Solved selected problems of Classical Electrodynamics - Hans Ohanian

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Chapter 4 - Dielectrics

Problems

Solution. 1. Equation (8) and (9) state that

$$\rho_P(\boldsymbol{x}') = -\nabla' \cdot \boldsymbol{P}(\boldsymbol{x}') \quad \text{and} \quad \sigma_P(\boldsymbol{x}') = \boldsymbol{P}(\boldsymbol{x}') \cdot \hat{\boldsymbol{n}}$$

We want to prove that the net polarization charge in the volume and on the surface is always zero so by integration we get that

$$\begin{aligned} Q_{net} &= \int \sigma_P(\boldsymbol{x}')\ dS' + \int \rho_P(\boldsymbol{x}')\ dV' \\ &= \int \boldsymbol{P}(\boldsymbol{x}') \cdot \hat{\boldsymbol{n}}\ dS' - \int \nabla' \cdot \boldsymbol{P}(\boldsymbol{x}')\ dV' \\ &= \int \boldsymbol{P}(\boldsymbol{x}') \cdot \hat{\boldsymbol{n}}\ dS' - \int \boldsymbol{P}(\boldsymbol{x}') \cdot \hat{\boldsymbol{n}}\ dS' \\ &= 0 \end{aligned}$$

Where we used Gauss' Theorem in the third step. $\square$

Solution. 3. Let two long, conducting cylinders of sheet metal have radii R and $3R$, respectively. The space between them is filled with a gas of dielectric constant ε and the potential difference between the cylinders is V_0. From Gauss' law in a dielectric we know that

$$\int \boldsymbol{D} \cdot d\boldsymbol{S} = Q_F$$

Where Q_F is the charge due to the free charges in the dielectric.
Let us take a cylindrical Gaussian surface at $R < r < 3R$ of length L.
By symmetry we have that $\boldsymbol{D} = D\hat{\boldsymbol{r}}$ then

$$\begin{aligned} \int_0^L \int_0^{2\pi} D\hat{\boldsymbol{r}} \cdot r dz d\phi \hat{\boldsymbol{r}} &= Q_F \\ D 2\pi r L &= Q_F \\ D &= \frac{\lambda_F}{2\pi r} \end{aligned}$$

Where in the last step we defined $\lambda_F = Q_F/L$ to be the linear free charges density in between cylinders.
Now since the dielectric constant of the gas is ε we get that

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\varepsilon} = \frac{\lambda_F}{2\pi\varepsilon r}\hat{\boldsymbol{r}}$$

But also we know that

$$V_0 = -\int \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_R^{3R} \frac{\lambda_F}{2\pi\varepsilon r}\hat{\boldsymbol{r}} \cdot dr\hat{\boldsymbol{r}} = -\frac{\lambda_F}{2\pi\varepsilon}\log(3)$$

Therefore we can replace $\lambda_F = -2\pi\varepsilon V_0/\log(3)$ to get

$$\boldsymbol{E} = -\frac{V_0}{\log(3) r}\hat{\boldsymbol{r}}$$

And

$$\boldsymbol{D} = -\frac{\varepsilon V_0}{\log(3) r}\hat{\boldsymbol{r}}$$

□

Solution. 4. Let two dielectrics with constants ε_1 and ε_2. The electric field with magnitude E_1 in the first dielectric makes an angle α_1 with the normal. We want to know the angle the electric field makes in the second dielectric and its magnitude.
Because of the boundary condition imposed on $\boldsymbol{E}$ must be that

$$E_{1\parallel} = E_{2\parallel}$$

Which implies that

$$\begin{aligned} E_1 \cos\left(\alpha_1 - \frac{\pi}{2}\right) &= E_2 \cos\left(\frac{\pi}{2} - \alpha_2\right) \\ E_1 \sin(\alpha_1) &= E_2 \sin(\alpha_2) \end{aligned}$$

But also from the boundary condition imposed on $\boldsymbol{D}$

$$\varepsilon_1 E_{1\perp} = D_{1\perp} = D_{2\perp} = \varepsilon_2 E_{2\perp}$$

We get that

$$\begin{aligned} \varepsilon_1 E_1 \sin\left(\alpha_1 - \frac{\pi}{2}\right) &= \varepsilon_2 E_2 \sin\left(\frac{\pi}{2} - \alpha_2\right) \\ \varepsilon_1 E_1 \cos(\alpha_1) &= \varepsilon_2 E_2 \cos(\alpha_2) \end{aligned}$$

So from the first boundary condition we get that

$$E_2 = E_1 \frac{\sin(\alpha_1)}{\sin(\alpha_2)}$$

Therefore from the second boundary condition we get that the angle α_2 is

$$\begin{aligned} E_2 \cos(\alpha_2) &= \frac{\varepsilon_1}{\varepsilon_2} E_1 \cos(\alpha_1) \\ E_1 \frac{\sin(\alpha_1)}{\sin(\alpha_2)} \cos(\alpha_2) &= \frac{\varepsilon_1}{\varepsilon_2} E_1 \cos(\alpha_1) \\ \frac{\cos(\alpha_2)}{\sin(\alpha_2)} &= \frac{\varepsilon_1}{\varepsilon_2} \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\ \tan(\alpha_2) &= \frac{\varepsilon_2}{\varepsilon_1} \tan(\alpha_1) \\ \alpha_2 &= \arctan\left(\frac{\varepsilon_2}{\varepsilon_1} \tan(\alpha_1)\right) \end{aligned}$$

And hence the magnitude E_2 is

$$E_2 = \frac{E_1 \sin(\alpha_1)}{\sin\left(\arctan\left(\frac{\varepsilon_2}{\varepsilon_1} \tan(\alpha_1)\right)\right)} = E_1 \sqrt{\left(\frac{\varepsilon_1}{\varepsilon_2} \cos\alpha_1\right)^2 + \sin^2\alpha_1}$$

$\square$

Solution. 6.

(a) From Gauss' law in a dielectric we know that

$$\int \boldsymbol{D} \cdot d\boldsymbol{S} = \varepsilon \int \boldsymbol{E} \cdot d\boldsymbol{S} = 4\pi q$$

Where q is the free charge in the dielectric.
Taking a spherical Gaussian surface around the point charge gives us

$$\begin{aligned} \varepsilon E \int dS &= 4\pi q \\ \varepsilon E 4\pi r^2 &= 4\pi q \\ E &= \frac{q}{\varepsilon r^2} \end{aligned}$$

And hence the electrostatic potential is

$$\Phi = -\int_{\infty}^{r} \boldsymbol{E} \cdot d\boldsymbol{r} = -\int_{\infty}^{r} \frac{q}{\varepsilon r^2} \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} dr = -\left[-\frac{q}{\varepsilon r} \right]_{\infty}^{r} = \frac{q}{\varepsilon r}$$

(b) We know that the force an electron orbiting the ion $+e$ inmerse in the dielectric is

$$F = -eE = -\frac{e^2}{\varepsilon r^2}$$

Making this force equal to the centripetal force we get that

$$\frac{m_e v^2}{r} = \frac{e^2}{\varepsilon r^2}$$

Where m_e is the mass of the electron.
On the other hand, we know Bohr requires the quantization of the angular momentum so

$$m_e v r = n\hbar$$

Hence the allower radius are given by

$$\begin{aligned} m_e^2 v^2 &= \frac{m_e e^2}{\varepsilon r} \\ \frac{n^2 \hbar^2}{r^2} &= \frac{m_e e^2}{\varepsilon r} \\ r &= \frac{n^2 \hbar^2 \varepsilon}{m_e e^2} \end{aligned}$$

Therefore putting in the electric charge and the mass of an electron and the dielectric constant of silicon, the allowed radius for $n = 1$ (the smallest orbit) is

$$r = \frac{\hbar^2 \varepsilon}{m_e e^2} = \frac{(1.0546 \times 10^{-27})^2 \cdot 11.7}{(9.109 \times 10^{-28}) \cdot (4.803 \times 10^{-10})^2} = 6.192 \times 10^{-8}\ cm = 6.19 \mathring{A}$$

Finally, the binding energy of the electron is

$$E = K + U = \frac{1}{2} m_e v^2 - \frac{e^2}{\varepsilon r} = \frac{1}{2} \frac{e^2}{\varepsilon r} - \frac{e^2}{\varepsilon r} = \frac{e^2}{2\varepsilon r}$$

Hence

$$E = \frac{(4.803 \times 10^{-10})^2}{2 \cdot 11.6 \cdot 6.192 \times 10^{-8}} = 1.592 \times 10^{-13}\ erg = 0.099\ eV$$

$\square$

Solution. 7. Let a dielectric sphere of radius R with dielectric constant ε and free charge Q distributed uniformly over its volume.
Let us consider first the case where $r \leq R$.
The charge per unit of volume is $3Q/4\pi R^3$ then the charge enclosed in the sphere of radius r is

$$\frac{3Q}{4\pi R^3}\frac{4\pi r^3}{3} = \frac{Qr^3}{R^3}$$

Also, we know that $\boldsymbol{D}(\boldsymbol{x}) = D\hat{\boldsymbol{r}}$ because of the symmetry of the sphere.
So from Gauss' law in a dielectric we have that

$$\begin{aligned}
\int \boldsymbol{D}\cdot d\boldsymbol{S} &= 4\pi\frac{Qr^3}{R^3}\\
\int_0^{2\pi}\int_0^{\pi} D\hat{\boldsymbol{r}}\cdot(r^2\sin\theta\; d\theta d\phi)\hat{\boldsymbol{r}} &= 4\pi\frac{Qr^3}{R^3}\\
Dr^2\int_0^{2\pi}\Big[-\cos\theta\Big]_0^{\pi} d\phi &= 4\pi\frac{Qr^3}{R^3}\\
2D\Big[\phi\Big]_0^{2\pi} &= 4\pi\frac{Qr}{R^3}
\end{aligned}$$

Hence

$$\boldsymbol{D} = \frac{Qr}{R^3}\hat{\boldsymbol{r}}$$

And since the dielectric constant is ε then

$$\boldsymbol{E} = \frac{Qr}{\varepsilon R^3}\hat{\boldsymbol{r}}$$

Now, for the case where $r > R$ the enclosed charge is Q so

$$\begin{aligned}
\int \boldsymbol{D}\cdot d\boldsymbol{S} &= 4\pi Q\\
\int_0^{2\pi}\int_0^{\pi} D\hat{\boldsymbol{r}}\cdot(r^2\sin\theta\; d\theta d\phi)\hat{\boldsymbol{r}} &= 4\pi Q\\
Dr^2\int_0^{2\pi}\Big[-\cos\theta\Big]_0^{\pi} d\phi &= 4\pi Q\\
2D\Big[\phi\Big]_0^{2\pi} &= 4\pi\frac{Q}{r^2}
\end{aligned}$$

Hence

$$\boldsymbol{D} = \frac{Q}{r^2}\hat{\boldsymbol{r}} \quad \text{and} \quad \boldsymbol{E} = \frac{Q}{r^2}\hat{\boldsymbol{r}}$$

Where we used that $\varepsilon = 1$ in empty space.
We know that $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$ so $\boldsymbol{P}$ for the case $r \leq R$ is

$$\boldsymbol{P} = \frac{1}{4\pi}(\boldsymbol{D}-\boldsymbol{E}) = \frac{1}{4\pi}\left(\frac{Qr}{R^3}\hat{\boldsymbol{r}} - \frac{Qr}{\varepsilon R^3}\hat{\boldsymbol{r}}\right) = \frac{Qr}{4\pi R^3}\left(1-\frac{1}{\varepsilon}\right)\hat{\boldsymbol{r}}$$

Then the polarization volume charge density is

$$\rho_P = -\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\frac{1}{r^2}\frac{\partial(r^2 P_r)}{\partial r} = \frac{3Q}{4\pi R^3}\left(\frac{1}{\varepsilon} - 1\right)$$

And the polarization surface charge density is

$$\sigma_P = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = \boldsymbol{P} \cdot \hat{\boldsymbol{r}} = \frac{QR}{4\pi R^3}\left(1 - \frac{1}{\varepsilon}\right) = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\varepsilon}\right)$$

Where we used that $r = R$ for the surface.
Finally, the net polarization volume charge is

$$\rho_P \cdot \frac{4}{3}\pi R^3 = Q\left(\frac{1}{\varepsilon} - 1\right)$$

And in the same way, the net polarization surface charge is

$$\sigma_P \cdot 4\pi R^2 = Q\left(1 - \frac{1}{\varepsilon}\right)$$

□

Solution. 8. Let a point charge q placed at a distance d above the surface of a large lake of dielectric liquid constant ε.
We want to use the method of images to determine the electrostatic potential in the region above and below de surface.
Let us place a charge $-q'$ at a distance d below the surface then the electrostatic potential of the two charges is

$$\Phi_{above} = \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q'}{\varepsilon\sqrt{x^2+y^2+(z+d)^2}}$$

Now if we drop the charge q and we place a charge q'' at a distance above the surface the electrostatic potential because of this charge is

$$\Phi_{below} = \frac{q''}{\sqrt{x^2+y^2+(z-d)^2}}$$

We see that

$$\begin{aligned} D_{above\perp} = E_{above,z} &= \frac{q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{q'(z+d)}{\varepsilon(x^2+y^2+(z+d)^2)^{3/2}} \\ D_{below\perp} = \varepsilon E_{below,z} &= \frac{\varepsilon q''(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} \end{aligned}$$

Because of the boundary condition $D_{above\perp} = D_{below\perp}$ must be true at any point of the plane $z=0$ then we can take the origin to get that

$$\begin{aligned} D_{above\perp}\Big|_{(0,0,0)} &= D_{below\perp}\Big|_{(0,0,0)} \\ -\frac{q}{d^2} - \frac{q'}{\varepsilon d^2} &= -\frac{\varepsilon q''}{d^2} \\ q + \frac{q'}{\varepsilon} &= \varepsilon q'' \\ q' &= \varepsilon(\varepsilon q'' - q) \end{aligned}$$

Now, to satisfy the boundary condition $E_{above\parallel} = E_{below\parallel}$ we take a point where $(x,y,z) = (x,0,0)$ then we get that

$$\begin{aligned} E_{above\parallel}\Big|_{(x,0,0)} &= E_{below\parallel}\Big|_{(x,0,0)} \\ \frac{qx}{(x^2+d^2)^{3/2}} - \frac{q'x}{\varepsilon(x^2+d^2)^{3/2}} &= \frac{q''x}{(x^2+d^2)^{3/2}} \\ q - \frac{q'}{\varepsilon} &= q'' \end{aligned}$$

Replacing q'' into the first boundary condition we get that

$$\begin{aligned} q' &= \varepsilon\left(\varepsilon\left(q - \frac{q'}{\varepsilon}\right) - q\right) \\ q' + \varepsilon q' &= \varepsilon^2 q - \varepsilon q \\ q' &= q\frac{\varepsilon(1-\varepsilon)}{1+\varepsilon} \end{aligned}$$

And hence q'' is

$$q'' = q - \frac{q}{\varepsilon}\frac{\varepsilon(1-\varepsilon)}{1+\varepsilon}$$
$$q'' = q\left(1 - \frac{1-\varepsilon}{1+\varepsilon}\right)$$
$$q'' = q\left(\frac{2\varepsilon}{1+\varepsilon}\right)$$

Finally, we replace the values we computed for q' and q'' to get the electrostatic potential above and below the surface as follows

$$\Phi_{above} = \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1-\varepsilon}{1+\varepsilon}\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}$$

And

$$\Phi_{below} = \frac{2\varepsilon}{1+\varepsilon}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}$$

□

Solution. 19.[Unfinished]
Let a dielectric sphere of radius R, the upper half of the sphere has a dielectric constant ε_1 and the lower half has a dielectric constant ε_2. The sphere is placed in an initially uniform electric field $\boldsymbol{E}_0$ in the vertical direction. We want to find the electrostatic potential inside and outside of the sphere.
We divide the problem to solve into three regions, we take the region 1 when $r < R, \theta < \pi/2$ (the upper half), region 2 when $r < R, \theta > \pi/2$ (the lower half) and region 3 when $r > R$.
The differential equation that Φ must satisfy in all the regions is

$$\nabla^2\Phi = 0$$

The potential associated with the field $\boldsymbol{E}_0$ is $-E_0 z = -E_0 r\cos\theta$ so must be that $\Phi \to -E_0 r\cos\theta$ as $r \to \infty$.
The other boundary conditions are

$$\left.\frac{\partial\Phi_1(r,\theta)}{\partial\theta}\right|_{r=R} = \left.\frac{\partial\Phi_2(r,\theta)}{\partial\theta}\right|_{r=R} = \left.\frac{\partial\Phi_3(r,\theta)}{\partial\theta}\right|_{r=R}$$

$$\left.\frac{\partial\Phi_1(r,\theta)}{\partial r}\right|_{\theta=\pi/2} = \left.\frac{\partial\Phi_2(r,\theta)}{\partial r}\right|_{\theta=\pi/2}$$

$$\varepsilon_1\left.\frac{\partial\Phi_1(r,\theta)}{\partial r}\right|_{r=R} = \varepsilon_2\left.\frac{\partial\Phi_2(r,\theta)}{\partial r}\right|_{r=R} = 1\cdot\left.\frac{\partial\Phi_3(r,\theta)}{\partial r}\right|_{r=R} \tag{1}$$

$$\varepsilon_1\left.\frac{\partial\Phi_1(r,\theta)}{\partial\theta}\right|_{\theta=\pi/2} = \varepsilon_2\left.\frac{\partial\Phi_2(r,\theta)}{\partial\theta}\right|_{\theta=\pi/2} \tag{2}$$

We saw that in the thoery that the first two conditions can be replaced by

$$\left.\Phi_1(r,\theta)\right|_{r=R} = \left.\Phi_2(r,\theta)\right|_{r=R} = \left.\Phi_3(r,\theta)\right|_{r=R} \tag{3}$$

$$\left.\Phi_1(r,\theta)\right|_{\theta=\pi/2} = \left.\Phi_2(r,\theta)\right|_{\theta=\pi/2} \tag{4}$$

From chapter 3 we know that the solution in spherical coordinates must be of the form

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}[A_l r^l P_l(\cos\theta) + B_l r^{-l-1}P_l(\cos\theta)]$$

For the third region we get that

$$\Phi_3(r,\theta) = -E_0 r\cos\theta + \sum_{l=1}^{\infty} A_l\frac{P_l(\cos\theta)}{r^{l+1}}$$

Where we excluded all higher positive powers of r because Φ_3 must tend to $-E_0 r\cos\theta$ as $r \to \infty$ and the term A_0/r has been excluded because it would indicate the presence of some net electric charge.
For the second region we get that

$$\Phi_2(r,\theta) = \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta)$$

Where we excluded all negative powers of r because the potential would diverge at $r = 0$. And in the same way for the first region

$$\Phi_1(r,\theta) = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta)$$

From the boundary condition (3) we get that

$$-E_0 R\cos\theta + \sum_{l=1}^{\infty} A_l \frac{P_l(\cos\theta)}{R^{l+1}} = \sum_{l=0}^{\infty} B_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} C_l R^l P_l(\cos\theta)$$

Which we know implies that

$$\begin{aligned} &B_0 = C_0 = 0 && -E_0 R + \frac{A_1}{R^2} = B_1 R = C_1 R \\ &\frac{A_2}{R^3} = B_2 R^2 = C_2 R^2 && \frac{A_3}{R^4} = B_3 R^3 = C_3 R^3 \\ &\quad\vdots && \quad\vdots \end{aligned}$$

Boundary condition (4) implies that

$$\sum_{l=0}^{\infty} B_l r^l P_l(0) = \sum_{l=0}^{\infty} C_l r^l P_l(0)$$

$$\sum_{l=0}^{\infty} (B_l - C_l) r^l P_l(0) = 0$$

Since for the even values of l, $P_l(0) \neq 0$ then

$$\begin{aligned} &B_0 = C_0 && B_2 = C_2 \\ &B_4 = C_4 && B_6 = C_6 \\ &\quad\vdots && \quad\vdots \end{aligned}$$

Now, boundary condition (1) give us

$$\varepsilon_1 \left[\sum_{l=0}^{\infty} l C_l R^{l-1} P_l(\cos\theta) \right] = \varepsilon_2 \left[\sum_{l=0}^{\infty} l B_l R^{l-1} P_l(\cos\theta) \right] = -E_0\cos\theta - \sum_{l=1}^{\infty} (l+1) A_l \frac{P_l(\cos\theta)}{R^{l+2}}$$

Which we know implies that

$$-E_0 - \frac{2A_1}{R^3} = \varepsilon_1 C_1 = \varepsilon_2 B_2 \qquad\qquad -\frac{3A_2}{R^4} = \varepsilon_1(2C_2R) = \varepsilon_2(2B_2R)$$
$$-\frac{4A_3}{R^5} = \varepsilon_1(3C_3R^2) = \varepsilon_2(3B_3R^2) \qquad\qquad ...$$

$\square$

Solution. 20.

(a) Let the electrostatic potential to have initially the following value

$$\Phi(r,\theta) = Ar^2\left(\frac{3\cos^2\theta - 1}{2}\right)$$

Then the electric field initially is

$$E_r = -\frac{\partial\Phi}{\partial r} = -Ar(3\cos^2\theta - 1)$$
$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 3Ar\sin\theta\cos\theta$$

(b) Suppose now we place a dielectric sphere of radius R and dielectric constant ε at the origin.

We know that when $r \to \infty$ then

$$\Phi \to Ar^2\left(\frac{3\cos^2\theta - 1}{2}\right)$$

The other boundary conditions in this case are

$$\left.\frac{\partial\Phi_1(r,\theta)}{\partial\theta}\right|_{r=R} = \left.\frac{\partial\Phi_2(r,\theta)}{\partial\theta}\right|_{r=R}$$

$$1\cdot\left.\frac{\partial\Phi_1(r,\theta)}{\partial r}\right|_{r=R} = \varepsilon\left.\frac{\partial\Phi_2(r,\theta)}{\partial r}\right|_{r=R}$$

Within the region $r > R$ the electrostatic potential has the form

$$\Phi_1(r,\theta) = Ar^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \sum_{l=1}^{\infty} A_l\frac{P_l(\cos\theta)}{r^{l+1}}$$

Where we excluded all higher powers of r since Φ_1 must tend to $Ar^2(3\cos^2\theta - 1)/2$ when $r \to \infty$ and the term A_0/r has been excluded as well since it would indicate the presence of some net electric charge on the dielectric sphere.

Within the region $r < R$ the electrostatic potential has the form

$$\Phi_2(r,\theta) = \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta)$$

Here negative powers of r have been excluded since otherwise the potential would diverge at $r = 0$.

We can also write the first boundary condition as

$$\left.\Phi_1(r,\theta)\right|_{r=R} = \left.\Phi_2(r,\theta)\right|_{r=R}$$

So replacing, we get that

$$AR^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \sum_{l=1}^{\infty} A_l \frac{P_l(\cos\theta)}{R^{l+1}} = \sum_{l=0}^{\infty} B_l R^l P_l(\cos\theta)$$

Multiplying the equation by $P_n(\mu)$ where $\mu = \cos\theta$ and integrating with respect to μ between -1 and 1 we get that

$$AR^2 \int_{-1}^{1} \left(\frac{3\mu^2 - 1}{2}\right) P_n(\mu)\ d\mu + \sum_{l=1}^{\infty} A_l \int_{-1}^{1} \frac{P_l(\mu)P_n(\mu)}{R^{l+1}}\ d\mu = \sum_{l=0}^{\infty} B_l R^l \int_{-1}^{1} P_l(\mu)P_n(\mu)\ d\mu$$

Let $n = 1$ then since $\int_{-1}^{1} P_l(\mu)P_n(\mu)\ d\mu = 0$ when $l \neq n$ then

$$AR^2 \int_{-1}^{1} P_2(\mu)P_1(\mu)\ d\mu + \frac{A_1}{R^2} \int_{-1}^{1} P_1(\mu)P_1(\mu)\ d\mu = B_1 R \int_{-1}^{1} P_1(\mu)P_1(\mu)\ d\mu$$

$$\begin{aligned} AR^2 \int_{-1}^{1} P_2(\mu)P_1(\mu)\ d\mu + \frac{2}{3}\frac{A_1}{R^2} &= \frac{2}{3} B_1 R \\ \frac{2}{3}\frac{A_1}{R^2} &= \frac{2}{3} B_1 R \\ \frac{A_1}{R^2} &= B_1 R \end{aligned}$$

Where we used that $\int_{-1}^{1} P_l(\mu)P_l(\mu)\ d\mu = 2/(2l+1)$ and that $(3\mu^2 - 1)/2 = P_2(\mu)$. Let now $n = 2$ then we get that

$$\begin{aligned} AR^2 \int_{-1}^{1} \left(\frac{3\mu^2 - 1}{2}\right)^2\ d\mu + \frac{2}{5}\frac{A_2}{R^3} &= \frac{2}{5} B_2 R^2 \\ \frac{2}{5} AR^2 + \frac{2}{5}\frac{A_2}{R^3} &= \frac{2}{5} B_2 R^2 \\ AR^2 + \frac{A_2}{R^3} &= B_2 R^2 \end{aligned}$$

For $n = 3$ we get that

$$\begin{aligned} AR^2 \int_{-1}^{1} P_2(\mu)P_3(\mu) + \frac{2}{7}\frac{A_3}{R^4} &= \frac{2}{7} B_3 R^3 \\ \frac{A_3}{R^4} &= B_3 R^3 \end{aligned}$$

We can continue this process to obtain the rest of the constants.

On the other hand, the second boundary condition demands

$$2AR\left(\frac{3\cos^2\theta - 1}{2}\right) - \sum_{l=1}^{\infty} (l+1) A_l \frac{P_l(\cos\theta)}{R^{l+2}} = \varepsilon\left(\sum_{l=0}^{\infty} l B_l R^{l-1} P_l(\cos\theta)\right)$$

Again, multiplying the equation by $P_n(\mu)$ where $\mu = \cos\theta$ and integrating with respect to μ between -1 and 1 we get that

$$2AR\int_{-1}^{1} P_2(\mu)P_n(\mu)\ d\mu - \sum_{l=1}^{\infty}(l+1)\frac{A_l}{R^{l+2}}\int_{-1}^{1} P_l(\mu)P_n(\mu)\ d\mu$$
$$= \varepsilon\bigg(\sum_{l=0}^{\infty} lB_lR^{l-1}\int_{-1}^{1} P_l(\mu)P_n(\mu)\ d\mu\bigg)$$

Following the same procedure we get that

$$\begin{aligned} -2\frac{A_1}{R^3} &= \varepsilon B_1 & 2AR - 3\frac{A_2}{R^4} &= \varepsilon 2B_2R \\ -4\frac{A_3}{R^5} &= \varepsilon 3B_3R^2 & -5\frac{A_4}{R^6} &= \varepsilon 4B_4R^3 \\ &\vdots & &\vdots \end{aligned}$$

Now we solve the system of linear equations, for A_1 we get that

$$\begin{aligned} -2\frac{A_1}{R^3} &= \varepsilon\frac{A_1}{R^3} \\ -A_1\bigg(\frac{2}{R^3} + \frac{\varepsilon}{R^3}\bigg) &= 0 \\ A_1 &= 0 \end{aligned}$$

And hence $B_1 = 0$. This is also the case for $A_3, A_4, ...$ and $B_3, B_4, ..$ but for A_2 and B_2 we get that

$$\begin{aligned} 2AR - 3\frac{A_2}{R^4} &= \varepsilon 2B_2R \\ 2AR^2 - 3\frac{A_2}{R^3} &= \varepsilon 2B_2R^2 \\ 2AR^2 - 3\frac{A_2}{R^3} &= \varepsilon\bigg(2AR^2 + 2\frac{A_2}{R^3}\bigg) \\ -\frac{A_2}{R^3}(3+2\varepsilon) &= 2AR^2(\varepsilon - 1) \\ A_2 &= \frac{2AR^5(1-\varepsilon)}{(3+2\varepsilon)} \end{aligned}$$

And hence

$$\begin{aligned} B_2R^2 &= AR^2 + \frac{2AR^5(1-\varepsilon)}{(3+2\varepsilon)}\frac{1}{R^3} \\ B_2 &= A + \frac{2A(1-\varepsilon)}{(3+2\varepsilon)} \\ B_2 &= \frac{3A + 2A\varepsilon + 2A - 2A\varepsilon}{(3+2\varepsilon)} \\ B_2 &= \frac{5A}{(3+2\varepsilon)} \end{aligned}$$

Therefore the equation for the region $r > R$ is

$$\begin{aligned}\Phi_1(r,\theta) &= Ar^2\left(\frac{3\cos^2\theta - 1}{2}\right) + A_2\frac{P_2(\cos\theta)}{r^3} \\ &= Ar^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \frac{2AR^5(1-\varepsilon)}{(3+2\varepsilon)r^3}\left(\frac{3\cos^2\theta - 1}{2}\right)\end{aligned}$$

And the equation for the region $r < R$ is

$$\Phi_2(r,\theta) = B_2 r^2 P_2(\cos\theta) = \frac{5Ar^2}{(3+2\varepsilon)}\left(\frac{3\cos^2\theta - 1}{2}\right)$$

(c) The net translational force only depends on the free charges and since there are no free charges then the net translational force is 0.

□

Solution. 21. Let a conducting sphere of radius R at potential zero be surrounded by a concentric spherical shell of dielectric material of inner radius R, outer radius $2R$ and dielectric constant $\varepsilon = 7/5$. Also, let us suppose that the sphere with its dielectric is placed in an initially uniform electric field $\boldsymbol{E}_0$.

(a) We will consider two regions to determine Φ, the region when $R < r < 2R$ where $\Phi = \Phi_1$ and the region where $r > 2R$ where $\Phi = \Phi_2$.
The potential associated with the field $\boldsymbol{E}_0$ is $-E_0 z = -E_0 r\cos\theta$ so must be that $\Phi \to -E_0 r\cos\theta$ as $r \to \infty$.
The other boundary conditions are

$$\left.\frac{\partial\Phi_1(r,\theta)}{\partial\theta}\right|_{r=2R} = \left.\frac{\partial\Phi_2(r,\theta)}{\partial\theta}\right|_{r=2R}$$

or

$$\left.\Phi_1(r,\theta)\right|_{r=2R} = \left.\Phi_2(r,\theta)\right|_{r=2R}$$

And

$$\left.\Phi_1(r,\theta)\right|_{r=R} = 0$$

$$\varepsilon\cdot\left.\frac{\partial\Phi_1(r,\theta)}{\partial r}\right|_{r=2R} = 1\cdot\left.\frac{\partial\Phi_2(r,\theta)}{\partial r}\right|_{r=2R}$$

Within the region $r > 2R$ the electrostatic potential has the form

$$\Phi_2(r,\theta) = -E_0 r\cos\theta + \sum_{l=1}^{\infty} A_l \frac{P_l(\cos\theta)}{r^{l+1}}$$

Within the region $R < r < 2R$ the electrostatic potential has the form

$$\Phi_1(r,\theta) = \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta) + C_l \frac{P_l(\cos\theta)}{r^{l+1}}$$

Now, using the first boundary condition we get that

$$-2E_0 R\cos\theta + \sum_{l=1}^{\infty} A_l \frac{P_l(\cos\theta)}{(2R)^{l+1}} = \sum_{l=0}^{\infty} B_l (2R)^l P_l(\cos\theta) + C_l \frac{P_l(\cos\theta)}{(2R)^{l+1}}$$

Multiplying the whole equation by $P_1(\mu)$ and integrating between -1 and 1 we get that

$$-2E_0 R \int_{-1}^{1} \mu P_1(\mu)\ d\mu + \frac{A_1}{4R^2}\int_{-1}^{1} (P_1(\mu))^2\ d\mu =$$

$$= 2B_1 R \int_{-1}^{1} (P_1(\mu))^2\ d\mu + \frac{C_1}{4R^2}\int_{-1}^{1} (P_1(\mu))^2\ d\mu$$

$$-2E_0 R \int_{-1}^{1} \mu^2\ d\mu + \frac{A_1}{4R^2}\frac{2}{3} = 2B_1 R\frac{2}{3} + \frac{C_1}{4R^2}\frac{2}{3}$$

$$-2E_0 R + \frac{A_1}{4R^2} = 2B_1 R + \frac{C_1}{4R^2}$$

Following the same method for $n = 2, 3, ...$ we get that

$$\begin{aligned}
\frac{A_2}{(2R)^3} &= B_2(2R)^2 + \frac{C_2}{(2R)^3} \\
\frac{A_3}{(2R)^4} &= B_3(2R)^3 + \frac{C_3}{(2R)^4} \\
&\vdots
\end{aligned}$$

Using that $\Phi_1|_{r=R} = 0$ we see that

$$\begin{aligned}
\sum_{l=0}^{\infty} B_l R^l P_l(\cos\theta) + C_l \frac{P_l(\cos\theta)}{R^{l+1}} &= 0 \\
\sum_{l=0}^{\infty} P_l(\cos\theta)\left(B_l R^l + \frac{C_l}{R^{l+1}}\right) &= 0
\end{aligned}$$

Then must be that

$$\begin{aligned}
B_l R^l + \frac{C_l}{R^{l+1}} &= 0 \\
B_l R^l &= -\frac{C_l}{R^{l+1}}
\end{aligned}$$

Using the last boundary condition we get that

$$\varepsilon\left[\sum_{l=0}^{\infty} B_l l(2R)^{l-1} P_l(\cos\theta) - (l+1)C_l \frac{P_l(\cos\theta)}{(2R)^{l+2}}\right] = -E_0\cos\theta - \sum_{l=1}^{\infty}(l+1)A_l \frac{P_l(\cos\theta)}{(2R)^{l+2}}$$

And following the same procedure to determine the value of the constants we get that

$$\begin{aligned}
\varepsilon\left[B_1 - \frac{2C_1}{(2R)^3}\right] &= -E_0 - \frac{2A_1}{(2R)^3} \\
\varepsilon\left[2B_2(2R) - \frac{3C_2}{(2R)^4}\right] &= -\frac{3A_2}{(2R)^4} \\
\varepsilon\left[3B_3(2R)^2 - \frac{4C_3}{(2R)^5}\right] &= -\frac{4A_3}{(2R)^5} \\
&\vdots
\end{aligned}$$

Let us repalce $B_1 = -C_1/R^3$ in the first equation we have for A_1, B_1 and C_1

$$\begin{aligned}
-2E_0 R + \frac{A_1}{4R^2} &= 2B_1 R + \frac{C_1}{4R^2} \\
-2E_0 R + \frac{A_1}{4R^2} &= -2\frac{C_1}{R^2} + \frac{C_1}{4R^2} \\
-2E_0 R + \frac{A_1}{4R^2} &= -7\frac{C_1}{4R^2}
\end{aligned}$$

Also, we see that

$$\begin{aligned}
\varepsilon\left[-\frac{C_1}{R^3}-\frac{C_1}{4R^3}\right] &= -E_0-\frac{2A_1}{(2R)^3}\\
\varepsilon\frac{5C_1}{4R^3} &= E_0+\frac{2A_1}{(2R)^3}\\
7C_1 &= 4E_0R^3+A_1\\
C_1 &= \frac{4}{7}E_0R^3+\frac{A_1}{7}
\end{aligned}$$

Then replacing C_1 gives us

$$\begin{aligned}
-2E_0R+\frac{A_1}{4R^2} &= -E_0R-\frac{A_1}{4R^2}\\
\frac{2A_1}{4R^2} &= E_0R\\
A_1 &= 2E_0R^3
\end{aligned}$$

This, also tells us that

$$C_1=\frac{6}{7}E_0R^3 \qquad B_1=-\frac{6}{7}E_0$$

For A_2 we get that

$$\begin{aligned}
\frac{A_2}{8R^3} &= 4B_2R^2+\frac{C_2}{8R^3}\\
\frac{A_2}{8R^3} &= -4\frac{C_2}{R^3}+\frac{C_2}{8R^3}\\
A_2 &= -31C_2
\end{aligned}$$

And hence

$$\begin{aligned}
\frac{7}{5}\left[4B_2R-\frac{3C_2}{16R^4}\right] &= -\frac{3A_2}{16R^4}\\
\frac{28}{5}C_2-\frac{21C_2}{80R^2} &= \frac{93C_2}{16R^2}\\
C_2\left(\frac{28}{5}-\frac{21}{80R^2}-\frac{93}{16R^2}\right) &= 0\\
C_2 &= 0
\end{aligned}$$

This also imply that $A_2=B_2=0$. We can show the same for A_3, B_3, C_3 and the rest of the constants.

Therefore Φ_1 and Φ_2 become

$$\begin{aligned}
\Phi_1(r,\theta) &= -\frac{6}{7}E_0\left(r-\frac{R^3}{r^2}\right)\cos\theta \qquad &&\text{when} \quad R<r<2R\\
\Phi_2(r,\theta) &= -E_0r\cos\theta+\frac{2E_0R^3}{r^2}\cos\theta \qquad &&\text{when} \quad r>2R
\end{aligned}$$

Where we used the only non-zero values we determined i.e. A_1, B_1 and C_1.

(b) We know that if there exist free surface charge density at the interface between the conducting sphere and the dielectic then

$$D_{diel\perp} - D_{cond\perp} = \varepsilon E_{diel\perp} - E_{cond\perp} = E_r(\varepsilon - 1) = 4\pi\sigma_F$$

where σ_F is the free surface charge density.
So we need to determine $E_\perp = E_r$ at R, hence

$$\begin{aligned} E_r\Big|_{r=R} &= \frac{\partial \Phi_1(r,\theta)}{\partial r}\Big|_{r=R} \\ &= \left[-\frac{6}{7}E_0\left(1 - \frac{2R^3}{r^3}\right)\cos\theta\right]_{r=R} \\ &= \frac{6}{7}E_0\cos\theta \end{aligned}$$

Then σ_F is given by

$$\begin{aligned} 4\pi\sigma_F &= E_r(\varepsilon - 1) \\ \sigma_F &= \frac{1}{4\pi}\frac{6}{7}E_0\cos\theta\left(\frac{7}{5} - 1\right) \\ \sigma_F &= \frac{1}{\pi}\frac{3}{35}E_0\cos\theta \end{aligned}$$

(c) To find the bound charge density induced on the surface of the dielectric at $r = R$ we need to find first P_r (the radial component of the polarization) since $\sigma_P = \boldsymbol{P}\cdot\hat{\boldsymbol{r}} = P_r$.
Since

$$\boldsymbol{P} = \frac{\varepsilon - 1}{4\pi}\boldsymbol{E}$$

Then

$$P_r = \frac{1}{10\pi}E_r = \frac{1}{\pi}\frac{3}{35}E_0\cos\theta$$

Therefore

$$\sigma_P = P_r = \frac{1}{\pi}\frac{3}{35}E_0\cos\theta$$

□