Solutions to selected problems on Differential Geometry and Lie Groups - Gallier.

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Solutions to selected problems on Differential Geometry and Lie Groups from different books, but mainly from Gallier.

Chapter 2 - The Matrix Exponential: Some Matrix and Lie Groups

Proof. **Problem 2.1.**

(a) Let the following symmetric matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Then AB is given by

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2+0 & 4+0 \\ 0+1 & 0+1/2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 1/2 \end{pmatrix}$$

Which is non-symmetric.

(b) Let the following skew symmetric matrices

$$A = \pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \pi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

We can work out a explicit formula for e^A and e^B by using Rodrigues formula. For A we see that A has the required form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

taking $a = \pi$ and $b = c = 0$ we have that $\theta = \pi$. Hence by Rodrigues formula we have that

$$
e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\pi^2-2}{\pi^2} & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix}
$$

In the case of B we have that $b = \pi$ and $a = c = 0$, hence $\theta = \pi$ then

$$
e^B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{\pi^2} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{\pi^2-2}{\pi^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix}
$$

Now, we compute $e^A e^B$ as follows

$$
e^A e^B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\pi^2-2}{\pi^2} & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix} \begin{pmatrix} \frac{\pi^2-2}{\pi^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\pi^2-2}{\pi^2} \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{\pi^2-2}{\pi^2} & 0 & 0 \\ 0 & \frac{\pi^2-2}{\pi^2} & 0 \\ 0 & 0 & \left(\frac{\pi^2-2}{\pi^2}\right)^2 \end{pmatrix}
$$

Finally, we compute e^{A+B} using Rodrigues formula as well since the matrix $A + B$ is skew symmetric as we want, where $a = b = \pi$ and

$c = 0$ then we have that $\theta = \sqrt{2}\pi$ and hence

$$e^{A+B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} + \frac{1 - \cos(\sqrt{2}\pi)}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}^2$$

$$e^{A+B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} + \frac{1 - \cos(\sqrt{2}\pi)}{2} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$e^{A+B} = \begin{pmatrix} 1 - \frac{1-\cos(\sqrt{2}\pi)}{2} & \frac{1-\cos(\sqrt{2}\pi)}{2} & \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \\ \frac{1-\cos(\sqrt{2}\pi)}{2} & 1 - \frac{1-\cos(\sqrt{2}\pi)}{2} & -\frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} \\ -\frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} & \frac{\sin(\sqrt{2}\pi)}{\sqrt{2}} & \cos(\sqrt{2}\pi) \end{pmatrix}$$

Therefore we see that $e^A e^B \neq e^{A+B}$.

(c) Let the following matrices

$$A = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix}$$

We see that

$$AB = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4\pi^2 & 2\pi i \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4\pi^2 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence $AB \neq BA$.

Now, we need to compute e^A so first we compute $A^2, A^3, ...$ as follows

$$A^2 = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -4\pi^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} -8i\pi^3 & 0 \\ 0 & 0 \end{pmatrix}$$

So we see that

$$A^n = (2\pi i)^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (2\pi i)^n J_A$$

Where

$$J_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$e^A = I_2 + \left((2\pi i)J_A + \frac{(2\pi i)^2}{2!}J_A + \frac{(2\pi i)^3}{3!}J_A + \frac{(2\pi i)^4}{4!}J_A + ... \right)$$

$$= I_2 + \sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} J_A$$

$$= I_2$$

To compute e^B we follow the same path, we see that

$$B^2 = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -4\pi^2 & 2\pi i \\ 0 & 0 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} 2\pi i & 1 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} -8i\pi^3 & -4\pi^2 \\ 0 & 0 \end{pmatrix}$$

So

$$B^n = (2\pi i)^n \begin{pmatrix} 1 & 1/(2\pi i) \\ 0 & 0 \end{pmatrix} = (2\pi i)^n J_B$$

Where

$$J_B = \begin{pmatrix} 1 & 1/(2\pi i) \\ 0 & 0 \end{pmatrix}$$

Hence

$$e^B = I_2 + \left((2\pi i)J_B + \frac{(2\pi i)^2}{2!} J_B + \frac{(2\pi i)^3}{3!} J_B + \frac{(2\pi i)^4}{4!} J_B + ... \right)$$

$$= I_2 + \sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} J_B$$

$$= I_2$$

Therefore

$$e^A e^B = I_2 \cdot I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But also we have that

$$A + B = \begin{pmatrix} 4\pi i & 1 \\ 0 & 0 \end{pmatrix}$$

$$(A + B)^2 = \begin{pmatrix} 4\pi i & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -16\pi^2 & 4\pi i \\ 0 & 0 \end{pmatrix}$$

$$(A + B)^3 = \begin{pmatrix} 4\pi i & 1 \\ 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} -64\pi^3 i & -16\pi^2 \\ 0 & 0 \end{pmatrix}$$

So we see that

$$(A + B)^n = (4\pi i)^n \begin{pmatrix} 1 & 1/(4\pi i) \\ 0 & 0 \end{pmatrix} = (4\pi i)^n J_{A+B}$$

Where

$$J_{A+B} = \begin{pmatrix} 1 & 1/(4\pi i) \\ 0 & 0 \end{pmatrix}$$

Hence

$$e^{A+B} = I_2 + \left((4\pi i)J_{A+B} + \frac{(4\pi i)^2}{2!} J_{A+B} + \frac{(4\pi i)^3}{3!} J_{A+B} + \frac{(4\pi i)^4}{4!} J_{A+B} + ... \right)$$

$$= I_2 + \sum_{n=1}^{\infty} \frac{(4\pi i)^n}{n!} J_{A+B}$$

$$= I_2$$

Therefore we obtain that $e^A e^B = e^{A+B}$.

$\square$

Proof. **Hubbard: Problem 1.5.10**

a. Let

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

Because of Proposition 2.1 from Gallier we know that the series is absolutely convergent this implies that the original series is convergent as well.

Also, it is shown that the series $|e^A|$ is bounded by the following series

$$e^{n\mu} = \sum_{k=0}^{\infty} \frac{(n\mu)^k}{k!}$$

where $\mu = \max\{|a_{ij}| : 1 \le i, j \le n\}$ and $A = (a_{ij})$.

b.   i. Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^3 = \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

$$\dots$$

$$A^k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

So we have that

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{a^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{b^k}{k!} \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

ii. Let

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

In this case, the matrix has a null trace so according to Gallier we can compute e^A as follows

$$e^A = I_2 + A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

iii. Let

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = -a \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then from Gallier we know that

$$e^A = \begin{bmatrix} \cos(-a) & -\sin(-a) \\ \sin(-a) & \cos(-a) \end{bmatrix} = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix}$$

c. 1. Let

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$$

Such that $a, b > 0$. Then

$$e^A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad e^B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

Because they are null trace matrices. So

$$e^A e^B = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 + ab & a \\ b & 1 \end{bmatrix}$$

But

$$A + B = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

Hence by what we saw on Gallier we get that

$$e^{A+B} = \cosh(\sqrt{ab})I_2 + \frac{\sinh(\sqrt{ab})}{\sqrt{ab}}(A + B)$$

$$= \begin{bmatrix} \cosh(\sqrt{ab}) & 0 \\ 0 & \cosh(\sqrt{ab}) \end{bmatrix} + \begin{bmatrix} 0 & \frac{a\sinh(\sqrt{ab})}{\sqrt{ab}} \\ \frac{b\sinh(\sqrt{ab})}{\sqrt{ab}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(\sqrt{ab}) & \frac{a\sinh(\sqrt{ab})}{\sqrt{ab}} \\ \frac{b\sinh(\sqrt{ab})}{\sqrt{ab}} & \cosh(\sqrt{ab}) \end{bmatrix}$$

Therefore $e^{A+B} \neq e^A e^B$.

2. We proved this on Gallier Proposition 2.5.

3. Let A be an $n \times n$ matrix. Using the sub-problem number 2 and given that A commutes with itself we have that

$$e^{A+A} = e^A e^A$$

i.e.

$$e^{2A} = (e^A)^2$$

$\square$

Proof. **Hall: 2.6.3** Let $A(t)$ and $B(t)$ be two smooth matrix-valued functions then by definition we have that

$$
\begin{aligned}
\frac{d}{dt}[A(t)B(t)] &= \lim_{h \to 0} \frac{A(t+h)B(t+h) - A(t)B(t)}{h} \\
&= \lim_{h \to 0} \frac{A(t+h)B(t+h) - A(t)B(t) + A(t)B(t+h) - A(t)B(t+h)}{h} \\
&= \lim_{h \to 0} \frac{B(t+h)(A(t+h) - A(t)) + A(t)(B(t+h) - B(t))}{h} \\
&= \lim_{h \to 0} \frac{B(t+h)(A(t+h) - A(t))}{h} + \lim_{h \to 0} \frac{A(t)(B(t+h) - B(t))}{h} \\
&= B(t)\frac{dA}{dt} + A(t)\frac{dB}{dt}
\end{aligned}
$$

Now we want to prove that $A(t)B(t)$ is smooth, we see that the element jk of $A(t)B(t)$ is given by

$$
(A(t)B(t))_{jk} = \sum_{i=0}^{n} A_{ji}(t)B_{ik}(t)
$$

Given that $A_{ji}(t)$ and $B_{ik}(t)$ are smooth functions from $\mathbb{R}$ to $\mathbb{R}$ then the product $A_{ji}(t)B_{ik}(t)$ is smooth and the sum of smooth functions is also smooth. Hence $(A(t)B(t))_{jk}$ is smooth and therefore $A(t)B(t)$ is smooth. $\quad\square$

Proof. **Hall: 2.6.4** Let A be an $n \times n$ complex matrix. If A is diagonalizable then we have that $\lim_{m \to \infty} A = A$ and hence we are done.

Suppose A is not diagonalizable, then we know it's similar to an upper triangular matrix U i.e.

$$A = PUP^{-1}$$

Then U has the form

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

Where $u_{11}, u_{22}, \dots, u_{nn}$ are the eigenvalues of U. Given that A is not diagonalizable then U is not diagonalizable then one or more eigenvalues are equal. Suppose we modify the eigenvalues of U as follows, we take u_{22} if it's equal to u_{11} we modify it as $u_{22} - 1/m$ then we take u_{33} if it's equal to u_{11} or u_{22} we modify it as $u_{33} - 2/m$ and we continue this process until u_{nn}. In the worst case, where all the eigenvalues are equal we end up with a matrix U_m as follows

$$U_m = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} - \frac{1}{m} & \dots & u_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & u_{nn} - \frac{n-1}{m} \end{bmatrix}$$

We see that U_m for any $m \in \mathbb{N}$ will be diagonalizable but also we see that $\lim_{m \to \infty} U_m = U$ hence

$$\lim_{m \to \infty} PU_m P^{-1} = PUP^{-1} = A$$

Therefore A is the limit of a sequence of diagonalizable matrices. $\square$

Proof. **Hall: 2.6.5** Let $a \neq d$ then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & ab + bd \\ 0 & d^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & b(a+d)\frac{a-d}{a-d} \\ 0 & d^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & b\frac{a^2-d^2}{a-d} \\ 0 & d^2 \end{pmatrix}$$

Also, we see that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^3 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2 \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & b\frac{a^2-d^2}{a-d} \\ 0 & d^2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & a^2 b + bd\frac{a^2-d^2}{a-d} \\ 0 & d^3 \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & b\frac{a^2(a-d)+da^2-d^3}{a-d} \\ 0 & d^3 \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & b\frac{a^3-d^3}{a-d} \\ 0 & d^3 \end{pmatrix}$$

We can continue this process m times so we can write that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \begin{pmatrix} a^m & b\frac{a^m-d^m}{a-d} \\ 0 & d^m \end{pmatrix}$$

Now we compute $\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ as follows

$$\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a^k & b\frac{a^k-d^k}{a-d} \\ 0 & d^k \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & \frac{b}{a-d} \sum_{k=0}^{\infty} \frac{1}{k!}(a^k - d^k) \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} d^k \end{pmatrix}$$

$$= \begin{pmatrix} e^a & b\frac{e^a-e^d}{a-d} \\ 0 & e^d \end{pmatrix}$$

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On the other hand, in the case $a = d$ we have that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^2 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}$$

And that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^3 = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{pmatrix}$$

So we can write that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^k = \begin{pmatrix} a^k & ka^{k-1}b \\ 0 & a^k \end{pmatrix}$$

Then $\exp \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ in this case gives us

$$\begin{aligned}
\exp \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a^k & ka^{k-1}b \\ 0 & a^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & b \sum_{k=0}^{\infty} \frac{ka^{k-1}}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} a^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & b \sum_{k=0}^{\infty} \frac{a^{k-1}}{(k-1)!} \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} a^k \end{pmatrix} \\
&= \begin{pmatrix} e^a & be^a \\ 0 & e^a \end{pmatrix}
\end{aligned}$$

Which also matches with the equation we determined before assuming that

$$\lim_{a \to d} \frac{e^a - e^d}{a - d} = e^a$$

Therefore for any $a, b, d \in \mathbb{C}$ we have that

$$\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} e^a & b\frac{e^a - e^d}{a - d} \\ 0 & e^d \end{pmatrix}$$

$\square$

Proof. **Hall: 2.6.6** From Gallier we know that if a matrix X has null trace then X has the form

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

And we saw that these matrices satisfy that

$$X^2 = (a^2 + bc)I = -\det(X)I$$

We saw that if $a^2 + bc = -\det(X) = 0$ then

$$e^X = I + X$$

which matches with the equation

$$e^X = \cos(\sqrt{\det x})I + \frac{\sin(\sqrt{\det X})}{\sqrt{\det X}}X$$

Where we used that $\cos(0) = 1$ and $\lim_{\theta \to 0} \sin\theta/\theta = 1$.
Also, we know that if $a^2 + bc = -\det(X) < 0$ then

$$e^X = \cos(\sqrt{\det X})I + \frac{\sin(\sqrt{\det X})}{\sqrt{\det X}}X$$

which is the equation we have.
Finally, if $a^2 + bc = -\det(X) > 0$ i.e. $\sqrt{\det(X)}$ is complex by Gallier we know that

$$e^X = \cosh(\sqrt{\det x})I + \frac{\sinh(\sqrt{\det X})}{\sqrt{\det X}}X$$

$$= \cos(\sqrt{\det x})I + \frac{i\sin(\sqrt{\det X})}{\sqrt{\det X}}X$$

So for any value of $\det(X)$ the exponential of X is given by the equation mentioned above.
Let

$$X_1 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

Then given that $\det(X) = a^2$ we have that

$$e^{X_1} = \cos a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin a}{a} \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & 0 \\ 0 & \cos a \end{pmatrix} + \begin{pmatrix} 0 & -\sin a \\ \sin a & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}$$

$\square$

Proof. **Hall: 2.6.7** Let

$$X = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$$

Then we can write X as $X = A + B$ where

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$$

We see that they commute cause A can be written as $A = 3I_2$ and I_2 commutes with every matrix then we can compute e^X as follows

$$e^X = e^{A+B} = e^A e^B$$

First, we need to compute e^A

$$e^A = \sum_{k \geq 0} \frac{(3I_2)^k}{k!} = I_2 \sum_{k \geq 0} \frac{3^k}{k!} = e^3 I_2$$

Now, we compute e^B by using what we have from problem 2.6.6 since B has $tr(B) = 0$ and $\det(B) = 2$ we have that

$$e^B = \cos(\sqrt{\det B})I_2 + \frac{\sin(\sqrt{\det B})}{\sqrt{\det B}} B$$
$$= \cos(\sqrt{2})I_2 + \frac{\sin(\sqrt{2})}{\sqrt{2}} B$$

Finally we have that

$$e^X = e^A e^B$$
$$= (e^3 I_2)(\cos(\sqrt{2})I_2 + \frac{\sin(\sqrt{2})}{\sqrt{2}} B)$$
$$= e^3 \cos(\sqrt{2})I_2 + e^3 \frac{\sin(\sqrt{2})}{\sqrt{2}} B$$
$$= \begin{pmatrix} e^3(\cos(\sqrt{2}) + \frac{\sin(\sqrt{2})}{\sqrt{2}}) & 3e^3 \frac{\sin(\sqrt{2})}{\sqrt{2}} \\ -e^3 \frac{\sin(\sqrt{2})}{\sqrt{2}} & e^3(\cos(\sqrt{2}) - \frac{\sin(\sqrt{2})}{\sqrt{2}}) \end{pmatrix}$$

$\square$

Proof. **Hall: 2.6.9**

(a) Let A be unipotent then $A - I$ is nilpotent hence there is $k \in \mathbb{N}$ such that $(A - I)^k = 0$. Also, by definition we know that

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - I)^m}{m}$$

Then we can write that

$$\log A = (A - I) - \frac{(A - I)^2}{2} + \frac{(A - I)^3}{3} - \dots + (-1)^{k+1} \frac{(A - I)^k}{k}$$

If we compute $(\log A)^2$ the smallest power of $A - I$ is $(A - I)^2$ but also we get $A - I$ powered to $k + 1$, $k + 2$, ..., $2k$ which are all zero cause $A - I$ is nilpotent. Then if we compute $(\log A)^k$ the smallest power of $A - I$ is going to be $(A - I)^k$ which is zero. Therefore $(\log A)^k = 0$ and hence $\log A$ is nilpotent.

(b) Let X be nilpotent then there is $k \in \mathbb{N}$ such that $X^k = 0$. Also, by definition we know that

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

But since X is nilpotent we can write that

$$e^X = \sum_{m=0}^{k} \frac{X^m}{m!}$$
$$= I + X + \frac{X^2}{2} + \dots + \frac{X^k}{k!}$$

Then $e^X - I$ is

$$e^X - I = X + \frac{X^2}{2} + \dots + \frac{X^k}{k!}$$

If we compute $(e^X - I)^2$ the smallest power of X is X^2 but also we get X powered to $k + 1$, $k + 2$, ..., $2k$ which are all zero cause X is nilpotent. Then if we compute $(e^X - I)^k$ the smallest power of X is going to be X^k which is zero. Therefore $(e^X - I)^k = 0$ and hence $e^X - I$ is nilpotent which implies that e^X is unipotent.

(c) Let $A(t) = I + t(A - I)$ be unipotent then $A(t) - I = t(A - I)$ is nilpotent hence there is some $k \in \mathbb{N}$ for which $t^k(A - I)^k = 0$ then

$$\log(A(t)) = \sum_{m=1}^{k} (-1)^{m+1} \frac{t^m (A - I)^m}{m}$$

From part (a) we know that $\log(A(t))$ is nilpotent so there is some $j \in \mathbb{N}$ such that $(\log(A(t)))^j = 0$. Hence we can write that

$$\exp(\log(A(t))) = I + \sum_{n=1}^{j} \frac{1}{n!} \left(\sum_{m=1}^{k} (-1)^{m+1} \frac{t^m (A - I)^m}{m} \right)^n$$

Then $\exp(\log(A(t)))$ depends polynomially on t and for a sufficiently small t we can get that $\|A(t) - I\| < 1$ then we can apply Theorem 2.8 and therefore

$$\exp(\log(A(t))) = A(t)$$

So we have proven that the polynomial $\exp(\log(A(t)))$ is equal to the polynomial $A(t)$ on an interval close to 0 then they must be equal. This implies that the result we got must be true for $t = 1$ hence

$$\exp(\log(A)) = \exp(\log(A(1))) = I + 1(A - I) = A$$

In the same way, let now $X(t) = tX$ be nilpotent then there is some $k \in \mathbb{N}$ such that $X(t)^k = (tX)^k = 0$ then

$$\exp(X(t)) = \sum_{m=0}^{k} \frac{t^m X^m}{m!}$$

From part (b) we know that $\exp(X(t))$ is unipotent so there is some $j \in \mathbb{N}$ such that $(\exp(X(t)) - I)^j = 0$. Hence we can write that

$$\log(\exp(X(t))) = \sum_{n=1}^{j} \frac{(-1)^{n+1}}{n} \left(\sum_{m=0}^{k} \frac{t^m X^m}{m!} - I \right)^n$$

Again $\log(\exp(X(t)))$ depends polynomially on t and for a sufficiently small t we can get that $\|X(t)\| < \log 2$ and hence $\|e^X - I\| < 1$ then we can apply Theorem 2.8 and therefore

$$\log(\exp(X(t))) = X(t)$$

Finally, again because the two polynomials are equal on an interval close to 0 then must be that

$$\log(\exp(X)) = \log(\exp(X(1))) = 1X = X$$

$\square$

Proof. **Hall: 2.6.10** Let A be an invertible $n \times n$ matrix then A is similar to a block-diagonal matrix D where each block is of the form $\lambda I + N_\lambda$ with N_λ nilpotent. Then

$$A = PDP^{-1}$$

for some invertible matrix P. Let D' be a matrix defined as

$$D' = \begin{pmatrix} \log(\lambda_1 I(I + N_{\lambda_1}/\lambda_1)) & 0 & \ldots & 0 \\ 0 & \log(\lambda_2 I(I + N_{\lambda_2}/\lambda_2)) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \log(\lambda_n I(I + N_{\lambda_n}/\lambda_n)) \end{pmatrix}$$

$$= \begin{pmatrix} \log(\lambda_1 I) + \log(I + N_{\lambda_1}/\lambda_1) & 0 & \ldots & 0 \\ 0 & \log(\lambda_2 I) + \log(I + N_{\lambda_2}/\lambda_2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \log(\lambda_n I) + \log(I + N_{\lambda_n}/\lambda_n) \end{pmatrix}$$

Where we used that $\lambda_1 I$ and $I + N_{\lambda_1}/\lambda_1$ commutes. So

$$e^{D'} = \begin{pmatrix} \exp(\log(\lambda_1 I) + \log(I + N_{\lambda_1}/\lambda_1)) & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & \exp(\log(\lambda_n I) + \log(I + N_{\lambda_n}/\lambda_n)) \end{pmatrix}$$

$$= \begin{pmatrix} \exp(\log(\lambda_1 I))\exp(\log(I + N_{\lambda_1}/\lambda_1)) & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & \exp(\log(\lambda_n I))\exp(\log(I + N_{\lambda_n}/\lambda_n)) \end{pmatrix}$$

Since $I + N_{\lambda_i}/\lambda_i$ is unipotent we know that $\exp(\log(I + N_{\lambda_i}/\lambda_i)) = I + N_{\lambda_i}/\lambda_i$. And for $\exp(\log(\lambda_i I))$ we see that

$$\exp(\log(\lambda_i I)) = \exp\left(\sum_{m=1}^{\infty} (-1)^{m+1} \frac{(\lambda_i I - I)^m}{m} \right)$$

$$= \exp\left(\sum_{m=1}^{\infty} (-1)^{m+1} \frac{(\lambda_i - 1)^m}{m} I \right)$$

$$= \exp(\log(\lambda_i)I)$$

$$= \sum_{m=0}^{\infty} \frac{(\log(\lambda_i)I)^m}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{\log(\lambda_i)^m}{m!} I$$

$$= \exp(\log(\lambda_i))I$$

$$= \lambda_i I$$

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Then

$$e^{D'} = \begin{pmatrix} \lambda_1 I(I + N_{\lambda_1}/\lambda_1) & 0 & \ldots & 0 \\ 0 & \lambda_2 I(I + N_{\lambda_2}/\lambda_2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_n I(I + N_{\lambda_n}/\lambda_n) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 I + N_{\lambda_1} & 0 & \ldots & 0 \\ 0 & \lambda_2 I + N_{\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_n I + N_{\lambda_n} \end{pmatrix}$$

Hence $D = e^{D'}$ and therefore

$$A = PDP^{-1} = Pe^{D'}P^{-1} = e^{PD'P^{-1}}$$

Which implies that A can be written as $A = e^X$ where $X = PD'P^{-1}$.

$\square$