Solved selected problems of Classical Mechanics - Taylor

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Chapter 1 - Newton's laws of Motion

Proof. **1.10** The cartesian components x and y for the vector r are

$$x = R\cos\theta \qquad y = R\sin\theta$$

we also know that $\theta = \omega t$ so we have that

$$\begin{aligned} \boldsymbol{r} &= \hat{\boldsymbol{x}}R\cos\theta + \hat{\boldsymbol{y}}R\sin\theta \\ &= \hat{\boldsymbol{x}}R\cos\omega t + \hat{\boldsymbol{y}}R\sin\omega t \end{aligned}$$

To find the particle's velocity and acceleration we must differetiate the $\boldsymbol{r}$ equation with respect to time.
For the velocity we have that

$$\begin{aligned} \frac{d\boldsymbol{r}}{dt} &= \frac{d}{dt}(\hat{\boldsymbol{x}}R\cos\omega t + \hat{\boldsymbol{y}}R\sin\omega t) \\ \boldsymbol{v} &= -\hat{\boldsymbol{x}}R\omega\sin\omega t + \hat{\boldsymbol{y}}R\omega\cos\omega t \end{aligned}$$

For the acceleration we need to differentiate the velocity again

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= \frac{d}{dt}(-\hat{\boldsymbol{x}}R\omega\sin\omega t + \hat{\boldsymbol{y}}R\omega\cos\omega t) \\ \boldsymbol{a} &= -\hat{\boldsymbol{x}}R\omega^2\cos\omega t - \hat{\boldsymbol{y}}R\omega^2\sin\omega t \end{aligned}$$

as noticed the direction of the acceleration is oposite to $\boldsymbol{r}$. Finally, the magnitude of the acceleration is

$$\begin{aligned} |\boldsymbol{a}| &= \sqrt{(-R\omega^2\cos\omega t)^2 + (-R\omega^2\sin\omega t)^2} \\ &= \sqrt{R^2\omega^4(\cos^2\omega t + \sin^2\omega t)} \\ &= R\omega^2 \end{aligned}$$

$\square$

Proof. **1.11** Given that the position of the particle is given by

$$\boldsymbol{r}(t) = \hat{\boldsymbol{x}} b \cos \omega t + \hat{\boldsymbol{y}} c \sin \omega t$$

Then the cartesian coordinates for the particle are $x = b \cos \omega t$ and $y = c \sin \omega t$

We can derive from the identity $\cos^2 \omega t + \sin^2 \omega t = 1$ the equation that involves both cartesian coordinates as follows.

$$\begin{aligned} \cos^2 \omega t + \sin^2 \omega t &= 1 \\ \frac{b^2}{b^2} \cos^2 \omega t + \frac{c^2}{c^2} \sin^2 \omega t &= 1 \\ \frac{x^2}{b^2} + \frac{y^2}{c^2} &= 1 \end{aligned}$$

Which is the equation for an ellipse centered at the origin. □

Proof. **1.12** In this case, the particle in addition to being rotating around an ellipse trajectory as we already know from the last problem is also moving in the positive direction around the z-axis so the resultant trajectory is a helicoid centered around z-axis. □

Proof. **1.26**

(a) Given a reference frame $\mathcal{S}$ where the x-axis point east and y-axis point north, if we kick a frictionless puck due north, from this reference frame, there is no x-axis movement for the puck and it moves to north with constant velocity, then the coordinates for the puck are

$$x = 0 \qquad y = vt$$

where v is the constant velocity at which the puck is moving.

(b) In a reference frame $\mathcal{S}'$ which is moving with constant velocity v' due east with respect to $\mathcal{S}$ the puck movement seen from $\mathcal{S}'$ has an added movement in the $-x$ direction since $\mathcal{S}'$ is moving away from the puck so the coordinates for the puck seen from $\mathcal{S}'$ are

$$x' = -v't \qquad y' = vt$$

this means that the puck has a trajectory given by $y' = -v/v'x'$ which is a line with a negative slope that passes through the origin.

(c) Lastly the reference frame $\mathcal{S}''$ is moving with constant acceleration due east with respect to $\mathcal{S}$ so the puck movement seen from $\mathcal{S}''$ has an added movement in the $-x$ direction since $\mathcal{S}''$ is moving away from the puck, therefore the coordinates for the puck seen from $\mathcal{S}''$ are

$$x'' = -(v''t + \frac{1}{2}a''t^2) \qquad y'' = vt$$

where v'' and a'' are the velocity and acceleration of $\mathcal{S}''$ respectively.

Since $\mathcal{S}$ and $\mathcal{S}'$ are not accelerating they are inertial reference frames, which is not the case for $\mathcal{S}''$. □

Proof. **1.27** The puck path observed by someone sitting at rest at the origin of the turntable is a spiral that grows indefinitely as the puck continues its path to infinity. □

Proof. **1.30** The total momentum of the system before the collision is given by

$$\boldsymbol{P} = m_1\boldsymbol{v} + m_2 0 = m_1\boldsymbol{v}$$

but after the collision the total momentum is given by

$$\boldsymbol{P} = m_1\boldsymbol{v'} + m_2\boldsymbol{v'}$$

since there is no external forces, then the total momentum of the system is constant, therefore

$$\begin{aligned} m_1\boldsymbol{v'} + m_2\boldsymbol{v'} &= m_1\boldsymbol{v} \\ \boldsymbol{v'}(m_1 + m_2) &= m_1\boldsymbol{v} \\ \boldsymbol{v'} &= \frac{m_1}{m_1 + m_2}\boldsymbol{v} \end{aligned}$$

□

Proof. **1.40**

(a) According to the Newton's second Law we have that $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ or if we write it in cartesian coordinates for two dimensions then

$$F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}} = m\ddot{x}\hat{\boldsymbol{x}} + m\ddot{y}\hat{\boldsymbol{y}}$$

given that the only force is the gravitational force and is in the y-direction we have that

$$m\ddot{x} = 0 \qquad m\ddot{y} = -mg$$

therefore integrating both equations we have that

$$\begin{aligned} \dot{x} &= \int 0\ dt \\ &= 0 + v_{0x} \\ x &= \int v_{0x}\ dt \\ &= v_{0x}t + x_0 \\ &= v_{0x}t \end{aligned}$$

where v_{0x} and x_0 are the initial velocity and position of the ball in the x-direction, note that $x_0 = 0$. Also we have that $v_{0x} = v_0\cos\theta$ so the final equation is

$$x = v_0 t\cos\theta$$

In the same way for the y-direction we have that

$$\begin{aligned}
\dot{y} &= \int -g \ dt \\
&= v_{0y} - gt \\
y &= \int v_{0y} - gt \ dt \\
&= y_0 + v_{0y}t - g\frac{t^2}{2} \\
&= v_{0y}t - g\frac{t^2}{2}
\end{aligned}$$

As we know from the x-direction $v_{0y} = v_0 \sin\theta$ so the final equation becomes

$$y = v_0 t \sin\theta - g\frac{t^2}{2}$$

(b) We write the r^2 (the magnitude of $\boldsymbol{r}$ squared) as

$$\begin{aligned}
r^2 &= (v_0 t \cos\theta)^2 + \left(v_0 t \sin\theta - g\frac{t^2}{2}\right)^2 \\
&= (v_0 t \cos\theta)^2 + (v_0 t \sin\theta)^2 - 2g\frac{t^2}{2} v_0 t \sin\theta + \left(-g\frac{t^2}{2}\right)^2 \\
&= (v_0 t)^2 (\cos^2\theta + \sin^2\theta) - g v_0 t^3 \sin\theta + \left(-g\frac{t^2}{2}\right)^2 \\
&= (v_0 t)^2 - g v_0 t^3 \sin\theta + \left(-g\frac{t^2}{2}\right)^2
\end{aligned}$$

To check whether a function increases or decreases we need to analyze the equation's derivative. Then

$$\dot{r^2} = 2v_0^2 t - 3g v_0 t^2 \sin\theta + g^2 t^3$$

What we want is that $2v_0^2 t - 3g v_0 t^2 \sin\theta + g^2 t^3 > 0$. The minimum of $\dot{r^2}$ can be found by differentiating this equation again and solving for $\ddot{r^2} = 0$, so

$$2v_0^2 - 6g v_0 t \sin\theta + 3g^2 t^2 = 0$$

$$\begin{aligned}
t_{min} &= \frac{6g v_0 \sin\theta \pm \sqrt{(-6g v_0 \sin\theta)^2 - 24g^2 v_0 2}}{6g^2} \\
&= \frac{6g v_0 \sin\theta \pm \sqrt{36g^2 v_0^2 \sin^2\theta - 24g^2 v_0 2}}{6g^2} \\
&= \frac{6g v_0 \sin\theta \pm 6g v_0 \sqrt{\sin^2\theta - 24/36}}{6g^2} \\
&= \frac{v_0}{g}(\sin\theta \pm \sqrt{\sin^2\theta - 6/9})
\end{aligned}$$

The next step would be to write $\dot{r}^2$ valued at t_{min} and find where this expression is > 0 to determine the value for θ.

□

Proof. **1.41** The Newton's second law in polar coordinates are

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$$F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

where in this case $\dot{\phi} = \omega$ and since ω is constant $\ddot{\phi} = 0$ also $r = R$ which is also constant then $\dot{r} = \ddot{r} = 0$, therefore

$$F_r = -mR\omega^2 \qquad F_\phi = 0$$

Since the tension is the only force in the r direction we have that

$$T = -mR\omega^2$$

□

Proof. **1.46**

(a) For an observer on the ground in a reference frame $\mathcal{S}$ with its origin O at the center of the turntable the puck will have polar coordinates given by

$$\boldsymbol{\phi} = 0$$

$$\boldsymbol{r} = (R - vt)\hat{\boldsymbol{r}}$$

where R is the radius of the turntable and v the velocity of the puck.

(b) In the case that the observer is fixed to the turntable with its origin O at the center of the turntable the puck will have the same $\boldsymbol{r}$ polar coordinate but now $\boldsymbol{\phi}$ is changing according to the rotation of the reference frame therefore

$$\boldsymbol{\phi'} = \omega t\hat{\boldsymbol{\phi}}$$

$$\boldsymbol{r'} = (R - vt)\hat{\boldsymbol{r}}$$

where ω is the angular velocity of the turntable. This is not an inertial reference frame since it's rotating.

□

Proof. **1.47**

(a) The expressions for ρ, ϕ, and z in terms of x, y, and z are

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \arctan(y/x)$$

$$z = z$$

In words, ρ is the distance to P' from the origin, where P' is the projection of P to the plane $(x, y, 0)$.
The use of r to name ρ in this case is unfortunate since we are naming r as the magnitude of the vector $\boldsymbol{r}$ which is the distance to P from the origin.

(b) The unit vector $\hat{\boldsymbol{\rho}}$ is the same unit vector we defined previously for two dimensions problems as $\hat{\boldsymbol{r}}$ and the same thing happening with $\hat{\boldsymbol{\phi}}$. In the case of $\hat{\boldsymbol{z}}$ is a unit vector that points in the direction of the z cartesian coordinate.
The position vector in this case is given by

$$\boldsymbol{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}$$

(c) Given that $\hat{\boldsymbol{z}}$ is the same unit vector as in cartesian coordinates then $d\hat{\boldsymbol{z}}/dt = 0$ also using the result we obtained for $d\hat{\boldsymbol{r}}/dt = d\hat{\boldsymbol{\rho}}/dt$ and $d\hat{\boldsymbol{\phi}}/dt$ we get that

$$\begin{aligned}\dot{\boldsymbol{r}} = \boldsymbol{v} &= \left(\dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt}\right) + \dot{z}\hat{\boldsymbol{z}} \\ &= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\boldsymbol{z}}\end{aligned}$$

Then

$$\begin{aligned}\ddot{\boldsymbol{r}} = \boldsymbol{a} &= \left(\ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt}\right) + \left((\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt}\right) + \ddot{z}\hat{\boldsymbol{z}} \\ &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\boldsymbol{z}}\end{aligned}$$

□

Proof. **1.48** The projection of $\hat{\boldsymbol{\rho}}$ to the x-axis is given by $\cos\phi$ and the projection to the y-axis is given by $\sin\phi$ then

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$$

since $\hat{\boldsymbol{\phi}}$ is perpendicular to $\hat{\boldsymbol{\rho}}$ then

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi$$

and since $\hat{\boldsymbol{z}}$ is the same as the one used in cartesian coordinates

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

The derivatives of the above equations given that $d\hat{\boldsymbol{x}}/dt = 0$, $d\hat{\boldsymbol{y}}/dt = 0$, and $d\hat{\boldsymbol{z}}/dt = 0$ are

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\rho}}}{dt} &= (\frac{d\hat{\boldsymbol{x}}}{dt}\cos\phi - \hat{\boldsymbol{x}}\dot{\phi}\sin\phi) + (\frac{d\hat{\boldsymbol{y}}}{dt}\sin\phi + \hat{\boldsymbol{y}}\dot{\phi}\cos\phi) \\
&= -\hat{\boldsymbol{x}}\dot{\phi}\sin\phi + \hat{\boldsymbol{y}}\dot{\phi}\cos\phi \\
&= \dot{\phi}(-\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi) \\
&= \dot{\phi}\hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\phi}}}{dt} &= -(\frac{d\hat{\boldsymbol{x}}}{dt}\sin\phi + \hat{\boldsymbol{x}}\dot{\phi}\cos\phi) + (\frac{d\hat{\boldsymbol{y}}}{dt}\cos\phi - \hat{\boldsymbol{y}}\dot{\phi}\sin\phi) \\
&= -\hat{\boldsymbol{x}}\dot{\phi}\cos\phi - \hat{\boldsymbol{y}}\dot{\phi}\sin\phi \\
&= -\dot{\phi}(\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi) \\
&= -\dot{\phi}\hat{\boldsymbol{\rho}}
\end{aligned}$$

$$\frac{d\hat{\boldsymbol{z}}}{dt} = 0$$

□

Proof. **1.49** Newton's second law equations in cylindrical coordtinates are

$$\begin{aligned}
F_\phi &= m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\
F_\rho &= m(\ddot{\rho} - \rho\dot{\phi}^2) \\
F_z &= m\ddot{z}
\end{aligned}$$

Given that the puck is fixed to a certain $\rho = R$ then $\dot{\rho} = \ddot{\rho} = 0$ also since the gravitational force is in the z direction, we have that

$$0 = mR\ddot{\phi} \tag{1}$$
$$N_1 - N_2 = N = -mR\dot{\phi}^2 \tag{2}$$
$$-mg = m\ddot{z} \tag{3}$$

where N_1 and N_2 are the two normal forces exerted by the two cylinders, we named the difference between them N.
Integrating the equation (1) we have that

$$\begin{aligned}\dot{\phi} = \omega &= \int 0dt \\ &= 0 + \omega_0\end{aligned}$$

where ω_0 is the angular velocity and integrating again we have that

$$\begin{aligned}\phi &= \int \omega_0 dt \\ &= \omega_0 t + \phi_0\end{aligned}$$

Then integrating the equation (3), we have that

$$\begin{aligned}\dot{z} = v_z &= \int -g dt \\ &= -gt + v_{z0}\end{aligned}$$

and integrating again we have that

$$\begin{aligned}z &= \int (-gt + v_{z0}) dt \\ &= -g\frac{t^2}{2} + v_{z0}t + z_0\end{aligned}$$

Then the movement in the z-direction is as a free falling object and in the ϕ-direction is a rotation with constant angular velocity. Therefore the final trajectory is a downward helix. □

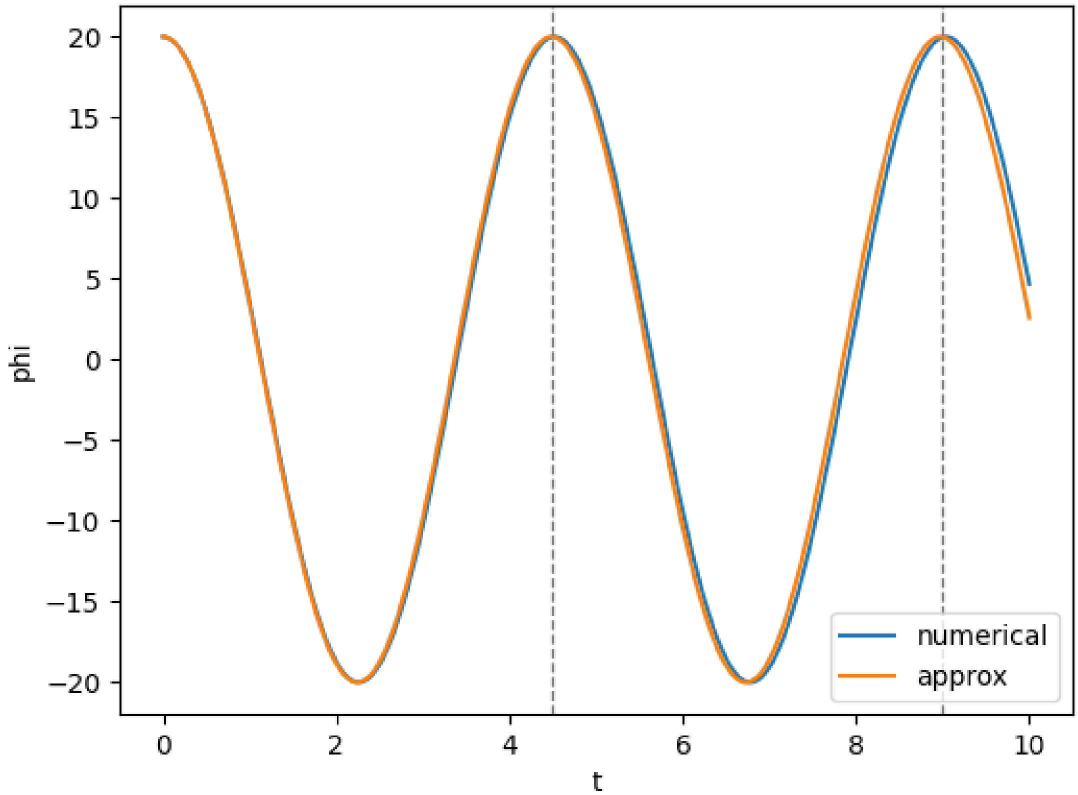
Proof. **1.50** We can transform the 2nd order equation

$$\ddot{\phi} = -\frac{g}{R}\sin\phi$$

into two 1st order equation as

$$\begin{aligned}\dot{\phi} &= \omega \\ \dot{\omega} &= -\frac{g}{R}\sin\phi\end{aligned}$$

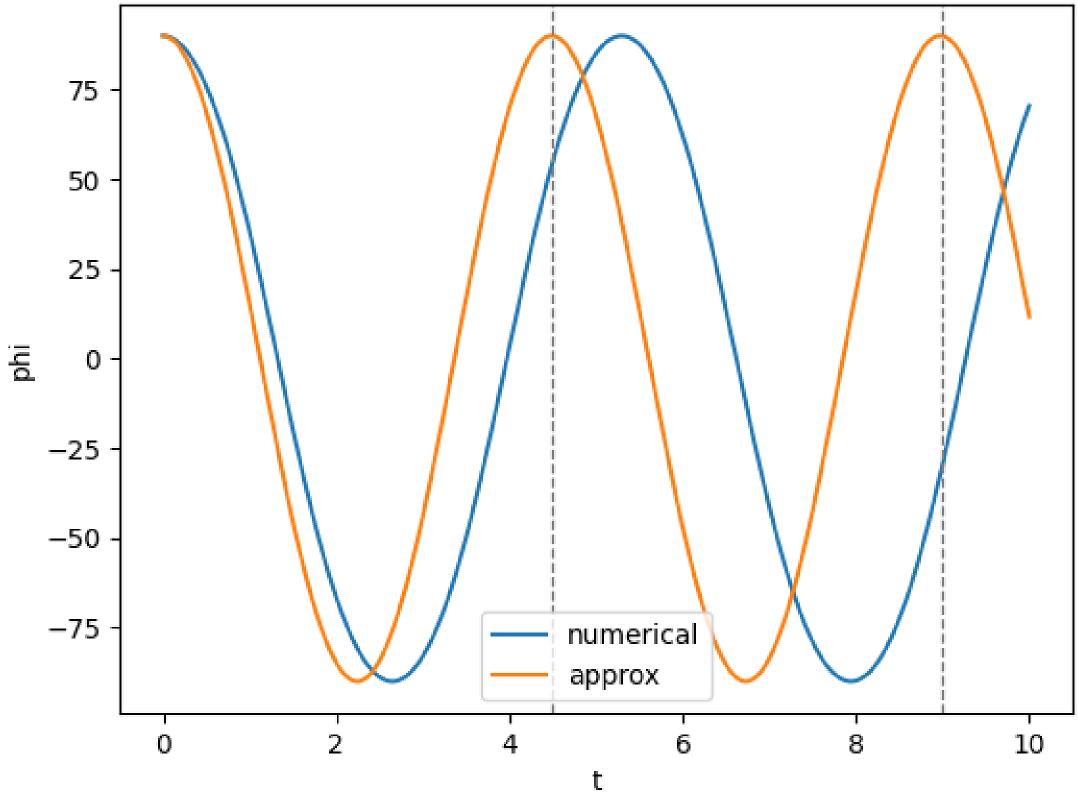
Solving this system of equations with the improved Euler method we get the following graph.



numerical
approx
phi
t
20
15
10
5
0
−5
−10
−15
−20
0
2
4
6
8
10

□

Proof. **1.51**



□