Solved selected problems of Classical Mechanics - Gregory

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Chapter 2 - Velocity, acceleration and scalar angular velocity

Proof. **2.3** Given that acceleration is constant then the instant acceleration a is equal to the average acceleration a_{avg} so assuming that we start from $t = 0$ at $x = 0$ and the final velocity is v then we have that

$$a = \frac{\Delta v}{\Delta t}$$

$$a = \frac{v-u}{t-0}$$

$$at = v - u$$

$$v = u + at \tag{1}$$

Now integrating both sides of the equation with respect to t:

$$\int_0^t v dt = \int_0^t u dt + \int_0^t a t dt$$

$$\int_0^t dx = \int_0^t u dt + \int_0^t a t dt$$

$$x(t) - x(0) = ut + \frac{1}{2}at^2$$

$$x = ut + \frac{1}{2}at^2 \tag{2}$$

From (1) we get that $t = \frac{v-u}{a}$ and replacing in (2) we have

$$x = u\frac{(v-u)}{a} + \frac{1}{2}a\frac{(v-u)^2}{a^2}$$

$$ax = uv - u^2 + \frac{1}{2}(v^2 - uv + u^2)$$

$$ax = \frac{v^2}{2} - \frac{u^2}{2}$$

$$2ax = v^2 - u^2$$
$$v^2 = u^2 + 2ax$$

Finally if we replace $t = 11.4$ s and $v = 116$ mph $= 168.96$ ft/s in (1) we get that $a = 14.82$ ft/s^2 but then if we replace the same values in (2) assuming $x = 0.25$ mi $= 1320$ ft we get that $a = 20.31$ ft/s^2 which means that the acceleration is not constant because otherwise it must satisfy both equations. □

Proof. **2.6** First we calculate the following

$$\begin{aligned} r &= be^{\Omega t} & \theta &= \Omega t \\ \dot{r} &= b\Omega e^{\Omega t} & \dot{\theta} &= \Omega \\ \ddot{r} &= b\Omega^2 e^{\Omega t} & \ddot{\theta} &= 0 \end{aligned}$$

Now replacing in $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + (r\dot{\theta})\hat{\boldsymbol{\theta}}$ and in $\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$ we get that

$$\boldsymbol{v} = b\Omega e^{\Omega t}\hat{\boldsymbol{r}} + (be^{\Omega t}\Omega)\hat{\boldsymbol{\theta}}$$

And

$$\begin{aligned} \boldsymbol{a} &= (b\Omega^2 e^{\Omega t} - b\Omega^2 e^{\Omega t})\hat{\boldsymbol{r}} + (be^{\Omega t}0 + 2b\Omega^2 e^{\Omega t})\hat{\boldsymbol{\theta}} \\ &= 2b\Omega^2 e^{\Omega t}\hat{\boldsymbol{\theta}} \end{aligned}$$

Then,

$$\begin{aligned} |v| &= \sqrt{2}b\Omega^2 e^{\Omega t} \\ |a| &= 2b\Omega^2 e^{\Omega t} \end{aligned}$$

So by using the formula for the dot product we compute $cos(\alpha) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|a||b|}$ as

$$\begin{aligned} cos(\alpha) &= \frac{2b^2\Omega^3 e^{2\Omega t}}{(\sqrt{2}b\Omega^2 e^{\Omega t})(2b\Omega^2 e^{\Omega t})} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Finally, this means that $\alpha = \pi/4$. □

Proof. **2.9** From $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + (r\dot{\theta})\hat{\boldsymbol{\theta}}$ we calculate the velocity vector where

$$\dot{r} = \frac{2b}{\tau} - \frac{2b}{\tau^2}t$$
$$\dot{\theta} = \frac{1}{\tau}$$

Then the velocity vector is

$$\boldsymbol{v} = \frac{2b}{\tau^2}(\tau - t)\hat{\boldsymbol{r}} + \frac{bt}{\tau^3}(2\tau - t)\hat{\boldsymbol{\theta}}$$

Now let's compute $|\boldsymbol{v}|^2$

$$\begin{aligned}|\boldsymbol{v}|^2 &= \frac{4b^2}{\tau^4}\tau^2 - \frac{8b^2\tau}{\tau^4}t + \frac{4b^2}{\tau^4}t^2 + \frac{4b^2\tau^2}{\tau^6}t^2 - \frac{4b^2\tau}{\tau^6}t^3 + \frac{b^2}{\tau^6}t^4 \\ &= \frac{4b^2}{\tau^2} - \frac{8b^2}{\tau^3}t + \frac{8b^2}{\tau^4}t^2 - \frac{4b^2}{\tau^5}t^3 + \frac{b^2}{\tau^6}t^4 \\ &= \frac{b^2}{\tau^6}(t^4 - 4\tau t^3 + 8\tau^2t^2 - 8\tau^3 t + 4\tau^4)\end{aligned}$$

To find the maximum value of $|\boldsymbol{v}|$ let's consider the time derivative of $|\boldsymbol{v}|^2$

$$\begin{aligned}\frac{d|\boldsymbol{v}|^2}{dt} &= \frac{b^2}{\tau^6}(4t^3 - 12\tau t^2 + 16\tau^2 t - 8\tau^3) \\ &= \frac{4b^2}{\tau^6}(t^3 - 3\tau t^2 + 4\tau^2 t - 2\tau^3) \\ &= \frac{4b^2}{\tau^6}(t^3 - \tau t^2 - 2\tau t^2 + 2\tau^2 t + 2\tau^2 t - 2\tau^3) \\ &= \frac{4b^2}{\tau^6}(t(t^2 - 2\tau t + 2\tau^2) - \tau(t^2 - 2\tau t + 2\tau^2)) \\ &= \frac{4b^2}{\tau^6}(t - \tau)(t^2 - 2\tau t + 2\tau^2)\end{aligned}$$

Since $t^2 - 2\tau t + 2\tau^2$ can be written as $\tau^2 + (t-\tau)^2$ we see that this term no matter the value of t is always positive so

$$\frac{d|\boldsymbol{v}|^2}{dt} = \begin{cases} < 0 & \text{if } t < \tau \\ = 0 & \text{if } t = \tau \\ > 0 & \text{if } t > \tau \end{cases}$$

Hence $|\boldsymbol{v}|$ achives a minimum value when $t = \tau$ so

$$|\boldsymbol{v}|^2 = \frac{b^2}{\tau^6}\tau^4$$

$$|\boldsymbol{v}| = \frac{b}{\tau}$$

Finally the acceleration is given by

$$\boldsymbol{a} = (-\frac{2b}{\tau^2} - \frac{2b}{\tau^3}t + \frac{b}{\tau^4}t^2)\hat{\boldsymbol{r}} + (\frac{4b}{\tau^2}(1 - \frac{t}{\tau}))\hat{\boldsymbol{\theta}}$$

and when $t = \tau$ we have that

$$\boldsymbol{a} = -\frac{3b}{\tau^2}\hat{\boldsymbol{r}}$$

$\square$

Proof. **2.10** The Lion's velocity can be calculated as

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

So calculating the speed of the Lion knowing that $\dot{\theta}$ is the angular velocity and can be written as $\dot{\theta} = \frac{u}{a}$ where u is the velocity of Daniel and a is the radius then we get that

$$\begin{aligned} U^2 &= \dot{r}^2 + (r\dot{\theta})^2 \\ &= \dot{r}^2 + (\frac{ru}{a})^2 \end{aligned}$$

which can be re-written as

$$\begin{aligned} \dot{r}^2 &= U^2 - (\frac{ru}{a})^2 \\ &= \frac{u^2}{a^2}(\frac{U^2a^2}{u^2} - r^2) \end{aligned}$$

This ODE can be solved as follows

$$\begin{aligned} \dot{r} &= \frac{u}{a}\sqrt{\frac{U^2a^2}{u^2} - r^2} \\ \frac{dr}{dt} &= \frac{u}{a}\sqrt{\frac{U^2a^2}{u^2} - r^2} \\ \int \frac{dr}{\sqrt{\frac{U^2a^2}{u^2} - r^2}} &= \int \frac{u}{a}dt \\ \int \frac{dr}{\sqrt{\frac{U^2a^2}{u^2} - r^2}} &= \frac{ut}{a} \\ sin^{-1}(\frac{ur}{Ua}) + C &= \frac{ut}{a} \end{aligned}$$

the constant of integration C can be determined by knowing that $r = 0$ when $t = 0$ then $C = 0$ and therefore

$$r = \frac{Ua}{u}sin(\frac{ut}{a})$$

For $t = \pi a/2u$ we get that $r = Ua/u$ and if $U \geq u$ then $\frac{U}{u} \geq 1$ so $r = \frac{Ua}{u} \geq a$ which means that the Lion will caught Daniel in $t = \pi a/2u$ if $U \geq u$.
In order to recognize the equation for r as a circle we multiply both sides by r so we get that

$$r^2 = \frac{Ua}{u}r\sin(\frac{ut}{a})$$

now knowing that $r^2 = x^2 + y^2$ and that $r\sin(\frac{ut}{a}) = y$ then

$$x^2 + y^2 = \frac{Ua}{u}y$$

$$y^2 - \frac{Ua}{u}y = -x^2$$

$$y^2 - \frac{2Ua}{2u}y + (\frac{Ua}{2u})^2 = -x^2 + (\frac{Ua}{2u})^2$$

$$(y - \frac{Ua}{2u})^2 = -x^2 + (\frac{Ua}{2u})^2$$

$$x^2 + (y - \frac{Ua}{2u})^2 = (\frac{Ua}{2u})^2$$

which is an equation of a circle with center at $(0, \frac{Ua}{2u})$ and radius $\frac{Ua}{2u}$. When $U = u$ we get that the Lion's path is described by

$$x^2 + (y - \frac{a}{2})^2 = (\frac{a}{2})^2$$

which is a circle with center at $(0, \frac{a}{2})$ and radius $\frac{a}{2}$. □

Proof. **2.11** We know that the velocity in a general form can be written as

$$\boldsymbol{v} = v\boldsymbol{t}$$

where $\boldsymbol{t}$ is the unit tangential vector, and the acceleration can be written as

$$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{t} + \frac{v^2}{\rho}\boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit normal vector, since v is constant then $\frac{dv}{dt} = 0$ so the acceleration can be written as

$$\boldsymbol{a} = \frac{v^2}{\rho}\boldsymbol{n}$$

which only has the normal component, therefore the acceleration is perpendicular to the velocity. □

Proof. **2.14** Knowing that

$$r = b\cosh(\Omega t)$$

then

$$\dot{r} = b\Omega\sinh(\Omega t)$$
$$\ddot{r} = b\Omega^2\cosh(\Omega t)$$

so from the polar velocity equation we have that

$$\begin{aligned}\boldsymbol{v} &= \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \\ &= b\Omega\sinh(\Omega t)\hat{\boldsymbol{r}} + b\Omega\cosh(\Omega t)\hat{\boldsymbol{\theta}} \\ &= b\Omega(\sinh(\Omega t)\hat{\boldsymbol{r}} + \cosh(\Omega t)\hat{\boldsymbol{\theta}})\end{aligned}$$

and the speed of the particle is then

$$\begin{aligned}|v|^2 &= b^2\Omega^2\sinh^2(\Omega t) + b^2\Omega^2\cosh^2(\Omega t)\\ &= b^2\Omega^2\cosh(2\Omega t)\end{aligned}$$

Finally, the acceleration is derived as follows

$$\begin{aligned}\boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + \ 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}\\ &= [b\Omega^2\cosh(\Omega t) - b\Omega^2\cosh(\Omega t)]\hat{\boldsymbol{r}} + \ [0 + 2b\Omega^2\sinh(\Omega t)]\hat{\boldsymbol{\theta}}\\ &= 2b\Omega^2\sinh(\Omega t)\hat{\boldsymbol{\theta}}\end{aligned}$$

Therefore the acceleration direction is circunferential and is given by $\hat{\boldsymbol{\theta}}$. □

Proof. **2.18** As the graph shows P is at $(b\sin(\Omega t), 0)$ from there and by using Pythagoras theorem we can derive that Q is at $(0, \sqrt{a^2 - b^2\sin^2(\Omega t)})$ then C, the center of the link, has coordinates given $(\frac{b}{2}\sin(\Omega t), \frac{\sqrt{a^2-b^2\sin^2(\Omega t)}}{2})$. If we now move the origin $(0,0)$ to C then the coordinates of P are given by $(\frac{b}{2}\sin(\Omega t), -\frac{\sqrt{a^2-b^2\sin^2(\Omega t)}}{2})$ so we have that the sine of the angle between the link and the negative Y-axis is given by

$$\sin(\theta) = \frac{\frac{b}{2}\sin(\Omega t)}{\frac{a}{2}} \quad \text{so} \quad \theta = \sin^{-1}\left(\frac{b\sin\Omega t}{a}\right)$$

now derivating this expression with respect to t we get the Angular velocity as

$$\dot{\theta} = \omega = \frac{b\Omega\cos(\Omega t)}{\sqrt{a^2 - b^2\sin^2(\Omega t)}}$$

Finally, since we know the coordinates for C we can calculate the velocity as

$$\boldsymbol{v} = \frac{\Omega b}{2}\cos(\Omega t)\boldsymbol{i} - \frac{\Omega b^2\sin(\Omega t)\cos(\Omega t)}{2\sqrt{a^2 - b^2 sin^2(\Omega t)}}\boldsymbol{j}$$

so now we can calculate the speed for the link center by

$$\begin{aligned}|v|^2 &= \frac{\Omega^2 b^2}{4}\cos^2(\Omega t) + \frac{\Omega^2 b^4\sin^2(\Omega t)\cos^2(\Omega t)}{4(a^2 - b^2 sin^2(\Omega t))}\\ &= \frac{\Omega^2 b^2}{4}\cos^2(\Omega t)\left(1 + \frac{b^2\sin^2(\Omega t)}{a^2 - b^2 sin^2(\Omega t)}\right)\\ &= \frac{\Omega^2 b^2}{4}\cos^2(\Omega t)\left(\frac{a^2}{a^2 - b^2 sin^2(\Omega t)}\right)\\ &= \frac{\Omega^2 b^2 a^2\cos^2(\Omega t)}{4(a^2 - b^2 sin^2(\Omega t))}\end{aligned}$$

Finally

$$|v| = \frac{\Omega ba\cos(\Omega t)}{2\sqrt{a^2 - b^2 sin^2(\Omega t)}}$$

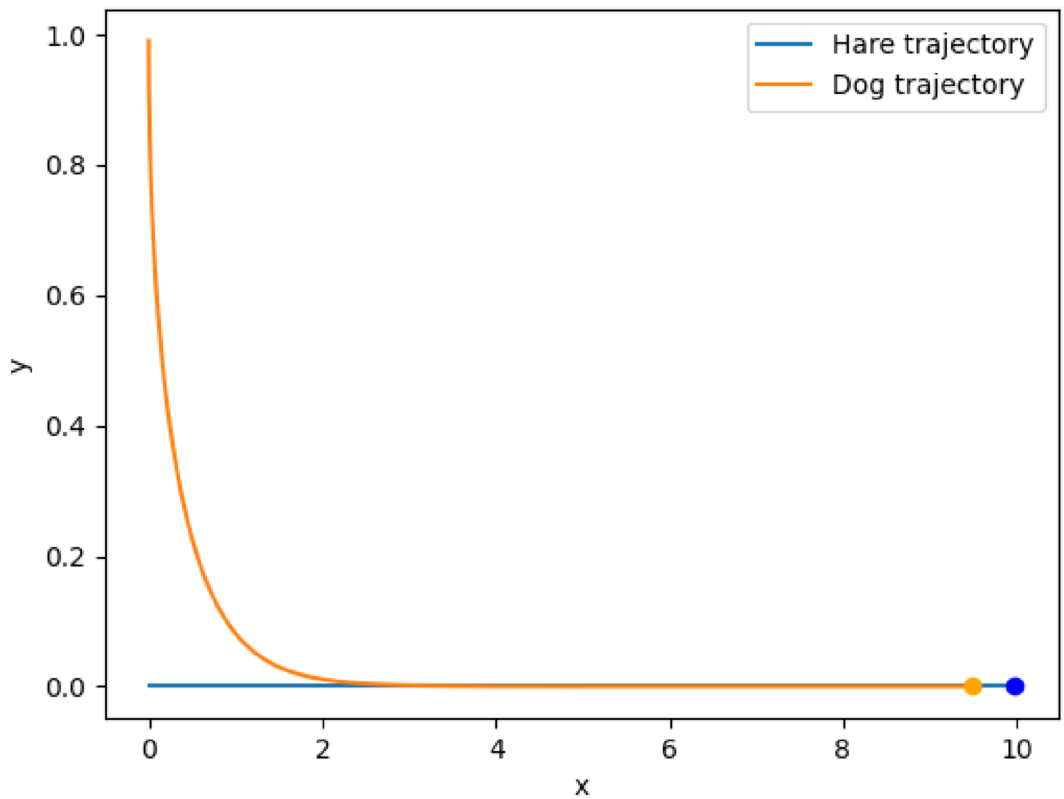
□

Proof. **2.22**

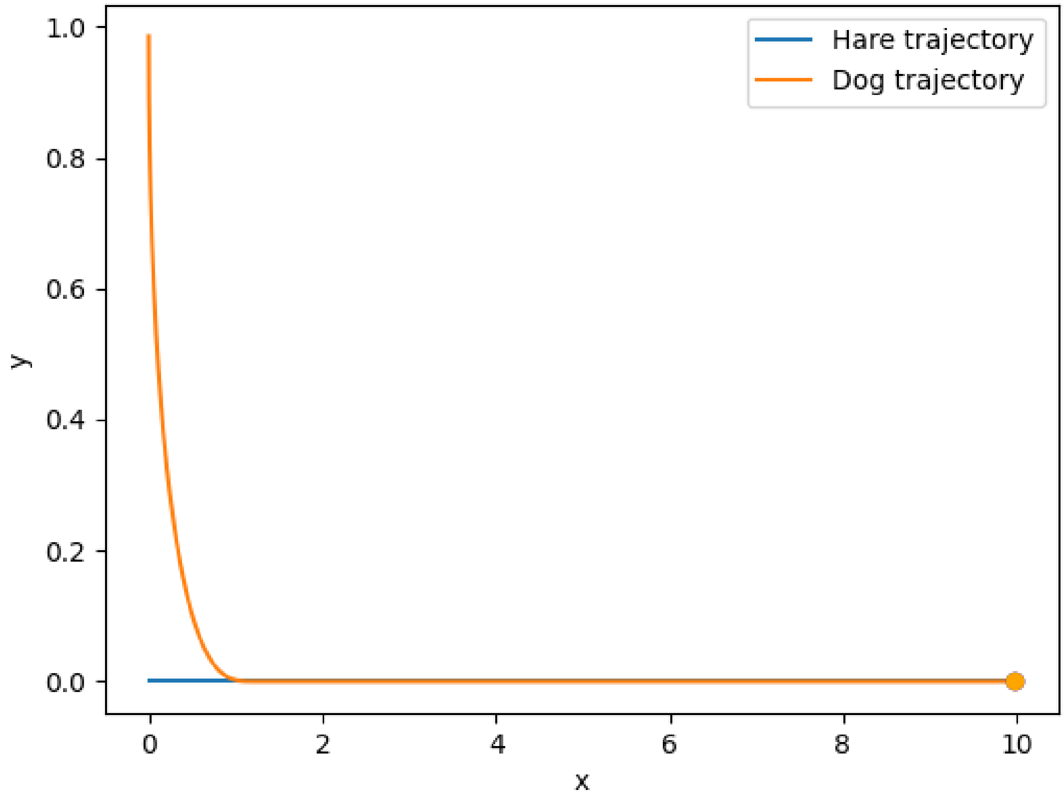
(i) Solving the equations given by computational means (Runge-kutta method)

$$\dot{X} = -\frac{v^D X}{\sqrt{X^2+Y^2}} - v_x^H \qquad \dot{Y} = -\frac{v^D Y}{\sqrt{X^2+Y^2}} - v_y^H$$

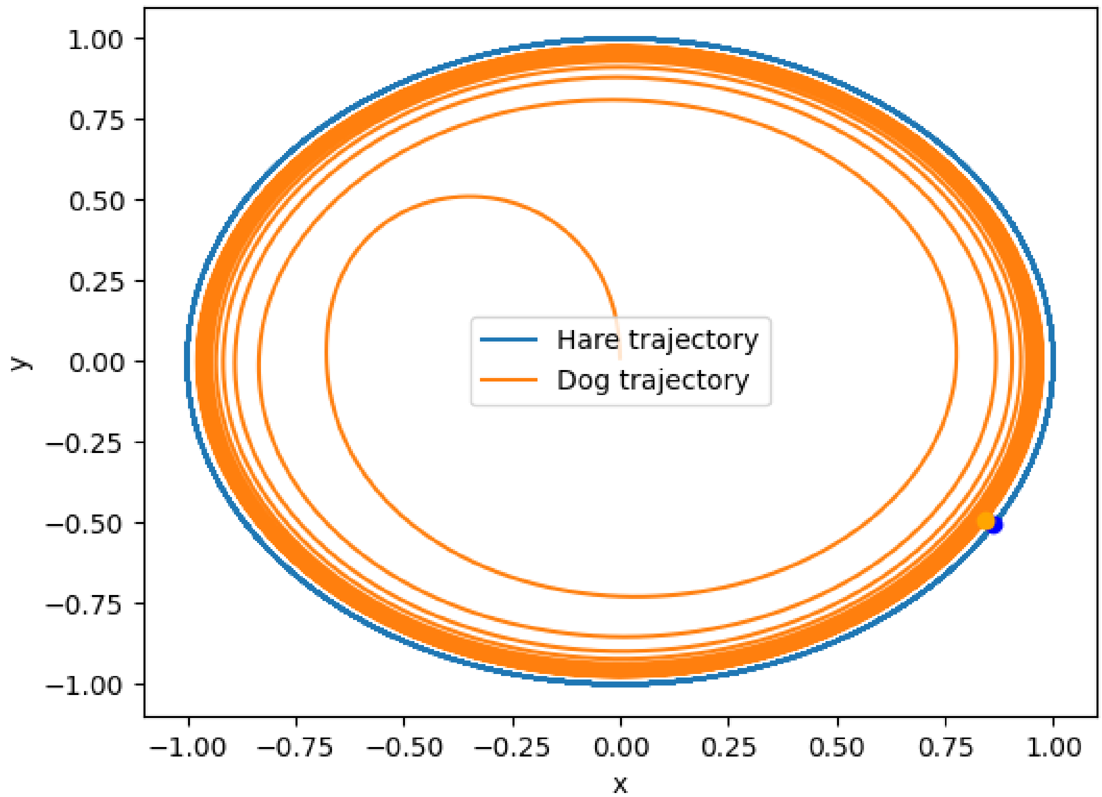
when $v^D = v^H$ we get the following plot



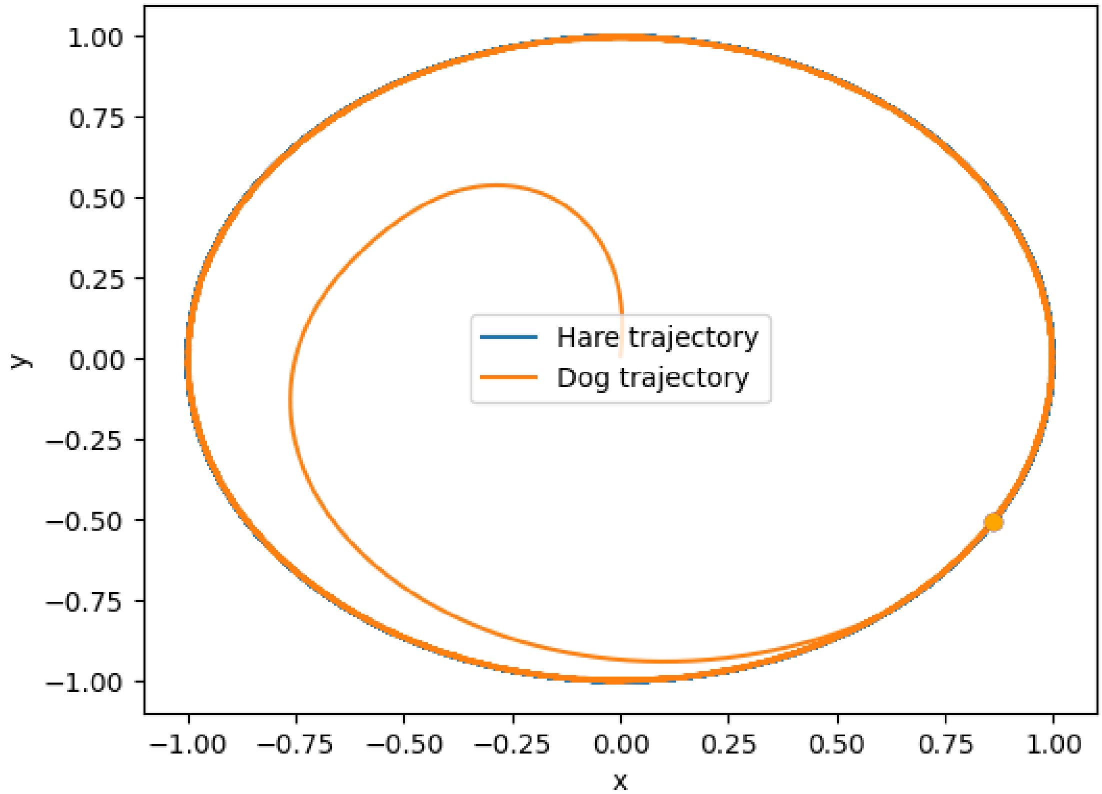
but if $v^D > v^H$



(ii) Now we solve the equations for the case where the Hare has a circular trajectory when $v^D = v^H$ we get the following plot



but if $v^D > v^H$


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
Hare trajectory
Dog trajectory
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x

□