Solved selected problems of Classical Electrodynamics - Hans Ohanian

Franco Zacco

Chapter 1 - Vector Calculus

Exercises

Solution. Exercise 1. Verification of $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1$

$$\begin{aligned}
\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} &= 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 1 \\
\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} &= 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 = 1 \\
\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} &= 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 = 1
\end{aligned}$$

Verification of $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} = 0$

$$\begin{aligned}
\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} &= 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0 \\
\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{z}} &= 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0 \\
\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} &= 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0
\end{aligned}$$

□

Solution. Exercise 2. We want to verify that the components of Eq. (15) agree with those of Eq. (13). Let us write the implicit summation for the 1st component as follows

$$\begin{aligned}\varepsilon^{1lm}x^l x'^m &= \varepsilon^{111}x^1x'^1 + \varepsilon^{112}x^1x'^2 + \varepsilon^{113}x^1x'^3 + \\ &\quad \varepsilon^{121}x^2x'^1 + \varepsilon^{122}x^2x'^2 + \varepsilon^{123}x^2x'^3 + \\ &\quad \varepsilon^{131}x^3x'^1 + \varepsilon^{132}x^3x'^2 + \varepsilon^{133}x^3x'^3 \\ &= \varepsilon^{123}x^2x'^3 + \varepsilon^{132}x^3x'^2 \\ &= x^2x'^3 - x^3x'^2\end{aligned}$$

We see that it agrees with the first component of Eq. (13). In the same way, for the 2nd and 3rd components, we have that

$$\begin{aligned}\varepsilon^{2lm}x^l x'^m &= \varepsilon^{211}x^1x'^1 + \varepsilon^{212}x^1x'^2 + \varepsilon^{213}x^1x'^3 + \\ &\quad \varepsilon^{221}x^2x'^1 + \varepsilon^{222}x^2x'^2 + \varepsilon^{223}x^2x'^3 + \\ &\quad \varepsilon^{231}x^3x'^1 + \varepsilon^{232}x^3x'^2 + \varepsilon^{233}x^3x'^3 \\ &= \varepsilon^{213}x^1x'^3 + \varepsilon^{231}x^3x'^1 \\ &= x^3x'^1 - x^1x'^3 \\ \varepsilon^{3lm}x^l x'^m &= \varepsilon^{311}x^1x'^1 + \varepsilon^{312}x^1x'^2 + \varepsilon^{313}x^1x'^3 + \\ &\quad \varepsilon^{321}x^2x'^1 + \varepsilon^{322}x^2x'^2 + \varepsilon^{323}x^2x'^3 + \\ &\quad \varepsilon^{331}x^3x'^1 + \varepsilon^{332}x^3x'^2 + \varepsilon^{333}x^3x'^3 \\ &= \varepsilon^{312}x^1x'^2 + \varepsilon^{321}x^2x'^1 \\ &= x^1x'^2 - x^2x'^1\end{aligned}$$

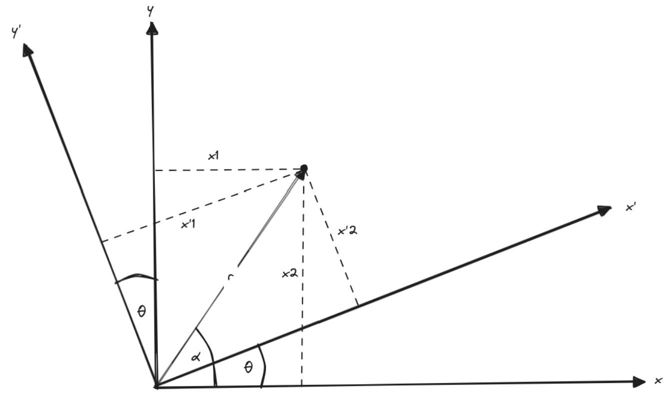
□

Solution. Exercise 3. We know that the k-th component of $\boldsymbol{x}\times\boldsymbol{x}'$ is given $[\boldsymbol{x}\times\boldsymbol{x}']^k = \varepsilon^{klm}x^lx'^m$. Also, we know that $\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = 1$ and that $\varepsilon^{213} = \varepsilon^{132} = \varepsilon^{321} = -1$ this implies that if we swap any two superscripts we invert the sign, so we have that $\varepsilon^{klm} = -\varepsilon^{kml}$. From this, we have that

$$\begin{aligned}[\boldsymbol{x}\times\boldsymbol{x}']^k &= \varepsilon^{klm}x^lx'^m \\ &= -\varepsilon^{kml}x^lx'^m \\ &= -\varepsilon^{kml}x'^mx^l \\ &= -[\boldsymbol{x}'\times\boldsymbol{x}]^k\end{aligned}$$

□

Solution. Exercise 4. Let us plot the position vector r as follows



Then we see that

$$
\begin{aligned}
x^1 &= r\cos\alpha \\
x^2 &= r\sin\alpha \\
x^3 &= 0
\end{aligned}
$$

Also, we have that

$$
\begin{aligned}
x'^1 &= r\cos(\alpha-\theta) \\
x'^2 &= r\sin(\alpha-\theta) \\
x'^3 &= 0
\end{aligned}
$$

Hence by the trigonometric identities, we get that

$$
\begin{aligned}
x'^1 &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\
x'^2 &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta
\end{aligned}
$$

Therefore

$$
\begin{aligned}
x'^1 &= x^1\cos\theta + x^2\sin\theta \\
x'^2 &= x^2\cos\theta - x^1\sin\theta \\
x'^3 &= x^3
\end{aligned}
$$

$\square$

Solution. Exercise 5. Let

$$x'^k = a^{kn}x^n$$

Where

$$\begin{aligned}
a^{11} &= \cos\theta & a^{12} &= \sin\theta & a^{13} &= 0 \\
a^{21} &= -\sin\theta & a^{22} &= \cos\theta & a^{23} &= 0 \\
a^{31} &= 0 & a^{32} &= 0 & a^{33} &= 1
\end{aligned}$$

Hence we have that

$$\begin{aligned}
x'^1 &= a^{11}x^1 + a^{12}x^2 + a^{13}x^3 \\
&= x^1\cos\theta + x^2\sin\theta
\end{aligned}$$

And in the same way

$$\begin{aligned}
x'^2 &= a^{21}x^1 + a^{22}x^2 + a^{23}x^3 \\
&= -x^1\sin\theta + x^2\cos\theta \\
x'^3 &= a^{31}x^1 + a^{32}x^2 + a^{33}x^3 \\
&= x^3
\end{aligned}$$

$\square$

Solution. Exercise 6. Let

$$a^{kn} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$b^{mk} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then c^{mn} is given by

$$\begin{aligned} c^{mn} &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & \cos\phi\sin\theta + \sin\phi\cos\theta & 0 \\ -\sin\phi\cos\theta - \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi+\theta) & \sin(\phi+\theta) & 0 \\ -\sin(\phi+\theta) & \cos(\phi+\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Where we used that $\sin(\phi \pm \theta) = \sin\phi\cos\theta \pm \cos\phi\sin\theta$ and that $\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta$. Finally, we expect to have the sum of the angles ϕ and θ since the entire transformation implies a final rotation of an angle $\phi + \theta$. □

Solution. Exercise 7. Let $x^l \neq 0$ and let $x^m = x^l$ then if $a^{km}a^{kl} \neq \delta^{ml}$ we have that

$$x^l x^m (a^{km}a^{kl} - \delta^{ml}) \neq 0$$

□

Solution. Exercise 8. Given that A^k and B^k are vectors then they transform under rotations as follows

$$A'^k = a^{kn}A^n \quad \text{and} \quad B'^k = a^{kn}B^n$$

Let us define $C^n = \alpha A^n + \beta B^n$ then by multiplying C^n by a^{kn} we get that

$$\begin{aligned} a^{kn}C^n &= a^{kn}(\alpha A^n + \beta B^n) \\ &= \alpha a^{kn}A^n + \beta a^{kn}B^n \\ &= \alpha A'^k + \beta B'^k \\ &= C'^k \end{aligned}$$

Hence C^n also transforms in the same way under rotations. This implies that $C^n = \alpha A^n + \beta B^n$ is also a vector. □

Solution. Exercise 9. Given that T^{kl} and Q^{kl} are tensors then they transform under rotations as follows

$$T'^{kl} = a^{kn}a^{lm}T^{nm} \quad \text{and} \quad Q'^{kl} = a^{kn}a^{lm}Q^{nm}$$

Let us define $C^{nm} = \alpha T^{nm} + \beta Q^{nm}$ then by multiplying C^{nm} by $a^{kn}a^{lm}$ we get that

$$\begin{aligned} a^{kn}a^{lm}C^{nm} &= a^{kn}a^{lm}(\alpha T^{nm} + \beta Q^{nm}) \\ &= \alpha a^{kn}a^{lm}T^{nm} + \beta a^{kn}a^{lm}Q^{nm} \\ &= \alpha T'^{kl} + \beta Q'^{kl} \\ &= C'^{kl} \end{aligned}$$

Hence C^{nm} also transforms in the same way under rotations. This implies that $C^{nm} = \alpha T^{nm} + \beta Q^{nm}$ is also a tensor. □

Solution. Exercise 10. Multiplying δ^{nm} by $a^{kn}a^{lm}$ we have that

$$a^{kn}a^{lm}\delta^{nm} = a^{km}a^{lm} = a^{km}(a^T)^{ml} = \delta^{kl}$$

where we used that $a^{kn}\delta^{nm} = a^{km}$ and that $a^{lm} = (a^T)^{ml}$. Therefore the Kronecker delta is a tensor. □

Solution. Exercise 11. We know that $(T'^T)^{kl} = T'^{lk}$ so we have that

$$(T'^T)^{kl} = T'^{lk} = a^{lm}a^{kn}T^{mn} = a^{kn}a^{lm}(T^T)^{nm}$$

Where we changed from $a^{lm}a^{kn}$ to $a^{kn}a^{lm}$ since the order of multiplication doesn't matter. This implies that the transpose of any tensor is a tensor. □

Solution. Exercise 12. Let T'^{kl} be a tensor and B'^{l} be a vector, we want to show $T'^{kl}B^{l}$ is a vector hence by using the transformation laws for tensors and vectors we have that

$$\begin{aligned} T'^{kl}B'^{l} &= a^{kn}a^{lm}T^{nm}a^{ld}B^{d} \\ &= (a^T)^{ml}a^{ld}a^{kn}T^{nm}B^{d} \\ &= a^{kn}T^{nm}\delta^{md}B^{d} \\ &= a^{kn}T^{nm}B^{m} \end{aligned}$$

Therefore $T'^{kl}B'^{l}$ transforms like a vector, and hence it's a vector. □

Solution. Exercise 13. Let T'^{kl} and Q'^{ln} be tensors, we want to show that $T'^{kl}Q'^{ln}$ is a tensor hence by using the transformation laws for tensors we have that

$$\begin{aligned} T'^{kl}Q'^{ln} &= a^{kf}a^{lm}T^{fm}a^{ld}a^{nb}Q^{db} \\ &= a^{kf}a^{nb}T^{fm}(a^T)^{ml}a^{ld}Q^{db} \\ &= a^{kf}a^{nb}T^{fm}\delta^{md}Q^{db} \\ &= a^{kf}a^{nb}T^{fm}Q^{mb} \end{aligned}$$

Therefore $T'^{kl}Q'^{ln}$ transforms like a tensor, and hence it's a tensor. □

Solution. Exercise 14. Let T'^{kl} be a tensor, we want to show that T'^{ll} is a scalar hence by using the transformation laws for tensors we have that

$$\begin{aligned}
T'^{ll} &= a^{ln}a^{lm}T^{nm} \\
&= (a^T)^{nl}a^{lm}T^{nm} \\
&= \delta^{nm}T^{nm} \\
&= T^{nn}
\end{aligned}$$

Therefore T'^{ll} remains unchanged under rotations so it is a scalar. □

Solution. Exercise 15. Let A^k, B^l and C^m be vectors then they transform as $A'^k = a^{kn}A^n$, $B'^l = a^{lr}B^r$ and $C'^m = a^{ms}C^s$ so by multiplying them we have that

$$A'^k B'^l C'^m = a^{kn}A^n a^{lr}B^r a^{ms}C^s = a^{kn}a^{lr}a^{ms}A^n B^r C^s$$

Then $A^k B^l C^m$ transform like a third rank tensor hence $A^k B^l C^m$ is a tensor. □

Solution. Exercise 16. Let ε^{klm} be the alternating tensor, we want to show it is a third-rank tensor i.e. we want to prove that

$$\varepsilon'^{klm} = a^{kn}a^{lr}a^{ms}\varepsilon^{nrs}$$

So let us evaluate explicitly the right side of the above equation for the indices $k = 1$, $l = 2$ and $m = 3$

$$\begin{aligned}
a^{1n}a^{2r}a^{3s}\varepsilon^{nrs} &= a^{11}a^{22}a^{33}\varepsilon^{123} + a^{12}a^{23}a^{31}\varepsilon^{231} + a^{13}a^{21}a^{32}\varepsilon^{312} \\
&\quad + a^{12}a^{21}a^{33}\varepsilon^{213} + a^{11}a^{23}a^{32}\varepsilon^{132} + a^{13}a^{22}a^{31}\varepsilon^{321} \\
&= \cos\theta \cdot \cos\theta \cdot 1 \cdot 1 + \sin\theta \cdot 0 \cdot 0 \cdot 1 + 0 \cdot (-\sin\theta) \cdot 0 \cdot 1 \\
&\quad + \sin\theta \cdot (-\sin\theta) \cdot 1 \cdot (-1) + \cos\theta \cdot 0 \cdot 0 \cdot (-1) \\
&\quad + 0 \cdot \cos\theta \cdot 0 \cdot (-1) \\
&= \cos^2\theta + \sin^2\theta \\
&= 1
\end{aligned}$$

In the same way for $k, l, m = 2, 3, 1$ and $k, l, m = 3, 1, 2$

$$\begin{aligned}
a^{2n}a^{3r}a^{1s}\varepsilon^{nrs} &= a^{21}a^{32}a^{13}\varepsilon^{123} + a^{22}a^{33}a^{11}\varepsilon^{231} + a^{23}a^{31}a^{12}\varepsilon^{312} \\
&\quad + a^{22}a^{31}a^{13}\varepsilon^{213} + a^{21}a^{33}a^{12}\varepsilon^{132} + a^{23}a^{32}a^{11}\varepsilon^{321} \\
&= 0 + \cos^2\theta + 0 + 0 + \sin^2\theta + 0 \\
&= \cos^2\theta + \sin^2\theta \\
&= 1 \\
a^{3n}a^{1r}a^{2s}\varepsilon^{nrs} &= a^{31}a^{12}a^{23}\varepsilon^{123} + a^{32}a^{13}a^{21}\varepsilon^{231} + a^{33}a^{11}a^{22}\varepsilon^{312} \\
&\quad + a^{32}a^{11}a^{23}\varepsilon^{213} + a^{31}a^{13}a^{22}\varepsilon^{132} + a^{33}a^{12}a^{21}\varepsilon^{321} \\
&= 0 + 0 + \cos^2\theta + 0 + 0 + \sin^2\theta \\
&= \cos^2\theta + \sin^2\theta \\
&= 1
\end{aligned}$$

Now we evaluate the equation for $k, l, m = 2, 1, 3$

$$\begin{aligned}
a^{2n}a^{1r}a^{3s}\varepsilon^{nrs} &= a^{21}a^{12}a^{33}\varepsilon^{123} + a^{22}a^{13}a^{31}\varepsilon^{231} + a^{23}a^{11}a^{32}\varepsilon^{312} \\
&\quad + a^{22}a^{11}a^{33}\varepsilon^{213} + a^{21}a^{13}a^{32}\varepsilon^{132} + a^{23}a^{12}a^{31}\varepsilon^{321} \\
&= -\sin^2\theta + 0 + 0 - \cos^2\theta + 0 + 0 \\
&= -(\cos^2\theta + \sin^2\theta) \\
&= -1
\end{aligned}$$

Thus for $k, l, m = 1, 3, 2$ and $k, l, m = 3, 2, 1$

$$\begin{aligned}
a^{1n}a^{3r}a^{2s}\varepsilon^{nrs} &= a^{11}a^{32}a^{23}\varepsilon^{123} + a^{12}a^{33}a^{21}\varepsilon^{231} + a^{13}a^{31}a^{22}\varepsilon^{312} \\
&\quad + a^{12}a^{31}a^{23}\varepsilon^{213} + a^{11}a^{33}a^{22}\varepsilon^{132} + a^{13}a^{32}a^{21}\varepsilon^{321} \\
&= 0 - \sin^2\theta + 0 + 0 - \cos^2\theta + 0 \\
&= -(\cos^2\theta + \sin^2\theta) \\
&= -1 \\
a^{3n}a^{2r}a^{1s}\varepsilon^{nrs} &= a^{31}a^{22}a^{13}\varepsilon^{123} + a^{32}a^{23}a^{11}\varepsilon^{231} + a^{33}a^{21}a^{12}\varepsilon^{312} \\
&\quad + a^{32}a^{21}a^{13}\varepsilon^{213} + a^{31}a^{23}a^{12}\varepsilon^{132} + a^{33}a^{22}a^{11}\varepsilon^{321} \\
&= 0 + 0 - \sin^2\theta + 0 + 0 - \cos^2\theta \\
&= -(\cos^2\theta + \sin^2\theta) \\
&= -1
\end{aligned}$$

Hence the equation matches the non-zero elements of ε'^{klm}.

Next, we want to show that elements of the form ε'^{kkm} are zero, hence let $k, l = 1, 1$ then

$$\begin{aligned}
a^{1n}a^{1r}a^{ms}\varepsilon^{nrs} &= a^{11}a^{12}a^{ms}\varepsilon^{12s} + a^{12}a^{11}a^{ms}\varepsilon^{21s} \\
&= a^{ms}(a^{11}a^{12}\varepsilon^{12s} + a^{12}a^{11}\varepsilon^{21s}) \\
&= a^{ms}(a^{11}a^{12} - a^{12}a^{11}) \\
&= 0
\end{aligned}$$

In the same way for $k, l = 2, 2$ we have that

$$\begin{aligned}
a^{2n}a^{2r}a^{ms}\varepsilon^{nrs} &= a^{21}a^{22}a^{ms}\varepsilon^{12s} + a^{22}a^{21}a^{ms}\varepsilon^{21s} \\
&= a^{ms}(a^{21}a^{22}\varepsilon^{12s} + a^{22}a^{21}\varepsilon^{21s}) \\
&= a^{ms}(a^{21}a^{22} - a^{22}a^{21}) \\
&= 0
\end{aligned}$$

and for $k, l = 3, 3$ since no index combination is different from zero we have that

$$a^{3n}a^{3r}a^{ms}\varepsilon^{nrs} = 0$$

The same can be shown for elements of the form ε^{kll} and ε^{mlm}. Therefore we have shown that

$$\varepsilon'^{klm} = a^{kn} a^{lr} a^{ms} \varepsilon^{nrs}$$

□

Solution. Exercise 17. Let A^l and B^m be vectors then they transform as $A'^l = a^{lr} A^r$ and $B'^m = a^{ms} B^s$ so we have that

$$\begin{aligned}
\varepsilon'^{klm} A'^l B'^m &= a^{kn} a^{lr} a^{ms} \varepsilon^{nrs} a^{lr} A^r a^{ms} B^s \\
&= a^{kn} a^{lr} (a^T)^{rl} a^{ms} (a^T)^{sm} \varepsilon^{nrs} A^r B^s \\
&= a^{kn} \delta^{ll} \delta^{mm} \varepsilon^{nrs} A^r B^s \\
&= a^{kn} \varepsilon^{nrs} A^r B^s
\end{aligned}$$

So we see that $\varepsilon^{klm} A^l B^m$ transforms like a vector as we wanted. □

Solution. Exercise 18. Let $\boldsymbol{B} = \nabla\Phi$ where Φ is a scalar field then by switching indices we have that

$$\begin{aligned}
\varepsilon^{klm} \frac{\partial}{\partial x^l} \frac{\partial \Phi}{\partial x^m} &= \varepsilon^{kml} \frac{\partial}{\partial x^m} \frac{\partial \Phi}{\partial x^l} \\
&= \varepsilon^{kml} \frac{\partial}{\partial x^l} \frac{\partial \Phi}{\partial x^m} \\
&= -\varepsilon^{klm} \frac{\partial}{\partial x^l} \frac{\partial \Phi}{\partial x^m}
\end{aligned}$$

Where we used that $\frac{\partial}{\partial x^m} \frac{\partial \Phi}{\partial x^l} = \frac{\partial}{\partial x^l} \frac{\partial \Phi}{\partial x^m}$. But this can only be true if

$$\varepsilon^{klm} \frac{\partial}{\partial x^l} \frac{\partial \Phi}{\partial x^m} = 0$$

Finally, we should be able to obtain the same result making explicit the equation $\nabla \times \nabla\Phi$ as follows

$$\begin{aligned}
\nabla \times \nabla\Phi = \hat{\boldsymbol{x}} &\left(\frac{\partial}{\partial x^2} \frac{\partial \Phi}{\partial x^3} - \frac{\partial}{\partial x^3} \frac{\partial \Phi}{\partial x^2} \right) + \hat{\boldsymbol{y}} \left(\frac{\partial}{\partial x^3} \frac{\partial \Phi}{\partial x^1} - \frac{\partial}{\partial x^1} \frac{\partial \Phi}{\partial x^3} \right) \\
&+ \hat{\boldsymbol{z}} \left(\frac{\partial}{\partial x^1} \frac{\partial \Phi}{\partial x^2} - \frac{\partial}{\partial x^2} \frac{\partial \Phi}{\partial x^1} \right) \\
&= \hat{\boldsymbol{x}} \cdot 0 + \hat{\boldsymbol{y}} \cdot 0 + \hat{\boldsymbol{z}} \cdot 0 \\
&= \boldsymbol{0}
\end{aligned}$$

□

Solution. Exercise 19. Let $\boldsymbol{B}$ be a vector then the divergence of the curl can be written as

$$\begin{aligned}
\frac{\partial}{\partial x^k}\varepsilon^{klm}\frac{\partial}{\partial x^l}B^m &= \varepsilon^{lkm}\frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}B^m \\
&= \varepsilon^{lkm}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}B^m \\
&= -\varepsilon^{klm}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}B^m \\
&= -\frac{\partial}{\partial x^k}\varepsilon^{klm}\frac{\partial}{\partial x^l}B^m
\end{aligned}$$

Where we used that $\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}B^m = \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}B^m$. But this can only be true if

$$\frac{\partial}{\partial x^k}\varepsilon^{klm}\frac{\partial}{\partial x^l}B^m = 0$$

Finally, we should be able to obtain the same result by making explicit the equation $\nabla\cdot(\nabla\times\boldsymbol{B}) = 0$ as follows

$$\begin{aligned}
\nabla\cdot(\nabla\times\boldsymbol{B}) = &\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z\right) \\
&+ \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x\right) \\
= &\frac{\partial}{\partial x}\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial x}\frac{\partial}{\partial z}B_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial y}\frac{\partial}{\partial x}B_z \\
&+ \frac{\partial}{\partial z}\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial z}\frac{\partial}{\partial y}B_x \\
= &\,0
\end{aligned}$$

□

Solution. Exercise 20.

$$\begin{aligned}
\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\phi} &= \hat{\boldsymbol{x}}\frac{\partial\cos\phi}{\partial\phi} + \hat{\boldsymbol{y}}\frac{\partial\sin\phi}{\partial\phi} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi = \hat{\boldsymbol{\phi}} \\
\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} &= -\hat{\boldsymbol{x}}\frac{\partial\sin\phi}{\partial\phi} + \hat{\boldsymbol{y}}\frac{\partial\cos\phi}{\partial\phi} = -\hat{\boldsymbol{x}}\cos\phi - \hat{\boldsymbol{y}}\sin\phi = -\hat{\boldsymbol{\rho}}
\end{aligned}$$

□

Solution. Exercise 21. We want to prove that

$$\nabla = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

So from this expression, we should be able to get ∇ in cartesian coordinates.

First, we need to compute a few partial derivatives that we will need later

$$\frac{\partial x}{\partial\rho} = \cos\phi \qquad \frac{\partial y}{\partial\rho} = \sin\phi$$
$$\frac{\partial x}{\partial\phi} = -\rho\sin\phi \qquad \frac{\partial y}{\partial\phi} = \rho\cos\phi$$

So replacing the partial derivatives we computed and using equation (81), we have that

$$\begin{aligned}
\nabla &= \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{\rho}}\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\rho}\right) + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\phi}\right) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{\rho}}\left(\frac{\partial}{\partial x}\cos\phi + \frac{\partial}{\partial y}\sin\phi\right) + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\left(-\frac{\partial}{\partial x}\rho\sin\phi + \frac{\partial}{\partial y}\rho\cos\phi\right) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= (\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi)\left(\frac{\partial}{\partial x}\cos\phi + \frac{\partial}{\partial y}\sin\phi\right) + \\
&\quad + (-\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi)\frac{1}{\rho}\left(-\frac{\partial}{\partial x}\rho\sin\phi + \frac{\partial}{\partial y}\rho\cos\phi\right) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\cos^2\phi + \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\sin\phi\cos\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\cos\phi\sin\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\sin^2\phi + \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\sin^2\phi - \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\sin\phi\cos\phi - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\cos\phi\sin\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\cos^2\phi + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{x}}\frac{\partial}{\partial x}(\cos^2\phi + \sin^2\phi) + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}(\cos^2\phi + \sin^2\phi) + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&= \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}
\end{aligned}$$

□

Solution. Exercise 22. We want to compute the following

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{\rho}}A_\rho + \hat{\boldsymbol{\phi}}A_\phi + \hat{\boldsymbol{z}}A_z\right)$$

So below we compute the distribution of the first term

$$\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\right)\cdot\left(\hat{\boldsymbol{\rho}}A_\rho\right) = \hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\rho}}\frac{\partial A_\rho}{\partial\rho} + A_\rho\hat{\boldsymbol{\rho}}\cdot\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\rho} = \frac{\partial A_\rho}{\partial\rho}$$
$$\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\right)\cdot\left(\hat{\boldsymbol{\phi}}A_\phi\right) = \hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\phi}}\frac{\partial A_\phi}{\partial\rho} + A_\phi\hat{\boldsymbol{\rho}}\cdot\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\rho} = 0$$
$$\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\right)\cdot\left(\hat{\boldsymbol{z}}A_z\right) = \hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{z}}\frac{\partial A_z}{\partial\rho} + A_z\hat{\boldsymbol{\rho}}\cdot\frac{\partial\hat{\boldsymbol{z}}}{\partial\rho} = 0$$

Where we used that $\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\phi}} = 0$ and that $\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{z}} = 0$. Also, the derivatives with respect to ρ of the unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$ are 0. In the same way for the second term, we have that

$$\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{\boldsymbol{\rho}}A_\rho\right) = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\rho}}\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi} + A_\rho\frac{1}{\rho}\hat{\boldsymbol{\phi}}\cdot\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\phi} = \frac{A_\rho}{\rho}$$
$$\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{\boldsymbol{\phi}}A_\phi\right) = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + A_\phi\frac{1}{\rho}\hat{\boldsymbol{\phi}}\cdot\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}$$
$$\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{\boldsymbol{z}}A_z\right) = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{z}}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} + A_z\frac{1}{\rho}\hat{\boldsymbol{\phi}}\cdot\frac{\partial\hat{\boldsymbol{z}}}{\partial\phi} = 0$$

In this case, additionally, we used that $\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\phi} = \hat{\boldsymbol{\phi}}$ and that $\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = -\hat{\boldsymbol{\rho}}$. Finally, for the third term, we have that

$$\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{\rho}}A_\rho\right) = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\rho}}\frac{\partial A_\rho}{\partial z} + A_\rho\hat{\boldsymbol{z}}\cdot\frac{\partial\hat{\boldsymbol{\rho}}}{\partial z} = 0$$
$$\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{\phi}}A_\phi\right) = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}}\frac{\partial A_\phi}{\partial z} + A_\phi\hat{\boldsymbol{z}}\cdot\frac{\partial\hat{\boldsymbol{\phi}}}{\partial z} = 0$$
$$\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{z}}A_z\right) = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}}\frac{\partial A_z}{\partial z} + A_z\hat{\boldsymbol{z}}\cdot\frac{\partial\hat{\boldsymbol{z}}}{\partial z} = \frac{\partial A_z}{\partial z}$$

Adding all these results we get that

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_\rho}{\partial\rho} + \frac{A_\rho}{\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

□

Solution. **Exercise 23.** By using the product rule we have that

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} &= \frac{1}{\rho}A_\rho\frac{\partial \rho}{\partial \rho} + \frac{1}{\rho}\rho\frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
&= \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
&= \boldsymbol{\nabla} \cdot \boldsymbol{A}
\end{aligned}
$$

□

Solution. Exercise 24. We want to compute the Laplacian by solving the following

$$\nabla^2 = \nabla \cdot \nabla = \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)$$

We will compute each term separately as follows

$$\begin{aligned}
\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\right) \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\right) &= \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}}\frac{\partial^2}{\partial \rho^2} + \hat{\boldsymbol{\rho}} \cdot \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \rho}\frac{\partial}{\partial \rho} = \frac{\partial^2}{\partial \rho^2} \\
\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\right) \cdot \left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right) &= \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial^2}{\partial \rho \partial \phi} + \hat{\boldsymbol{\rho}} \cdot \left(\frac{1}{\rho}\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \rho} - \frac{1}{\rho^2}\hat{\boldsymbol{\phi}}\right)\frac{\partial}{\partial \phi} = 0 \\
\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\right) \cdot \left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) &= \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{z}}\frac{\partial^2}{\partial \rho \partial z} + \hat{\boldsymbol{\rho}} \cdot \frac{\partial \hat{\boldsymbol{z}}}{\partial \rho}\frac{\partial}{\partial z} = 0
\end{aligned}$$

Where we used that $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = 0$ and that $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{z}} = 0$. Also, the derivatives with respect to ρ of the unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$ are 0. In the same way for the second term, we have that

$$\begin{aligned}
\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right) \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\right) &= \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\rho}}\frac{1}{\rho}\frac{\partial^2}{\partial \phi \partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho} \cdot \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\frac{\partial}{\partial \rho} = \frac{1}{\rho}\frac{\partial}{\partial \rho} \\
\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right) \cdot \left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right) &= \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \hat{\boldsymbol{\phi}}\frac{1}{\rho^2} \cdot \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}\frac{\partial}{\partial \phi} = \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \\
\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right) \cdot \left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) &= \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{z}}\frac{1}{\rho}\frac{\partial^2}{\partial \phi \partial z} + \hat{\boldsymbol{\phi}}\frac{1}{\rho} \cdot \frac{\partial \hat{\boldsymbol{z}}}{\partial \phi}\frac{\partial}{\partial z} = 0
\end{aligned}$$

In this case, additionally, we used that $\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi} = \hat{\boldsymbol{\phi}}$ and that $\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\boldsymbol{\rho}}$. Finally, for the third term, we have that

$$\begin{aligned}
\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho}\right) &= \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\rho}}\frac{\partial^2}{\partial z \partial \rho} + \hat{\boldsymbol{z}} \cdot \frac{\partial \hat{\boldsymbol{\rho}}}{\partial z}\frac{\partial}{\partial \rho} = 0 \\
\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right) &= \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial^2}{\partial z \partial \phi} + \hat{\boldsymbol{z}} \cdot \frac{1}{\rho}\frac{\partial \hat{\boldsymbol{\phi}}}{\partial z}\frac{\partial}{\partial \phi} = 0 \\
\left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) &= \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}}\frac{\partial^2}{\partial z^2} + \hat{\boldsymbol{z}} \cdot \frac{\partial \hat{\boldsymbol{z}}}{\partial z}\frac{\partial}{\partial z} = \frac{\partial^2}{\partial z^2}
\end{aligned}$$

Adding all these results we get that

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

□

Solution. Exercise 25. By using the product rule we have that

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}=\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}=\nabla^2$$

□

Solution. Exercise 26. Knowing that

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}}\sin\theta\cos\phi+\hat{\boldsymbol{y}}\sin\theta\sin\phi+\hat{\boldsymbol{z}}\cos\theta\\
\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{x}}\cos\theta\cos\phi+\hat{\boldsymbol{y}}\cos\theta\sin\phi-\hat{\boldsymbol{z}}\sin\theta\\
\hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{x}}\sin\phi+\hat{\boldsymbol{y}}\cos\phi
\end{aligned}$$

We have that

$$\begin{aligned}
\frac{\partial\hat{\boldsymbol{r}}}{\partial\theta} &= \hat{\boldsymbol{x}}\cos\theta\cos\phi+\hat{\boldsymbol{y}}\sin\theta\sin\phi-\hat{\boldsymbol{z}}\sin\theta=\hat{\boldsymbol{\theta}}\\
\frac{\partial\hat{\boldsymbol{r}}}{\partial\phi} &= -\hat{\boldsymbol{x}}\sin\theta\sin\phi+\hat{\boldsymbol{y}}\sin\theta\cos\phi=\hat{\boldsymbol{\phi}}\sin\theta\\
\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} &= -\hat{\boldsymbol{x}}\sin\theta\cos\phi-\hat{\boldsymbol{y}}\sin\theta\sin\phi-\hat{\boldsymbol{z}}\cos\theta=-\hat{\boldsymbol{r}}\\
\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} &= -\hat{\boldsymbol{x}}\cos\theta\sin\phi+\hat{\boldsymbol{y}}\cos\theta\cos\phi=\hat{\boldsymbol{\phi}}\cos\theta
\end{aligned}$$

Finally, we have that

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi}=-\hat{\boldsymbol{x}}\cos\phi-\hat{\boldsymbol{y}}\sin\phi$$

But also we know that

$$\begin{aligned}
-\hat{\boldsymbol{r}}\sin\theta-\hat{\boldsymbol{\theta}}\cos\theta &= -\hat{\boldsymbol{x}}\sin^2\theta\cos\phi-\hat{\boldsymbol{y}}\sin^2\theta\sin\phi-\hat{\boldsymbol{z}}\sin\theta\cos\theta\\
&\quad-\hat{\boldsymbol{x}}\cos^2\theta\cos\phi-\hat{\boldsymbol{y}}\cos^2\theta\sin\phi+\hat{\boldsymbol{z}}\sin\theta\cos\theta\\
&= -\hat{\boldsymbol{x}}\cos\phi(\sin^2\theta+\cos^2\theta)-\hat{\boldsymbol{y}}\sin\phi(\sin^2\theta+\cos^2\theta)\\
&= -\hat{\boldsymbol{x}}\cos\phi-\hat{\boldsymbol{y}}\sin\phi
\end{aligned}$$

Therefore

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi}=-\hat{\boldsymbol{r}}\sin\theta-\hat{\boldsymbol{\theta}}\cos\theta$$

□

Solution. Exercise 27. We want to prove that

$$\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

So from this expression, we should be able to get ∇ in cartesian coordinates.

First, we need to compute a few partial derivatives that we will need later

$$\begin{aligned}
&\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad &&\frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad &&\frac{\partial z}{\partial r} = \cos\theta \\
&\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad &&\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad &&\frac{\partial z}{\partial \theta} = -r\sin\theta \\
&\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad &&\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad &&\frac{\partial z}{\partial \phi} = 0
\end{aligned}$$

So replacing the partial derivatives we computed and using equation (100), we have that

$$\begin{aligned}
\nabla &= \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\
&= \hat{\boldsymbol{r}}\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r}\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \theta}\right) + \\
&\quad + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \phi}\right) \\
&= \hat{\boldsymbol{r}}\left(\frac{\partial}{\partial x}\sin\theta\cos\phi + \frac{\partial}{\partial y}\sin\theta\sin\phi + \frac{\partial}{\partial z}\cos\theta\right) + \\
&\quad + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{\partial}{\partial x}r\cos\theta\cos\phi + \frac{\partial}{\partial y}r\cos\theta\sin\phi - \frac{\partial}{\partial z}r\sin\theta\right) + \\
&\quad + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\left(-\frac{\partial}{\partial x}r\sin\theta\sin\phi + \frac{\partial}{\partial y}r\sin\theta\cos\phi\right) \\
&= (\hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta)\left(\frac{\partial}{\partial x}\sin\theta\cos\phi + \frac{\partial}{\partial y}\sin\theta\sin\phi + \frac{\partial}{\partial z}\cos\theta\right) + \\
&\quad + (\hat{\boldsymbol{x}}\cos\theta\cos\phi + \hat{\boldsymbol{y}}\cos\theta\sin\phi - \hat{\boldsymbol{z}}\sin\theta)\frac{1}{r}\left(\frac{\partial}{\partial x}r\cos\theta\cos\phi + \frac{\partial}{\partial y}r\cos\theta\sin\phi - \frac{\partial}{\partial z}r\sin\theta\right) + \\
&\quad + (-\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi)\frac{1}{r\sin\theta}\left(-\frac{\partial}{\partial x}r\sin\theta\sin\phi + \frac{\partial}{\partial y}r\sin\theta\cos\phi\right)
\end{aligned}$$

$$
\begin{aligned}
&= \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\sin^2\theta\cos^2\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\sin^2\theta\sin\phi\cos\phi + \hat{\boldsymbol{z}}\frac{\partial}{\partial x}\sin\theta\cos\phi\cos\theta \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\sin^2\theta\cos\phi\sin\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\sin^2\theta\sin^2\phi + \hat{\boldsymbol{z}}\frac{\partial}{\partial y}\sin\theta\sin\phi\cos\theta \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial z}\sin\theta\cos\phi\cos\theta + \hat{\boldsymbol{y}}\frac{\partial}{\partial z}\sin\theta\sin\phi\cos\theta + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\cos^2\theta \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\cos^2\theta\cos^2\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\cos^2\theta\sin\phi\cos\phi - \hat{\boldsymbol{z}}\frac{\partial}{\partial x}\sin\theta\cos\theta\cos\phi \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\cos^2\theta\cos\phi\sin\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\cos^2\theta\sin^2\phi - \hat{\boldsymbol{z}}\frac{\partial}{\partial y}\sin\theta\cos\theta\sin\phi \\
&\quad - \hat{\boldsymbol{x}}\frac{\partial}{\partial z}\cos\theta\cos\phi\sin\theta - \hat{\boldsymbol{y}}\frac{\partial}{\partial z}\cos\theta\sin\phi\sin\theta + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\sin^2\theta \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\sin^2\phi - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\cos\phi\sin\phi - \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\sin\phi\cos\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\cos^2\phi \\
&= \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\cos^2\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\sin\phi\cos\phi + \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\cos\phi\sin\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\sin^2\phi + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
&\quad + \hat{\boldsymbol{x}}\frac{\partial}{\partial x}\sin^2\phi - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\cos\phi\sin\phi - \hat{\boldsymbol{x}}\frac{\partial}{\partial y}\sin\phi\cos\phi + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\cos^2\phi \\
&= \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}
\end{aligned}
$$

□

Solution. Exercise 29. Let us compute the rest of the surface integrals over the other faces of the cube. For the faces at $y = y_0 + L$ and $y = y_0$ we have that

$$\iint A_y\bigg|_{y=y_0+L} dxdz - \iint A_y\bigg|_{y=y_0} dxdz$$

Where the minus sign at $y = y_0$ arises because the $\hat{\boldsymbol{n}}$ vector points in the opposite direction to the $\hat{\boldsymbol{y}}$ vector.

In the same way for the faces at $z = z_0 + L$ and $z = z_0$ we have that

$$\iint A_z\bigg|_{z=z_0+L} dxdy - \iint A_z\bigg|_{z=z_0} dxdy$$

□

Solution. Exercise 30. The gradient of $\Phi(\boldsymbol{x})$ is given by

$$\boldsymbol{\nabla}\Phi(\boldsymbol{x}) = \frac{\partial\Phi(\boldsymbol{x})}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\Phi(\boldsymbol{x})}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\Phi(\boldsymbol{x})}{\partial z}\hat{\boldsymbol{z}}$$

So we compute the $\hat{\boldsymbol{x}}$ component as follows

$$\frac{\partial\Phi(\boldsymbol{x})}{\partial x} = \frac{\partial}{\partial x}\left(\int_{x_0}^{x} A_x dx' + \int_{y_0}^{y} A_y dy' + \int_{z_0}^{z} A_z dz'\right) = A_x$$

Where we used the Fundamental Theorem of Calculus to get that

$$\frac{\partial}{\partial x}\left(\int_{x_0}^{x} A_x dx'\right) = A_x$$

But also, we see that

$$\frac{\partial}{\partial x}\left(\int_{y_0}^{y} A_y dy' + \int_{z_0}^{z} A_z dz'\right) = 0$$

Similarly, we have that

$$\frac{\partial\Phi(\boldsymbol{x})}{\partial y} = A_y \qquad \frac{\partial\Phi(\boldsymbol{x})}{\partial z} = A_z$$

Which implies that

$$\boldsymbol{\nabla}\Phi(\boldsymbol{x}) = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}} = \boldsymbol{A}(\boldsymbol{x})$$

$\square$