Solved selected problems of Covariant Physics by Moataz Emam

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Solution. **Exercise 1.1**

2. For the oblate spheroidal coordinates system if we fix μ to different values we get a set of concentric spheres as we show below If we fix ν we get the following surfaces Finally if we fix ϕ to different values we get that Now, joining all together we get that

$\square$

Solution. **Exercise 1.2**
Let us consider a point in the x', y', z' coordinate system as shown below Then we see that

$$\begin{aligned} x &= x' \\ y &= y' + z' \sin\alpha \\ z &= z' \cos\alpha \end{aligned}$$

Or

$$\begin{aligned} x' &= x \\ y' &= y - z \tan\alpha \\ z' &= \frac{z}{\cos\alpha} \end{aligned}$$

□

Solution. **Exercise 1.3**

1. We know that in spherical coordinates

$$\begin{aligned}x^1 &= r\sin\theta\cos\phi\\ x^2 &= r\sin\theta\sin\phi\\ x^3 &= r\cos\theta\end{aligned}$$

Then

$$\begin{aligned}dx^1 &= d(r\sin\theta\cos\phi)\\ &= dr\sin\theta\cos\phi + rd(\sin\theta)\cos\phi + r\sin\theta d(\cos\phi)\\ &= dr\sin\theta\cos\phi + rd\theta\cos\theta\cos\phi - rd\phi\sin\theta\sin\phi\end{aligned}$$

$$\begin{aligned}dx^2 &= d(r\sin\theta\sin\phi)\\ &= dr\sin\theta\sin\phi + rd(\sin\theta)\sin\phi + r\sin\theta d(\sin\phi)\\ &= dr\sin\theta\sin\phi + rd\theta\cos\theta\sin\phi + rd\phi\sin\theta\cos\phi\end{aligned}$$

$$\begin{aligned}dx^3 &= d(r\cos\theta)\\ &= dr\cos\theta + rd(\cos\theta)\\ &= dr\cos\theta - rd\theta\sin\theta\end{aligned}$$

Let us compute each component $(dx^i)^2$ separately

$$\begin{aligned}(dx^1)^2 &= (dr\sin\theta\cos\phi + rd\theta\cos\theta\cos\phi - rd\phi\sin\theta\sin\phi)^2\\ &= dr^2\sin^2\theta\cos^2\phi + r^2d\theta^2\cos^2\theta\cos^2\phi + r^2d\phi^2\sin^2\theta\sin^2\phi\\ &\quad + 2rdrd\theta\sin\theta\cos\theta\cos^2\phi - 2rdrd\phi\sin^2\theta\cos\phi\sin\phi\\ &\quad - 2r^2d\theta d\phi\cos\theta\cos\phi\sin\theta\sin\phi\end{aligned}$$

$$\begin{aligned}(dx^2)^2 &= (dr\sin\theta\sin\phi + rd\theta\cos\theta\sin\phi + rd\phi\sin\theta\cos\phi)^2\\ &= dr^2\sin^2\theta\sin^2\phi + r^2d\theta^2\cos^2\theta\sin^2\phi + r^2d\phi^2\sin^2\theta\cos^2\phi\\ &\quad + 2rdrd\theta\sin\theta\cos\theta\sin^2\phi + r^2d\theta d\phi\cos\theta\sin\phi\sin\theta\cos\phi\\ &\quad + rdrd\phi\sin^2\theta\sin\phi\cos\phi\end{aligned}$$

$$\begin{aligned}(dx^3)^2 &= (dr\cos\theta - rd\theta\sin\theta)^2\\ &= dr^2\cos^2\theta + r^2d\theta^2\sin^2\theta - 2rdrd\theta\cos\theta\sin\theta\end{aligned}$$

Now we sum all components

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

Where we see that all terms with two different differentials cancel out.

2. We know that the spherical coordinates relate to the cylindrical coordinates as follows

$$\begin{aligned} \rho &= r \sin\theta \\ z &= r\cos\theta \\ \phi &= \phi \end{aligned}$$

Then

$$\begin{aligned} d\rho &= dr \sin\theta + r d\theta \cos\theta \\ dz &= dr\cos\theta - r d\theta \sin\theta \\ d\phi &= d\phi \end{aligned}$$

Also, we know that the line element in cylindrical coordinates is given by

$$dl^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

Hence replacing we get that

$$\begin{aligned} dl^2 &= (dr\sin\theta + r d\theta\cos\theta)^2 + r^2\sin^2\theta d\phi^2 + (dr\cos\theta - r d\theta \sin\theta)^2 \\ &= dr^2 \sin^2\theta + r^2 d\theta^2 \cos^2\theta + 2r dr d\theta \sin\theta\cos\theta \\ &\quad + r^2\sin^2\theta d\phi^2 + dr^2\cos^2\theta + r^2 d\theta^2 \sin^2\theta - 2r dr d\theta \cos\theta\sin\theta \\ &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \end{aligned}$$

5. We know that in the skew coordinate system we have

$$\begin{aligned} x &= x' \\ y &= y' + z'\sin\alpha \\ z &= z'\cos\alpha \end{aligned}$$

Then

$$\begin{aligned} dx &= dx' \\ dy &= dy' + dz'\sin\alpha \\ dz &= dz'\cos\alpha \end{aligned}$$

Hence the line element in the skew coordinate system is

$$\begin{aligned} dl^2 &= dx^2 + dy^2 + dz^2 \\ &= dx'^2 + (dy' + dz'\sin\alpha)^2 + \cos^2\alpha dz'^2 \\ &= dx'^2 + dy'^2 + \sin^2\alpha dz'^2 + 2dy'dz'\sin\alpha + \cos^2\alpha dz'^2 \\ &= dx'^2 + dy'^2 + dz'^2 + 2dy'dz'\sin\alpha \end{aligned}$$

We see that the metric is not diagonal as expected.

$\square$

Solution. **Exercise 1.4**

3. The Oblate spheroidal coordinates are defined by

$$\begin{aligned} x &= a\cosh\mu\cos\nu\cos\varphi \\ y &= a\cosh\mu\cos\nu\sin\varphi \\ z &= a\sinh\mu\sin\nu \end{aligned}$$

Then the scale factors are

$$\begin{aligned} h_\mu^2 &= a^2\sinh^2\mu\cos^2\nu\cos^2\varphi + a^2\sinh^2\mu\cos^2\nu\sin^2\varphi + a^2\cosh^2\mu\sin^2\nu \\ &= a^2\sinh^2\mu\cos^2\nu(\cos^2\varphi+\sin^2\varphi) + a^2\cosh^2\mu\sin^2\nu \\ &= a^2\sinh^2\mu\cos^2\nu + a^2\cosh^2\mu\sin^2\nu \\ &= a^2(\sinh^2\mu\cos^2\nu + (\sinh^2\mu+1)\sin^2\nu) \\ &= a^2(\sinh^2\mu+\sin^2\nu) \end{aligned}$$

$$\begin{aligned} h_\nu^2 &= a^2\cosh^2\mu\sin^2\nu\cos^2\varphi + a^2\cosh^2\mu\sin^2\nu\sin^2\varphi + a^2\sinh^2\mu\cos^2\nu \\ &= a^2\cosh^2\mu\sin^2\nu + a^2\sinh^2\mu\cos^2\nu \\ &= a^2(\sinh^2\mu+\sin^2\nu) \end{aligned}$$

$$\begin{aligned} h_\varphi^2 &= a^2\cosh^2\mu\cos^2\nu\sin^2\varphi + a^2\cosh^2\mu\cos^2\nu\cos^2\varphi + 0 \\ &= a^2\cosh^2\mu\cos^2\nu \end{aligned}$$

Hence the line element is given by

$$\begin{aligned} dl^2 &= a^2(\sinh^2\mu+\sin^2\nu)d\mu^2 + a^2(\sinh^2\mu+\sin^2\nu)d\nu^2 + a^2\cosh^2\mu\cos^2\nu d\varphi^2 \\ &= a^2(\sinh^2\mu+\sin^2\nu)(d\mu^2+d\nu^2) + a^2\cosh^2\mu\cos^2\nu d\varphi^2 \end{aligned}$$

□

Solution. **Exercise 1.6**
Equation (1.57) states that

$$dl^2 = \delta_{ij} dx^i dx^j$$

If we expand it we get that

$$\begin{aligned} dl^2 &= \delta_{11} dx^1 dx^1 + \delta_{12} dx^1 dx^2 + \delta_{13} dx^1 dx^3 + \delta_{21} dx^2 dx^1 + \delta_{22} dx^2 dx^2 \\ &\quad + \delta_{23} dx^2 dx^3 + \delta_{31} dx^3 dx^1 + \delta_{32} dx^3 dx^2 + \delta_{33} dx^3 dx^3 \\ &= \delta_{11} (dx^1)^2 + \delta_{12} dx^1 dx^2 + \delta_{13} dx^1 dx^3 + \delta_{21} dx^2 dx^1 + \delta_{22} (dx^2)^2 \\ &\quad + \delta_{23} dx^2 dx^3 + \delta_{31} dx^3 dx^1 + \delta_{32} dx^3 dx^2 + \delta_{33} (dx^3)^2 \end{aligned}$$

but we know that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ then

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Which is equation (1.15). □

Solution. **Exercise 1.7**

1.

$$(\boldsymbol{A}\cdot\boldsymbol{B})(\boldsymbol{C}\cdot\boldsymbol{D}) = (A^iB^j\delta_{ij})(C^kD^l\delta_{kl}) = (A^iB_i)(C^kD_k)$$

2.

$$\begin{aligned}(\boldsymbol{A}-\boldsymbol{B})\cdot\hat{\boldsymbol{n}} &= 0\\ (A^i-B^i)n^j\delta_{ij} &= 0\\ (A^i-B^i)n_i &= 0\end{aligned}$$

The condition for $\hat{\boldsymbol{n}}$ to have unit magnitude is $n^in^j\delta_{ij} = 1$.

3.

$$\cos\theta = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \frac{A^iB^j\delta_{ij}}{\sqrt{A^iA^j\delta_{ij}}\sqrt{B^kB^l\delta_{kl}}}$$

4.

$$\boldsymbol{E} = k\frac{\boldsymbol{r}}{r^3} = k\frac{x^i\hat{\boldsymbol{e}}_i}{(\sqrt{x^ix^j\delta_{ij}})^3} = k\frac{x^i\hat{\boldsymbol{e}}_i}{(x^ix^j\delta_{ij})^{3/2}}$$

5.

$$dW = \boldsymbol{F}\cdot d\boldsymbol{r} = F^idx^j\delta_{ij}$$

6.

$$\frac{1}{2}mv^2 = \frac{1}{2}m(v^iv^j\delta_{ij})$$

□

Solution. **Exercise 1.8**

1.

$$A_3^1 B_1^2 + A_3^2 B_2^2 + A_3^3 B_3^2 = A_3^i B_i^2$$

2.

$$A_{11}^1 + A_{12}^2 + A_{13}^3 = A_{1i}^i$$

3. The three expressions

$$\begin{aligned} A^{11}B_1 + A^{12}B_2 + A^{13}B_3 &= C^1 \\ A^{21}B_1 + A^{22}B_2 + A^{23}B_3 &= C^2 \\ A^{31}B_1 + A^{32}B_2 + A^{33}B_3 &= C^3 \end{aligned}$$

can be reduced to

$$A^{ji}B_i = C^j$$

□

Solution. **Exercise 1.9**

1. Assuming each vector has 3 components we get that

$$\begin{aligned} A_i^j B_j C^i &= A_1^1 B_1 C^1 + A_2^1 B_1 C^2 + A_3^1 B_1 C^3 \\ &\quad + A_1^2 B_2 C^1 + A_2^2 B_2 C^2 + A_3^2 B_2 C^3 \\ &\quad + A_1^3 B_3 C^1 + A_2^3 B_3 C^2 + A_3^3 B_3 C^3 \end{aligned}$$

2. Assuming each vector has 3 components we get that

$$\begin{aligned} A_i B_k^j C^k D_j^i &= A_1 B_1^1 C^1 D_1^1 + A_2 B_1^1 C^1 D_1^2 + A_3 B_1^1 C^1 D_1^3 \\ &\quad + A_1 B_1^2 C^1 D_2^1 + A_2 B_1^2 C^1 D_2^2 + A_3 B_1^2 C^1 D_2^3 \\ &\quad + A_1 B_1^3 C^1 D_3^1 + A_2 B_1^3 C^1 D_3^2 + A_3 B_1^3 C^1 D_3^3 \\ &\quad + A_1 B_2^1 C^2 D_1^1 + A_2 B_2^1 C^2 D_1^2 + A_3 B_2^1 C^2 D_1^3 \\ &\quad + A_1 B_2^2 C^2 D_2^1 + A_2 B_2^2 C^2 D_2^2 + A_3 B_2^2 C^2 D_2^3 \\ &\quad + A_1 B_2^3 C^2 D_3^1 + A_2 B_2^3 C^2 D_3^2 + A_3 B_2^3 C^2 D_3^3 \\ &\quad + A_1 B_3^1 C^3 D_1^1 + A_2 B_3^1 C^3 D_1^2 + A_3 B_3^1 C^3 D_1^3 \\ &\quad + A_1 B_3^2 C^3 D_2^1 + A_2 B_3^2 C^3 D_2^2 + A_3 B_3^2 C^3 D_2^3 \\ &\quad + A_1 B_3^3 C^3 D_3^1 + A_2 B_3^3 C^3 D_3^2 + A_3 B_3^3 C^3 D_3^3 \end{aligned}$$

$\square$

Solution. **Exercise 1.10**

1. To fix $\boldsymbol{B}\cdot\boldsymbol{C} = B^iC_j$ we need to replace the index j by i because of the definition we have in index notation for the dot product i.e. should be $\boldsymbol{B}\cdot\boldsymbol{C} = B^iC_i$.

2. In the case of $A^{ij} = B^{ik}C^?_k$ since on the LHS we have two free indexes then on the RHS we should have the same then must be that
$$A^{ij} = B^{ik}C^j_k$$

3. The expression $N_i = R^k_i D^p_k C_?$ is fixed putting p where the question mark is i.e.
$$N_i = R^k_i D^p_k C_p$$
Since the only free index in the LHS is i.

4. The expression $L^i_k D^k = R^n_m N^?_n P^?$ is fixed as follows
$$L^i_k D^k = R^n_m N^i_n P^m$$
or as
$$L^i_k D^k = R^n_m N^m_n P^i$$
Since the only free index in the LHS is i.

5. The expression $\delta^{ij}A_jB^k_i = A^?B^k_i = C^?$ is fixed as follows
$$\delta^{ij}A_jB^k_i = A^iB^k_i = C^k$$
Since we are rasing the index in the first equality and we are leaving only k as a free index in the last equality.

6. The expression $N_i = R^k_i\delta^p_k C_? = R^?_? C^?$ is fixed as follows
$$N_i = R^k_i\delta^p_k C_p = R^k_i C^k$$
Since we are raising the index of $C_?$ in the last equality.

$\square$

Solution. **Exercise 1.12**

1.
$$\boldsymbol{A} \cdot (\vec{\nabla} f) = A^i \partial_i f$$

2.
$$\vec{\nabla} \cdot \vec{\nabla} f = \partial^i \partial_i f$$

3.
$$\boldsymbol{A}(\vec{\nabla} \cdot \boldsymbol{B}) = A^i \partial_j B^j \hat{\boldsymbol{e}}_i$$

4.
$$\nabla^2 f - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \partial_i \partial^i f - \frac{1}{v^2} \partial_t^2 f = 0$$

□

Solution. **Exercise 1.13**

1. Using the product rule we see that

$$\begin{aligned}\partial_k(a_{ij}A^iA^j) &= a_{ij}\partial_k(A^iA^j)\\ &= a_{ij}(A^j(\partial_kA^i) + A^i(\partial_kA^j))\\ &= 2a_{ij}A^i(\partial_kA^j)\end{aligned}$$

   In the third steps we changed the indices because the summation isn't altered by swapping the indices.

2. We know that the notation $\partial_i x_j$ implies $\partial_i x_j = \partial x_j/\partial x_i$ so if $j = i$ then $\partial x_j/\partial x_i = 1$ and if $j \neq i$ we get that $\partial x_j/\partial x_i = 0$ so we can write that $\partial_i x_j = \delta_{ij}$.
   In the case of $\partial_i x^j$ we must note first that $x^j = \delta^{kj}x_k$ then

$$\partial_i x^j = \partial_i \delta^{kj}x_k = \delta^{kj}\partial_i x_k = \delta^{kj}\delta_{ik} = \delta_i^j$$

   Therefore if $i = j$ then $\partial_i x^j = 1$ and if $i \neq j$ we get that $\partial_i x^j = 0$.

3. We see that

$$\nabla \cdot \boldsymbol{r} = \partial_i x^i = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

4. Using that $|\boldsymbol{r}| = \sqrt{x_i x^i}$, the chain rule and the product rule we see that

$$\begin{aligned}\nabla(\ln|\boldsymbol{r}|) &= \partial^j(\ln\sqrt{x_ix^i})\hat{\boldsymbol{e}}_j\\ &= \frac{1}{2}\partial^j(\ln(x_ix^i))\hat{\boldsymbol{e}}_j\\ &= \frac{1}{2}\left(\frac{\partial^j x_ix^i}{x_kx^k}\right)\hat{\boldsymbol{e}}_j\\ &= \frac{1}{2}\left(\frac{x^i\partial^j x_i + x_i\partial^j x^i}{x_kx^k}\right)\hat{\boldsymbol{e}}_j\\ &= \frac{1}{2}\left(\frac{x^i\delta_i^j + x_i\delta^{ji}}{x_kx^k}\right)\hat{\boldsymbol{e}}_j\\ &= \frac{1}{2}\left(\frac{x^j + x^j}{x_kx^k}\right)\hat{\boldsymbol{e}}_j\\ &= \frac{x^j\hat{\boldsymbol{e}}_j}{x_kx^k}\\ &= \frac{\boldsymbol{r}}{r^2}\end{aligned}$$

5. Let us note that $\nabla^2 = \nabla \cdot \nabla$, so reusing the result from part 4 we see that

$$\begin{aligned}
\nabla^2(\ln|\boldsymbol{r}|) &= \nabla \cdot \nabla(\ln|\boldsymbol{r}|) \\
&= \partial_j \frac{x^j}{x_k x^k} \\
&= x^j \partial_j \frac{1}{x_k x^k} + \frac{1}{x_k x^k} \partial_j x^j \\
&= -x^j \frac{\partial_j x_k x^k}{(x_k x^k)^2} + \frac{3}{x_k x^k} \\
&= -x^j \frac{x^k \partial_j x_k + x_k \partial_j x^k}{(x_k x^k)^2} + \frac{3}{x_k x^k} \\
&= -x^j \frac{x^k \delta_{jk} + x_k \delta_j^k}{(x_k x^k)^2} + \frac{3}{x_k x^k} \\
&= -\frac{2x_j x^j}{(x_k x^k)^2} + \frac{3}{x_k x^k} \\
&= -\frac{2r^2}{(r^2)^2} + \frac{3}{r^2} \\
&= -\frac{2}{r^2} + \frac{3}{r^2} \\
&= \frac{1}{r^2}
\end{aligned}$$

Where we used that $\partial_j x^j = 3$ from part 3.

$\square$

Solution. **Exercise 1.14**

1. Equation 1.99 states that

$$\boldsymbol{C} = \delta^{ni}\varepsilon_{ijk}A^j B^k \hat{\boldsymbol{e}}_n$$

Expanding the equation we get that

$$\begin{aligned}
\boldsymbol{C} &= \delta^{11}\varepsilon_{123}A^2B^3\hat{\boldsymbol{e}}_1 + \delta^{11}\varepsilon_{132}A^3B^2\hat{\boldsymbol{e}}_1 + \\
&\quad + \delta^{22}\varepsilon_{213}A^1B^3\hat{\boldsymbol{e}}_2 + \delta^{22}\varepsilon_{231}A^3B^1\hat{\boldsymbol{e}}_2 + \\
&\quad + \delta^{33}\varepsilon_{321}A^2B^1\hat{\boldsymbol{e}}_3 + \delta^{33}\varepsilon_{312}A^1B^2\hat{\boldsymbol{e}}_3 \\
&= \varepsilon_{123}A^2B^3\hat{\boldsymbol{e}}_1 - \varepsilon_{123}A^3B^2\hat{\boldsymbol{e}}_1 - \varepsilon_{231}A^1B^3\hat{\boldsymbol{e}}_2 + \varepsilon_{231}A^3B^1\hat{\boldsymbol{e}}_2 \\
&\quad - \varepsilon_{312}A^2B^1\hat{\boldsymbol{e}}_3 + \varepsilon_{312}A^1B^2\hat{\boldsymbol{e}}_3 \\
&= \varepsilon_{123}(A^2B^3 - A^3B^2)\hat{\boldsymbol{e}}_1 + \varepsilon_{231}(A^3B^1 - A^1B^3)\hat{\boldsymbol{e}}_2 \\
&\quad + \varepsilon_{312}(A^1B^2 - A^2B^1)\hat{\boldsymbol{e}}_3 \\
&= (A^2B^3 - A^3B^2)\hat{\boldsymbol{e}}_1 + (A^3B^1 - A^1B^3)\hat{\boldsymbol{e}}_2 + (A^1B^2 - A^2B^1)\hat{\boldsymbol{e}}_3
\end{aligned}$$

Where we only wrote the non-zero terms and we used the rules of the Levi-Civita symbol.
Equations (1.100) and (1.101) are essentially the same calculations.

3. Equation (1.106) states that

$$\varepsilon_{ijk}\varepsilon^{mnl} = \delta_i^m(\delta_j^n\delta_k^l - \delta_j^l\delta_k^n) - \delta_i^n(\delta_j^m\delta_k^l - \delta_j^l\delta_k^m) + \delta_i^l(\delta_j^m\delta_k^n - \delta_j^n\delta_k^m)$$

Then if $l = k$ we get that

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon^{mnk} &= \delta_i^m(\delta_j^n\delta_k^k - \delta_j^k\delta_k^n) - \delta_i^n(\delta_j^m\delta_k^k - \delta_j^k\delta_k^m) + \delta_i^k(\delta_j^m\delta_k^n - \delta_j^n\delta_k^m) \\
&= \delta_i^m(3\delta_j^n - \delta_j^n) - \delta_i^n(3\delta_j^m - \delta_j^m) + \delta_i^k(\delta_j^m\delta_k^n - \delta_j^n\delta_k^m) \\
&= 2\delta_i^m\delta_j^n - 2\delta_i^n\delta_j^m + \delta_i^k\delta_j^m\delta_k^n - \delta_i^k\delta_j^n\delta_k^m \\
&= 2\delta_i^m\delta_j^n - 2\delta_i^n\delta_j^m + \delta_j^m\delta_i^n - \delta_j^n\delta_i^m \\
&= \delta_i^m\delta_j^n - \delta_i^n\delta_j^m
\end{aligned}$$

If $n = j$ and $l = k$ we get that

$$\varepsilon_{ijk}\varepsilon^{mjk} = \delta_i^m\delta_j^j - \delta_i^j\delta_j^m = 3\delta_i^m - \delta_i^m = 2\delta_i^m$$

And if $m = i$, $n = j$ and $l = k$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 2\delta_i^i = 2 \cdot 3 = 6$$

4. Our guess for the formula equivalent to (1.106) for the case of the two-dimensional Levi-Civita symbol is

$$\varepsilon_{ij}\varepsilon^{mn} = \delta_i^m \delta_j^n - \delta_i^n \delta_j^m$$

We check this guess by computing the components.

$$\begin{aligned}
\varepsilon_{12}\varepsilon^{12} &= 1 = 1 - 0 = \delta_1^1 \delta_2^2 - \delta_1^2 \delta_2^1 \\
\varepsilon_{21}\varepsilon^{12} &= -1 = 0 - 1 = \delta_2^1 \delta_1^2 - \delta_2^2 \delta_1^1 \\
\varepsilon_{12}\varepsilon^{21} &= -1 = 0 - 1 = \delta_1^2 \delta_2^1 - \delta_1^1 \delta_2^2 \\
\varepsilon_{21}\varepsilon^{21} &= 1 = 1 - 0 = \delta_2^2 \delta_1^1 - \delta_2^1 \delta_1^2
\end{aligned}$$

For the components where $i = j$ we have that

$$\varepsilon_{ii}\varepsilon^{mn} = 0 = \delta_i^m \delta_i^n - \delta_i^n \delta_i^m$$

And for the components where $m = n$ we get that

$$\varepsilon_{ij}\varepsilon^{mm} = 0 = \delta_i^m \delta_j^m - \delta_i^m \delta_j^m$$

Therefore our guess is correct.
Let us compute the identities similar to (1.107) as follows

$$\begin{aligned}
\varepsilon_{ij}\varepsilon^{mj} &= \delta_i^m \delta_j^j - \delta_i^j \delta_j^m = 3\delta_i^m - \delta_i^m = 2\delta_i^m \\
\varepsilon_{ij}\varepsilon^{ij} &= 2\delta_i^i = 6
\end{aligned}$$

$\square$

Solution. **Exercise 1.15**
Let A^{ij} be any matrix then we can write it as

$$A^{ij} = \frac{1}{2}(A^{ij} + A^{ji}) + \frac{1}{2}(A^{ij} - A^{ji})$$

Let us consider the two terms as independent matrices.
If we define $B^{ij} = \frac{1}{2}(A^{ij} + A^{ji})$ we see that

$$B^{ji} = \frac{1}{2}(A^{ji} + A^{ij}) = \frac{1}{2}(A^{ij} + A^{ji}) = B^{ij}$$

So we see that B^{ij} is a totally symmetric matrix.
On the other hand, let us define $C^{ij} = \frac{1}{2}(A^{ij} - A^{ji})$, then we see that

$$C^{ji} = \frac{1}{2}(A^{ji} - A^{ij}) = -\frac{1}{2}(A^{ij} - A^{ji}) = -C^{ij}$$

So C^{ij} is a totally antisymmetric matrix.
Therfore we may write A^{ij} as

$$A^{ij} = A^{(ij)} + A^{[ij]}$$

Where $B^{ij} = A^{(ij)}$ and $C^{ij} = A^{[ij]}$ and hence

$$A^{(ij)} = \frac{1}{2}(A^{ij} + A^{ji}) \qquad A^{[ij]} = \frac{1}{2}(A^{ij} - A^{ji})$$

$\square$

Solution. **Exercise 1.16**
To decompose the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & -3 \\ 4 & 3 & 3 \end{bmatrix}$$

We need to compute each element of the totally symmetric matrix A_s and the totally antisymmetric matrix A_a, for this we use the equations (1.113) as follows

$$A_s^{(11)} = \frac{1}{2}(A_s^{11} + A_s^{11}) = \frac{1}{2}(1+1) = 1$$
$$A_s^{(12)} = \frac{1}{2}(A_s^{12} + A_s^{21}) = \frac{1}{2}(1-3) = -1$$
$$A_s^{(13)} = \frac{1}{2}(A_s^{13} + A_s^{31}) = \frac{1}{2}(2+4) = 3$$

Given that A_s is totally symmetric we know that $A_s^{(21)} = A_s^{(12)}$ and $A_s^{(31)} = A_s^{(13)}$ hence the only element left to compute is $A_s^{(23)} = A_s^{(32)}$

$$A_s^{(23)} = \frac{1}{2}(A_s^{23} + A_s^{32}) = \frac{1}{2}(-3+3) = 0$$

Then the totally symmetric matrix is

$$A_s = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

For the totally antisymmetric matrix A_a we compute what follows

$$A_a^{[11]} = \frac{1}{2}(A_a^{11} - A_a^{11}) = \frac{1}{2}(1-1) = 0$$
$$A_a^{[12]} = \frac{1}{2}(A_a^{12} - A_a^{21}) = \frac{1}{2}(1+3) = 2$$
$$A_a^{[13]} = \frac{1}{2}(A_a^{13} - A_a^{31}) = \frac{1}{2}(2-4) = -1$$

And computing the element $A_a^{[23]}$ we have all we need to get A_a

$$A_a^{[23]} = \frac{1}{2}(A_a^{23} - A_a^{32}) = \frac{1}{2}(-3-3) = -3$$

Then the matrix A_a is

$$A_a = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$$

Finally, we can compute the sum $A_s + A_a$ to check that the matrices are correct, we should get A.

$$A_s + A_a = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & -3 \\ 4 & 3 & 3 \end{bmatrix} = A$$

$\square$

Solution. **Exercise 1.17**

1. Let $\varepsilon_{ijk}A^k = B_{ij}$ then multiplying both sides by ε^{ijl} we get that

$$\begin{aligned}\varepsilon_{ijk}\varepsilon^{ijl}A^k &= \varepsilon^{ijl}B_{ij}\\ 2\delta^l_k A^k &= \varepsilon^{ijl}B_{ij}\\ 2A^l &= \varepsilon^{ijl}B_{ij}\\ A^l &= \frac{1}{2}\varepsilon^{ijl}B_{ij}\end{aligned}$$

2. Let now A^{jk} be totally antisymmetric and let also $\varepsilon_{ijk}A^{jk} = B_i$ then

$$\begin{aligned}\varepsilon^{inm}\varepsilon_{ijk}A^{jk} &= \varepsilon^{inm}B_i\\ (\delta^n_j\delta^m_k - \delta^n_k\delta^m_j)A^{jk} &= \varepsilon^{inm}B_i\\ \delta^n_j\delta^m_k A^{jk} - \delta^n_k\delta^m_j A^{jk} &= \varepsilon^{inm}B_i\\ A^{nm} - A^{mn} &= \varepsilon^{inm}B_i\end{aligned}$$

   But A is totally antisymmetric so $A^{mn} = -A^{nm}$ and hence

$$\begin{aligned}A^{nm} + A^{nm} &= \varepsilon^{inm}B_i\\ A^{nm} &= \frac{1}{2}\varepsilon^{inm}B_i\end{aligned}$$

3. Let now A^{jk} be totally symmetric and let also $\varepsilon_{ijk}A^{jk} = B_i$ then

$$\begin{aligned}\varepsilon^{inm}\varepsilon_{ijk}A^{jk} &= \varepsilon^{inm}B_i\\ \varepsilon^{nmi}\varepsilon_{jki}A^{jk} &= \varepsilon^{inm}B_i\\ (\delta^n_j\delta^m_k - \delta^m_j\delta^n_k)A^{jk} &= \varepsilon^{inm}B_i\\ \delta^n_j\delta^m_k A^{jk} - \delta^m_j\delta^n_k A^{jk} &= \varepsilon^{inm}B_i\\ A^{nm} - A^{mn} &= \varepsilon^{inm}B_i\\ A^{nm} - A^{nm} &= \varepsilon^{inm}B_i\\ 0 &= B_i\end{aligned}$$

   Where we used that A is totally symmetric so $A^{mn} = A^{nm}$, then since $B_i = 0$ we have that $\varepsilon_{ijk}A^{jk} = 0$.

4. If A^{jk} is neither totally symmetric nor totally antisymmetric then we get that

$$A^{nm} - A^{mn} = \varepsilon^{inm}B_i$$

But we can write A^{nm} as the sum of a totally symmetric and a totally antisymmetric matrix so

$$\begin{aligned}
A^{(nm)} + A^{[nm]} - A^{(mn)} - A^{[mn]} &= \varepsilon^{inm} B_i \\
A^{(nm)} + A^{[nm]} - A^{(nm)} + A^{[nm]} &= \varepsilon^{inm} B_i \\
2A^{[nm]} &= \varepsilon^{inm} B_i \\
A^{[nm]} &= \frac{1}{2}\varepsilon^{inm} B_i
\end{aligned}$$

So as we saw in part 2 the antisymmetric matrix will be of the form $\frac{1}{2}\varepsilon^{inm}B_i$ but we cannot say anything about the symmetric part.

5. Let $\delta^{ik}\varepsilon_{ijn}A^n = B_j^k$ then

$$\begin{aligned}
\delta^{ik}\varepsilon_{ijn}A^n &= B_j^k \\
\varepsilon^k{}_{jn}A^n &= B_j^k \\
\varepsilon_k{}^{jl}\varepsilon^k{}_{jn}A^n &= \varepsilon_k{}^{jl}B_j^k \\
2\delta_n^l A^n &= \varepsilon_k{}^{jl}B_j^k \\
A^l &= \frac{1}{2}\varepsilon_k{}^{jl}B_j^k
\end{aligned}$$

□