Solved selected problems of Classical Mechanics - Gregory

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Chapter 16 - Vector angular velocity and rigid body kinematics

Solution. 16.1 The vector $\hat{\boldsymbol{n}}$ in this case points in the positive z direction i.e. $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$ then the angular velocity vector is given by

$$\boldsymbol{\omega} = 2\hat{\boldsymbol{k}} \text{ radians per second}$$

To determine the instantaneous velocity of $P = (4, -3, 7)$ we first write that $\boldsymbol{r} = 4\hat{\boldsymbol{i}} + -3\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$ and hence

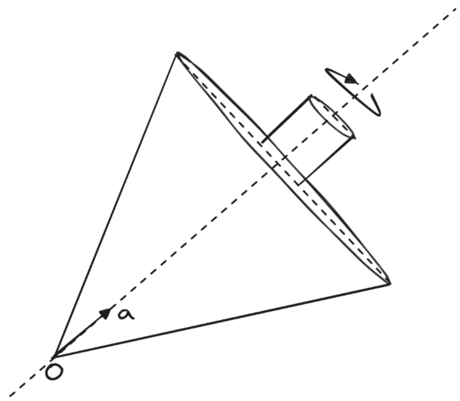
$$\begin{aligned}
\boldsymbol{v} &= (2\hat{\boldsymbol{k}}) \times (4\hat{\boldsymbol{i}} + -3\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}) \\
&= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 0 & 2 \\ 4 & -3 & 7 \end{vmatrix} \\
&= 6\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} \ m/s
\end{aligned}$$

The speed is therefore $|\boldsymbol{v}| = \sqrt{6^2 + 8^2} = 10 \ m/s$ Finally, the acceleration of P can be found using that $\boldsymbol{a} = \boldsymbol{\omega} \times \dot{\boldsymbol{r}}$ hence

$$\begin{aligned}
\boldsymbol{a} &= \boldsymbol{\omega} \times \boldsymbol{v} \\
&= (2\hat{\boldsymbol{k}}) \times (6\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}) \\
&= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 0 & 2 \\ 6 & 8 & 0 \end{vmatrix} \\
&= -16\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}} \ m/s^2
\end{aligned}$$

□

Solution. 16.3 In this case, the system looks like the following



We know that the particle at the origin is fixed so Theorem 16.1 applies and hence the velocity of any particle of the spinning top is given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

In particular, this must be true for any particle lying on the axis of symmetry. These particles have a position vector given by $\beta\boldsymbol{a}$ where β is a scalar and $\boldsymbol{a}$ is the unit vector pointing along the axis of symmetry, hence we have that

$$\beta\dot{\boldsymbol{a}} = \boldsymbol{\omega} \times (\beta\boldsymbol{a})$$

Which implies that

$$\dot{\boldsymbol{a}} = \boldsymbol{\omega} \times \boldsymbol{a}$$

On taking the cross product of this equation with $\boldsymbol{a}$, we obtain

$$\begin{aligned}\boldsymbol{a} \times \dot{\boldsymbol{a}} &= \boldsymbol{a} \times (\boldsymbol{\omega} \times \boldsymbol{a}) \\ &= (\boldsymbol{a} \cdot \boldsymbol{a})\boldsymbol{\omega} - (\boldsymbol{a} \cdot \boldsymbol{\omega})\boldsymbol{a}\end{aligned}$$

Since $\boldsymbol{a}$ is a unit vector we have that $\boldsymbol{a} \cdot \boldsymbol{a} = 1$ and hence

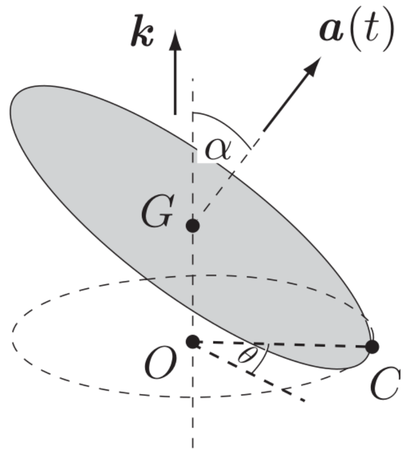
$$\boldsymbol{\omega} = \boldsymbol{a} \times \dot{\boldsymbol{a}} + (\boldsymbol{a} \cdot \boldsymbol{\omega})\boldsymbol{a}$$

It follows that $\boldsymbol{\omega}$ must have the following form

$$\boldsymbol{\omega} = \boldsymbol{a} \times \dot{\boldsymbol{a}} + \lambda\boldsymbol{a}$$

Where λ is a scalar function of the time. $\square$

Solution. 16.4 In this case, we are considering a system that looks like the following



Suppose we view the motion of the penny about a rotating reference frame with respect to the $\hat{\boldsymbol{k}}$ axis, then the angular velocity $\boldsymbol{\Omega}$ of the frame is given by

$$\boldsymbol{\Omega} = \dot{\theta}\hat{\boldsymbol{k}}$$

From this rotating reference frame, the axis $\hat{\boldsymbol{a}}$ is fixed so the penny rotates about $\hat{\boldsymbol{a}}$ with an angular velocity

$$\boldsymbol{\omega'} = \lambda\hat{\boldsymbol{a}}$$

for some function λ. Then using the result of the angular velocity addition theorem, stated in Chapter 17, we get that

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\Omega} + \boldsymbol{\omega'} \\ &= \dot{\theta}\hat{\boldsymbol{k}} + \lambda\hat{\boldsymbol{a}} \end{aligned}$$

So by taking the dot product of the equation with $\hat{\boldsymbol{a}}$ we have that

$$\boldsymbol{\omega} \cdot \hat{\boldsymbol{a}} = \dot{\theta}\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{a}} + \lambda\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{a}} = \dot{\theta}\cos\alpha + \lambda$$

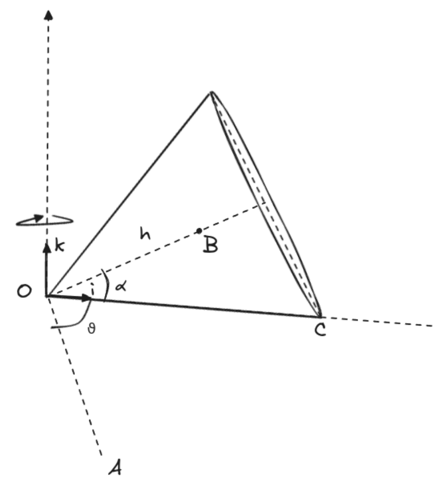
Instantaneously the point C because of the rolling condition is fixed but also G is fixed then $\boldsymbol{\omega}$ must be perpendicular to $\hat{\boldsymbol{a}}$ and hence

$$\begin{aligned} \dot{\theta}\cos\alpha + \lambda = 0 \\ \lambda = -\dot{\theta}\cos\alpha \end{aligned}$$

Therefore $\boldsymbol{\omega} = \dot{\theta}(\hat{\boldsymbol{k}} - \cos\alpha\hat{\boldsymbol{a}})$.

Finally, from particle G (which is fixed) the velocity of the highest particle is given by $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ but the vector $\boldsymbol{r}$ instantaneously from what we saw earlier is in the same direction as $\boldsymbol{\omega}$ therefore $\boldsymbol{v} = 0$ for the highest particle. □

Solution. 16.5 In this case, we are considering a system that looks like the following



First, because the cone rolls without slipping we know that every point in the line OC (the line in contact with the table) must have $v = 0$ this includes the vertex O of the cone so it never moves.

Let us consider now a particle in the line OC then given that the point O is fixed then the particle has a velocity given by $\boldsymbol{v} = \boldsymbol{\omega} \times d\boldsymbol{i}$ where d is the distance from O to the particle and $\boldsymbol{i}$ is the unit vector in the OC direction. But we know that particles in the OC line have $\boldsymbol{v} = 0$ so we get that $\boldsymbol{\omega} \times d\boldsymbol{i} = 0$ which implies that $\boldsymbol{\omega}$ is parallel to $\boldsymbol{i}$ hence we can write $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = \lambda\boldsymbol{i}$ for some function λ.

On the other hand, let us consider a particle in the axis of symmetry at some point B at a distance r from the origin O then this particle has a velocity of

$$\begin{aligned}
\boldsymbol{v} &= \boldsymbol{\omega} \times \boldsymbol{r} \\
&= \lambda\boldsymbol{i} \times (r\sin\alpha\boldsymbol{k} + r\cos\alpha\boldsymbol{i}) \\
&= \lambda\boldsymbol{i} \times r\sin\alpha\boldsymbol{k} \\
&= \lambda r\sin\alpha\ (\boldsymbol{i} \times \boldsymbol{k})
\end{aligned}$$

Where $\boldsymbol{k}$ is the unit vector pointing in the z direction as shown. Also, we could see the particle B as a particle rotating around the z axis with a velocity of magnitude $v = -\dot{\theta}(r\cos\alpha)$ where we take a minus sign since the angle θ is decreasing so by joining these results we get that

$$\begin{aligned}
\lambda r\sin\alpha &= -\dot{\theta} r\cos\alpha \\
\lambda &= -\dot{\theta}\frac{\cos\alpha}{\sin\alpha} \\
\lambda &= -\dot{\theta}\cot\alpha
\end{aligned}$$

Therefore the angular velocity is given by

$$\boldsymbol{\omega} = -(\dot{\theta} \cot \alpha)\boldsymbol{i}$$

Finally, from what we saw the particle with the maximum speed will be the one the furthest perpendicularly from the OC axis. From trigonometry, we know that $OC = h/\cos\alpha$ then the furthest point perpendicularly is at a distance

$$OC \sin 2\alpha = \frac{h \sin 2\alpha}{\cos\alpha} = \frac{2h \sin\alpha \cos\alpha}{\cos\alpha} = 2h \sin\alpha$$

Therefore the speed of this point is

$$|v| = 2h|\dot{\theta}| \cot\alpha \sin\alpha = 2h|\dot{\theta}| \cos\alpha$$

□