Solved selected problems of From Calculus to Chaos - Acheson

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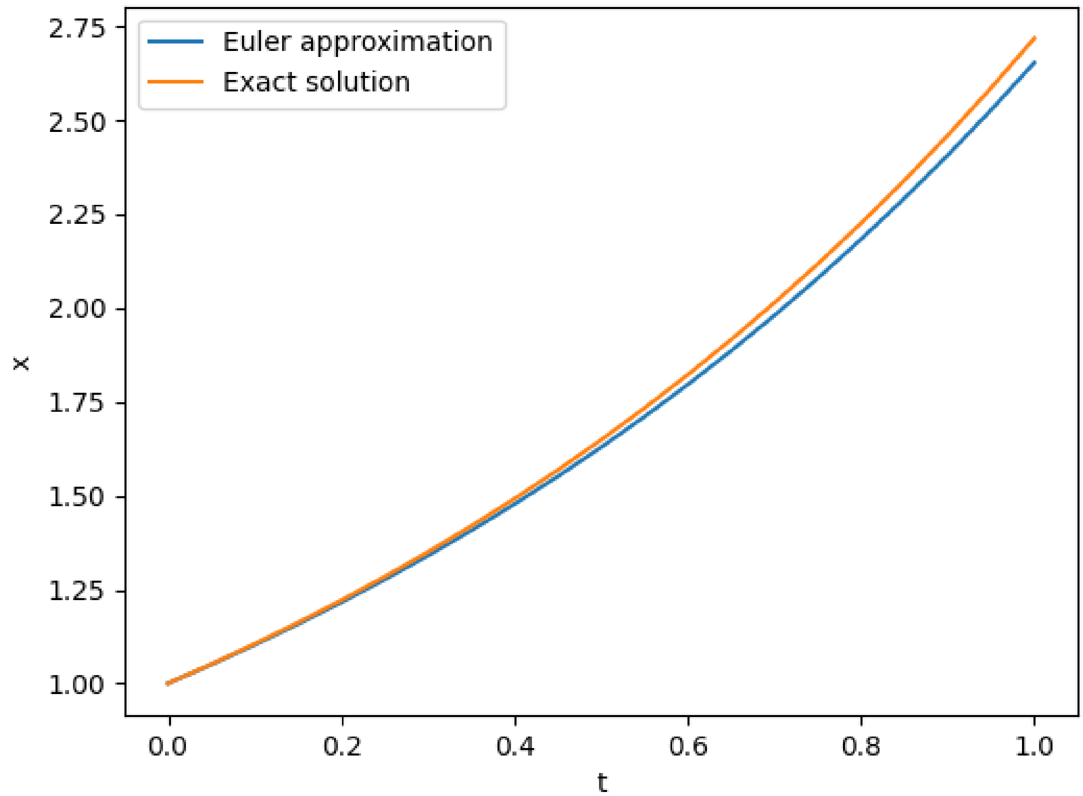
Chapter 4 - Computer solution methods

Proof. **4.1**
The code to generate the graphs is shown below.

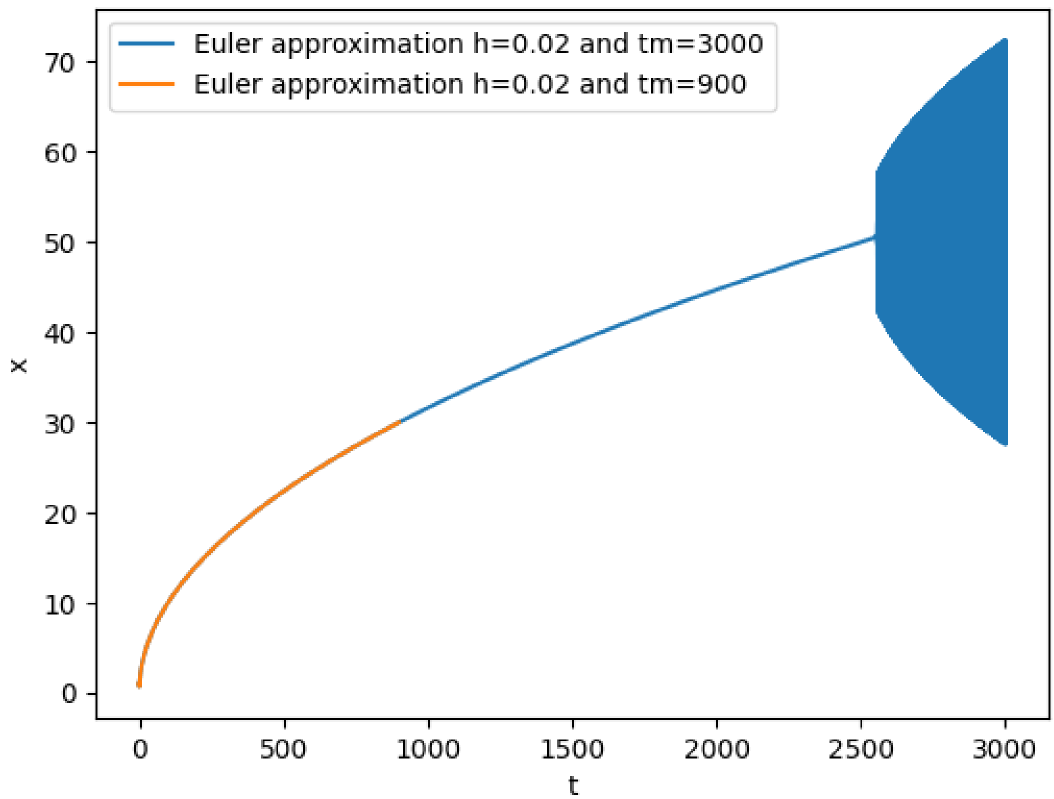
```
import matplotlib
matplotlib.use('TkAgg')
import matplotlib.pyplot as plt
import math


def main():
    plt.figure()
    ax = plt.axes()
    t, x = linear_euler(x0=1, t0=0)
    plot(ax, t, x, "Euler approximation")
    plot(ax, t, [math.e**i for i in t], "Exact solution")
    plt.savefig("euler.png")
    plt.show()


def linear_euler(x0, t0, h=0.05, tm=1):
    x, t = [x0], [t0]
    while abs(t[-1] - tm) > (h / 2):
        x.append(x[-1] + (h * x[-1]))
        t.append(t[-1] + h)
    print("Result: ", x[-1])
    return t, x


def plot(ax, t, x, label):
    ax.plot(t, x, label=label)
    ax.legend()
    ax.set_xlabel("t")
    ax.set_ylabel("x")
```

```
if __name__ == "__main__":
    main())
```

Below is shown the obtained graph



Finally, we show below that the numerical stability in Euler's method is lost when we increase again the time for which we try to approximate the solution.



□

Proof. **4.3** Below is shown the functions used to calculate both the improved Euler's method and the Runge-Kutta method.

```
def imprvd_euler(x0, t0, h, tm=1):
    x, t = x0, t0
    while abs(t - tm) > (h / 2):
        c1 = h * x
        c2 = h * (x + c1)
        x = x + 0.5 * (c1 + c2)
        t = t + h
    print(f"h={h}, x(1)={x}, error={math.e - x}")


def runge_kutta(x0, t0, h, tm=1):
    x, t = x0, t0
    while abs(t - tm) > (h / 2):
        c1 = h * x
        c2 = h * (x + (0.5 * c1))
        c3 = h * (x + (0.5 * c2))
        c4 = h * (x + c3)
        x = x + 1/6 * (c1 + (2 * c2) + (2 * c3) + c4)
        t = t + h
    print(f"h={h}, x(1)={x}, error={math.e - x}")
```

And here is the output after looping through the different h values

```
Improved Euler method
h=0.1, x(1)=2.714080846608224, error=0.004200981850821073
h=0.01, x(1)=2.7182368625599573, error=4.496589908775661e-05
h=0.001, x(1)=2.7182813757517628, error=4.5270728232793545e-07

Runge-Kutta method
h=0.1, x(1)=2.718279744135166, error=2.0843238792700447e-06
h=0.01, x(1)=2.7182818282344035, error=2.2464163862423447e-10
h=0.001, x(1)=2.7182818284590247, error=2.042810365310288e-14
```

□

Proof. **4.4** Let's compute one step of Runge-Kutta method for $f(x,t) = x$ with $x_0 = 1$ then

$$\begin{aligned}
c_1 &= hx_0 = h \\
c_2 &= h(x_0 + \frac{c_1}{2}) = h + \frac{h^2}{2} \\
c_3 &= h(x_0 + \frac{c_2}{2}) = h + \frac{h^2}{2} + \frac{h^3}{4} \\
c_4 &= h(x_0 + c_3) = h + h^2 + \frac{h^3}{2} + \frac{h^4}{4} \\
x &= x_0 + \frac{(c_1 + 2c_2 + 2c_3 + c_4)}{6}
\end{aligned}$$

By replacing the values of x_0, c_1, c_2, c_3 and c_4 we get that

$$\begin{aligned}
x = 1 &+ \frac{h + (2h + h^2) + (2h + h^2 + \frac{h^3}{2}) + (h + h^2 + \frac{h^3}{2} + \frac{h^4}{4})}{6} \\
&= 1 + \frac{6h + 3h^2 + h^3 + \frac{h^4}{4}}{6} \\
&= 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} \\
&= 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!}
\end{aligned}$$

□

Proof. **4.6** Expanding $x(t)$ as a taylor series we see that

$$x(t) = x(t_0) + (t - t_0)\dot{x}(t_0) + \frac{(t-t_0)^2}{2!}\ddot{x}(t_0)$$

Let $h = t - t_0$ where $t_0 = 0$ then $x(h) = x(t - t_0) = x(t - 0) = x(t)$ then

$$x(h) = x(0) + h\dot{x}(0) + \frac{h^2}{2!}\ddot{x}(0)$$

since $\dot{x} = f(x)$ then by the chain rule $\ddot{x} = f'(x)\dot{x}$ and also we know that $x(0) = x_0$ so replacing we have that

$$x(h) = x_0 + hf'(x_0) + \frac{h^2}{2!}f'(x_0)f(x_0)$$

Now let's compute one step of the *Improved Euler method* when $x = x_0$ is the starting point

$$\begin{aligned}
c_1 &= hf(x_0) \\
c_2 &= hf(x_0 + c_1) \\
x &= x_0 + \frac{(c_1 + c_2)}{2}
\end{aligned}$$

Expanding one term of the Taylor series of $f(x_0 + c_1)$ around x_0 we get that

$$\begin{aligned}
f(x_0 + c_1) &= f(x_0) + ((x_0 + c_1) - x_0)f'(x_0) \\
&= f(x_0) + c_1 f'(x_0) \\
&= f(x_0) + hf(x_0)f'(x_0)
\end{aligned}$$

Then replacing c_1 and c_2 values using the taylor series expansion for $f(x_0 + c_1)$

$$\begin{aligned}
x &= x_0 + \frac{hf(x_0)}{2} + \frac{h[f(x_0) + hf(x_0)f'(x_0)]}{2} \\
&= x_0 + \frac{hf(x_0)}{2} + \frac{hf(x_0)}{2} + \frac{h^2 f(x_0)f'(x_0)}{2} \\
&= x_0 + hf(x_0) + \frac{h^2}{2}f(x_0)f'(x_0)
\end{aligned}$$

□