Solved selected problems of From Calculus to Chaos - Acheson

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Chapter 3 - Ordinary differential equations

Proof. **3.2** The equation presented can be classified as a nonlinear non-autonomous differential equation, and the equation is of the form

$$\frac{dx}{dt} = g(x)h(t)$$

where $g(x) = x^2$ and $h(t) = \frac{1}{1+t}$ which can be rewritten to the form

$$\frac{1}{g(x)}\frac{dx}{dt} = h(t)$$

as

$$\frac{1}{x^2}\frac{dx}{dt} = \frac{1}{1+t}$$

then integrating both sides with respect to t

$$\int \frac{1}{x^2}\frac{dx}{dt}dt = \int \frac{1}{1+t}dt$$

$$\int \frac{1}{x^2}dx = \int \frac{1}{1+t}dt$$

$$-\frac{1}{x} = log(1+t) + C$$

$$x = \frac{1}{C - log(1+t)}$$

Subject to $x = 1$ when $t = 0$ then $C = 1$ so we have

$$x = \frac{1}{1 - log(1+t)}$$

therefore the "blow up" of the solution happens when t approaches $e-1$. $\square$

Proof. **3.4** Since we know that $x = x_1 u$ then

$$\dot{x} = \dot{x_1} u + \dot{u} x_1$$
$$\ddot{x} = \ddot{x_1} u + 2\dot{x_1}\dot{u} + x_1 \ddot{u}$$

by replacing in $a\ddot{x} + b\dot{x} + cx = 0$ we have that:

$$a(\ddot{x_1} u + 2\dot{x_1}\dot{u} + x_1\ddot{u}) + b(\dot{x_1} u + \dot{u} x_1) + c x_1 u = 0$$
$$u(a\ddot{x} + b\dot{x} + cx) + \dot{u}(2a\dot{x_1} + b x_1) + \ddot{u} a x_1 = 0$$

since $a\ddot{x} + b\dot{x} + cx = 0$ then

$$\dot{u}(2a\dot{x_1} + b x_1) + \ddot{u} a x_1 = 0$$

defining $z = \dot{u}$ then

$$\dot{z} a x_1 + z(2a\dot{x_1} + b x_1) = 0 \tag{1}$$

which is an equation of first-order for z.
Now this method is going to be used to solve $\ddot{x} - 2\dot{x} + x = 0$ since the solution we have is $x_1 = e^t$ then $\dot{x_1} = e^t$ and replacing in the equation (1) with the values $a = 1$ and $b = -2$ we have that

$$\dot{z} e^t + z(2e^t - 2e^t) = 0$$
$$\dot{z} e^t = 0$$
$$\dot{z} = 0$$

then $z = B$ where B is a constant, but $\dot{u} = z$ so $\dot{u} = B$ and then $u = Bt + A$. Since we are assuming that $x = u x_1$ then the general solution is of the form

$$x = (Bt + A)e^t$$

then $\dot{x} = Be^t + (Bt + A)e^t$ and by applying the initial conditions we obtain that $A = 1$ and $0 = B + A$ and then $B = -1$.
Therefore the other solution is

$$x = (1 - t)e^t$$

□