Solutions to selected problems on Differential Geometry and Lie Groups - Gallier.

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Solutions to selected problems on Differential Geometry and Lie Groups from different books, but mainly from Gallier.

Chapter 2 - The Matrix Exponential: Some Matrix and Lie Groups

2.1 The Exponential Map

Proof. **Problem 2.1.**

(a) Let the following symmetric matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Then AB is given by

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2+0 & 4+0 \\ 0+1 & 0+1/2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 1/2 \end{pmatrix}$$

Which is non-symmetric.

(b) Let the following skew symmetric matrices

$$A = \pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \pi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

We can work out a explicit formula for e^A and e^B by using Rodrigues formula. For A we see that A has the required form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

taking $a = \pi$ and $b = c = 0$ we have that $\theta = \pi$. Hence by Rodrigues formula we have that

$$\begin{aligned}
e^A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\pi^2-2}{\pi^2} & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix}
\end{aligned}$$

In the case of B we have that $b = \pi$ and $a = c = 0$, hence $\theta = \pi$ then

$$\begin{aligned}
e^B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\pi^2-2}{\pi^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix}
\end{aligned}$$

Now, we compute $e^A e^B$ as follows

$$\begin{aligned}
e^A e^B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\pi^2-2}{\pi^2} & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix} \begin{pmatrix} \frac{\pi^2-2}{\pi^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\pi^2-2}{\pi^2} & 0 & 0 \\ 0 & \frac{\pi^2-2}{\pi^2} & 0 \\ 0 & 0 & \left(\frac{\pi^2-2}{\pi^2}\right)^2 \end{pmatrix}
\end{aligned}$$

Finally, we compute e^{A+B} using Rodrigues formula as well since the matrix $A + B$ is skew symmetric as we want, where $a = b = \pi$ and

$c = 0$ then we have that $\theta = \sqrt{2}\pi$ and hence

$$e^{A+B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} + \frac{1-\cos(\sqrt{2}\pi)}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}^2$$

$$e^{A+B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} + \frac{1-\cos(\sqrt{2}\pi)}{2} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$e^{A+B} = \begin{pmatrix} 1 - \frac{1-\cos(\sqrt{2}\pi)}{2} & \frac{1-\cos(\sqrt{2}\pi)}{2} & \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \\ \frac{1-\cos(\sqrt{2}\pi)}{2} & 1 - \frac{1-\cos(\sqrt{2}\pi)}{2} & -\frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \\ -\frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} & \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} & \cos(\sqrt{2}\pi) \end{pmatrix}$$

Therefore we see that $e^A e^B \neq e^{A+B}$.

(c) Let the following matrices

$$A = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix}$$

We see that

$$AB = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4\pi^2 & 2\pi i \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4\pi^2 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence $AB \neq BA$.

Now, we need to compute e^A so first we compute $A^2, A^3, \ldots$ as follows

$$A^2 = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -4\pi^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} -8i\pi^3 & 0 \\ 0 & 0 \end{pmatrix}$$

So we see that

$$A^n = (2\pi i)^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (2\pi i)^n J_A$$

Where

$$J_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$\begin{aligned} e^A &= I_2 + \left((2\pi i) J_A + \frac{(2\pi i)^2}{2!} J_A + \frac{(2\pi i)^3}{3!} J_A + \frac{(2\pi i)^4}{4!} J_A + \ldots \right) \\ &= I_2 + \sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} J_A \\ &= I_2 \end{aligned}$$

To compute e^B we follow the same path, we see that

$$B^2 = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -4\pi^2 & 2\pi i \\ 0 & 0 \end{pmatrix}$$
$$B^3 = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} -8i\pi^3 & -4\pi^2 \\ 0 & 0 \end{pmatrix}$$

So

$$B^n = (2\pi i)^n \begin{pmatrix} 1 & 1/(2\pi i) \\ 0 & 0 \end{pmatrix} = (2\pi i)^n J_B$$

Where

$$J_B = \begin{pmatrix} 1 & 1/(2\pi i) \\ 0 & 0 \end{pmatrix}$$

Hence

$$\begin{aligned} e^B &= I_2 + \left((2\pi i) J_B + \frac{(2\pi i)^2}{2!} J_B + \frac{(2\pi i)^3}{3!} J_B + \frac{(2\pi i)^4}{4!} J_B + ... \right) \\ &= I_2 + \sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} J_B \\ &= I_2 \end{aligned}$$

Therefore

$$e^A e^B = I_2 \cdot I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But also we have that

$$A + B = \begin{pmatrix} 4\pi i & 1 \\ 0 & 0 \end{pmatrix}$$
$$(A+B)^2 = \begin{pmatrix} 4\pi i & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -16\pi^2 & 4\pi i \\ 0 & 0 \end{pmatrix}$$
$$(A+B)^3 = \begin{pmatrix} 4\pi i & 1 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} -64\pi^3 i & -16\pi^2 \\ 0 & 0 \end{pmatrix}$$

So we see that

$$(A+B)^n = (4\pi i)^n \begin{pmatrix} 1 & 1/(4\pi i) \\ 0 & 0 \end{pmatrix} = (4\pi i)^n J_{A+B}$$

Where

$$J_{A+B} = \begin{pmatrix} 1 & 1/(4\pi i) \\ 0 & 0 \end{pmatrix}$$

Hence

$$\begin{aligned} e^{A+B} &= I_2 + \left((4\pi i) J_{A+B} + \frac{(4\pi i)^2}{2!} J_{A+B} + \frac{(4\pi i)^3}{3!} J_{A+B} + \frac{(4\pi i)^4}{4!} J_{A+B} + ... \right) \\ &= I_2 + \sum_{n=1}^{\infty} \frac{(4\pi i)^n}{n!} J_{A+B} \\ &= I_2 \end{aligned}$$

Therefore we obtain that $e^A e^B = e^{A+B}$.

$\square$

Proof. **Hubbard: Problem 1.5.10**

a. Let

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + ...$$

Because of Proposition 2.1 from Gallier we know that the series is absolutely convergent this implies that the original series is convergent as well.

Also, it is shown that the series $|e^A|$ is bounded by the following series

$$e^{n\mu} = \sum_{k=0}^{\infty} \frac{(n\mu)^k}{k!}$$

where $\mu = \max\{|a_{ij}| : 1 \leq i, j \leq n\}$ and $A = (a_{ij})$.

b. i. Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^3 = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

$$\dots$$

$$A^k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

So we have that

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{a^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{b^k}{k!} \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

ii. Let

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

In this case, the matrix has a null trace so according to Gallier we can compute e^A as follows

$$e^A = I_2 + A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

iii. Let

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = -a \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then from Gallier we know that

$$e^A = \begin{bmatrix} \cos(-a) & -\sin(-a) \\ \sin(-a) & \cos(-a) \end{bmatrix} = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix}$$

c. 1. Let

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$$

Such that $a, b > 0$. Then

$$e^A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad e^B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

Because they are null trace matrices. So

$$e^A e^B = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 + ab & a \\ b & 1 \end{bmatrix}$$

But

$$A + B = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

Hence by what we saw on Gallier we get that

$$\begin{aligned} e^{A+B} &= \cosh(\sqrt{ab}) I_2 + \frac{\sinh(\sqrt{ab})}{\sqrt{ab}}(A + B) \\ &= \begin{bmatrix} \cosh(\sqrt{ab}) & 0 \\ 0 & \cosh(\sqrt{ab}) \end{bmatrix} + \begin{bmatrix} 0 & \frac{a \sinh(\sqrt{ab})}{\sqrt{ab}} \\ \frac{b \sinh(\sqrt{ab})}{\sqrt{ab}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cosh(\sqrt{ab}) & \frac{a \sinh(\sqrt{ab})}{\sqrt{ab}} \\ \frac{b \sinh(\sqrt{ab})}{\sqrt{ab}} & \cosh(\sqrt{ab}) \end{bmatrix} \end{aligned}$$

Therefore $e^{A+B} \neq e^A e^B$.

2. We proved this on Gallier Proposition 2.5.
3. Let A be an $n \times n$ matrix. Using the sub-problem number 2 and given that A commutes with itself we have that

$$e^{A+A} = e^A e^A$$

i.e.

$$e^{2A} = (e^A)^2$$

$\square$

Proof. **Hall: 2.6.3** Let $A(t)$ and $B(t)$ be two smooth matrix-valued functions then by definition we have that

$$\begin{aligned}
\frac{d}{dt}[A(t)B(t)] &= \lim_{h\to 0}\frac{A(t+h)B(t+h)-A(t)B(t)}{h}\\
&= \lim_{h\to 0}\frac{A(t+h)B(t+h)-A(t)B(t)+A(t)B(t+h)-A(t)B(t+h)}{h}\\
&= \lim_{h\to 0}\frac{B(t+h)(A(t+h)-A(t))+A(t)(B(t+h)-B(t))}{h}\\
&= \lim_{h\to 0}\frac{B(t+h)(A(t+h)-A(t))}{h}+\lim_{h\to 0}\frac{A(t)(B(t+h)-B(t))}{h}\\
&= B(t)\frac{dA}{dt}+A(t)\frac{dB}{dt}
\end{aligned}$$

Now we want to prove that $A(t)B(t)$ is smooth, we see that the element jk of $A(t)B(t)$ is given by

$$(A(t)B(t))_{jk} = \sum_{i=0}^{n} A_{ji}(t)B_{ik}(t)$$

Given that $A_{ji}(t)$ and $B_{ik}(t)$ are smooth functions from $\mathbb{R}$ to $\mathbb{R}$ then the product $A_{ji}(t)B_{ik}(t)$ is smooth and the sum of smooth functions is also smooth. Hence $(A(t)B(t))_{jk}$ is smooth and therefore $A(t)B(t)$ is smooth. $\square$

Proof. **Hall: 2.6.4** Let A be an $n \times n$ complex matrix. If A is diagonalizable then we have that $\lim_{m\to\infty} A = A$ and hence we are done.

Suppose A is not diagonalizable, then we know it's similar to an upper triangular matrix U i.e.

$$A = PUP^{-1}$$

Then U has the form

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

Where $u_{11}, u_{22}, \dots, u_{nn}$ are the eigenvalues of U. Given that A is not diagonalizable then U is not diagonalizable then one or more eigenvalues are equal. Suppose we modify the eigenvalues of U as follows, we take u_{22} if it's equal to u_{11} we modify it as $u_{22} - 1/m$ then we take u_{33} if it's equal to u_{11} or u_{22} we modify it as $u_{33} - 2/m$ and we continue this process until u_{nn}. In the worst case, where all the eigenvalues are equal we end up with a matrix U_m as follows

$$U_m = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} - \frac{1}{m} & \dots & u_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & u_{nn} - \frac{n-1}{m} \end{bmatrix}$$

We see that U_m for any $m \in \mathbb{N}$ will be diagonalizable but also we see that $\lim_{m\to\infty} U_m = U$ hence

$$\lim_{m\to\infty} PU_mP^{-1} = PUP^{-1} = A$$

Therefore A is the limit of a sequence of diagonalizable matrices. $\square$

Proof. **Hall: 2.6.5** Let $a \neq d$ then

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2 &= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\
&= \begin{pmatrix} a^2 & ab + bd \\ 0 & d^2 \end{pmatrix} \\
&= \begin{pmatrix} a^2 & b(a+d)\frac{a-d}{a-d} \\ 0 & d^2 \end{pmatrix} \\
&= \begin{pmatrix} a^2 & b\frac{a^2-d^2}{a-d} \\ 0 & d^2 \end{pmatrix}
\end{aligned}$$

Also, we see that

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^3 &= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2 \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\
&= \begin{pmatrix} a^2 & b\frac{a^2-d^2}{a-d} \\ 0 & d^2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\
&= \begin{pmatrix} a^3 & a^2b + bd\frac{a^2-d^2}{a-d} \\ 0 & d^3 \end{pmatrix} \\
&= \begin{pmatrix} a^3 & b\frac{a^2(a-d)+da^2-d^3}{a-d} \\ 0 & d^3 \end{pmatrix} \\
&= \begin{pmatrix} a^3 & b\frac{a^3-d^3}{a-d} \\ 0 & d^3 \end{pmatrix}
\end{aligned}$$

We can continue this process m times so we can write that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \begin{pmatrix} a^m & b\frac{a^m-d^m}{a-d} \\ 0 & d^m \end{pmatrix}$$

Now we compute $\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ as follows

$$\begin{aligned}
\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a^k & b\frac{a^k-d^k}{a-d} \\ 0 & d^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & \frac{b}{a-d} \sum_{k=0}^{\infty} \frac{1}{k!}(a^k - d^k) \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} d^k \end{pmatrix} \\
&= \begin{pmatrix} e^a & b\frac{e^a-e^d}{a-d} \\ 0 & e^d \end{pmatrix}
\end{aligned}$$

On the other hand, in the case $a = d$ we have that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^2 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}$$

And that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^3 = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{pmatrix}$$

So we can write that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^k = \begin{pmatrix} a^k & ka^{k-1}b \\ 0 & a^k \end{pmatrix}$$

Then $\exp \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ in this case gives us

$$\begin{aligned}
\exp \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a^k & ka^{k-1}b \\ 0 & a^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & b \sum_{k=0}^{\infty} \frac{ka^{k-1}}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} a^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & b \sum_{k=0}^{\infty} \frac{a^{k-1}}{(k-1)!} \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} a^k \end{pmatrix} \\
&= \begin{pmatrix} e^a & be^a \\ 0 & e^a \end{pmatrix}
\end{aligned}$$

Which also matches with the equation we determined before assuming that

$$\lim_{a \to d} \frac{e^a - e^d}{a - d} = e^a$$

Therefore for any $a, b, d \in \mathbb{C}$ we have that

$$\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} e^a & b \frac{e^a - e^d}{a - d} \\ 0 & e^d \end{pmatrix}$$

$\square$

Proof. **Hall: 2.6.6** From Gallier we know that if a matrix X has null trace then X has the form

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

And we saw that these matrices satisfy that

$$X^2 = (a^2 + bc)I = -\det(X)I$$

We saw that if $a^2 + bc = -\det(X) = 0$ then

$$e^X = I + X$$

which matches with the equation

$$e^X = \cos(\sqrt{\det x})I + \frac{\sin(\sqrt{\det X})}{\sqrt{\det X}}X$$

Where we used that $\cos(0) = 1$ and $\lim_{\theta \to 0} \sin\theta/\theta = 1$.
Also, we know that if $a^2 + bc = -\det(X) < 0$ then

$$e^X = \cos(\sqrt{\det X})I + \frac{\sin(\sqrt{\det X})}{\sqrt{\det X}}X$$

which is the equation we have.
Finally, if $a^2 + bc = -\det(X) > 0$ i.e. $\sqrt{\det(X)}$ is complex by Gallier we know that

$$\begin{aligned} e^X &= \cosh(\sqrt{\det x})I + \frac{\sinh(\sqrt{\det X})}{\sqrt{\det X}}X \\ &= \cos(\sqrt{\det x})I + \frac{i\sin(\sqrt{\det X})}{\sqrt{\det X}}X \end{aligned}$$

So for any value of $\det(X)$ the exponential of X is given by the equation mentioned above.
Let

$$X_1 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

Then given that $\det(X) = a^2$ we have that

$$\begin{aligned} e^{X_1} &= \cos a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin a}{a} \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos a & 0 \\ 0 & \cos a \end{pmatrix} + \begin{pmatrix} 0 & -\sin a \\ \sin a & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \end{aligned}$$

□

Proof. **Hall: 2.6.7** Let

$$X = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$$

Then we can write X as $X = A + B$ where

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$$

We see that they commute cause A can be written as $A = 3I_2$ and I_2 commutes with every matrix then we can compute e^X as follows

$$e^X = e^{A+B} = e^A e^B$$

First, we need to compute e^A

$$e^A = \sum_{k\geq 0} \frac{(3I_2)^k}{k!} = I_2 \sum_{k\geq 0} \frac{3^k}{k!} = e^3 I_2$$

Now, we compute e^B by using what we have from problem 2.6.6 since B has $tr(B) = 0$ and $\det(B) = 2$ we have that

$$\begin{aligned} e^B &= \cos(\sqrt{\det B}) I_2 + \frac{\sin(\sqrt{\det B})}{\sqrt{\det B}} B \\ &= \cos(\sqrt{2}) I_2 + \frac{\sin(\sqrt{2})}{\sqrt{2}} B \end{aligned}$$

Finally we have that

$$\begin{aligned} e^X &= e^A e^B \\ &= (e^3 I_2)(\cos(\sqrt{2}) I_2 + \frac{\sin(\sqrt{2})}{\sqrt{2}} B) \\ &= e^3 \cos(\sqrt{2}) I_2 + e^3 \frac{\sin(\sqrt{2})}{\sqrt{2}} B \\ &= \begin{pmatrix} e^3(\cos(\sqrt{2}) + \frac{\sin(\sqrt{2})}{\sqrt{2}}) & 3e^3 \frac{\sin(\sqrt{2})}{\sqrt{2}} \\ -e^3 \frac{\sin(\sqrt{2})}{\sqrt{2}} & e^3(\cos(\sqrt{2}) - \frac{\sin(\sqrt{2})}{\sqrt{2}}) \end{pmatrix} \end{aligned}$$

$\square$

Proof. **Hall: 2.6.9**

(a) Let A be unipotent then $A - I$ is nilpotent hence there is $k \in \mathbb{N}$ such that $(A - I)^k = 0$. Also, by definition we know that

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - I)^m}{m}$$

Then we can write that

$$\log A = (A - I) - \frac{(A - I)^2}{2} + \frac{(A - I)^3}{3} - ... + (-1)^{k+1} \frac{(A - I)^k}{k}$$

If we compute $(\log A)^2$ the smallest power of $A - I$ is $(A - I)^2$ but also we get $A - I$ powered to $k + 1$, $k + 2$, ..., $2k$ which are all zero cause $A - I$ is nilpotent. Then if we compute $(\log A)^k$ the smallest power of $A - I$ is going to be $(A - I)^k$ which is zero. Therefore $(\log A)^k = 0$ and hence $\log A$ is nilpotent.

(b) Let X be nilpotent then there is $k \in \mathbb{N}$ such that $X^k = 0$. Also, by definition we know that

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

But since X is nilpotent we can write that

$$\begin{aligned} e^X &= \sum_{m=0}^{k} \frac{X^m}{m!} \\ &= I + X + \frac{X^2}{2} + ... + \frac{X^k}{k!} \end{aligned}$$

Then $e^X - I$ is

$$e^X - I = X + \frac{X^2}{2} + ... + \frac{X^k}{k!}$$

If we compute $(e^X - I)^2$ the smallest power of X is X^2 but also we get X powered to $k + 1$, $k + 2$, ..., $2k$ which are all zero cause X is nilpotent. Then if we compute $(e^X - I)^k$ the smallest power of X is going to be X^k which is zero. Therefore $(e^X - I)^k = 0$ and hence $e^X - I$ is nilpotent which implies that e^X is unipotent.

(c) Let $A(t) = I + t(A - I)$ be unipotent then $A(t) - I = t(A - I)$ is nilpotent hence there is some $k \in \mathbb{N}$ for which $t^k(A - I)^k = 0$ then

$$\log(A(t)) = \sum_{m=1}^{k} (-1)^{m+1} \frac{t^m (A - I)^m}{m}$$

From part (a) we know that $\log(A(t))$ is nilpotent so there is some $j \in \mathbb{N}$ such that $(\log(A(t)))^j = 0$. Hence we can write that

$$\exp(\log(A(t))) = I + \sum_{n=1}^{j} \frac{1}{n!} \left(\sum_{m=1}^{k} (-1)^{m+1} \frac{t^m (A - I)^m}{m} \right)^n$$

Then $\exp(\log(A(t)))$ depends polynomially on t and for a sufficiently small t we can get that $\|A(t) - I\| < 1$ then we can apply Theorem 2.8 and therefore

$$\exp(\log(A(t))) = A(t)$$

So we have proven that the polynomial $\exp(\log(A(t)))$ is equal to the polynomial $A(t)$ on an interval close to 0 then they must be equal. This implies that the result we got must be true for $t = 1$ hence

$$\exp(\log(A)) = \exp(\log(A(1))) = I + 1(A - I) = A$$

In the same way, let now $X(t) = tX$ be nilpotent then there is some $k \in \mathbb{N}$ such that $X(t)^k = (tX)^k = 0$ then

$$\exp(X(t)) = \sum_{m=0}^{k} \frac{t^m X^m}{m!}$$

From part (b) we know that $\exp(X(t))$ is unipotent so there is some $j \in \mathbb{N}$ such that $(\exp(X(t)) - I)^j = 0$. Hence we can write that

$$\log(\exp(X(t))) = \sum_{n=1}^{j} \frac{(-1)^{n+1}}{n} \left(\sum_{m=0}^{k} \frac{t^m X^m}{m!} - I \right)^n$$

Again $\log(\exp(X(t)))$ depends polynomially on t and for a sufficiently small t we can get that $\|X(t)\| < \log 2$ and hence $\|e^X - I\| < 1$ then we can apply Theorem 2.8 and therefore

$$\log(\exp(X(t))) = X(t)$$

Finally, again because the two polynomials are equal on an interval close to 0 then must be that

$$\log(\exp(X)) = \log(\exp(X(1))) = 1X = X$$

$\square$

Proof. **Hall: 2.6.10** Let A be an invertible $n \times n$ matrix then A is similar to a block-diagonal matrix D where each block is of the form $\lambda I + N_\lambda$ with N_λ nilpotent. Then

$$A = PDP^{-1}$$

for some invertible matrix P. Let D' be a matrix defined as

$$D' = \begin{pmatrix} \log(\lambda_1 I(I + N_{\lambda_1}/\lambda_1)) & 0 & \dots & 0 \\ 0 & \log(\lambda_2 I(I + N_{\lambda_2}/\lambda_2)) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \log(\lambda_n I(I + N_{\lambda_n}/\lambda_n)) \end{pmatrix}$$
$$= \begin{pmatrix} \log(\lambda_1 I) + \log(I + N_{\lambda_1}/\lambda_1) & 0 & \dots & 0 \\ 0 & \log(\lambda_2 I) + \log(I + N_{\lambda_2}/\lambda_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \log(\lambda_n I) + \log(I + N_{\lambda_n}/\lambda_n) \end{pmatrix}$$

Where we used that $\lambda_1 I$ and $I + N_{\lambda_1}/\lambda_1$ commutes. So

$$e^{D'} = \begin{pmatrix} \exp(\log(\lambda_1 I) + \log(I + N_{\lambda_1}/\lambda_1)) & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \exp(\log(\lambda_n I) + \log(I + N_{\lambda_n}/\lambda_n)) \end{pmatrix}$$
$$= \begin{pmatrix} \exp(\log(\lambda_1 I)) \exp(\log(I + N_{\lambda_1}/\lambda_1)) & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \exp(\log(\lambda_n I)) \exp(\log(I + N_{\lambda_n}/\lambda_n)) \end{pmatrix}$$

Since $I + N_{\lambda_i}/\lambda_i$ is unipotent we know that $\exp(\log(I + N_{\lambda_i}/\lambda_i)) = I + N_{\lambda_i}/\lambda_i$. And for $\exp(\log(\lambda_i I))$ we see that

$$\begin{aligned} \exp(\log(\lambda_i I)) &= \exp\left(\sum_{m=1}^{\infty} (-1)^{m+1} \frac{(\lambda_i I - I)^m}{m} \right) \\ &= \exp\left(\sum_{m=1}^{\infty} (-1)^{m+1} \frac{(\lambda_i - 1)^m}{m} \, I \right) \\ &= \exp(\log(\lambda_i) I) \\ &= \sum_{m=0}^{\infty} \frac{(\log(\lambda_i) I)^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{\log(\lambda_i)^m}{m!} \, I \\ &= \exp(\log(\lambda_i)) I \\ &= \lambda_i I \end{aligned}$$

Then

$$e^{D'} = \begin{pmatrix} \lambda_1 I(I + N_{\lambda_1}/\lambda_1) & 0 & ... & 0 \\ 0 & \lambda_2 I(I + N_{\lambda_2}/\lambda_2) & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \lambda_n I(I + N_{\lambda_n}/\lambda_n) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 I + N_{\lambda_1} & 0 & ... & 0 \\ 0 & \lambda_2 I + N_{\lambda_2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \lambda_n I + N_{\lambda_n} \end{pmatrix}$$

Hence $D = e^{D'}$ and therefore

$$A = PDP^{-1} = Pe^{D'}P^{-1} = e^{PD'P^{-1}}$$

Which implies that A can be written as $A = e^X$ where $X = PD'P^{-1}$.

$\square$

2.2 The Lie Groups $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$, the Lie Algebras $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{o}(n)$, $\mathfrak{so}(n)$, and the Exponential Map

Proof. **Problem 2.2.** Let

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$$

also, let $\omega^2 = a^2 + bc$ such that $\omega \neq 0$. We see that B^2 is

$$B^2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix} = (a^2+bc)I_2$$

Then

$$\begin{aligned} e^B &= I_2 + \frac{B}{1!} + \frac{\omega^2}{2!}I_2 + \frac{\omega^2}{3!}B + \frac{\omega^4}{4!}I_2 + \frac{\omega^4}{5!}B + ... \\ &= \left(1 + \frac{\omega^2}{2!} + \frac{\omega^4}{4!} + ...\right)I_2 + \left(1 + \frac{\omega^2}{3!} + \frac{\omega^4}{5!} + ...\right)B \\ &= \left(1 + \frac{\omega^2}{2!} + \frac{\omega^4}{4!} + ...\right)I_2 + \left(\omega + \frac{\omega^3}{3!} + \frac{\omega^5}{5!} + ...\right)\frac{B}{\omega} \\ &= \cosh\omega I_2 + \frac{\sinh\omega}{\omega}B \end{aligned}$$

Suppose now that $\omega^2 = 0 = a^2 + bc$ then $B^2 = 0$ and hence

$$e^B = I_2 + \frac{B}{1!} = I_2 + B$$

Finally, suppose the matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is the exponential of a matrix in $\mathfrak{sl}(2,\mathbb{C})$, we want to arrive to a contradiction. Then must be that

$$\exp\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

If $\omega \neq 0$ where $\omega^2 = a^2 + bc$ then must be that

$$\begin{pmatrix} \cosh\omega + \frac{\sinh\omega}{\omega}a & \frac{\sinh\omega}{\omega}b \\ \frac{\sinh\omega}{\omega}c & \cosh\omega - \frac{\sinh\omega}{\omega}a \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then $c = 0$, $b = \frac{\omega}{\sin\omega}$ and

$$\begin{aligned} \cosh\omega + \frac{a}{b} &= -1 \\ \cosh\omega - \frac{a}{b} &= -1 \end{aligned}$$

But there is no solution to these equations. So there must be that $\omega = 0$ and hence

$$\begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then $c = 0$, $b = 1$ and

$$\begin{aligned} 1 + a &= -1 \\ 1 - a &= -1 \end{aligned}$$

But again there is no solution to these equations.
Therefore we arrived at a contradiction and must be that there is no exponential matrix A in $\mathfrak{sl}(2, \mathbb{C})$ such that

$$e^A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Which implies that the map $\exp : \mathfrak{sl}(2, \mathbb{C}) \to \boldsymbol{SL}(2, \mathbb{C})$ is not surjective. $\square$

Proof. **Problem 2.3.**

(a) Let $B = \log(I - N)$. We see that $(I - N) - I = -N$ is nilpotent, this implies that $I - N$ is unipotent then by the problem "Hall: 2.6.9(c)" we know that $\exp(\log(I - N)) = I - N$ therefore

$$\exp(B) = \exp(\log(I - N)) = I - N = A$$

(b) Let $A \in \boldsymbol{GL}(n, \mathbb{C})$ then A is a complex invertible $n \times n$ matrix and by "Hall: 2.6.10" we know there is $B \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^B = A$. Thus, the exponential map $\exp : \mathfrak{gl}(n, \mathbb{C}) \to \boldsymbol{GL}(n, \mathbb{C})$ is surjective.

$\square$

Proof. **Problem 2.4.**

(a) Let

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

Then

$$\begin{aligned}
A^2 &= \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \\
&= \begin{pmatrix} -c^2 - b^2 & ba & ac \\ ba & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{pmatrix} \\
&= \begin{pmatrix} a^2 - a^2 - b^2 - c^2 & ba & ac \\ ba & b^2 - a^2 - b^2 - c^2 & bc \\ ac & bc & c^2 - a^2 - b^2 - c^2 \end{pmatrix} \\
&= \begin{pmatrix} -a^2 - b^2 - c^2 & 0 & 0 \\ 0 & -a^2 - b^2 - c^2 & 0 \\ 0 & 0 & -a^2 - b^2 - c^2 \end{pmatrix} + \begin{pmatrix} a^2 & ba & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \\
&= (-a^2 - b^2 - c^2)I + B \\
&= -\theta^2 I + B
\end{aligned}$$

Where $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ba & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

On the other hand, we see that

$$\begin{aligned}
AB &= \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \cdot \begin{pmatrix} a^2 & ba & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \\
&= \begin{pmatrix} -abc + abc & -b^2c + b^2c & -bc^2 + bc^2 \\ a^2c - a^2c & abc - abc & ac^2 - ac^2 \\ -a^2b + a^2b & -ab^2 + ab^2 & -abc + abc \end{pmatrix} \\
&= 0
\end{aligned}$$

Also

$$\begin{aligned}
BA &= \begin{pmatrix} a^2 & ba & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \\
&= \begin{pmatrix} abc - abc & -a^2c + a^2c & ba^2 - ba^2 \\ b^2c - b^2c & -abc + abc & ab^2 - ab^2 \\ b^2c - b^2c & -ac^2 + ac^2 & abc - abc \end{pmatrix} \\
&= 0
\end{aligned}$$

Therefore $AB = BA = 0$. Finally, we compute A^3 as follows

$$A^3 = A(-\theta^2 I + B) = -A\theta^2 I + AB = -\theta^2 A$$

(b) This is proved in Proposition 2.7.

(c) Since $\det(e^A) = e^{\mathrm{tr}(A)}$ and we see that $\mathrm{tr}(A) = 0$ then

$$\det(e^A) = e^0 = 1$$

On the other hand, we know that $(e^A)^T = e^{A^T}$ also, $A^T = -A$ and hence A and A^T commute then

$$(e^A)^T e^A = e^{A^T} e^A = e^{A^T + A} = e^{-A+A} = e^0 = I$$

And

$$e^A (e^A)^T = e^A e^{A^T} = e^{A+A^T} = e^{A-A} = e^0 = I$$

Therefore e^A is an orthogonal matrix of determinant $+1$.

(d) (2) Let $R \in \mathbf{SO}(3)$ as

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And let $B \in \mathfrak{so}(3)$ such that B is of the form

$$\begin{aligned} B &= \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \ : \ 1 + 2\cos\theta = \mathrm{tr}(R) \right\} \\ &= \frac{\theta}{2\sin\theta} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{\theta}{2\sin\theta} \begin{pmatrix} 0 & -2\sin\phi & 0 \\ 2\sin\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Given that $1 + 2\cos\theta = \mathrm{tr}(R)$ we get that $1 + 2\cos\theta = 2\cos\phi + 1$ hence $\theta = \phi$ then we write B as

$$B = \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then B^2 is

$$B^2 = \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\theta^2 & 0 & 0 \\ 0 & -\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we compute e^B as follows

$$\begin{aligned}
e^B &= I_3 + \frac{\sin\theta}{\theta}B + \frac{(1-\cos\theta)}{\theta^2}B^2 \\
&= I_3 + \frac{\sin\theta}{\theta}\begin{pmatrix}0 & -\theta & 0\\ \theta & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{(1-\cos\theta)}{\theta^2}\begin{pmatrix}-\theta^2 & 0 & 0\\ 0 & -\theta^2 & 0\\ 0 & 0 & 0\end{pmatrix} \\
&= I_3 + \begin{pmatrix}0 & -\sin\theta & 0\\ \sin\theta & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}\cos\theta - 1 & 0 & 0\\ 0 & \cos\theta - 1 & 0\\ 0 & 0 & 0\end{pmatrix} \\
&= \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}
\end{aligned}$$

Therefore we have proven that $e^B = R$.

(3) Let now $R \neq I$ and $\mathrm{tr}(R) = -1$ then from the trace equation we see that

$$\begin{aligned}
2\cos\phi + 1 &= -1 \\
\cos\phi + 1 &= 0 \\
\phi &= \arccos(-1) = \pi
\end{aligned}$$

We can compute the eigenvalues of R from the characteristic polynomial as follows

$$\begin{aligned}
\det(R - \lambda I) &= 0 \\
\begin{vmatrix}\cos\phi - \lambda & -\sin\phi & 0\\ \sin\phi & \cos\phi - \lambda & 0\\ 0 & 0 & 1-\lambda\end{vmatrix} &= 0 \\
(1-\lambda)(\cos\phi - \lambda)^2 + \sin^2\phi(1-\lambda) &= 0 \\
(1-\lambda)(\cos^2\phi - 2\lambda\cos\phi + \lambda^2 + \sin^2\phi) &= 0 \\
(1-\lambda)(\lambda^2 - 2\lambda\cos\phi + 1) &= 0 \\
(1-\lambda)(\lambda^2 + 2\lambda + 1) &= 0
\end{aligned}$$

From the first factor we see that one eigenvalue of R is 1 and from the second factor we have a double eigenvalue at -1.
Given that $\phi = \pi$ then R becomes

$$R = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Then we see that $R = R^T$ and also R^2 is

$$R^2 = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = I_3$$

Let $S = \frac{1}{2}(R - I)$ then replacing R and solving we get that

$$S = \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So S is symmetric and since it's a diagonal matrix the eigenvalues of S are $-1, -1, 0$.
Suppose there is a skew symmetric matrix U defined as follows

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

Then U^2 is

$$\begin{aligned} U^2 &= \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix} \\ &= \begin{pmatrix} -c^2 - d^2 & bc & bd \\ bc & -b^2 - d^2 & cd \\ bd & cd & -b^2 - c^2 \end{pmatrix} \end{aligned}$$

So if $b = c = 0$ and $d = -1$ we get that

$$U^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = S$$

If $U^2 = S$ then we see that $\operatorname{tr}(S) = -2$ then must be that

$$\begin{aligned} -c^2 - d^2 - b^2 - d^2 - b^2 - c^2 &= -2 \\ -2b^2 - 2c^2 - 2d^2 &= -2 \\ b^2 + c^2 + d^2 &= 1 \end{aligned}$$

Finally, let

$$B = \left\{ (2k+1)\pi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : k \in \mathbb{Z} \right\}$$

We want to prove that $e^B = R$ then

$$\begin{aligned}
e^B &= I_3 + \frac{\sin\theta}{\theta} B + \frac{(1-\cos\theta)}{\theta^2} B^2 \\
&= I_3 + \frac{2}{\pi^2} B^2 \\
&= I_3 + \frac{2}{\pi^2} (2k+1)^2 \pi^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= (2k+1)^2 \begin{pmatrix} 1-2 & 0 & 0 \\ 0 & 1-2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Where we took $k = 0$ in the last step.

$\square$

Proof. **Duistermaat: 2.4** Let $A \in \mathrm{End}(\mathbb{R}^n)$ be an orthogonal transformation i.e. $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$

(i) Let $Ax = 0$ then $\|Ax\| = 0$ but this implies that $\|x\| = 0$ hence must be that $x = 0$. Then A^{-1} exists.

Now, let $x_1, x_2 \in \mathbb{R}^n$ and suppose $Ax_1 = Ax_2$ then $A(x_1 - x_2) = 0$ since $A \in \mathrm{End}(\mathbb{R}^n)$ but this implies that $x_1 - x_2 = 0$ by what we proved before then $x_1 = x_2$ and therefore A is injective.

Given that $A : \mathbb{R}^n \to \mathbb{R}^n$ then $\dim(\mathscr{D}(A)) = \dim(\mathscr{R}(A)) = n$ hence $\dim(\mathscr{N}(A)) = 0$ because of the rank-nullity theorem. Also, given that A^{-1} exists this implies that $\mathscr{R}(A) = \mathbb{R}^n$ i.e. A is surjective.

Therefore A is bijective and hence $A \in \mathrm{Aut}(\mathbb{R}^n)$.

Finally, let us take $x = A^{-1}y$ then

$$\|Ax\| = \|AA^{-1}y\| = \|y\| = \|A^{-1}y\| = \|x\|$$

Therefore A^{-1} is an orthogonal transformation.

(ii) Let us note first that $\|Ax + Ay\| = \|A(x + y)\| = \|x + y\|$ then by the Polarization identity we have that

$$\begin{aligned}
\langle Ax, Ay\rangle &= \frac{1}{4}(\|Ax + Ay\|^2 - \|Ax - Ay\|^2) \\
&= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \\
&= \langle x, y\rangle
\end{aligned}$$

On the other hand, let us name $Ax = z$ then by the adjoint linear operator characteristic property we have that

$$\langle A^t z, y\rangle = \langle z, (A^t)^t y\rangle = \langle z, Ay\rangle$$

so replacing z again we have that

$$\langle A^t Ax, y\rangle = \langle Ax, Ay\rangle$$

Therefore joining both results we see that

$$\langle A^t Ax, y\rangle = \langle Ax, Ay\rangle = \langle x, y\rangle$$

(iii) From the previous result we see that

$$\langle A^tAx - x, y\rangle = \langle A^tAx, y\rangle - \langle x, y\rangle = 0$$

for all x and y in $\mathbb{R}^n$. So taking $y = A^tAx - x$ we see by the Polarization identity that

$$\langle A^tAx - x, A^tAx - x\rangle = \|A^tA(x - x)\|^2 = 0$$

Hence by the properties of the norm must be that

$$A^tAx - x = 0$$
$$A^tAx = x$$

Therefore $A^tA = I$.

By the properties of the determinant we see that

$$\det(A^tA) = \det(A^t)\det(A) = \det(A)\det(A) = \det(A)^2$$

But also we know that $\det(I) = 1$ hence $\det(A)^2 = 1$ so must be that $\det(A) = \pm 1$.

From (i) we know that A^{-1} exists so we have that $A^{-1}A = I$ but then $A^{-1}A = A^tA$ so $A^{-1} = A^t$ and we saw in (i) that A^{-1} is orthogonal hence A^t is orthogonal. In the same way, as above we see that $\langle (A^t)^tA^tx - x, y\rangle = \langle AA^tx - x, y\rangle = 0$ for all $x, y \in \mathbb{R}^n$ therefore this implies that $AA^t = I$.

Let $(e_1, ..., e_n)$ be the standard basis for $\mathbb{R}^n$ then by what we proved in (ii) we have that $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ but since A^t is orthogonal then it has the same property i.e. $\langle A^te_i, A^te_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ So joining results we have that

$$\langle Ae_i, Ae_j\rangle = \langle A^te_i, A^te_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$$

Therefore since $\|Ae_i\| = \|A^te_i\| = \|e_i\| = 1$ and

$$\langle Ae_i, Ae_j\rangle = \langle A^te_i, A^te_j\rangle = 0$$

if $i \neq j$ then both column and row vectors of A form an orthonormal basis for $\mathbb{R}^n$.

(iv) If $A \in \mathbf{GL}(\mathbf{n}, \mathbb{R})$ and A is orthogonal from (ii) and (iii) we get that $A^t A = I$ i.e. $A \in \mathbf{O}(\mathbf{n}, \mathbb{R})$.

We see that the ij element of $AA^t = I$ is given by

$$a_{i1}a^t_{1j} + a_{i2}a^t_{2j} + ... + a_{in}a^t_{nj} = \sum_{1 \leq k \leq n} a_{ik}a^t_{kj} = \delta_{ij}$$

Where a^t_{kj} are the elements of A^t. Also, we see that $a^t_{kj} = a_{jk}$ then

$$a_{i1}a_{j1} + a_{i2}a_{j2} + ... + a_{in}a_{jn} = \sum_{1 \leq k \leq n} a_{ik}a_{jk} = \delta_{ij}$$

On the other hand, if we compute $A^t A = I$ the ij element is given by

$$a^t_{i1}a_{1j} + a^t_{i2}a_{2j} + ... + a^t_{in}a_{nj} = \sum_{1 \leq k \leq n} a^t_{ik}a_{kj} = \delta_{ij}$$

But again since $a^t_{ik} = a_{ki}$ we have that

$$a_{1i}a_{1j} + a_{2i}a_{2j} + ... + a_{ni}a_{nj} = \sum_{1 \leq k \leq n} a_{ki}a_{kj} = \delta_{ij}$$

Finally, joining both results we have that

$$\sum_{1 \leq k \leq n} a_{ki}a_{kj} = \sum_{1 \leq k \leq n} a_{ik}a_{jk} = \delta_{ij}$$

$\square$

Proof. **Duistermaat: 2.5** Let $R \in \boldsymbol{SO}(3, \mathbb{R})$, $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq \pi$ and $a \in \mathbb{R}^3$ with $\|a\| = 1$.
Following the hint given, let $b \in N_a$ with $\|b\| = 1$ and define $c = a \times b$.

First, we want to check that $Rb \in N_a$ so we want that $\langle Rb, a\rangle = 0$. We see that

$$\langle Rb, a\rangle = \langle Rb, Ra\rangle = \langle b, a\rangle$$

Where we used in the first equality that $Ra = a$ and in the second equality we used the property determined in the previous problem. But $b \in N_a$ hence must be that $\langle b, a\rangle = 0$ therefore $\langle Rb, a\rangle = 0$ and $Rb \in N_a$.

Let $\|Rb\| = 1$ then since $Rb \in N_a$ we can write that $Rb = Cb + Dc$ for some scalars C, D so we have that

$$\begin{aligned} \|Rb\|^2 &= 1 \\ \langle Rb, Rb\rangle &= 1 \\ \langle Cb + Dc, Cb + Dc\rangle &= 1 \\ C^2\langle b, b\rangle + CD\langle b, c\rangle + DC\langle c, b\rangle + D^2\langle c, c\rangle &= 1 \\ C^2 + D^2 &= 1 \end{aligned}$$

Where we used that $\langle b, b\rangle = 1$ and $\langle b, c\rangle = 0$. Therefore must be that C and D are of the form $C = \cos\alpha$ and $D = \sin\alpha$ and hence there exists $0 \leq \alpha \leq \pi$ such that $Rb = (\cos\alpha)b + (\sin\alpha)c$.

On the other hand, let $Rc \in N_a$, $\|Rc\| = 1$, $Rc \perp Rb$ and $\det R = 1$. Then we can write $Rc = Ab + Bc$ for some scalars A, B. By the same procedure we showed above we see that $A^2 + B^2 = 1$ but since $Rc \perp Rb$ must be that $A = -\sin\alpha$ and $B = \cos\alpha$ therefore we have that $Rc = -(\sin\alpha)b + (\cos\alpha)c$.

Finally, we want to show that for any $y \in N_a$ we have that $\det(a\ y\ Ry) > 0$. If we take the basis (a, b, c) for $\mathbb{R}^3$ then we have that

$$\begin{aligned} &\det\begin{pmatrix} a & y_1 & y_1 \\ 0 & y_2 & y_2\cos\alpha - y_3\sin\alpha \\ 0 & y_3 & y_2\sin\alpha + y_3\cos\alpha \end{pmatrix} = \\ &= ay_2(y_2\sin\alpha + y_3\cos\alpha) - ay_3(y_2\cos\alpha - y_3\sin\alpha) \\ &= a\sin\alpha(y_2^2 + y_3^2) \end{aligned}$$

Then since we can take $a > 0$ and $0 \leq \alpha \leq \pi$ we have that

$$\det(a\ y\ Ry) > 0$$

□

Proof. **Duistermaat: 4.22**

(i) Let $x \in \mathbb{R}^3$ and y be component of x perpendicular to a. Let us also define b in the direction of y such that $\|b\| = 1$ then $a \perp b$ and hence we can write that $y = \|y\|b$.

Then we can write that

$$x = Ca + \|y\|b + 0 \cdot (a \times b) = Ca + y$$

So with respect to the basis $(a, b, a \times b) \in \mathbb{R}^3$ by definition we have that

$$\begin{aligned}\langle x, a\rangle &= C \cdot 1 + \|y\| \cdot 0 + 0 \\ C &= \langle x, a\rangle\end{aligned}$$

Therefore

$$x = \langle x, a\rangle a + y$$

(ii) By definition $a \times x$ is a vector such that $\langle a, a \times x\rangle = \det(a\ a\ x)$ but then by the properties of determinants since two columns are the same this implies that $\langle a, a \times x\rangle = 0$ then $a \times x$ is in N_a.

Also, since $a \times x = a \times y$ we have that $\langle y, a \times y\rangle = \det(y\ a\ y) = 0$ which implies that $a \times x$ is also perpendicular to y.

Finally, from Section 5.3 we know that

$$\langle a, y\rangle^2 + \|a \times y\|^2 = \|a\|^2\|y\|^2$$

But $\langle a, y\rangle = 0$ since y is perpendicular to a and $\|a\| = 1$ therefore

$$\|a \times y\| = \|y\|$$

(iii) Let us compute $R_{\alpha,a}y$ as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \|y\| \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ (\cos\alpha)\|y\| \\ (\sin\alpha)\|y\| \end{pmatrix}$$

Hence since $a \times x$ is a vector of length $\|y\|$ perpendicular to y we can write that

$$R_{\alpha,a}y = (\cos\alpha)y + (\sin\alpha)a \times x$$

Also, since $y = x - \langle x, a\rangle a$ we have that

$$\begin{aligned}R_{\alpha,a}(x - \langle x, a\rangle a) &= (\cos\alpha)x - (\cos\alpha)\langle x, a\rangle a + (\sin\alpha)a \times x \\ R_{\alpha,a}x &= R_{\alpha,a}\langle x, a\rangle a + (\cos\alpha)x - (\cos\alpha)\langle x, a\rangle a + (\sin\alpha)a \times x \\ R_{\alpha,a}x &= \langle x, a\rangle a + (\cos\alpha)x - (\cos\alpha)\langle x, a\rangle a + (\sin\alpha)a \times x \\ R_{\alpha,a}x &= (1 - \cos\alpha)\langle x, a\rangle a + (\cos\alpha)x + (\sin\alpha)a \times x\end{aligned}$$

Where we used that $R_{\alpha,a}\langle x,a\rangle a = \langle x,a\rangle a$.

Let a_1, a_2, a_3 be the components of a in the standard basis of $\mathbb{R}^3$. We name the basis vectors of the standard basis as e_1, e_2, e_3.

Let us compute $R_{\alpha,a}e_1$ as follows

$$\begin{aligned}
R_{\alpha,a}e_1 &= (1-\cos\alpha)\langle e_1,a\rangle a + (\cos\alpha)e_1 + (\sin\alpha)a\times e_1\\
&= (1-\cos\alpha)a_1(a_1e_1+a_2e_2+a_3e_3) + (\cos\alpha)e_1\\
&\quad + (\sin\alpha)(a_1e_1+a_2e_2+a_3e_3)\times e_1\\
&= (\cos\alpha + a_1^2(1-\cos\alpha))e_1 + (a_3(\sin\alpha)+a_1a_2(1-\cos\alpha))e_2\\
&\quad + (-a_2(\sin\alpha)+a_1a_3(1-\cos\alpha))e_3\\
&= (\cos\alpha + a_1^2c(\alpha))e_1 + (a_3(\sin\alpha)+a_1a_2c(\alpha))e_2\\
&\quad + (-a_2(\sin\alpha)+a_1a_3c(\alpha))e_3
\end{aligned}$$

Where $c(\alpha) = 1-\cos\alpha$. In the same way, we compute $R_{\alpha,a}e_2$ and $R_{\alpha,a}e_3$ as follows

$$\begin{aligned}
R_{\alpha,a}e_2 &= c(\alpha)\langle e_2,a\rangle a + (\cos\alpha)e_2 + (\sin\alpha)a\times e_2\\
&= c(\alpha)a_2(a_1e_1+a_2e_2+a_3e_3) + (\cos\alpha)e_2\\
&\quad + (\sin\alpha)(a_1e_1+a_2e_2+a_3e_3)\times e_2\\
&= (-a_3\sin\alpha + a_1a_2c(\alpha))e_1 + (\cos\alpha + a_2^2c(\alpha))e_2\\
&\quad + (a_1\sin\alpha + a_2a_3c(\alpha))e_3
\end{aligned}$$

$$\begin{aligned}
R_{\alpha,a}e_3 &= c(\alpha)\langle e_3,a\rangle a + (\cos\alpha)e_3 + (\sin\alpha)a\times e_3\\
&= c(\alpha)a_3(a_1e_1+a_2e_2+a_3e_3) + (\cos\alpha)e_3\\
&\quad + (\sin\alpha)(a_1e_1+a_2e_2+a_3e_3)\times e_3\\
&= (a_2\sin\alpha + a_1a_3c(\alpha))e_1 + (-a_1\sin\alpha + a_2a_3c(\alpha))e_2\\
&\quad + (\cos\alpha + a_3^2c(\alpha))e_3
\end{aligned}$$

Then $R_{\alpha,a}$ with respect to the standard basis for $\mathbb{R}^3$ is given by

$$R_{\alpha,a} = \begin{pmatrix} \cos\alpha + a_1^2c(\alpha) & -a_3\sin\alpha + a_1a_2c(\alpha) & a_2\sin\alpha + a_1a_3c(\alpha)\\ a_3\sin\alpha + a_1a_2c(\alpha) & \cos\alpha + a_2^2c(\alpha) & -a_1\sin\alpha + a_2a_3c(\alpha)\\ -a_2\sin\alpha + a_1a_3c(\alpha) & a_1\sin\alpha + a_2a_3c(\alpha) & \cos\alpha + a_3^2c(\alpha) \end{pmatrix}$$

Suppose now that $R_{\alpha,a} = R_{\beta,b}$ then must be that $\mathrm{tr}(R_{\alpha,a}) = \mathrm{tr}(R_{\beta,b})$ hence

$$\begin{aligned}
3\cos\alpha + a_1^2c(\alpha) + a_2^2c(\alpha) + a_3^2c(\alpha) &= 3\cos\beta + b_1^2c(\beta) + b_2^2c(\beta) + b_3^2c(\beta)\\
3(\cos\alpha - \cos\beta) + a_1^2c(\alpha) - b_1^2c(\beta) + a_2^2c(\alpha) - b_2^2c(\beta) + a_3^2c(\alpha) - b_3^2c(\beta) &= 0
\end{aligned}$$

If $\alpha = \beta = 0$ with a, b arbitrary the equation holds.

If $0 < \alpha = \beta < \pi$ and $a = b$ then the equation also holds.

Finally if $\alpha = \beta = \pi$ we have that

$$\begin{aligned} a_1^2 c(\pi) - b_1^2 c(\pi) + a_2^2 c(\pi) - b_2^2 c(\pi) + a_3^2 c(\pi) - b_3^2 c(\pi) &= 0 \\ 2a_1^2 - 2b_1^2 + 2a_2^2 - 2b_2^2 + 2a_3^2 - 2b_3^2 &= 0 \\ a_1^2 - b_1^2 + a_2^2 - b_2^2 + a_3^2 - b_3^2 &= 0 \end{aligned}$$

So the only way for this equation to hold is that $a = \pm b$.

(iv) We know that $\operatorname{tr}(R_{\alpha,a}) = 1 + 2\cos\alpha$ hence

$$\begin{aligned} 2\cos\alpha &= \operatorname{tr}(R_{\alpha,a}) - 1 \\ \cos\alpha &= \frac{1}{2}(\operatorname{tr}(R_{\alpha,a}) - 1) \end{aligned}$$

Also, $R^t_{\alpha,a}$ is

$$R^t_{\alpha,a} = \begin{pmatrix} \cos\alpha + a_1^2 c(\alpha) & a_3 \sin\alpha + a_1 a_2 c(\alpha) & -a_2 \sin\alpha + a_1 a_3 c(\alpha) \\ -a_3 \sin\alpha + a_1 a_2 c(\alpha) & \cos\alpha + a_2^2 c(\alpha) & a_1 \sin\alpha + a_2 a_3 c(\alpha) \\ a_2 \sin\alpha + a_1 a_3 c(\alpha) & -a_1 \sin\alpha + a_2 a_3 c(\alpha) & \cos\alpha + a_3^2 c(\alpha) \end{pmatrix}$$

Then

$$\begin{aligned} &R_{\alpha,a} - R^t_{\alpha,a} = \\ &\begin{pmatrix} \cos\alpha + a_1^2 c(\alpha) & -a_3 \sin\alpha + a_1 a_2 c(\alpha) & a_2 \sin\alpha + a_1 a_3 c(\alpha) \\ a_3 \sin\alpha + a_1 a_2 c(\alpha) & \cos\alpha + a_2^2 c(\alpha) & -a_1 \sin\alpha + a_2 a_3 c(\alpha) \\ -a_2 \sin\alpha + a_1 a_3 c(\alpha) & a_1 \sin\alpha + a_2 a_3 c(\alpha) & \cos\alpha + a_3^2 c(\alpha) \end{pmatrix} \\ &- \begin{pmatrix} \cos\alpha + a_1^2 c(\alpha) & a_3 \sin\alpha + a_1 a_2 c(\alpha) & -a_2 \sin\alpha + a_1 a_3 c(\alpha) \\ -a_3 \sin\alpha + a_1 a_2 c(\alpha) & \cos\alpha + a_2^2 c(\alpha) & a_1 \sin\alpha + a_2 a_3 c(\alpha) \\ a_2 \sin\alpha + a_1 a_3 c(\alpha) & -a_1 \sin\alpha + a_2 a_3 c(\alpha) & \cos\alpha + a_3^2 c(\alpha) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2a_3 \sin\alpha & 2a_2 \sin\alpha \\ 2a_3 \sin\alpha & 0 & -2a_1 \sin\alpha \\ -2a_2 \sin\alpha & 2a_1 \sin\alpha & 0 \end{pmatrix} \\ &= 2\sin\alpha \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \end{aligned}$$

On the other hand, suppose there is an angle $\beta \in \mathbb{R}$ where $0 < \beta < \pi$ and an axis of rotation $b \in \mathbb{R}^3$ such that $R_{\alpha,a} = R_{\beta,b}$. Since the trace does not depend on the representation of $R_{\alpha,a}$ must be that

$$\cos\alpha = \frac{1}{2}(\operatorname{tr}(R_{\alpha,a}) - 1) = \frac{1}{2}(\operatorname{tr}(R_{\beta,b}) - 1) = \cos\beta$$

This implies that $\alpha = \beta$. Also, we see that $r_a = (R_{\alpha,a} - R^t_{\alpha,a})/2\sin\alpha$ and since $0 < \alpha < \pi$ then $\sin\alpha \neq 0$ so

$$r_a = \frac{(R_{\alpha,a} - R^t_{\alpha,a})}{2\sin\alpha} = \frac{(R_{\beta,b} - R^t_{\beta,b})}{2\sin\beta} = r_b$$

Which implies that $a = b$ since the entries of r_a and r_b are the components of a and b respectively.

Let now, $\alpha = 0$ then

$$R_{0,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $a \in \mathbb{R}^3$ so in this case is not uniquely determined by $R_{0,a}$.

Finally, let $\alpha = \beta = \pi$ then as mentioned if we write $R_{\pi,a}$ as (r_{ij}) and $R_{\pi,b}$ as (s_{ij}) then $(r_{ij}) = (s_{ij})$. Also, we know that $2a_i^2 = 1 + r_{ii}$ and that $2a_ia_j = r_{ij}$ then must be that

$$2a_i^2 = 1 + r_{ii} = 1 + s_{ii} = 2b_i^2$$

Which implies that $a_i = \pm b_i$. Also, must be that

$$a_ia_j = r_{ij} = s_{ij} = b_ib_j$$

Since at least one $a_i \neq 0$ let $a_1 \neq 0$ and suppose $a_1 = b_1$ then from the second equation we get that $a_j = b_j$. But if we take $a_1 = -b_1$ then we get that $a_j = -b_j$.

Therefore $R_{\pi,a}$ determines a to within a factor of ± 1.

$\square$

Proof. **Hall: 1.6.13** Let us select our basis such that $v \in \mathbb{R}^n$ is a linear combination of e_1 and e_2 then we can write that $v = (\cos\theta, \sin\theta, 0, ..., 0)$. On the other hand, let us we define $R(t) \in \boldsymbol{SO}(n)$ as follows

$$R(t) = \begin{pmatrix} \cos(\theta t) & \sin(\theta t) & 0 & ... & 0 \\ -\sin(\theta t) & \cos(\theta t) & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$$

Then we see that $R(0) = I_n$ but also we have that

$$R(1)v = \begin{pmatrix} \cos^2\theta + \sin^2\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1$$

Let $R \in \boldsymbol{SO}(n)$ then taking $R(t)R$ as our path map we see that

$$R(0)R = I_n R = R$$

Also, the first column of R is given by Re_1 let us name it r_1 then by what we have proven we see that

$$R(1)Re_1 = R(1)r_1 = e_1$$

So $R(t)R$ connects R to a block-diagonal matrix of the form

$$\begin{pmatrix} 1 & \\ & R_1 \end{pmatrix}$$

Where $R_1 \in \boldsymbol{SO}(n-1)$ since $R(t)$ and R are in $\boldsymbol{SO}(n)$.
Therefore by induction we can connect any matrix $R \in \boldsymbol{SO}(n)$, and hence $\boldsymbol{SO}(n)$ is connected. $\square$

Proof. **Hall: 1.6.14** Let $R \in \boldsymbol{SO}(3)$ then with respect to the basis (a, b, c) we can write R as

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Let v be a vector in the a direction then v can be written as

$$v = \begin{pmatrix} \|v\| \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$Rv = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \|v\| \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \|v\| \\ 0 \\ 0 \end{pmatrix} = v$$

So we can write that $Rv - v = 0$ or $(R - I)v = 0$. This implies that v is an eigenvector of R with eigenvalue 1.

Now let $w \in N_v$ where N_v is the plane orthogonal to v this implies that $\langle w, v\rangle = 0$ then we see that

$$\langle Rw, v\rangle = \langle Rw, Rv\rangle = \langle w, v\rangle = 0$$

So R takes any element of the plane orthogonal to v into itself.

Finally, let w be in the direction of b (orthogonal to v) so

$$w = \begin{pmatrix} 0 \\ \|w\| \\ 0 \end{pmatrix}$$

Then

$$Rw = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \|w\| \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \|w\|\cos\theta \\ \|w\|\sin\theta \end{pmatrix} = \|w\|(\cos\theta b + \sin\theta c)$$

Therefore we get an equation showing the rotation of w by an angle of θ and hence applying R we get a rotation by some angle θ around the axis v. $\square$

Proof. **Hall: 1.6.15** Let $R \in \boldsymbol{SO}(n)$

(a) Let $v \in \mathbb{C}^n$ be an eigenvector of R with eigenvalue $\lambda \in \mathbb{C}$ and suppose λ is not real. Let $V \subset \mathbb{R}^n$ be the two-dimensional span of $u = (v+\overline{v})/2$ and $w = (v - \overline{v})/(2i)$.

Let us compute $R(au + bw)$ where $a, b \in \mathbb{R}$ as follows

$$\begin{aligned} R(au + bw) &= R\left(a\frac{v+\overline{v}}{2} + b\frac{v-\overline{v}}{2i}\right) \\ &= a\frac{Rv + \overline{Rv}}{2} + b\frac{Rv - \overline{Rv}}{2i} \\ &= a\frac{\lambda v + \overline{\lambda v}}{2} + b\frac{\lambda v - \overline{\lambda v}}{2i} \end{aligned}$$

Let us write $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$, $v = u + iw$ and $\overline{v} = u - iw$ then

$$\begin{aligned} R(au + bw) = a&\frac{(\alpha + i\beta)(u + iw) + (\alpha - i\beta)(u - iw)}{2} + \\ &+ b\frac{(\alpha + i\beta)(u + iw) - (\alpha - i\beta)(u - iw)}{2i} \\ &= \frac{a}{2}[\alpha(u + iw) + i\beta(u + iw) + \alpha(u - iw) - i\beta(u - iw)] - \\ &- \frac{ib}{2}[\alpha(u + iw) + i\beta(u + iw) - \alpha(u - iw) + i\beta(u - iw)] \\ &= \frac{a}{2}[\alpha u + \alpha iw + i\beta u - \beta w + \alpha u - \alpha iw - i\beta u - \beta w] - \\ &- \frac{ib}{2}[\alpha u + \alpha iw + i\beta u - \beta w - \alpha u + \alpha iw + i\beta u + \beta w] \\ &= \frac{a}{2}[2\alpha u - 2\beta w] - \frac{ib}{2}[2i\alpha w + 2i\beta u] \\ &= a\alpha u - a\beta w + b\alpha w + b\beta u \\ &= (a\alpha + b\beta)u + (b\alpha - a\beta)w \end{aligned}$$

Therefore $R(au + bw)$ is a linear combination of u and w and hence $R(au + bw) \in V$.

Finally, let the restriction of R to V be R' then $R' \in \boldsymbol{SO}(2)$ since V is two-dimensional. So R' by definition has determinant 1.

(b) Let $V \subset \mathbb{R}^n$ be invariant under both R and R^{-1} so if $v \in V$ then $Rv \in V$ and $R^{-1}v \in V$. We know that

$$V^{\perp} = \{w \in \mathbb{R}^n : \langle w, v \rangle = 0 \text{ for all } v \in V\}$$

We want to prove that $V^{\perp}$ is also invariant under R and R^{-1}.

Let $w \in V^{\perp}$ we want to prove that $Rw \in V^{\perp}$. Let also $u \in V$ which can be written as $u = Rv$ for some $v \in V$ then

$$\langle u, Rw \rangle = \langle Rv, Rw \rangle = \langle v, w \rangle = 0$$

Where we used the property determined in Duistermaat 2.4(ii). Hence $Rw \in V^{\perp}$ so R is invariant under $V^{\perp}$.

Finally, we want to prove that $R^{-1}w \in V^{\perp}$ but first let us note that $R^{-1} = R^t$ then if $u = R^t v \in V$ then

$$\langle u, R^t w \rangle = \langle R^t v, R^t w \rangle = \langle v, w \rangle = 0$$

Where we used the property determined in Duistermaat 2.4(iii). Hence $R^t w \in V^{\perp}$ so $R^t = R^{-1}$ is invariant under $V^{\perp}$.

(c) By part (a) and (b) we know that if $R \in \boldsymbol{SO}(n)$ with an eigenvalue $\lambda \in \mathbb{C}^n$ and λ is not real then we can form V and $W = V^{\perp}$ which are invariant under R.

Let $n = 2k$ we want to prove by induction on k that R has the required form.

For the base case, suppose we take $R \in \boldsymbol{SO}(2)$ then we know that R has the form

$$R = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

So in a 2-dimensional space R has the effect of a planar rotation.

Now, by the induction hypotesis, $R \in \boldsymbol{SO}(2(k-1))$ has the form

$$R = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & \\ \sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_{k-1} & -\sin\theta_{k-1} \\ & & & \sin\theta_{k-1} & \cos\theta_{k-1} \end{pmatrix}$$

With respect to a basis $\{v_1, v_2, w_1, ..., w_{2k-4}\} \subset V_{2(k-1)} \cup W_{2(k-1)}$ where $V_{2(k-1)}$ is the 2-dimensional space spanned as in part (a) in $\boldsymbol{SO}(2(k-1))$ and $W_{2(k-1)} = V_{2(k-1)}^{\perp}$.

Then in $\boldsymbol{SO}(2k)$ must be that $W_{2k} = V_{2(k-1)} \cup W_{2(k-1)}$ and hence V_{2k} is orthogonal to W_{2k}, so R must act as a planar rotation with respect to V_{2k} hence we must add a rotation block to R and therefore R must be of the form

$$R = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & \\ \sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_{k} & -\sin\theta_{k} \\ & & & \sin\theta_{k} & \cos\theta_{k} \end{pmatrix}$$

Let now $n = 2k + 1$ then for the base case, we take $R \in \boldsymbol{SO}(3)$ that we know R has the form

$$R = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Again R in a 3-dimensional space has the effect of a planar rotation leaving one the components unchanged.

So in the same way, in $\boldsymbol{SO}(2k)$ must be that $W_{2k} = V_{2(k-1)} \cup W_{2(k-1)}$ where V_{2k} is orthogonal to W_{2k}, so R must act as a planar rotation with respect to V_{2k} hence we must add a rotation block to R and therefore R must be of the form

$$R = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_k & -\sin\theta_k & \\ & & & \sin\theta_k & \cos\theta_k & \\ & & & & & 1 \end{pmatrix}$$

Now, suppose $\lambda \in \mathbb{R}^n$ then $\lambda = 1$ so we see that

$$\begin{aligned} R\left(a\frac{v+\overline{v}}{2} + b\frac{v-\overline{v}}{2i}\right) &= a\frac{Rv+\overline{Rv}}{2} + b\frac{Rv-\overline{Rv}}{2i} \\ &= a\frac{\lambda v+\overline{\lambda v}}{2} + b\frac{\lambda v-\overline{\lambda v}}{2i} \\ &= a\frac{v+\overline{v}}{2} + b\frac{v-\overline{v}}{2i} \end{aligned}$$

So V is also invariant under R. Then for $n = 2k+1$ the same proof given above applies.

In the case $n = 2k$ and $\lambda = 1$ we see that in the base case must be that $\theta_1 = 0$ so R is of the form

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence it has an eigenvalue of 1. But the same proof given above (for the case $n = 2k$) still applies and hence R is of the form

$$R = \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & \cos\theta_k & -\sin\theta_k \\ & & & \sin\theta_k & \cos\theta_k \end{pmatrix}$$

□

2.3 Symmetric Matrices, Symmetric Positive Definite Matrices, and the Exponential Map

1) Show that $\exp : \boldsymbol{S}(n) \to \boldsymbol{SPD}(n)$ is a homeomorphism

Proof. First, we want to show that the exponential map is continuous.
Let $A \in \boldsymbol{S}(n)$ then we can take $M \geq 0$ such that $\|A\| < M$ also, we see that $e^M = \sum_{n=0}^{\infty} M^n/n! < \infty$. Then by the Weierstrass M-test the series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

converges uniformly on the ball $B(0, M)$ and since $A^n/n!$ is continuous for each $n \geq 0$ then e^A is continuous on $B(0, M)$.
Finally, since each $B(0, M)$ is open then e^A is continuous everywhere.

TODO: Prove that the inverse is continuous.

□