Solved selected problems of Classical Mechanics - Gregory

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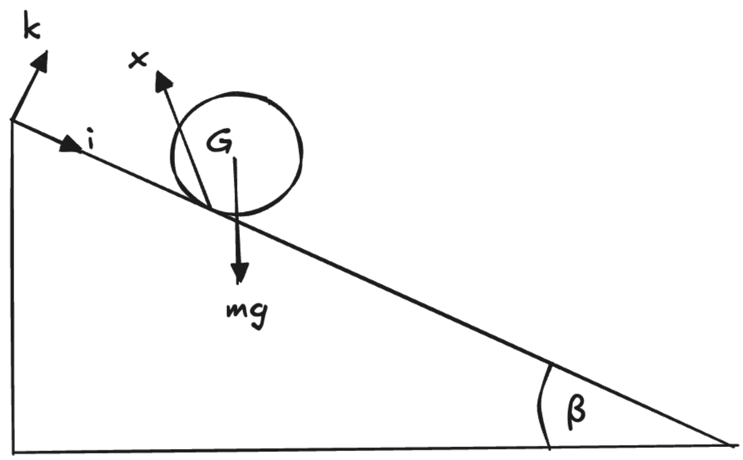
Chapter 19 - Problems in rigid body dynamics

Solution. 19.1 Let us consider the following system



Then from the governing equations we have that

$$\begin{aligned} m\dot{\boldsymbol{V}} &= \boldsymbol{X} + mg\sin\beta\boldsymbol{i} - mg\cos\beta\boldsymbol{k} \\ \dot{\boldsymbol{L}}_G &= (-b\boldsymbol{k}) \times \boldsymbol{X} \end{aligned}$$

Where $\boldsymbol{X}$ is the reaction of the table exerted on the ball. Using that $\boldsymbol{L}_G = A\boldsymbol{\omega}$ and eliminating the reaction $\boldsymbol{X}$ we get that

$$\begin{aligned} A\dot{\boldsymbol{\omega}} &= (-b\boldsymbol{k}) \times (m\dot{\boldsymbol{V}} - mg\sin\beta\boldsymbol{i} + mg\cos\beta\boldsymbol{k}) \\ &= b(m\dot{\boldsymbol{V}} - mg\sin\beta\boldsymbol{i} + mg\cos\beta\boldsymbol{k}) \times \boldsymbol{k} \\ &= mb\dot{\boldsymbol{V}} \times \boldsymbol{k} - mgb\sin\beta\boldsymbol{i} \times \boldsymbol{k} \\ &= mb\dot{\boldsymbol{V}} \times \boldsymbol{k} + mgb\sin\beta\boldsymbol{j} \end{aligned}$$

On integrating with respect to t we get that

$$A\boldsymbol{\omega} + mb\boldsymbol{k} \times \boldsymbol{V} = \boldsymbol{C} + mgbt\sin\beta\boldsymbol{j}$$

Where $\boldsymbol{C}$ is a constant vector. In particular if we take the scalar product of this equation with $\boldsymbol{k}$, we have that

$$A\boldsymbol{\omega} \cdot \boldsymbol{k} + mb(\boldsymbol{k} \times \boldsymbol{V}) \cdot \boldsymbol{k} = \boldsymbol{C} \cdot \boldsymbol{k} + mgbt \sin \beta \boldsymbol{j} \cdot \boldsymbol{k}$$
$$A\boldsymbol{\omega} \cdot \boldsymbol{k} = n$$

Where n is a constant. Also, we used that $\boldsymbol{j} \cdot \boldsymbol{k} = 0$ and the triple product is also 0. Therefore we see that the component of $\boldsymbol{\omega}$ in the direction of $\boldsymbol{k}$ (perpendicular to the inclined plane) is constant independent of the motion of the ball.

If we now consider that the ball is rolling, then the particle C in contact with the plane has zero velocity, so the rolling condition give us

$$\boldsymbol{V} + b\boldsymbol{k} \times \boldsymbol{\omega} = \boldsymbol{0}$$

Now, by cross-multiplying the conservation principle by $\boldsymbol{k}$ we have that

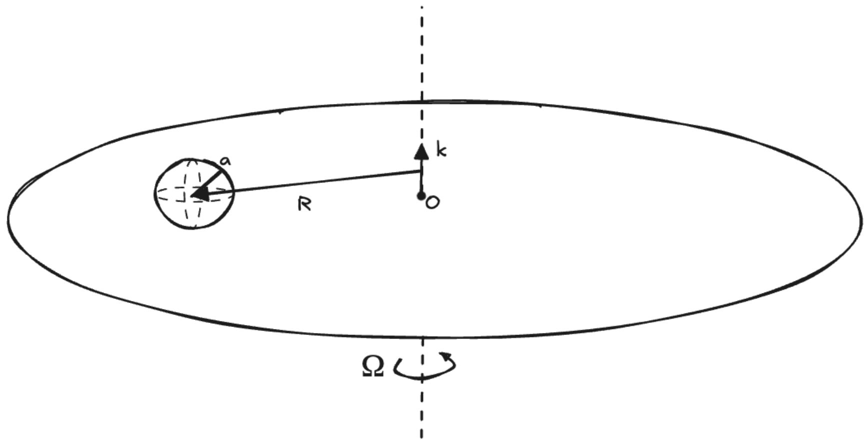
$$\begin{aligned} A\boldsymbol{k} \times \boldsymbol{\omega} &= \boldsymbol{k} \times \boldsymbol{C} + mgbt \sin \beta \boldsymbol{k} \times \boldsymbol{j} - mb\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{V}) \\ &= \boldsymbol{k} \times \boldsymbol{C} - mgbt \sin \beta \boldsymbol{i} - mb((\boldsymbol{k} \cdot \boldsymbol{V})\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{k})\boldsymbol{V}) \\ &= \boldsymbol{k} \times \boldsymbol{C} - mgbt \sin \beta \boldsymbol{i} + mb\boldsymbol{V} \end{aligned}$$

Therefore by joining this equation and the rolling condition equation we get that

$$\boldsymbol{V} + \frac{mb^2}{A}\boldsymbol{V} = -\frac{b}{a}\boldsymbol{k} \times \boldsymbol{C} + mgbt \sin \beta \boldsymbol{i}$$
$$\boldsymbol{V} + \frac{mb^2}{A}\boldsymbol{V} = \frac{b}{a}\boldsymbol{C} \times \boldsymbol{k} + mgbt \sin \beta \boldsymbol{i}$$

This shows that $\boldsymbol{V}$ is dependent on time. If we integrate again this equation we see that the rolling motion depends on t^2 and therefore the path of the ball must be a parabola. □

Solution. 19.2 Let us consider the following system



Then from the governing equations we have that

$$m\dot{\boldsymbol{V}} = \boldsymbol{X} - mg\boldsymbol{k}$$
$$\dot{\boldsymbol{L}}_G = (-a\boldsymbol{k}) \times \boldsymbol{X}$$

Where $\boldsymbol{X}$ is the reaction of the turntable exerted on the ball. Using that $\boldsymbol{L}_G = A\boldsymbol{\omega}$ where $A = \frac{2}{5}ma^2$ is the moment of inertia of the ball, then eliminating the reaction $\boldsymbol{X}$ we get that

$$\begin{aligned} A\dot{\boldsymbol{\omega}} &= (-a\boldsymbol{k}) \times (m\dot{\boldsymbol{V}} + mg\boldsymbol{k}) \\ &= a(m\dot{\boldsymbol{V}} + mg\boldsymbol{k}) \times \boldsymbol{k} \\ &= ma\dot{\boldsymbol{V}} \times \boldsymbol{k} \end{aligned}$$

On integrating with respect to t we get that

$$A\boldsymbol{\omega} + ma\boldsymbol{k} \times \boldsymbol{V} = \boldsymbol{C}$$

Where $\boldsymbol{C}$ is a constant vector. In particular if we take the scalar product of this equation with $\boldsymbol{k}$, we have that

$$\begin{aligned} A\boldsymbol{\omega} \cdot \boldsymbol{k} + ma(\boldsymbol{k} \times \boldsymbol{V}) \cdot \boldsymbol{k} &= \boldsymbol{C} \cdot \boldsymbol{k} \\ A\boldsymbol{\omega} \cdot \boldsymbol{k} &= n \end{aligned}$$

Where n is a constant. Also, we used that the triple product is also 0. Therefore we see that in any motion of the ball, the vertical spin $\boldsymbol{\omega} \cdot \boldsymbol{k}$ is constant.

If we now consider that the ball is rolling, then the particle C in contact with the turntable has zero velocity, so the rolling condition give us

$$\boldsymbol{V}_C = \boldsymbol{V} + a\boldsymbol{k} \times \boldsymbol{\omega} = \boldsymbol{0}$$

Also, from the equation for $A\dot{\boldsymbol{\omega}}$ we derived above, replacing the value of A we have that

$$\frac{2}{5}ma^2\dot{\boldsymbol{\omega}} = ma\dot{\boldsymbol{V}} \times \boldsymbol{k}$$
$$\frac{2}{5}a\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{V}} \times \boldsymbol{k}$$

And cross-multiplying this equiation by $\boldsymbol{k}$ we obtain an expression for $\dot{\boldsymbol{V}}$ as follows

$$\frac{2}{5}a\boldsymbol{k} \times \dot{\boldsymbol{\omega}} = \boldsymbol{k} \times (\dot{\boldsymbol{V}} \times \boldsymbol{k})$$
$$\frac{2}{5}a\boldsymbol{k} \times \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{V}}(\boldsymbol{k} \cdot \boldsymbol{k}) - \boldsymbol{k}(\boldsymbol{k} \cdot \dot{\boldsymbol{V}})$$
$$\frac{2}{5}a\boldsymbol{k} \times \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{V}}$$

If we assume the position of the ball's center of mass is determined by a vector $\boldsymbol{R}$ from the axis $\{O, \boldsymbol{k}\}$ then the velocity $\boldsymbol{V}_C$ of the particle C is also

$$\boldsymbol{V}_C = \Omega\boldsymbol{k} \times \boldsymbol{R}$$

Hence by joining the rolling condition and this equation we have that

$$\boldsymbol{V} + a\boldsymbol{k} \times \boldsymbol{\omega} = \Omega\boldsymbol{k} \times \boldsymbol{R}$$

Derivating this expression and replacing the value for $a\boldsymbol{k} \times \dot{\boldsymbol{\omega}}$ we finnally get that

$$\dot{\boldsymbol{V}} + \frac{5}{2}\dot{\boldsymbol{V}} = \Omega\boldsymbol{k} \times \boldsymbol{V}$$
$$\frac{7}{2}\dot{\boldsymbol{V}} = \Omega\boldsymbol{k} \times \boldsymbol{V}$$
$$\dot{\boldsymbol{V}} = \frac{2}{7}\Omega\boldsymbol{k} \times \boldsymbol{V}$$

Integrating this equation we get that

$$\dot{\boldsymbol{R}} = \frac{2}{7}\Omega\boldsymbol{k} \times \boldsymbol{R} + \boldsymbol{C}$$

Where $\boldsymbol{C}$ is some constant vector.

Finally, let us suppose that the unit vector $\boldsymbol{i}$ is in the direction of $\boldsymbol{R}$ at the moment the ball is released. Then by the initial conditions we see that $\boldsymbol{R} = b\boldsymbol{i}$ and $\dot{\boldsymbol{R}} = \Omega b\boldsymbol{j}$ since the ball is at rest with respect to the turntable so the constant of integration becomes

$$\Omega b\boldsymbol{j} = \frac{2}{7}\Omega b\boldsymbol{j} + \boldsymbol{C}$$
$$\boldsymbol{C} = \Omega b\left(1 - \frac{2}{7}\right)\boldsymbol{j}$$
$$\boldsymbol{C} = \frac{5}{7}\Omega b\boldsymbol{j}$$

Therefore

$$\dot{\boldsymbol{R}} = \frac{2}{7}\Omega \boldsymbol{k} \times \boldsymbol{R} + \frac{5}{7}\Omega b \boldsymbol{j}$$

To solve this equation let us introduce a new variable $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = (\boldsymbol{R} + \frac{5}{2}b\boldsymbol{i})$$

Hence

$$\dot{\boldsymbol{A}} = \frac{2}{7}\Omega \boldsymbol{k} \times \boldsymbol{A}$$

So now we can solve this equation by assumming that $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j}$ and $\dot{\boldsymbol{A}} = \dot{A}_x\boldsymbol{i} + \dot{A}_y\boldsymbol{j}$ as follows

$$\begin{aligned}
\dot{A}_x\boldsymbol{i} + \dot{A}_y\boldsymbol{j} &= \frac{2}{7}\Omega \boldsymbol{k} \times (A_x\boldsymbol{i} + A_y\boldsymbol{j}) \\
\dot{A}_x\boldsymbol{i} + \dot{A}_y\boldsymbol{j} &= \frac{2}{7}\Omega A_x\boldsymbol{j} - \frac{2}{7}\Omega A_y\boldsymbol{i}
\end{aligned}$$

Therefore we get the following system of differential equations

$$\begin{aligned}
\dot{A}_x &= -\frac{2}{7}\Omega A_y \\
\dot{A}_y &= \frac{2}{7}\Omega A_x
\end{aligned}$$

Where the solution is

$$\begin{aligned}
A_x &= C_1 \cos\left(\frac{2}{7}\Omega t\right) - C_2 \sin\left(\frac{2}{7}\Omega t\right) \\
A_y &= C_1 \sin\left(\frac{2}{7}\Omega t\right) - C_2 \cos\left(\frac{2}{7}\Omega t\right)
\end{aligned}$$

When we apply the initial conditions we get that

$$\frac{7}{2}b = C_1 \qquad 0 = C_2$$

Therefore

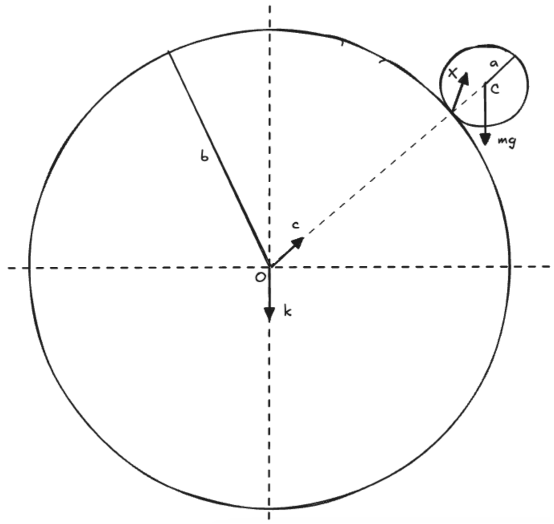
$$\begin{aligned}
A_x &= \frac{7}{2}b\cos\left(\frac{2}{7}\Omega t\right) \\
A_y &= \frac{7}{2}b\sin\left(\frac{2}{7}\Omega t\right)
\end{aligned}$$

But returning to the original variable we get that

$$\begin{aligned}
R_x &= \frac{7}{2}b\cos\left(\frac{2}{7}\Omega t\right) - \frac{5}{2}b \\
R_y &= \frac{7}{2}b\sin\left(\frac{2}{7}\Omega t\right)
\end{aligned}$$

These are the equations for a circular path which has a centre displaced $\frac{5}{2}b$ from O and has a radius of $\frac{7}{2}b$. □

Solution. 19.3 Let us consider the following system



Then from the governing equations we have that

$$m\dot{\boldsymbol{V}} = \boldsymbol{X} - mg\boldsymbol{k}$$
$$\dot{\boldsymbol{L}}_C = (-a\boldsymbol{c}) \times \boldsymbol{X}$$

Where $\boldsymbol{X}$ is the reaction of the sphere exerted on the ball. Using that $\boldsymbol{L}_C = A\boldsymbol{\omega}$ where $A = \frac{2}{5}ma^2$ is the moment of inertia of the ball and eliminating the reaction $\boldsymbol{X}$ we get that

$$\begin{aligned} A\dot{\boldsymbol{\omega}} &= (-a\boldsymbol{c}) \times (m\dot{\boldsymbol{V}} + mg\boldsymbol{k}) \\ &= a(m\dot{\boldsymbol{V}} + mg\boldsymbol{k}) \times \boldsymbol{c} \\ &= ma(\dot{\boldsymbol{V}} \times \boldsymbol{c} + g\boldsymbol{k} \times \boldsymbol{c}) \end{aligned}$$

Dot-multiplying the equation by $\boldsymbol{c}$ we obtain that

$$\begin{aligned} A\boldsymbol{c} \cdot \dot{\boldsymbol{\omega}} &= ma(\boldsymbol{c} \cdot (\dot{\boldsymbol{V}} \times \boldsymbol{c}) + g\boldsymbol{c} \cdot (\boldsymbol{k} \times \boldsymbol{c})) \\ &= ma(0 + g \cdot 0) \\ &= 0 \end{aligned}$$

If we now consider that the ball is rolling, then the particle in contact with the sphere has zero velocity, so the rolling condition give us

$$\boldsymbol{V} + a\boldsymbol{c} \times \boldsymbol{\omega} = \boldsymbol{0}$$
$$\boldsymbol{V} = a\boldsymbol{\omega} \times \boldsymbol{c}$$

Cross-multiplying this equation by $\boldsymbol{c}$ we get that

$$\begin{aligned} \boldsymbol{c} \times \boldsymbol{V} &= a\boldsymbol{c} \times (\boldsymbol{\omega} \times \boldsymbol{c}) \\ \boldsymbol{c} \times \boldsymbol{V} &= a(\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{c})\boldsymbol{c}) \\ a\boldsymbol{\omega} &= (\boldsymbol{c} \times \boldsymbol{V}) + a(\boldsymbol{\omega} \cdot \boldsymbol{c})\boldsymbol{c} \\ a\boldsymbol{\omega} &= (\boldsymbol{c} \times \boldsymbol{V}) + a\lambda\boldsymbol{c} \end{aligned}$$

Where λ is a scalar function of time. Also, we know that $\boldsymbol{V} = (a+b)\dot{\boldsymbol{c}}$ hence

$$a\boldsymbol{\omega} = (a+b)\boldsymbol{c} \times \dot{\boldsymbol{c}} + a\lambda\boldsymbol{c}$$

And derivating this equation we find that

$$\begin{aligned} a\dot{\boldsymbol{\omega}} &= (a+b)(\dot{\boldsymbol{c}} \times \dot{\boldsymbol{c}} + \boldsymbol{c} \times \ddot{\boldsymbol{c}}) + a(\dot{\lambda}\boldsymbol{c} + \lambda\dot{\boldsymbol{c}}) \\ &= (a+b)\boldsymbol{c} \times \ddot{\boldsymbol{c}} + a(\dot{\lambda}\boldsymbol{c} + \lambda\dot{\boldsymbol{c}}) \end{aligned}$$

Finally, dot-multypling this equation by $\boldsymbol{c}$ we obtain

$$\begin{aligned} a\boldsymbol{c} \cdot \dot{\boldsymbol{\omega}} &= (a+b)\boldsymbol{c} \cdot (\boldsymbol{c} \times \ddot{\boldsymbol{c}}) + a\boldsymbol{c} \cdot (\dot{\lambda}\boldsymbol{c} + \lambda\dot{\boldsymbol{c}}) \\ 0 &= 0 + a(\dot{\lambda} + 0) \end{aligned}$$

Therefore $\dot{\lambda} = 0$ which implies that $\lambda = \boldsymbol{\omega} \cdot \boldsymbol{c} = n$ where n is constant. Which implies that the radial spin is conserved.

On the other hand, by derivating the rolling condition we have that

$$\dot{\boldsymbol{V}} = a\dot{\boldsymbol{\omega}} \times \boldsymbol{c} + a\boldsymbol{\omega} \times \dot{\boldsymbol{c}}$$

Replacing these expressions and the value of A in the equation for $A\dot{\boldsymbol{\omega}}$ we get that

$$\begin{aligned} \frac{2}{5}ma^2\dot{\boldsymbol{\omega}} &= ma(a(\dot{\boldsymbol{\omega}} \times \boldsymbol{c} + \boldsymbol{\omega} \times \dot{\boldsymbol{c}}) \times \boldsymbol{c} + g\boldsymbol{k} \times \boldsymbol{c}) \\ 2a\dot{\boldsymbol{\omega}} &= 5a(\dot{\boldsymbol{\omega}} \times \boldsymbol{c}) \times \boldsymbol{c} + 5a(\boldsymbol{\omega} \times \dot{\boldsymbol{c}}) \times \boldsymbol{c} + 5g\boldsymbol{k} \times \boldsymbol{c} \\ 2a\dot{\boldsymbol{\omega}} &= 5a(-(\boldsymbol{c} \cdot \boldsymbol{c})\dot{\boldsymbol{\omega}} + (\boldsymbol{c} \cdot \dot{\boldsymbol{\omega}})\boldsymbol{c}) + 5a(-(\boldsymbol{c} \cdot \dot{\boldsymbol{c}})\boldsymbol{\omega} + (\boldsymbol{c} \cdot \boldsymbol{\omega})\dot{\boldsymbol{c}}) + 5g\boldsymbol{k} \times \boldsymbol{c} \\ 2a\dot{\boldsymbol{\omega}} &= -5a\dot{\boldsymbol{\omega}} + 0 - 0 + 5an\dot{\boldsymbol{c}} + 5g\boldsymbol{k} \times \boldsymbol{c} \\ 7a\dot{\boldsymbol{\omega}} &= 5an\dot{\boldsymbol{c}} + 5g\boldsymbol{k} \times \boldsymbol{c} \end{aligned}$$

$$7(a+b)\boldsymbol{c} \times \ddot{\boldsymbol{c}} + 2an\dot{\boldsymbol{c}} - 5g\boldsymbol{k} \times \boldsymbol{c} = 0$$
$$7(a+b)\boldsymbol{c} \times \ddot{\boldsymbol{c}} + 2an\dot{\boldsymbol{c}} + 5g\boldsymbol{c} \times \boldsymbol{k} = 0$$

Where we used that $\boldsymbol{c} \cdot \boldsymbol{c} = 1$, $\boldsymbol{c} \cdot \boldsymbol{\omega} = n$ and $\boldsymbol{c} \cdot \dot{\boldsymbol{c}} = 0$.

This equation matches with the equation of a spinning top which has a stable motion therefore the ball must have a stable motion too so it can roll on the spherical surface without ever falling off.

Finally, at the highest point of the sphere the ball will be stable when

$$C^2n^2 > 4Amgh$$

Which is the equation that determines when a spinning top will be stable in the vertically upright position but we apply it to this case as follows

$$\begin{aligned} (2am)^2n^2 &> 4(7m(a+b))mg5 \\ a^2n^2 &> 35g(a+b) \\ n^2 &> \frac{35g(a+b)}{a^2} \end{aligned}$$

□

Solution. 19.4 We know that steady precession can only happen if

$$A\cos\alpha\Omega^2 - Cn\Omega + Mgh = 0$$

In this case, since the axis of the top moves in the horizontal plane through O we get that $\alpha = \pi/2$ and hence $\cos\alpha = 0$, then the equation becomes

$$-Cn\Omega + Mgh = 0$$

So there is only one rate of precession given by

$$\Omega = \frac{Mgh}{Cn}$$

□

Solution. 19.5 We know the Lagrangian for the top is given by

$$L = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A(\dot{\phi}\sin\theta)^2 + \frac{1}{2}Cw + \dot{\phi}\cos\theta)^2 - Mgh\cos\theta$$

So the Lagrange equation for θ give us

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \\ A\ddot{\theta} - A\dot{\phi}^2\sin\theta\cos\theta + C\dot{\psi}\dot{\phi}\sin\theta + C\dot{\phi}^2\sin\theta\cos\theta - Mgh\sin\theta = 0 \\ A\ddot{\theta} - A\dot{\phi}^2\sin\theta\cos\theta + C\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) - Mgh\sin\theta = 0 \end{aligned}$$

Also, we know that

$$\begin{aligned} A\dot{\phi}\sin^2\theta + Cn\cos\theta = L_z \\ C(\dot{\psi} + \dot{\phi}\cos\theta) = Cn \end{aligned}$$

But given that the top is spinning upright we see that $L_z = Cn$ so we have that

$$\begin{aligned} A\dot{\phi}\sin^2\theta = Cn(1-\cos\theta) \\ \dot{\phi} = \frac{Cn(1-\cos\theta)}{A\sin^2\theta} \end{aligned}$$

Replacing this value in the Lagrange equation we get that

$$\begin{aligned} A\ddot{\theta} - A\frac{C^2n^2(1-\cos\theta)^2}{A^2\sin^4\theta}\sin\theta\cos\theta + Cn\frac{Cn(1-\cos\theta)}{A\sin^2\theta}\sin\theta - Mgh\sin\theta = 0 \\ A\ddot{\theta} - \frac{C^2n^2(1-\cos\theta)}{A\sin^2\theta}\left(\frac{(1-\cos\theta)}{\sin^2\theta}\cos\theta - 1\right)\sin\theta - Mgh\sin\theta = 0 \\ A\ddot{\theta} - \frac{C^2n^2(1-\cos\theta)}{A\sin^2\theta}\left(\frac{\cos\theta - \cos^2\theta - \sin^2\theta}{\sin^2\theta}\right)\sin\theta - Mgh\sin\theta = 0 \\ A\ddot{\theta} + \frac{C^2n^2(1-\cos\theta)}{A\sin^2\theta}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\sin\theta - Mgh\sin\theta = 0 \\ A\ddot{\theta} + \frac{C^2n^2(1-\cos\theta)^2}{A\sin^4\theta}\sin\theta - Mgh\sin\theta = 0 \end{aligned}$$

If we consider θ to be small we can approximate $(1-\cos\theta)^2/(\sin^4\theta)$ as

$$\frac{(1-\cos\theta)^2}{\sin^4\theta} \approx \frac{(1-(1-\theta^2/2))^2}{\theta^4} = \frac{\theta^4/4}{\theta^4} = \frac{1}{4}$$

Therefore, we can write that

$$A\ddot{\theta} + \left(\frac{C^2n^2}{4A} - Mgh\right)\theta = 0$$

Which is the equation for a Simple Harmonic Motion and hence for the motion to be stable it must happen that $\frac{C^2n^2}{4A} - Mgh$ is positive i.e.

$$C^2n^2 > 4AMgh$$

□

Solution. 19.6 Given that a pencil is an axisymmetric body and we know from Problem 19.5 that a body of this type will be stable in the upright position if

$$C^2 n^2 > 4AMgh$$

In this case we know that $C = \frac{1}{2}Mr^2$ and $A = \frac{1}{4}Mr^2 + \frac{1}{12}Ml^2 + \frac{Ml^2}{4}$ where r is the radius and l is the length of the pencil. Also, we used the parallel axis theorem to determine A with respect to O and we assumed that the pencil is a uniform cylinder. Then

$$\begin{aligned}\left(\frac{1}{4}M^2r^4\right)n^2 &> 4\left(\frac{1}{4}Mr^2 + \frac{1}{12}Ml^2 + \frac{Ml^2}{4}\right)Mgh \\ \frac{1}{4}r^4n^2 &> gh\left(r^2 + \frac{1}{3}l^2 + l^2\right) \\ n &> \sqrt{4gh\left(\frac{1}{r^2} + \frac{4l^2}{3r^4}\right)}\end{aligned}$$

Hence n must be

$$\begin{aligned}n &> \sqrt{4 \cdot 980\ cm/s^2 \cdot 7.5\ cm \cdot \left(\frac{1}{(0.35\ cm)^2} + \frac{4 \cdot (15\ cm)^2}{3 \cdot (0.35\ cm)^4}\right)} \\ n &> 24248.61\ rad/s\end{aligned}$$

Therefore for the pencil to remain stable in the upright position n must be at least 3859.28 rev/s

□

Solution. 19.7

(i) Taking the uniform solid ball as an axisymmetric body from Problem 19.5 we know that the ball can be stable rotating in the finger if

$$C^2 n^2 > 4AMgh$$

In this case we know we have that $C = \frac{2}{5}M(\frac{d}{2})^2$ and $A = \frac{2}{5}M(\frac{d}{2})^2 + M(\frac{d}{2})^2$ where d is the diameter of the ball. Also, we used the parallel axis theorem to determine A with respect to O which is where the ball touches the finger. Then

$$\left(\frac{1}{100}M^2 d^4\right)n^2 > 4\left(\frac{1}{10}Md^2 + M\frac{d^2}{4}\right)Mgh$$

$$\frac{1}{100}d^4 n^2 > gh\left(\frac{2}{5}d^2 + d^2\right)$$

$$n > \sqrt{140\ \frac{gh}{d^2}}$$

$$n > \sqrt{70\ \frac{g}{d}}$$

Hence n must be

$$n > \sqrt{70 \cdot \frac{980\ cm/s^2}{20\ cm}}$$

$$n > 58.56\ rad/s$$

Therefore for the uniform solid ball to remain stable n must be at least $9.32\ rev/s$.

(ii) In the same way, for a uniform thin hollow ball, taking $C = \frac{2}{3}M(\frac{d}{2})^2$ and $A = \frac{2}{3}M(\frac{d}{2})^2 + M(\frac{d}{2})^2$ we have that

$$\left(\frac{1}{36}M^2 d^4\right)n^2 > 4\left(\frac{1}{6}Md^2 + M\frac{d^2}{4}\right)Mgh$$

$$\frac{1}{36}d^4 n^2 > gh\left(\frac{2}{3}d^2 + d^2\right)$$

$$n > \sqrt{60\ \frac{gh}{d^2}}$$

$$n > \sqrt{30\ \frac{g}{d}}$$

Hence n must be

$$n > \sqrt{30 \cdot \frac{980\ cm/s^2}{20\ cm}}$$

$$n > 38.34\ rad/s$$

Therefore for the uniform thin hollow ball to remain stable n must be at least $6.10\ rev/s$.

Finally, given that most balls are not solid, the juggler should use the determined n for the uniform thin hollow ball. $\square$

Solution. 19.9 From the equation of motion we see that

$$\dot{\boldsymbol{L}}_G = -K\boldsymbol{\omega}$$

Where $K\boldsymbol{\omega}$ is the frictional couple. So from the equation of motion for a free axisymmetric body we have that

$$\begin{aligned}\frac{d}{dt}(A\boldsymbol{a} \times \dot{\boldsymbol{a}} + C\lambda\boldsymbol{a}) &= -K\boldsymbol{\omega}\\ A\boldsymbol{a} \times \ddot{\boldsymbol{a}} + C(\dot{\lambda}\boldsymbol{a} + \lambda\dot{\boldsymbol{a}}) &= -K\boldsymbol{\omega}\end{aligned}$$

If we now take the scalar product of this equation with $\boldsymbol{a}$, we obtain

$$\begin{aligned}A\boldsymbol{a} \cdot (\boldsymbol{a} \times \ddot{\boldsymbol{a}}) + C(\dot{\lambda}(\boldsymbol{a} \cdot \boldsymbol{a}) + \lambda(\boldsymbol{a} \cdot \dot{\boldsymbol{a}})) &= -K\boldsymbol{a} \cdot \boldsymbol{\omega}\\ C\dot{\lambda} &= -K\boldsymbol{a} \cdot \boldsymbol{\omega}\\ \dot{\lambda} &= -\frac{K}{C}\lambda\end{aligned}$$

Then replacing this value of $\dot{\lambda}$ we get that

$$\begin{aligned}A\boldsymbol{a} \times \ddot{\boldsymbol{a}} - K\lambda\boldsymbol{a} + C\lambda\dot{\boldsymbol{a}} &= -K\boldsymbol{\omega}\\ A\boldsymbol{a} \times \ddot{\boldsymbol{a}} + K(\boldsymbol{\omega} - \lambda\boldsymbol{a}) + C\lambda\dot{\boldsymbol{a}} &= 0\end{aligned}$$

But we know that the velocity of $\boldsymbol{a}$ is $\dot{\boldsymbol{a}} = \boldsymbol{\omega} \times \boldsymbol{a}$ and hence by taking the cross product of this equation with $\boldsymbol{a}$ we have that

$$\begin{aligned}\boldsymbol{a} \times \dot{\boldsymbol{a}} &= \boldsymbol{a} \times (\boldsymbol{\omega} \times \boldsymbol{a})\\ &= (\boldsymbol{a} \cdot \boldsymbol{a})\boldsymbol{\omega} - (\boldsymbol{a} \cdot \boldsymbol{\omega})\boldsymbol{a}\\ &= \boldsymbol{\omega} - \lambda\boldsymbol{a}\end{aligned}$$

Hence replacing this value we get that

$$A\boldsymbol{a} \times \ddot{\boldsymbol{a}} + K\boldsymbol{a} \times \dot{\boldsymbol{a}} + C\lambda\dot{\boldsymbol{a}} = 0$$

Taking the cross product of this equation with $\dot{\boldsymbol{a}}$ we get that

$$\begin{aligned}A\dot{\boldsymbol{a}} \times (\boldsymbol{a} \times \ddot{\boldsymbol{a}}) + K\dot{\boldsymbol{a}} \times (\boldsymbol{a} \times \dot{\boldsymbol{a}}) + C\lambda\dot{\boldsymbol{a}} \times \dot{\boldsymbol{a}} &= 0\\ A((\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}})\boldsymbol{a} - (\dot{\boldsymbol{a}} \cdot \boldsymbol{a})\ddot{\boldsymbol{a}}) + K((\dot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}})\boldsymbol{a} - (\dot{\boldsymbol{a}} \cdot \boldsymbol{a})\boldsymbol{a}) &= 0\\ A(\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}})\boldsymbol{a} + K|\dot{\boldsymbol{a}}|^2\boldsymbol{a} &= 0\\ A\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}} + K|\dot{\boldsymbol{a}}|^2 &= 0\end{aligned}$$

But also we know that

$$\frac{d}{dt}|\dot{\boldsymbol{a}}|^2 = \frac{d}{dt}(\dot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}}) = \ddot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}} + \dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}} = 2\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}}$$

Hence

$$A\frac{d}{dt}|\dot{\boldsymbol{a}}|^2 + 2K|\dot{\boldsymbol{a}}|^2 = 0$$

The solution to this differential equation gives us

$$|\dot{\boldsymbol{a}}| = \frac{A}{2K} D e^{-\frac{K}{A}t}$$

Where D is a constant.

Finally, to determine the angle θ between $\boldsymbol{\omega}$ and $\boldsymbol{a}$ we can compute $\tan\theta$ as follows

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{|\boldsymbol{\omega}||\boldsymbol{a}|\sin\theta}{|\boldsymbol{\omega}||\boldsymbol{a}|\cos\theta} = \frac{|\boldsymbol{\omega}\times\boldsymbol{a}|}{\boldsymbol{\omega}\cdot\boldsymbol{a}} = \frac{|\dot{\boldsymbol{a}}|}{\lambda}$$

So we need an equation to determine λ. By solving the equation $\dot{\lambda}+\frac{K}{C}\lambda = 0$ which we derived above we get that

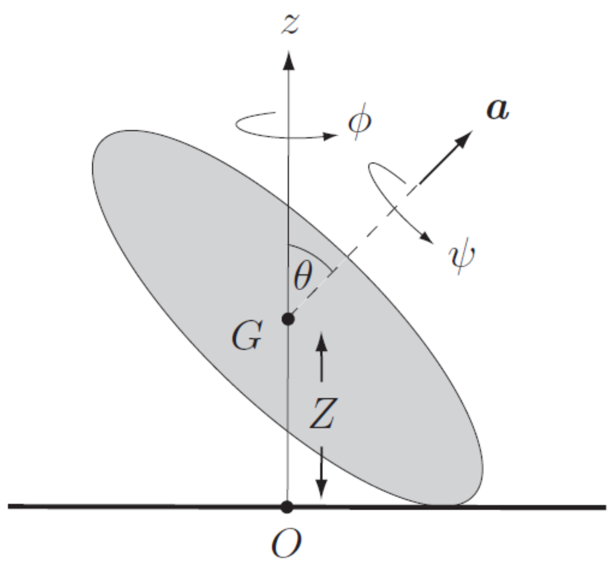
$$\lambda = E e^{-\frac{K}{C}t}$$

Where E is another constant. Then

$$\begin{aligned} \tan\theta &= \frac{ADe^{-\frac{K}{A}t}}{2KEe^{-\frac{K}{C}t}} \\ \tan\theta &= \frac{AD}{2KE} e^{(\frac{K}{C}-\frac{K}{A})t} \\ \theta &= \arctan\left(\frac{AD}{2KE} e^{K(\frac{1}{C}-\frac{1}{A})t}\right) \end{aligned}$$

Hence if $C > A$ the function $e^{K(\frac{1}{C}-\frac{1}{A})t}$ decreases with time and therefore θ decreases. $\square$

Solution. 19.10 Let us consider a spinning hoop as shown below



The kinetic energy in terms of the Euler angles is

$$T = \frac{1}{2}M\dot{Z}^2 + \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A(\dot{\phi}\sin\theta)^2 + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

Where A and C are the principal moments of inertia with respect to G. By geometry we see that

$$Z = a\sin\theta$$

and hence $\dot{Z} = a\dot{\theta}\cos\theta$ so the kinetic energy becomes

$$\begin{aligned} T &= \frac{1}{2}M(a\dot{\theta}\cos\theta)^2 + \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A(\dot{\phi}\sin\theta)^2 + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 \\ &= \frac{1}{2}\dot{\theta}^2(Ma^2\cos^2\theta + A) + \frac{1}{2}A(\dot{\phi}\sin\theta)^2 + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 \end{aligned}$$

Also, the gravitational potential energy is given by

$$V = MgZ = Mga\sin\theta$$

Then the Lagrangian for the spinning hoop is

$$L = \frac{1}{2}\dot{\theta}^2(Ma^2\cos^2\theta + A) + \frac{1}{2}A(\dot{\phi}\sin\theta)^2 + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mga\sin\theta$$

Since the Lagrangian doesn't depend on ϕ or ψ then the generalized momenta p_ϕ and p_ψ are conserved. Hence

$$\begin{aligned} A\dot{\phi}\sin^2\theta + C(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta &= L_z \\ C(\dot{\psi} + \dot{\phi}\cos\theta) &= Cn \end{aligned}$$

Where L_z and n are constants. Let us compute now the Lagrange equation for θ, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

Hence

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\dot{\theta}(Ma^2\cos^2\theta + A)] - [-\dot{\theta}^2 Ma^2 \sin\theta\cos\theta + A\dot{\phi}^2\sin\theta\cos\theta - C\dot{\psi}\dot{\phi}\sin\theta \\
&\quad - C\dot{\phi}^2\sin\theta\cos\theta - Mga\cos\theta] = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}[\dot{\theta}(Ma^2\cos^2\theta + A)] + [\dot{\theta}^2 Ma^2\sin\theta\cos\theta - A\dot{\phi}^2\sin\theta\cos\theta \\
&\quad + Cn\dot{\phi}\sin\theta + Mga\cos\theta] = 0
\end{aligned}$$

If we consider that the angle between the hoop and the floor is a constant α and we assume that Ω is the rate of steady precession the Lagrange equation for θ becomes

$$-A\Omega^2\sin\alpha\cos\alpha + Cn\Omega\sin\alpha + Mga\cos\alpha = 0$$

Now by replacing the values of $A = 1/2Ma^2$ and $C = Ma^2$ we get that

$$\begin{aligned}
-\frac{1}{2}Ma^2\Omega^2\sin\alpha\cos\alpha + Ma^2n\Omega\sin\alpha + Mga\cos\alpha = 0 \\
-\Omega^2\sin\alpha\cos\alpha + 2n\Omega\sin\alpha + 2\frac{g}{a}\cos\alpha = 0 \\
\Omega^2\cos\alpha - 2n\Omega - 2\frac{g}{a}\cot\alpha = 0
\end{aligned}$$

The solution to this equation in terms of Ω is

$$\begin{aligned}
\Omega &= \frac{2n \pm \sqrt{4n^2 + 8g\cos\alpha\cot\alpha/a}}{2\cos\alpha} \\
&= \frac{2n}{2\cos\alpha}\left(1 \pm \sqrt{1 + \frac{8g\cos\alpha\cot\alpha}{4an^2}}\right)
\end{aligned}$$

If we assume that $8g\cos\alpha\cot\alpha/4an^2$ is small we can apply the binomial approximation as follows

$$\Omega = \frac{n}{\cos\alpha}\left(1 \pm \left(1 + \frac{g\cos\alpha\cot\alpha}{an^2}\right)\right)$$

So the fast and slow solutions for the precession Ω are

$$\begin{aligned}
\Omega^F &= \frac{n}{\cos\alpha}\left(2 + \frac{g\cos\alpha\cot\alpha}{an^2}\right) = \frac{2n}{\cos\alpha} + \frac{g\cot\alpha}{an} \\
\Omega^S &= \frac{n}{\cos\alpha}\left(-\frac{g\cos\alpha\cot\alpha}{an^2}\right) = -\frac{g\cot\alpha}{an}
\end{aligned}$$

Where the fast precession goes in the "same way" as n and the slower one goes in the "opposite direction" (because of the minus sign). □

Solution. 19.12 Let an unsymmetrical body to be in steady rotation about the principal axis Gx_1 through G and suppose we disturb the body slightly. In the initial motion we can assume that $\omega_2 = \omega_3 = 0$ and $\omega_1 = \Lambda$ for some Λ constant. But after some small disturbance we get that $\omega_1 = \Lambda + a_1$ and that ω_2 and ω_3 are small but not zero.

The Euler's equations state the following

$$\begin{aligned} A\dot{\omega}_1 - (B-C)\omega_2\omega_3 &= 0 \\ B\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_3 - (A-B)\omega_1\omega_2 &= 0 \end{aligned}$$

After linealization and taking into account the variation of ω_1 we get that

$$\begin{aligned} A\dot{a}_1 &= 0 \\ B\dot{\omega}_2 - (C-A)(\Lambda + a_1)\omega_3 &= 0 \\ C\dot{\omega}_3 - (A-B)(\Lambda + a_1)\omega_2 &= 0 \end{aligned}$$

But given that a_1 is a small variation and ω_3 and ω_2 are small too, we can drop the products to get the following

$$\begin{aligned} A\dot{\omega}_1 &= 0 \\ B\dot{\omega}_2 - (C-A)\Lambda\omega_3 &= 0 \\ C\dot{\omega}_3 - (A-B)\Lambda\omega_2 &= 0 \end{aligned}$$

Where we wrote back that $\dot{\omega}_1 = \dot{a}_1$. Derivating the second equation we get that

$$B\ddot{\omega}_2 - (C-A)\Lambda\dot{\omega}_3 = 0$$

And replacing $\dot{\omega}_3$ from the third equation we get that

$$\begin{aligned} B\ddot{\omega}_2 - (C-A)\Lambda\frac{(A-B)}{C}\Lambda\omega_2 &= 0 \\ \ddot{\omega}_2 + \frac{(A-C)(A-B)}{BC}\Lambda^2\omega_2 &= 0 \end{aligned}$$

This is the equation for a Simple Harmonic Motion and hence for the motion to be stable it must happen that

$$\frac{(A-C)(A-B)}{BC}\Lambda^2 > 0$$

Therefore must be that $A > C$ and $A > B$ or $A < C$ and $A < B$ i.e. A is the largest or smallest moment of inertia.

Doing the same for ω_3 we get that

$$\begin{aligned} C\ddot{\omega}_3 - (A-B)\Lambda\frac{(C-A)}{B}\Lambda\omega_2 &= 0 \\ \ddot{\omega}_2 + \frac{(A-C)(A-B)}{BC}\Lambda^2\omega_2 &= 0 \end{aligned}$$

Which tells us the same restrictions about A i.e. A is the largest or smallest moment of inertia.

We could do the same perturbation analysis for ω_2 and ω_3 but the analysis is analogous and we will get that for the unsymmetrical body to be stable while rotating about Gx_2 we need that B to be the largest or smallest moment of inertia. The same is true for when the body is rotating about Gx_3 and the moment of inertia C needs to be the largest or smallest. $\square$

Solution. 19.13 Given that the Frisbee is an axisymmetric body the Euler's equations become

$$\begin{aligned} A\dot{\omega}_1 - (A-C)\omega_2\omega_3 &= -K\omega_1 \\ A\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= -K\omega_2 \\ C\dot{\omega}_3 &= -K\omega_3 \end{aligned}$$

From the third equation we have that

$$\omega_3 = C_1 e^{-\frac{K}{C}t}$$

Where C_1 is a constant. Multiplying the first equiation by ω_1 and the second by ω_2 we get that

$$\begin{aligned} A\dot{\omega}_1\omega_1 - (A-C)\omega_3\omega_2\omega_1 &= -K\omega_1^2 \\ A\dot{\omega}_2\omega_2 + (A-C)\omega_3\omega_2\omega_1 &= -K\omega_2^2 \end{aligned}$$

Now we sum both equations

$$\begin{aligned} -K\omega_1^2 - K\omega_2^2 &= A\dot{\omega}_1\omega_1 + A\dot{\omega}_2\omega_2 \\ \omega_1^2 + \omega_2^2 &= -\frac{A}{K}(\dot{\omega}_1\omega_1 + \dot{\omega}_2\omega_2) \end{aligned}$$

Let us note that

$$\frac{d}{dt}(\omega_1^2 + \omega_2^2) = 2(\dot{\omega}_1\omega_1 + \dot{\omega}_2\omega_2)$$

So we can write the differential equation as

$$\frac{d}{dt}(\omega_1^2 + \omega_2^2) + 2\frac{K}{A}(\omega_1^2 + \omega_2^2) = 0$$

The solution to this equation is

$$\omega_1^2 + \omega_2^2 = C_2 e^{-2\frac{K}{A}t}$$

To compute the angle θ between $\boldsymbol{\omega}$ and $\boldsymbol{a}$ we follow the same procedure as before. Let us observe that in this case, the unit vector $\boldsymbol{a}$ is in the direction of ω_3 then

$$\begin{aligned} \tan\theta &= \frac{|\boldsymbol{\omega} \times \boldsymbol{a}|}{\boldsymbol{\omega} \cdot \boldsymbol{a}} \\ \tan\theta &= \frac{\sqrt{(\omega_2 a_3)^2 + (-\omega_1 a_3)^2}}{\omega_3} \\ \tan\theta &= \frac{\sqrt{C_2 e^{-2\frac{K}{A}t}}}{C_1 e^{-\frac{K}{C}t}} \\ \theta &= \arctan\Big(C_3 e^{K(\frac{1}{C}-\frac{1}{A})t}\Big) \end{aligned}$$

Where C_3 is another constant. Hence if $C > A$ the function $e^{K(\frac{1}{C}-\frac{1}{A})t}$ decreases with time and therefore θ decreases. $\square$

Solution. 19.16 We know that the Lagrangian of the top is

$$L = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A(\dot{\phi}\sin\theta)^2 + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgh\cos\theta$$

Then the Lagrange's equation for θ is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \\
A\ddot{\theta} - \left[A\dot{\phi}^2\sin\theta\cos\theta + C(-\dot{\psi}\dot{\phi}\sin\theta - \dot{\phi}^2\sin\theta\cos\theta) + Mgh\sin\theta\right] = 0 \\
A\ddot{\theta} - A\dot{\phi}^2\sin\theta\cos\theta + C(\dot{\psi}\dot{\phi}\sin\theta + \dot{\phi}^2\sin\theta\cos\theta) - Mgh\sin\theta = 0
\end{aligned}$$

For ϕ we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \\
\frac{d}{dt}(A\dot{\phi}\sin^2\theta + C(\dot{\psi}\cos\theta + \dot{\phi}\cos^2\theta)) = 0 \\
A(\ddot{\phi}\sin^2\theta + 2\dot{\phi}\dot{\theta}\sin\theta\cos\theta) + C(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta + \ddot{\phi}\cos^2\theta \\
-2\dot{\phi}\dot{\theta}\sin\theta\cos\theta) = 0
\end{aligned}$$

And for ψ we get that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0 \\
\frac{d}{dt}(C(\dot{\psi} + \dot{\phi}\cos\theta)) = 0 \\
C(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) = 0
\end{aligned}$$

Let us name $\dot{\theta} = u$, $\dot{\phi} = v$ and $\dot{\psi} = w$ then we can write the equation for θ as

$$\dot{u} = v^2\sin\theta\cos\theta - \frac{C}{A}(wv\sin\theta + v^2\sin\theta\cos\theta) + \frac{Mgh}{A}\sin\theta$$

We can write the equation for ψ as

$$\begin{aligned}
C(\dot{w} + \dot{v}\cos\theta - vu\sin\theta) = 0 \\
C(\dot{w}\cos\theta + \dot{v}\cos^2\theta - vu\sin\theta\cos\theta) = 0
\end{aligned}$$

Where we multiplied by $\cos\theta$ in the end so we can subtract it from the equation for ϕ which states the following

$$\begin{aligned}
&A(\dot{v}\sin^2\theta + 2vu\sin\theta\cos\theta) + C(\dot{w}\cos\theta - wu\sin\theta + \dot{v}\cos^2\theta \\
&- 2vu\sin\theta\cos\theta) = 0
\end{aligned}$$

hence subracting them we get that

$$A(\dot{v}\sin^2\theta + 2vu\sin\theta\cos\theta) - C(wu\sin\theta + vu\sin\theta\cos\theta) = 0$$

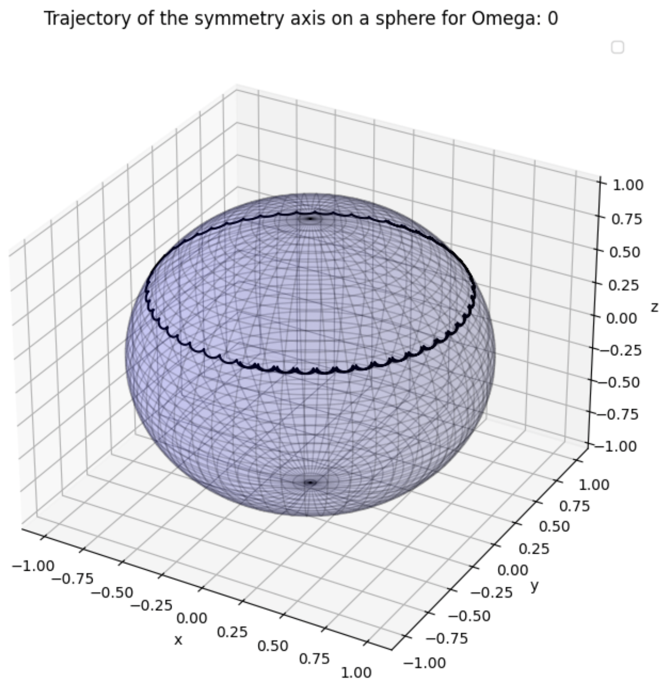
Which implies that

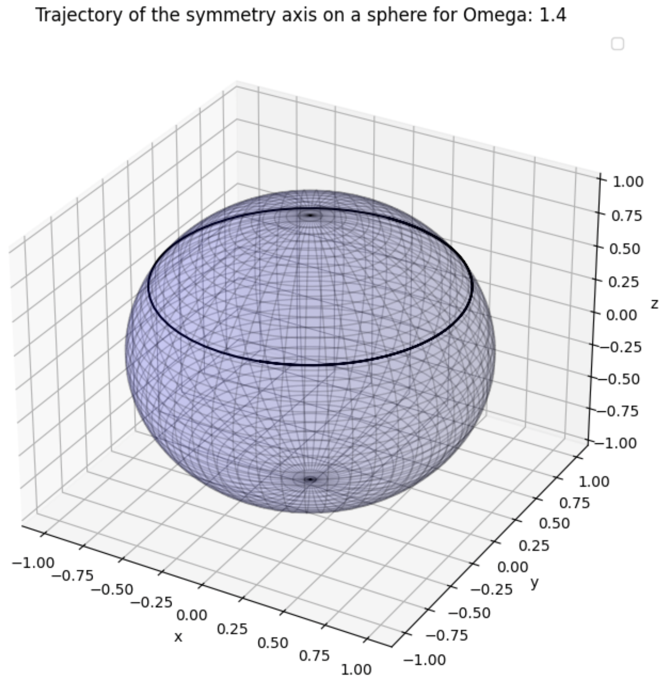
$$\dot{v} = \frac{C/A(wu\sin\theta + vu\sin\theta\cos\theta) - 2vu\sin\theta\cos\theta}{\sin^2\theta}$$

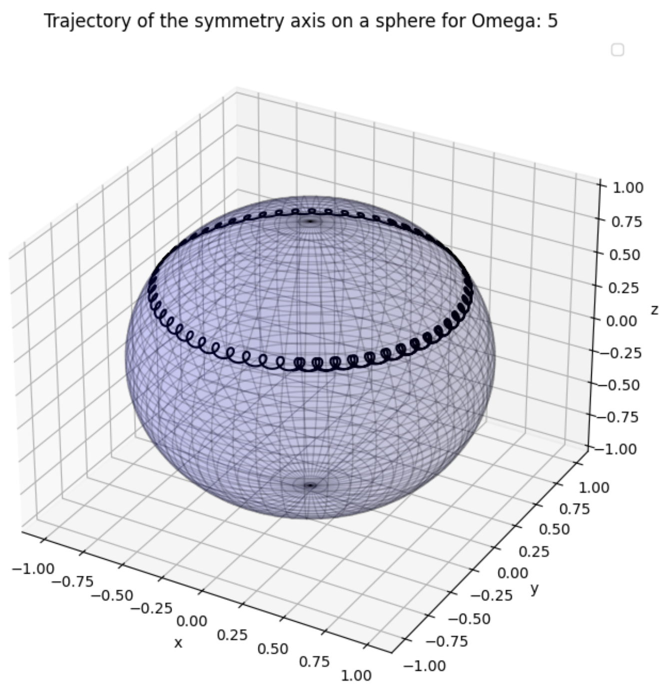
Finally we can replace this value in the equation for ψ to get an equation for $\dot{w}$ as follows

$$\dot{w} = vu\sin\theta - \frac{C/A(wu\sin\theta + vu\sin\theta\cos\theta) - 2vu\sin\theta\cos\theta}{\sin^2\theta}\cos\theta$$

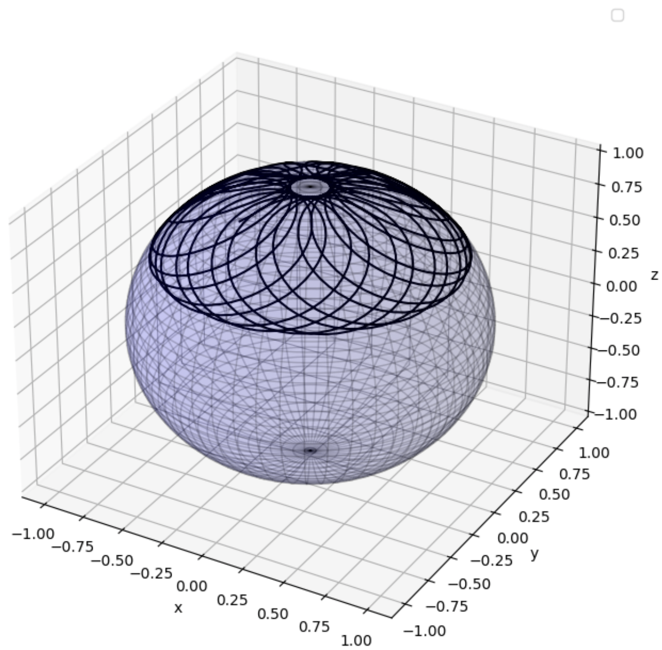
Now we can solve the system of first-order differential equations using the Runge-Kutta method. Below we show some results




Trajectory of the symmetry axis on a sphere for Omega: 1.4
x
y
z


Trajectory of the symmetry axis on a sphere for Omega: 5
x
y
z

Trajectory of the symmetry axis on a sphere for Omega: 78



□

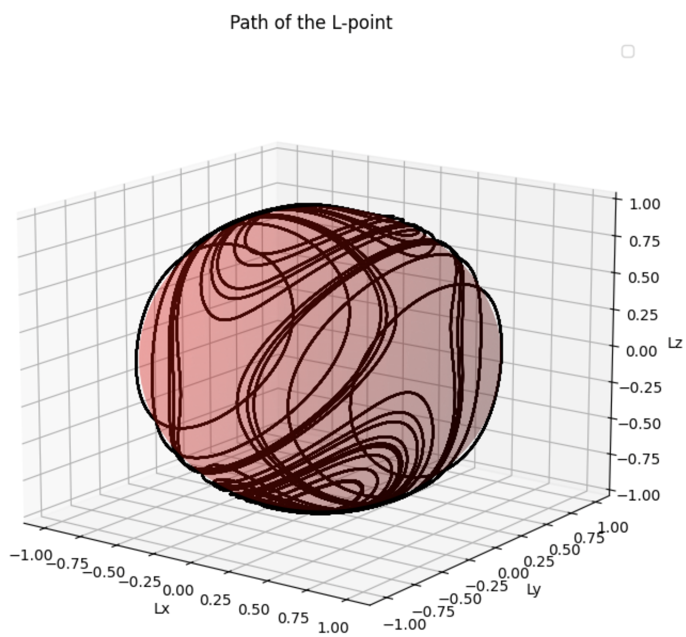
Solution. 19.17 We want to obtain the paths of the $\boldsymbol{L}$-point for an unsymmetrical body. Euler equations in terms of the components of $\boldsymbol{L}$ in the frame $Gxyz$ give us the following

$$\dot{L}_x = \frac{(B-C)}{BC} L_y L_z$$
$$\dot{L}_y = \frac{(C-A)}{AC} L_z L_x$$
$$\dot{L}_z = \frac{(A-B)}{AB} L_x L_y$$

Also, the equations for $\dot{\boldsymbol{e}}_{\boldsymbol{1}}$, $\dot{\boldsymbol{e}}_{\boldsymbol{2}}$ and $\dot{\boldsymbol{e}}_{\boldsymbol{3}}$ gives us

$$\dot{e}_{1x} = 0 \qquad \dot{e}_{2x} = -\omega_z = -\frac{L_z}{C} \qquad \dot{e}_{3x} = \omega_y = \frac{L_y}{B}$$
$$\dot{e}_{1y} = \omega_z = \frac{L_z}{C} \qquad \dot{e}_{2y} = 0 \qquad \dot{e}_{3y} = -\omega_x = -\frac{L_x}{A}$$
$$\dot{e}_{1z} = -\omega_y = -\frac{L_y}{B} \qquad \dot{e}_{2z} = \omega_x = \frac{L_x}{A} \qquad \dot{e}_{3z} = 0$$

Finally, we can solve the system of twelve first-order differential equations using the Runge-Kutta method. Below we show the result



□