

Solved selected problems of Classical Mechanics - Taylor

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Chapter 1 - Newton's laws of Motion

Solution. 1.10 The cartesian components x and y for the vector \mathbf{r} are

$$x = R \cos \theta \quad y = R \sin \theta$$

we also know that $\theta = \omega t$ so we have that

$$\begin{aligned}\mathbf{r} &= \hat{\mathbf{x}}R \cos \theta + \hat{\mathbf{y}}R \sin \theta \\ &= \hat{\mathbf{x}}R \cos \omega t + \hat{\mathbf{y}}R \sin \omega t\end{aligned}$$

To find the particle's velocity and acceleration we must differentiate the \mathbf{r} equation with respect to time.

For the velocity we have that

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(\hat{\mathbf{x}}R \cos \omega t + \hat{\mathbf{y}}R \sin \omega t) \\ \mathbf{v} &= -\hat{\mathbf{x}}R\omega \sin \omega t + \hat{\mathbf{y}}R\omega \cos \omega t\end{aligned}$$

For the acceleration we need to differentiate the velocity again

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{d}{dt}(-\hat{\mathbf{x}}R\omega \sin \omega t + \hat{\mathbf{y}}R\omega \cos \omega t) \\ \mathbf{a} &= -\hat{\mathbf{x}}R\omega^2 \cos \omega t - \hat{\mathbf{y}}R\omega^2 \sin \omega t\end{aligned}$$

as noticed the direction of the acceleration is opposite to \mathbf{r} . Finally, the magnitude of the acceleration is

$$\begin{aligned}|\mathbf{a}| &= \sqrt{(-R\omega^2 \cos \omega t)^2 + (-R\omega^2 \sin \omega t)^2} \\ &= \sqrt{R^2\omega^4(\cos^2 \omega t + \sin^2 \omega t)} \\ &= R\omega^2\end{aligned}$$

□

Solution. 1.11 Given that the position of the particle is given by

$$\mathbf{r}(t) = \hat{\mathbf{x}}b \cos \omega t + \hat{\mathbf{y}}c \sin \omega t$$

Then the cartesian coordinates for the particle are $x = b \cos \omega t$ and $y = c \sin \omega t$

We can derive from the identity $\cos^2 \omega t + \sin^2 \omega t = 1$ the equation that involves both cartesian coordinates as follows.

$$\begin{aligned} \cos^2 \omega t + \sin^2 \omega t &= 1 \\ \frac{b^2}{b^2} \cos^2 \omega t + \frac{c^2}{c^2} \sin^2 \omega t &= 1 \\ \frac{x^2}{b^2} + \frac{y^2}{c^2} &= 1 \end{aligned}$$

Which is the equation for an ellipse centered at the origin. \square

Solution. 1.12 In this case, the particle in addition to being rotating around an ellipse trajectory as we already know from the last problem is also moving in the positive direction around the z -axis so the resultant trajectory is a helicoid centered around z -axis. \square

Solution. 1.26

- (a) Given a reference frame \mathcal{S} where the x -axis point east and y -axis point north, if we kick a frictionless puck due north, from this reference frame, there is no x -axis movement for the puck and it moves to north with constant velocity, then the coordinates for the puck are

$$x = 0 \quad y = vt$$

where v is the constant velocity at which the puck is moving.

- (b) In a reference frame \mathcal{S}' which is moving with constant velocity v' due east with respect to \mathcal{S} the puck movement seen from \mathcal{S}' has an added movement in the $-x$ direction since \mathcal{S}' is moving away from the puck so the coordinates for the puck seen from \mathcal{S}' are

$$x' = -v't \quad y' = vt$$

this means that the puck has a trajectory given by $y' = -v/v'x'$ which is a line with a negative slope that passes through the origin.

- (c) Lastly the reference frame \mathcal{S}'' is moving with constant acceleration due east with respect to \mathcal{S} so the puck movement seen from \mathcal{S}'' has an added movement in the $-x$ direction since \mathcal{S}'' is moving away from the puck, therefore the coordinates for the puck seen from \mathcal{S}'' are

$$x'' = -(v''t + \frac{1}{2}a''t^2) \quad y'' = vt$$

where v'' and a'' are the velocity and acceleration of \mathcal{S}'' respectively.

Since \mathcal{S} and \mathcal{S}' are not accelerating they are inertial reference frames, which is not the case for \mathcal{S}'' . \square

Solution. 1.27 The puck path observed by someone sitting at rest at the origin of the turntable is a spiral that grows indefinitely as the puck continues its path to infinity. \square

Solution. 1.30 The total momentum of the system before the collision is given by

$$\mathbf{P} = m_1\mathbf{v} + m_2\mathbf{0} = m_1\mathbf{v}$$

but after the collision the total momentum is given by

$$\mathbf{P} = m_1\mathbf{v}' + m_2\mathbf{v}'$$

since there is no external forces, then the total momentum of the system is constant, therefore

$$\begin{aligned} m_1\mathbf{v}' + m_2\mathbf{v}' &= m_1\mathbf{v} \\ \mathbf{v}'(m_1 + m_2) &= m_1\mathbf{v} \\ \mathbf{v}' &= \frac{m_1}{m_1 + m_2}\mathbf{v} \end{aligned}$$

\square

Solution. 1.40

- (a) According to the Newton's second Law we have that $\mathbf{F} = m\ddot{\mathbf{r}}$ or if we write it in cartesian coordinates for two dimensions then

$$F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} = m\ddot{x}\hat{\mathbf{x}} + m\ddot{y}\hat{\mathbf{y}}$$

given that the only force is the gravitational force and is in the y-direction we have that

$$m\ddot{x} = 0 \quad m\ddot{y} = -mg$$

therefore integrating both equations we have that

$$\begin{aligned} \dot{x} &= \int 0 \, dt \\ &= 0 + v_{0x} \\ x &= \int v_{0x} \, dt \\ &= v_{0x}t + x_0 \\ &= v_{0x}t \end{aligned}$$

where v_{0x} and x_0 are the initial velocity and position of the ball in the x-direction, note that $x_0 = 0$. Also we have that $v_{0x} = v_0 \cos \theta$ so the final equation is

$$x = v_0 t \cos \theta$$

In the same way for the y -direction we have that

$$\begin{aligned}
 \dot{y} &= \int -g \, dt \\
 &= v_{0y} - gt \\
 y &= \int v_{0y} - gt \, dt \\
 &= y_0 + v_{0y}t - g\frac{t^2}{2} \\
 &= v_{0y}t - g\frac{t^2}{2}
 \end{aligned}$$

As we know from the x -direction $v_{0y} = v_0 \sin \theta$ so the final equation becomes

$$y = v_0 t \sin \theta - g\frac{t^2}{2}$$

(b) We write the r^2 (the magnitude of \mathbf{r} squared) as

$$\begin{aligned}
 r^2 &= (v_0 t \cos \theta)^2 + \left(v_0 t \sin \theta - g\frac{t^2}{2} \right)^2 \\
 &= (v_0 t \cos \theta)^2 + (v_0 t \sin \theta)^2 - 2g\frac{t^2}{2}v_0 t \sin \theta + \left(-g\frac{t^2}{2} \right)^2 \\
 &= (v_0 t)^2 (\cos^2 \theta + \sin^2 \theta) - gv_0 t^3 \sin \theta + \left(-g\frac{t^2}{2} \right)^2 \\
 &= (v_0 t)^2 - gv_0 t^3 \sin \theta + \left(-g\frac{t^2}{2} \right)^2
 \end{aligned}$$

To check whether a function increases or decreases we need to analyze the equation's derivative. Then

$$\dot{r}^2 = 2v_0^2 t - 3gv_0 t^2 \sin \theta + g^2 t^3$$

What we want is that $2v_0^2 t - 3gv_0 t^2 \sin \theta + g^2 t^3 > 0$. The minimum of \dot{r}^2 can be found by differentiating this equation again and solving for $\ddot{r}^2 = 0$, so

$$\begin{aligned}
 2v_0^2 - 6gv_0 t \sin \theta + 3g^2 t^2 &= 0 \\
 t_{min} &= \frac{6gv_0 \sin \theta \pm \sqrt{(-6gv_0 \sin \theta)^2 - 24g^2 v_0^2}}{6g^2} \\
 &= \frac{6gv_0 \sin \theta \pm \sqrt{36g^2 v_0^2 \sin^2 \theta - 24g^2 v_0^2}}{6g^2} \\
 &= \frac{6gv_0 \sin \theta \pm 6gv_0 \sqrt{\sin^2 \theta - 24/36}}{6g^2} \\
 &= \frac{v_0}{g} (\sin \theta \pm \sqrt{\sin^2 \theta - 6/9})
 \end{aligned}$$

The next step would be to write \dot{r}^2 valued at t_{min} and find where this expression is > 0 to determine the value for θ .

□

Solution. 1.41 The Newton's second law in polar coordinates are

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$$F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

where in this case $\dot{\phi} = \omega$ and since ω is constant $\ddot{\phi} = 0$ also $r = R$ which is also constant then $\dot{r} = \ddot{r} = 0$, therefore

$$F_r = -mR\omega^2 \quad F_\phi = 0$$

Since the tension is the only force in the r direction we have that

$$T = -mR\omega^2$$

□

Solution. 1.46

- (a) For an observer on the ground in a reference frame \mathcal{S} with its origin O at the center of the turntable the puck will have polar coordinates given by

$$\phi = 0$$

$$\mathbf{r} = (R - vt)\hat{\mathbf{r}}$$

where R is the radius of the turntable and v the velocity of the puck.

- (b) In the case that the observer is fixed to the turntable with its origin O at the center of the turntable the puck will have the same \mathbf{r} polar coordinate but now ϕ is changing according to the rotation of the reference frame therefore

$$\phi' = \omega t \hat{\phi}$$

$$\mathbf{r}' = (R - vt)\hat{\mathbf{r}}$$

where ω is the angular velocity of the turntable. This is not an inertial reference frame since it's rotating.

□

Solution. 1.47

- (a) The expressions for ρ , ϕ , and z in terms of x , y , and z are

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \arctan(y/x)$$

$$z = z$$

In words, ρ is the distance to P' from the origin, where P' is the projection of P to the plane $(x, y, 0)$.

The use of r to name ρ in this case is unfortunate since we are naming r as the magnitude of the vector \mathbf{r} which is the distance to P from the origin.

- (b) The unit vector $\hat{\rho}$ is the same unit vector we defined previously for two dimensions problems as \hat{r} and the same thing happening with $\hat{\phi}$. In the case of \hat{z} is a unit vector that points in the direction of the z cartesian coordinate.

The position vector in this case is given by

$$\mathbf{r} = \rho\hat{\rho} + z\hat{z}$$

- (c) Given that \hat{z} is the same unit vector as in cartesian coordinates then $d\hat{z}/dt = 0$ also using the result we obtained for $d\hat{r}/dt = d\hat{\rho}/dt$ and $d\hat{\phi}/dt$ we get that

$$\begin{aligned}\dot{\mathbf{r}} = \mathbf{v} &= \left(\dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} \right) + \dot{z}\hat{z} \\ &= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}\end{aligned}$$

Then

$$\begin{aligned}\ddot{\mathbf{r}} = \mathbf{a} &= \left(\ddot{\rho}\hat{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} \right) + \left((\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt} \right) + \ddot{z}\hat{z} \\ &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}\end{aligned}$$

□

Solution. 1.48 The projection of $\hat{\rho}$ to the x-axis is given by $\cos \phi$ and the projection to the y-axis is given by $\sin \phi$ then

$$\hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi$$

since $\hat{\phi}$ is perpendicular to $\hat{\rho}$ then

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi$$

and since \hat{z} is the same as the one used in cartesian coordinates

$$\hat{z} = \hat{z}$$

The derivatives of the above equations given that $d\hat{x}/dt = 0$, $d\hat{y}/dt = 0$, and $d\hat{z}/dt = 0$ are

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= \left(\frac{d\hat{x}}{dt} \cos \phi - \hat{x} \dot{\phi} \sin \phi\right) + \left(\frac{d\hat{y}}{dt} \sin \phi + \hat{y} \dot{\phi} \cos \phi\right) \\ &= -\hat{x} \dot{\phi} \sin \phi + \hat{y} \dot{\phi} \cos \phi \\ &= \dot{\phi}(-\hat{x} \sin \phi + \hat{y} \cos \phi) \\ &= \dot{\phi} \hat{\phi} \end{aligned}$$

$$\begin{aligned} \frac{d\hat{\phi}}{dt} &= -\left(\frac{d\hat{x}}{dt} \sin \phi + \hat{x} \dot{\phi} \cos \phi\right) + \left(\frac{d\hat{y}}{dt} \cos \phi - \hat{y} \dot{\phi} \sin \phi\right) \\ &= -\hat{x} \dot{\phi} \cos \phi - \hat{y} \dot{\phi} \sin \phi \\ &= -\dot{\phi}(\hat{x} \cos \phi + \hat{y} \sin \phi) \\ &= -\dot{\phi} \hat{\rho} \end{aligned}$$

$$\frac{d\hat{z}}{dt} = 0$$

□

Solution. 1.49 Newton's second law equations in cylindrical coordinates are

$$F_\phi = m(\rho \ddot{\phi} + 2\dot{\rho} \dot{\phi})$$

$$F_\rho = m(\ddot{\rho} - \rho \dot{\phi}^2)$$

$$F_z = m\ddot{z}$$

Given that the puck is fixed to a certain $\rho = R$ then $\dot{\rho} = \ddot{\rho} = 0$ also since the gravitational force is in the z direction, we have that

$$0 = mR\ddot{\phi} \tag{1}$$

$$N_1 - N_2 = N = -mR\dot{\phi}^2 \tag{2}$$

$$-mg = m\ddot{z} \tag{3}$$

where N_1 and N_2 are the two normal forces exerted by the two cylinders, we named the difference between them N . Integrating the equation (1) we have that

$$\begin{aligned}\dot{\phi} &= \omega = \int 0 dt \\ &= 0 + \omega_0\end{aligned}$$

where ω_0 is the angular velocity and integrating again we have that

$$\begin{aligned}\phi &= \int \omega_0 dt \\ &= \omega_0 t + \phi_0\end{aligned}$$

Then integrating the equation (3), we have that

$$\begin{aligned}\dot{z} = v_z &= \int -g dt \\ &= -gt + v_{z0}\end{aligned}$$

and integrating again we have that

$$\begin{aligned}z &= \int (-gt + v_{z0}) dt \\ &= -g \frac{t^2}{2} + v_{z0}t + z_0\end{aligned}$$

Then the movement in the z -direction is as a free falling object and in the ϕ -direction is a rotation with constant angular velocity. Therefore the final trajectory is a downward helix. \square

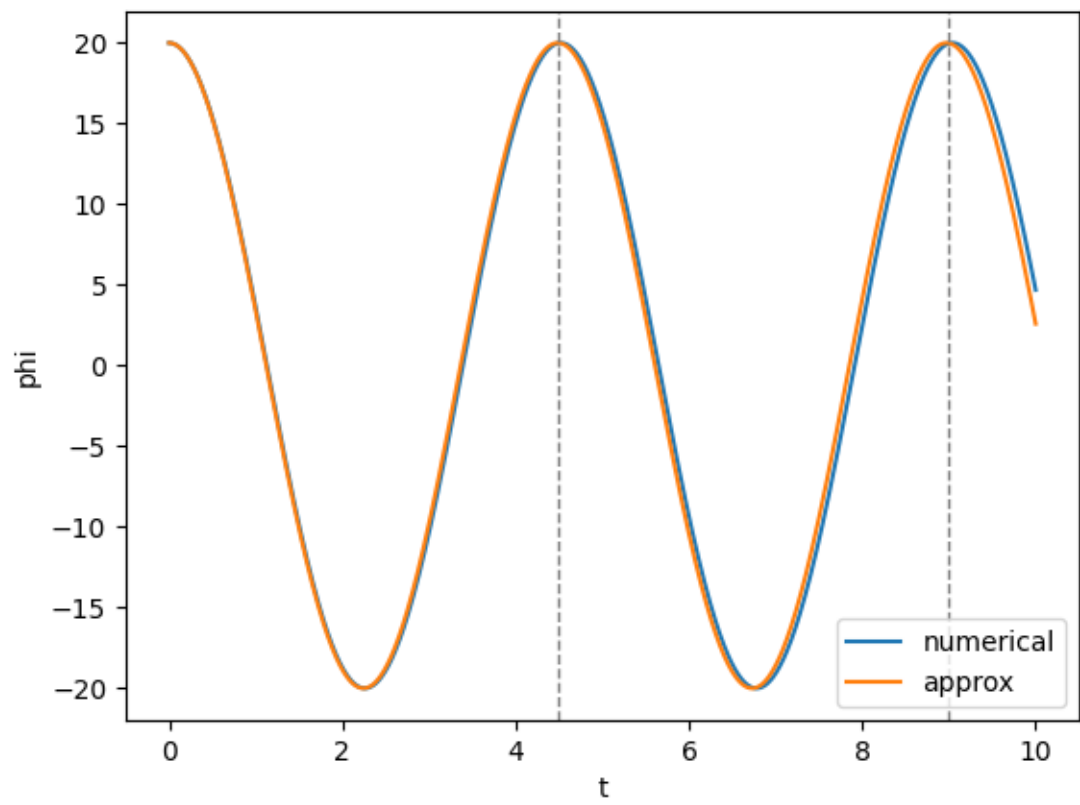
Solution. 1.50 We can transform the 2nd order equation

$$\ddot{\phi} = -\frac{g}{R} \sin \phi$$

into two 1st order equation as

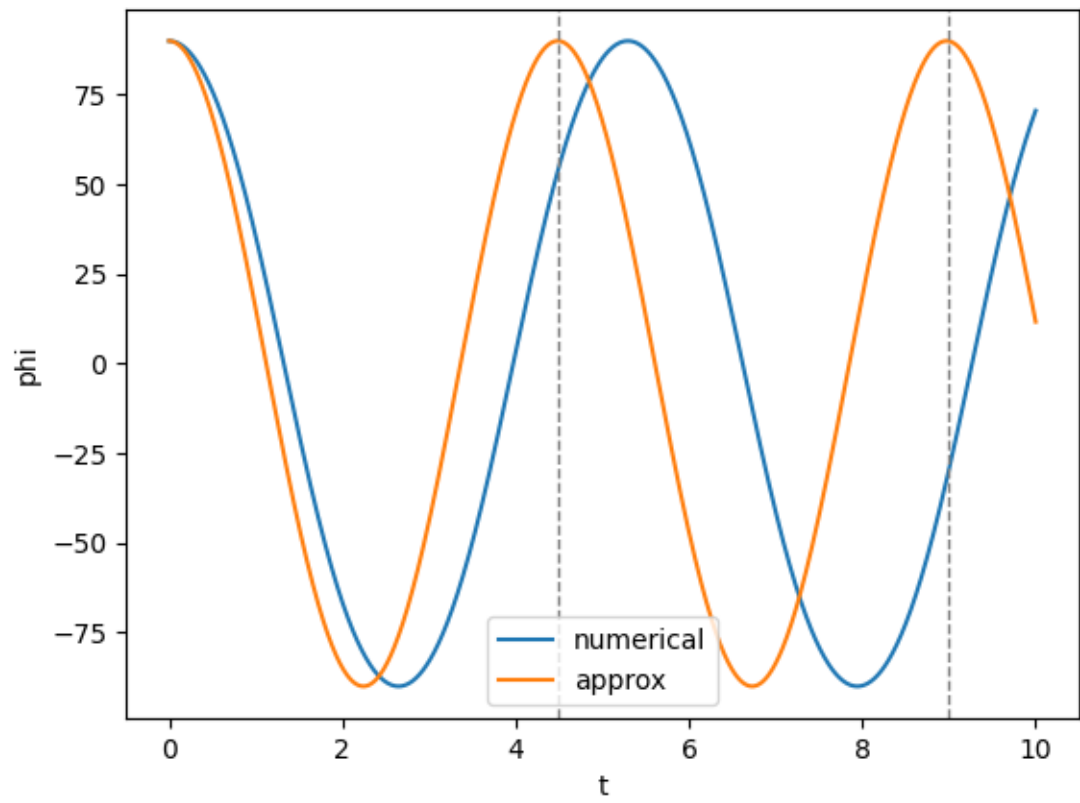
$$\begin{aligned}\dot{\phi} &= \omega \\ \dot{\omega} &= -\frac{g}{R} \sin \phi\end{aligned}$$

Solving this system of equations with the improved Euler method we get the following graph.



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Solution. **1.51**



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