Solved selected problems of Classical Electrodynamics - Hans Ohanian

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Chapter 2 - Electrostatics

Problems

Solution. 1. Let us suppose we have a proton and an electron at a distance r then the magnitude of the gravitational force exerted on the proton by the electron is

$$\begin{aligned} F_{\text{grav}} &= G\frac{m_e m_p}{r^2} \\ &= 6.674 \times 10^{-8}\ \frac{dyne \cdot cm^2}{g^2} \cdot \frac{9.10938 \times 10^{-28}\ g \cdot 1.67262 \times 10^{-24}\ g}{r^2} \end{aligned}$$

On the other hand, the magnitude of the electric force exerted on the proton by the electron is

$$\begin{aligned} F_{\text{electric}} &= \frac{|q_e q_p|}{r^2} \\ &= \frac{(4.803207 \times 10^{-10}\ esu)^2}{r^2} \end{aligned}$$

So the ratio between the gravitational and the electrical force is

$$\frac{F_{\text{electric}}}{F_{\text{grav}}} = \frac{2.307079 \times 10^{-19}}{1.01688 \times 10^{-58}} = 2.26876 \times 10^{39}$$

$\square$

Solution. **4.** Let us consider a system that looks like the following

We know the rods carry a uniformly distributed charge Q so the charge per unit of length is Q/L then if we consider a small piece of rod dx' it will carry a charge of $(Q/L)dx'$ so at a distance of $x' + x + d$ this small charge contributes an electric field of magnitude

$$dE = \frac{Q}{L}\frac{dx}{(x' + x + d)^2}$$

So by integration, we have that

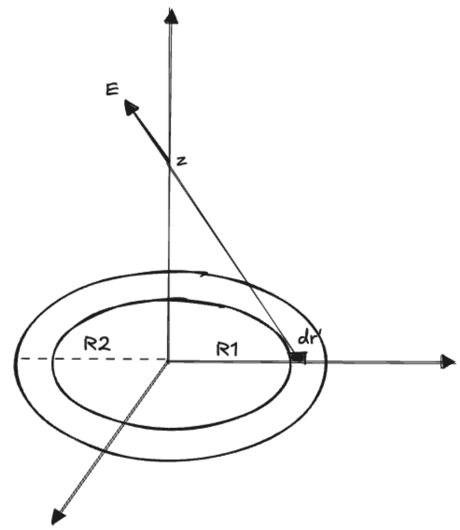
$$\begin{aligned}
E &= \int_0^L \frac{Q}{L}\frac{dx'}{(x' + x + d)^2} \\
&= \frac{Q}{L}\left[-\frac{1}{x' + x + d}\right]_0^L \\
&= \frac{Q}{L}\left[-\frac{1}{L + x + d} + \frac{1}{x + d}\right] \\
&= \frac{Q}{L}\frac{L}{(x + d)(x + d + L)} \\
&= \frac{Q}{(x + d)(x + d + L)}
\end{aligned}$$

Therefore the force on a small charge $(Q/L)dx$ on the other rod is $(Q/L)E\ dx$ so again by integration we have that

$$\begin{aligned}
F &= \int_0^L \frac{Q}{L}E\ dx \\
&= \int_0^L \frac{Q^2}{L(x + d)(x + d + L)}\ dx \\
&= \frac{Q^2}{L}\int_0^L \frac{1}{(x + d)(x + d + L)}\ dx \\
&= \frac{Q^2}{L}\left[\frac{\log(x + d) - \log(x + d + L)}{L}\right]_0^L \\
&= \frac{Q^2}{L^2}\Big[\log(L + d) - \log(d + 2L) - \log(d) + \log(d + L)\Big] \\
&= \frac{Q^2}{L^2}\Big[2\log(L + d) - \log(d + 2L) - \log(d)\Big]
\end{aligned}$$

$\square$

Solution. 6. Let us consider a system that looks like the following



Let us consider a small piece of area in polar coordinates $r'dr'd\phi'$ then this small piece carries a charge $\sigma r'dr'd\phi'$ where $\sigma = 0$ if $r' < R_1$. Then at a height z above the annulus this charge contributes an electric field of magnitude

$$dE = \frac{\sigma r'dr'd\phi'}{z^2 + r'^2}$$

This electric field has both a vertical and a horizontal component. In view of the symmetry of the charge distribution, the horizontal components cancel upon integration. Hence we need to consider only the vertical components

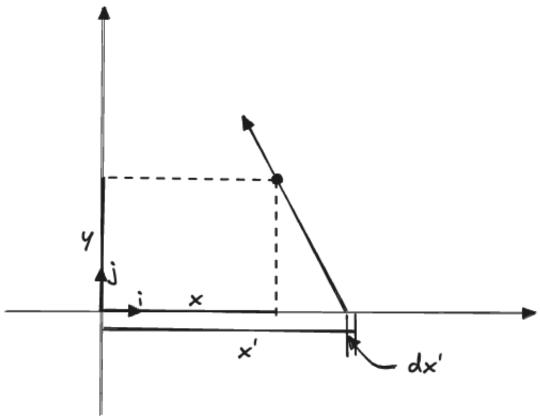
$$\begin{aligned} dE_z &= \cos\theta \frac{\sigma r'dr'd\phi'}{z^2 + r'^2} \\ &= \frac{z}{\sqrt{z^2 + r'^2}} \frac{\sigma r'dr'd\phi'}{z^2 + r'^2} \\ &= \frac{\sigma z r'dr'd\phi'}{(z^2 + r'^2)^{3/2}} \end{aligned}$$

Now by integration we have that

$$\begin{aligned} E &= \sigma \int_{R_1}^{R_2} \int_0^{2\pi} \frac{zr'dr'd\phi'}{(z^2 + r'^2)^{3/2}} \\ E &= 2\pi\sigma \int_{R_1}^{R_2} \frac{zr'dr'}{(z^2 + r'^2)^{3/2}} \\ E &= 2\pi\sigma \Big[- \frac{z}{\sqrt{z^2 + r'^2}} \Big]_{R_1}^{R_2} \\ E &= 2\pi\sigma z \Big[\frac{1}{\sqrt{z^2 + R_1^2}} - \frac{1}{\sqrt{z^2 + R_2^2}} \Big] \end{aligned}$$

□

Solution. 7. Let us take a point (x, y) on the x-y plane. First, we evaluate the contribution of the rod on the x axis as shown below



then a dx' element on the x-axis rod will contribute an electric field of magnitude

$$dE = \frac{\lambda dx'}{(x - x')^2 + y^2}$$

This electric field has both a vertical and a horizontal component. In view of the symmetry of the charge distribution, the horizontal components between 0 and $2x$ cancel out upon integration. Hence we need to consider only the vertical components in the segment $0 \le x' \le 2x$ where dE_y is given by

$$\begin{aligned} dE_y &= \cos\theta \frac{\lambda dx'}{(x - x')^2 + y^2} \\ &= \frac{y}{\sqrt{(x - x')^2 + y^2}} \frac{\lambda dx'}{(x - x')^2 + y^2} \\ &= \frac{\lambda y dx'}{((x - x')^2 + y^2)^{3/2}} \end{aligned}$$

For the rest of the x-axis we do have an horizontal component which is given by

$$\begin{aligned} dE_x &= \sin\theta \frac{\lambda dx'}{(x - x')^2 + y^2} \\ &= \frac{(x - x')}{\sqrt{(x - x')^2 + y^2}} \frac{\lambda dx'}{(x - x')^2 + y^2} \\ &= \frac{\lambda (x - x') dx'}{((x - x')^2 + y^2)^{3/2}} \end{aligned}$$

Now, by integration we compute E^x where the superscript implies that the electric field correspond to the x-axis rod.

$$\begin{aligned}
E^x &= \int_{2x}^{\infty} \frac{\lambda(x-x')dx'}{((x-x')^2+y^2)^{3/2}}\boldsymbol{i} + \int_{0}^{\infty} \frac{\lambda y dx'}{((x-x')^2+y^2)^{3/2}}\boldsymbol{j} \\
&= \lambda\left[\left[\frac{1}{\sqrt{(x-x')^2+y^2}}\right]_{2x}^{\infty}\boldsymbol{i} + y\left[\frac{(x'-x)}{y^2\sqrt{(x-x')^2+y^2}}\right]_{0}^{\infty}\boldsymbol{j}\right] \\
&= \lambda\left[\frac{1}{\sqrt{x^2+y^2}}\boldsymbol{i} + y\left[\frac{1}{y^2} + \frac{x}{y^2\sqrt{x^2+y^2}}\right]\boldsymbol{j}\right] \\
&= \lambda\left[\frac{1}{\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{\sqrt{x^2+y^2}+x}{y\sqrt{x^2+y^2}}\boldsymbol{j}\right]
\end{aligned}$$

Analogously, we can do the same thing for the rod on the y-axis where we get that

$$\begin{aligned}
E^y &= \int_{0}^{\infty} \frac{\lambda x dy'}{(x^2+(y-y')^2)^{3/2}}\boldsymbol{i} + \int_{2y}^{\infty} \frac{\lambda(y-y')dx'}{(x^2+(y-y')^2)^{3/2}}\boldsymbol{j} \\
&= \lambda\left[\frac{\sqrt{x^2+y^2}+y}{x\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{1}{\sqrt{x^2+y^2}}\boldsymbol{j}\right]
\end{aligned}$$

Adding both contributions we get that

$$\begin{aligned}
E &= E^x + E^y \\
&= \lambda\left[\left(\frac{1}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}+y}{x\sqrt{x^2+y^2}}\right)\boldsymbol{i} + \left(\frac{1}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}+x}{y\sqrt{x^2+y^2}}\right)\boldsymbol{j}\right] \\
&= \lambda\left[\frac{\sqrt{x^2+y^2}+x+y}{x\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{\sqrt{x^2+y^2}+x+y}{y\sqrt{x^2+y^2}}\boldsymbol{j}\right]
\end{aligned}$$

Therefore E is the electric field at any point in the x-y plane. $\square$

Solution. 8. We know because of Excercise 2 that the electric field inside a sphere is

$$\boldsymbol{E}(r) = \frac{Qr}{R^3}\hat{\boldsymbol{r}}$$

In this case, the uniformly distributed charge Q is equal to the positive charge e. If we consider the electron to be at the center of the sphere then the force on the electron is

$$\boldsymbol{F} = -e\boldsymbol{E} = -\frac{e^2}{R^3}r\hat{\boldsymbol{r}}$$

But from the Newton's second law we have that

$$m\frac{\partial^2 r}{\partial t^2} = -\frac{e^2}{R^3}r$$

Defining $k = \frac{e^2}{R^3}$ we see that this is the equation of a simple harmonic motion which has an angular frequency of

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{e^2}{mR^3}}$$

Therefore by replacing the value of $R = 0.5\text{Å}$ and the values of e and m for the electron we have that

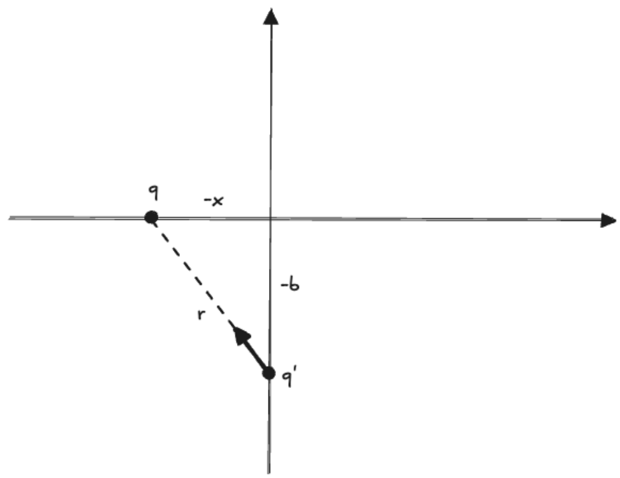
$$\omega = \sqrt{\frac{(4.803 \times 10^{-10})^2}{(9.109 \times 10^{-28})(5 \times 10^{-9})^3}} = 4.501 \times 10^{16} \text{ rad/s}$$

And hence the oscillation frequency is given by

$$f = \frac{\omega}{2\pi} = 7.164 \times 10^{15} \text{ Hz}$$

□

Solution. 10. The system described implies that



Then the charge q' produces an electric field

$$\boldsymbol{E} = \frac{q'}{r^2}\hat{\boldsymbol{r}} = \frac{q'}{(b^2+x^2)}\hat{\boldsymbol{r}}$$

but we are only interested in the y component of the electric field hence

$$E_y = E\cos\theta = \frac{q'}{(b^2+x^2)}\cos\theta = \frac{q'}{(b^2+x^2)}\frac{b}{\sqrt{b^2+x^2}}$$

Where we used that $\cos\theta = b/r$ then the integral $\int E_y dx$ becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} E_y dx &= \int_{-\infty}^{\infty} \frac{q'}{(b^2+x^2)}\frac{b}{\sqrt{b^2+x^2}}dx \\
&= bq'\int_{-\infty}^{\infty}\frac{\sqrt{b^2+x^2}}{(b^2+x^2)^2}dx \\
&= bq'\left[\frac{x}{b^2\sqrt{b^2+x^2}}\right]_{-\infty}^{\infty} \\
&= bq'\left[\frac{1}{b^2}+\frac{1}{b^2}\right] \\
&= \frac{2q'}{b}
\end{aligned}$$

Therefore the transverse impulse is

$$\int_{-\infty}^{\infty} F_y dt = \frac{q}{v}\int_{-\infty}^{\infty} E_y dx = \frac{2qq'}{vb}$$

From Newton's second law it must be that $ma_y = F_y$ then

$$\begin{aligned}
\int_{-\infty}^{\infty} m\frac{\mathrm{d}v_y}{\mathrm{d}t}dt &= \frac{2qq'}{vb} \\
m\int_{-\infty}^{\infty} dv_y &= \frac{2qq'}{vb} \\
v_y(\infty) - v_y(-\infty) &= \frac{2qq'}{mvb} \\
v_y(\infty) &= \frac{2qq'}{mvb}
\end{aligned}$$

The transverse deflection angle is the same angle the velocity is deflected so must be that

$$\sin\alpha = \frac{v_y}{v} = \frac{2qq'}{mv^2b}$$

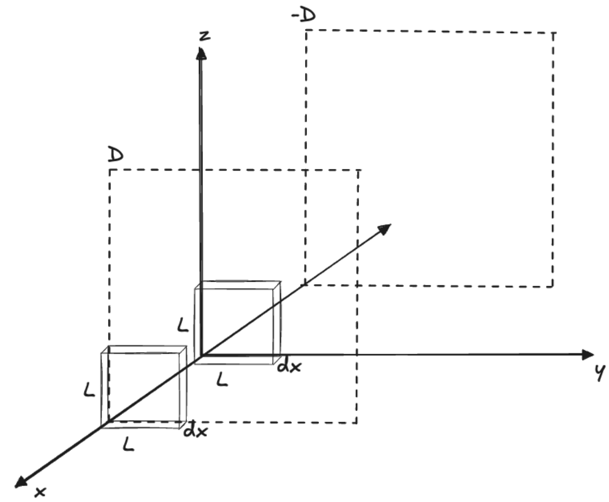
but when α is small we have that $\alpha \approx \sin\alpha = 2qq'/mv^2b$ which is the transverse deflection angle we wanted.

Finally, we want to compute the angular deflection for an alpha particle of $mv^2/2 = 6.0\ MeV$ passing by a lead nucleus with and impact parameter of $b = 10^{-10}\ cm$ then

$$\begin{aligned}
\alpha = \frac{qq'}{\frac{mv^2}{2}b} &= \frac{(2\cdot 4.8032\times 10^{-10}\ esu)\cdot(82\cdot 4.8032\times 10^{-10}\ esu)}{(9.6131\times 10^{-6}\ erg)\cdot(10^{-10}\ cm)} \\
&= 0.03935\ radian
\end{aligned}$$

$\square$

Solution. 12. Let us consider the following system



(a) We want to compute first the electric field when $-D < x < D$. Let us consider a differential volume $dV = l^2 dx$ between $-D$ and D then the electric field on the volume must be in the x direction by symmetry. The charge inside this differential volume is

$$dQ = \rho l^2 dx$$

Hence by integration we get that

$$Q = \int_0^x \rho l^2 dx = \rho l^2 x$$

Now to solve $\int E \cdot dS$ we see that only the front and back faces of the volume contribute to the integral so $\int E \cdot dS = 2E_x l^2$ therefore by Gauss' law we have that

$$\begin{aligned} 2E_x l^2 &= 4\pi(2\rho l^2 x) \\ E_x &= 4\pi \rho x \end{aligned}$$

Where we also used that to account for the charge on the section from 0 to $-D$ we need to muliply Q by 2.

To compute the electric field at $x \geq D$ we consider another differential volume placed at $x = D$ as shown in the figure. Then the charge inside this differential volume is

$$dQ = \rho l^2 dx$$

And hence by integration we get that

$$Q = \int_0^D \rho l^2 dx = \rho l^2 D$$

Again the front and back faces are the only ones that contribute to the electrical field so by Gauss' law we have that

$$\begin{aligned} 2E_x l^2 &= 4\pi(2\rho l^2 D) \\ E_x &= 4\pi\rho D \end{aligned}$$

Finally, for a differential volume placed at $x = -D$ what changes is the integration of dQ where we integrate from 0 to $-D$ to obtain the following

$$Q = \int_0^{-D} \rho l^2 dx = -\rho l^2 D$$

Theferore in this case, we get that

$$E_x = -4\pi\rho D$$

(b) We know that the electrostatic potential is given by

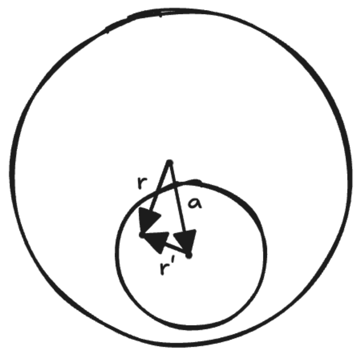
$$\boldsymbol{\phi(x)} = -\int_{x_0}^{x} \boldsymbol{E(x') \cdot dx'}$$

where x_0 is some reference point. So taking $x_0 = -D$ and $x = D$ to get the potential difference between these points we can write that

$$\begin{aligned} \phi(x) &= -\int_{-D}^{D} 4\pi\rho x \ dx \\ &= -4\pi\rho\left[\frac{D^2}{2} - \frac{(-D)^2}{2}\right] \\ &= 0 \end{aligned}$$

$\square$

Solution. 14. Let us consider the following section of the cylinder described



Assuming the cylindrical cavity doesn't exists. Let us consider a point at a distance $\boldsymbol{r}$ from the center of the big cylinder of radius R then the electric field at this point will be in the direction of $\boldsymbol{r}$ and we can determine the magnitude of it by considering a cylindrical Gaussian surface of radius r, hence by Gauss' law we get that

$$\int \boldsymbol{E} \cdot d\boldsymbol{S} = 4\pi Q$$

$$E_{cyl} \cdot 2\pi r L = 4\pi \frac{\lambda L r^2}{R^2}$$

$$E_{cyl} = \frac{2\lambda r}{R^2}$$

Where we used that the charge enclosed by the Gaussian surface is $\lambda L r^2/R^2$. In vector form we get that

$$\boldsymbol{E}_{cyl} = \frac{2\lambda r}{R^2}\hat{\boldsymbol{r}} = \frac{2\lambda}{R^2}\boldsymbol{r}$$

Now if we consider the same point but from the cavity's centre, assuming the cavity has a charge $-\lambda L r'^2/R^2$ the electric field is given by

$$E_{hole} \cdot 2\pi r' L = -4\pi \frac{\lambda L r'^2}{R^2}$$

$$E_{hole} = -\frac{2\lambda r'}{R^2}$$

Or in vector form

$$\boldsymbol{E}_{hole} = -\frac{2\lambda r'}{R^2}\hat{\boldsymbol{r}}' = -\frac{2\lambda}{R^2}\boldsymbol{r'}$$

Therefore the total electric field is the sum of both electric fields i.e.

$$\boldsymbol{E} = \boldsymbol{E}_{cyl} + \boldsymbol{E}_{hole}$$
$$\boldsymbol{E} = \frac{2\lambda}{R^2}\boldsymbol{r} - \frac{2\lambda}{R^2}\boldsymbol{r'}$$
$$\boldsymbol{E} = \frac{2\lambda}{R^2}\boldsymbol{a}$$

Where $\boldsymbol{a}$ is the distance from the big cylinder centre to the cavity's centre.

□

Solution. 15. Lets first compute the electric field inside the cylinder using Gauss' law

$$\begin{aligned}\int \boldsymbol{E} \cdot d\boldsymbol{S} &= 4\pi Q \\ E_{inside} \cdot 2\pi r L &= 4\pi \frac{\lambda L r^2}{R_1^2} \\ E_{inside} &= \frac{2\lambda r}{R_1^2}\end{aligned}$$

Also, the electric field in the space between the cylinder and the shell is

$$\begin{aligned}\int \boldsymbol{E} \cdot d\boldsymbol{S} &= 4\pi Q \\ E_{outside} \cdot 2\pi r L &= 4\pi \lambda L \\ E_{outside} &= \frac{2\lambda}{r}\end{aligned}$$

Since the enclosed charge is the charge of the cylinder. Now we determine the electrostatic potential using

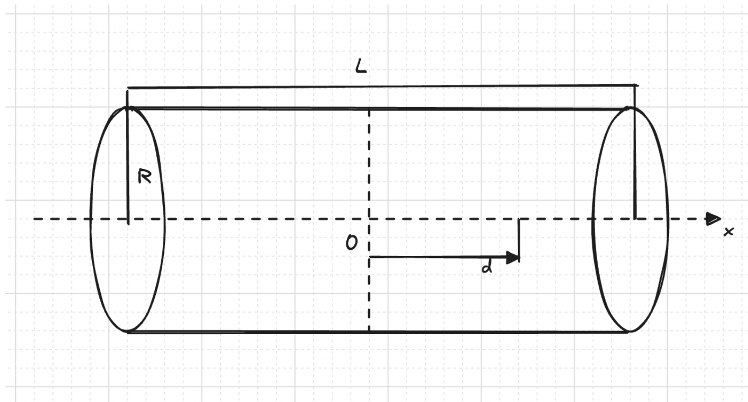
$$\Phi(\boldsymbol{x}) = -\int_{x_0}^{x} E(\boldsymbol{x}') \cdot d\boldsymbol{x}'$$

Hence

$$\begin{aligned}\Phi(r) &= -\left[\int_0^{R_1} \frac{2\lambda r}{R_1^2} dr + \int_{R_1}^{R_2} \frac{2\lambda}{r} dr\right] \\ &= -\left[\frac{2\lambda}{R_1^2}\frac{R_1^2}{2} + 2\lambda \log\left(\frac{R_2}{R_1}\right)\right] \\ &= -\lambda + 2\lambda \log\left(\frac{R_1}{R_2}\right) \\ &= \lambda\left(2\log\left(\frac{R_1}{R_2}\right) - 1\right)\end{aligned}$$

□

Solution. 19. Let us consider a point at a distance d from the center of the cylinder as shown below



Lets consider a small piece in cylindrical coordinates $r'dr'd\phi'dx'$. Since the charge per unit of volume is $Q/\pi R^2 L$ then the electric field because of this small piece at a distance d from the center is

$$dE = \frac{Qr'dr'd\phi'dx'}{\pi R^2 L}\frac{1}{(d-x')^2 + r'^2}$$

But because of the symmetry the only component we care about is the horizontal component in the x direction hence

$$\begin{aligned} dE_x &= \frac{Qr'dr'd\phi'dx'}{\pi R^2 L}\frac{1}{(d-x')^2 + r'^2}\cos\theta \\ &= \frac{Qr'dr'd\phi'dx'}{\pi R^2 L}\frac{1}{(d-x')^2 + r'^2}\frac{d-x'}{\sqrt{(d-x')^2 + r'^2}} \\ &= \frac{Qr'dr'd\phi'dx'}{\pi R^2 L}\frac{d-x'}{((d-x')^2 + r'^2)^{3/2}} \end{aligned}$$

Then by integration we have that

$$\begin{aligned} E_x &= \frac{Q}{\pi R^2 L}\int_{-L/2}^{L/2}\int_0^{2\pi}\int_0^R \frac{(d-x')r'}{((d-x')^2 + r'^2)^{3/2}} dr'd\phi'dx' \\ &= \frac{Q}{\pi R^2 L}\int_{-L/2}^{L/2}\int_0^{2\pi}\left[\frac{1}{(d-x')} - \frac{1}{\sqrt{(d-x')^2 + R^2}}\right](d-x')d\phi'dx' \\ &= \frac{2Q}{R^2 L}\int_{-L/2}^{L/2}\left[1 - \frac{(d-x')}{\sqrt{(d-x')^2 + R^2}}\right]dx' \\ &= \frac{2Q}{R^2 L}\left[L + \sqrt{(d-L/2)^2 + R^2} - \sqrt{(d+L/2)^2 + R^2}\right] \end{aligned}$$

Now we compute the electrostatic potential as follows

$$
\begin{aligned}
\Phi &= \frac{2Q}{R^2L}\int_0^d L + \sqrt{(x'-L/2)^2+R^2} - \sqrt{(x'+L/2)^2+R^2}\ dx' \\
&= \frac{2Q}{R^2L}\Bigg[\frac{1}{2}\Bigg[2Lx' + (x'-L)\sqrt{(x'-L/2)^2+R^2} - (x'+L)\sqrt{(x'+L/2)^2+R^2} \\
&\quad - R^2\text{arctanh}\left(\frac{L-x'}{\sqrt{(x'-L/2)^2+R^2}}\right) - R^2\text{arctanh}\left(\frac{L+x'}{\sqrt{(x'+L/2)^2+R^2}}\right)\Bigg]\Bigg]_0^d \\
&= \frac{2Q}{R^2L}\Bigg[\frac{1}{2}\Bigg[2Ld + (d-L)\sqrt{(d-L/2)^2+R^2} - (d+L)\sqrt{(d+L/2)^2+R^2} \\
&\quad - R^2\text{arctanh}\left(\frac{L-d}{\sqrt{(d-L/2)^2+R^2}}\right) - R^2\text{arctanh}\left(\frac{L+d}{\sqrt{(d+L/2)^2+R^2}}\right)\Bigg] \\
&- \frac{1}{2}\Bigg[-L\sqrt{L^2/4+R^2} - L\sqrt{L^2/4+R^2} - 2R^2\text{arctanh}\left(\frac{L}{\sqrt{L^2/4+R^2}}\right)\Bigg]\Bigg]
\end{aligned}
$$

□

Solution. 20.

(a) Let us consider a spherical Gaussian surface enclosing the spherically symmetric charge, then $\boldsymbol{E} = E\hat{\boldsymbol{r}}$ and hence by Gauss' law we have that

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} E\ r^2 \sin\theta d\theta d\phi &= 4\pi \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho\ r^2 \sin\theta d\theta d\phi dr \\
4\pi E r^2 &= 4\pi k \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^n\ r^2 \sin\theta d\theta d\phi dr \\
E r^2 &= 4\pi k \int_0^{r} r^{n+2} dr
\end{aligned}$$

We know that $n > -3$ then $n + 2 \geq 0$ so knowing this we can solve the integral as follows

$$\begin{aligned}
E r^2 &= 4\pi k\ \frac{r^{n+3}}{n+3} \\
E &= 4\pi k\ \frac{r^{n+1}}{n+3}
\end{aligned}$$

(b) To find the potential difference between the points $r = a$ and $r = b$ we solve the following integral

$$\Phi(r) = -\int_a^b E(r) dr$$

Since E only depends on r hence

$$\begin{aligned}
\Phi(r) &= -4\pi k \int_a^b \frac{r^{n+1}}{n+3}\ dr \\
&= -\frac{4\pi k}{n+3}\int_a^b r^{n+1}\ dr
\end{aligned}$$

Therefore

$$\Phi(r) = \begin{cases} -\frac{4\pi k}{n+3}\log\left(\frac{b}{a}\right) & \text{if } n = -2 \\ -\frac{4\pi k}{(n+3)(n+2)}\left[b^{n+2} - a^{n+2}\right] & \text{if } n \geq -1 \end{cases}$$

$\square$

Solution. 21. Let the potential in some region be

$$\Phi(x, y, z) = ax^2 + by^3$$

By the Poisson's equation we know that

$$\nabla^2\Phi(\boldsymbol{x}) = -4\pi\rho(\boldsymbol{x})$$

Then the charge density is given by

$$\begin{aligned} -4\pi\rho(x, y, z) &= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \\ \rho(x, y, z) &= -\frac{1}{4\pi}\big(2a + 6by\big) \end{aligned}$$

□

Solution. 23. From the hint we have we can compute the following

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{E_r}{E_\theta} \\
&= \frac{2pr^3\cos\theta}{pr^3\sin\theta} \\
&= \frac{2\cos\theta}{\sin\theta} \\
&= 2\cot\theta
\end{aligned}$$

Hence by integration

$$\begin{aligned}
\int dr &= 2\int \cot\theta \; d\theta \\
r &= 2\log(\sin\theta) + C
\end{aligned}$$

Which is the polar equation for the electric field lines. $\square$

Solution. 25.

(a) According to equation (61) the force that a point charge q' exerts on a dipole of moment p is

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}(\boldsymbol{x})$$

A point charge q' produces an electric field $\boldsymbol{E}(r) = q' \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$ then

$$\begin{aligned}\boldsymbol{F} &= p(\hat{\boldsymbol{z}} \cdot \nabla) q' \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \\ &= p\frac{\partial}{\partial z}\left(q' \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}}{(\rho^2+z^2)^{3/2}}\right) \\ &= pq' \left[-\frac{3z\rho}{(\rho^2+z^2)^{5/2}}\hat{\boldsymbol{\rho}} + \frac{\rho^2 - 2z^2}{(\rho^2+z^2)^{5/2}}\hat{\boldsymbol{z}}\right]\end{aligned}$$

But given that the charge is in the x-y plane we can set $z = 0$ hence

$$\boldsymbol{F} = \frac{pq'}{\rho^3}\hat{\boldsymbol{z}}$$

(b) According to Coulomb's law the force a dipole exerts on a point charge q' is

$$\boldsymbol{F} = q'\boldsymbol{E}$$

Where $\boldsymbol{E}$ is the electric field produced by the dipole, hence

$$\boldsymbol{F} = q'\left(2p\frac{\cos\theta}{\rho^3}\hat{\boldsymbol{\rho}} + p\frac{\sin\theta}{\rho^3}\hat{\boldsymbol{\theta}}\right)$$

But cosidering that the charge is in the x-y plane we have that $\theta = \pi/2$ hence

$$\boldsymbol{F} = \frac{pq'}{\rho^3}\hat{\boldsymbol{\theta}}$$

Also, in this position the spherical unit vector $\hat{\boldsymbol{\theta}}$ matches with the rectangular unit vector $-\hat{\boldsymbol{z}}$ therefore

$$\boldsymbol{F} = -\frac{pq'}{\rho^3}\hat{\boldsymbol{z}}$$

Which is in agreement with Newton's Third Law.

$\square$

Solution. 26. According to the equation (65) the torque that a point charge q' exerts on a dipole of moment p is

$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$$

Where $\boldsymbol{E}$ is the electric field generated by q'. Hence in cylindrical coordinates we have that

$$\begin{aligned}\boldsymbol{\tau} &= p\hat{\boldsymbol{z}} \times q' \frac{\rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}}{(\rho^2 + z^2)^{3/2}} \\ &= \frac{pq'\rho}{(\rho^2 + z^2)^{3/2}}\hat{\boldsymbol{\theta}}\end{aligned}$$

But if we change to spherical coordinates we get that

$$\boldsymbol{\tau} = \frac{pq' \sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

Where we used that $r^2 = (\rho^2 + z^2)$ and that $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$.

On the other hand, from problem 25 we know that the force a dipole exerts on a point charge in spherical coordinates is

$$\boldsymbol{F} = q' \left(2p\frac{\cos\theta}{r^3}\hat{\boldsymbol{r}} + p\frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \right)$$

Then the torque exerted by the dipole on the point charge is

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{r} \times \boldsymbol{F} \\ &= r\hat{\boldsymbol{r}} \times q' \left(2p\frac{\cos\theta}{r^3}\hat{\boldsymbol{r}} + p\frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \right) \\ &= \frac{pq' \sin\theta}{r^2}\hat{\boldsymbol{\phi}}\end{aligned}$$

□

Solution. 28. We know by definition that

$$E_x = -\frac{\partial \Phi}{\partial x} \qquad E_y = -\frac{\partial \Phi}{\partial y} \qquad E_z = -\frac{\partial \Phi}{\partial z}$$

Then

$$\begin{aligned}
\nabla^2 E_x &= \nabla^2 \left(-\frac{\partial \Phi}{\partial x}\right) \\
&= -\left[\frac{\partial^2}{\partial x^2}\frac{\partial \Phi}{\partial x} + \frac{\partial^2}{\partial y^2}\frac{\partial \Phi}{\partial x} + \frac{\partial^2}{\partial z^2}\frac{\partial \Phi}{\partial x}\right] \\
&= -\left[\frac{\partial}{\partial x}\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial}{\partial x}\frac{\partial^2 \Phi}{\partial z^2}\right] \\
&= -\frac{\partial}{\partial x}\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right] \\
&= -\frac{\partial}{\partial x}\nabla^2 \Phi \\
&= 0
\end{aligned}$$

Where we used that $\nabla^2 \Phi = 0$ in a charge-free region. The same can be shown for E_y and E_z. Hence the Mean-Value Theorem can be applied to E_x, E_y and E_z. The net force a spherical charge distribution experiences because of the electric field is

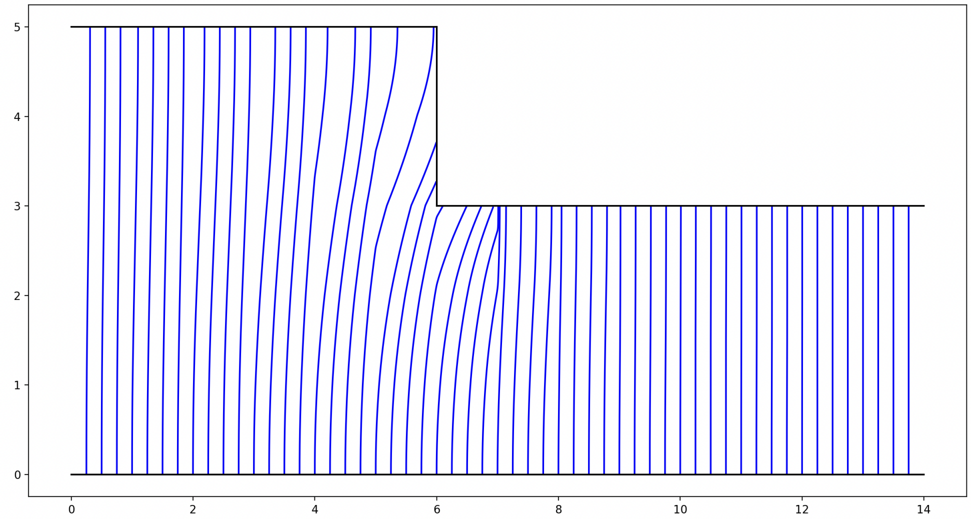
$$\boldsymbol{F} = \int_S \rho \, (\boldsymbol{i}E_x + \boldsymbol{j}E_y + \boldsymbol{k}E_z) \, dS$$

But for the center applying the Mean-Value Theorem we have that

$$\begin{aligned}
\boldsymbol{F} &= 4\pi r^2 \rho \left(\frac{\boldsymbol{i}}{4\pi r^2}\int_S E_x dS + \frac{\boldsymbol{j}}{4\pi r^2}\int_S E_y dS + \frac{\boldsymbol{k}}{4\pi r^2}\int_S E_z dS\right) \\
&= \rho \int_S (\boldsymbol{i}E_x + \boldsymbol{j}E_y + \boldsymbol{k}E_z) dS \\
&= \int_S \rho \, (\boldsymbol{i}E_x + \boldsymbol{j}E_y + \boldsymbol{k}E_z) dS
\end{aligned}$$

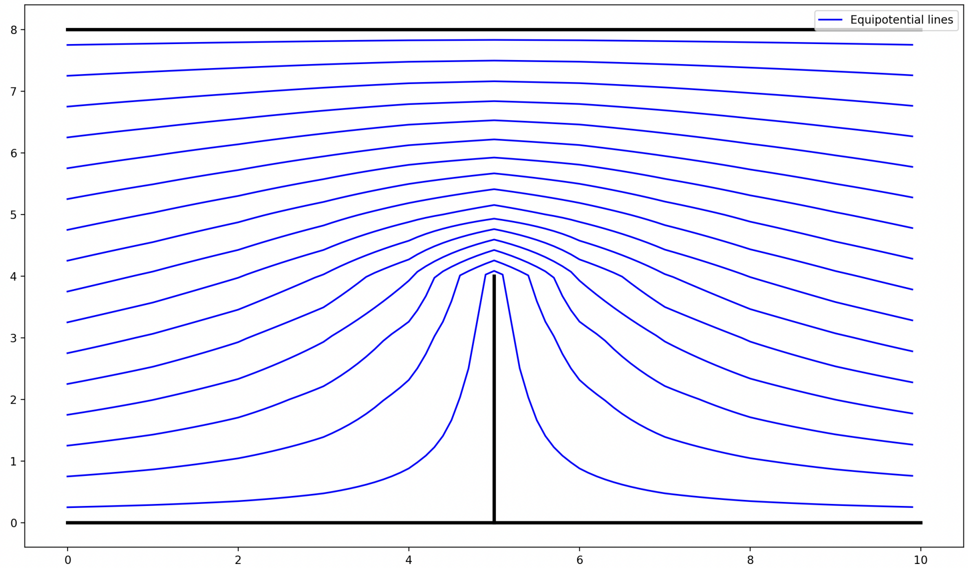
Therefore the net force experienced by the spherical distribution is as though all of the charge were concentrated at the center of the distribution. □

Solution. 29. The electric field lines look like the following

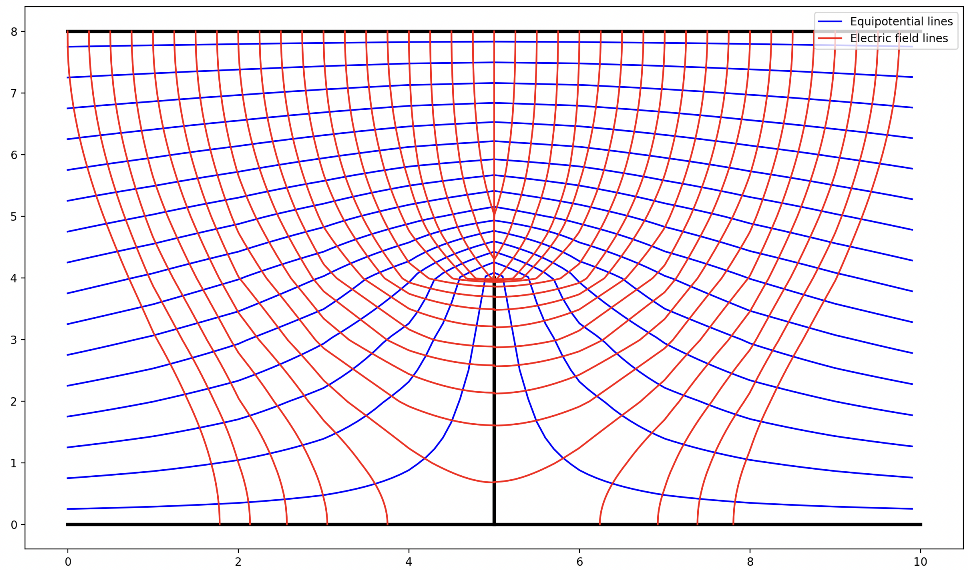


The code used to generate it is at ch2-29.py ☐

Solution. 31. The Equipotential lines look like the following

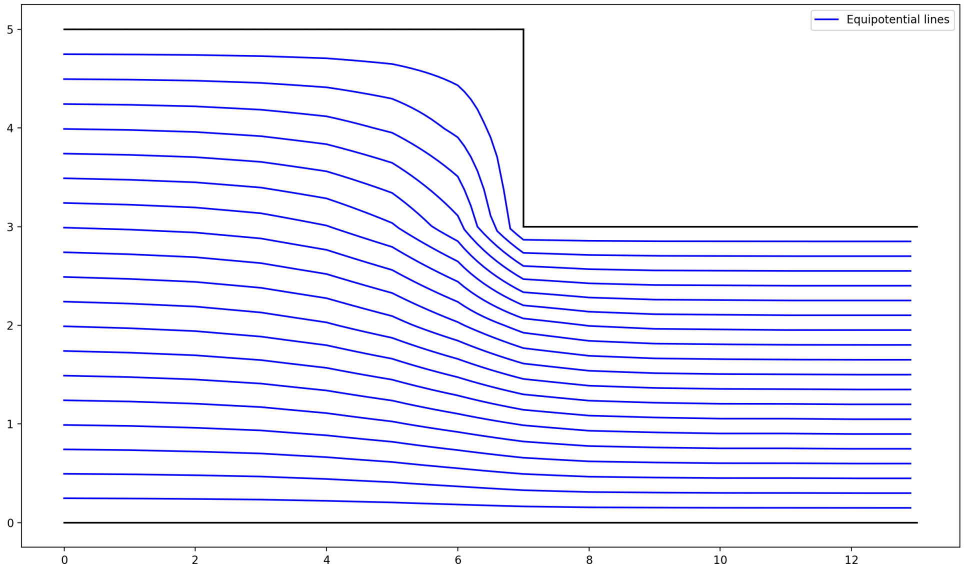


And the Electric field lines plus the Equipotential lines all together give us

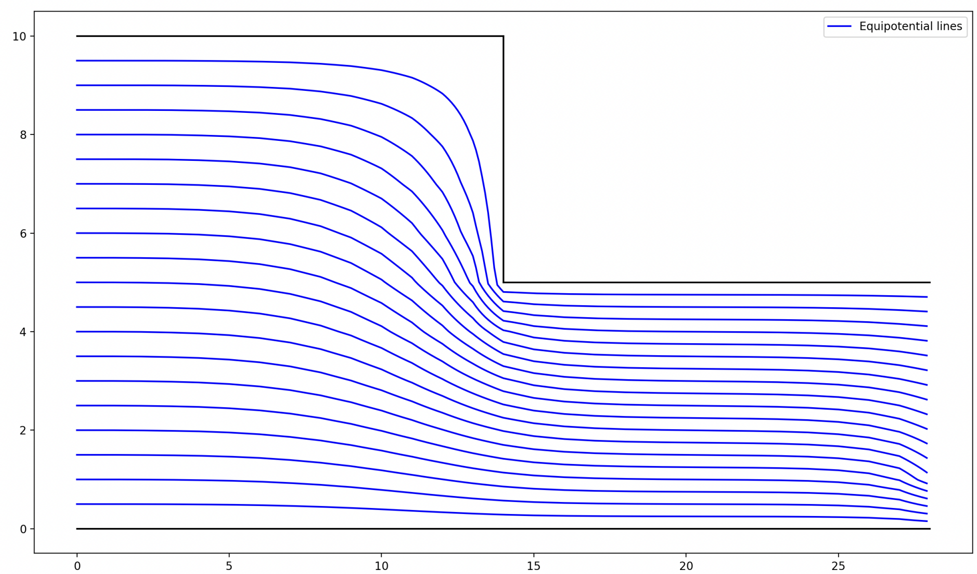


The code used to generate it is at ch2-31.py □

Solution. 32. Equipotential lines with the original grid look like the following

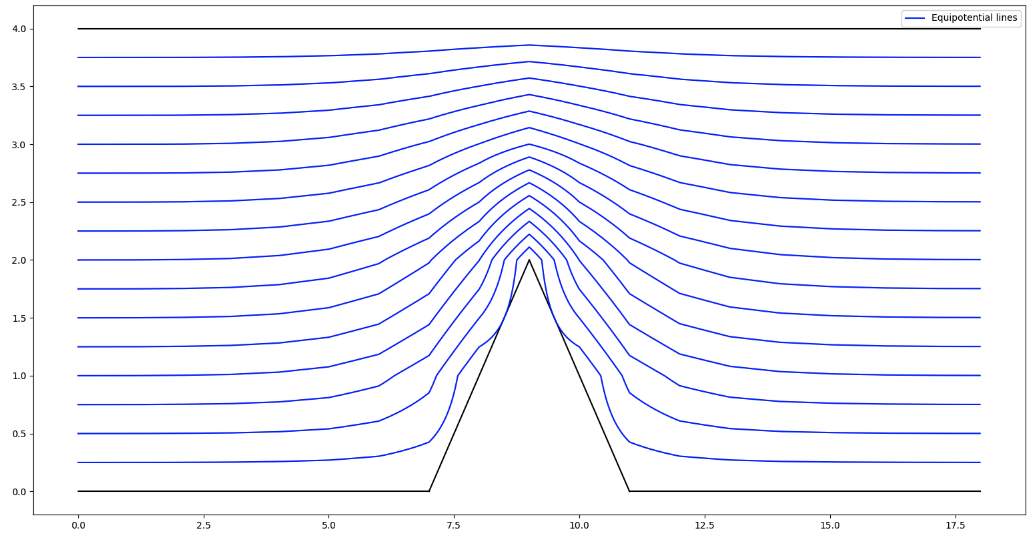


And with the finer grid look like the following



The code used to generate these plots is at ch2-32.py □

Solution. 33. The Equipotential lines look like the following ins this case



The code used to generate this plot is at ch2-33.py □