Solved selected problems of Classical Mechanics - Gregory

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Chapter 4 - Problems in Particle Dynamics

Proof. **4.1** The blocks are being towed at constant speed, then the acceleration is 0 and since both masses are identical the frictional forces (F) are identical too then we have the following two equations

$$0 = T_0 - T - F \qquad 0 = T - F$$

$$T = T_0 - F \qquad T = F$$

the first equation correspond to the block that is being towed and the second equation correspond to the other block. Then if we sum both equations we get that

$$2T = T_0 - F + F$$

therefore

$$T = \frac{T_0}{2}$$

but also by subtracting both equations we have that

$$F = \frac{T_0}{2}$$

If now the tension of the towed block changes to $4T_0$ instantaneously we have the following changes in the equations

$$Ma = 4T_0 - T - F \qquad Ma = T - F$$

by subtracting both equations we get that $0 = 4T_0 - T - F - T + F$ so

$$T = 2T_0$$

also if we consider that F does not changes at that moment then from the second equation and doing the needed replacements we have that

$$a = \frac{T - F}{M} = \frac{2T_0 - T_0/2}{M} = \frac{3T_0}{2M}$$

□

Proof. **4.3** For this problem we can replace the spheres by particles which are $1m$ apart. We want to measure the time it takes for them to move $3cm$. The acceleration in this case is not constant since it grows as the spheres approach, so we can calculate the initial acceleration as

$$a = \frac{mG}{R^2} = \frac{5000 \cdot 6.67 \times 10^{-11}}{1^2} = 3.335 \times 10^{-7}$$

and from $x = 1/2at^2$ we can calculate the collision time (an approximation) as if the acceleration were constant, then

$$t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{2 \cdot 0.03}{3.335 \times 10^{-7}}} = 424.15\text{s}$$

Since the accelearation grows as the spheres approach, then the time is going to decrease because of the inverse depencence with the acceleration, therefore the real collision time is even less than the one provided, i.e. the approximated time is an upper bound. □

Proof. **4.4** For this problem we have the following equations already derived

$$ma = mg\sin\alpha - F \qquad 0 = N - mg\cos\alpha$$

so $N = mg\cos\alpha$ and we know that F is defined as $F = \mu N$ then

$$\begin{aligned} ma &= mg\sin\alpha - F \\ ma &= mg\sin\alpha - \mu mg\cos\alpha \\ a &= g(\sin\alpha - \mu\cos\alpha) \end{aligned}$$

but we can write the same equation for a as

$$a = g\cos\alpha(\tan\alpha - \mu)$$

so if $\mu < \tan\alpha$ then $a > 0$ so the block should slide down
and if $\mu > \tan\alpha$ then $a < 0$ so the block should come to rest at some point. □

Proof. **4.10** The motion in this case is only happening in the y-axis then the equation of motion is given by

$$m\frac{dv}{dt} = -\frac{GmM}{r^2}$$

where r is the distance OP.
Also we know that

$$\frac{dv}{dt} = \frac{dr}{dt} \times \frac{dv}{dr} = v\frac{dv}{dr}$$

then we have that

$$\begin{aligned} v\frac{dv}{dr} &= -\frac{GM}{r^2} \\ \int vdv &= -\int \frac{GM}{r^2}dr \\ \frac{v^2}{2} &= \frac{GM}{r} + C \end{aligned}$$

Where C is a constant that can be determined using the initial conditions. If $v = \sqrt{2GM/a}$ when $r = a$ then we have that

$$C = \frac{GM}{a} - \frac{\sqrt{2GM/a}^2}{2} = 0$$

so

$$\begin{aligned} v^2 &= \frac{2GM}{r} \\ \left(\frac{dr}{dt}\right)^2 &= \frac{2GM}{r} \\ \frac{dr}{dt} &= \sqrt{\frac{2GM}{r}} \\ \int \sqrt{r}dr &= \int \sqrt{2GM}dt \\ \frac{2\sqrt{r^3}}{3} &= \sqrt{2GM}t + C \end{aligned}$$

where C is a constant that can be determined by knowing that $r = a$ when $t = 0$, then

$$C = \frac{2\sqrt{a^3}}{3}$$

Finally we can write that

$$\begin{aligned} \frac{2\sqrt{r^3}}{3} &= \sqrt{2GM}t + \frac{2\sqrt{a^3}}{3} \\ \sqrt{r^3} &= \frac{3}{2}\sqrt{2GM}t + \sqrt{a^3} \\ r &= \left(\frac{3}{2}\sqrt{2GM}t + \sqrt{a^3}\right)^{2/3} \end{aligned}$$

Then r tends to infinity when t goes to infinity, therefore the particle P escapes from M. □

Proof. **4.11** The motion in this case is only happening in the y-axis then the equation of motion is given by

$$m\frac{dv}{dt} = -\frac{m\gamma}{r^3}$$

where r is the distance OP.
Also we know that

$$\frac{dv}{dt} = \frac{dr}{dt} \times \frac{dv}{dr} = v\frac{dv}{dr}$$

then we have that

$$\begin{aligned} v\frac{dv}{dr} &= -\frac{\gamma}{r^3} \\ \int vdv &= -\int \frac{\gamma}{r^3}dr \\ \frac{v^2}{2} &= \frac{\gamma}{2r^2} + C \end{aligned}$$

Where C is a constant that can be determined using the initial conditions. If $v = u$ when $r = a$ then we have that

$$C = \frac{u^2}{2} - \frac{\gamma}{2a^2}$$

so

$$\begin{aligned}
\frac{v^2}{2} &= \frac{\gamma}{2r^2} + \frac{u^2}{2} - \frac{\gamma}{2a^2} \\
v^2 &= \frac{\gamma}{r^2} + u^2 - \frac{\gamma}{a^2}
\end{aligned}$$

hence if $u^2 > \frac{\gamma}{a^2}$ we can define $V^2 = u^2 - \frac{\gamma}{a^2}$ where V^2 is a positive constant then

$$v^2 = \frac{\gamma}{r^2} + V^2 > V^2$$

since the velocity of P is bigger than V then P escapes from O.

For the case $u^2 = \gamma/2a^2$ we have that

$$v^2 = \frac{\gamma}{r^2} - \frac{\gamma}{2a^2}$$

and the maximum distance is reached when $v = 0$ then

$$\begin{aligned}
0 &= \frac{\gamma}{r_{max}^2} - \frac{\gamma}{2a^2} \\
\frac{\gamma}{r_{max}^2} &= \frac{\gamma}{2a^2} \\
r_{max}^2 &= 2a^2 \\
r_{max} &= \sqrt{2}a
\end{aligned}$$

To find the time it takes to reach that altitude we need to get the trajectory from the equation of motion, then

$$\begin{aligned}
v^2 &= \frac{\gamma}{r^2} - \frac{\gamma}{2a^2} \\
\left(\frac{dr}{d\tau}\right)^2 &= \frac{\gamma}{r^2} - \frac{\gamma}{2a^2} \\
\frac{dr}{d\tau} &= \sqrt{\frac{\gamma}{r^2} - \frac{\gamma}{2a^2}} \\
\int_a^{\sqrt{2}a} \frac{dr}{\sqrt{\frac{\gamma}{r^2} - \frac{\gamma}{2a^2}}} &= \int_0^t d\tau \\
\left[\frac{r\sqrt{\frac{\gamma}{r^2} - \frac{\gamma}{2a^2}}}{\frac{\gamma}{2a^2}}\right]_a^{\sqrt{2}a} &= t
\end{aligned}$$

in this case we are establishing the initial and final conditions directly into the integral

$$\left[-\frac{\sqrt{2}a\sqrt{\frac{\gamma}{2a^2}-\frac{\gamma}{2a^2}}}{\frac{\gamma}{2a^2}}+\frac{a\sqrt{\frac{\gamma}{a^2}-\frac{\gamma}{2a^2}}}{\frac{\gamma}{2a^2}}\right]=t$$

$$\frac{a\sqrt{\frac{\gamma}{2a^2}}}{\frac{\gamma}{2a^2}}=t$$

$$\frac{\sqrt{\frac{\gamma}{2}}}{\frac{\gamma}{2a^2}}=t$$

$$a^2\sqrt{\frac{2}{\gamma}}=t$$

□

Proof. **4.12** The equation of motion in this case is happening in the y-axis direction, assuming y-axis is pointing from Sun to Earth, we have that

$$m\frac{dv}{dt}=-\frac{GmM}{r^2}$$

we are assuming M and m are the masses of the Sun and Earth respectively. Also we know that

$$\frac{dv}{dt}=\frac{dr}{dt}\times\frac{dv}{dr}=v\frac{dv}{dr}$$

then we can replace dv/dt

$$v\frac{dv}{dr}=-\frac{GM}{r^2}$$

$$\int vdv=\int-\frac{GM}{r^2}dr$$

$$\frac{v^2}{2}=\frac{GM}{r}+C$$

where C is a constant that can be determined using the initial conditions. If $v=0$ when $r=R$ (where R is the distance at that moment from earth to the sun) then we have that

$$C=-\frac{GM}{R}$$

so we have

$$\frac{v^2}{2}=\frac{GM}{r}-\frac{GM}{R}$$

$$v^2=2GM\left(\frac{1}{r}-\frac{1}{R}\right)$$

now replacing v with dr/dt we have

$$\left(\frac{dr}{dt}\right)^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$$
$$\int_R^0 \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} = \int_0^T \sqrt{2GM}dt$$

in this case we have added the initial and final conditions directly to the integral (T is the time taken for Earth to collide with the Sun), then

$$\begin{aligned} T &= \frac{1}{\sqrt{2GM}} \int_R^0 \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} \\ &= \frac{1}{\sqrt{2GM}} \frac{\pi R^{3/2}}{2} \\ &= \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}} \end{aligned}$$

On the other hand we know that the orbital period of the earth is calculated as $T' = 2\pi\sqrt{\frac{R^3}{GM}}$ so we could wirte T in terms of T' as

$$T = \frac{T'}{4\sqrt{2}}$$

since $T' = 365$ days, then $T = 64.52$ days. □

Proof. **4.13** For the particle Q to remain at rest we need to ensure that

$$T = Mg$$

where T is the tension on the string and Mg is the weight of Q.
On the other hand, the particle P has only a possible moving trajectory which is a circular motion around O (the origin is fixed at the hole), so the equations of motion for P in polar coordinates is given by

$$m\left[-\frac{v^2}{b}\hat{\boldsymbol{r}} + \dot{v}\hat{\boldsymbol{\theta}}\right] = -T\hat{\boldsymbol{r}} = -Mg\hat{\boldsymbol{r}}$$

where m is the mass of P and $v = b\ddot{\theta}$ is the circumferential velocity.
Which taking components in the radial and transverse direction, gives

$$-\frac{mv^2}{b} = -Mg \tag{1}$$
$$\dot{v} = 0 \tag{2}$$

from the equation (2) we see that the velocity should be constant and from (1) we have that

$$v^2 = \frac{Mgb}{m}$$

□

Proof. **4.15** We can write the equation of motion for P in polar coordianates as

$$m[(\ddot{r} - r\Omega^2)\hat{\boldsymbol{r}} + (r\dot{\Omega} + 2\dot{r}\Omega)\hat{\boldsymbol{\theta}}] = -N\hat{\boldsymbol{\theta}}$$

where m is the mass of P and $\Omega = \dot{\theta}$.
Which taking components in the radial and transverse direction, gives

$$\ddot{r} - r\Omega^2 = 0 \tag{1}$$

$$r\dot{\Omega} + 2\dot{r}\Omega = N \tag{2}$$

then equations we wanted.
The equation (1) is second-order linear differential equation, for which we propose a solution to the form

$$r(t) = C_1 e^{\Omega t} + C_2 e^{-\Omega t}$$

where C_1 and C_2 are constants that should be determined by the initial conditions. Since $r = a$ when $t = 0$ we have that

$$C_1 + C_2 = a$$

and we also know that P is at rest at the beginning, so derivating r equation we get

$$\frac{dr}{dt} = C_1 \Omega e^{\Omega t} - C_2 \Omega e^{-\Omega t}$$

and applying the initial conditions $dr/dt = 0$ at $t = 0$ we get

$$0 = C_1\Omega - C_2\Omega$$

then $C_1 = C_2 = a/2$ from the previous equation. Therefore the equation for r is given by

$$r(t) = \frac{a}{2}(e^{\Omega t} + e^{-\Omega t})$$

On the other hand, replacing dr/dt in equation (2) we have that

$$N = ma\Omega^2(e^{\Omega t} - e^{-\Omega t})$$

□

Proof. **4.27** With the linear resistance term included, the vector equation of motion for the projected body becomes

$$m\frac{d\boldsymbol{v}}{dt} = -mK\boldsymbol{v} - mg\boldsymbol{k}$$

the direction of $\boldsymbol{v}$ is given by $\boldsymbol{v} = u\cos\alpha\boldsymbol{i} + u\sin\alpha\boldsymbol{k}$. We could propose a different magnitude and direction for the gravity, writing the new direction for our new gravity as $\boldsymbol{w}$, we get that

$$\begin{aligned} g'\boldsymbol{w} &= uK\cos\alpha\boldsymbol{i} + uK\sin\alpha\boldsymbol{k} + g\boldsymbol{k} \\ &= uK\cos\alpha\boldsymbol{i} + (uK\sin\alpha + g)\boldsymbol{k} \end{aligned}$$

then we have that the equation of motion is given by

$$m\frac{d\boldsymbol{v}}{dt} = -mg'\boldsymbol{w}$$

which is equivalent to the original equation of motion but using our new gravity.

On the other hand, we want to deduce that it's possible for the body to return to its starting point. By continuity, it's possible to end before and after the starting point, therefore, it's possible to end up exactly at the starting point, the shape of the path, in that case, would be an inclined line. □

Proof. **4.29** The vector equation of motion in this case is given by

$$\begin{aligned} m[(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}] &= -T\sin\alpha\hat{\boldsymbol{r}} + (T\cos\alpha - mg)\hat{\boldsymbol{z}} \\ -ma\sin\alpha\dot{\theta}^2\hat{\boldsymbol{r}} &= -T\sin\alpha\hat{\boldsymbol{r}} + (T\cos\alpha - mg)\hat{\boldsymbol{z}} \end{aligned}$$

where m is the mass of the particle and T is the tension of the string. Which taking components in $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{z}}$ direction, gives

$$\begin{aligned} ma\sin\alpha\dot{\theta}^2 &= T\sin\alpha \qquad (1) \\ 0 &= T\cos\alpha - mg \qquad (2) \end{aligned}$$

from the equation (2) we have that $T = mg/\cos\alpha$ and replacing on equation (1) we get that

$$\begin{aligned} ma\sin\alpha\dot{\theta}^2 &= mg\frac{\sin\alpha}{\cos\alpha} \\ (a\sin\alpha)^2\dot{\theta}^2 &= ga\sin\alpha\tan\alpha \\ u^2 &= ga\sin\alpha\tan\alpha \end{aligned}$$

□