

Solved selected problems of Classical Mechanics - Gregory

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Chapter 13 - The Calculus of Variations and Hamilton's Principle

Solution. 13.1 By definition, extremals are solutions of the Euler-Lagrange equation. In this case $F = \dot{x}^2/t^3$ so we have that

$$\frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial \dot{x}} = \frac{2\dot{x}}{t^3}$$

Hence the Euler-Lagrange equation takes the form

$$\begin{aligned} \frac{d}{dt} \left(\frac{2\dot{x}}{t^3} \right) - 0 &= 0 \\ \frac{2\ddot{x}}{t^3} - \frac{6\dot{x}}{t^4} &= 0 \\ \ddot{x} - \frac{3\dot{x}}{t} &= 0 \end{aligned}$$

Now we solve the equation by setting $v = \dot{x}$ hence

$$\begin{aligned} \frac{dv}{dt} - \frac{3v}{t} &= 0 \\ \int \frac{dv}{3v} &= \int \frac{dt}{t} \\ \frac{\log(v)}{3} &= \log(t) + C \\ v &= Ct^3 \end{aligned}$$

by replacing again and solving the equation we have that

$$\begin{aligned} \frac{dx}{dt} &= Ct^3 \\ \int dx &= C \int t^3 dt \\ x &= Ct^4 + D \end{aligned}$$

The admissible extremals are those that satisfy the conditions $x(1) = 3$ and $x(2) = 18$ this way we find the values of $C = 1$ and $D = 2$ so the only admissible extremal of $J[x]$ is given by

$$\hat{x} = t^4 + 2$$

Finally, we want to show that this extremal provides a global minimum of $J[x]$. Let h be any admissible variation and consider the variation in J that it produces

$$\begin{aligned}
J[\hat{x} + h] - J[\hat{x}] &= \int_1^2 \frac{(4t^3 + \dot{h})^2}{t^3} dt - \int_1^2 \frac{(4t^3)^2}{t^3} dt \\
&= \int_1^2 16t^3 + 8\dot{h} + \frac{\dot{h}^2}{t^3} dt - \int_1^2 16t^3 dt \\
&= 8 \left[h \right]_{t=1}^{t=2} + \int_1^2 \frac{\dot{h}^2}{t^3} dt \\
&= \int_1^2 \frac{\dot{h}^2}{t^3} dt
\end{aligned}$$

Where we used that h is an admissible extremal hence it must satisfy $h(1) = h(2) = 0$. Since the integral of a positive function must be positive we see that

$$J[\hat{x} + h] - J[\hat{x}] = \int_1^2 \frac{\dot{h}^2}{t^3} dt \geq 0$$

Thus \hat{x} provides the global minimum of $J[x]$. The global minimum value of J is therefore $J[t^4 + 2] = 60$.

□

Solution. 13.3 We want to find the extremals of the path functional

$$L[y] = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

By definition, extremals are solutions of the Euler-Lagrange equation. In this case $F(y, \dot{y}) = \sqrt{1 + \dot{y}^2}$ so there is no explicit dependence on x this implies that if we satisfy the equation

$$\dot{y} \frac{\partial F}{\partial \dot{y}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned} \dot{y} \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} - \sqrt{1 + \dot{y}^2} &= C \\ \frac{\dot{y}^2 - 1 - \dot{y}^2}{\sqrt{1 + \dot{y}^2}} &= C \\ \sqrt{1 + \dot{y}^2} &= -\frac{1}{C} \\ \dot{y}^2 &= -1 + \frac{1}{C^2} \\ \int \dot{y} &= \frac{\sqrt{1 - C^2}}{C^2} \int dx \\ y &= \frac{\sqrt{1 - C^2}}{C^2} x + D \end{aligned}$$

Where D is a constant of integration. Now applying the given initial and end conditions we get that $C = 1$ and $D = 0$ so the admissible extremal is

$$\hat{y} = 0$$

But this is a constant solution so it may or may not be a solution to the Euler-Lagrange equation, we must check, hence we have that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{y}} \right) - \frac{\partial F}{\partial y} &= 0 \\ \frac{d}{dt} \left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} \right) - 0 &= 0 \\ \frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}} &= 0 \end{aligned}$$

So $\hat{y} = 0$ is a solution to the Euler-Lagrange equation and therefore an extremal.

Finally, we want to check \hat{y} provides a global minimum for L . Let h be any admissible variation and consider the variation in L that it produces

$$\begin{aligned} L[\hat{y} + h] &= \int_0^1 \sqrt{1 + (\dot{\hat{y}} + \dot{h})^2} dx \\ &= \int_0^1 \sqrt{1 + \dot{h}^2} dx \end{aligned}$$

Also, we know that

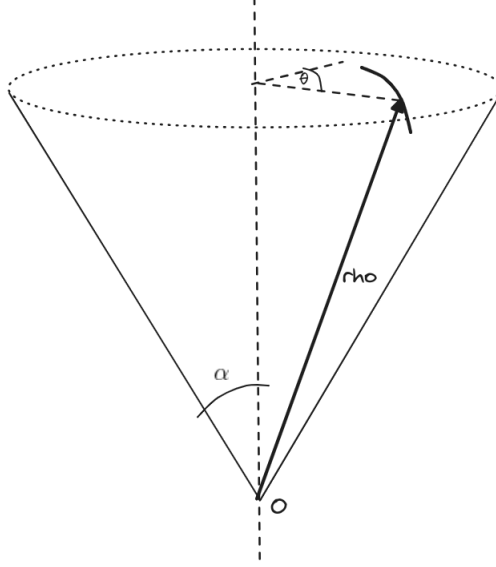
$$\int_0^1 \sqrt{1 + \dot{y}^2} dx = \int_0^1 dx = 1$$

And we have that

$$\int_0^1 \sqrt{1 + \dot{h}^2} dx \geq 1 = \int_0^1 dx$$

Therefore \hat{y} provides a local minimum for L . □

Solution. 13.5 Let us analyze the following system



From this, we can write the differential length as

$$(ds)^2 = (d\rho)^2 + (\rho \sin \alpha d\theta)^2$$

So integrating we get the length functional as follows

$$L = \int_{-\pi/2}^{\pi/2} ds = \int_{-\pi/2}^{\pi/2} \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + (\rho \sin \alpha)^2} d\theta$$

In this case, we have that $F(\rho, \dot{\rho}) = \sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2}$ so there is no explicit dependence on θ this implies that if we satisfy the equation

$$\dot{\rho} \frac{\partial F}{\partial \dot{\rho}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned} \dot{\rho} \frac{\dot{\rho}}{\sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2}} - \sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2} &= C \\ \frac{\dot{\rho}^2 - \dot{\rho}^2 - (\rho \sin \alpha)^2}{\sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2}} &= C \\ \sqrt{\dot{\rho}^2 + (\rho \sin \alpha)^2} &= -\frac{(\rho \sin \alpha)^2}{C} \\ \dot{\rho}^2 &= -(\rho \sin \alpha)^2 + \frac{(\rho \sin \alpha)^4}{C^2} \\ \frac{d\rho}{d\theta} &= (\rho \sin \alpha) \sqrt{\frac{(\rho \sin \alpha)^2}{C^2} - 1} \end{aligned}$$

Now we can integrate the equation as follows

$$\begin{aligned}
\int \frac{d\rho}{(\rho \sin \alpha) \sqrt{\frac{(\rho \sin \alpha)^2}{C^2} - 1}} &= \int d\theta \\
\frac{\arctan\left(\sqrt{\frac{(\rho \sin \alpha)^2}{C^2} - 1}\right)}{\sin \alpha} &= \theta + D \\
\arctan\left(\sqrt{\frac{(\rho \sin \alpha)^2}{C^2} - 1}\right) &= (\theta + D) \sin \alpha \\
\frac{(\rho \sin \alpha)^2}{C^2} - 1 &= \tan^2((\theta + D) \sin \alpha) \\
\rho &= \frac{C}{\sin \alpha} \sqrt{\tan^2((\theta + D) \sin \alpha) + 1} \\
\rho &= \frac{C}{\sin \alpha} \sec((\theta + D) \sin \alpha)
\end{aligned}$$

This is the equation for the extremals of L . Now applying the given initial and end conditions we get that

$$a = \frac{C}{\sin \alpha} \sec(-(\pi/2 - D) \sin \alpha) = \frac{C}{\sin \alpha} \sec((\pi/2 + D) \sin \alpha)$$

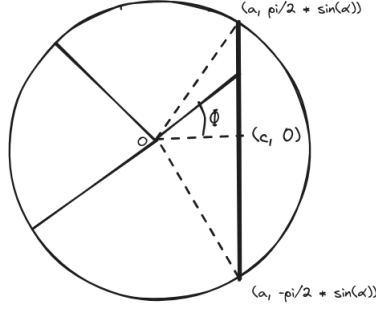
which implies that $D = 0$ and C is

$$\begin{aligned}
C \sec\left(\frac{\pi \sin \alpha}{2}\right) &= a \sin \alpha \\
C &= \frac{a \sin \alpha}{\sec\left(\frac{\pi \sin \alpha}{2}\right)}
\end{aligned}$$

Replacing the values for C and D we get that the admissible extremal is

$$\begin{aligned}
\rho &= \frac{a \sec(\theta \sin \alpha)}{\sec\left(\frac{\pi \sin \alpha}{2}\right)} \\
\rho &= \frac{a \cos\left(\frac{\pi \sin \alpha}{2}\right)}{\cos(\theta \sin \alpha)}
\end{aligned}$$

Finally, we want to verify that this extremal is the same as the shortest path that would be obtained by developing the cone onto a plane. If we develop the cone onto a plane, we get that the path is a vertical line in the plane as shown below



where the angle in the plane is given by $\phi = \theta \sin \alpha$. We know that the equation of a straight vertical line in polar coordinates is given by

$$\rho \cos(\phi) = c$$

where c is the value of ρ when $\phi = 0$ and from the drawing above we have that $\cos(\pi/2 \sin(\alpha)) = c/a$ hence

$$\rho \cos(\theta \sin \alpha) = a \cos\left(\frac{\pi \sin(\alpha)}{2}\right)$$

Which is the admissible extremal that we have such that it satisfies the Euler-Lagrange equation. \square

Solution. 13.7 We want to find the extremals of

$$J[y] = \int_{-a}^a y \sqrt{1 + \dot{y}^2} dx$$

In this case, we have that $F(y, \dot{y}) = y \sqrt{1 + \dot{y}^2}$ so there is no explicit dependence on x this implies that if we satisfy the equation

$$\dot{y} \frac{\partial F}{\partial \dot{y}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned} \dot{y} \frac{y \dot{y}}{\sqrt{1 + \dot{y}^2}} - y \sqrt{1 + \dot{y}^2} &= C \\ \frac{\dot{y} y^2 - (y(1 + \dot{y}^2))}{\sqrt{1 + \dot{y}^2}} &= C \\ \frac{y}{\sqrt{1 + \dot{y}^2}} &= -C \\ y^2 &= C^2(1 + \dot{y}^2) \\ \sqrt{\frac{y^2}{C^2} - 1} &= \dot{y} \end{aligned}$$

Now we can solve this equation by separation as follows

$$\begin{aligned} \int \frac{dy}{\sqrt{\frac{y^2}{C^2} - 1}} &= \int dx \\ C \int \frac{dy}{\sqrt{y^2 - C^2}} &= \int dx \\ \cosh^{-1} \left(\frac{y}{C} \right) &= \frac{x}{C} + D \\ y &= C \cosh \left(\frac{x}{C} + D \right) \end{aligned}$$

Which is the form of the extremals of $J[y]$ we wanted.

To determine the values of C and D we use the initial and end conditions. We know that $y(-a) = y(a) = b$ hence we have that

$$\begin{aligned} C \cosh \left(\frac{a}{C} + D \right) &= C \cosh \left(-\frac{a}{C} + D \right) \\ \cosh \left(\frac{a}{C} + D \right) &= \cosh \left(\frac{a}{C} - D \right) \\ \frac{a}{C} + D &= \frac{a}{C} - D \\ D &= 0 \end{aligned}$$

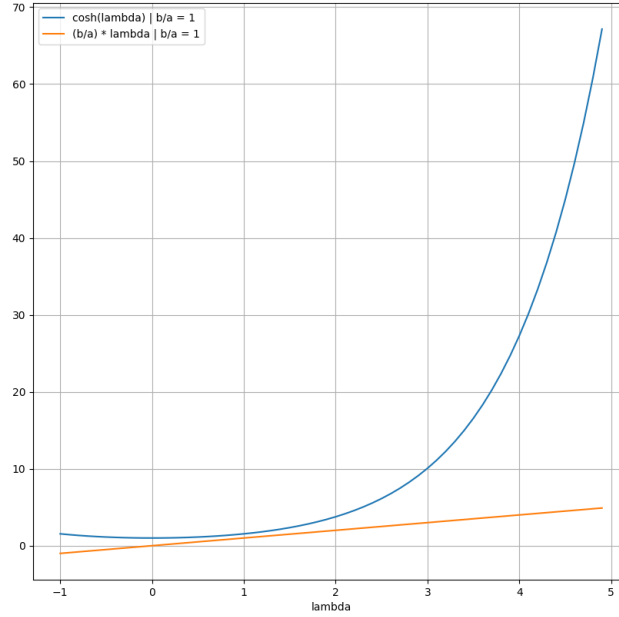
So the initial and end conditions are satisfied only if $D = 0$.

Naming $\lambda = a/C$ we see that

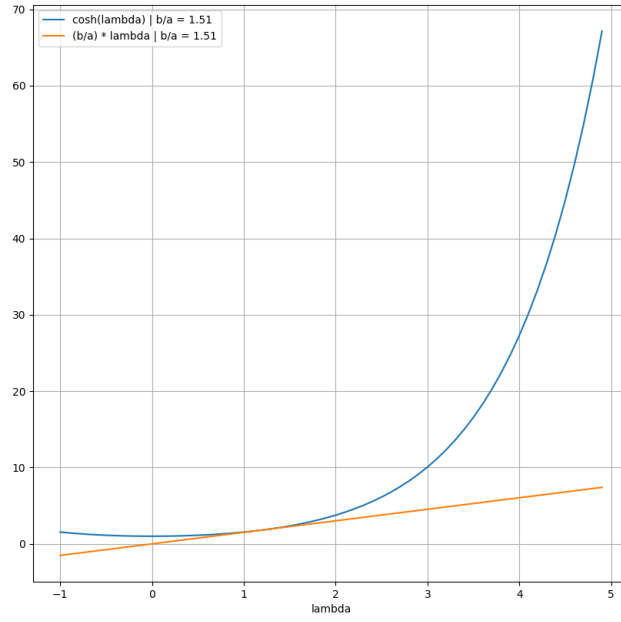
$$\cosh\left(\frac{a}{C}\right) = \frac{b}{C}$$

$$\cosh(\lambda) = \frac{b}{a}\lambda$$

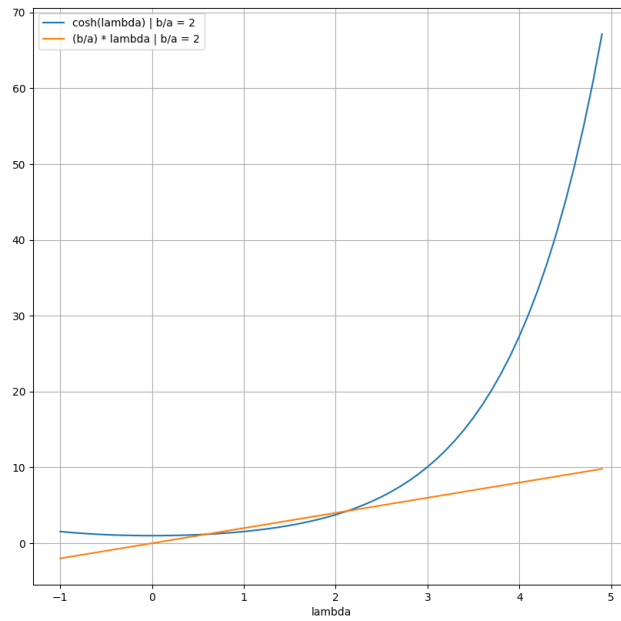
We see that depending the value of b/a the curve $\cosh(\lambda)$ will or will not intersect the line $(b/a)\lambda$ for example if we take $b/a = 1$ and we plot both curves we have that



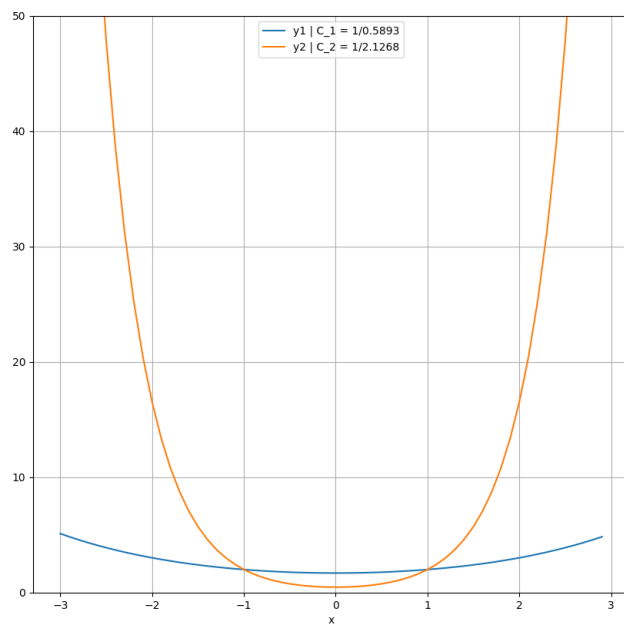
Where we see that the curves don't intersect hence there is no solution to the equation so no value of C satisfies the initial and end conditions and therefore there are no admissible extremals for this case. If we set $b/a = 1.51$ we see that the curves do intersect and this value seems to be quite close to the critical point as shown below.



If we choose $b/a = 2$ the curves intersect at two points $\lambda_1 = 0.5893$ and $\lambda_2 = 2.1268$ as shown below



Now taking $a = 1$ we plot both extremals where we use $C_1 = 1/\lambda_1$ and $C_2 = 1/\lambda_2$



By the shape of the curve, we can guess that the blue curve represents the soap film. □

Solution. 13.9 We want to determine Fermat's time functional, for this, we need first, the line element in cylindrical polar coordinates i.e.

$$ds^2 = dr^2 + r^2 d\theta^2$$

So Fermat's time functional is given by

$$\begin{aligned} T[r] &= c^{-1} \int n ds \\ &= c^{-1} \int n \sqrt{dr^2 + r^2 d\theta^2} \\ &= c^{-1} \int_{\theta_0}^{\theta_1} n \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta \end{aligned}$$

Which is the expression we wanted. Using the calculus of variations we note that $F(r, \dot{r}) = n\sqrt{\dot{r}^2 + r^2}$ which does not depends on θ so if we satisfy the equation

$$\dot{r} \frac{\partial F}{\partial \dot{r}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned} \dot{r} \frac{n\dot{r}}{\sqrt{\dot{r}^2 + r^2}} - n\sqrt{\dot{r}^2 + r^2} &= C \\ n \left(\frac{\dot{r}^2}{\sqrt{\dot{r}^2 + r^2}} - \sqrt{\dot{r}^2 + r^2} \right) &= C \\ n \left(\frac{\dot{r}^2 - (\dot{r}^2 + r^2)}{\sqrt{\dot{r}^2 + r^2}} \right) &= C \\ \frac{nr^2}{\sqrt{\dot{r}^2 + r^2}} &= C \end{aligned}$$

Let now $\dot{r} = r \tan \psi$ then we get that

$$\begin{aligned} \frac{nr^2}{\sqrt{(r \tan \psi)^2 + r^2}} &= C \\ \frac{nr}{\sqrt{\tan^2 \psi + 1}} &= C \\ \frac{nr}{\frac{1}{\cos \psi} \sqrt{\sin^2 \psi + \cos^2 \psi}} &= C \\ nr \cos \psi &= C \end{aligned}$$

Which is Snell's law for this case as we wanted. Finally, if we consider a circular ray with center at the origin then must be that $\psi = 0$ hence we get that $nr = a$ where we named the constant as a then $n = a/r$. \square

Solution. 13.10 Let a particle of mass 2 kg move under uniform gravity where $g = 10 \text{ m/s}^2$ along the z -axis, which points vertically downwards. The lagrangian in this case is given by

$$\begin{aligned} L(z, \dot{z}) &= T(\dot{z}) - V(z) \\ &= \frac{1}{2}m\dot{z}^2 - (-mgz) \\ &= \dot{z}^2 + 20z \end{aligned}$$

Then the action functional for a time interval $[0, 2]$ is given by

$$S[z] = \int_0^2 L(z, \dot{z}) \, dt = \int_0^2 (\dot{z}^2 + 20z) dt$$

Because of Hamilton's principle, $z(t) = 5t^2$ makes stationary the action functional. Let us compute the value of $S[z]$

$$\begin{aligned} S[z] &= \int_0^2 100t^2 + 100t^2 \, dt \\ &= \left[\frac{200t^3}{3} \right]_{t=0}^{t=2} \\ &= \frac{1600}{3} \end{aligned}$$

Now let us consider an admissible variation of $z(t)$ as follows

$$\begin{aligned} S[z + h] &= \int_0^2 ((10t + \dot{h})^2 + 20(5t^2 + h)) \, dt \\ &= \int_0^2 200t^2 + 20t\dot{h} + \dot{h}^2 + 20h \, dt \\ &= \left[\frac{200t^3}{3} \right]_{t=0}^{t=2} + [20th]_{t=0}^{t=2} + \int_0^2 \dot{h}^2 \, dt \\ &= \frac{1600}{3} + 0 + \int_0^2 \dot{h}^2 \, dt \end{aligned}$$

We used that h is an admissible variation and hence $h(0) = h(2) = 0$. So we see that $S[z + h] = S[z] + \int_0^2 \dot{h}^2 \, dt$ and since the integral of a positive function \dot{h}^2 must be positive then

$$S[z + h] = S[z] + \int_0^2 \dot{h}^2 \, dt \geq S[z]$$

so $z(t) = 5t^2$ provides a global minimum for $S[z]$. □

Solution. 13.11 In this case, we are given the Lagrangian of the system which is

$$L = \dot{q}^2 - 4q^2$$

So the Lagrange equation in this case is

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} &= 0 \\ \ddot{q} + 4q &= 0 \end{aligned}$$

We want to verify that $q^* = \sin 2t$ is a motion of the oscillator then it must be a solution to the Lagrange equation we got, we check this below

$$\frac{d^2 \sin(2t)}{dt^2} + 4 \sin(2t) = -4 \sin(2t) + 4 \sin(2t) = 0$$

So we see q^* is a solution to the Lagrange equation and therefore a motion of the oscillator.

The action functional is given by

$$S[q] = \int_0^\tau \dot{q}^2 - 4q^2 dt$$

and since q^* is a solution to the Lagrange equation it must make S stationary, we check this below. Let $h(t)$ be an admissible variation then

$$\begin{aligned} S[q^* + h] - S[q^*] &= \int_0^\tau (2 \cos 2t + \dot{h})^2 - 4(\sin 2t + h)^2 dt \\ &\quad - \int_0^\tau (2 \cos 2t)^2 - 4(\sin 2t)^2 dt \\ &= 4 \int_0^\tau \cos^2 2t + \dot{h} \cos 2t + \frac{\dot{h}^2}{4} - \sin^2 2t - 2h \sin 2t - h^2 dt \\ &\quad - 4 \int_0^\tau \cos^2 2t - \sin^2 2t dt \\ &= 4 \int_0^\tau \cos 4t + \dot{h} \cos 2t + \frac{\dot{h}^2}{4} - 2h \sin 2t - h^2 dt \\ &\quad - 4 \int_0^\tau \cos 4t dt \\ &= 4 \left[h \cos 2t \right]_0^\tau + 4 \int_0^\tau \left(\frac{\dot{h}^2}{4} - h^2 \right) dt \\ &= \int_0^\tau (\dot{h}^2 - 4h^2) dt \end{aligned}$$

We used that $h(\tau) = h(0) = 0$. It follows that

$$\begin{aligned}
|S[q^* + h] - S[q^*]| &= \left| \int_0^\tau (\dot{h}^2 - 4h^2) dt \right| \\
&\leq \int_0^\tau |\dot{h}^2 - 4h^2| dt \\
&\leq \int_0^\tau |\dot{h}|^2 + |4h|^2 dt \\
&\leq \int_0^\tau (\max |\dot{h}|)^2 + (\max |4h|)^2 dt \\
&= \tau((\max |\dot{h}|)^2 + (\max |4h|)^2) \\
&\leq \tau(\max |\dot{h}| + 4 \max |h|)^2 = \tau \|h\|^2
\end{aligned}$$

Hence $|S[q^* + h] - S[q^*]| = O(\|h\|^2)$ which by definition, means that q^* makes the action functional $S[q]$ stationary.

Let us take now a time interval $0 \leq t \leq \pi$ then

$$S[q^*] = 4 \int_0^\pi \cos(4t) dt = 0$$

If $h = \epsilon \sin(4t)$ we have that

$$\begin{aligned}
S[q^* + h] - S[q^*] &= \int_0^\pi ((4\epsilon \cos 4t)^2 - 4(\epsilon \sin 4t)^2) dt \\
&= 4\epsilon^2 \int_0^\pi 4 \cos^2 4t - \sin^2 4t dt \\
&= \epsilon^2 \left[\frac{5}{4} \sin(8t) + 6t \right]_{t=0}^{t=\pi} \\
&= \epsilon^2 [6\pi - 0] = \epsilon^2 6\pi
\end{aligned}$$

So we see that $S[q^* + h] = S[q^*] + \epsilon^2 6\pi \geq S[q^*] = 0$.

On the other hand, if $h = \epsilon \sin(t)$ in the same way we have that

$$\begin{aligned}
S[q^* + h] - S[q^*] &= \int_0^\pi ((\epsilon \cos t)^2 - 4(\epsilon \sin t)^2) dt \\
&= \epsilon^2 \int_0^\pi \cos^2 t - 4 \sin^2 t dt \\
&= \epsilon^2 \left[\frac{1}{2} (5 \sin(t) \cos(t) - 3t) \right]_{t=0}^{t=\pi} \\
&= \epsilon^2 \left[-\frac{3\pi}{2} - 0 \right] = -\epsilon^2 \frac{3\pi}{2}
\end{aligned}$$

So we see that $S[q^* + h] = S[q^*] - \epsilon^2 \frac{3\pi}{2} \leq S[q^*] = 0$.

Therefore we have that q^* does not make S a minimum or a maximum. \square

Solution. 13.12 Given that the particle is constrained to move over a smooth fixed surface under no forces other than the force of constraint, and the former do no work the lagrangian for the system is $L = \frac{1}{2}m\dot{\mathbf{r}}^2$. We want to check if the path of a particle must be a geodesic i.e. makes the Lenght functional

$$L[\mathbf{r}] = \int_{t_0}^{t_1} |\dot{\mathbf{r}}| dt$$

stationary. We know from the conservation of energy that $T = \text{constant}$ which implies that a particle that moves along the surface must have a constant velocity so

$$L[\mathbf{r}] = |\dot{\mathbf{r}}| \Delta t$$

On the other hand, we know from Hamilton's principle that a motion of a particle on the surface must satisfy the Lagrange equations and hence it must make the action functional stationary.

So let \mathbf{r}^* be the actual path of a particle and h be an admissible variation such that $\mathbf{r} = \mathbf{r}^* + h$, we know that

$$S[\mathbf{r}^* + h] - S[\mathbf{r}^*] = O(h^2)$$

hence

$$\begin{aligned} \frac{1}{2}m \int_{t_0}^{t_1} \dot{\mathbf{r}}^2 dt &= \frac{1}{2}m \int_{t_0}^{t_1} \dot{\mathbf{r}}^{*2} dt + O(h^2) \\ &= \frac{1}{2}m |\dot{\mathbf{r}}^*|^2 \Delta t + O(h^2) \end{aligned}$$

Where we used that $\dot{\mathbf{r}}^*$ is constant since \mathbf{r}^* is the actual path of a particle. Given the above equation must be true for any path we can traverse $\mathbf{r} = \mathbf{r}^* + h$ with constant velocity i.e. we can take out of the integral $\dot{\mathbf{r}}^2$ then we have that

$$\frac{1}{2}m |\dot{\mathbf{r}}|^2 \Delta t = \frac{1}{2}m |\dot{\mathbf{r}}^*|^2 \Delta t + O(h^2)$$

Finally, by replacing $L[\mathbf{r}]$ we get that

$$\frac{1}{2}m \frac{L[\mathbf{r}]^2}{\Delta t} = \frac{1}{2}m \frac{L[\mathbf{r}^*]^2}{\Delta t} + O(h^2)$$

Which implies that $L[\mathbf{r}] = L[\mathbf{r}^*] + O(h^2)$. Therefore the Length functional is stationary too. \square

Solution. 13.13 Let $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ be the Lagrangian of the system then the motion of the system \mathbf{q}^* will satisfy the Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = 0$$

but also \mathbf{q}^* makes the action functional stationary. We want to check that a modification to the Lagrangian of the system to $L' = L + \frac{d}{dt}g(\mathbf{q}, t)$ still preserves \mathbf{q}^* as a motion of the system hence it must make the action functional with the new Lagrangian stationary.

Let us compute the action functional for this case

$$\begin{aligned} S'[\mathbf{q}] &= \int_{t_0}^{t_1} L'(\mathbf{q}, \dot{\mathbf{q}}, t) dt \\ S'[\mathbf{q}] &= \int_{t_0}^{t_1} L + \frac{d}{dt}g(\mathbf{q}, t) dt \\ S'[\mathbf{q}] &= S[\mathbf{q}] + \int_{t_0}^{t_1} dg(\mathbf{q}, t) \\ S'[\mathbf{q}] &= S[\mathbf{q}] + \left[g(\mathbf{q}, t) \right]_{t_0}^{t_1} \\ S'[\mathbf{q}] &= S[\mathbf{q}] + [g(\mathbf{q}, t_1) - g(\mathbf{q}, t_0)] \end{aligned}$$

So we see that $S'[\mathbf{q}]$ differs from $S[\mathbf{q}]$ only on a constant value, so $S'[\mathbf{q}]$ will be still stationary for \mathbf{q}^* . Therefore if \mathbf{q}^* is a motion of the system it still will be if we change the Lagrangian to L' . \square

Solution. 13.14 Let \mathcal{C} be a path lying on the surface that connects P and Q . We want to determine the length of \mathcal{C} . We know that the line element in cylindrical coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Given that P and Q are separated by π radians in cylindrical coordinates we integrate the length functional between $-\pi/2$ and $\pi/2$ as follows

$$\begin{aligned} L[r] &= \int_{-\pi/2}^{\pi/2} ds \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{dr^2 + r^2 d\theta^2 + dz^2} \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 + \left(\frac{dz}{d\theta}\right)^2} d\theta \end{aligned}$$

Also, we know that the surface of the paraboloid is given by $z = x^2 + y^2$ so in cylindrical coordinates, we have that $z = (r \cos \theta)^2 + (r \sin \theta)^2 = r^2$ which implies that $dz/d\theta = 2r\dot{r}$ where $\dot{r} = dr/d\theta$ hence

$$\begin{aligned} L[r] &= \int_{-\pi/2}^{\pi/2} \sqrt{\dot{r}^2 + r^2 + (2r\dot{r})^2} d\theta \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2 + (1 + 4r^2)\dot{r}^2} d\theta \end{aligned}$$

We see that $F(r, \dot{r}) = \sqrt{r^2 + (1 + 4r^2)\dot{r}^2}$ so the Euler-Lagrange equation is given by

$$\begin{aligned} \frac{d}{d\theta} \left(\frac{\partial F}{\partial \dot{r}} \right) - \frac{\partial F}{\partial r} &= 0 \\ \frac{d}{d\theta} \left(\frac{(4r^2 + 1)\dot{r}}{\sqrt{r^2 + (1 + 4r^2)\dot{r}^2}} \right) - \frac{(4\dot{r}^2 + 1)r}{\sqrt{r^2 + (1 + 4r^2)\dot{r}^2}} &= 0 \end{aligned}$$

And we see that

$$\begin{aligned} \frac{d}{d\theta} \left(\frac{(4r^2 + 1)\dot{r}}{\sqrt{r^2 + (1 + 4r^2)\dot{r}^2}} \right) &= \frac{2r(4r^2(4\dot{r} + 1) + 4\dot{r} - 1)\dot{r}^2 + (4r^2 + 1)(r^2(4\dot{r} + 2) + \dot{r})\ddot{r}}{2(r^2 + (1 + 4r^2)\dot{r}^2)^{3/2}} \\ &= \frac{4r(4r^2 + 1)\dot{r}^4 + r(4r^2 - 1)\dot{r}^2 + r^2(4r^2 + 1)\ddot{r}}{(r^2 + (1 + 4r^2)\dot{r}^2)^{3/2}} \end{aligned}$$

So the Euler-Lagrange equation is

$$\begin{aligned} \frac{(4\dot{r}^2 + 1)r}{\sqrt{r^2 + (1 + 4r^2)\dot{r}^2}} &= \frac{4r(4r^2 + 1)\dot{r}^4 + r(4r^2 - 1)\dot{r}^2 + r^2(4r^2 + 1)\ddot{r}}{(r^2 + (1 + 4r^2)\dot{r}^2)^{3/2}} \\ (4r\dot{r}^2 + r)(r^2 + (1 + 4r^2)\dot{r}^2) &= 4r(4r^2 + 1)\dot{r}^4 + r(4r^2 - 1)\dot{r}^2 + r^2(4r^2 + 1)\ddot{r} \\ r(4r^2 + 1)\ddot{r} &= (4\dot{r}^2 + 1)(r^2 + (1 + 4r^2)\dot{r}^2) - 4(4r^2 + 1)\dot{r}^4 - (4r^2 - 1)\dot{r}^2 \end{aligned}$$

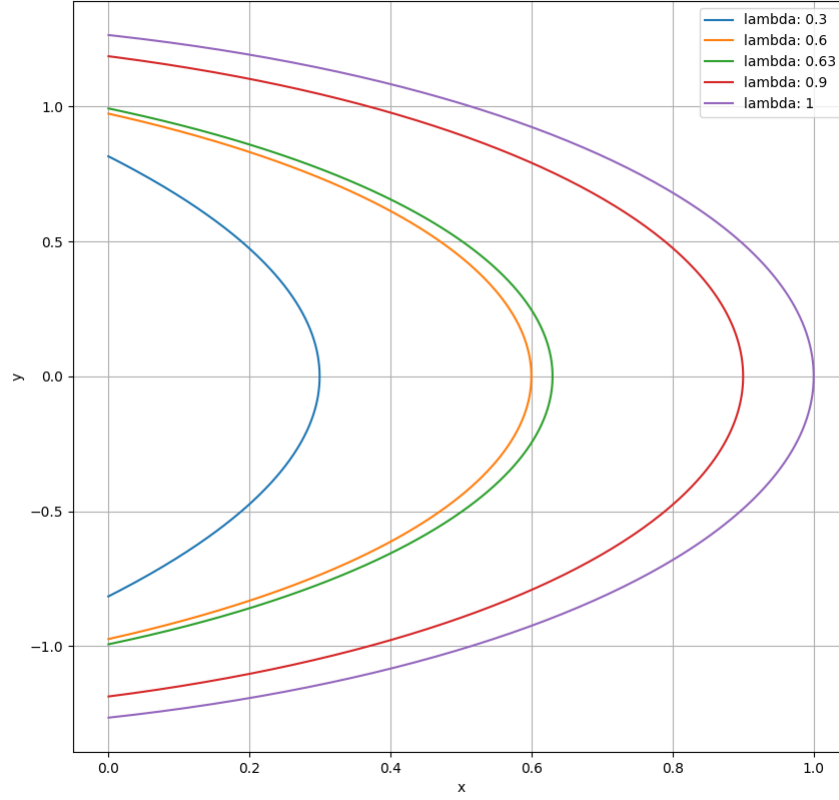
So finally we get that

$$\ddot{r} = \frac{(4\dot{r}^2 + 1)(r^2 + (1 + 4r^2)\dot{r}^2) - 4(4r^2 + 1)\dot{r}^4 - (4r^2 - 1)\dot{r}^2}{r(4r^2 + 1)}$$

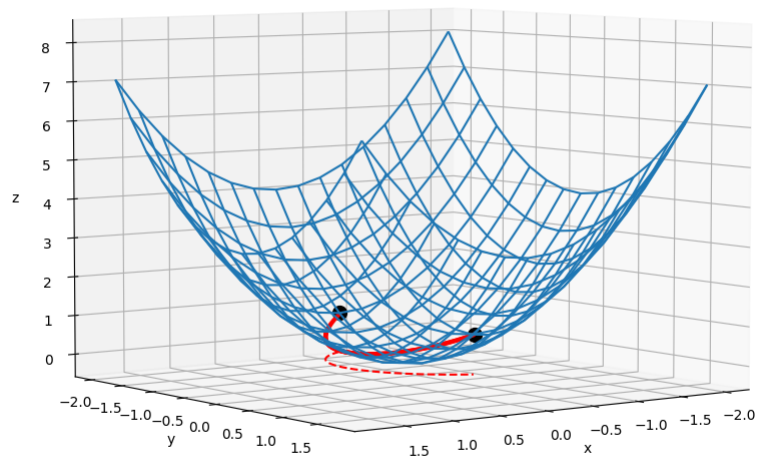
Now naming $\dot{r} = u$ we have that

$$\dot{u} = \frac{(4u^2 + 1)(r^2 + (1 + 4r^2)u^2) - 4(4r^2 + 1)u^4 - (4r^2 - 1)u^2}{r(4r^2 + 1)}$$

Hence we have a system of differential equations that we can solve numerically assuming that $r(0) = \lambda$ and $u(0) = \dot{r}(0) = 0$ in this way, we get that



Which is the projection of the curve over the $z = 0$ plane. Observing this plot we see that the right value for λ is around 0.63. Finally, we have the 3d plot



Where the red line is the path that starts from the black dot $Q = (0, -1, 1)$ and ends on $P = (0, 1, 1)$. \square