial}{\partial x^r} E^s) \\
&= (\delta^{kr}\delta^{ls} - \delta^{ks}\delta^{lr})(p^l \frac{\partial}{\partial x^r} E^s) \\
&= (p^l \frac{\partial}{\partial x^k} E^l) - (p^l \frac{\partial}{\partial x^l} E^k) \\
&= \frac{\partial}{\partial x^k} (p^l E^l) - (p^l \frac{\partial}{\partial x^l} E^k) \\
&= \nabla(\boldsymbol{p} \cdot \boldsymbol{E}) - (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}
\end{aligned}$$

If $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ we get that

$$\begin{aligned}
\nabla(\boldsymbol{p} \cdot \boldsymbol{E}) - (\boldsymbol{p} \cdot \nabla)\boldsymbol{E} &= 0 \\
\nabla(\boldsymbol{p} \cdot \boldsymbol{E}) &= (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}
\end{aligned}$$

Which implies that the equations (61) and (64) are equivalent. $\square$

Solution. Exercise 7. Let $U = -pE\cos\theta$ then by differentiation we have that

$$\frac{\partial U}{\partial \theta} = pE\sin\theta$$

Hence this is a torque in the normal direction to $\boldsymbol{p}$ and $\boldsymbol{E}$ i.e.

$$\boldsymbol{\tau} = pE\sin\theta\ \boldsymbol{n} = \boldsymbol{p}\times\boldsymbol{E}$$

$\square$