Solved selected problems of Classical Mechanics - Gregory

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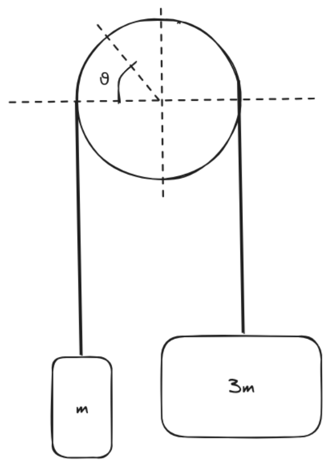
Chapter 12 - Lagrange's equations and conservation principles

Solution. 12.2 The system described looks like the following



The system is a standard system and the forces are conservative so we can apply the Lagrange method for conservative systems where θ is our generalized coordinate. In this case, the kinetic energy is given by

$$T = \frac{1}{2}(mr^2)\dot{\theta}^2 + \frac{1}{2}m(\dot{\theta}r)^2 + \frac{1}{2}3m(\dot{\theta}r)^2$$
$$T = \frac{5}{2}mr^2\dot{\theta}^2$$

Where the first term corresponds to the pulley and the last 2 to the masses hanging. Also, r is the radius of the pulley.

For the potential energy, we have that

$$V = mg(\theta r) - 3mg(\theta r)$$
$$V = -2mg(\theta r)$$

Now we compute the partial derivatives as follows

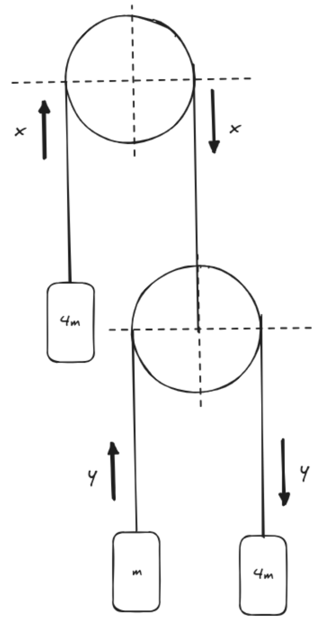
$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial T}{\partial \dot{\theta}} = 5mr^2\dot{\theta} \qquad \frac{\partial V}{\partial \theta} = -2mgr$$

So the Lagrange equation is given by

$$\begin{aligned} \frac{d}{dt}(5mr^2\dot{\theta}) - 0 &= 2mgr \\ r\ddot{\theta} &= \frac{2}{5}g \end{aligned}$$

Therefore the upward acceleration of the mass m is $a = \frac{2}{5}g$ since the mass is experiencing the same acceleration as the tangential acceleration of the pulley because of the no slipping restriction. $\square$

Solution. 12.3 The system described looks like the following



The system is a standard system and the forces are conservative so we can apply the Lagrange method for conservative systems where x and y (the displacements with respect to each pulley) are our generalized coordinates. In this case, the kinetic energy is given by

$$T = \frac{1}{2}4m\dot{x}^2 + \frac{1}{2}m(-\dot{x}+\dot{y})^2 + \frac{1}{2}4m(-\dot{x}-\dot{y})^2$$
$$T = \frac{1}{2}m(4\dot{x}^2 + \dot{x}^2 - 2\dot{x}\dot{y} + \dot{y}^2 + 4(\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2))$$
$$T = \frac{1}{2}m(9\dot{x}^2 + 6\dot{x}\dot{y} + 5\dot{y}^2)$$

For the potential energy, we have that

$$V = 4mgx + mg(y-x) - 4mg(y+x)$$
$$V = -mgx - 3mgy$$

Now we compute the partial derivatives as follows

$$\frac{\partial T}{\partial x} = 0 \qquad \frac{\partial T}{\partial \dot{x}} = m(9\dot{x}+3\dot{y}) \qquad \frac{\partial V}{\partial x} = -mg$$

$$\frac{\partial T}{\partial y} = 0 \qquad \frac{\partial T}{\partial \dot{y}} = m(5\dot{y}+3\dot{x}) \qquad \frac{\partial V}{\partial y} = -3mg$$

So the Lagrange equation for $q_1 = x$ is given by

$$\begin{aligned} \frac{d}{dt}(m(9\dot{x} + 3\dot{y})) - 0 &= mg \\ 9\ddot{x} + 3\ddot{y} &= g \end{aligned}$$

And for $q_2 = y$ we have that

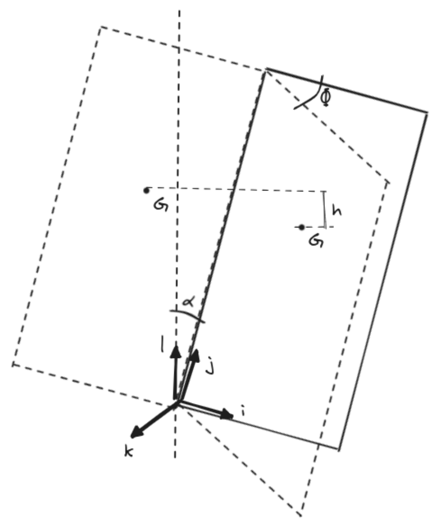
$$\begin{aligned} \frac{d}{dt}(m(5\dot{y} + 3\dot{x})) - 0 &= 3mg \\ 5\ddot{y} + 3\ddot{x} &= 3g \end{aligned}$$

Therefore the generalized coordinates accelerations are

$$\ddot{y} = \frac{2}{3}g \qquad \ddot{x} = -\frac{g}{9}$$

So the acceleration for the mass $4m$ of the upward pulley is $\ddot{x} = -g/9$, the acceleration for the mass m of the downward pulley is $\ddot{y} - \ddot{x} = 7g/9$ and the acceleration for the mass $4m$ of the downward pulley is $-\ddot{y} - \ddot{x} = -5g/9$. $\square$

Solution. 12.4 The system described looks like the following



Where our generalized coordinate is ϕ the angle of rotation from the door equilibrium position.

First, we need to compute the moment of inertia of the door with respect to the hinges, let $\rho = m/2a$ be the linear density of mass then a differential of mass is given by $dm = \rho dx$ hence

$$\begin{aligned} I &= \int_0^{2a} \frac{mx^2}{2a} dx \\ &= \frac{m}{2a}\left[\frac{x^3}{3}\right]_0^{2a} \\ &= \frac{4}{3}ma^2 \end{aligned}$$

So in this case, the kinetic energy is

$$T = \frac{1}{2}\left(\frac{4}{3}ma^2\right)\dot{\phi}^2 = \frac{2}{3}ma^2\dot{\phi}^2$$

Now we have to compute the potential energy. Let us define a set of coordinates $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ where $\boldsymbol{i}$ and $\boldsymbol{j}$ are in the plane of the door equilibrium position and $\boldsymbol{k}$ is perpendicular to them. We need to determine first the height gain of the center of mass when the door is opened. Let the center of mass in the equilibrium position be $G = a\boldsymbol{i} + b\boldsymbol{j}$ then the center of mass will be at $G = a\cos\phi\boldsymbol{i} + b\boldsymbol{j} + a\sin\phi\boldsymbol{k}$ when the door is opened at an angle ϕ hence the center of mass of the door has displaced

$$\Delta G = a(\cos\phi - 1)\boldsymbol{i} + a\sin\phi\boldsymbol{k}$$

But we are interested in the vertical component of this quantity so we multiply this expression by $\boldsymbol{l}$ our vertical unit vector, so we get that

$$\begin{aligned} h = \Delta G \cdot \boldsymbol{l} &= a(\cos\phi - 1)\boldsymbol{i}\cdot\boldsymbol{l} + a\sin\phi\boldsymbol{k}\cdot\boldsymbol{l} \\ &= a(\cos\phi - 1)\boldsymbol{i}\cdot\boldsymbol{l} + 0 \\ &= a(\cos\phi - 1)\cos(\alpha + \pi/2) \\ &= -a(\cos\phi - 1)\sin(\alpha) \end{aligned}$$

Then the potential energy is given by:

$$V = -mgh = mga(\cos\phi - 1)\sin(\alpha)$$

Now we compute the partial derivatives as follows

$$\frac{\partial T}{\partial \phi} = 0 \qquad \frac{\partial T}{\partial \dot{\phi}} = \frac{4}{3}ma^2\dot{\phi} \qquad \frac{\partial V}{\partial \phi} = -mga\sin(\alpha)\sin\phi$$

So the Lagrange equation for $q_1 = \phi$ is given by

$$\begin{aligned} \frac{d}{dt}(\frac{4}{3}ma^2\dot{\phi}) - 0 &= mga\sin(\alpha)\sin\phi \\ \frac{4}{3}a\ddot{\phi} &= g\sin(\alpha)\sin\phi \\ \ddot{\phi} &= \frac{3}{4}\frac{g\sin(\alpha)}{a}\sin\phi \\ \ddot{\phi} - \frac{3}{4}\frac{g\sin(\alpha)}{a}\sin\phi = 0 \end{aligned}$$

But when ϕ is small we can assume $\sin\phi \approx \phi$ so we get

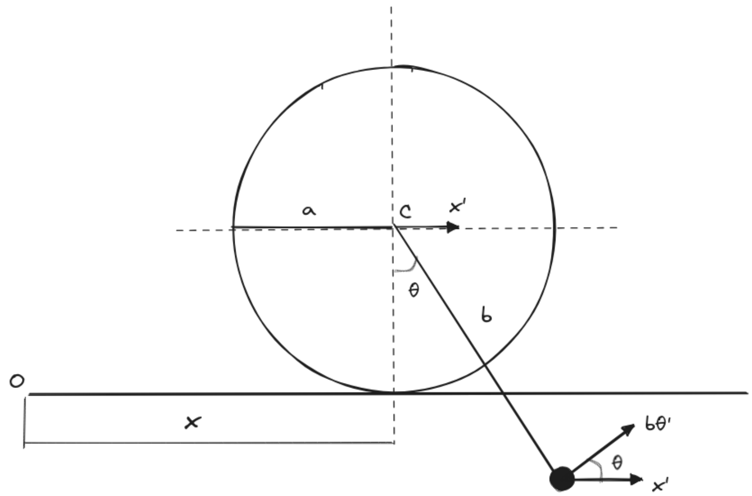
$$\ddot{\phi} - \frac{3}{4}\frac{g\sin(\alpha)}{a}\phi = 0$$

Which is the equation for a simple harmonic motion of the form $\ddot{x} + \Omega^2 x = 0$ where we know the period is given by $\tau = 2\pi/\Omega$ therefore the period of small oscillations is

$$\tau = \frac{2\pi}{\sqrt{\frac{3}{4}\frac{g\sin(\alpha)}{a}}} = 2\pi\sqrt{\frac{4a}{3g\sin\alpha}}$$

□

Solution. 12.6 The system described looks like the following



Where our generalized coordinates are x and θ. The kinetic energy in this case is given by

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\left(\frac{\dot{x}}{a}\right)^2 + \frac{1}{2}m\left(b\dot{\theta} + \dot{x}\right)^2$$
$$T = \frac{3}{4}M\dot{x}^2 + \frac{1}{2}m\left(b^2\dot{\theta}^2 + 2b\dot{\theta}\dot{x}\cos\theta + \dot{x}^2\right)$$

Where we used that the angular velocity of the disk is $\dot{x}/a$ because of the rolling condition. Now we determine the potential energy, there is no change in the height for the center of mass of the disk but there is for the hanging mass hence

$$V = -mgb\cos\theta$$

Then the Lagrangian is

$$L = T - V = \frac{3}{4}M\dot{x}^2 + \frac{1}{2}m\left(b^2\dot{\theta}^2 + 2b\dot{\theta}\dot{x}\cos\theta + \dot{x}^2\right) + mgb\cos\theta$$

We see that L is not a function of x hence x is a cyclic coordinate. The generalized momentum p_x is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{3}{2}M\dot{x} + m\left(b\dot{\theta}\cos\theta + \dot{x}\right)$$

And it's not the horizontal linear momentum which is given by

$$M\dot{x} + m\left(b\dot{\theta}\cos\theta + \dot{x}\right)$$

Finally, we compute the Lagrangian partial derivatives left

$$\frac{\partial L}{\partial \dot\theta} = mb(b\dot\theta + \dot x\cos\theta)$$
$$\frac{\partial L}{\partial \theta} = -mb\dot\theta\dot x\sin\theta - mbg\sin\theta$$

So now we compute the Lagrangian form for the two generalized coordinates as follows

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} &= 0\\ mb(b\ddot\theta + \ddot x\cos\theta - \dot x\dot\theta\sin\theta) + mb\dot\theta\dot x\sin\theta + mbg\sin\theta &= 0\\ b\ddot\theta + \ddot x\cos\theta + g\sin\theta &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - 0 &= 0\\ \frac{3}{2}M\ddot x + m\left(b(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) + \ddot x\right) &= 0\\ \ddot x\left(\frac{3}{2}M + m\right) + mb(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) &= 0\\ -\frac{mb}{\frac{3}{2}M + m}(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) &= \ddot x\end{aligned}$$

By replacing $\ddot x$ we get that

$$\begin{aligned}b\ddot\theta - \frac{mb\cos\theta}{\frac{3}{2}M + m}(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) + g\sin\theta &= 0\\ \ddot\theta - \frac{2m\cos\theta}{(3M + 2m)}(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) + \frac{g}{b}\sin\theta &= 0\\ \ddot\theta\left(1 - \frac{2m\cos^2\theta}{(3M + 2m)}\right) + \frac{2m\cos\theta\sin\theta}{(3M + 2m)}\dot\theta^2 + \frac{g}{b}\sin\theta &= 0\\ \ddot\theta(3M + 2m\sin^2\theta) + 2m\cos\theta\sin\theta\dot\theta^2 + \frac{g(3M + 2m)}{b}\sin\theta &= 0\end{aligned}$$

Now we approximate this equation to a linear equation by neglecting quadratic terms (i.e. θ^2 and $\dot\theta^2$) and assuming $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ hence the linearized equation is

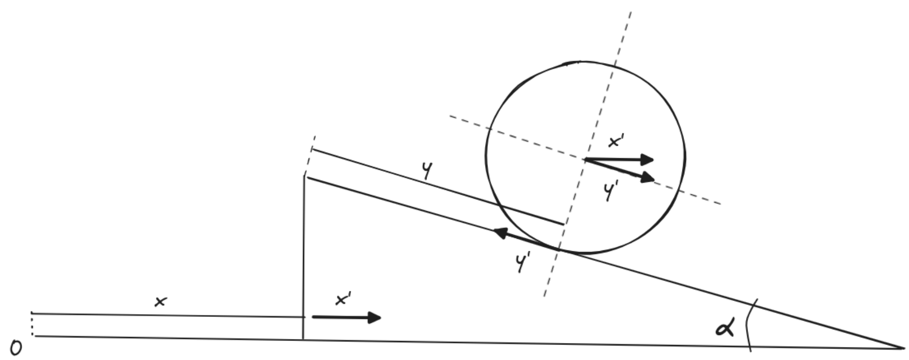
$$\ddot\theta + \frac{g(3M + 2m)}{3Mb}\theta = 0$$

Therefore the period of small oscillations is given by

$$\tau = 2\pi\sqrt{\frac{3Mb}{g(3M + 2m)}}$$

□

Solution. 12.7 The system described looks like the following



Where our generalized coordinates are x and y hence we have 2 degrees of freedom. The kinetic energy in this case is given by

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{2}{5}ma^2\right)\left(\frac{\dot{y}}{a}\right)^2 + \frac{1}{2}m\left(\dot{y} + \dot{x}\right)^2$$
$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{5}m\dot{y}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{y}\cos\alpha + \dot{y}^2)$$

Where we used that the angular velocity of the ball is $\dot{y}/a$ because of the rolling condition. Then the potential energy is

$$V = -mgy\sin\alpha$$

Now we compute the partial derivatives of T and V as follows

$$\frac{\partial T}{\partial x} = 0 \qquad \frac{\partial T}{\partial \dot{x}} = \dot{x}(M+m) + m\dot{y}\cos\alpha$$
$$\frac{\partial T}{\partial y} = 0 \qquad \frac{\partial T}{\partial \dot{y}} = \frac{7}{5}m\dot{y} + m\dot{x}\cos\alpha$$
$$\frac{\partial V}{\partial x} = 0 \qquad \frac{\partial V}{\partial y} = -mg\sin\alpha$$

So the Lagrange equation for x gives us

$$\ddot{x}(M+m) + m\ddot{y}\cos\alpha = 0$$

And for y

$$\frac{7}{5}m\ddot{y} + m\ddot{x}\cos\alpha = mg\sin\alpha$$
$$\frac{7}{5}\ddot{y} + \ddot{x}\cos\alpha = g\sin\alpha$$

If we assume now $M = 3m/2$ we get from the x-Lagrange equation that

$$\frac{5}{2}\ddot{x} + \ddot{y}\cos\alpha = 0$$

the second Lagrange equation does not depend on M so it's left untouched. Then by replacing the values, we get that the acceleration of the wedge is

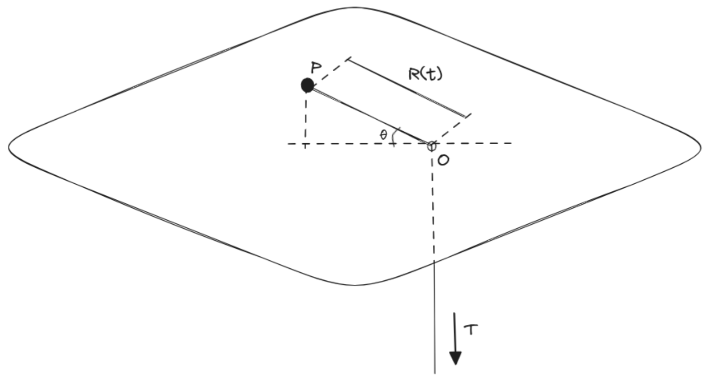
$$\begin{aligned}
-\frac{7}{2}\frac{\ddot{x}}{\cos\alpha} + \ddot{x}\cos\alpha &= g\sin\alpha \\
-\frac{7}{2}\ddot{x} + \ddot{x}\cos^2\alpha &= g\sin\alpha\cos\alpha \\
\ddot{x} &= \frac{2g\sin\alpha\cos\alpha}{2\cos^2\alpha - 7}
\end{aligned}$$

In the same way, the acceleration of the ball with respect to the wedge is

$$\begin{aligned}
\ddot{y} &= -\frac{5}{2\cos\alpha}\frac{2g\sin\alpha\cos\alpha}{2\cos^2\alpha - 7} \\
\ddot{y} &= \frac{5g\sin\alpha}{7 - 2\cos^2\alpha}
\end{aligned}$$

□

Solution. 12.9 The system described looks like the following



Where our generalized coordinate is θ. We know that the velocity in polar coordinates is given by

$$v = \dot{R(t)}\hat{r} + R(t)\dot{\theta}\hat{\theta}$$

hence the kinetic energy in this case is

$$T = \frac{1}{2}m(\dot{R(t)}\hat{r} + R(t)\dot{\theta}\hat{\theta})^2$$
$$T = \frac{1}{2}m[\dot{R(t)}^2 + R(t)^2\dot{\theta}^2]$$

The kinetic energy is not conserved since the tension force is doing work on the particle. There is no change in the height of P therefore the potential energy is $V = 0$. Then the Lagrangian is

$$L = T - V = \frac{1}{2}m[\dot{R(t)}^2 + R(t)^2\dot{\theta}^2]$$

We see that L is not a function of θ hence θ is a cyclic coordinate. The generalized momentum p_θ is given by

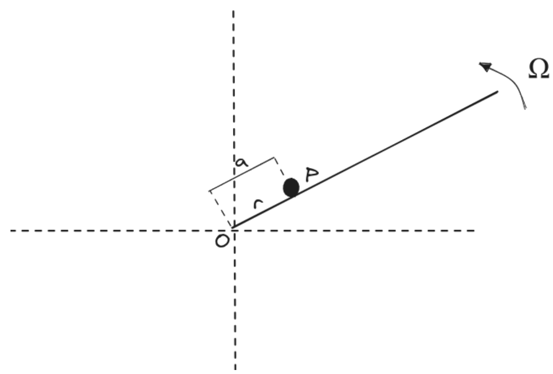
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR(t)^2\dot{\theta}$$

From Newton's equations, we know that $F = ma$ where F is the tension in the string so if we write the acceleration in polar coordinates and we replace $\dot{\theta} = L/R(t)^2$ (since $mL = mR(t)^2\dot{\theta}$) we have that

$$F = m[(\ddot{R} - R\dot{\theta}^2)\hat{r} + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\theta}]$$
$$F = m\left[\left(\ddot{R} - R\left(\frac{L^2}{R^4}\right)\right)\hat{r} + \left(-R\frac{2\dot{R}L}{R^3} + 2\dot{R}\frac{L}{R^2}\right)\hat{\theta}\right]$$
$$F = m\left(\ddot{R} - \left(\frac{L^2}{R^3}\right)\right)\hat{r}$$

□

Solution. 12.10 The system described looks like the following



Where our generalized coordinate is r. The kinetic energy in this case is

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(\Omega r)^2$$

Since gravity is not present then the potential energy for P is 0 hence the lagrangian is equal to T i.e.

$$L = \frac{1}{2}m(\dot{r}^2 + \Omega^2 r^2)$$

Then the Lagrange equation is given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{d}{dt}\left(m\dot{r}\right) - m\Omega^2 r &= 0 \\ \ddot{r} - \Omega^2 r &= 0 \end{aligned}$$

The solution for this differential equation and its derivative are

$$\begin{aligned} r &= Ae^{\Omega t} + Be^{-\Omega t} \\ \dot{r} &= A\Omega e^{\Omega t} - B\Omega e^{-\Omega t} \end{aligned}$$

we know from the initial conditions that $r = a$ and $\dot{r} = 0$ when $t = 0$ hence

$$\begin{aligned} a &= A + B \\ 0 &= A - B \end{aligned}$$

This implies that $B = A = a/2$ therefore the position of the particle at time t is

$$r = \frac{a}{2}\left(e^{\Omega t} + e^{-\Omega t}\right) = a\cosh(\Omega t)$$

Finally, the h function is given by

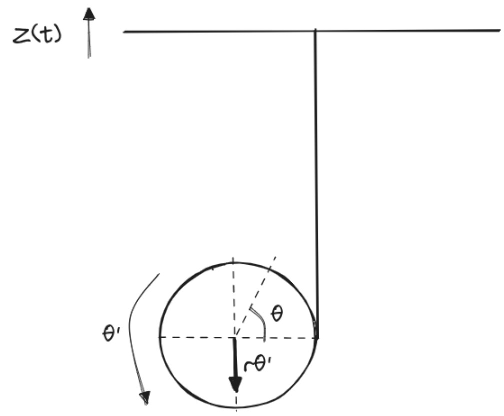
$$
\begin{aligned}
h &= \frac{\partial L}{\partial \dot{r}}\dot{r} - L \\
h &= m\dot{r}^2 - \frac{1}{2}m(\dot{r}^2 + \Omega^2 r^2) \\
h &= \frac{1}{2}m(\dot{r}^2 - \Omega^2 r^2)
\end{aligned}
$$

If we replace r and $\dot{r}$ we have that

$$
\begin{aligned}
h &= \frac{1}{2}m((\Omega a\sinh(\Omega t))^2 - \Omega^2(a\cosh(\Omega t))^2) \\
h &= \frac{1}{2}m\Omega^2 a^2(\sinh^2(\Omega t) - \cosh^2(\Omega t)) \\
h &= -\frac{1}{2}m\Omega^2 a^2
\end{aligned}
$$

We see that $-\frac{1}{2}m\Omega^2 a^2$ is a constant. Therefore the energy function is conserved. $\square$

Solution. 12.11 The system described looks like the following



Where our generalized coordinate is θ. Using the rolling condition of the yo-yo the kinetic energy in this case is

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{\boldsymbol{Z}} - \boldsymbol{r}\dot{\boldsymbol{\theta}})^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\dot{\boldsymbol{\theta}}^2 \\
T &= \frac{1}{2}m(\dot{Z}^2 - 2r\dot{\theta}\dot{Z} + (r\dot{\theta})^2) + \frac{1}{4}m(r\dot{\theta})^2 \\
T &= \frac{1}{2}m(\dot{Z}^2 - 2r\dot{\theta}\dot{Z}) + \frac{3}{4}m(r\dot{\theta})^2
\end{aligned}
$$

The potential energy in this case is

$$V = mg(Z - r\theta)$$

Then the Lagrange equation is given by

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} &= -\frac{\partial V}{\partial \theta} \\
\frac{d}{dt}\left[-mr\dot{Z} + \frac{3}{2}mr^2\dot{\theta}\right] - 0 &= mgr \\
-\ddot{Z} + \frac{3}{2}r\ddot{\theta} &= g
\end{aligned}
$$

So the acceleration of the yo-yo is

$$r\ddot{\theta} = \frac{2}{3}(\ddot{Z} + g)$$

The acceleration that the support must have so the center of the yo-yo can remain at rest can be determined by derivating the velocity of the yo-yo where $v = \dot{Z} - r\dot{\theta}$ i.e.

$$
\begin{aligned}
a &= \ddot{Z} - r\ddot{\theta} \\
&= \ddot{Z} - \frac{2}{3}(\ddot{Z} + g) \\
&= \frac{\ddot{Z} - 2g}{3}
\end{aligned}
$$

This implies that the support must have an acceleration of $\ddot{Z} = 2g$ so that the center of the yo-yo can remain at rest.

If the whole system starts from rest then by integrating the yo-yo acceleration twice we get that

$$r\dot{\theta} = \frac{2}{3}(\dot{Z} + gt) + C_1 \qquad r\theta = \frac{2}{3}\left(Z + \frac{1}{2}gt^2\right) + C_2$$

and since $\theta = \dot{\theta} = Z = \dot{Z} = 0$ when $t = 0$ we see that $C_1 = C_2 = 0$ hence

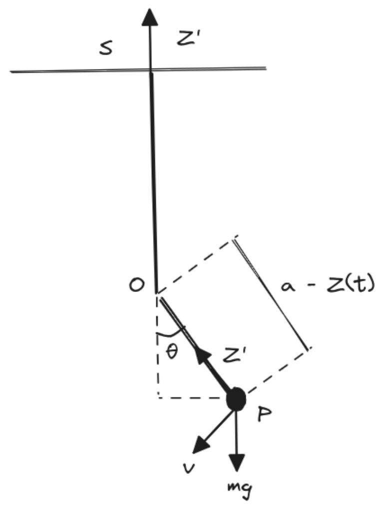
$$r\dot{\theta} = \frac{2}{3}(\dot{Z} + gt) \qquad r\theta = \frac{2}{3}\left(Z + \frac{1}{2}gt^2\right)$$

Now if we replace these values in T and V we get that $E = T + V$ for some time t is given by

$$E = \frac{1}{2}m(\dot{Z}^2 - 2r\dot{\theta}\dot{Z}) + \frac{3}{4}m(r\dot{\theta})^2 + mg(Z - r\theta)$$
$$E = \frac{1}{2}m(\dot{Z}^2 - \frac{4}{3}(\dot{Z} + gt)\dot{Z}) + \frac{3}{4}m(\frac{2}{3}(\dot{Z} + gt))^2 + mg\left(Z - \frac{2}{3}\left(Z + \frac{1}{2}gt^2\right)\right)$$
$$E = \frac{1}{2}m(-\frac{1}{3}\dot{Z}^2 - \frac{4}{3}gt\dot{Z}) + \frac{1}{3}m(\dot{Z}^2 + 2\dot{Z}gt + (gt)^2) + mg\left(\frac{1}{3}Z - \frac{1}{3}gt^2\right)$$
$$E = m\left[-\frac{1}{6}\dot{Z}^2 - \frac{2}{3}gt\dot{Z} + \frac{1}{3}\dot{Z}^2 + \frac{2}{3}\dot{Z}gt + \frac{1}{3}(gt)^2 + \frac{1}{3}gZ - \frac{1}{3}g^2t^2\right]$$
$$E = m\left[\frac{1}{6}\dot{Z}^2 + \frac{1}{3}gZ\right] = m\left[\frac{\dot{Z}^2 + 2gZ}{6}\right]$$

$\square$

Solution. 12.12 The system described looks like the following



Where our generalized coordinate is θ. At some time t the kinetic energy is given by

$$T = \frac{1}{2}m((\boldsymbol{a}-\boldsymbol{Z})\dot{\boldsymbol{\theta}} + \dot{\boldsymbol{Z}})^2$$
$$T = \frac{1}{2}m[(a-Z)^2\dot{\theta}^2 + \dot{Z}^2]$$

and the potential energy with respect to the ring is

$$V = -mg(a-Z)\cos\theta$$

then the Lagrangian is given by

$$L = \frac{1}{2}m((a-Z)^2\dot{\theta}^2 + \dot{Z}^2) + mg(a-Z)\cos\theta$$

The Lagrange equation is

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{\theta}}\right] - \frac{\partial T}{\partial\theta} = -\frac{\partial V}{\partial\theta}$$
$$\frac{d}{dt}\left[m(a-Z)^2\dot{\theta}\right] - 0 = -mg(a-Z)\sin\theta$$
$$(a-Z)^2\ddot{\theta} + 2\dot{Z}(Z-a)\dot{\theta} = -g(a-Z)\sin\theta$$
$$(a-Z)\ddot{\theta} - 2\dot{Z}\dot{\theta} = -g\sin\theta$$

On the other hand $E = T + V$ is

$$E = \frac{1}{2}m((a-Z)^2\dot{\theta}^2 + \dot{Z}^2) - mg(a-Z)\cos\theta$$

Finally, using the Lagrangian and replacing E at the end we compute the energy function h as follows

$$
\begin{aligned}
h &= m(a-Z)^2\dot{\theta}^2 - \frac{1}{2}m((a-Z)^2\dot{\theta}^2 + \dot{Z}^2) - mg(a-Z)\cos\theta \\
h &= \frac{1}{2}m((a-Z)^2\dot{\theta}^2 - \dot{Z}^2) - mg(a-Z)\cos\theta \\
h &= E - m\dot{Z}^2
\end{aligned}
$$

Neither h nor E are conserved because L depends on Z and Z depends on time. □

Solution. 12.14 We want to show that the force $\boldsymbol{F} = f(t)\boldsymbol{\nabla}W(\boldsymbol{r})$ can be represented by the time-dependent potential $U = -f(t)W(\boldsymbol{r})$.

Let $\boldsymbol{r} = (q_1, ..., q_n)$ be the vector of generalized coordinates so if each force coordinate can be written as follows

$$
F_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \left(\frac{\partial U}{\partial q_j}\right)
$$

then $\boldsymbol{F}$ can be represented by the time-dependent potential U. For the j-th generalized coordinate, we have that

$$
\begin{aligned}
F_j &= 0 - \left(-f(t)\frac{\partial W(\boldsymbol{r})}{\partial q_j}\right) \\
F_j &= f(t)\frac{\partial W(\boldsymbol{r})}{\partial q_j}
\end{aligned}
$$

Therefore $\boldsymbol{F}$ is given by

$$
\boldsymbol{F} = f(t)\left(\frac{\partial W(\boldsymbol{r})}{\partial q_1}, ..., \frac{\partial W(\boldsymbol{r})}{\partial q_n}\right) = f(t)\boldsymbol{\nabla}W(\boldsymbol{r})
$$

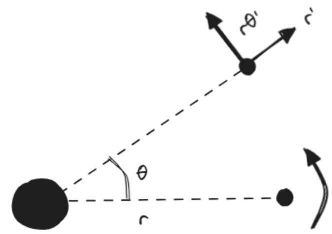
which implies that $\boldsymbol{F}$ can be represented by the time-dependent potential U.

Finally, if $\boldsymbol{F} = f(t)\boldsymbol{i}$ we have that U must be $U = -f(t)x\boldsymbol{i}$ hence

$$
\begin{aligned}
F_x &= 0 - \left(-f(t)\frac{\partial x}{\partial x}\boldsymbol{i}\right) \\
F_x &= f(t)\boldsymbol{i}
\end{aligned}
$$

where we assumed that $\boldsymbol{F}$ depends on a position vector $\boldsymbol{r} = (x\boldsymbol{i}, y\boldsymbol{j}, z\boldsymbol{k})$ where $y = z = 0$. □

Solution. 12.17 The system described looks like the following



Where our generalized coordinates are r and θ. The kinetic energy in this case is given by

$$
\begin{aligned}
T &= \frac{1}{2}m(\boldsymbol{r}\dot{\boldsymbol{\theta}} + \dot{\boldsymbol{r}})^2 \\
T &= \frac{1}{2}m((r\dot{\theta})^2 + 2r\dot{\theta}\dot{r}\cos(\pi/2) + \dot{r}^2) \\
T &= \frac{1}{2}m((r\dot{\theta})^2 + \dot{r}^2)
\end{aligned}
$$

and for the potential energy, we have that

$$
V = -G\frac{mM}{r}
$$

Hence the Lagrange equations are

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} &= -\frac{\partial V}{\partial \theta} \\
\frac{d}{dt}(mr^2\dot{\theta}) - 0 &= 0 \\
m2r\dot{r}\dot{\theta} + mr^2\ddot{\theta} &= 0 \\
2r\dot{r}\dot{\theta} + r^2\ddot{\theta} &= 0
\end{aligned}
$$

And for r we have that

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{r}}\right] - \frac{\partial T}{\partial r} &= -\frac{\partial V}{\partial r} \\
m\ddot{r} - mr\dot{\theta}^2 &= -G\frac{mM}{r^2} \\
\ddot{r} - r\dot{\theta}^2 &= -G\frac{M}{r^2}
\end{aligned}
$$

On the other hand, the Lagrangian is given by

$$
L = \frac{1}{2}m(r\dot{\theta})^2 + G\frac{mM}{r}
$$

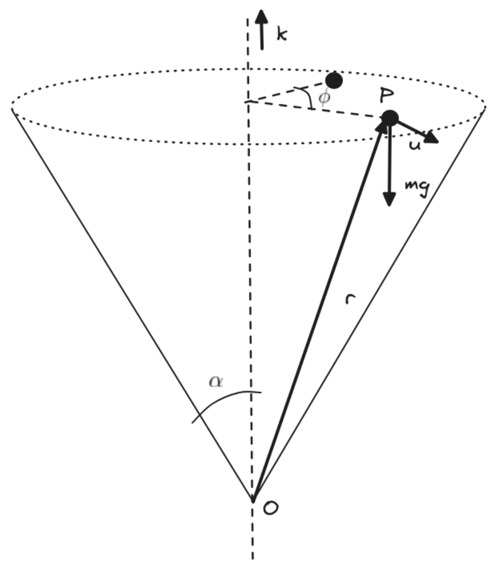
Where θ does not appear, hence θ is a cyclic coordinate.

Finally, the generalized momentum p_θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = mrv$$

where $v = r\dot{\theta}$ is the tangential velocity and p_θ is the component of the angular momentum about O perpendicular to the plane of motion. $\square$

Solution. 12.18 The system in this case looks like the following



Where our generalized coordinates are r and ϕ. The kinetic energy in this case is given by

$$T = \frac{1}{2}m(\boldsymbol{r\sin\alpha\dot{\phi}} + \dot{\boldsymbol{r}})^2$$
$$T = \frac{1}{2}m((r\sin\alpha\dot{\phi})^2 + \dot{r}^2)$$

and for the potential energy, we have that

$$V = mg(r\cos\alpha)$$

Hence the Lagrange equations are

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{\phi}}\right] - \frac{\partial T}{\partial\phi} = -\frac{\partial V}{\partial\phi}$$
$$\frac{d}{dt}(m(r\sin\alpha)^2\dot{\phi}) - 0 = 0$$
$$m2r\sin^2\alpha\dot{r}\dot{\phi} + m(r\sin\alpha)^2\ddot{\phi} = 0$$
$$2\sin\alpha\dot{r}\dot{\phi} + r\sin\alpha\ddot{\phi} = 0$$

And for r we have that

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{r}}\right] - \frac{\partial T}{\partial r} = -\frac{\partial V}{\partial r}$$
$$\frac{d}{dt}(m\dot{r}) - m(\sin\alpha\dot{\phi})^2 r = -mg\cos\alpha$$
$$\ddot{r} - (\sin\alpha\dot{\phi})^2 r = -g\cos\alpha$$

On the other hand, the Lagrangian is given by

$$L = \frac{1}{2}m((r\sin\alpha\dot{\phi})^2 + \dot{r}^2) - mg(r\cos\alpha)$$

Where ϕ does not appear, hence ϕ is a cyclic coordinate.

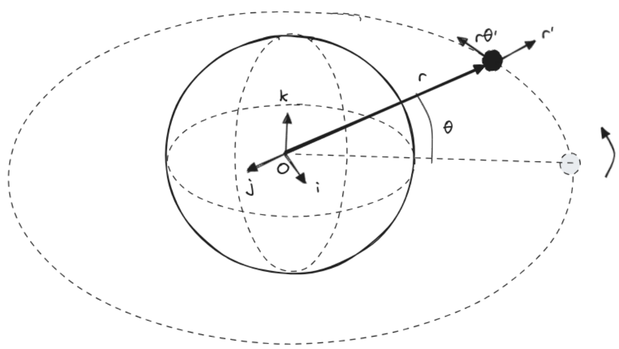
Finally, the generalized momentum p_ϕ is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m(r\sin\alpha)^2\dot{\phi}$$

p_ϕ is the component of the angular momentum about O in the $\boldsymbol{k}$ direction hence perpendicular to the plane of motion. $\square$

Solution. 12.20

(i) The system in this case looks like the following



From Chapter 3 we know that the gravitational force exerted to a particle from a sphere is given by

$$F = -G\frac{mM}{r^2}\boldsymbol{r}$$

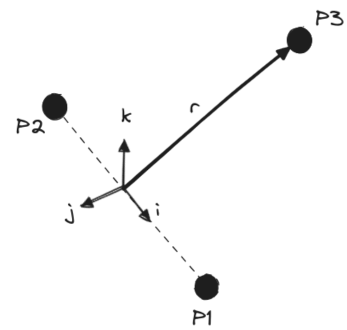
Where M is the mass of the sphere and m is the mass of the particle. So the Potential Energy is given by

$$V = G\frac{mM}{r}$$

From Theorem 12.2 given that the system (the particle) can be rotated about any of the following axes $\{O, \boldsymbol{k}\}$, $\{O, \boldsymbol{i}\}$ or $\{O, \boldsymbol{j}\}$ and V is unchanged since it does not depend on any of the spherical coordinates, we get that the full angular momentum $\boldsymbol{L}$ must be conserved.

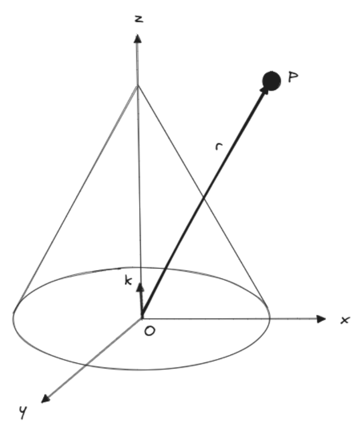
(ii) In this case let us examine the half x-y plane i.e. the set $\{(x, y, z) : x \geq 0 \text{ and } z = 0\}$ since the particle could be anywhere in the space the only movement we can do without affecting V is a translation in the y direction hence the conserved quantity is the linear momentum in the y direction P_y.

(iii) The system in this case looks like the following



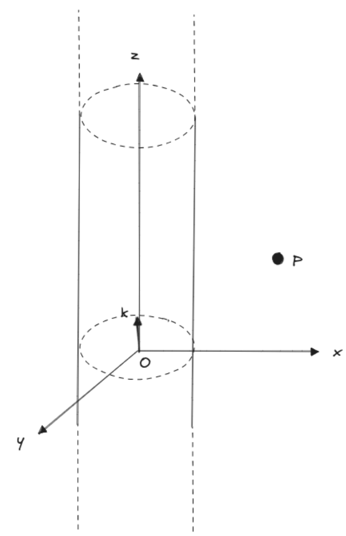
where the particle P_1 and P_2 are fixed and P_3 can move freely in the space. From Theorem 12.2 given that the particle can be rotated about the axis $\{O, \boldsymbol{i}\}$ and V is unchanged we get that the conserved quantity is the i component of the angular momentum i.e. L_x is conserved.

(iv) The system in this case looks like the following



where the right cone has its base fixed at the origin and P can move freely. The particle can be rotated about the axis $\{O, \boldsymbol{k}\}$ independent of the position vector $\boldsymbol{r}$ and V is unchanged so we get that the conserved quantity is the z component of the angular momentum i.e. L_z is conserved.

(v) Finally, the system in this case looks like the following



where the infinite uniform circular cylinder extends in the z direction and P, the particle, can move freely in the space. The particle can be rotated about the axis $\{O, \boldsymbol{k}\}$ and/or translated in the z direction and V would be unchanged so we get that the conserved quantities are the z component of the angular momentum L_z and the z component of the linear momentum P_z.

$\square$

Solution. 12.21 Joining the results we got from Theorem 12.1 and 12.2 we see that in our case it must happen that

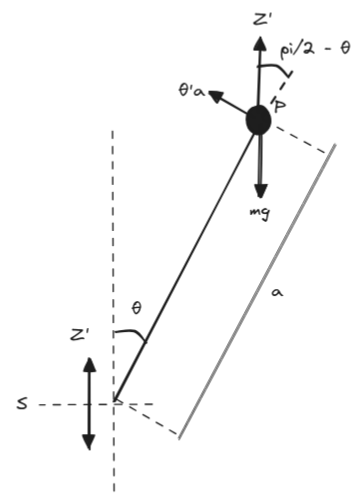
$$\frac{\partial \boldsymbol{r}_i^\lambda}{\partial \lambda} = c\boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{r}_i^\lambda$$

Since it's the combination of two displacements a $c\lambda$ translation in the $\boldsymbol{k}$ direction and a rotation of λ with respect to the axis $\boldsymbol{k}$. When $\lambda = 0$ we have that $\boldsymbol{r}_i^\lambda = \boldsymbol{r}_i$ hence

$$\left[\frac{\partial \boldsymbol{r}_i^\lambda}{\partial \lambda}\right]_{\lambda=0} = c\boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{r}_i$$

Therefore this implies that the conserved quantity must be $cP_z + L_z$ where P_z is the linear momentum in the $\boldsymbol{k}$ direction and L_z is the angular momentum about the axis $\{O, \boldsymbol{k}\}$. $\square$

Solution. 12.22 The system in this case looks like the following



Where our generalized coordinate is θ. The kinetic energy in this case is given by

$$T = \frac{1}{2}m(\boldsymbol{a\dot{\theta}} + \boldsymbol{\dot{Z}})^2$$
$$T = \frac{1}{2}m[(a\dot{\theta})^2 + 2a\dot{\theta}\dot{Z}\cos(\pi/2 - \theta) + \dot{Z}^2]$$
$$T = \frac{1}{2}m[(a\dot{\theta})^2 + 2a\dot{\theta}\dot{Z}\sin(\theta) + \dot{Z}^2]$$

And for the potential energy, we have that

$$V = mg(a\cos\theta + Z)$$

The Lagrange equation is then

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{\theta}}\right] - \frac{\partial T}{\partial\theta} = -\frac{\partial V}{\partial\theta}$$
$$m\frac{d}{dt}\left[a^2\dot{\theta} + a\dot{Z}\sin(\theta)\right] - ma\dot{\theta}\dot{Z}\cos(\theta) = mga\sin(\theta)$$
$$a^2\ddot{\theta} + a\ddot{Z}\sin(\theta) + a\dot{Z}\dot{\theta}\cos(\theta) - a\dot{\theta}\dot{Z}\cos(\theta) = ga\sin(\theta)$$
$$a\ddot{\theta} - (g - \ddot{Z})\sin(\theta) = 0$$

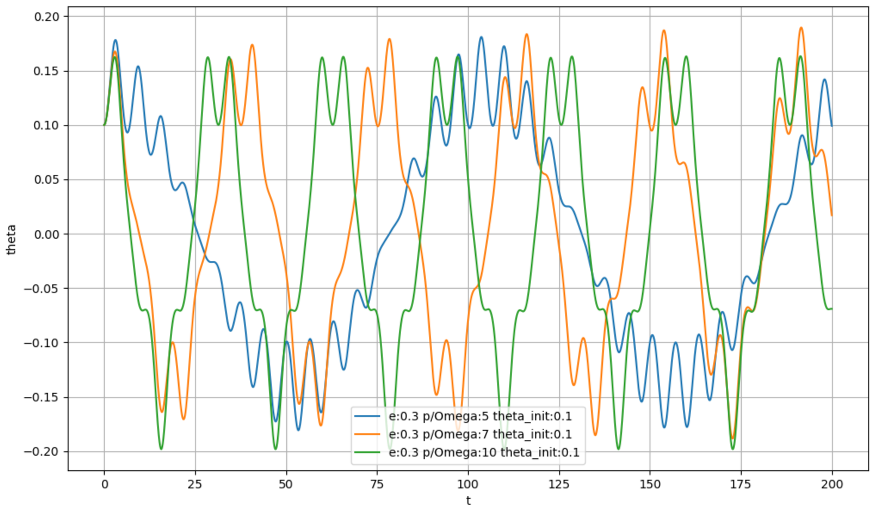
We know that $Z = \epsilon a\cos pt$ hence $\ddot{Z} = -p^2\epsilon a\cos pt$ so by replacing this value and setting $\Omega^2 = g/a$, we have that

$$\ddot{\theta} - \left(\frac{g}{a} + \epsilon p^2\cos pt\right)\sin(\theta) = 0$$
$$\ddot{\theta} - \left(\Omega^2 + \epsilon p^2\cos pt\right)\sin(\theta) = 0$$

Setting a dimensionless time $\tau = pt$ we have that

$$\frac{d^2\theta}{d\tau^2} - \left(\frac{\Omega^2}{p^2} + \epsilon\cos\tau\right)\sin(\theta) = 0$$

Now we solve the equation numerically to obtain $\theta(\tau)$ for different parameters as follows



$\square$