Solved selected problems of Classical Mechanics - Gregory

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Chapter 13 - The Calculus of Variations and Hamilton's Principle

Solution. 13.1 By definition, extremals are solutions of the Euler-Lagrange equation. In this case $F = \dot{x}^2/t^3$ so we have that

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = \frac{2\dot{x}}{t^3}$$

Hence the Euler-Lagrange equation takes the form

$$\begin{aligned}\frac{d}{dt}\left(\frac{2\dot{x}}{t^3}\right) - 0 &= 0\\ \frac{2\ddot{x}}{t^3} - \frac{6\dot{x}}{t^4} &= 0\\ \ddot{x} - \frac{3\dot{x}}{t} &= 0\end{aligned}$$

Now we solve the equation by setting $v = \dot{x}$ hence

$$\begin{aligned}\frac{dv}{dt} - \frac{3v}{t} &= 0\\ \int \frac{dv}{3v} &= \int \frac{dt}{t}\\ \frac{\log(v)}{3} &= \log(t) + C\\ v &= Ct^3\end{aligned}$$

by replacing again and solving the equation we have that

$$\begin{aligned}\frac{dx}{dt} &= Ct^3\\ \int dx &= C\int t^3 dt\\ x &= Ct^4 + D\end{aligned}$$

The admissible extremals are those that satisfy the conditions $x(1) = 3$ and $x(2) = 18$ this way we find the values of $C = 1$ and $D = 2$ so the only admissible extremal of $J[x]$ is given by

$$\hat{x} = t^4 + 2$$

Finally, we want to show that this extremal provides a global minimum of $J[x]$. Let h be any admissible variation and consider the variation in J that it produces

$$
\begin{aligned}
J[\hat{x}+h]-J[\hat{x}] &= \int_1^2 \frac{(4t^3+\dot{h})^2}{t^3}dt - \int_1^2 \frac{(4t^3)^2}{t^3}dt \\
&= \int_1^2 16t^3 + 8\dot{h} + \frac{\dot{h}^2}{t^3}dt - \int_1^2 16t^3 dt \\
&= 8\Big[h\Big]_{t=1}^{t=2} + \int_1^2 \frac{\dot{h}^2}{t^3}dt \\
&= \int_1^2 \frac{\dot{h}^2}{t^3}dt
\end{aligned}
$$

Where we used that h is an admissible extremal hence it must satisfy $h(1) = h(2) = 0$. Since the integral of a positive function must be positive we see that

$$J[\hat{x}+h]-J[\hat{x}] = \int_1^2 \frac{\dot{h}^2}{t^3}dt \geq 0$$

Thus $\hat{x}$ provides the global minimum of $J[x]$. The global minimum value of J is therefore $J[t^4+2] = 60$.

$\square$

Solution. 13.3 We want to find the extremals of the path functional

$$L[y] = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

By definition, extremals are solutions of the Euler-Lagrange equation. In this case $F(y, \dot{y}) = \sqrt{1 + \dot{y}^2}$ so there is no explicit dependence on x this implies that if we satisfy the equation

$$\dot{y}\frac{\partial F}{\partial \dot{y}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned}
\dot{y}\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} - \sqrt{1+\dot{y}^2} &= C \\
\frac{\dot{y}^2 - 1 - \dot{y}^2}{\sqrt{1+\dot{y}^2}} &= C \\
\sqrt{1+\dot{y}^2} &= -\frac{1}{C} \\
\dot{y}^2 &= -1 + \frac{1}{C^2} \\
\int y &= \frac{\sqrt{1-C^2}}{C^2} \int dx \\
y &= \frac{\sqrt{1-C^2}}{C^2} x + D
\end{aligned}$$

Where D is a constant of integration. Now applying the given initial and end conditions we get that $C = 1$ and $D = 0$ so the admissible extremal is

$$\hat{y} = 0$$

But this is a constant solution so it may or may not be a solution to the Euler-Lagrange equation, we must check, hence we have that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} &= 0 \\
\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) - 0 &= 0 \\
\frac{\ddot{y}}{(1+\dot{y}^2)^{3/2}} &= 0
\end{aligned}$$

So $\hat{y} = 0$ is a solution to the Euler-Lagrange equation and therefore an extremal.

Finally, we want to check $\hat{y}$ provides a global minimum for L. Let h be any admissible variation and consider the variation in L that it produces

$$\begin{aligned} L[\hat{y}+h] &= \int_0^1 \sqrt{1+(\dot{\hat{y}}+\dot{h})^2}dx \\ &= \int_0^1 \sqrt{1+\dot{h}^2}dx \end{aligned}$$

Also, we know that

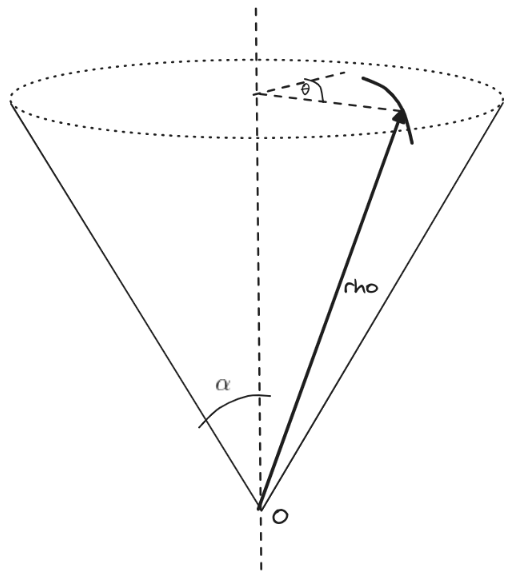
$$\int_0^1 \sqrt{1+\dot{\hat{y}}^2}dx = \int_0^1 dx = 1$$

And we have that

$$\int_0^1 \sqrt{1+\dot{h}^2}dx \geq 1 = \int_0^1 dx$$

Therefore $\hat{y}$ provides a local minimum for L. □

Solution. 13.5 Let us analyze the following system



From this, we can write the differential length as

$$(ds)^2 = (d\rho)^2 + (\rho \sin \alpha d\theta)^2$$

So integrating we get the length functional as follows

$$L = \int_{-\pi/2}^{\pi/2} ds = \int_{-\pi/2}^{\pi/2} \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + (\rho \sin \alpha)^2}\ d\theta$$

In this case, we have that $F(\rho, \dot{\rho}) = \sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2}$ so there is no explicit dependence on θ this implies that if we satisfy the equation

$$\dot{\rho}\frac{\partial F}{\partial \dot{\rho}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned}
\dot{\rho}\frac{\dot{\rho}}{\sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2}} - \sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2} &= C \\
\frac{\dot{\rho}^2 - \dot{\rho}^2 - (\rho \sin \alpha)^2}{\sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2}} &= C \\
\sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2} &= -\frac{(\rho \sin \alpha)^2}{C} \\
\dot{\rho}^2 &= -(\rho \sin \alpha)^2 + \frac{(\rho \sin \alpha)^4}{C^2} \\
\frac{d\rho}{d\theta} &= (\rho \sin \alpha)\sqrt{\frac{(\rho \sin \alpha)^2}{C^2} - 1}
\end{aligned}$$

Now we can integrate the equation as follows

$$
\begin{aligned}
\int \frac{d\rho}{(\rho \sin\alpha)\sqrt{\frac{(\rho\sin\alpha)^2}{C^2} - 1}} &= \int d\theta \\
\frac{\arctan\left(\sqrt{\frac{(\rho\sin\alpha)^2}{C^2} - 1}\right)}{\sin\alpha} &= \theta + D \\
\arctan\left(\sqrt{\frac{(\rho\sin\alpha)^2}{C^2} - 1}\right) &= (\theta + D)\sin\alpha \\
\frac{(\rho\sin\alpha)^2}{C^2} - 1 &= \tan^2\left((\theta + D)\sin\alpha\right) \\
\rho &= \frac{C}{\sin\alpha}\sqrt{\tan^2\left((\theta + D)\sin\alpha\right) + 1} \\
\rho &= \frac{C}{\sin\alpha}\sec((\theta + D)\sin\alpha)
\end{aligned}
$$

This is the equation for the extremals of L. Now applying the given initial and end conditions we get that

$$
a = \frac{C}{\sin\alpha}\sec(-(\pi/2 - D)\sin\alpha) = \frac{C}{\sin\alpha}\sec((\pi/2 + D)\sin\alpha)
$$

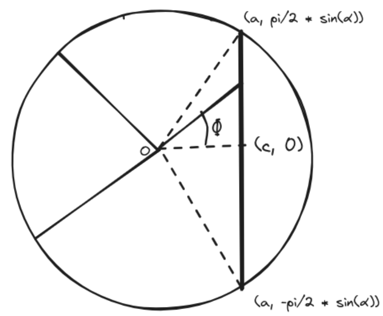
which implies that $D = 0$ and C is

$$
\begin{aligned}
C\sec\left(\frac{\pi\sin\alpha}{2}\right) &= a\sin\alpha \\
C &= \frac{a\sin\alpha}{\sec\left(\frac{\pi\sin\alpha}{2}\right)}
\end{aligned}
$$

Replacing the values for C and D we get that the admissible extremal is

$$
\begin{aligned}
\rho &= \frac{a\sec(\theta\sin\alpha)}{\sec\left(\frac{\pi\sin\alpha}{2}\right)} \\
\rho &= \frac{a\cos\left(\frac{\pi\sin\alpha}{2}\right)}{\cos(\theta\sin\alpha)}
\end{aligned}
$$

Finally, we want to verify that this extremal is the same as the shortest path that would be obtained by developing the cone onto a plane. If we develop the cone onto a plane, we get that the path is a vertical line in the plane as shown below



where the angle in the plane is given by $\phi = \theta \sin \alpha$. We know that the equation of a straight vertical line in polar coordinates is given by

$$\rho \cos(\phi) = c$$

where c is the value of ρ when $\phi = 0$ and from the drawing above we have that $\cos(\pi/2 \sin(\alpha)) = c/a$ hence

$$\rho \cos(\theta \sin \alpha) = a \cos\left(\frac{\pi \sin(\alpha)}{2}\right)$$

Which is the admissible extremal that we have such that it satisfies the Euler-Lagrange equation. $\square$