Solved selected problems of Classical Mechanics - Gregory

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Chapter 8 - Non-linear oscillations and phase space

Solution. **8.2** Let us define $s = \omega(\epsilon)t$ and change variables so the equation is written in terms of s then

$$(\omega(\epsilon))^2 x'' + x + \epsilon x^5 = 0$$

By Lindstedt's method, we know there is a solution to this equation in the form of the perturbation series

$$x(s, \epsilon) = x_0(s) + \epsilon x_1(s) + \epsilon^2 x_2(s) + ...$$

and the same for $\omega(\epsilon)$ which is also part of the solution

$$\omega(\epsilon) = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + ...$$

So by replacing these we get that

$$\begin{aligned}(1 + \epsilon\omega_1 + &\epsilon^2\omega_2 + ...)^2(x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + ...) + \\ &(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...) + \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...)^5 = 0\end{aligned}$$

Now we can equate coefficients of powers of ϵ so we get a succession of ODEs and initial contitions, the first two of which are as follows

$$x_0'' + x_0 = 0 \qquad \text{(zero-order)}$$

With $x_0 = 1$ and $x_0' = 0$ when $s = 0$ also we have

$$x_1'' + x_1 = -2\omega x_0'' - x_0^5 \qquad \text{(first-order)}$$

With $x_1 = 0$ and $x_1' = 0$ when $s = 0$. The solution to the zero-order equation is

$$x_0(s) = \cos(s)$$

And we can substitute this in the first-order equation as follows

$$\begin{aligned}x_1'' + x_1 &= 2\omega_1 \cos(s) - \cos^5(s)\\ x_1'' + x_1 &= \frac{1}{8}(16\omega_1 - 5)\cos(s) - \frac{5}{16}\cos(3s) - \frac{1}{16}\cos(5s)\end{aligned}$$

Where we used the identity $16\cos^5(s) = 10\cos(s) + 5\cos(3s) + \cos(5s)$. Also, we see that the first term of the right-hand side must be 0 otherwise $x_1(s)$ wouldn't be periodic then

$$\omega_1 = \frac{5}{16}$$

This equation can now be solved by standard methods and we get that

$$x_1(s) = A\sin(s) + B\cos(s) + \frac{5}{128}\cos(3s) + \frac{1}{384}\cos(5s)$$

Now by replacing with the initial conditions we get that the constant B is

$$\begin{aligned} 0 &= B + \frac{5}{128} + \frac{1}{384} \\ B &= -\frac{1}{24} \end{aligned}$$

And computing $x_1'(s)$ we can get the constant A as

$$\begin{aligned} x_1'(s) &= A\cos(s) - B\sin(s) - \frac{15}{128}\sin(3s) - \frac{5}{384}\sin(5s) \\ 0 &= A\cos(0) - B\sin(0) - \frac{15}{128}\sin(0) - \frac{5}{384}\sin(0) \\ A &= 0 \end{aligned}$$

The solution to the first-order equation is then

$$x_1(s) = -\frac{1}{24}\cos(s) + \frac{5}{128}\cos(3s) + \frac{1}{384}\cos(5s)$$

Finally, we have that when ϵ is small then

$$\omega(\epsilon) = 1 + \frac{5}{16}\epsilon + O(\epsilon^2)$$

and

$$x(s) = \cos(s) + \epsilon\left(-\frac{1}{24}\cos(s) + \frac{5}{128}\cos(3s) + \frac{1}{384}\cos(5s)\right) + O(\epsilon^2)$$

where $s = (1 + \frac{5}{16}\epsilon + O(\epsilon^2))t$. $\square$

Solution. **8.5** Let

$$\dot{x_1} = F_1(x_1, x_2, t) \qquad \dot{x_2} = F_2(x_1, x_2, t)$$

Also, we know that $r = \sqrt{x_1^2 + x_2^2}$ then

$$\dot{r} = \frac{x_1\dot{x_1} + x_2\dot{x_2}}{\sqrt{x_1^2 + x_2^2}}$$
$$\dot{r} = \frac{x_1 F_1 + x_2 F_2}{r}$$

On the other hand, we know that $\tan\theta = \sin\theta / \cos\theta$ then

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta}$$
$$\theta = \tan^{-1}\frac{x_2}{x_1}$$

And the derivative of θ is

$$\dot{\theta} = \frac{x_1\dot{x_2} - x_2\dot{x_1}}{x_1^2 + x_2^2}$$
$$\dot{\theta} = \frac{x_1 F_2 - x_2 F_1}{r^2}$$

Now let us convert the following system to the polar form

$$\dot{x} = -x + y$$
$$\dot{y} = -x - y$$

Then by using the equations we just derived we have that

$$\dot{r} = \frac{x(-x+y) + y(-x-y)}{r}$$
$$\dot{r} = \frac{-(x^2+y^2)}{r} = \frac{-r^2}{r} = -r$$

and

$$\dot{\theta} = \frac{x(-x-y) - y(-x+y)}{r^2}$$
$$\dot{\theta} = \frac{-x^2 - y^2}{r} = \frac{-r^2}{r^2} = -1$$

So we have two ODEs that we can solve with standard methods to obtain

$$r = Ae^{-t} \qquad \theta = -t + B$$

As t goes to infinity we see that θ tends to $-\infty$ and r tends to 0 so the phase path must encircle the origin and end up there no matter which are the initial conditions. Also, the phase paths rotate clockwise because the negative sign we see for $\dot{\theta}$. □

Solution. **8.7** From the damped oscillator equation

$$\ddot{x} + \dot{x} + x = 0$$

we can get two first-order ODEs by replacing variables as follows

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y - x \end{aligned}$$

Now we can derive the polar equations by replacing values in the equations we determined in problem 5. We also use that $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\begin{aligned} \dot{r} &= \frac{xy + y(-y-x)}{r} = \frac{-y^2}{r} \\ \dot{r} &= -\frac{r^2\sin^2\theta}{r} = -r\sin^2\theta \end{aligned}$$

On the other hand for θ we get that

$$\begin{aligned} \dot{\theta} &= \frac{x(-y-x) - y^2}{r^2} = -\frac{x^2 + xy + y^2}{r^2} \\ \dot{\theta} &= -\frac{r^2 + xy}{r^2} = -(1 + \cos\theta\sin\theta) \\ \dot{\theta} &= -\left(1 + \frac{1}{2}\sin 2\theta\right) \end{aligned}$$

Now we want to check that the phase paths encircle the origin infinitely many times clockwise, i.e. we want to show that $\theta \to -\infty$ and $r \to 0$ when $t \to \infty$. From the equation of $\dot{\theta}$ we see that $\dot{\theta} \leq -1/2$ then this implies that

$$\theta \leq -1/2t + A$$

So when $t \to \infty$ we have that $\theta \to -\infty$ no matter the initial conditions i.e. the phase paths encircle the origin clockwise.

Finally, let us check that $r \to 0$ when $t \to \infty$ by solving the equation. So we see that

$$\begin{aligned} \int \frac{dr}{r} &= -\int \sin^2\theta \; dt \\ -\log(r) &= \int \sin^2\theta \; dt \\ -\log(r) &= \int \frac{\sin^2\theta}{\dot{\theta}} \; d\theta \end{aligned}$$

Since $\dot{\theta} \leq -1/2$ then we can do the following analysis

$$\begin{aligned} -\log(r) &= \int \frac{\sin^2\theta}{\dot{\theta}} \; d\theta \\ &\geq -2\int \sin^2\theta \; d\theta \\ &= \frac{1}{2}\sin(2\theta) - \theta + C \end{aligned}$$

Since when $t \to \infty$ then $\theta \to -\infty$ we see that $\frac{1}{2}\sin(2\theta) - \theta + C \to \infty$ no matter the initial conditions (i.e. the value of C). Therefore $-\log(r) \to \infty$ when $t \to \infty$ then $r \to 0$. $\square$

Solution. **8.10** Consider the symmetrical predator-prey equations

$$\dot{x} = x - xy$$
$$\dot{y} = xy - y$$

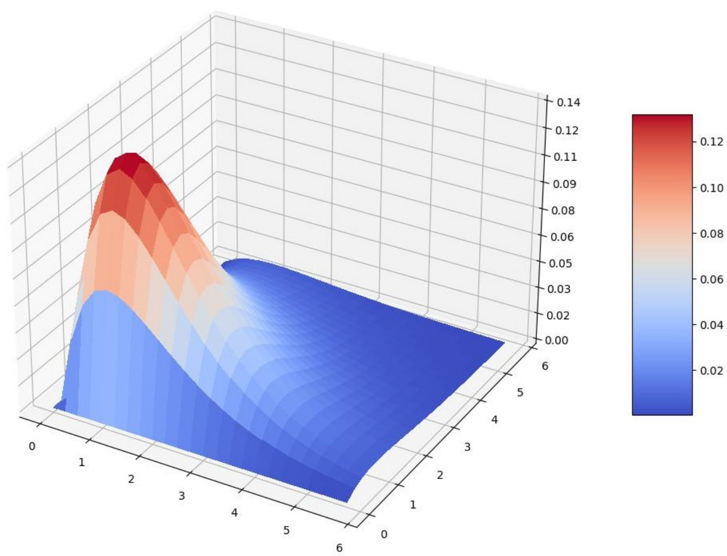
We want to determine the phase path equation so we write the path gradient as follows

$$\frac{dx}{dy} = \frac{x - xy}{xy - y} = \frac{x(1-y)}{y(x-1)}$$

This is a separable ODE so we can solve it

$$\begin{aligned} \int \frac{x-1}{x}\, dx &= \int \frac{1-y}{y}\, dy \\ x - \log(x) &= \log(y) - y + B \\ e^{x-\log(x)} &= e^{\log(y)-y+B} \\ x^{-1}e^{x} &= e^{B} y e^{-y} \\ A &= (ye^{-y})(xe^{-x}) \end{aligned}$$

Where we named $A = 1/e^{B}$. Let us now consider the equation $z = (ye^{-y})(xe^{-x})$ and let us plot it using a computer program. We get the following plot



Where we see that if we intersect the plot with a horizontal plane, we get a closed curve that encircles the equilibrium point (1,1) as we wanted. $\square$