Solved selected problems of Classical Mechanics - Gregory

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Chapter 9 - The energy principle and energy conservation

Solution. **9.1** First, we want to calculate the potential energy that comes from the uniform gravity which is given by

$$V = mgZ_Q + MgZ_P$$

Where Z_Q and Z_P are the vertical displacement of P and Q against O. Then we have that

$$Z_P = a\cos(\theta) \qquad Z_Q = a\cos\Big(\frac{\pi}{2} - \theta\Big)$$

Then

$$V = ga(m\cos\Big(\frac{\pi}{2} - \theta\Big) + M\cos(\theta))$$

We must now show that the constraint forces do no work. The rate of working of the constraint force $\boldsymbol{R}$ that the cylinder exerts on each of the particles is $\boldsymbol{R}\cdot\boldsymbol{v^Q} + \boldsymbol{R}\cdot\boldsymbol{v^P}$ where $\boldsymbol{R}$ is perpendicular to both $\boldsymbol{v^Q}$ and $\boldsymbol{v^P}$ therefore the rate of working of $\boldsymbol{R}$ is zero. Also, the internal constraint forces that enforce the rigidity of the light inextensible string does not work in total. Hence, the constraint forces do no work in total.

Therefore energy conservation applies in the form

$$T + ga(m\cos\Big(\frac{\pi}{2} - \theta\Big) + M\cos(\theta)) = E$$

The equilibrium position happens when $V' = 0$ then it should happen that

$$m\sin\Big(\frac{\pi}{2} - \theta\Big) - M\sin(\theta) = 0$$

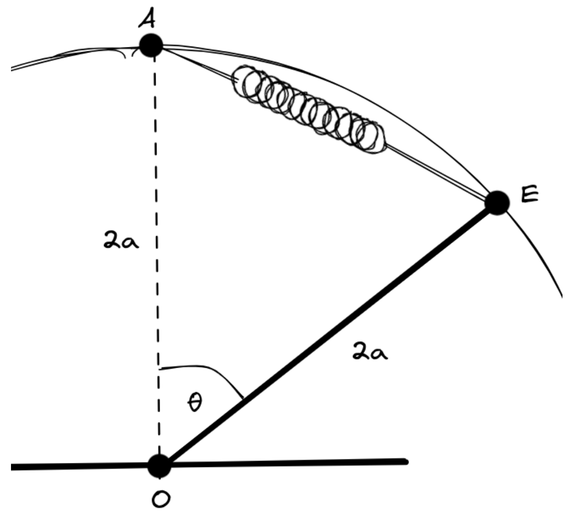
and this is true when $\theta = \tan^{-1}(m/M)$

It follows that the equilibrium position will be stable if V has a minimum there, to check this we calculate V'' and we replace the value of θ we got there, then

$$\begin{aligned}
V'' &= -ga(m\sin(\theta) + M\cos(\theta))\\
&= -ga(m\sin\Big(\arctan\Big(\frac{m}{M}\Big)\Big) + M\cos\Big(\arctan\Big(\frac{m}{M}\Big)\Big))\\
&= -ga\sqrt{m^2 + M^2}
\end{aligned}$$

Since $V'' < 0$ then V does not have a minimum there. Therefore the equilibrium point is not stable. $\square$

Solution. **9.2** Just to clarify, the system looks like this



We want to calculate the potential energy of the system. There are two contributions to the potential energy, the uniform gravity exerted on the rod and the spring elasticity. Then we have that

$$V = mgZ + \frac{k}{2}X^2$$

Where Z is the vertical displacement of the rod's center of mass, X is the net displacement of the spring over or under the natural length a, and $k = \frac{mg}{2a}$ is the spring constant. From the image, we see that AE is a chord of the circular trajectory into which the rod is moving so we have that

$$AE = 4a\sin\left(\frac{\theta}{2}\right)$$

Then $X = 4a\sin\left(\frac{\theta}{2}\right) - a = a(4\sin\left(\frac{\theta}{2}\right) - 1)$. On the other hand $Z = a\cos\theta$ then we have that

$$\begin{aligned}
V &= mga\cos\theta + \frac{1}{4}mga\left(16\sin^2\left(\frac{\theta}{2}\right) - 8\sin\left(\frac{\theta}{2}\right) + 1\right)\\
&= \frac{1}{4}mga\left(4\cos\theta + 16\sin^2\left(\frac{\theta}{2}\right) - 8\sin\left(\frac{\theta}{2}\right) + 1\right)\\
&= \frac{1}{4}mga\left(4 - 8\sin^2\left(\frac{\theta}{2}\right) + 16\sin^2\left(\frac{\theta}{2}\right) - 8\sin\left(\frac{\theta}{2}\right) + 1\right)\\
&= \frac{1}{4}mga\left(8\sin^2\left(\frac{\theta}{2}\right) - 8\sin\left(\frac{\theta}{2}\right) + 5\right)
\end{aligned}$$

The equilibrium point will happen when $V' = 0$ then

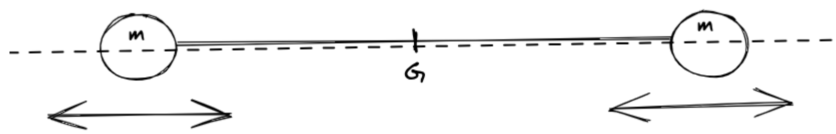
$$4\cos\left(\frac{\theta}{2}\right)\left(2\sin\left(\frac{\theta}{2}\right) - 1\right) = 0$$

This means that the equilibrium points happen at $\theta = \pi$ and $\theta = \pi/3$. To find out if they are stable we need to check that V has a minimum at these points so we compute V'' and we check the signs of V'' then

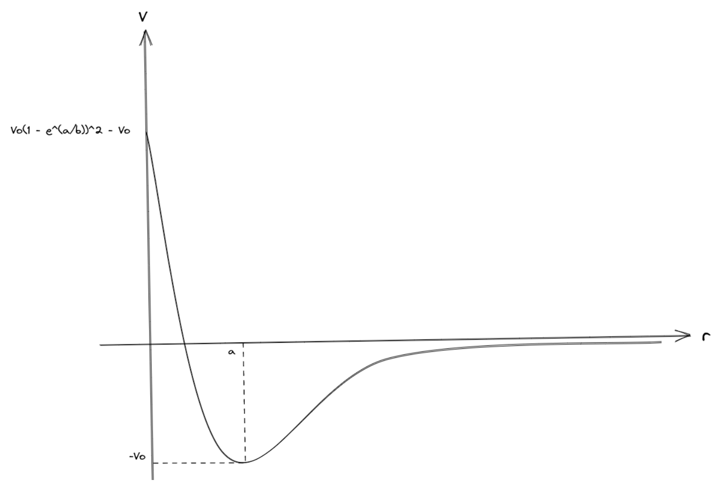
$$V'' = 2(2\cos^2(\theta/2) + \sin(\theta/2) - 2\sin^2(\theta/2))$$

When $\theta = \pi$ we have that $V''(\pi) = 2(0+1-2) = -2$ and when $\theta = \pi/3$ we get that $V''(\pi/3) = 2(3/2 + 1/2 - 1/2) = 3$. Therefore for $\theta = \pi$ the system is unstable (the point E is opposite to A) and for $\theta = \pi/3$ the system is stable. $\square$

Solution. **9.3** The system we are presented with looks like this



And here is a sketch of the Morse potential



Now, we want to show that there is a single equilibrium configuration and that it's stable. The equilibrium point will happen when $dV/dr = 0$ then we calculate the following

$$\frac{dV}{dr} = \frac{2V_0}{b}(e^{\frac{a-r}{b}})(1 - e^{\frac{a-r}{b}})$$

and we want that

$$\begin{aligned}(e^{\frac{a-r}{b}})(1 - e^{\frac{a-r}{b}}) &= 0 \\ e^{\frac{a-r}{b}} - e^{\frac{2(a-r)}{b}} &= 0\end{aligned}$$

This equation is satisfied only when $r = a$ so the system has only one equilibrium point. Let's now show that this is a stable point by showing that $d^2V/dt^2 > 0$ i.e. it's a minimum, we have that

$$\frac{d^2V}{dr^2} = -\frac{2V_0}{b^2}(e^{\frac{a-2r}{b}})(e^{\frac{r}{b}} - 2e^{\frac{a}{b}})$$

so

$$\frac{d^2V(a)}{dr^2} = \frac{2V_0}{b^2}$$

Therefore we see that $d^2V(a)/dr^2 > 0$ since V_0 and b are positive constants. From the sketch we did we see this result is correct.

Finally, we want to calculate the angular frequency of the particle. Let us calculate a Taylor series approximation to the force applied to the particle assuming the Morse potential energy is the only contribution then

$$F(a) = -\frac{dV}{dr}(a) \approx 0 - \frac{2V_0}{b^2}(r-a)$$

Then the approximated equation of motion from Hooke's law is given by

$$\begin{aligned} -k(r-a) &= -\frac{2V_0}{b^2}(r-a) \\ k &= \frac{2V_0}{b^2} \end{aligned}$$

where k is the spring constant and $r-a$ is the displacement because of small oscillation. So with this value, we can calculate the angular frequency ω as follows

$$\omega = \sqrt{\frac{2V_0}{mb^2}}$$

□

Solution. **9.4** Let us suppose that a sphere of radius r was already built, then we can equal the densities and calculate the mass of the built sphere

$$\begin{aligned} \frac{m}{4/3\pi r^3} &= \frac{M}{4/3\pi R^3} \\ m &= M\left(\frac{r}{R}\right)^3 \end{aligned}$$

Now suppose we want to add a thin layer of radius dr with mass dm then in the same way

$$\begin{aligned} \frac{dm}{4\pi r^2 dr} &= \frac{M}{4/3\pi R^3} \\ dm &= \frac{3Mr^2}{R^3}dr \end{aligned}$$

To bring this extra mass from infinity to the already-built sphere we need to do $Gmdm/r$ work. So to calculate the work of forming the entire sphere from nothing we need to replace the values we have for dm and m and integrate the expression from 0 to R. The self-energy is the negative of the work done.

$$\begin{aligned} V &= -\int_0^R \frac{GM}{r}\frac{r^3}{R^3}\frac{3Mr^2}{R^3}dr \\ &= -\frac{3GM^2}{R^6}\int_0^R r^4 dr \\ &= -\frac{3GM^2}{R^6}\left[\frac{r^5}{5}\right]_0^R \\ &= -\frac{3GM^2}{5R} \end{aligned}$$

□

Solution. **9.5** The kinetic energy of the system is given by:

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}mv^2$$

where v is the velocity with which the two blocks move along each of the inclined planes.

The gravitational potential energy is given by

$$V = mgx\sin\beta - Mgx\sin\alpha = xg(m\sin\beta - M\sin\alpha)$$

Where x is the displacement of the block and since they are connected by a light inextensible string they move the same amount.

We must now dispose of the constraint forces. Since there is no slippage between the string and the two material bodies of the system, the total work done by the string on the bodies must be equal and opposite to the total work done by the bodies on the string. Hence the constraint forces do no work in total.

Energy conservation, therefore, applies in the form

$$\frac{1}{2}(M+m)v^2 + xg(m\sin\beta - M\sin\alpha) = E$$

Where E is the total energy. If we now differentiate with respect to t we get the acceleration of the blocks as follows

$$\begin{aligned}(M+m)v\frac{dv}{dt} + vg(m\sin\beta - M\sin\alpha) &= 0\\ \frac{dv}{dt} &= g\left(\frac{M\sin\alpha - m\sin\beta}{M+m}\right)\end{aligned}$$

Finally, to get the tension T in the string we apply Newton's second law

$$\begin{aligned}T - mg\sin\beta &= mdv/dt\\ T &= m(dv/dt + g\sin\beta)\\ T &= mg\left(\frac{M\sin\alpha - m\sin\beta}{M+m} + \sin\beta\right)\\ T &= mMg\left(\frac{\sin\alpha + \sin\beta}{M+m}\right)\end{aligned}$$

□

Solution. **9.6** From problem 9.1 we know that the potential energy for the system is given by

$$V = ga\left(m\sin(\theta) + M\cos(\theta)\right)$$
$$V = mga\left(\sin(\theta) + 2\cos(\theta)\right)$$

Now we need to determine the kinetic energy T so

$$T = \frac{1}{2}m(a\dot{\theta})^2 + m(a\dot{\theta})^2$$
$$T = \frac{3}{2}m(a\dot{\theta})^2$$

Then the energy conservation equation is given by

$$\frac{3}{2}m(a\dot{\theta})^2 + mga\left(\sin(\theta) + 2\cos(\theta)\right) = E$$

And by using the initial conditions where $\theta = \pi/4$ and $\dot{\theta} = 0$ when $t = 0$ we can determine that $E = 3mga/\sqrt{2}$. Therefore the energy equation in terms of θ is given by

$$\frac{3}{2}a\dot{\theta}^2 = \frac{3}{\sqrt{2}}g - g\left(\sin(\theta) + 2\cos(\theta)\right)$$
$$\dot{\theta}^2 = \frac{g}{3a}\left(3\sqrt{2} - 2\sin(\theta) - 4\cos(\theta)\right)$$

Now we want to find the normal reaction of the cylinder on each of the particles. By using Newton's second law in the radial direction we can determine the cylinder reaction on particle P as follows

$$2mg\cos(\theta) - N_P = 2m\frac{(a\dot{\theta})^2}{a}$$

Then we have that

$$N_P = 2mg\cos(\theta) - 2m\frac{(a\dot{\theta})^2}{a}$$
$$N_P = 2mg\cos(\theta) - mg\frac{2}{3}\left(3\sqrt{2} - 2\sin(\theta) - 4\cos(\theta)\right)$$
$$N_P = \frac{2}{3}mg\left(7\cos(\theta) + 2\sin(\theta) - 3\sqrt{2}\right)$$

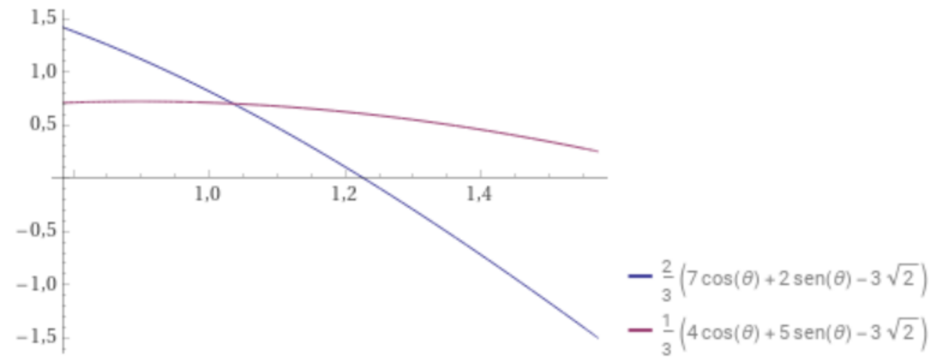
For the particle Q we have that

$$mg\sin(\theta) - N_Q = m\frac{(a\dot{\theta})^2}{a}$$

And in the same way

$$N_Q = mg\sin(\theta) - m\frac{(a\dot{\theta})^2}{a}$$
$$N_Q = mg\sin(\theta) - \frac{mg}{3}\left(3\sqrt{2} - 2\sin(\theta) - 4\cos(\theta)\right)$$
$$N_Q = \frac{mg}{3}\left(4\cos(\theta) + 5\sin(\theta) - 3\sqrt{2}\right)$$

When $N_P = 0$ and $N_Q = 0$ the particles left the cylinder, so by solving the equations we get that the particle Q leaves the cylinder when $\theta \approx 99°$ and particle P leaves the cylinder when $\theta \approx 70°$. Here we leave a plot of N_P and N_Q against θ assuming $mg = 1$.



$\square$

Solution. **9.7** The system we are examining looks like this

Let us first determine the kinetic energy of the rope, since the rope is heavy and uniform we can assume that each particle in the rope has the same velocity $v = \dot{x}$ then we have that

$$T = \frac{1}{2}M\dot{x}^2$$

where M is the total mass of the rope.

The only contribution to the potential energy comes from the uniform gravity exerted on the rope so if we suppose the rope is displaced a distance x then the mass added to the right side is given by $Mx/2a$ and also the center of mass is displaced a distance x, then

$$V = -\left(\frac{Mx}{2a}\right)gx$$

We must now show that the constraint forces do no work. The reactions exerted by the thin smooth peg on the particles of the rope are always perpendicular to the velocities of these particles, these reactions therefore do no work. Also, the tension forces exerted by each segment of the inextensible string do no work in total. Hence, the constraint forces do no work in total.

Therefore energy conservation can be applied in the form

$$\frac{1}{2}M\dot{x}^2 - \frac{Mg}{2a}x^2 = E$$

and applying the initial conditions $x = 0$ and $\dot{x} = 0$ when $t = 0$ impliest that $E = 0$ then

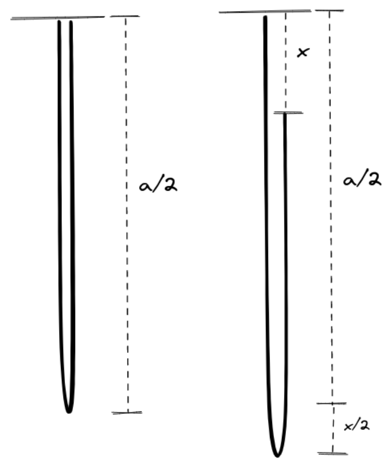
$$\dot{x} = \sqrt{\frac{g}{a}}x$$

We want to determine the speed of the rope when it finally leaves the peg and this will happen when $x = a$ then

$$\dot{x} = \sqrt{ga}$$

□

Solution. **9.8** The system we are examining looks like this



Let us first determine the kinetic energy of the rope, the mass of the right side of the rope (the one that falls) changes with the distance x then the kinetic energy is given by

$$T = \frac{1}{2}\left(\frac{M(a/2 - x/2)}{a}\right)v^2$$
$$T = \frac{1}{4}M\left(1 - \frac{x}{a}\right)v^2$$

where M is the total mass of the rope.

The only contribution to the potential energy comes from the uniform gravity exerted on the rope so if we suppose the rope is displaced a distance x on the right side then

$$V = -\left(\frac{M(a/2 + x/2)}{a}\right)g\left(\frac{a/2 + x/2}{2}\right) - \left(\frac{M(a/2 - x/2)}{a}\right)g\left(\frac{a/2 - x/2}{2} + x\right)$$
$$V = -\frac{Mg}{a}\left[\left(\frac{a}{2} + \frac{x}{2}\right)\left(\frac{a}{4} + \frac{x}{4}\right) + \left(\frac{a}{2} - \frac{x}{2}\right)\left(\frac{a}{4} + \frac{3x}{4}\right)\right]$$
$$V = -\frac{Mg}{4a}\left(a^2 + 2ax - x^2\right)$$

Assuming the constraint forces do not work the energy conservation can be applied in the form

$$\frac{1}{4}M\left(1 - \frac{x}{a}\right)v^2 - \frac{Mg}{4a}\left(a^2 + 2ax - x^2\right) = E$$

and applying the initial conditions $x = 0$ and $v = 0$ when $t = 0$ implies that $E = -Mga/4$ then we have that the velocity of the free end when it has descended by a distance x is

$$\frac{1}{4}M\left(1-\frac{x}{a}\right)v^2-\frac{Mg}{4a}\left(a^2+2ax-x^2\right)=-\frac{Mga}{4}$$
$$\left(1-\frac{x}{a}\right)v^2-\frac{g}{a}\left(a^2+2ax-x^2\right)=-ga$$
$$\left(1-\frac{x}{a}\right)v^2=-ga+\frac{g}{a}\left(a^2+2ax-x^2\right)$$
$$v^2=\frac{gx(2a-x)}{a-x}$$

On differentiating again we get the acceleration as follows

$$\frac{dv}{dt}=\frac{g(2a^2-2ax+x^2)}{2(a-x)^2}$$

If we subtract g from both sides we get that

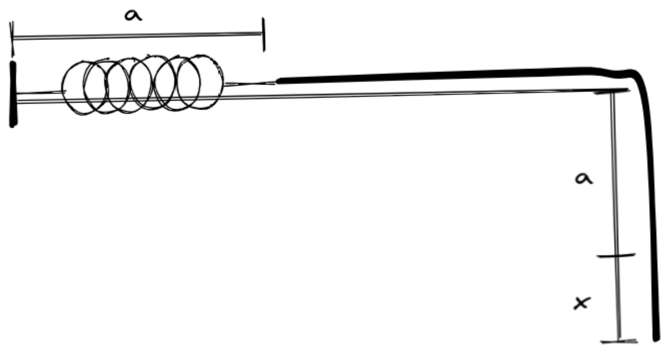
$$\frac{dv}{dt}-g=g\left(\frac{(2a^2-2ax+x^2)}{2(a-x)^2}-1\right)$$
$$\frac{dv}{dt}-g=g\left(\frac{x(2a-x)}{2(a-x)^2}\right)$$

Where $\frac{x(2a-x)}{2(a-x)^2}>0$ for $0<x<a$ then the acceleration always exceeds g. Finally, if $dv/dt=5g$ we get that

$$5g=\frac{g(2a^2-2ax+x^2)}{2(a-x)^2}$$
$$0=8a^2-18ax+9x^2$$
$$x=\frac{2a}{3}\quad\text{and}\quad x=\frac{4a}{3}$$

But given that x cannot exceed a it must happen that $x=2a/3$. So the free end has fallen $2a/3$ when the acceleration is $5g$. □

Solution. **9.9** The system we are examining looks like this



Let us first determine the kinetic energy of the rope, since the rope is heavy and uniform we can assume that each particle in the rope has the same velocity $v = \dot{x}$ then we have that

$$T = \frac{1}{2}M\dot{x}^2$$

where M is the mass of the rope.

The contributions to the potential energy come from the uniform gravity exerted on the free end of the rope and from the potential energy stored in the spring then

$$V = -\left(\frac{Mx}{4a}\right)g\left(a + \frac{x}{2}\right) + \frac{1}{2}\left(\frac{Mg}{2a}\right)x^2$$
$$V = \left(\frac{Mg}{4a}\right)\left(-xa - \frac{x^2}{2} + x^2\right)$$
$$V = \left(\frac{Mg}{8a}\right)x(x - 2a)$$

where $\frac{Mg}{2a}$ is the spring constant and we are supposing a small displacement x as shown above, which implies a negative gravitational potential energy and positive potential energy stored in the spring.

Assuming the table is smooth and the spring is light then the constraint forces do not work and so the energy conservation can be applied in the form

$$\frac{1}{2}M\dot{x}^2 + \left(\frac{Mg}{8a}\right)x(x - 2a) = E$$

Applying the initial conditions $x = 0$ and $\dot{x} = 0$ when $t = 0$ implies that $E = 0$ then we have that the velocity of the free end when it has descended by a distance x is

$$\begin{aligned}
\frac{1}{2}M\dot{x}^2 + \left(\frac{Mg}{8a}\right)x(x-2a) = 0 \\
\dot{x}^2 + \left(\frac{g}{4a}\right)x(x-2a) = 0 \\
\dot{x}^2 = \left(\frac{g}{4a}\right)x(2a-x)
\end{aligned}$$

By differentiating this expression we can get the motion equation as follows

$$\begin{aligned}
2\frac{dx}{dt}\frac{d^2x}{dt^2} &= \left(\frac{g}{4a}\right)(2a-2x)\frac{dx}{dt} \\
\frac{d^2x}{dt^2} &= \left(\frac{g}{4a}\right)(a-x) \\
\frac{d^2(x-a)}{dt^2} + \left(\frac{g}{4a}\right)(x-a) = 0
\end{aligned}$$

The equation is the one of a simple harmonic motion around the initial point a as we wanted. Assuming $\Omega^2 = g/4a$ the period is given by

$$\tau = 4\pi\sqrt{\frac{a}{g}}$$

Finally, the solution to the above equation is of the form

$$x - a = A\cos\Omega t + B\sin\Omega t$$

From the initial conditions $x = 0$ when $t = 0$ we have that $A = -a$ and from the initial condition $\dot{x} = 0$ when $t = 0$ we have that $B = 0$ then the final equation is given by

$$x - a = -a\cos\Omega t$$

Therefore the amplitude of the motion is a. $\square$

Solution. **9.10** We want to compute first the energy equation before the circular hoop loses h altitude then the kinetic energy is given by

$$
\begin{aligned}
T_i &= \frac{1}{2}Mv_i^2 + \frac{1}{2}I\omega_i^2 \\
T_i &= \frac{1}{2}Mv_i^2 + \frac{1}{2}Mb^2\left(\frac{v_i}{b}\right)^2 \\
T_i &= Mv_i^2
\end{aligned}
$$

Where M is the mass of the hoop, $I = Mb^2$ is the moment of inertia (from the appendix) and $\omega_i = v_i/b$ is the initial angular velocity (b is the hoop's radius).

The contribution to the potential energy come from the uniform gravity exerted to the hoop then.

$$V_i = Mgh$$

Assuming the hoop is rolling without slipping the velocity of the bottom point i.e the one in contact with the surface is 0 and in consequence there is no displacement so the work done by the constraint force is 0 and therefore energy conservation for the initial state can be applied in the form

$$Mv_i^2 + Mgh = E_i$$

On the other hand, we can compute the energy conservation equation for the final state as follows

$$Mv_f^2 = E_f$$

Where in this case the kinetic energy is computed with the final velocity v_f and since we took as a reference point the final state, the potential energy is 0 in this case. Finally, since we assume the energy is conserved we have that E_i must be equal to E_f then the final velocity is given by

$$
\begin{aligned}
Mv_i^2 + Mgh &= Mv_f^2 \\
v_f^2 &= v_i^2 + gh
\end{aligned}
$$

□

Solution. **9.11** Suppose that, at time t, the ball has displacement x down the plane (from some reference configuration) and that the center of mass G of the ball has velocity v down the plane. The angular velocity ω of the ball is then determined by $\omega = v/b$ where b is the radius of the ball. The kinetic energy of the ball is therefore

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$
$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}Mb^2\right)\left(\frac{v}{b}\right)^2$$
$$T = \frac{7}{10}Mv^2$$

Where we assumed the ball is a sphere with $I = \frac{2}{5}Mb^2$.

The contributions to the potential energy come from the uniform gravity exerted on the ball then

$$V = -Mgx\sin(\alpha)$$

We must now dispose of the constraint forces. The reaction forces that the inclined plane exerts on the ball act on particles of the ball which, because of the rolling condition, have zero velocity. These reaction forces, therefore, do no work. Also, the internal forces that keep the ball rigid do no work in total. Hence the constraint forces do no work in total.

So the energy conservation equation is given by

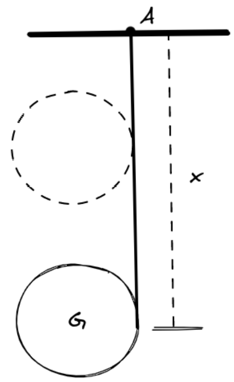
$$\frac{7}{10}Mv^2 - Mgx\sin(\alpha) = E$$

Finally, if we differentiate the energy conservation equation we get the acceleration as we wanted

$$\frac{7}{5}Mv\frac{dv}{dt} - Mgv\sin(\alpha) = 0$$
$$\frac{dv}{dt} = \frac{5g\sin(\alpha)}{7}$$

□

Solution. **9.12** The system we are examining looks like this



The string does no work on the yo-yo because the yo-yo descends without slipping so the yo-yo particles in contact with the string have 0 velocity so they have 0 displacement and hence they do no work.

Suppose that, at time t, the cylinder has displacement x down the fixed point A and that the center of mass G of the cylinder has velocity v. Then the angular velocity ω of the cylinder is then determined by $\omega = v/a$ where a is the radius of the cylinder. The kinetic energy of the cylinder is therefore

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\omega^2$$
$$T = \frac{1}{2}Mv^2 + \frac{1}{4}Ma^2\left(\frac{v}{a}\right)^2$$
$$T = \frac{3}{4}Mv^2$$

Where we assumed the yo-yo is a circular disk with a moment of inertia of $I = \frac{1}{2}Ma^2$.

The contribution to the potential energy come from the uniform gravity exerted on the yo-yo then

$$V = -Mgx$$

As we said, the reaction forces of the string because of the rolling condition do not work. Also, the internal forces that keep the cylinder rigid do no work in total. Hence the constraint forces do no work in total.

So the energy conservation equation is given by

$$\frac{3}{4}Mv^2 - Mgx = E$$

Finally, if we differentiate the energy conservation equation we get the acceleration as we wanted

$$\frac{3}{2}Mv\frac{dv}{dt} - Mgv = 0$$
$$\frac{dv}{dt} = \frac{2}{3}g$$

$\square$

Solution. **9.13** Let us first compute the kinetic energy at the initial state of the system where the radius is a (we are assuming the roll of paper is rolling without slipping) then

$$T_a = \frac{1}{2}MV^2 + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\left(\frac{V}{a}\right)^2$$
$$T_a = \frac{3}{4}MV^2$$

Where we used that M is the initial mass of the roll, the moment of inertia is $I = \frac{1}{2}Ma^2$ and the angular velocity of the roll is $\omega = V/a$.

Also, we determine the gravitational potential energy for the initial state of the system as

$$V_a = Mga$$

Now we are interested in determining what happens when the roll has increased its radius to b then the kinetic energy is given by

$$T_b = \frac{1}{2}\left(M\frac{b^2}{a^2}\right)v^2 + \frac{1}{2}\left(\frac{1}{2}\left(M\frac{b^2}{a^2}\right)b^2\right)\left(\frac{v}{b}\right)^2$$
$$T_b = \frac{3}{4}\left(M\frac{b^2}{a^2}\right)v^2$$

And the potential energy of the system is

$$V_b = M\frac{b^3}{a^2}g$$

Where we used that the mass of the roll when the radius has increased to b is Mb^2/a^2.

Given that the roll is rolling without slipping the particles in contact with the horizontal floor have zero velocity and therefore zero displacements, this is saying that the reaction forces do not work. Also, the internal forces of the roll do not work and hence the constraint forces do not work in total.

So the energy conservation equation can be applied and it must be conserved between the two states we are considering hence from the equation we can compute the final velocity as follows

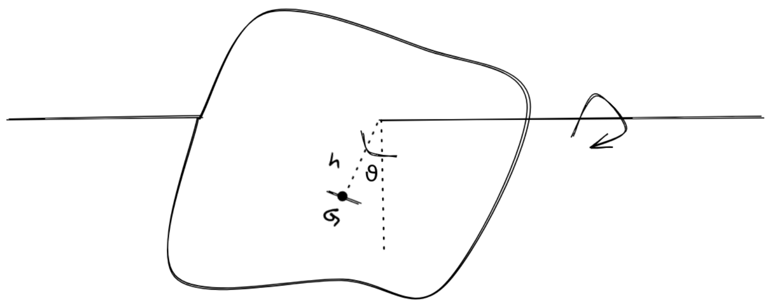
$$\frac{3}{4}MV^2 + Mga = \frac{3}{4}\left(M\frac{b^2}{a^2}\right)v^2 + M\frac{b^3}{a^2}g$$
$$\left(\frac{b^2}{a^2}\right)v^2 = V^2 + \frac{4}{3}ga - \frac{4}{3}\frac{b^3}{a^2}g$$
$$v^2 = \frac{a^2}{b^2}V^2 + \frac{4}{3}\frac{a^3}{b^2}g - \frac{4}{3}bg$$
$$v^2 = \frac{a^2}{b^2}V^2 + g\frac{4}{3}\left(\frac{a^3}{b^2} - b\right)$$

Finally, when the roll comes to rest the radius is given by

$$
\begin{aligned}
0 &= \frac{a^2}{b^2}V^2 + g\frac{4}{3}\left(\frac{a^3}{b^2} - b\right) \\
\frac{b^3 - a^3}{b^2} &= \frac{1}{g}\frac{3}{4}\frac{a^2}{b^2}V^2 \\
b^3 &= \frac{a^2}{g}\frac{3}{4}V^2 + a^3 \\
b &= a\left(\frac{3}{4}\frac{V^2}{ag} + 1\right)^{1/3}
\end{aligned}
$$

$\square$

Solution. **9.14** The system we are examining looks like this



where G is the center of mass of the body and theta is the angular displacement from the vertical line.

In this configuration, the kinetic energy looks like

$$T = \frac{1}{2} I \dot{\theta}^2$$

where $\dot{\theta}$ is the angular velocity and I is the moment of inertia of the body. Also, the gravitational potential energy is given by

$$V = -mgh \cos\theta$$

Given that the body rotates freely, there are no constraint forces and therefore the energy conservation equation is given by

$$\frac{1}{2} I \dot{\theta}^2 - mgh \cos\theta = E$$

By differentiating this equation we get that

$$\begin{aligned} I\dot{\theta}\ddot{\theta} + mgh\dot{\theta}\sin\theta &= 0 \\ I\ddot{\theta} + mgh\sin\theta &= 0 \end{aligned}$$

Since we are assuming a set of small oscillations we can replace $\sin\theta \approx \theta$ and then the equation becomes

$$\ddot{\theta} + \frac{mgh}{I}\theta = 0$$

Finally, we know that the oscillation's period in this case is given by

$$\tau = 2\pi \left(\frac{I}{mgh} \right)^{1/2}$$

where we used that $\Omega^2 = \frac{mgh}{I}$.

Now we want to analyze a system analogous to the previous general case where the body is a rod of length $2a$ and the distance from it's center of mass (the middle of the rod) to the axis of rotation is b.

First, we need to compute the moment of inertia as follows

$$\begin{aligned}
I &= \int_{-a+b}^{a+b} \frac{m}{2a} r^2 dr \\
&= \frac{m}{2a}\left[\frac{2}{3}(a^3 + 3ab^2)\right] \\
&= \frac{m}{3}(a^2 + 3b^2)
\end{aligned}$$

Here, we assumed the density of the rod is $m/2a$.

Finally, we can replace the values in the previous equation and compute the period of small oscillations for the rod case as follows.

$$\tau = 2\pi\left(\frac{a^2 + 3b^2}{3gb}\right)^{1/2}$$

$\square$