Solved selected problems of Classical Mechanics - Gregory

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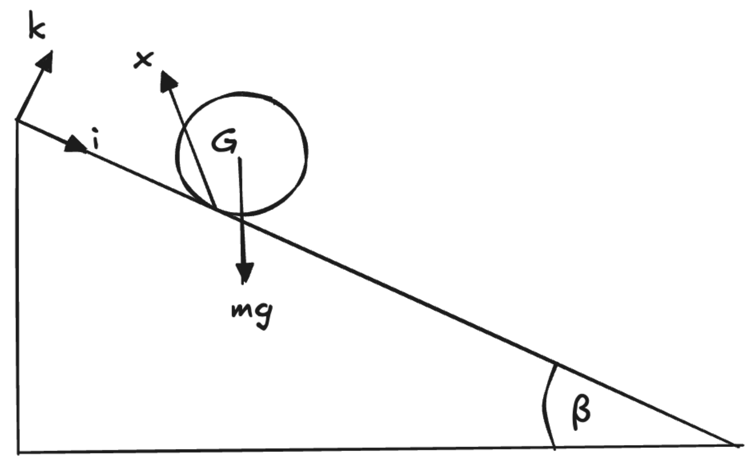
Chapter 19 - Problems in rigid body dynamics

Solution. 19.1 Let us consider the following system



Then from the governing equations we have that

$$\begin{aligned}
m\dot{\boldsymbol{V}} &= \boldsymbol{X} + mg\sin\beta\boldsymbol{i} - mg\cos\beta\boldsymbol{k}\\
\dot{\boldsymbol{L}}_G &= (-b\boldsymbol{k}) \times \boldsymbol{X}
\end{aligned}$$

Where $\boldsymbol{X}$ is the reaction of the table exerted on the ball. Using that $\boldsymbol{L}_G = A\boldsymbol{\omega}$ and eliminating the reaction $\boldsymbol{X}$ we get that

$$\begin{aligned}
A\dot{\boldsymbol{\omega}} &= (-b\boldsymbol{k}) \times (m\dot{\boldsymbol{V}} - mg\sin\beta\boldsymbol{i} + mg\cos\beta\boldsymbol{k})\\
&= b(m\dot{\boldsymbol{V}} - mg\sin\beta\boldsymbol{i} + mg\cos\beta\boldsymbol{k}) \times \boldsymbol{k}\\
&= mb\dot{\boldsymbol{V}} \times \boldsymbol{k} - mgb\sin\beta\boldsymbol{i} \times \boldsymbol{k}\\
&= mb\dot{\boldsymbol{V}} \times \boldsymbol{k} + mgb\sin\beta\boldsymbol{j}
\end{aligned}$$

On integrating with respect to t we get that

$$A\boldsymbol{\omega} + mb\boldsymbol{k} \times \boldsymbol{V} = \boldsymbol{C} + mgbt\sin\beta\boldsymbol{j}$$

Where $\boldsymbol{C}$ is a constant vector. In particular if we take the scalar product of this equation with $\boldsymbol{k}$, we have that

$$\begin{gathered} A\boldsymbol{\omega}\cdot\boldsymbol{k} + mb(\boldsymbol{k}\times\boldsymbol{V})\cdot\boldsymbol{k} = \boldsymbol{C}\cdot\boldsymbol{k} + mgbt\sin\beta\boldsymbol{j}\cdot\boldsymbol{k} \\ A\boldsymbol{\omega}\cdot\boldsymbol{k} = n \end{gathered}$$

Where n is a constant. Also, we used that $\boldsymbol{j}\cdot\boldsymbol{k} = 0$ and the triple product is also 0. Therefore we see that the component of $\boldsymbol{\omega}$ in the direction of $\boldsymbol{k}$ (perpendicular to the inclined plane) is constant independent of the motion of the ball.

If we now consider that the ball is rolling, then the particle C in contact with the plane has zero velocity, so the rolling condition give us

$$\boldsymbol{V} + b\boldsymbol{k}\times\boldsymbol{\omega} = \boldsymbol{0}$$

Now, by cross-multiplying the conservation principle by $\boldsymbol{k}$ we have that

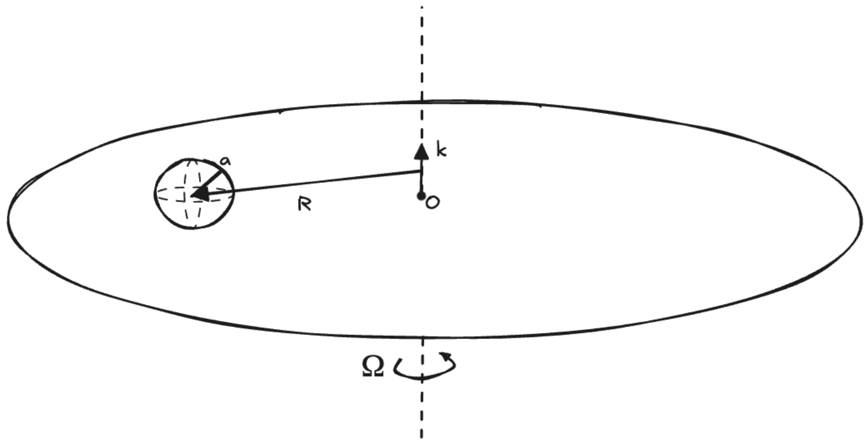
$$\begin{aligned} A\boldsymbol{k}\times\boldsymbol{\omega} &= \boldsymbol{k}\times\boldsymbol{C} + mgbt\sin\beta\boldsymbol{k}\times\boldsymbol{j} - mb\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{V}) \\ &= \boldsymbol{k}\times\boldsymbol{C} - mgbt\sin\beta\boldsymbol{i} - mb((\boldsymbol{k}\cdot\boldsymbol{V})\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{V}) \\ &= \boldsymbol{k}\times\boldsymbol{C} - mgbt\sin\beta\boldsymbol{i} + mb\boldsymbol{V} \end{aligned}$$

Therefore by joining this equation and the rolling condition equation we get that

$$\begin{gathered} \boldsymbol{V} + \frac{mb^2}{A}\boldsymbol{V} = -\frac{b}{a}\boldsymbol{k}\times\boldsymbol{C} + mgbt\sin\beta\boldsymbol{i} \\ \boldsymbol{V} + \frac{mb^2}{A}\boldsymbol{V} = \frac{b}{a}\boldsymbol{C}\times\boldsymbol{k} + mgbt\sin\beta\boldsymbol{i} \end{gathered}$$

This shows that $\boldsymbol{V}$ is dependent on time. If we integrate again this equation we see that the rolling motion depends on t^2 and therefore the path of the ball must be a parabola. $\square$

Solution. 19.2 Let us consider the following system



Then from the governing equations we have that

$$
\begin{aligned}
m\dot{\boldsymbol{V}} &= \boldsymbol{X} - mg\boldsymbol{k} \\
\dot{\boldsymbol{L}}_G &= (-a\boldsymbol{k}) \times \boldsymbol{X}
\end{aligned}
$$

Where $\boldsymbol{X}$ is the reaction of the turntable exerted on the ball. Using that $\boldsymbol{L}_G = A\boldsymbol{\omega}$ where $A = \frac{2}{5}ma^2$ is the moment of inertia of the ball, then eliminating the reaction $\boldsymbol{X}$ we get that

$$
\begin{aligned}
A\dot{\boldsymbol{\omega}} &= (-a\boldsymbol{k}) \times (m\dot{\boldsymbol{V}} + mg\boldsymbol{k}) \\
&= a(m\dot{\boldsymbol{V}} + mg\boldsymbol{k}) \times \boldsymbol{k} \\
&= ma\dot{\boldsymbol{V}} \times \boldsymbol{k}
\end{aligned}
$$

On integrating with respect to t we get that

$$
A\boldsymbol{\omega} + ma\boldsymbol{k} \times \boldsymbol{V} = \boldsymbol{C}
$$

Where $\boldsymbol{C}$ is a constant vector. In particular if we take the scalar product of this equation with $\boldsymbol{k}$, we have that

$$
\begin{aligned}
A\boldsymbol{\omega} \cdot \boldsymbol{k} + ma(\boldsymbol{k} \times \boldsymbol{V}) \cdot \boldsymbol{k} &= \boldsymbol{C} \cdot \boldsymbol{k} \\
A\boldsymbol{\omega} \cdot \boldsymbol{k} &= n
\end{aligned}
$$

Where n is a constant. Also, we used that the triple product is also 0. Therefore we see that in any motion of the ball, the vertical spin $\boldsymbol{\omega} \cdot \boldsymbol{k}$ is constant.

If we now consider that the ball is rolling, then the particle C in contact with the turntable has zero velocity, so the rolling condition give us

$$
\boldsymbol{V}_C = \boldsymbol{V} + a\boldsymbol{k} \times \boldsymbol{\omega} = \boldsymbol{0}
$$

Also, from the equation for $A\dot{\boldsymbol{\omega}}$ we derived above, replacing the value of A we have that

$$\frac{2}{5}ma^2\dot{\boldsymbol{\omega}} = ma\dot{\boldsymbol{V}} \times \boldsymbol{k}$$
$$\frac{2}{5}a\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{V}} \times \boldsymbol{k}$$

And cross-multiplying this equiation by $\boldsymbol{k}$ we obtain an expression for $\dot{\boldsymbol{V}}$ as follows

$$\frac{2}{5}a\boldsymbol{k} \times \dot{\boldsymbol{\omega}} = \boldsymbol{k} \times (\dot{\boldsymbol{V}} \times \boldsymbol{k})$$
$$\frac{2}{5}a\boldsymbol{k} \times \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{V}}(\boldsymbol{k} \cdot \boldsymbol{k}) - \boldsymbol{k}(\boldsymbol{k} \cdot \dot{\boldsymbol{V}})$$
$$\frac{2}{5}a\boldsymbol{k} \times \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{V}}$$

If we assume the position of the ball's center of mass is determined by a vector $\boldsymbol{R}$ from the axis $\{O, \boldsymbol{k}\}$ then the velocity $\boldsymbol{V}_C$ of the particle C is also

$$\boldsymbol{V}_C = \Omega\boldsymbol{k} \times \boldsymbol{R}$$

Hence by joining the rolling condition and this equation we have that

$$\boldsymbol{V} + a\boldsymbol{k} \times \boldsymbol{\omega} = \Omega\boldsymbol{k} \times \boldsymbol{R}$$

Derivating this expression and replacing the value for $a\boldsymbol{k} \times \dot{\boldsymbol{\omega}}$ we finnally get that

$$\dot{\boldsymbol{V}} + \frac{5}{2}\dot{\boldsymbol{V}} = \Omega\boldsymbol{k} \times \boldsymbol{V}$$
$$\frac{7}{2}\dot{\boldsymbol{V}} = \Omega\boldsymbol{k} \times \boldsymbol{V}$$
$$\dot{\boldsymbol{V}} = \frac{2}{7}\Omega\boldsymbol{k} \times \boldsymbol{V}$$

Integrating this equation we get that

$$\dot{\boldsymbol{R}} = \frac{2}{7}\Omega\boldsymbol{k} \times \boldsymbol{R} + \boldsymbol{C}$$

Where $\boldsymbol{C}$ is some constant vector.

Finally, let us suppose that the unit vector $\boldsymbol{i}$ is in the direction of $\boldsymbol{R}$ at the moment the ball is released. Then by the initial conditions we see that $\boldsymbol{R} = b\boldsymbol{i}$ and $\dot{\boldsymbol{R}} = \Omega b\boldsymbol{j}$ since the ball is at rest with respect to the turntable so the constant of integration becomes

$$\Omega b\boldsymbol{j} = \frac{2}{7}\Omega b\boldsymbol{j} + \boldsymbol{C}$$
$$\boldsymbol{C} = \Omega b\left(1 - \frac{2}{7}\right)\boldsymbol{j}$$
$$\boldsymbol{C} = \frac{5}{7}\Omega b\boldsymbol{j}$$

Therefore

$$\dot{\boldsymbol{R}} = \frac{2}{7}\Omega \boldsymbol{k} \times \boldsymbol{R} + \frac{5}{7}\Omega b \boldsymbol{j}$$

To solve this equation let us introduce a new variable $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = (\boldsymbol{R} + \frac{5}{2}b\boldsymbol{i})$$

Hence

$$\dot{\boldsymbol{A}} = \frac{2}{7}\Omega \boldsymbol{k} \times \boldsymbol{A}$$

So now we can solve this equation by assumming that $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j}$ and $\dot{\boldsymbol{A}} = \dot{A}_x\boldsymbol{i} + \dot{A}_y\boldsymbol{j}$ as follows

$$\begin{aligned}
\dot{A}_x\boldsymbol{i} + \dot{A}_y\boldsymbol{j} &= \frac{2}{7}\Omega \boldsymbol{k} \times (A_x\boldsymbol{i} + A_y\boldsymbol{j}) \\
\dot{A}_x\boldsymbol{i} + \dot{A}_y\boldsymbol{j} &= \frac{2}{7}\Omega A_x\boldsymbol{j} - \frac{2}{7}\Omega A_y\boldsymbol{i}
\end{aligned}$$

Therefore we get the following system of differential equations

$$\begin{aligned}
\dot{A}_x &= -\frac{2}{7}\Omega A_y \\
\dot{A}_y &= \frac{2}{7}\Omega A_x
\end{aligned}$$

Where the solution is

$$\begin{aligned}
A_x &= C_1 \cos\left(\frac{2}{7}\Omega t\right) - C_2 \sin\left(\frac{2}{7}\Omega t\right) \\
A_y &= C_1 \sin\left(\frac{2}{7}\Omega t\right) - C_2 \cos\left(\frac{2}{7}\Omega t\right)
\end{aligned}$$

When we apply the initial conditions we get that

$$\frac{7}{2}b = C_1 \qquad 0 = C_2$$

Therefore

$$\begin{aligned}
A_x &= \frac{7}{2}b\cos\left(\frac{2}{7}\Omega t\right) \\
A_y &= \frac{7}{2}b\sin\left(\frac{2}{7}\Omega t\right)
\end{aligned}$$

But returning to the original variable we get that

$$\begin{aligned}
R_x &= \frac{7}{2}b\cos\left(\frac{2}{7}\Omega t\right) - \frac{5}{2}b \\
R_y &= \frac{7}{2}b\sin\left(\frac{2}{7}\Omega t\right)
\end{aligned}$$

These are the equations for a circular path which has a centre displaced $\frac{5}{2}b$ from O and has a radius of $\frac{7}{2}b$.

$\square$