

Solved selected problems of From Calculus to Chaos - Acheson

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Chapter 3 - Ordinary differential equations

Solution. 3.2 The equation presented can be classified as a nonlinear non-autonomous differential equation, and the equation is of the form

$$\frac{dx}{dt} = g(x)h(t)$$

where $g(x) = x^2$ and $h(t) = \frac{1}{1+t}$ which can be rewritten to the form

$$\frac{1}{g(x)} \frac{dx}{dt} = h(t)$$

as

$$\frac{1}{x^2} \frac{dx}{dt} = \frac{1}{1+t}$$

then integrating both sides with respect to t

$$\int \frac{1}{x^2} \frac{dx}{dt} dt = \int \frac{1}{1+t} dt$$

$$\int \frac{1}{x^2} dx = \int \frac{1}{1+t} dt$$

$$-\frac{1}{x} = \log(1+t) + C$$

$$x = \frac{1}{C - \log(1+t)}$$

Subject to $x = 1$ when $t = 0$ then $C = 1$ so we have

$$x = \frac{1}{1 - \log(1+t)}$$

therefore the "blow up" of the solution happens when t approaches $e - 1$. \square

Solution. **3.4** Since we know that $x = x_1 u$ then

$$\begin{aligned}\dot{x} &= \dot{x}_1 u + \dot{u} x_1 \\ \ddot{x} &= \ddot{x}_1 u + 2\dot{x}_1 \dot{u} + x_1 \ddot{u}\end{aligned}$$

by replacing in $a\ddot{x} + b\dot{x} + cx = 0$ we have that:

$$\begin{aligned}a(\ddot{x}_1 u + 2\dot{x}_1 \dot{u} + x_1 \ddot{u}) + b(\dot{x}_1 u + \dot{u} x_1) + c x_1 u &= 0 \\ u(a\ddot{x}_1 + b\dot{x}_1 + cx) + \dot{u}(2ax_1 + bx_1) + \ddot{u}ax_1 &= 0\end{aligned}$$

since $a\ddot{x} + b\dot{x} + cx = 0$ then

$$\dot{u}(2ax_1 + bx_1) + \ddot{u}ax_1 = 0$$

defining $z = \dot{u}$ then

$$\dot{z}ax_1 + z(2ax_1 + bx_1) = 0 \tag{1}$$

which is an equation of first-order for z .

Now this method is going to be used to solve $\ddot{x} - 2\dot{x} + x = 0$ since the solution we have is $x_1 = e^t$ then $\dot{x}_1 = e^t$ and replacing in the equation (1) with the values $a = 1$ and $b = -2$ we have that

$$\begin{aligned}\dot{z}e^t + z(2e^t - 2e^t) &= 0 \\ \dot{z}e^t &= 0 \\ \dot{z} &= 0\end{aligned}$$

then $z = B$ where B is a constant, but $\dot{u} = z$ so $\dot{u} = B$ and then $u = Bt + A$. Since we are assuming that $x = ux_1$ then the general solution is of the form

$$x = (Bt + A)e^t$$

then $\dot{x} = Be^t + (Bt + A)e^t$ and by applying the initial conditions we obtain that $A = 1$ and $0 = B + A$ and then $B = -1$.

Therefore the other solution is

$$x = (1 - t)e^t$$

□