Solved selected problems of Classical Electrodynamics - Hans Ohanian

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Chapter 3 - The Boundary-Value Problem

Exercises

Solution. Exercise 1. Let's take a metal like copper then the electric field due to equation (17) is given by

$$\begin{aligned} E &= \frac{Mg}{Ze} \\ &= \frac{1.0552 \times 10^{-22}\ g \cdot 981\ cm/s^2}{29 \cdot 4.803 \times 10^{-10}\ \text{esu}} \\ &= 0.743 \times 10^{-11}\ \text{statvolt/cm} \end{aligned}$$

Where we used the following units conversion

$$\frac{statvolt}{cm} = \frac{erg}{esu \cdot cm} = \frac{dyn}{esu} = \frac{g \cdot cm/s^2}{esu}$$

Therefore the field is about 10^{-11} statvolt/cm $\square$

Solution. Exercise 2. We know that the potential for the system of two charges is

$$\Phi(\boldsymbol{x}) = \frac{q}{\sqrt{x^2 + y^2 + (z-b)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+b)^2}}$$

then if we take $z = 0$ as it is for the x-y plane we get that

$$\Phi(\boldsymbol{x}) = \frac{q}{\sqrt{x^2 + y^2 + (-b)^2}} - \frac{q}{\sqrt{x^2 + y^2 + b^2}} = 0$$

Therefore regardless of the point $\boldsymbol{x} = (x, y, 0)$ we take the potential is always the same (zero) i.e. the x-y plane is an equipotential surface. $\square$

Solution. Exercise 3. Equations (22)-(24) state that

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = \alpha^2 \quad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = \beta^2 \quad \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \gamma^2$$

Since they are of the same form we solve only the first differential equation but the solution applies to all of them.

Let us take $X(x) = e^{\pm\alpha x}$ we will show it's a solution to the first equation. We see that

$$\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2} = \alpha^2 e^{\pm\alpha x} = \alpha X(x)$$

Therefore the $X(x)$ we took satisfies the equation. The same can be shown for Y and Z. $\square$

Solution. **Exercise 4.** Let $m \neq n$ then

$$\begin{aligned}
\int_0^b e^{2\pi i(n-m)y/b}\, dy &= \left[-\frac{ibe^{2\pi i(n-m)y/b}}{2\pi(n-m)}\right]_0^b \\
&= \left[-\frac{ibe^{2\pi i(n-m)}}{2\pi(n-m)} + \frac{ibe^0}{2\pi(n-m)}\right] \\
&= \frac{ib}{2\pi(n-m)}\left[1 - e^{2\pi i(n-m)}\right] \\
&= \frac{ib}{2\pi(n-m)}\left[1 - (\cos(2\pi(n-m)) + i\sin(2\pi(n-m)))\right] \\
&= \frac{ib}{2\pi(n-m)}\left[1 - (1+0)\right] \\
&= 0
\end{aligned}$$

Where we used that $\sin(2\pi(n-m)) = 0$ and $\cos(2\pi(n-m)) = 1$ no matter the values of n or m.

If $n = m$ then we have that

$$\int_0^b e^{2\pi i(n-m)y/b}\, dy = \int_0^b e^0\, dy = \int_0^b\, dy = [y]_0^b = b$$

Therefore

$$\int_0^b e^{2\pi i(n-m)y/b}\, dy = \begin{cases} 0 & \text{if } n \neq m \\ b & \text{if } n = m \end{cases}$$

$\square$

Solution. Exercise 5. Equation (36) states that

$$A_m = \frac{1}{b}\int_0^b \Phi(y,0)e^{-2\pi imy/b}\ dy$$

Where $\Phi(y,0)$ is

$$\Phi(y,0) = \begin{cases} V_0 & \text{if } \ 0 \le y \le b/2 \\ -V_0 & \text{if } \ b/2 < y \le b \end{cases}$$

Then the integral becomes

$$\begin{aligned}
A_m &= \frac{V_0}{b}\left[\int_0^{b/2} e^{-2\pi imy/b}\ dy - \int_{b/2}^{b} e^{-2\pi imy/b}\ dy\right] \\
&= \frac{V_0}{b}\left[-\frac{be^{-2\pi imy/b}}{2\pi im}\right]_0^{b/2} - \left[-\frac{be^{-2\pi imy/b}}{2\pi im}\right]_{b/2}^{b} \\
&= -\frac{V_0}{2\pi im}\left[e^{-2\pi imy/b}\right]_0^{b/2} - \left[e^{-2\pi imy/b}\right]_{b/2}^{b} \\
&= -\frac{V_0}{2\pi im}\left[e^{-\pi im} - 1\right] - \left[e^{-2\pi im} - e^{-\pi im}\right] \\
&= -\frac{V_0}{2\pi im}\left[2e^{-\pi im} - 1 - e^{-2\pi im}\right] \\
&= -\frac{V_0}{2\pi im}\left[2e^{-\pi im} - 2\right] \\
&= -\frac{V_0}{\pi im}\left[e^{-\pi im} - 1\right]
\end{aligned}$$

$\square$

Solution. Exercise 6. Let us consider the i-th term of equation (38)

$$-\frac{V_0}{\pi i n}(e^{-\pi i n}-1)e^{2\pi i n y/b}\frac{e^{-2\pi n h/b}e^{2\pi n z/b}-e^{2\pi n h/b}e^{-2\pi n z/b}}{e^{-2\pi n h/b}-e^{2\pi n h/b}}$$

We see that

$$\frac{e^{-2\pi n h/b}e^{2\pi n z/b}-e^{2\pi n h/b}e^{-2\pi n z/b}}{e^{-2\pi n h/b}-e^{2\pi n h/b}}$$

is real so we want to prove the first part is real. Also, we see that $e^{-\pi i n}-1=0$ for even n and $e^{\pm\pi i n}-1=-2$ for odd n. So let us consider the sum

$$\begin{aligned}
&\sum_{n=-\infty}^{n=\infty}-\frac{V_0}{\pi i n}(e^{-\pi i n}-1)e^{2\pi i n y/b}\\
&=\sum_{\substack{n=1\\ n\text{ is odd}}}^{\infty}-\frac{V_0}{\pi i n}(e^{-\pi i n}-1)e^{2\pi i n y/b}+\frac{V_0}{\pi i n}(e^{\pi i n}-1)e^{-2\pi i n y/b}\\
&=\sum_{\substack{n=1\\ n\text{ is odd}}}^{\infty}\frac{V_0}{\pi i n}2e^{2\pi i n y/b}-\frac{V_0}{\pi i n}2e^{-2\pi i n y/b}\\
&=\sum_{\substack{n=1\\ n\text{ is odd}}}^{\infty}\frac{2V_0}{\pi i n}(e^{2\pi i n y/b}-e^{-2\pi i n y/b})\\
&=\sum_{\substack{n=1\\ n\text{ is odd}}}^{\infty}\frac{4V_0}{\pi n}\sin(2\pi n y/b)
\end{aligned}$$

Where we used that $\sin(x)=(e^{ix}-e^{-ix})/2i$. Therefore the equation (38) is a real function.

On the other hand, we can write that

$$\begin{aligned}
\frac{e^{-2\pi n h/b}e^{2\pi n z/b}-e^{2\pi n h/b}e^{-2\pi n z/b}}{e^{-2\pi n h/b}-e^{2\pi n h/b}}&=\\
&=\frac{e^{2\pi n(z-h)/b}-e^{-2\pi n(z-h)/b}}{e^{-2\pi n h/b}-e^{2\pi n h/b}}\\
&=\frac{\frac{e^{2\pi n(z-h)/b}-e^{-2\pi n(z-h)/b}}{2}i}{-\frac{e^{2\pi n h/b}-e^{-2\pi n h/b}}{2}i}\\
&=-\frac{\sin(i2\pi n(z-h)/b)}{\sin(i2\pi n h/b)}\\
&=-\frac{\sinh(2\pi n(z-h)/b)}{\sinh(2\pi n h/b)}
\end{aligned}$$

Therefore we can write equation (38) in terms of sines as follows

$$\Phi(y,z)=\sum_{\substack{n=1\\ n\text{ is odd}}}^{\infty}-\frac{V_0}{\pi n}\sin(2\pi n y/b)\frac{\sinh(2\pi n(z-h)/b)}{\sinh(2\pi n h/b)}$$

$\square$

Solution. Exercise 7. Let us multiply both numerator and denominator of the expression

$$\frac{e^{-2\pi nh/b}e^{2\pi nz/b} - e^{2\pi nh/b}e^{-2\pi nz/b}}{e^{-2\pi nh/b} - e^{2\pi nh/b}}$$

By $e^{-2\pi nh/b}$ then

$$\frac{e^{-2\pi nh/b}e^{2\pi nz/b} - e^{2\pi nh/b}e^{-2\pi nz/b}}{e^{-2\pi nh/b} - e^{2\pi nh/b}} = \frac{e^{-4\pi nh/b}e^{2\pi nz/b} - e^{-2\pi nz/b}}{e^{-4\pi nh/b} - 1}$$

So applying the limit as $h/b \to \infty$ gives us

$$\lim_{h/b\to\infty} \frac{e^{-4\pi nh/b}e^{2\pi nz/b} - e^{-2\pi nz/b}}{e^{-4\pi nh/b} - 1} = \frac{0 - e^{-2\pi nz/b}}{0 - 1} = e^{-2\pi nz/b}$$

Therefore the potential as $h/b \to \infty$ becomes

$$\begin{aligned}\lim_{h/b\to\infty} \Phi(y,z) &= \lim_{h/b\to\infty} \sum_{n=-\infty}^{n=\infty} -\frac{V_0}{\pi in}(e^{-\pi in} - 1)e^{2\pi iny/b}\frac{e^{-2\pi nh/b}e^{2\pi nz/b} - e^{2\pi nh/b}e^{-2\pi nz/b}}{e^{-2\pi nh/b} - e^{2\pi nh/b}} \\ &= \sum_{n=-\infty}^{n=\infty} -\frac{V_0}{\pi in}(e^{-\pi in} - 1)e^{2\pi iny/b}e^{-2\pi nz/b}\end{aligned}$$

□

Solution. Exercise 8. Let

$$\Phi(y,z) = \frac{4V_0}{\pi}\sin\frac{2\pi y}{b}e^{-2\pi z/b}$$

Then we compute E_y and E_z as follows

$$\begin{aligned}E_y &= -\frac{\partial\Phi(y,z)}{\partial y} = -\frac{8V_0}{b}\cos\frac{2\pi y}{b}e^{-2\pi z/b} \\ E_z &= -\frac{\partial\Phi(y,z)}{\partial z} = \frac{8V_0}{b}\sin\frac{2\pi y}{b}e^{-2\pi z/b}\end{aligned}$$

Therefore both components fall off exponentially with z. □

Solution. Exercise 9. Let $\Phi(\rho,\phi) = R(\rho)Q(\phi)$ then Laplace's equation becomes

$$\begin{aligned} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)Q(\phi)}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 R(\rho)Q(\phi)}{\partial\phi^2} &= 0 \\ Q(\phi)\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{R(\rho)}{\rho}\frac{\partial^2 Q(\phi)}{\partial\phi^2} &= 0 \\ \frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} &= 0 \end{aligned}$$

Each of these terms must be equal to a constant since the sum must remain equal to 0, then we have that

$$\begin{aligned} \frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) &= k^2 \\ \frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} &= m^2 \end{aligned}$$

But since $k^2 + m^2 = 0$ then must be that $m^2 = -k^2$ hence

$$\begin{aligned} \frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) &= k^2 \\ \frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} &= -k^2 \end{aligned}$$

□

Solution. Exercise 10. Let $k \neq 0$ then we can write equation (45) as

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} = -k^2 Q$$

This is the equation of a Simple Harmonic Oscillator for which we know the solution is

$$Q(\phi) = \cos(k\phi) \quad \text{or} \quad \sin(k\phi)$$

If we let $k = 0$ then equation (45) becomes

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} = 0$$

Hence the first derivative of Q must be a constant i.e.

$$\frac{\mathrm{d}Q}{\mathrm{d}\phi} = a_1$$

But for this to happen Q must be a linear function, then

$$Q = a_0 + a_1\phi$$

□

Solution. Exercise 11. Let $n \neq 0$ and let us define $s = \log \rho$. Also, note that for any function $f(\rho)$ we have that

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}\rho} &= \frac{\mathrm{d}f}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\rho} \\
\frac{\mathrm{d}f}{\mathrm{d}\rho} &= \frac{\mathrm{d}f}{\mathrm{d}s}\frac{1}{\rho} \\
\rho\frac{\mathrm{d}f}{\mathrm{d}\rho} &= \frac{\mathrm{d}f}{\mathrm{d}s}
\end{aligned}
$$

Then equation (48) becomes

$$
\begin{aligned}
\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) &= n^2 R \\
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) &= n^2 R \\
\frac{\mathrm{d}^2 R}{\mathrm{d}s^2} &= n^2 R
\end{aligned}
$$

The solution to this differential equation is of the form $R(s) = e^{\pm ns}$ but replacing s again we get that

$$
R(\rho) = e^{\pm n \log \rho} = (e^{\log \rho})^{\pm n} = \rho^{\pm n}
$$

Now, if $n = 0$ and we replace again $s = \log \rho$, equation (48) becomes

$$
\begin{aligned}
\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) &= 0 \\
\frac{\mathrm{d}^2 R}{\mathrm{d}s^2} &= 0
\end{aligned}
$$

for which we know the solution is $b_0 + b_1 s$ where b_0 and b_1 are constants, then replacing again s we get that

$$
R(\rho) = b_0 + b_1 \log \rho
$$

$\square$

Solution. Exercise 12. Let us compute the electric field components as follows

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = E_0 \cos\phi + E_0 \frac{R^2}{\rho^2} \cos\phi$$

And

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\frac{1}{\rho}\left(E_0 \rho \sin\phi - E_0 \frac{R^2}{\rho} \sin\phi\right) = -E_0 \sin\phi + E_0 \frac{R^2}{\rho^2} \sin\phi$$

□

Solution. Exercise 13. Expression (64) states that

$$J_n(\xi) = \left(\frac{\xi}{2}\right)^n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n}$$

And equation (63) states that

$$\begin{aligned}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}R}{\mathrm{d}\xi}\right) + \left(1-\frac{n^2}{\xi^2}\right)R = 0\\ \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}R}{\mathrm{d}\xi}\right) + R = \frac{n^2}{\xi^2}R\end{aligned}$$

Let us replace expression (64) into the LHS as follows

$$\begin{aligned}
&\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}J_n}{\mathrm{d}\xi}\right) + J_n = \\
&\quad= \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n}\right)\right) + \sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&\quad= \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_{j=0}^{\infty}\frac{(-1)^j(2j+n)}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n}\right) + \sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&\quad= \frac{1}{\xi}\sum_{j=0}^{\infty}\frac{(-1)^j(2j+n)^2}{j!(j+n)!2^{2j+n}}\xi^{2j+n-1} + \sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&\quad= \sum_{j=0}^{\infty}\frac{(-1)^j(2j+n)^2}{j!(j+n)!2^{2j+n}}\xi^{2j+n-2} + \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(j-1)!(j+n-1)!2^{2j+n-2}}\xi^{2j+n-2} \\
&\quad= \frac{n^2}{n!\xi^2}\frac{\xi^n}{2^n} + \sum_{j=1}^{\infty}\frac{(-1)^j(2j+n)^2}{j!(j+n)!2^{2j+n}}\xi^{2j+n-2} - \sum_{j=1}^{\infty}\frac{(-1)^j}{(j-1)!(j+n-1)!2^{2j+n-2}}\xi^{2j+n-2} \\
&\quad= \frac{n^2}{n!\xi^2}\frac{\xi^n}{2^n} + \sum_{j=1}^{\infty}\left(\frac{(2j+n)^2}{\xi^2} - \frac{4j!(j+n)!}{(j-1)!(j+n-1)!\xi^2}\right)\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&\quad= \frac{n^2}{n!\xi^2}\frac{\xi^n}{2^n} + \sum_{j=1}^{\infty}\left(\frac{(2j+n)^2}{\xi^2} - \frac{4j(j+n)}{\xi^2}\right)\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&\quad= \frac{n^2}{n!\xi^2}\frac{\xi^n}{2^n} + \sum_{j=1}^{\infty}\frac{n^2}{\xi^2}\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&\quad= \sum_{j=0}^{\infty}\frac{n^2}{\xi^2}\frac{(-1)^j}{j!(j+n)!}\left(\frac{\xi}{2}\right)^{2j+n} \\
&= \frac{n^2}{\xi^2}J_n
\end{aligned}$$

Therefore equation (64) satisfies equation (63). $\square$

Solution. **Exercise 14.** Let $\mu = \cos\theta$ and let us note that

$$\begin{aligned} \frac{\partial f}{\partial \mu}\frac{\partial \mu}{\partial \theta} &= \frac{\partial f}{\partial \theta} \\ -\frac{\partial f}{\partial \mu}\sin\theta &= \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \mu} &= -\frac{1}{\sin\theta}\frac{\partial f}{\partial \theta} \end{aligned}$$

Also, we have that $\sin^2\theta = 1 - \cos^2\theta = 1 - \mu^2$.
Then replacing in equation (67) we get that

$$\begin{aligned} \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} &= 0 \\ \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) - \frac{1}{r^2}\frac{\partial}{\partial\mu}\left(-\sin^2\theta\frac{\partial\Phi}{\partial\mu}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} &= 0 \\ \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right) + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2\Phi}{\partial\phi^2} &= 0 \end{aligned}$$

□

Solution. **Exercise 15.** Let us define $\Phi(r,\mu) = F(r)P(\mu)$ then Laplace's equation becomes

$$\begin{aligned} \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right] &= 0 \\ \frac{P}{r}\frac{\partial^2}{\partial r^2}(rF) + \frac{F}{r^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial P}{\partial\mu}\right] &= 0 \\ \frac{r}{F}\frac{\partial^2}{\partial r^2}(rF) + \frac{1}{P}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial P}{\partial\mu}\right] &= 0 \end{aligned}$$

Where in the last step we multiplied the equation by r^2/FP. Then we see that each term must be equal to a constant since the first term only depends on r and the second term only depends on μ.
Setting the constant to $l(l+1)$ for some l then must be that

$$\frac{r}{F}\frac{\partial^2}{\partial r^2}(rF) = l(l+1) \quad \text{and} \quad \frac{1}{P}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial P}{\partial\mu}\right] = -l(l+1)$$

□

Solution. Exercise 16. Let $F(r) = r^l, r^{-l-1}$ then equation (72) for the first case gives us

$$
\begin{aligned}
\frac{r}{F}\frac{\partial^2}{\partial r^2}(rF) &= \frac{r}{r^l}\frac{\partial^2}{\partial r^2}\left(r^{l+1}\right) \\
&= \frac{(l+1)}{r^{l-1}}\frac{\partial}{\partial r}\left(r^l\right) \\
&= \frac{l(l+1)}{r^{l-1}}r^{l-1} \\
&= l(l+1)
\end{aligned}
$$

And for the second case we get that

$$
\begin{aligned}
\frac{r}{F}\frac{\partial^2}{\partial r^2}(rF) &= \frac{r}{r^{-l-1}}\frac{\partial^2}{\partial r^2}\left(r^{-l}\right) \\
&= \frac{-l}{r^{-l-2}}\frac{\partial}{\partial r}\left(r^{-l-1}\right) \\
&= \frac{-l(-l-1)}{r^{-l-2}}r^{-l-2} \\
&= l(l+1)
\end{aligned}
$$

Therefore both r^l and r^{-l-1} are solutions to the equation (72). $\square$

Solution. Exercise 17. Let us apply the product rule to $[(\mu^2-1)u']^{(4)}$ as follows

$$\begin{aligned}
[(\mu^2-1)u']^{(4)} &= [2\mu u' + (\mu^2-1)u'']''' \\
&= [2u' + 2\mu u'' + 2\mu u'' + (\mu^2-1)u''']'' \\
&= [2u'' + 2u'' + 2\mu u''' + 2u'' + 2\mu u''' + 2\mu u''' + (\mu^2-1)u'''']' \\
&= [6u'' + 6\mu u''' + (\mu^2-1)u^{(4)}]' \\
&= 6u''' + 6u''' + 6\mu u^{(4)} + 2\mu u^{(4)} + (\mu^2-1)u^{(5)} \\
&= 12u''' + 8\mu u^{(4)} + (\mu^2-1)u^{(5)}
\end{aligned}$$

Then we see that if we apply it l-times to $[(\mu^2-1)u']^{(l)}$ we get that

$$\begin{aligned}
[(\mu^2-1)u']^{(l)} &= [2\mu u' + (\mu^2-1)u'']^{(l-1)} \\
&= [2u' + 2\mu u'' + 2\mu u'' + (\mu^2-1)u''']^{(l-2)} \\
&= [2u' + 4\mu u'' + (\mu^2-1)u''']^{(l-2)} \\
&= [2u'' + 4u'' + 4\mu u''' + 2\mu u''' + (\mu^2-1)u^{(4)}]^{(l-3)} \\
&= [6u'' + 6\mu u''' + (\mu^2-1)u^{(4)}]^{(l-3)} \\
&\dots \\
&= l(l-1)u^{(l-1)} + 2l\mu u^{(l)} + (\mu^2-1)u^{(l+1)}
\end{aligned}$$

In the same way, let us compute the l-derivative of $2l\mu u$ as follows

$$\begin{aligned}
[2l\mu u]^{(l)} &= [2lu + 2l\mu u']^{(l-1)} \\
&= [2lu' + 2lu' + 2l\mu u'']^{(l-2)} \\
&= [4lu' + 2l\mu u'']^{(l-2)} \\
&= [4lu'' + 2lu'' + 2l\mu u''']^{(l-3)} \\
&= [6lu'' + 2l\mu u''']^{(l-3)} \\
&\dots \\
&= 2l^2u^{(l-1)} + 2l\mu u^{(l)}
\end{aligned}$$

$\square$

Solution. Exercise 18. Let $v(\mu) = (\mu + 1)^l$ and $w(\mu) = (\mu - 1)^l$ then using the general Leibniz rule we have that

$$\frac{\mathrm{d}^l}{\mathrm{d}\mu^l} v(\mu)w(\mu) = \frac{\mathrm{d}^l}{\mathrm{d}\mu^l}(\mu+1)^l(\mu-1)^l = \sum_{k=0}^{l} \binom{l}{k} v(\mu)^{(l-k)} w(\mu)^{(k)}$$

We see that the only non-zero derivative of $w(\mu)$ valued at $\mu = 1$ is when $k = l$ so above equation for $\mu = 1$ becomes

$$\sum_{k=0}^{l} \binom{l}{k} v(\mu)^{(l-k)} w(\mu)^{(k)} \bigg|_{\mu=1} = 2^l l!$$

Where we used that $\binom{l}{l} = 1$ and that $w(\mu)^{(l)}|_{\mu=1} = l!$.
Therefore

$$P_l(1) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}\mu^l}(\mu+1)^l(\mu-1)^l \bigg|_{\mu=1} = 1$$

In the same way, the only non-zero derivative of $v(\mu)$ valued at $\mu = -1$ is when $k = 0$ so the equation for $\mu = -1$ becomes

$$\sum_{k=0}^{l} \binom{l}{k} v(\mu)^{(l-k)} w(\mu)^{(k)} \bigg|_{\mu=-1} = l!(-2)^l$$

Therefore

$$P_l(-1) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}\mu^l}(\mu+1)^l(\mu-1)^l \bigg|_{\mu=-1} = (-1)^l$$

$\square$

Solution. **Exercise 19.** We want to prove by induction on l that

$$P_l(-\mu) = (-1)^l P_l(\mu)$$

Then for $l = 1$ we see that

$$P_1(-\mu) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\nu}(\nu+1)(\nu-1)\bigg|_{\nu=-\mu} = -\mu$$

And that

$$P_1(\mu) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\nu}(\nu+1)(\nu-1)\bigg|_{\nu=\mu} = \mu$$

Then $P_1(-\mu) = -\mu = (-1)^1 P(\mu)$ so the base case holds.
Now, for the induction step, suppose the following is true

$$P_l(-\mu) = (-1)^l P_l(\mu)$$

We know that the recurrence relation for Legendre polynomials is

$$P_{l+1}(\mu) = \frac{(2l+1)\mu P_l(\mu) - lP_{l-1}(\mu)}{l+1}$$

Then replacing $-\mu$ we get that

$$\begin{aligned} P_{l+1}(-\mu) &= \frac{(2l+1)\mu P_l(-\mu) - lP_{l-1}(-\mu)}{l+1} \\ &= \frac{-(2l+1)\mu(-1)^l P_l(\mu) - l(-1)^{l-1}P_{l-1}(\mu)}{l+1} \\ &= \frac{(-1)^l}{l+1}\Big[-(2l+1)\mu P_l(\mu) - l(-1)^{-1}P_{l-1}(\mu)\Big] \\ &= (-1)^{l+1}\frac{(2l+1)\mu P_l(\mu) - lP_{l-1}(\mu)}{l+1} \\ &= (-1)^{l+1}P_{l+1}(\mu) \end{aligned}$$

Therefore, the induction holds as well, and we have proven by induction that

$$P_l(-\mu) = (-1)^l P_l(\mu)$$

□

Solution. Exercise 20. Let us compute $P_0(\mu)$, $P_1(\mu)$, $P_2(\mu)$, and $P_3(\mu)$ as follows

$$
\begin{aligned}
P_0(\mu) &= \frac{1}{2^0 \cdot 0!}\frac{\mathrm{d}^0}{\mathrm{d}\mu^0}\left(\mu^2-1\right)^0 = \frac{1}{1\cdot 1}\cdot 1 = 1\\
P_1(\mu) &= \frac{1}{2^1 \cdot 1!}\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\mu^2-1\right)^1 = \frac{1}{2}\cdot 2\mu = \mu\\
P_2(\mu) &= \frac{1}{2^2 \cdot 2!}\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\left(\mu^2-1\right)^2 = \frac{1}{8}\cdot 12\mu^2 - 4 = \frac{3\mu^2-1}{2}\\
P_3(\mu) &= \frac{1}{2^3 \cdot 3!}\frac{\mathrm{d}^3}{\mathrm{d}\mu^3}\left(\mu^2-1\right)^3 = \frac{1}{48}\cdot 120\mu^3 - 72\mu = \frac{24(5\mu^3-3\mu)}{48} = \frac{5\mu^3-3\mu}{2}
\end{aligned}
$$

□

Solution. Exercise 21. Let $l > n$ then

$$
\begin{aligned}
\int_{-1}^{1} P_l(\mu)P_n(\mu)\ d\mu &= \frac{1}{2^l l!}\frac{1}{2^n n!}\int_{-1}^{1}\frac{\mathrm{d}^l}{\mathrm{d}\mu^l}\left(\mu^2-1\right)^l\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}\left(\mu^2-1\right)^n\ d\mu\\
&= \frac{(-1)^l}{2^{l+n}l!n!}\int_{-1}^{1}(\mu^2-1)^l\frac{\mathrm{d}^{l+n}}{\mathrm{d}\mu^{l+n}}\left(\mu^2-1\right)^n\ d\mu\\
&= 0
\end{aligned}
$$

Where we used that this expression is zero because, for $l > n$, the $(l+n)$th derivative of a polynomial of order $2n$ vanishes.
In the same way, if $n > l$ we have that

$$
\begin{aligned}
\int_{-1}^{1} P_n(\mu)P_l(\mu)\ d\mu &= \frac{1}{2^n n!}\frac{1}{2^l l!}\int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}\left(\mu^2-1\right)^n\frac{\mathrm{d}^l}{\mathrm{d}\mu^l}\left(\mu^2-1\right)^l\ d\mu\\
&= \frac{(-1)^n}{2^{l+n}l!n!}\int_{-1}^{1}(\mu^2-1)^n\frac{\mathrm{d}^{l+n}}{\mathrm{d}\mu^{l+n}}\left(\mu^2-1\right)^l\ d\mu\\
&= 0
\end{aligned}
$$

Where we integrated by parts repeated n times and we used that the $(l+n)$th derivative of a polynomial of order $2l$ vanishes.
Therefore

$$
\int_{-1}^{1} P_l(\mu)P_n(\mu)\ d\mu = 0 \quad \text{for} \quad l \neq n
$$

□

Solution. Exercise 22. We want to prove that

$$\int_{-1}^{1} P_l(\mu)P_l(\mu)\ d\mu = \frac{2}{2l+1}$$

Applying integration by parts l times we get that

$$\begin{aligned}\int_{-1}^{1} P_l(\mu)P_l(\mu)\ d\mu &= \frac{1}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{\mathrm{d}^l}{\mathrm{d}\mu^l}\left(\mu^2-1\right)^l\frac{\mathrm{d}^l}{\mathrm{d}\mu^l}\left(\mu^2-1\right)^l\ d\mu\\ &= \frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^{1}(\mu^2-1)^l\frac{\mathrm{d}^{2l}}{\mathrm{d}\mu^{2l}}\left(\mu^2-1\right)^l\ d\mu\\ &= \frac{(-1)^l(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(\mu^2-1)^l\ d\mu\end{aligned}$$

Where we used that the $2l$ derivative of a polynomial of grade $2l$ is $(2l)!$. Let us now define $\mu=\cos\theta$ then $d\mu=-\sin\theta d\theta$ hence

$$\begin{aligned}\int_{-1}^{1} P_l(\mu)P_l(\mu)\ d\mu &= \frac{(-1)^l(2l)!}{2^{2l}(l!)^2}\int_{\pi}^{0}-(\cos^2\theta-1)^l\sin\theta\ d\theta\\ &= \frac{(-1)^l(2l)!}{2^{2l}(l!)^2}\int_{\pi}^{0}-(-1)^l(1-\cos^2\theta)^l\sin\theta\ d\theta\\ &= \frac{(2l)!}{2^{2l}(l!)^2}\int_{0}^{\pi}\sin^{2l}\theta\sin\theta\ d\theta\\ &= \frac{(2l)!}{2^{2l}(l!)^2}\int_{0}^{\pi}\sin^{2l+1}\theta\ d\theta\end{aligned}$$

Now, let $u=\sin^{2l}\theta$ and $v'=\sin\theta$ then integrating by parts we get that

$$\begin{aligned}I &= \int_0^{\pi}\sin^{2l+1}\theta\ d\theta\\ &= \Big[-\sin^{2l}\theta\cos\theta\Big]_0^{\pi}+\int_0^{\pi}2l\cos^2\theta\sin^{2l-1}\theta\ d\theta\\ &= 2l\int_0^{\pi}\cos^2\theta\sin^{2l-1}\theta\ d\theta\\ &= 2l\int_0^{\pi}(1-\sin^2\theta)\sin^{2l-1}\theta\ d\theta\\ &= 2l\Big[\int_0^{\pi}\sin^{2l-1}\theta\ d\theta-\int_0^{\pi}\sin^{2l+1}\theta\ d\theta\Big]\\ &= 2l\Big[\int_0^{\pi}\sin^{2l-1}\theta\ d\theta-I\Big]\end{aligned}$$

So

$$I(2l+1) = 2l\int_0^{\pi}\sin^{2l-1}\theta\ d\theta$$

We get then the following recursive equation

$$I_{2l+1} = \frac{2l}{2l+1} I_{2l-1}$$

Then if we apply it l times we get that

$$I_{2l+1} = \frac{2l}{2l+1} \frac{2(l-1)}{2l-1} \frac{2(l-2)}{2l-3} \cdots \frac{2}{3} I_1 = \frac{2^l l!}{(2l+1)\frac{(2l)!}{2^l l!}} I_1 = \frac{2 \cdot 2^{2l} (l!)^2}{(2l+1)(2l)!}$$

Where we used that $I_1 = \int_0^\pi \sin\theta \; d\theta = 2$. Joining the results we get that

$$\int_{-1}^{1} P_l(\mu) P_l(\mu) \; d\mu = \frac{(2l)!}{2^{2l}(l!)^2} \frac{2 \cdot 2^{2l}(l!)^2}{(2l+1)(2l)!} = \frac{2}{2l+1}$$

$\square$

Solution. Exercise 23. We want to compute

$$\frac{1}{2^m m!}\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}\mu^{m-1}}\left(\mu^2-1\right)^m\right]_{-1}^{0}$$

First, let us note that

$$(\mu^2-1)^m = \sum_{k=0}^{m}\frac{m!}{(m-k)!k!}(-1)^k\mu^{2(m-k)}$$

After we derivate $m-1$ times the only terms that survive are the terms above $m-1$ but, when we replace $\mu=0$ the term that survives is the term that involves μ^{m-1} before the derivation, and upon derivations leaves us the following coefficient

$$\frac{m!}{(m-k)!k!}(-1)^k(m-1)!$$

Where must be that $2(m-k)=m-1$ then we get that $k=m/2+1/2$, so replacing we have that

$$\begin{aligned}
\frac{1}{2^m m!}\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}\mu^{m-1}}\left(\mu^2-1\right)^m\right]_{-1}^{0} &= \frac{1}{2^m m!}\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}\mu^{m-1}}\left(\mu^2-1\right)^m\bigg|_{\mu=0}-0\right] \\
&= \frac{1}{2^m m!}\cdot\frac{(-1)^{m/2+1/2}(m-1)!m!}{(m/2-1/2)!(m/2+1/2)!} \\
&= \frac{1}{2^m}\cdot\frac{(-1)^{m/2+1/2}(m-1)!}{(m/2-1/2)!(m/2+1/2)!}\cdot\frac{m(m+1)}{m(m+1)} \\
&= \frac{1}{2^m}\frac{(-1)^{m/2+1/2}}{(m/2-1/2)!(m/2+1/2)!}\frac{(m+1)!}{m\cdot 2(m/2+1/2)} \\
&= \frac{1}{2^{m+1}}\frac{(-1)^{m/2+1/2}}{[(m/2+1/2)!]^2}\frac{(m+1)!}{m}
\end{aligned}$$

Where we multiplied numerator and denominator by $m(m+1)$ and we used that $(m/2-1/2)!\cdot(m/2+1/2)=(m/2+1/2)!$ □

Solution. **Exercise 24.** Equation (107) states that

$$\frac{q}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$$

Let $r < r'$, when $r = 0$ the LHS becomes q/r', which doesn't blow up. Then B_l in this case must be 0, otherwise the RHS blow up at $r = 0$. Let us consider the special case $\cos\theta = 1$, since $P_l(1) = 1$ we get that

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr'}} = \frac{1}{|r - r'|} = \frac{1}{r' - r} = \sum_{l=0}^{\infty} A_l r^l$$

On the other hand, the series expansion for $1/r' - r$ when $r < r'$ gives us

$$\frac{1}{r' - r} = \frac{1}{r'}\frac{1}{1 - \frac{r}{r'}} = \frac{1}{r'}\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n$$

Comparing with the previous equation must be that $A_l = 1/(r')^{l+1}$. Therefore replacing in the general equation we get that

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \sum_{l=0}^{\infty}\frac{r^l}{(r')^{l+1}}P_l(\cos\theta)$$

□

Solution. Exercise 25. In this case, the quadrupole-moment tensor is given by

$$
\begin{aligned}
Q^{ij} &= \int (3x'^i x'^j - \delta^{ij} r'^2)\rho dV' \\
&= q\left(3\frac{b}{2}\frac{b}{2} - \left(\frac{b}{2}\right)^2\right) + (-q)\left(3\frac{(-b)}{2}\frac{(-b)}{2} - \left(\frac{(-b)}{2}\right)^2\right) \\
&= 2q\left(\frac{b}{2}\right)^2 - 2q\left(\frac{b}{2}\right)^2 \\
&= 0
\end{aligned}
$$

$\square$

Solution. Exercise 26. Suppose the center of the dipole with charges $\pm q$ is at (x_0, y_0, z_0) in a coordinate system x, y, z. Then the charge $+q$ is at $(x_0, y_0, z_0 + b/2)$ and the charge $-q$ is at $(x_0, y_0, z_0 - b/2)$ then the dipole moment components are given by

$$
\begin{aligned}
p_z &= q\left(z_0 + \frac{b}{2}\right) + (-q)\left(z_0 - \frac{b}{2}\right) = qb \\
p_x &= qx_0 + (-q)x_0 = 0 \\
p_y &= qy_0 + (-q)y_0 = 0
\end{aligned}
$$

Therefore we see that the dipole moment is independent of the position of the center of the dipole.

For the general case of a charge distribution with zero net charge, suppose $\boldsymbol{x_0} = (x_0, y_0, z_0)$ is a point inside the charge distribution in a coordinate system x, y, z.
Then the coordinates of a point $\boldsymbol{x}$ in the charge distribution can be written as $\boldsymbol{x} = \boldsymbol{x_0} + \boldsymbol{x'}$ where $\boldsymbol{x'}$ are the coordinates of the point in a coordinate system x', y', z' centered at $\boldsymbol{x_0}$.
In the coordinate system x', y', z' the dipole moment is

$$
\boldsymbol{p} = \int \boldsymbol{x'}\rho(\boldsymbol{x'})dV'
$$

And in the coordinate system x, y, z the dipole moment is

$$
\boldsymbol{p} = \int \boldsymbol{x}\rho(\boldsymbol{x})dV
$$

But we can also write

$$
\begin{aligned}
\boldsymbol{p} &= \int (\boldsymbol{x_0} + \boldsymbol{x'})\rho(\boldsymbol{x_0} + \boldsymbol{x'})dV \\
&= \boldsymbol{x_0} \int \rho(\boldsymbol{x_0} + \boldsymbol{x'})dV + \int \boldsymbol{x'}\rho(\boldsymbol{x_0} + \boldsymbol{x'})dV \\
&= 0 + \int \boldsymbol{x'}\rho(\boldsymbol{x_0} + \boldsymbol{x'})dV \\
&= \int \boldsymbol{x'}\rho(\boldsymbol{x'})dV'
\end{aligned}
$$

Where we used in the second step that $\boldsymbol{x_0}$ is a fixed coordinate so we can take it out of the integral and hence the integral over the charge distribution is 0 because it has zero net charge. Finally in the last step we integrate over V' instead of V and hence numerically $\rho(\boldsymbol{x_0} + \boldsymbol{x'})$ is equal to $\rho(\boldsymbol{x'})$.
Therefore the dipole moment is independent of the coordinate system taken.

$\square$

Solution. Exercise 27. The potential of an ideal dipole was derived in equation (121) and it states the following

$$\Phi(\boldsymbol{x}) = \frac{1}{r^3}\boldsymbol{x} \cdot \boldsymbol{p}$$

This equation was derived assuming the center of the dipole was at the origin and hence it computes the potential at a point $\boldsymbol{x}$ at a distance r from the origin.

So, to compute the potential of a dipole centered at the point $\boldsymbol{x}'$ we need to compute first, the vector from the center of the dipole to the point where we want to compute the potential, this vector is $\boldsymbol{x} - \boldsymbol{x}'$.

Then, replacing the vector in the equation gives us

$$\Phi(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^3}(\boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{p}$$

Where we used that now the length of the vector from the center of the dipole to $\boldsymbol{x}$ is $|\boldsymbol{x} - \boldsymbol{x}'|$ instead of r.

On the other hand, let us compute $\boldsymbol{\nabla}(1/|\boldsymbol{x} - \boldsymbol{x}'|)$ as follows

$$\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \begin{bmatrix} \frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ \frac{\partial}{\partial y}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ \frac{\partial}{\partial z}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \end{bmatrix} = \begin{bmatrix} -\frac{x-x'}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3} \\ -\frac{y-y'}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3} \\ -\frac{z-z'}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{x-x'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\ -\frac{y-y'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\ -\frac{z-z'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \end{bmatrix} = -\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}$$

Therefore

$$\Phi(\boldsymbol{x}) = \boldsymbol{p} \cdot \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} = -\boldsymbol{p} \cdot \boldsymbol{\nabla}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

Also, we have that

$$\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \begin{bmatrix} \frac{\partial}{\partial x'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ \frac{\partial}{\partial y'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ \frac{\partial}{\partial z'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \end{bmatrix} = \begin{bmatrix} \frac{x-x'}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3} \\ \frac{y-y'}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3} \\ \frac{z-z'}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{x-x'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\ \frac{y-y'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \\ \frac{z-z'}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \end{bmatrix} = \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}$$

Therefore we also have

$$\Phi(\boldsymbol{x}) = \boldsymbol{p} \cdot \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} = \boldsymbol{p} \cdot \boldsymbol{\nabla}'\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

$\square$