Solved selected problems of Classical Electrodynamics - Hans Ohanian

Franco Zacco

Chapter 4 - Dielectrics

Exercises

Solution. Exercise 1. From the Gauss' law for Dielectrics in its integral form we know that

$$\int \boldsymbol{\nabla} \cdot \boldsymbol{D}\, dV = \int \boldsymbol{D} \cdot d\boldsymbol{S} = 4\pi \int \rho_F\, dV$$

But in this case, since we are assuming a pillbox where the sides of the box can be ignored as $h \to 0$, Gauss' equation becomes

$$\int \boldsymbol{D} \cdot d\boldsymbol{S} = AD_{2\perp} - AD_{1\perp} = 4\pi\sigma_F A$$

Where A is the area of the top and bottom sides of the pillbox, and we assumed that σ_F is constant inside the pillbox. Therefore the boundary condition becomes

$$D_{2\perp} - D_{1\perp} = 4\pi\sigma_F$$

□

Solution. Exercise 2. We want to compute the integral $\int \boldsymbol{E} \cdot d\boldsymbol{S}$ so taking a pill box that is half in the disk-cavity and half in the dielectric and assuming the width of the pill box h tends to 0 we get that

$$\int \boldsymbol{E} \cdot d\boldsymbol{S} = A(E_{\perp \mathrm{ext}} - E_{\perp \mathrm{int}}) = A(E_0 - \varepsilon E_0) = AE_0(1 - \varepsilon)$$

Where A is the area of the face of the pill-box. But also by Gauss' law we have that

$$\int \boldsymbol{E} \cdot d\boldsymbol{S} = 4\pi\sigma_P A$$

where we used σ_P since there are no free charges.
Therefore the (bound) surface charge density is given by

$$\sigma_P = \frac{E_0}{4\pi}(1 - \varepsilon)$$

Let us be sufficiently near the surface of the cavity, there the edge effects are unimportant and we may regard the surface as infinite. So the electric field generated by such an infinite plane has a magnitude $2\pi\sigma_P$ i.e.

$$2\pi\sigma_P = \frac{E_0}{2}(1 - \varepsilon)$$

But we must multiply this expression by two since we have two surfaces (each face of the disk) and hence the surface charge density contributes an electric field of $E_0(1 - \varepsilon)$ to the electric field inside the cavity. $\square$

Solution. Exercise 3. Equation (42) states that

$$-E_0 R\cos\theta + \sum_{l=1}^{\infty} A_l \frac{P_l(\cos\theta)}{R^{l+1}} = \sum_{l=0}^{\infty} B_l R^l P_l(\cos\theta)$$

If equation (42) is to be valid for all values of θ then if $\theta = \pi/2$ we get that

$$\sum_{l=1}^{\infty} A_l \frac{P_l(0)}{R^{l+1}} = \sum_{l=0}^{\infty} B_l R^l P_l(0) = B_0 + \sum_{l=1}^{\infty} B_l R^l P_l(0)$$

If l is odd then $P_l(0) = 0$ so we get that

$$B_0 = 0$$

but if l is even $P_l(0) \neq 0$ so we see that

$$\sum_{\substack{l=1 \\ l \text{ even}}}^{\infty} A_l \frac{P_l(0)}{R^{l+1}} - B_l R^l P_l(0) = \sum_{\substack{l=1 \\ l \text{ even}}}^{\infty} P_l(0)\left(\frac{A_l}{R^{l+1}} - B_l R^l\right) = 0$$

So for this to be equal to 0 must be that $A_l/R^{l+1} - B_l R^l = 0$ then we see that

$$\begin{aligned} \frac{A_2}{R^3} &= B_2 R^2 \\ \frac{A_4}{R^5} &= B_4 R^4 \\ &\vdots \end{aligned}$$

Now, let us multiply equation (42) by $P_n(\mu)$ where $\mu = \cos\theta$ and let us integrate with respect to μ between -1 and 1

$$-E_0 R \int_{-1}^{1} \mu P_n(\mu)\, d\mu + \sum_{l=1}^{\infty} \int_{-1}^{1} A_l \frac{P_l(\mu)P_n(\mu)}{R^{l+1}}\, d\mu = \sum_{l=0}^{\infty} \int_{-1}^{1} B_l R^l P_l(\mu)P_n(\mu)\, d\mu$$

Let $n = 1$ then since $\int_{-1}^{1} P_l(\mu)P_n(\mu)\, d\mu = 0$ when $l \neq n$ we get that

$$\begin{aligned} -E_0 R \int_{-1}^{1} \mu P_1(\mu)\, d\mu + \frac{2}{3}\frac{A_1}{R^2} &= \frac{2}{3} B_1 R \\ -E_0 R \int_{-1}^{1} \mu^2\, d\mu + \frac{2}{3}\frac{A_1}{R^2} &= \frac{2}{3} B_1 R \\ -\frac{2}{3} E_0 R + \frac{2}{3}\frac{A_1}{R^2} &= \frac{2}{3} B_1 R \\ -E_0 R + \frac{A_1}{R^2} &= B_1 R \end{aligned}$$

Where we used that $\int_{-1}^{1} P_l(\mu)P_l(\mu)\ d\mu = 2/(2l+1)$ and that $P_1(\mu) = \mu$. If we let $n = 3$ we get that

$$\begin{aligned}
-E_0 R \int_{-1}^{1} \mu P_3(\mu)\ d\mu + \frac{2}{7}\frac{A_3}{R^4} &= \frac{2}{7} B_3 R^3 \\
-E_0 R \int_{-1}^{1} \frac{5\mu^4 - 3\mu^2}{2}\ d\mu + \frac{2}{7}\frac{A_3}{R^4} &= \frac{2}{7} B_3 R^3 \\
-E_0 R \left[\frac{5}{2}\frac{\mu^5}{5}\bigg|_{-1}^{1} - \frac{3}{2}\frac{\mu^3}{3}\bigg|_{-1}^{1} \right] + \frac{2}{7}\frac{A_3}{R^4} &= \frac{2}{7} B_3 R^3 \\
-E_0 R \left[\left(\frac{1}{2} - \frac{1}{2} \right) - \left(\frac{1}{2} - \frac{1}{2} \right) \right] + \frac{2}{7}\frac{A_3}{R^4} &= \frac{2}{7} B_3 R^3 \\
\frac{2}{7}\frac{A_3}{R^4} &= \frac{2}{7} B_3 R^3 \\
\frac{A_3}{R^4} &= B_3 R^3
\end{aligned}$$

This process can be continued for each odd value of l. $\square$

Solution. Exercise 4. Equation (44) states that

$$\varepsilon_1\Big[-E_0\cos\theta-\sum_{l=1}^{\infty}(l+1)A_l\frac{P_l(\cos\theta)}{R^{l+2}}\Big]=\varepsilon_2\sum_{l=0}^{\infty}lB_lR^{l-1}P_l(\cos\theta)$$

So as we did in Exercise 3 let us multiply the equation by $P_n(\mu)$ where $\mu=\cos\theta$ and let us integrate with respect to μ between -1 and 1

$$\varepsilon_1\Big[-E_0\int_{-1}^{1}\mu P_n(\mu)\ d\mu-\sum_{l=1}^{\infty}(l+1)\frac{A_l}{R^{l+2}}\int_{-1}^{1}P_l(\mu)P_n(\mu)\ d\mu\Big]$$
$$=\varepsilon_2\sum_{l=0}^{\infty}lB_lR^{l-1}\int_{-1}^{1}P_l(\mu)P_n(\mu)\ d\mu$$

Then setting $n=1$ we get that

$$\varepsilon_1\Big[-E_0\int_{-1}^{1}\mu P_1(\mu)\ d\mu-\frac{2A_1}{R^3}\int_{-1}^{1}P_1(\mu)P_1(\mu)\ d\mu\Big]=\varepsilon_2B_1\int_{-1}^{1}P_1(\mu)P_1(\mu)\ d\mu$$
$$\varepsilon_1\Big[-E_0\int_{-1}^{1}\mu^2\ d\mu-\frac{2A_1}{R^3}\frac{2}{3}\Big]=\varepsilon_2B_1\frac{2}{3}$$
$$\varepsilon_1\Big[-E_0\frac{2}{3}-\frac{2A_1}{R^3}\frac{2}{3}\Big]=\varepsilon_2B_1\frac{2}{3}$$
$$\varepsilon_1\Big[-E_0-\frac{2A_1}{R^3}\Big]=\varepsilon_2B_1$$

If $n=2$ we get that

$$\varepsilon_1\Big[-E_0\int_{-1}^{1}\mu P_2(\mu)\ d\mu-\frac{3A_2}{R^4}\int_{-1}^{1}P_2(\mu)P_2(\mu)\ d\mu\Big]=2\varepsilon_2B_2R\int_{-1}^{1}P_2(\mu)P_2(\mu)\ d\mu$$
$$\varepsilon_1\Big[-\frac{E_0}{2}\int_{-1}^{1}\mu(3\mu^2-1)\ d\mu-\frac{3A_2}{R^4}\frac{2}{5}\Big]=2\varepsilon_2B_2R\frac{2}{5}$$
$$\varepsilon_1\Big[-\frac{E_0}{2}\Big[\frac{3\mu^4}{4}-\frac{\mu^2}{2}\Big]_{-1}^{1}-\frac{3A_2}{R^4}\frac{2}{5}\Big]=2\varepsilon_2B_2R\frac{2}{5}$$
$$\varepsilon_1\Big[-\frac{3A_2}{R^4}\Big]=2\varepsilon_2B_2R$$

And if $n=3$

$$\varepsilon_1\Big[-E_0\int_{-1}^{1}\mu P_3(\mu)\ d\mu-\frac{4A_3}{R^5}\int_{-1}^{1}P_3(\mu)P_3(\mu)\ d\mu\Big]=3\varepsilon_2B_3R^2\int_{-1}^{1}P_3(\mu)P_3(\mu)\ d\mu$$
$$\varepsilon_1\Big[-\frac{E_0}{2}\int_{-1}^{1}\mu(5\mu^3-3\mu)\ d\mu-\frac{4A_3}{R^5}\frac{2}{7}\Big]=3\varepsilon_2B_3R^2\frac{2}{7}$$
$$\varepsilon_1\Big[-\frac{E_0}{2}\Big[\mu^5-\mu^3\Big]_{-1}^{1}-\frac{4A_3}{R^5}\frac{2}{7}\Big]=3\varepsilon_2B_3R^2\frac{2}{7}$$
$$\varepsilon_1\Big[-\frac{4A_3}{R^5}\Big]=3\varepsilon_2B_3R^2$$

We can continue this process in the same way for any value of l. □

Solution. Exercise 5. From equations (43) and (45) for A_1 and B_1 we know that

$$-E_0R + \frac{A_1}{R^2} = B_1R \quad \text{and} \quad B_1 = \frac{\varepsilon_1}{\varepsilon_2}\left(- E_0 - \frac{2A_1}{R^3}\right)$$

Then replacing B_1 in the first equation we get that

$$\begin{aligned}
-E_0R + \frac{A_1}{R^2} &= \frac{\varepsilon_1}{\varepsilon_2}\left(- E_0R - \frac{2A_1}{R^2}\right)\\
-E_0R\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right) + \frac{A_1}{R^2}\left(1 + \frac{2\varepsilon_1}{\varepsilon_2}\right) &= 0\\
A_1\left(\frac{\varepsilon_2 + 2\varepsilon_1}{\varepsilon_2}\right) &= E_0R^3\left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}\right)\\
A_1 &= E_0R^3\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + 2\varepsilon_1}
\end{aligned}$$

Now, we replace this value of A_1 in the equation we have for B_1 as follows

$$\begin{aligned}
B_1 &= \frac{\varepsilon_1}{\varepsilon_2}\left(- E_0 - \frac{2E_0R^3}{R^3}\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + 2\varepsilon_1}\right)\\
B_1 &= -E_0\frac{\varepsilon_1}{\varepsilon_2}\left(1 + \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2 + 2\varepsilon_1}\right)\\
B_1 &= -E_0\left(\frac{\varepsilon_1}{\varepsilon_2} + \frac{2\varepsilon_1(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2(\varepsilon_2 + 2\varepsilon_1)}\right)\\
B_1 &= -E_0\left(\frac{\varepsilon_1(\varepsilon_2 + 2\varepsilon_1) + 2\varepsilon_1(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2(\varepsilon_2 + 2\varepsilon_1)}\right)\\
B_1 &= -E_0\left(\frac{3\varepsilon_1\varepsilon_2}{\varepsilon_2(\varepsilon_2 + 2\varepsilon_1)}\right)\\
B_1 &= -E_0\left(\frac{3\varepsilon_1}{\varepsilon_2 + 2\varepsilon_1}\right)
\end{aligned}$$

For A_2 and B_2 we see that

$$\frac{A_2}{R^3} = B_2R^2 \quad \text{and} \quad B_2 = \frac{\varepsilon_1}{\varepsilon_2}\left(\frac{-3A_2}{2R^5}\right)$$

Then replacing B_2 we get that

$$\begin{aligned}
\frac{A_2}{R^3} &= R^2\frac{\varepsilon_1}{\varepsilon_2}\left(\frac{-3A_2}{2R^5}\right)\\
A_2 &= -\frac{3\varepsilon_1A_2}{2\varepsilon_2}\\
A_2\left(1 + \frac{3\varepsilon_1}{2\varepsilon_2}\right) &= 0\\
A_2 &= 0
\end{aligned}$$

And hence $B_2 = 0$.
Fianlly, since the equations of A_i, B_i where $i \geq 2$ have the same form, we get that

$$A_2 = A_3 = ... = 0$$
$$B_2 = B_3 = ... = 0$$

$\square$

Solution. Exercise 6. We know that the total surface charge density on the interface is

$$\sigma_P = \left(1 - \frac{1 - 1/\varepsilon_1}{1 - 1/\varepsilon_2}\right) \frac{\varepsilon_2 - 1}{4\pi} \frac{3\varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \cos\theta$$

We want to consider the special case of an empty spherical cavity ($\varepsilon_2 = 1$), but first we need to re-write the equation as follows

$$\sigma_P = \left(1 - \frac{\varepsilon_2(\varepsilon_1 - 1)}{\varepsilon_1(\varepsilon_2 - 1)}\right) \frac{\varepsilon_2 - 1}{4\pi} \frac{3\varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \cos\theta$$
$$\sigma_P = \left(\varepsilon_2 - 1 - \frac{\varepsilon_2(\varepsilon_1 - 1)}{\varepsilon_1}\right) \frac{1}{4\pi} \frac{3\varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \cos\theta$$

So now, replacing $\varepsilon_2 = 1$ we get that

$$\begin{aligned}
\sigma_P &= \left(1 - 1 - \frac{\varepsilon_1 - 1}{\varepsilon_1}\right) \frac{1}{4\pi} \frac{3\varepsilon_1}{1 + 2\varepsilon_1} \cos\theta \\
&= -\frac{\varepsilon_1 - 1}{\varepsilon_1} \frac{1}{4\pi} \frac{3\varepsilon_1}{1 + 2\varepsilon_1} \cos\theta \\
&= -\frac{1}{4\pi} \frac{3(\varepsilon_1 - 1)}{1 + 2\varepsilon_1} \cos\theta
\end{aligned}$$

□