Solved selected problems of Classical Mechanics - Gregory

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Chapter 13 - The Calculus of Variations and Hamilton's Principle

Solution. 13.1 By definition, extremals are solutions of the Euler-Lagrange equation. In this case $F = \dot{x}^2/t^3$ so we have that

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = \frac{2\dot{x}}{t^3}$$

Hence the Euler-Lagrange equation takes the form

$$\begin{aligned} \frac{d}{dt}\left(\frac{2\dot{x}}{t^3}\right) - 0 &= 0 \\ \frac{2\ddot{x}}{t^3} - \frac{6\dot{x}}{t^4} &= 0 \\ \ddot{x} - \frac{3\dot{x}}{t} &= 0 \end{aligned}$$

Now we solve the equation by setting $v = \dot{x}$ hence

$$\begin{aligned} \frac{dv}{dt} - \frac{3v}{t} &= 0 \\ \int \frac{dv}{3v} &= \int \frac{dt}{t} \\ \frac{\log(v)}{3} &= \log(t) + C \\ v &= Ct^3 \end{aligned}$$

by replacing again and solving the equation we have that

$$\begin{aligned} \frac{dx}{dt} &= Ct^3 \\ \int dx &= C\int t^3 dt \\ x &= Ct^4 + D \end{aligned}$$

The admissible extremals are those that satisfy the conditions $x(1) = 3$ and $x(2) = 18$ this way we find the values of $C = 1$ and $D = 2$ so the only admissible extremal of $J[x]$ is given by

$$\hat{x} = t^4 + 2$$

Finally, we want to show that this extremal provides a global minimum of $J[x]$. Let h be any admissible variation and consider the variation in J that it produces

$$
\begin{aligned}
J[\hat{x}+h]-J[\hat{x}] &= \int_1^2 \frac{(4t^3+\dot{h})^2}{t^3}dt - \int_1^2 \frac{(4t^3)^2}{t^3}dt \\
&= \int_1^2 16t^3 + 8\dot{h} + \frac{\dot{h}^2}{t^3}dt - \int_1^2 16t^3 dt \\
&= 8\Big[h\Big]_{t=1}^{t=2} + \int_1^2 \frac{\dot{h}^2}{t^3}dt \\
&= \int_1^2 \frac{\dot{h}^2}{t^3}dt
\end{aligned}
$$

Where we used that h is an admissible extremal hence it must satisfy $h(1) = h(2) = 0$. Since the integral of a positive function must be positive we see that

$$J[\hat{x}+h]-J[\hat{x}] = \int_1^2 \frac{\dot{h}^2}{t^3}dt \geq 0$$

Thus $\hat{x}$ provides the global minimum of $J[x]$. The global minimum value of J is therefore $J[t^4+2] = 60$.

$\square$

Solution. 13.3 We want to find the extremals of the path functional

$$L[y] = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

By definition, extremals are solutions of the Euler-Lagrange equation. In this case $F(y, \dot{y}) = \sqrt{1 + \dot{y}^2}$ so there is no explicit dependence on x this implies that if we satisfy the equation

$$\dot{y}\frac{\partial F}{\partial \dot{y}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned}
\dot{y}\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} - \sqrt{1+\dot{y}^2} &= C \\
\frac{\dot{y}^2 - 1 - \dot{y}^2}{\sqrt{1+\dot{y}^2}} &= C \\
\sqrt{1+\dot{y}^2} &= -\frac{1}{C} \\
\dot{y}^2 &= -1 + \frac{1}{C^2} \\
\int y &= \frac{\sqrt{1-C^2}}{C^2} \int dx \\
y &= \frac{\sqrt{1-C^2}}{C^2} x + D
\end{aligned}$$

Where D is a constant of integration. Now applying the given initial and end conditions we get that $C = 1$ and $D = 0$ so the admissible extremal is

$$\hat{y} = 0$$

But this is a constant solution so it may or may not be a solution to the Euler-Lagrange equation, we must check, hence we have that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} &= 0 \\
\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) - 0 &= 0 \\
\frac{\ddot{y}}{(1+\dot{y}^2)^{3/2}} &= 0
\end{aligned}$$

So $\hat{y} = 0$ is a solution to the Euler-Lagrange equation and therefore an extremal.

Finally, we want to check $\hat{y}$ provides a global minimum for L. Let h be any admissible variation and consider the variation in L that it produces

$$\begin{aligned} L[\hat{y}+h] &= \int_0^1 \sqrt{1+(\dot{\hat{y}}+\dot{h})^2} dx \\ &= \int_0^1 \sqrt{1+\dot{h}^2} dx \end{aligned}$$

Also, we know that

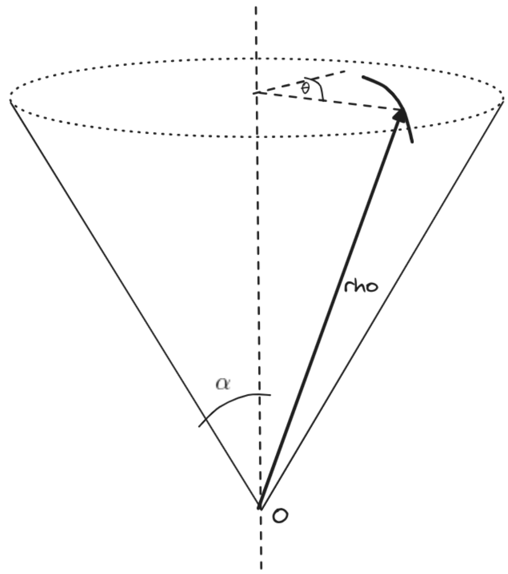
$$\int_0^1 \sqrt{1+\dot{\hat{y}}^2} dx = \int_0^1 dx = 1$$

And we have that

$$\int_0^1 \sqrt{1+\dot{h}^2} dx \geq 1 = \int_0^1 dx$$

Therefore $\hat{y}$ provides a local minimum for L. $\square$

Solution. 13.5 Let us analyze the following system



From this, we can write the differential length as

$$(ds)^2 = (d\rho)^2 + (\rho \sin\alpha d\theta)^2$$

So integrating we get the length functional as follows

$$L = \int_{-\pi/2}^{\pi/2} ds = \int_{-\pi/2}^{\pi/2} \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + (\rho\sin\alpha)^2}\, d\theta$$

In this case, we have that $F(\rho, \dot{\rho}) = \sqrt{\dot{\rho}^2 + (\rho\sin\alpha)^2}$ so there is no explicit dependence on θ this implies that if we satisfy the equation

$$\dot{\rho}\frac{\partial F}{\partial \dot{\rho}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned}
\dot{\rho}\frac{\dot{\rho}}{\sqrt{\dot{\rho}^2 + (\rho\sin\alpha)^2}} - \sqrt{\dot{\rho}^2 + (\rho\sin\alpha)^2} &= C \\
\frac{\dot{\rho}^2 - \dot{\rho}^2 - (\rho\sin\alpha)^2}{\sqrt{\dot{\rho}^2 + (\rho\sin\alpha)^2}} &= C \\
\sqrt{\dot{\rho}^2 + (\rho\sin\alpha)^2} &= -\frac{(\rho\sin\alpha)^2}{C} \\
\dot{\rho}^2 &= -(\rho\sin\alpha)^2 + \frac{(\rho\sin\alpha)^4}{C^2} \\
\frac{d\rho}{d\theta} &= (\rho\sin\alpha)\sqrt{\frac{(\rho\sin\alpha)^2}{C^2} - 1}
\end{aligned}$$

Now we can integrate the equation as follows

$$
\begin{aligned}
\int \frac{d\rho}{(\rho\sin\alpha)\sqrt{\frac{(\rho\sin\alpha)^2}{C^2}-1}} &= \int d\theta \\
\frac{\arctan\left(\sqrt{\frac{(\rho\sin\alpha)^2}{C^2}-1}\right)}{\sin\alpha} &= \theta + D \\
\arctan\left(\sqrt{\frac{(\rho\sin\alpha)^2}{C^2}-1}\right) &= (\theta + D)\sin\alpha \\
\frac{(\rho\sin\alpha)^2}{C^2} - 1 &= \tan^2\left((\theta+D)\sin\alpha\right) \\
\rho &= \frac{C}{\sin\alpha}\sqrt{\tan^2\left((\theta+D)\sin\alpha\right)+1} \\
\rho &= \frac{C}{\sin\alpha}\sec((\theta+D)\sin\alpha)
\end{aligned}
$$

This is the equation for the extremals of L. Now applying the given initial and end conditions we get that

$$
a = \frac{C}{\sin\alpha}\sec(-(\pi/2 - D)\sin\alpha) = \frac{C}{\sin\alpha}\sec((\pi/2 + D)\sin\alpha)
$$

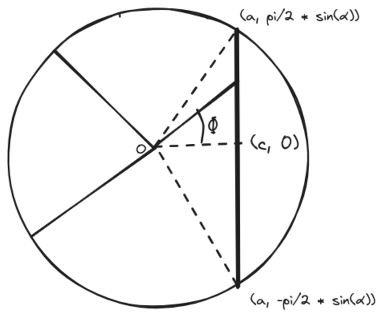
which implies that $D = 0$ and C is

$$
\begin{aligned}
C\sec\left(\frac{\pi\sin\alpha}{2}\right) &= a\sin\alpha \\
C &= \frac{a\sin\alpha}{\sec\left(\frac{\pi\sin\alpha}{2}\right)}
\end{aligned}
$$

Replacing the values for C and D we get that the admissible extremal is

$$
\begin{aligned}
\rho &= \frac{a\sec(\theta\sin\alpha)}{\sec\left(\frac{\pi\sin\alpha}{2}\right)} \\
\rho &= \frac{a\cos\left(\frac{\pi\sin\alpha}{2}\right)}{\cos(\theta\sin\alpha)}
\end{aligned}
$$

Finally, we want to verify that this extremal is the same as the shortest path that would be obtained by developing the cone onto a plane. If we develop the cone onto a plane, we get that the path is a vertical line in the plane as shown below



where the angle in the plane is given by $\phi = \theta \sin \alpha$. We know that the equation of a straight vertical line in polar coordinates is given by

$$\rho \cos(\phi) = c$$

where c is the value of ρ when $\phi = 0$ and from the drawing above we have that $\cos(\pi/2 \sin(\alpha)) = c/a$ hence

$$\rho \cos(\theta \sin \alpha) = a \cos\left(\frac{\pi \sin(\alpha)}{2}\right)$$

Which is the admissible extremal that we have such that it satisfies the Euler-Lagrange equation. $\square$

Solution. 13.7 We want to find the extremals of

$$J[y] = \int_{-a}^{a} y\sqrt{1+\dot{y}^2}\ dx$$

In this case, we have that $F(y,\dot{y}) = y\sqrt{1+\dot{y}^2}$ so there is no explicit dependence on x this implies that if we satisfy the equation

$$\dot{y}\frac{\partial F}{\partial \dot{y}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned}
\dot{y}\frac{y\dot{y}}{\sqrt{1+\dot{y}^2}} - y\sqrt{1+\dot{y}^2} &= C \\
\frac{\dot{y}y^2 - (y(1+\dot{y}^2))}{\sqrt{1+\dot{y}^2}} &= C \\
\frac{y}{\sqrt{1+\dot{y}^2}} &= -C \\
y^2 &= C^2(1+\dot{y}^2) \\
\sqrt{\frac{y^2}{C^2} - 1} &= \dot{y}
\end{aligned}$$

Now we can solve this equation by separation as follows

$$\begin{aligned}
\int \frac{dy}{\sqrt{\frac{y^2}{C^2} - 1}} &= \int dx \\
C\int \frac{dy}{\sqrt{y^2 - C^2}} &= \int dx \\
\cosh^{-1}\left(\frac{y}{C}\right) &= \frac{x}{C} + D \\
y &= C\cosh\left(\frac{x}{C} + D\right)
\end{aligned}$$

Which is the form of the extremals of $J[y]$ we wanted.

To determine the values of C and D we use the initial and end conditions. We know that $y(-a) = y(a) = b$ hence we have that

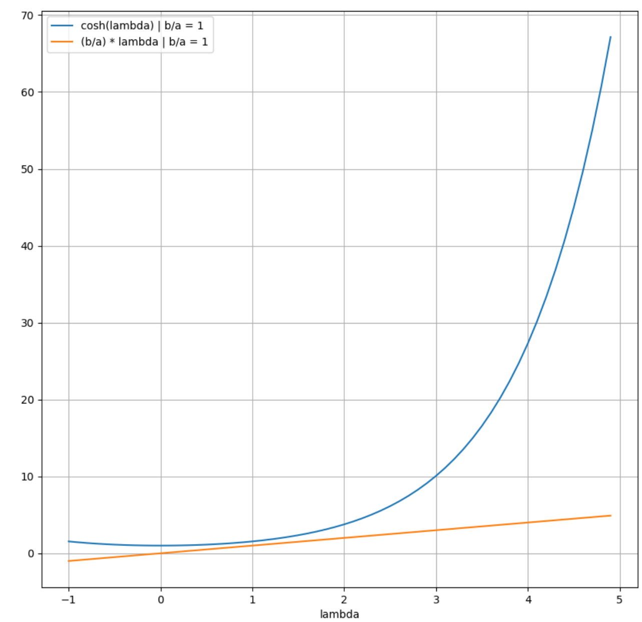
$$\begin{aligned}
C\cosh\left(\frac{a}{C} + D\right) &= C\cosh\left(-\frac{a}{C} + D\right) \\
\cosh\left(\frac{a}{C} + D\right) &= \cosh\left(\frac{a}{C} - D\right) \\
\frac{a}{C} + D &= \frac{a}{C} - D \\
D &= 0
\end{aligned}$$

So the initial and end conditions are satisfied only if $D = 0$.

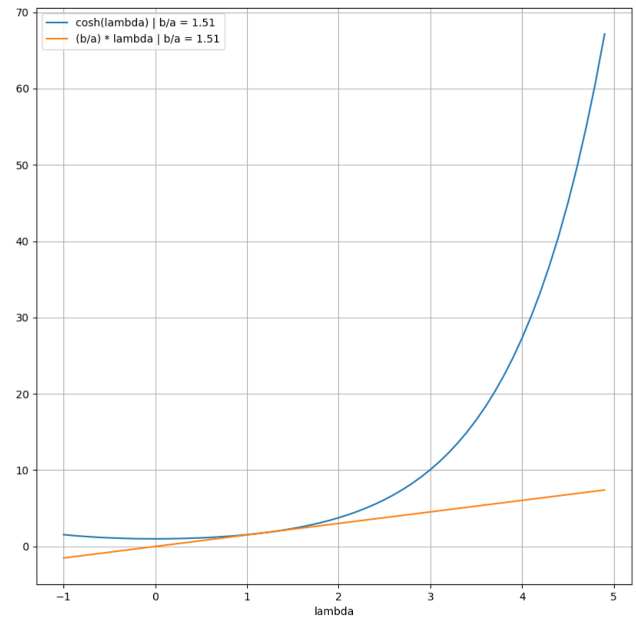
Naming $\lambda = a/C$ we see that

$$\cosh\left(\frac{a}{C}\right) = \frac{b}{C}$$
$$\cosh(\lambda) = \frac{b}{a}\lambda$$

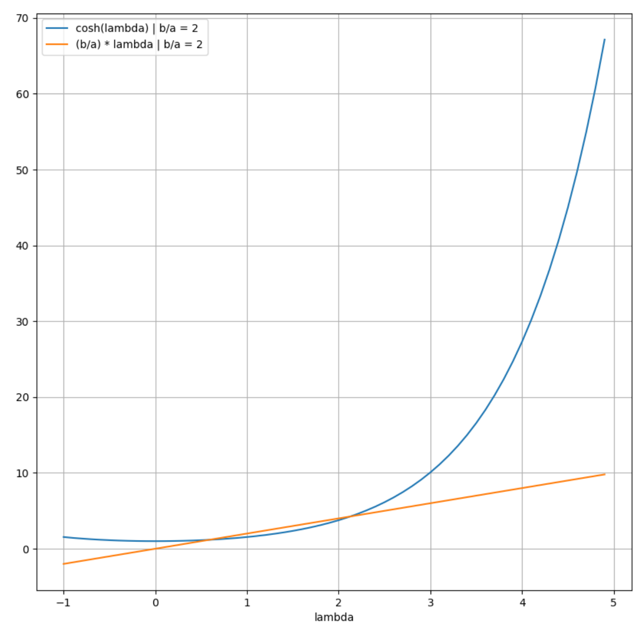
We see that depending the value of b/a the curve $\cosh(\lambda)$ will or will not intersect the line $(b/a)\lambda$ for example if we take $b/a = 1$ and we plot both curves we have that



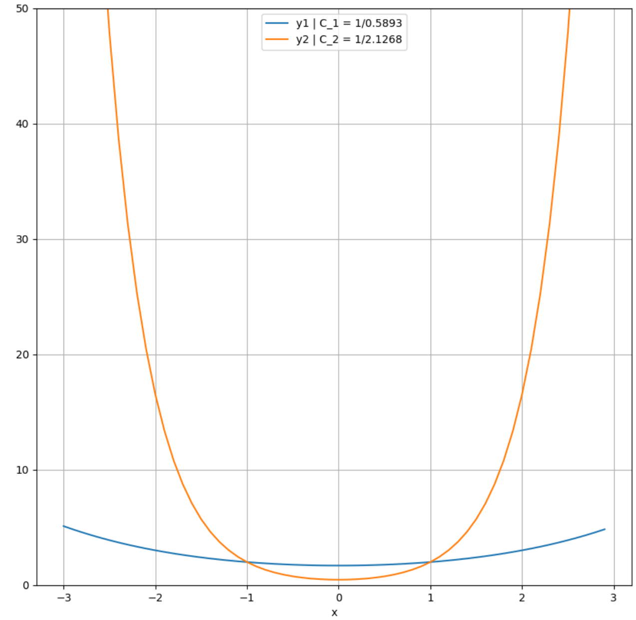
Where we see that the curves don't intersect hence there is no solution to the equation so no value of C satisfies the initial and end conditions and therefore there are no admissible extremals for this case. If we set $b/a = 1.51$ we see that the curves do intersect and this value seems to be quite close to the critical point as shown below.



If we choose $b/a = 2$ the curves intersect at two points $\lambda_1 = 0.5893$ and $\lambda_2 = 2.1268$ as shown below



Now taking $a = 1$ we plot both extremals where we use $C_1 = 1/\lambda_1$ and $C_2 = 1/\lambda_2$



By the shape of the curve, we can guess that the blue curve represents the soap film. $\square$

Solution. 13.9 We want to determine Feremat's time functional, for this, we need first, the line element in cylindrical polar coordinates i.e.

$$ds^2 = dr^2 + r^2 d\theta^2$$

So Fermat's time functional is given by

$$\begin{aligned}
T[r] &= c^{-1} \int n ds \\
&= c^{-1} \int n \sqrt{dr^2 + r^2 d\theta^2} \\
&= c^{-1} \int_{\theta_0}^{\theta_1} n \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\ d\theta
\end{aligned}$$

Which is the expression we wanted. Using the calculus of variations we note that $F(r, \dot{r}) = n\sqrt{\dot{r}^2 + r^2}$ which does not depends on θ so if we satisfy the equation

$$\dot{r}\frac{\partial F}{\partial \dot{r}} - F = C$$

then we satisfy the Euler-Lagrange equation too, hence we have that

$$\begin{aligned}
\dot{r}\ \frac{n\dot{r}}{\sqrt{\dot{r}^2 + r^2}} - n\sqrt{\dot{r}^2 + r^2} &= C \\
n\left(\frac{\dot{r}^2}{\sqrt{\dot{r}^2 + r^2}} - \sqrt{\dot{r}^2 + r^2}\right) &= C \\
n\left(\frac{\dot{r}^2 - (\dot{r}^2 + r^2)}{\sqrt{\dot{r}^2 + r^2}}\right) &= C \\
\frac{nr^2}{\sqrt{\dot{r}^2 + r^2}} &= C
\end{aligned}$$

Let now $\dot{r} = r\tan\psi$ then we get that

$$\begin{aligned}
\frac{nr^2}{\sqrt{(r\tan\psi)^2 + r^2}} &= C \\
\frac{nr}{\sqrt{\tan^2\psi + 1}} &= C \\
\frac{nr}{\frac{1}{\cos\psi}\sqrt{\sin^2\psi + \cos^2\psi}} &= C \\
nr\cos\psi &= C
\end{aligned}$$

Which is Snell's law for this case as we wanted. Finally, if we consider a circular ray with center at the origin then must be that $\psi = 0$ hence we get that $nr = a$ where we named the constant as a then $n = a/r$. $\square$

Solution. 13.10 Let a particle of mass 2 kg move under uniform gravity where $g = 10\ m/s^2$ along the z-axis, which points vertically downwards. The lagrangian in this case is given by

$$\begin{aligned} L(z, \dot{z}) &= T(\dot{z}) - V(z) \\ &= \frac{1}{2} m\dot{z}^2 - (-mgz) \\ &= \dot{z}^2 + 20z \end{aligned}$$

Then the action functional for a time interval $[0, 2]$ is given by

$$S[z] = \int_0^2 L(z, \dot{z})\ dt = \int_0^2 (\dot{z}^2 + 20z) dt$$

Because of Hamilton's principle, $z(t) = 5t^2$ makes stationary the action functional. Let us compute the value of $S[z]$

$$\begin{aligned} S[z] &= \int_0^2 100t^2 + 100t^2\ dt \\ &= \left[\frac{200t^3}{3}\right]_{t=0}^{t=2} \\ &= \frac{1600}{3} \end{aligned}$$

Now let us consider an admissible variation of $z(t)$ as follows

$$\begin{aligned} S[z + h] &= \int_0^2 ((10t + \dot{h})^2 + 20(5t^2 + h))\ dt \\ &= \int_0^2 200t^2 + 20t\dot{h} + \dot{h}^2 + 20h\ dt \\ &= \left[\frac{200t^3}{3}\right]_{t=0}^{t=2} + [20th]_{t=0}^{t=2} + \int_0^2 \dot{h}^2\ dt \\ &= \frac{1600}{3} + 0 + \int_0^2 \dot{h}^2\ dt \end{aligned}$$

We used that h is an admissible variation and hence $h(0) = h(2) = 0$. So we see that $S[z + h] = S[z] + \int_0^2 \dot{h}^2\ dt$ and since the integral of a positive function $\dot{h}^2$ must be positive then

$$S[z + h] = S[z] + \int_0^2 \dot{h}^2\ dt \geq S[z]$$

so $z(t) = 5t^2$ provides a global minimum for $S[z]$. $\square$

Solution. 13.11 In this case, we are given the Lagrangian of the system which is

$$L = \dot{q}^2 - 4q^2$$

So the Lagrange equation in this case is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} &= 0 \\ \ddot{q} + 4q &= 0 \end{aligned}$$

We want to verify that $q^* = \sin 2t$ is a motion of the oscillator then it must be a solution to the Lagrange equation we got, we check this below

$$\frac{d^2 \sin(2t)}{dt^2} + 4\sin(2t) = -4\sin(2t) + 4\sin(2t) = 0$$

So we see q^* is a solution to the Lagrange equation and therefore a motion of the oscillator.

The action functional is given by

$$S[q] = \int_0^\tau \dot{q}^2 - 4q^2 \; dt$$

and since q^* is a solution to the Lagrange equation it must make S stationary, we check this below. Let $h(t)$ be an admissible variation then

$$\begin{aligned} S[q^* + h] - S[q^*] &= \int_0^\tau (2\cos 2t + \dot{h})^2 - 4(\sin 2t + h)^2 \; dt \\ &\quad - \int_0^\tau (2\cos 2t)^2 - 4(\sin 2t)^2 \; dt \\ &= 4\int_0^\tau \cos^2 2t + \dot{h}\cos 2t + \frac{\dot{h}^2}{4} - \sin^2 2t - 2h\sin 2t - h^2 \; dt \\ &\quad - 4\int_0^\tau \cos^2 2t - \sin^2 2t \; dt \\ &= 4\int_0^\tau \cos 4t + \dot{h}\cos 2t + \frac{\dot{h}^2}{4} - 2h\sin 2t - h^2 \; dt \\ &\quad - 4\int_0^\tau \cos 4t \; dt \\ &= 4\Big[h\cos 2t\Big]_0^\tau + 4\int_0^\tau \left(\frac{\dot{h}^2}{4} - h^2\right) \; dt \\ &= \int_0^\tau \left(\dot{h}^2 - 4h^2\right) \; dt \end{aligned}$$

We used that $h(\tau) = h(0) = 0$. It follows that

$$\begin{aligned}
|S[q^* + h] - S[q^*]| &= \left| \int_0^\tau \left(\dot{h}^2 - 4h^2\right) \, dt \right| \\
&\leq \int_0^\tau \left|\dot{h}^2 - 4h^2\right| \, dt \\
&\leq \int_0^\tau |\dot{h}|^2 + |4h|^2 \, dt \\
&\leq \int_0^\tau (\max|\dot{h}|)^2 + (\max|4h|)^2 \, dt \\
&= \tau((\max|\dot{h}|)^2 + (\max|4h|)^2) \\
&\leq \tau(\max|\dot{h}| + 4\max|h|)^2 = \tau\|h\|^2
\end{aligned}$$

Hence $|S[q^* + h] - S[q^*]| = O(\|h\|^2)$ which by definition, means that q^* makes the action functional $S[q]$ stationary.

Let us take now a time interval $0 \leq t \leq \pi$ then

$$S[q^*] = 4\int_0^\pi \cos(4t) = 0$$

If $h = \epsilon\sin(4t)$ we have that

$$\begin{aligned}
S[q^* + h] - S[q^*] &= \int_0^\pi \left((4\epsilon\cos 4t)^2 - 4(\epsilon\sin 4t)^2\right) \, dt \\
&= 4\epsilon^2 \int_0^\pi 4\cos^2 4t - \sin^2 4t \; dt \\
&= \epsilon^2 \left[\frac{5}{4}\sin(8t) + 6t\right]_{t=0}^{t=\pi} \\
&= \epsilon^2[6\pi - 0] = \epsilon^2 6\pi
\end{aligned}$$

So we see that $S[q^* + h] = S[q^*] + \epsilon^2 6\pi \geq S[q^*] = 0$.

On the other hand, if $h = \epsilon\sin(t)$ in the same way we have that

$$\begin{aligned}
S[q^* + h] - S[q^*] &= \int_0^\pi \left((\epsilon\cos t)^2 - 4(\epsilon\sin t)^2\right) \, dt \\
&= \epsilon^2 \int_0^\pi \cos^2 t - 4\sin^2 t \; dt \\
&= \epsilon^2 \left[\frac{1}{2}(5\sin(t)\cos(t) - 3t)\right]_{t=0}^{t=\pi} \\
&= \epsilon^2 \left[-\frac{3\pi}{2} - 0\right] = -\epsilon^2 \frac{3\pi}{2}
\end{aligned}$$

So we see that $S[q^* + h] = S[q^*] - \epsilon^2 \frac{3\pi}{2} \leq S[q^*] = 0$.

Therefore we have that q^* does not make S a minimum or a maximum.

$\square$

Solution. 13.12 Given that the particle is constrained to move over a smooth fixed surface under no forces other than the force of constraint, and the former do no work the lagrangian for the system is $L = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. We want to check if the path of a particle must be a geodesic i.e. makes the Lenght functional

$$L[\boldsymbol{r}] = \int_{t_0}^{t_1} |\dot{\boldsymbol{r}}| dt$$

stationary. We know from the conservation of energy that $T =$ constant which implies that a particle that moves along the surface must have a constant velocity so

$$L[\boldsymbol{r}] = |\dot{\boldsymbol{r}}|\Delta t$$

On the other hand, we know from Hamilton's principle that a motion of a particle on the surface must satisfy the Lagrange equations and hence it must make the action functional stationary.

So let $\boldsymbol{r}^*$ be the actual path of a particle and h be an admissible variation such that $\boldsymbol{r} = \boldsymbol{r}^* + h$, we know that

$$S[\boldsymbol{r}^* + h] - S[\boldsymbol{r}^*] = O(h^2)$$

hence

$$\begin{aligned}\frac{1}{2}m\int_{t_0}^{t_1} \dot{\boldsymbol{r}}^2 dt &= \frac{1}{2}m\int_{t_0}^{t_1} \dot{\boldsymbol{r}}^{*2} dt + O(h^2)\\ &= \frac{1}{2}m|\dot{\boldsymbol{r}}^*|^2\Delta t + O(h^2)\end{aligned}$$

Where we used that $\dot{\boldsymbol{r}}^{\boldsymbol{*}}$ is constant since $\boldsymbol{r}^{\boldsymbol{*}}$ is the actual path of a particle. Given the above equation must be true for any path we can traverse $\boldsymbol{r} = \boldsymbol{r}^{\boldsymbol{*}} + h$ with constant velocity i.e. we can take out of the integral $\dot{\boldsymbol{r}}^2$ then we have that

$$\frac{1}{2}m|\dot{\boldsymbol{r}}|^2\Delta t = \frac{1}{2}m|\dot{\boldsymbol{r}}^*|^2\Delta t + O(h^2)$$

Finally, by replacing $L[\boldsymbol{r}]$ we get that

$$\frac{1}{2}m\frac{L[\boldsymbol{r}]^2}{\Delta t} = \frac{1}{2}m\frac{L[\boldsymbol{r}^{\boldsymbol{*}}]^2}{\Delta t} + O(h^2)$$

Which implies that $L[\boldsymbol{r}] = L[\boldsymbol{r}^*] + O(h^2)$. Therefore the Length functional is stationary too. $\square$

Solution. 13.13 Let $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ be the Lagrangian of the system then the motion of the system $\boldsymbol{q}^*$ will satisfy the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = 0$$

but also $\boldsymbol{q}^*$ makes the action functional stationary. We want to check that a modification to the Lagrangian of the system to $L' = L + \frac{d}{dt}g(\boldsymbol{q}, t)$ still preserves $\boldsymbol{q}^*$ as a motion of the system hence it must make the action functional with the new Lagrangian stationary.

Let us compute the action functional for this case

$$\begin{aligned}
S'[\boldsymbol{q}] &= \int_{t_0}^{t_1} L'(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ dt \\
S'[\boldsymbol{q}] &= \int_{t_0}^{t_1} L + \frac{d}{dt}g(\boldsymbol{q}, t)\ dt \\
S'[\boldsymbol{q}] &= S[\boldsymbol{q}] + \int_{t_0}^{t_1} dg(\boldsymbol{q}, t) \\
S'[\boldsymbol{q}] &= S[\boldsymbol{q}] + \Big[g(\boldsymbol{q}, t)\Big]_{t_0}^{t_1} \\
S'[\boldsymbol{q}] &= S[\boldsymbol{q}] + [g(\boldsymbol{q}, t_1) - g(\boldsymbol{q}, t_0)]
\end{aligned}$$

So we see that $S'[\boldsymbol{q}]$ differs from $S[\boldsymbol{q}]$ only on a constant value, so $S'[\boldsymbol{q}]$ will be still stationary for $\boldsymbol{q}^*$. Therefore if $\boldsymbol{q}^*$ is a motion of the system it still will be if we change the Lagrangian to L'. $\square$

Solution. 13.14 Let $\mathcal{C}$ be a path lying on the surface that connects P and Q. We want to determine the length of $\mathcal{C}$. We know that the line element in cylindrical coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Given that P and Q are separated by π radians in cylindrical coordinates we integrate the length functional between $-\pi/2$ and $\pi/2$ as follows

$$\begin{aligned} L[r] &= \int_{-\pi/2}^{\pi/2} ds \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{dr^2 + r^2 d\theta^2 + dz^2} \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 + \left(\frac{dz}{d\theta}\right)^2}\ d\theta \end{aligned}$$

Also, we know that the surface of the paraboloid is given by $z = x^2 + y^2$ so in cylindrical coordinates, we have that $z = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$ which implies that $dz/d\theta = 2r\dot{r}$ where $\dot{r} = dr/d\theta$ hence

$$\begin{aligned} L[r] &= \int_{-\pi/2}^{\pi/2} \sqrt{\dot{r}^2 + r^2 + (2r\dot{r})^2}\ d\theta \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2 + (1+4r^2)\dot{r}^2}\ d\theta \end{aligned}$$

We see that $F(r, \dot{r}) = \sqrt{r^2 + (1+4r^2)\dot{r}^2}$ so the Euler-Lagrange equation is given by

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{\partial F}{\partial \dot{r}}\right) - \frac{\partial F}{\partial r} &= 0 \\ \frac{d}{d\theta}\left(\frac{(4r^2+1)\dot{r}}{\sqrt{r^2 + (1+4r^2)\dot{r}^2}}\right) - \frac{(4\dot{r}^2+1)r}{\sqrt{r^2 + (1+4r^2)\dot{r}^2}} &= 0 \end{aligned}$$

And we see that

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{(4r^2+1)\dot{r}}{\sqrt{r^2 + (1+4r^2)\dot{r}^2}}\right) &= \frac{2r(4r^2(4\dot{r}+1) + 4\dot{r} - 1)\dot{r}^2 + (4r^2+1)(r^2(4\dot{r}+2) + \dot{r})\ddot{r}}{2(r^2 + (1+4r^2)\dot{r}^2)^{3/2}} \\ &= \frac{4r(4r^2+1)\dot{r}^4 + r(4r^2-1)\dot{r}^2 + r^2(4r^2+1)\ddot{r}}{(r^2 + (1+4r^2)\dot{r}^2)^{3/2}} \end{aligned}$$

So the Euler-Lagrange equation is

$$\begin{aligned} \frac{(4\dot{r}^2+1)r}{\sqrt{r^2 + (1+4r^2)\dot{r}^2}} &= \frac{4r(4r^2+1)\dot{r}^4 + r(4r^2-1)\dot{r}^2 + r^2(4r^2+1)\ddot{r}}{(r^2 + (1+4r^2)\dot{r}^2)^{3/2}} \\ (4r\dot{r}^2 + r)(r^2 + (1+4r^2)\dot{r}^2) &= 4r(4r^2+1)\dot{r}^4 + r(4r^2-1)\dot{r}^2 + r^2(4r^2+1)\ddot{r} \\ r(4r^2+1)\ddot{r} &= (4\dot{r}^2+1)(r^2 + (1+4r^2)\dot{r}^2) - 4(4r^2+1)\dot{r}^4 - (4r^2-1)\dot{r}^2 \end{aligned}$$

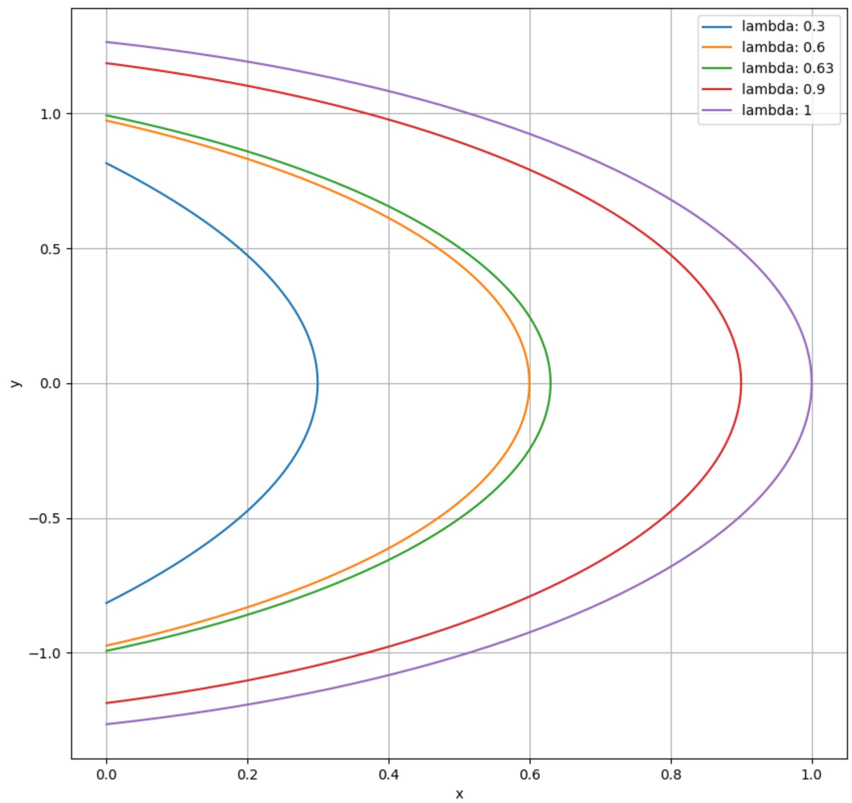
So finally we get that

$$\ddot{r} = \frac{(4\dot{r}^2+1)(r^2+(1+4r^2)\dot{r}^2) - 4(4r^2+1)\dot{r}^4 - (4r^2-1)\dot{r}^2}{r(4r^2+1)}$$

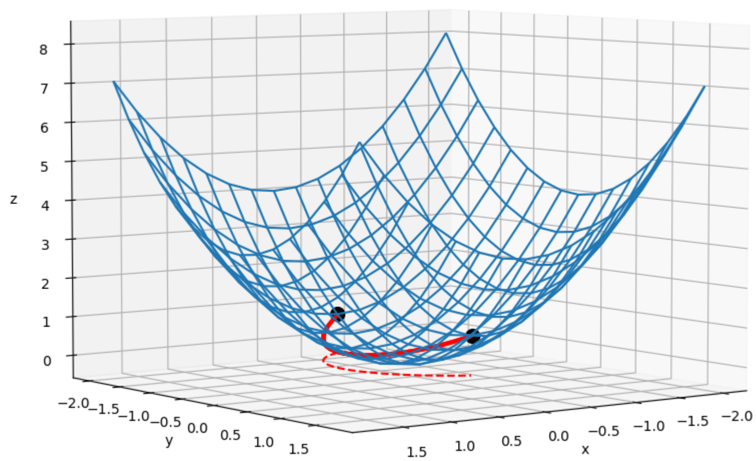
Now naming $\dot{r} = u$ we have that

$$\dot{u} = \frac{(4u^2+1)(r^2+(1+4r^2)u^2) - 4(4r^2+1)u^4 - (4r^2-1)u^2}{r(4r^2+1)}$$

Hence we have a system of differential equations that we can solve numerically assuming that $r(0) = \lambda$ and $u(0) = \dot{r}(0) = 0$ in this way, we get that



Which is the projection of the curve over the $z = 0$ plane. Observing this plot we see that the right value for λ is around 0.63. Finally, we have the 3d plot



Where the red line is the path that starts from the black dot $Q = (0, -1, 1)$ and ends on $P = (0, 1, 1)$. $\square$

Solution. 13.15 We want to solve the problem of finding the path of quickest descent for a skier from the point $P(x_0, y_0, z_0)$ to the point $Q(x_1, y_1, z_1)$ on a snow-covered mountain whose profile is given by $z = G(x, y)$, where G is a known function. We know that Energy is conserved so

$$\begin{aligned} \frac{1}{2}m\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + mgG(x,y) &= mgG(x_0, y_0) \\ \left(\frac{d\boldsymbol{r}}{dt}\right)^2 &= 2g(G(x_0, y_0) - G(x, y)) \end{aligned}$$

Where we used that the skier starts from rest i.e. $d\boldsymbol{r}_0/dt = 0$. Also, we know that

$$\begin{aligned} d\boldsymbol{r}^2 &= dx^2 + dy^2 + dz^2 \\ &= dx^2 + dy^2 + dG(x, y)^2 \end{aligned}$$

So we can write the conservation of energy equation as

$$\begin{aligned} dt^2 &= \frac{dx^2 + dy^2 + dG(x,y)^2}{2g(G(x_0, y_0) - G(x, y))} \\ \int_0^T dt &= \int_{P(x_0,y_0,z_0)}^{Q(x_1,y_1,z_1)} \sqrt{\frac{dx^2 + dy^2 + dG(x,y)^2}{2g(G(x_0, y_0) - G(x, y))}} \\ T &= \frac{1}{\sqrt{2g}} \int_{x_0}^{y_0} \sqrt{\frac{1 + (dy/dx)^2 + (dG(x,y)/dx)^2}{G(x_0, y_0) - G(x, y)}}\, dx \\ T &= \frac{1}{\sqrt{2g}} \int_{x_0}^{y_0} \sqrt{\frac{1 + \dot{y}^2 + (G_x(x,y) + G_y(x,y)\dot{y})^2}{G(x_0, y_0) - G(x, y)}}\, dx \end{aligned}$$

Let us take now $G(x, y) = x^2$ and let us solve the Euler-Lagrange equation for the following function

$$F(x, y, \dot{y}) = \sqrt{\frac{1 + \dot{y}^2 + 4x^2}{x_0^2 - x^2}}$$

Hence

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} &= 0 \\ \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{(x_0^2 - x^2)(1 + \dot{y}^2 + 4x^2)}}\right) - 0 &= 0 \\ \frac{\ddot{y}}{\sqrt{(x_0^2 - x^2)(1 + \dot{y}^2 + 4x^2)}} - \frac{\dot{y}((x_0^2 - x^2)(2\dot{y}\ddot{y} + 8x) - 2x(1 + \dot{y}^2 + 4x^2))}{2((x_0^2 - x^2)(1 + \dot{y}^2 + 4x^2))^{3/2}} &= 0 \end{aligned}$$

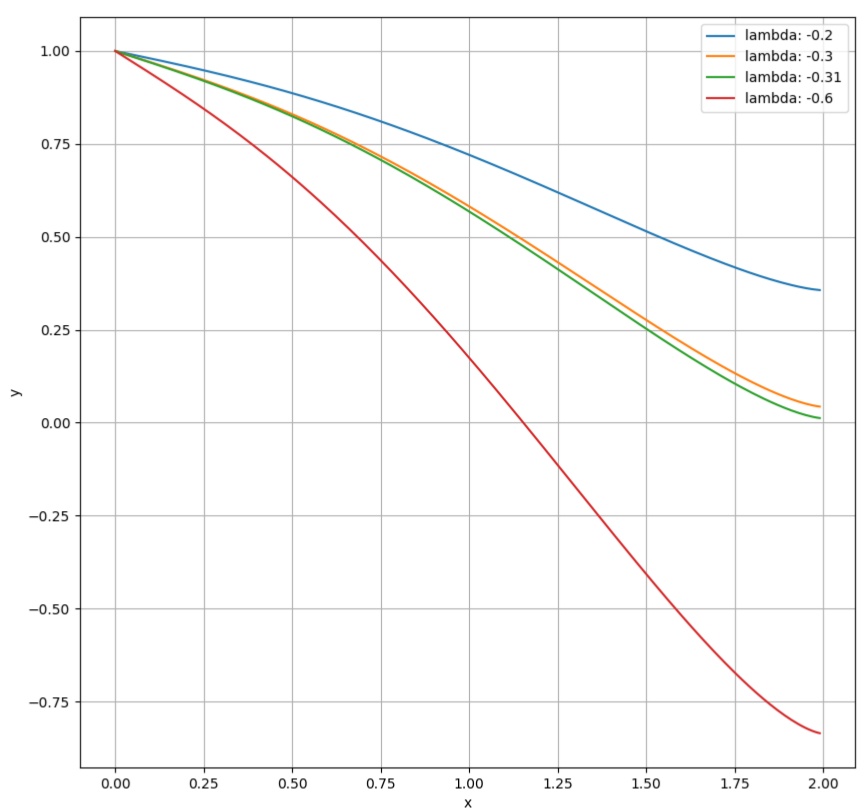
Then

$$\begin{aligned}
\frac{\ddot{y}}{\sqrt{(x_0^2-x^2)(1+\dot{y}^2+4x^2)}} &= \frac{\dot{y}((x_0^2-x^2)(2\dot{y}\ddot{y}+8x)-2x(1+\dot{y}^2+4x^2))}{2((x_0^2-x^2)(1+\dot{y}^2+4x^2))^{3/2}} \\
\ddot{y} &= \frac{\dot{y}(2x_0^2\dot{y}\ddot{y}+8x_0^2x-2x^2\dot{y}\ddot{y}-8x^3-(2x+2x\dot{y}^2+8x^3))}{2((x_0^2-x^2)(1+\dot{y}^2+4x^2))} \\
(x_0^2-x^2)(1+\dot{y}^2+4x^2)\ddot{y} &= x_0^2\dot{y}^2\ddot{y}+4x_0^2x\dot{y}-x^2\dot{y}^2\ddot{y}-4x^3\dot{y}-x\dot{y}-x\dot{y}^3-4x^3\dot{y} \\
(x_0^2-x^2)(1+\dot{y}^2+4x^2)\ddot{y} &= x\dot{y}(4x_0^2-8x^2-1-\dot{y}^2)+\ddot{y}\dot{y}^2(x_0^2-x^2) \\
\ddot{y} &= \frac{x\dot{y}(4x_0^2-8x^2-1-\dot{y}^2)}{(x_0^2-x^2)(1+4x^2)}
\end{aligned}$$

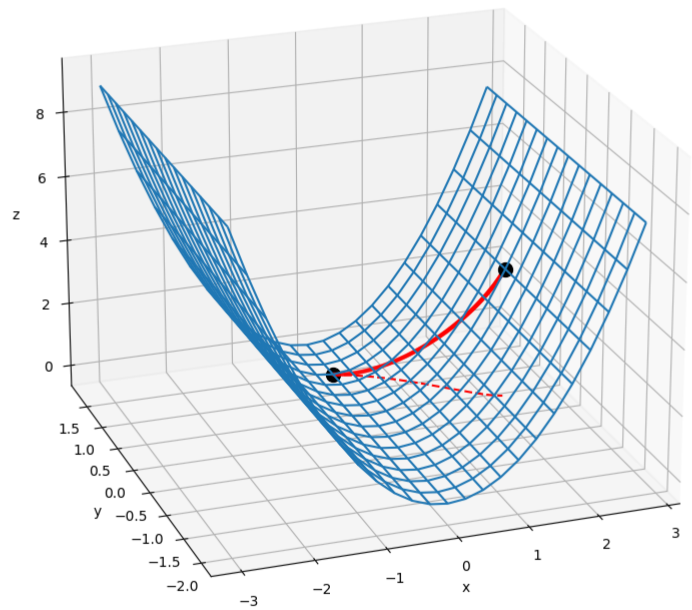
Now naming $\dot{y}=u$ we have that

$$\dot{u} = \frac{xu(4x_0^2-8x^2-1-u^2)}{(x_0^2-x^2)(1+4x^2)}$$

Finally, we can solve the system of differential equations numerically from $P=(2,0,4)$ to $Q=(0,1,0)$ where $y'(0)=\lambda$ and λ must be determined so the path passes through P. We determine λ to be close to -0.31 as the following plot indicates



Finally, we have the 3d plot



Where the red line is the path that starts from the black dot $P = (2, 0, 4)$ and ends on $Q = (0, 1, 0)$. $\square$