Solved selected problems of Classical Mechanics - Gregory

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Chapter 14 - Hamilton's equations and phase space

Solution. 14.1 We want to find the Legendre transform $G(v_1, v_2, w)$ of the function

$$F(u_1, u_2, w) = 2u_1^2 - 3u_1u_2 + u_2^2 + 3wu_1$$

where w is a passive variable. Then we have that

$$v_1 = \frac{\partial F}{\partial u_1} = 4u_1 - 3u_2 + 3w$$
$$v_2 = \frac{\partial F}{\partial u_2} = -3u_1 + 2u_2$$

So by replacing $u_2 = v_2/2 + 3u_1/2$ in the formula for v_1 we get that the inverse formula for u_1 is

$$u_1 = -2v_1 - 3v_2 + 6w$$

Plugging this value in the formula for u_2 gives us

$$u_2 = -3v_1 - 4v_2 + 9w$$

Using the Legendre transform equation we have that

$$\begin{aligned} G(v_1, v_2, w) &= u_1v_1 + u_2v_2 - F(u_1, u_2, w) \\ &= (-2v_1 - 3v_2 + 6w)v_1 + (-3v_1 - 4v_2 + 9w)v_2 \\ &\quad - (2(-2v_1 - 3v_2 + 6w)^2 \\ &\quad - 3(-2v_1 - 3v_2 + 6w)(-3v_1 - 4v_2 + 9w) \\ &\quad + (-3v_1 - 4v_2 + 9w)^2 \\ &\quad + 3w(-2v_1 - 3v_2 + 6w)) \\ &= -v_1^2 - 3v_1v_2 - 2v_2^2 + 6v_1w + 9v_2w - 9w^2 \end{aligned}$$

Finally, we want to check that $\partial F/\partial w = -\partial G/\partial w$ hence

$$\frac{\partial F}{\partial w} = 3u_1 = -6v_1 - 9v_2 + 18w$$
$$-\frac{\partial G}{\partial w} = -(6v_1 + 9v_2 - 18w)$$

□

Solution. 14.2 First, we need to determine the Lagrangian of the system. We know the coordinates of the particle at each point so we can compute the velocity of the particle P as follows

$$\begin{aligned}
v^2 &= (-a\dot{\theta}\sin\theta)^2 + (a\dot{\theta}\cos\theta)^2 + (b\dot{\theta})^2 \\
v^2 &= \dot{\theta}^2(a^2(\sin^2\theta + \cos^2\theta) + b^2) \\
v^2 &= \dot{\theta}^2(a^2 + b^2)
\end{aligned}$$

Now we can write the Lagrangian of the system as

$$L = \frac{1}{2}m\dot{\theta}^2(a^2 + b^2) - mgb\theta$$

So taking θ as our generalized coordinate we can compute the generalized momenta as follows

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}m(a^2 + b^2)$$

Inverting the formula we get that

$$\dot{\theta} = \frac{p_\theta}{m(a^2 + b^2)}$$

The Hamiltonian H is then given by

$$\begin{aligned}
H &= \dot{\theta} \cdot p_\theta - L \\
&= \frac{p_\theta^2}{m(a^2 + b^2)} - \frac{1}{2}m\left(\frac{p_\theta}{m(a^2 + b^2)}\right)^2 (a^2 + b^2) + mgb\theta \\
&= \frac{p_\theta^2}{2m(a^2 + b^2)} + mgb\theta
\end{aligned}$$

Finally, from H we can find Hamilton's equations as follows

$$\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m(a^2 + b^2)} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mgb
\end{aligned}$$

$\square$

Solution. 14.3 In Cartesian coordinates the velocity of a projectile is given by $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ where x, y and z are the projectile coordinates. Since the system is conservative and standard we can compute the Hamiltonian as $H = T + V$. On the other hand, the Lagrangian of a projectile of mass m moving under uniform gravity is

$$L = \frac{1}{2}mv^2 - mgz = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

So the generalized momenta are given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{hence} \quad \dot{x} = \frac{p_x}{m} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad \text{hence} \quad \dot{y} = \frac{p_y}{m} \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \text{hence} \quad \dot{z} = \frac{p_z}{m} \end{aligned}$$

Then the Hamiltonian can be computed as follows

$$\begin{aligned} H &= T + V \\ &= \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + mgz \\ &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz \end{aligned}$$

We can find now Hamilton's equations using that

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Hence we get that

$$\begin{aligned} \dot{x} &= \frac{p_x}{m} \qquad \dot{y} = \frac{p_y}{m} \qquad \dot{z} = \frac{p_z}{m} \\ \dot{p}_x &= 0 \qquad \dot{p}_y = 0 \qquad \dot{p}_z = -mg \end{aligned}$$

Finally, since x and y do not appear in the Hamiltonian they are the cyclic coordinates. □

Solution. 14.4 From the velocity diagram shown in Figure 11.7 the velocity of the spherical pendulum is given by $v^2 = (a\dot{\theta})^2 + (a\sin\theta\dot{\phi})^2$ and the potential energy is $V = -mga\cos\theta$. So the Lagrangian of the spherical pendulum of mass m moving under uniform gravity is

$$L = \frac{1}{2}mv^2 - (-mga\cos\theta) = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mga\cos\theta$$

So the generalized momenta are given by

$$\begin{aligned} p_\theta = \frac{\partial L}{\partial\dot{\theta}} &= ma^2\dot{\theta} \quad \text{hence} \quad \dot{\theta} = \frac{p_\theta}{ma^2} \\ p_\phi = \frac{\partial L}{\partial\dot{\phi}} &= ma^2\sin^2\theta\dot{\phi} \quad \text{hence} \quad \dot{\phi} = \frac{p_\phi}{ma^2\sin^2\theta} \end{aligned}$$

The system is conservative and standard so we can compute the Hamiltonian as $H = T + V$ then

$$\begin{aligned} H &= T + V \\ &= \frac{1}{2}ma^2\left(\left(\frac{p_\theta}{ma^2}\right)^2 + \sin^2\theta\left(\frac{p_\phi}{ma^2\sin^2\theta}\right)^2\right) - mga\cos\theta \\ &= \frac{p_\theta^2}{2ma^2} + \frac{p_\phi^2}{2ma^2\sin^2\theta} - mga\cos\theta \end{aligned}$$

We can find now Hamilton's equations from

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Hence we get that

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{ma^2} \qquad \dot{\phi} = \frac{p_\phi}{ma^2\sin^2\theta} \\ \dot{p}_\theta &= \frac{p_\phi^2\cos\theta}{ma^2\sin^3\theta} - mga\sin\theta \qquad \dot{p}_\phi = 0 \end{aligned}$$

Finally, since ϕ doesn't appear in the Hamiltonian it is a cyclic coordinate. Therefore p_ϕ is conserved. $\square$

Solution. 14.6 The velocity in this case has two components, the tangential velocity $(\boldsymbol{a}-\boldsymbol{Z})\dot{\boldsymbol{\theta}}$ and the velocity at which the string is shortening $\dot{\boldsymbol{Z}}$ hence the total squared velocity is given by

$$\begin{aligned} v^2 &= ((\boldsymbol{a}-\boldsymbol{Z})\dot{\boldsymbol{\theta}} + \dot{\boldsymbol{Z}})^2 \\ &= (a-Z)^2\dot{\theta}^2 + 2((\boldsymbol{a}-\boldsymbol{Z})\dot{\boldsymbol{\theta}})\cdot(\dot{\boldsymbol{Z}}) + \dot{Z}^2 \end{aligned}$$

but we know that

$$((\boldsymbol{a}-\boldsymbol{Z})\dot{\boldsymbol{\theta}})\cdot(\dot{\boldsymbol{Z}}) = \|(a-Z)\dot{\theta}\cdot\dot{Z}\|\cos(\pi/2) = 0$$

Hence $v^2 = (a-Z)^2\dot{\theta}^2 + \dot{Z}^2$. On the other hand, the potential energy is given by $V = -mg(a-Z)\cos\theta$. So the Lagrangian of the pendulum with a shortening string of mass m moving under uniform gravity is

$$\begin{aligned} L &= \frac{1}{2}mv^2 - (-mg(a-Z)\cos\theta) \\ &= \frac{1}{2}m((a-Z)^2\dot{\theta}^2 + \dot{Z}^2) + mg(a-Z)\cos\theta \end{aligned}$$

Then the generalized momenta is given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m(a-Z)^2\dot{\theta} \quad \text{hence} \quad \dot{\theta} = \frac{p_\theta}{m(a-Z)^2}$$

The system in this case is non-conservative since the moving constraint forces do work so we cannot compute the Hamiltonian as the total energy hence we compute it as follows

$$\begin{aligned} H &= \dot{\theta}p_\theta - L \\ &= \frac{p_\theta^2}{m(a-Z)^2} - \frac{1}{2}m\left(\frac{p_\theta^2}{m^2(a-Z)^2} + \dot{Z}^2\right) - mg(a-Z)\cos\theta \\ &= \frac{p_\theta^2}{2m(a-Z)^2} - \frac{1}{2}m\dot{Z}^2 - mg(a-Z)\cos\theta \end{aligned}$$

Now, we can find Hamilton's equations from

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Hence we have that

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{m(a-Z)^2} \\ \dot{p}_\theta &= -mg(a-Z)\sin\theta \end{aligned}$$

Finally, the Hamiltonian H is not conserved since it is dependent on time through $Z(t)$. $\square$

Solution. 14.9 Let

$$J[\boldsymbol{q}(t), \boldsymbol{p}(t)] = \int_{t_0}^{t_1} \big(H(\boldsymbol{q}, \boldsymbol{p}, t) - \dot{\boldsymbol{q}} \cdot \boldsymbol{p}\big)\ dt$$

and let us suppose $\boldsymbol{q}^*$ and $\boldsymbol{p}^*$ are extremals of J hence they make J stationary and they satisfy the following Euler-Lagrange equations simultaneously

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\boldsymbol{q}}}\big(H(\boldsymbol{q}, \boldsymbol{p}, t) - \dot{\boldsymbol{q}} \cdot \boldsymbol{p}\big)\right) - \frac{\partial}{\partial \boldsymbol{q}}\big(H(\boldsymbol{q}, \boldsymbol{p}, t) - \dot{\boldsymbol{q}} \cdot \boldsymbol{p}\big) &= 0 \\ -\frac{d\boldsymbol{p}}{dt} - \frac{\partial}{\partial \boldsymbol{q}} H(\boldsymbol{q}, \boldsymbol{p}, t) &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\boldsymbol{p}}}\big(H(\boldsymbol{q}, \boldsymbol{p}, t) - \dot{\boldsymbol{q}} \cdot \boldsymbol{p}\big)\right) - \frac{\partial}{\partial \boldsymbol{p}}\big(H(\boldsymbol{q}, \boldsymbol{p}, t) - \dot{\boldsymbol{q}} \cdot \boldsymbol{p}\big) &= 0 \\ -\frac{\partial}{\partial \boldsymbol{p}} H(\boldsymbol{q}, \boldsymbol{p}, t) + \dot{\boldsymbol{q}} &= 0 \end{aligned}$$

Hence $\boldsymbol{q}^*$ and $\boldsymbol{p}^*$ satisfy

$$\begin{aligned} \dot{\boldsymbol{p}} &= -\frac{\partial}{\partial \boldsymbol{q}} H(\boldsymbol{q}, \boldsymbol{p}, t) \\ \dot{\boldsymbol{q}} &= \frac{\partial}{\partial \boldsymbol{p}} H(\boldsymbol{q}, \boldsymbol{p}, t) \end{aligned}$$

Which are Hamilton's equations. □

Solution. 14.10 Suppose there is a point $\mathcal{P}$ in the phase space that attracts points in a nearby region $\mathcal{R}$. Then eventually the points that lay in $\mathcal{R}$ must lie in a $2n$-dimensional "sphere" of a small "radius" enclosing the point $\mathcal{P}$. But the volume of this sphere will decrease as time passes since the point will attract the nearby points "closer" so that the original volume of $\mathcal{R}$ cannot be preserved. This is contrary to Liouville's theorem and so asymptotically stable equilibrium points cannot occur in Hamiltonian dynamics. □

Solution. 14.11 Let $\dot{\boldsymbol{x}} = (\dot{q}, \dot{v})$ represent a two dimensional vector with coordinates $\dot{q}$ and $\dot{v}$. We want to determine the area $a(t)$ of a region $\mathcal{R}_t$ that evolves with time. From Liouville's theorem proof, we know that

$$\frac{da}{dt} = \int \boldsymbol{\nabla} \cdot \dot{\boldsymbol{x}} \; dqdv$$

Hence

$$\begin{aligned}
\frac{da}{dt} &= \int_{\mathcal{R}_t} \left[\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{v}}{\partial v}\right] \; dqdv \\
&= \int_{\mathcal{R}_t} \left[\frac{\partial}{\partial q}(v) + \frac{\partial}{\partial v}(-3q - 4v)\right] \; dqdv \\
&= \int_{\mathcal{R}_t} -4 \; dqdv \\
&= -4a
\end{aligned}$$

Where we used that an area element is given by $da = dqdv$, thus

$$\begin{aligned}
\frac{da}{dt} &= -4a \\
\int \frac{da}{a} &= -4 \int dt \\
\log(a) &= -4t + C \\
a(t) &= a(0)e^{-4t}
\end{aligned}$$

Where we used that when $t = 0$ we have that $a(0) = e^{C}$.

Finally, we see that the area shrinks with time because of the exponentially decreasing factor. Also, the equation has a damping term which implies that the motion doesn't come from a Lagrangian so the fact that the area changes with time doesn't contradict Liouville's theorem since Liouville's theorem can only be applied to Hamiltonian systems. $\square$

Solution. 14.12 Statistical Equilibrium in this case means that a region of $(\boldsymbol{q}, \boldsymbol{p})$-space is "well mixed" meaning that if we split the region in half then both sides have as many points as the other otherwise is not "well mixed".

On the other hand, we have that the integrals defined are actually the mass of points of the regions $\mathcal{R}_0$ and $\mathcal{R}_t$ respectively so they must be equal, otherwise, since the volume is preserved (because it is a Hamiltonian system) the number of points must change between them which contradicts the fact that the system is in Statistical Equilibrium.

Finally, if we set $\rho(\boldsymbol{q}, \boldsymbol{p}) = \rho_0$ where ρ_0 is a constant i.e. it's a uniform density then we have that

$$\rho_0 \int_{\mathcal{R}_0} dv = \rho_0 \int_{\mathcal{R}_t} dv$$

which is true since we are dealing with a Hamiltonian system and hence the volume between $\mathcal{R}_0$ and $\mathcal{R}_t$ must be preserved. $\square$

Solution. 14.13

(i) In the two-body gravitation problem where $E < 0$ the origin is set at some point O and we know that the center of mass of the two-body system is moving away from O at a constant speed, then the generalized coordinates $\boldsymbol{q}$ of the masses and the center of mass with respect to O increase without limit i.e. they are not bounded. Therefore the energy surfaces are not bounded.

(ii) The two-body gravitation system viewed from the zero momentum frame implies that we are seeing the system from the center of mass and hence the two masses are moving in bounded orbits where both $\boldsymbol{q}$ and $\boldsymbol{p}$ are bounded. Therefore the energy surfaces are bounded.

(iii) In the three-body gravitation system there is no analytical solution available which implies that the system behaves chaotically so even though we see the system from the zero momentum frame (i.e. the center of mass) we cannot be certain that both the generalized coordinates $\boldsymbol{q}$ and the generalized momenta $\boldsymbol{p}$ are going to be bounded. Therefore the energy surfaces are not bounded.

Given that the solar system is a multi-body problem then the energy surfaces are not bounded and therefore the solar system doesn't have the recurrence property.

$\square$

Solution. 14.14
Let $u(\boldsymbol{q},\boldsymbol{p}), v(\boldsymbol{q},\boldsymbol{p})$ be any two functions of position in the phase space $(\boldsymbol{q},\boldsymbol{p})$. We want to prove the following properties
Algebraic properties

(i) We want to prove that $[u,u]=0$ hence by definition

$$\begin{aligned}[u,u] &= \text{grad}_{\boldsymbol{q}} u \cdot \text{grad}_{\boldsymbol{p}} u - \text{grad}_{\boldsymbol{p}} u \cdot \text{grad}_{\boldsymbol{q}} u \\ &= 0\end{aligned}$$

(ii) We want to prove that $[u,v]=-[v,u]$ hence using the definition we have that

$$\begin{aligned}[v,u] &= \text{grad}_{\boldsymbol{q}} v \cdot \text{grad}_{\boldsymbol{p}} u - \text{grad}_{\boldsymbol{p}} v \cdot \text{grad}_{\boldsymbol{q}} u \\ &= -(\text{grad}_{\boldsymbol{q}} u \cdot \text{grad}_{\boldsymbol{p}} v - \text{grad}_{\boldsymbol{p}} u \cdot \text{grad}_{\boldsymbol{q}} v) \\ &= -[u,v]\end{aligned}$$

(ii) We want to prove that $[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1[u_1,v]+\lambda_2[u_2,v]$ hence we have that

$$\begin{aligned}[\lambda_1 u_1 + \lambda_2 u_2, v] &= \text{grad}_{\boldsymbol{q}}(\lambda_1 u_1 + \lambda_2 u_2) \cdot \text{grad}_{\boldsymbol{p}} v \\ &\quad - \text{grad}_{\boldsymbol{p}}(\lambda_1 u_1 + \lambda_2 u_2) \cdot \text{grad}_{\boldsymbol{q}} v \\ &= (\lambda_1 \text{grad}_{\boldsymbol{q}} u_1 + \lambda_2 \text{grad}_{\boldsymbol{q}} u_2) \cdot \text{grad}_{\boldsymbol{p}} v \\ &\quad - (\lambda_1 \text{grad}_{\boldsymbol{p}} u_1 + \lambda_2 \text{grad}_{\boldsymbol{p}} u_2) \cdot \text{grad}_{\boldsymbol{q}} v \\ &= \lambda_1(\text{grad}_{\boldsymbol{q}} u_1 \cdot \text{grad}_{\boldsymbol{p}} v) + \lambda_2(\text{grad}_{\boldsymbol{q}} u_2 \cdot \text{grad}_{\boldsymbol{p}} v) \\ &\quad - \lambda_1(\text{grad}_{\boldsymbol{p}} u_1 \cdot \text{grad}_{\boldsymbol{q}} v) - \lambda_2(\text{grad}_{\boldsymbol{p}} u_2 \cdot \text{grad}_{\boldsymbol{q}} v) \\ &= \lambda_1(\text{grad}_{\boldsymbol{q}} u_1 \cdot \text{grad}_{\boldsymbol{p}} v - \text{grad}_{\boldsymbol{p}} u_1 \cdot \text{grad}_{\boldsymbol{q}} v) \\ &\quad + \lambda_2(\text{grad}_{\boldsymbol{q}} u_2 \cdot \text{grad}_{\boldsymbol{p}} v - \text{grad}_{\boldsymbol{p}} u_2 \cdot \text{grad}_{\boldsymbol{q}}) \\ &= \lambda_1[u_1,v] + \lambda_2[u_2,v]\end{aligned}$$

Fundamental Poisson brackets

(i) We want to prove that $[q_j, q_k] = 0$. We know that the only partial derivatives that will have a value different from 0 are $\partial q_j / \partial q_j$ and $\partial q_k / \partial q_k$ hence we only write the explicit summation for the indexes j and k as follows

$$\begin{aligned}
[q_j, q_k] &= \frac{\partial q_j}{\partial q_j}\frac{\partial q_k}{\partial p_j} - \frac{\partial q_j}{\partial p_j}\frac{\partial q_k}{\partial q_j} + \frac{\partial q_j}{\partial q_k}\frac{\partial q_k}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial q_k}{\partial q_k} \\
&= 1 \cdot 0 - 0 \cdot 0 + 0 \cdot 0 - 0 \cdot 1 \\
&= 0
\end{aligned}$$

In the case where $j = k$ we only have two terms but the sum is still 0 as shown below

$$\begin{aligned}
[q_j, q_j] &= \frac{\partial q_j}{\partial q_j}\frac{\partial q_j}{\partial p_j} - \frac{\partial q_j}{\partial p_j}\frac{\partial q_j}{\partial q_j} \\
&= 1 \cdot 0 - 0 \cdot 1 \\
&= 0
\end{aligned}$$

(ii) We want to prove that $[p_j, p_k] = 0$. In the same way as we did for point (i) we have that

$$\begin{aligned}
[p_j, p_k] &= \frac{\partial p_j}{\partial q_j}\frac{\partial p_k}{\partial p_j} - \frac{\partial p_j}{\partial p_j}\frac{\partial p_k}{\partial q_j} + \frac{\partial p_j}{\partial q_k}\frac{\partial p_k}{\partial p_k} - \frac{\partial p_j}{\partial p_k}\frac{\partial p_k}{\partial q_k} \\
&= 0 \cdot 0 - 1 \cdot 0 + 0 \cdot 1 - 0 \cdot 0 \\
&= 0
\end{aligned}$$

And in the same way as before $[p_j, p_k]$ is still 0 when $j = k$.

(iii) We want to prove that $[q_j, p_k] = \delta_{jk}$. Let $j \neq k$ then

$$\begin{aligned}
[q_j, p_k] &= \frac{\partial q_j}{\partial q_j}\frac{\partial p_k}{\partial p_j} - \frac{\partial q_j}{\partial p_j}\frac{\partial p_k}{\partial q_j} + \frac{\partial q_j}{\partial q_k}\frac{\partial p_k}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial p_k}{\partial q_k} \\
&= 1 \cdot 0 - 0 \cdot 0 + 0 \cdot 1 - 0 \cdot 0 \\
&= 0
\end{aligned}$$

but if $j = k$ we have that

$$\begin{aligned}
[q_j, p_j] &= \frac{\partial q_j}{\partial q_j}\frac{\partial p_j}{\partial p_j} - \frac{\partial q_j}{\partial p_j}\frac{\partial p_j}{\partial q_j} \\
&= 1 \cdot 1 - 0 \cdot 0 \\
&= 1
\end{aligned}$$

So we have that $[q_j, p_k] = 0$ if $j \neq k$ and $[q_j, p_k] = 1$ if $j = k$. Therefore $[q_j, p_k] = \delta_{jk}$ where δ_{jk} is the Kronecker delta.

Hamilton's equations

(i) We want to prove that $[q_j, H] = \dot{q}_j$. We know that the only partial derivative that will have a value different from 0 is $\partial q_j/\partial q_j$ hence we only write the explicit summation for the index j as follows

$$\begin{aligned}[q_j, H] &= \frac{\partial q_j}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial q_j}{\partial p_j}\frac{\partial H}{\partial q_j} \\ &= 1 \cdot \frac{\partial H}{\partial p_j} - 0 \cdot \frac{\partial H}{\partial q_j} \\ &= \dot{q}_j\end{aligned}$$

Where we used that $\dot{q}_j = \frac{\partial H}{\partial p_j}$.

(ii) Now we want to prove that $[p_j, H] = \dot{p}_j$. So in the same way as before we have that

$$\begin{aligned}[p_j, H] &= \frac{\partial p_j}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_j}{\partial p_j}\frac{\partial H}{\partial q_j} \\ &= 0 \cdot \frac{\partial H}{\partial p_j} - 1 \cdot \frac{\partial H}{\partial q_j} \\ &= \dot{p}_j\end{aligned}$$

Where we used that $\dot{p}_j = -\frac{\partial H}{\partial q_j}$.

Constants of the motion

(i) By the chain rule we have that

$$\frac{du}{dt} = \text{grad}_{\boldsymbol{q}} u \cdot \dot{\boldsymbol{q}} + \text{grad}_{\boldsymbol{p}} u \cdot \dot{\boldsymbol{p}}$$

But using that $\dot{\boldsymbol{q}} = \text{grad}_{\boldsymbol{p}} H$ and that $\dot{\boldsymbol{p}} = -\text{grad}_{\boldsymbol{q}} H$ we get that

$$\begin{aligned}\frac{du}{dt} &= \text{grad}_{\boldsymbol{q}} u \cdot \dot{\boldsymbol{q}} + \text{grad}_{\boldsymbol{p}} u \cdot \dot{\boldsymbol{p}} \\ &= \text{grad}_{\boldsymbol{q}} u \cdot \text{grad}_{\boldsymbol{p}} H - \text{grad}_{\boldsymbol{p}} u \cdot \text{grad}_{\boldsymbol{q}} H \\ &= [u, H]\end{aligned}$$

Also, if $[u, H] = 0$ then $du/dt = 0$ hence u is a constant of the motion.

(ii) Let u and v be constants of the motion of $\mathcal{S}$ then $[u, H] = 0$ and $[v, H] = 0$. We want to show that $[u, v]$ is also a constant of the motion i.e. $[[u, v], H] = 0$. By the Jacobi's identity we know that

$$\begin{aligned}[[u, v], H] + [[H, u], v] + [[v, H], u] &= 0 \\ [[u, v], H] + [-[u, H], v] + [[v, H], u] &= 0 \\ [[u, v], H] + [0, v] + [0, u] &= 0 \\ [[u, v], H] + 0 + 0 &= 0 \\ [[u, v], H] &= 0\end{aligned}$$

Therefore $[u, v]$ is a constant of the motion too.

Finally, we cannot keep on finding more and more new constants of the motion what we will find are combinations of the constants of the motion we already know which are no new constants of the motion.

$\square$

Solution. 14.15 Let $\mathcal{S}$ be an autonomous system with n degrees of freedom and $n-1$ cyclic coordinates. Let $q_1, q_2, ..., q_{n-1}$ be the cyclic coordinates, then $p_1, p_2, ..., p_{n-1}$ are constants of the motion. We know that $[p_j, p_k] = 0$ for $1 \leq j, k, \leq n-1$ because of what we proved in problem 14.14 hence the $n-1$ constants of the motion we have commute as we want. So we need to find one more constant of the motion that commutes so we can say that the system is integrable according to Liouville's theorem.

We know that the system is autonomous so H is a constant of the motion. But also since $p_1, p_2, ..., p_j$ are constants of the motion by definition we have that

$$[p_j, H] = 0$$

This implies that H commutes with all of the other constants of the motion as we wanted. Therefore H is a new constant of the motion that commutes and because of Liouville's theorem the system $\mathcal{S}$ is integrable. $\square$

Solution. 14.16 Given that the primaries move around a fixed circle of radius a and they are on opposite sides of the circle then they will have coordinates $(a\cos\omega t, a\sin\omega t)$ and $(-a\cos\omega t, -a\sin\omega t)$. Then the small mass P has a kinetic energy of

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

And a potential energy of

$$U = -GmM\left(\frac{1}{\sqrt{(x-a\cos\omega t)^2+(y-a\sin\omega t)^2}} + \frac{1}{\sqrt{(x+a\cos\omega t)^2+(y+a\sin\omega t)^2}}\right)$$

Then the lagrangian of the system is given by

$$\begin{aligned} L = &\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ &+ \frac{GmM}{\sqrt{(x-a\cos\omega t)^2+(y-a\sin\omega t)^2}} \\ &+ \frac{GmM}{\sqrt{(x+a\cos\omega t)^2+(y+a\sin\omega t)^2}} \end{aligned}$$

So we determine the generalized momenta as

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \end{aligned}$$

Then the Hamiltonian is

$$\begin{aligned} \mathcal{H} = &\frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 \\ &- GmM\left(\frac{1}{\sqrt{(x-a\cos\omega t)^2+(y-a\sin\omega t)^2}} + \frac{1}{\sqrt{(x+a\cos\omega t)^2+(y+a\sin\omega t)^2}}\right) \end{aligned}$$

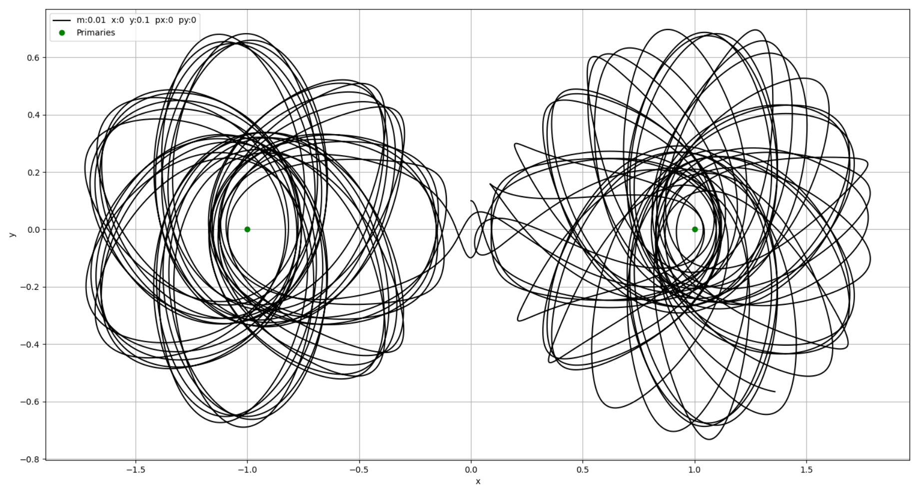
Finally, we can compute the equations of motion as follows

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \\ \dot{y} &= \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \end{aligned}$$

And

$$
\begin{aligned}
\dot{p_x} &= -\frac{\partial \mathcal{H}}{\partial x} \\
&= -GmM\Bigg(\frac{x - a\cos(wt)}{((x - a\cos\omega t)^2 + (y - a\sin\omega t)^2)^{3/2}} \\
&\qquad + \frac{x + a\cos(wt)}{((x + a\cos\omega t)^2 + (y + a\sin\omega t)^2)^{3/2}}\Bigg) \\
\dot{p_y} &= -\frac{\partial \mathcal{H}}{\partial y} \\
&= -GmM\Bigg(\frac{y - a\sin(wt)}{((x - a\cos\omega t)^2 + (y - a\sin\omega t)^2)^{3/2}} \\
&\qquad + \frac{y + a\sin(wt)}{((x + a\cos\omega t)^2 + (y + a\sin\omega t)^2)^{3/2}}\Bigg)
\end{aligned}
$$

Finally, we can solve this set of first-order differential equations numerically. We plot below the path of P seen from a rotating frame at the origin.



$\square$