Solved selected problems of Introductory functional analysis with applications - Erwin Kreyszig

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Chapter 2 - Normed Spaces. Banach Spaces

2.1 - Vector Space

Proof. **2** We want to prove (1) and (2). We want to prove first that $0x = \theta$. Since $0x$ is a vector we have that $0x + \theta = 0x$ and $0x + (-0x) = \theta$ hence

$$\begin{aligned}
0x + (0x + (-0x)) &= 0x \\
(0x + 0x) + (-0x) &= 0x \\
(0 + 0)x + (-0x) &= 0x \\
0x + (-0x) &= 0x \\
\theta &= 0x
\end{aligned}$$

Therefore we have that $0x = \theta$.

Now we want to prove that $\alpha\theta = \theta$. Since $\alpha\theta$ is a vector we know that $\alpha\theta + \theta = \alpha\theta$ and $\alpha\theta + (-\alpha\theta) = \theta$ hence

$$\begin{aligned}
\alpha\theta + (\alpha\theta + (-\alpha\theta)) &= \alpha\theta \\
\alpha(\theta + \theta) + (-\alpha\theta) &= \alpha\theta \\
\alpha\theta + (-\alpha\theta) &= \alpha\theta \\
\theta &= \alpha\theta
\end{aligned}$$

Where we used that $\theta + \theta = \theta$ since θ is also a vector. Therefore we have that $\alpha\theta = \theta$.

Finally, we want to prove that $(-1)x = -x$. We know that $(\alpha + \beta)x = \alpha x + \beta x$ using that $\alpha = -1$ and $\beta = 1$ we get that

$$\begin{aligned}
(1 + (-1))x &= 1x + (-1)x \\
0x &= x + (-1)x \\
\theta &= x + (-1)x \\
x + (-x) &= x + (-1)x
\end{aligned}$$

Where we used that $0x = \theta$ and that $x + (-x) = \theta$. This implies that $(-1)x = -x$ as we wanted. $\square$

Proof. **3** Let $M = \{(1,1,1),(0,0,2)\} \subset \mathbb{R}^3$ so the span of M is given by

$$\begin{aligned} \operatorname{span} M &= \{\alpha(1,1,1) + \beta(0,0,2) : \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha, \alpha, \alpha + 2\beta) : \alpha, \beta \in \mathbb{R}\} \end{aligned}$$

Therefore we have that

$$\operatorname{span} M = \{(x,y,z) \in \mathbb{R}^3 : x = y\}$$

□

Proof. **4** We want to check if the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$. This is true if given $x, y \in X$ where X is the subset in question then $\alpha x + \beta y \in X$ for all scalars α, β.

(a) Let X be "all $x \in \mathbb{R}^3$ such that $\xi_1 = \xi_2$ and $\xi_3 = 0$". So if $x, y \in X$ we have that

$$\begin{aligned} \alpha x + \beta y &= \alpha(\xi_1, \xi_1, 0) + \beta(\eta_1, \eta_1, 0) \\ &= (\alpha\xi_1 + \beta\eta_1, \alpha\xi_1 + \beta\eta_1, 0) \end{aligned}$$

We see that if $\alpha x + \beta y = (\mu_1, \mu_2, \mu_3)$ then $\mu_1 = \mu_2 = \alpha\xi_1 + \beta\eta_1$ and $\mu_3 = 0$. Also we see that $(0,0,0) \in X$ as well. Therefore we have that $\alpha x + \beta y \in X$ and X is a subspace of $\mathbb{R}^3$.

(b) Let X be "all $x \in \mathbb{R}^3$ such that $\xi_1 = \xi_2 + 1$". So if $x, y \in X$ we have that

$$\begin{aligned} \alpha x + \beta y &= \alpha(\xi_2 + 1, \xi_2, \xi_3) + \beta(\eta_2 + 1, \eta_2, \eta_3) \\ &= (\alpha(\xi_2 + 1) + \beta(\eta_2 + 1), \alpha\xi_2 + \beta\eta_2, \alpha\xi_3 + \beta\eta_3) \\ &= ((\alpha\xi_2 + \beta\eta_2) + \beta + \alpha, \alpha\xi_2 + \beta\eta_2, \alpha\xi_3 + \beta\eta_3) \end{aligned}$$

So if $\alpha x + \beta y$ is in X then it must happen that

$$(\alpha\xi_2 + \beta\eta_2) + \beta + \alpha = (\alpha\xi_2 + \beta\eta_2) + 1$$

hence it must be that $\alpha + \beta = 1$ which is not true for every choice of α, β. Therefore $\alpha x + \beta y \notin X$ and X is not a subspace of $\mathbb{R}^3$.

(c) Let X be "all $x \in \mathbb{R}^3$ such that ξ_1, ξ_2, ξ_3 are positive". So if $x, y \in X$ and $\alpha = \beta = -1$ we have that

$$\begin{aligned} \alpha x + \beta y &= -1(\xi_1, \xi_2, \xi_3) + -1(\eta_1, \eta_2, \eta_3) \\ &= (-\xi_1 - \eta_1, -\xi_2 - \eta_2, -\xi_3 - \eta_3) \end{aligned}$$

And since $\eta_i, \xi_i > 0$ then $-\xi_i - \eta_i < 0$ for $i = 1, 2, 3$. Therefore $\alpha x + \beta y \notin X$ for every α, β and X is not a subspace of $\mathbb{R}^3$.

(d) Let X be "all $x \in \mathbb{R}^3$ such that $\xi_1 - \xi_2 + \xi_3 = k$ where k is a constant". So if $x, y \in X$ we have that

$$\begin{aligned} \alpha x + \beta y &= \alpha(\xi_1, \xi_2, \xi_3) + \beta(\eta_1, \eta_2, \eta_3) \\ &= (\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2, \alpha\xi_3 + \beta\eta_3) \end{aligned}$$

Now, let us compute the following

$$\begin{aligned} (\alpha\xi_1 + \beta\eta_1) - (\alpha\xi_2 + \beta\eta_2) + (\alpha\xi_3 + \beta\eta_3) \\ &= \alpha(\xi_1 - \xi_2 + \xi_3) + \beta(\eta_1 - \eta_2 + \eta_3) \\ &= \alpha k + \beta k \end{aligned}$$

Therefore if $\alpha x + \beta y = (\mu_1, \mu_2, \mu_3)$ we see that $\mu_1 - \mu_2 + \mu_3 = \alpha k + \beta k$ so for $\alpha k + \beta k = k$ must happen that $\alpha + \beta = 1$ for $k \neq 0$ which is not the case for every α, β. Let $k = 0$ and $x = (0, 0, 0)$ then $0 - 0 + 0 = 0$ so $(0, 0, 0) \in X$. Hence $\alpha x + \beta y \notin X$ if $k \neq 0$ but $\alpha x + \beta y \in X$ if $k = 0$. Finally, X is a subspace of $\mathbb{R}^3$ only when $k = 0$.

□

Proof. **5** Let $\{x_1, x_2, ..., x_n\}$ be a set of $C[a, b]$ where $x_i(t) = t^i$, we want to prove that this is a linearly independent set. Let us suppose the set is not linearly independent, we want to arrive at a contradiction. Then there are constants $\alpha_1, \alpha_2, ..., \alpha_n$ where at least one of them is different from 0 such that

$$\begin{aligned} \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0 \\ \alpha_1 t^1 + \alpha_2 t^2 + ... + \alpha_n t^n = 0 \end{aligned}$$

If $\alpha_n \neq 0$ then this is a polynomial of degree n which has at most n roots and hence it cannot be equal to 0 at every point of $[a, b]$ then we have a contradiction and must be that $\alpha_n = 0$. We can follow the same arguments against any α_i and hence it must be that $\alpha_i = 0$ for $i = 1, 2, ..., n$. Therefore the set $\{x_1, x_2, ..., x_n\}$ is a linearly independent set of $C[a, b]$. □

Proof. **6** Let X be an n-dimensional vector space and let $x \in X$ then x can be expressed as a linear combination of basis vectors $e_1, ..., e_n$, we want to show that this representation is unique. Suppose there are two representations of x in terms of the basis vectors, we want to arrive at a contradiction, then we have that

$$x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$
$$x = \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n$$

So

$$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n$$

and hence

$$(\alpha_1 - \beta_1)e_1 + (\alpha_2 - \beta_2)e_2 + ... + (\alpha_n - \beta_n)e_n = 0$$

But we know that the basis vectors are a linearly independent set of vectors so the coefficients that solve the above equation can only be 0 i.e. $\alpha_i - \beta_i = 0$ for every $i = 1, ..., n$ which implies that $\alpha_i = \beta_i$ and therefore that x has a unique representation. $\square$

Proof. **7** Let $\{e_1, ..., e_n\}$ be a basis for a complex vector space X. We want to find a basis for X regarded as a real vector space. Let $x \in X$ then we have that x can be written as

$$x = \alpha_1 e_1 + ... + \alpha_n e_n$$

Where $\alpha_1, ..., \alpha_n \in \mathbb{C}$ hence each α_j can be written as $\alpha_j = a_j + ib_j$ where $a_j, b_j \in \mathbb{R}$ so we can write that

$$\begin{aligned} x &= (a_1 + ib_1)e_1 + ... + (a_n + ib_n)e_n \\ &= a_1 e_1 + b_1 ie_1 + ... + a_n e_n + b_n ie_n \end{aligned}$$

Then the set $\{e_1, ..., e_n, ie_1, ..., ie_n\}$ can be defined as a basis for X when X is regarded as a real vector space.

The dimension of X as a complex vector space is $\dim X = n$ and as a real vector space is $\dim X = 2n$. $\square$

Proof. **10** Let Y and Z be subspaces of a vector space X. We want to show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be.

We know that Y and Z are non-empty since they are subspaces of X then let $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ and let us take $\alpha = \beta = 0$ then $\alpha y_1 + \beta y_2 = 0 \in Y$ and $\alpha z_1 + \beta z_2 = 0 \in Z$ this implies that $0 \in Y \cap Z$ and thus $Y \cap Z$ is non-empty.

Let $y_1, z_1 \in Y \cap Z$ and let us compute $\alpha y_1 + \beta z_1$ where α, β are scalars. We know that $y_1, z_1 \in Y$ and that $y_1, z_1 \in Z$ by definition, then $\alpha y_1 + \beta z_1 \in Y$ and $\alpha y_1 + \beta z_1 \in Z$ since they are subspaces. But then by the definition of intersection, we have that $\alpha y_1 + \beta z_1 \in Y \cap Z$. Therefore $Y \cap Z$ is a subspace of X.

Let us define $Y = \mathbb{R}$, $Z = i\mathbb{R}$ both considered as subspaces of $\mathbb{C}$. Then we see that $Y \cap Z = \{0\}$ which is also a subspace of $\mathbb{C}$.

On the other hand, let $1 \in Y$ and $i \in Z$ and let $\alpha = \beta = 1$ hence we see that $1 \cdot 1 + 1 \cdot i = 1 + i \notin Y$ and $1 + i \notin Z$ therefore $1 + i \notin Y \cup Z$ and thus $Y \cup Z$ is not a subspace of $\mathbb{C}$. $\square$

Proof. **11** Let $M \neq \emptyset$ be any subset of a vector space X, we want to prove that $\operatorname{span} M$ is a subspace of X. Let $m_1, m_2 \in M$ and let α, β be scalars. We see that $m_1, m_2 \in \operatorname{span} M$ then $\alpha m_1 + \beta m_2 \in \operatorname{span} M$ since $\alpha m_1 + \beta m_2$ is a linear combination of m_1 and m_2. Therefore $\operatorname{span} M$ is a subspace of X. $\square$

2.2 - Normed Space. Banach Space

Proof. **1** Let $x \in X$ where X is a normed space then the distance from x to 0 is by definition $d(x, 0)$ hence by the definition of the metric induced by the norm we have that $d(x, 0) = \|x - 0\| = \|x\|$ $\square$

Proof. **3** We want to prove that $|\|y\| - \|x\|| \leq \|y - x\|$. From property (N4) we have that

$$\|(y - x) + x\| \leq \|y - x\| + \|x\|$$

Hence

$$\|y\| - \|x\| \leq \|y - x\|$$

But also we have that

$$\begin{aligned} \|(y - x) - y\| &\leq \|y - x\| + \|y\| \\ \|x\| - \|y\| &\leq \|y - x\| \\ -\|y - x\| &\leq \|y\| - \|x\| \end{aligned}$$

Therefore by the properties of the absolute value, we have that

$$|\|y\| - \|x\|| \leq \|y - x\|$$

$\square$

Proof. **11** Let $\tilde{B}(0,1) = \{x \in X : \|x\| \leq 1\}$ be the closed unit ball in a normed space X, we want to show that $\tilde{B}(0,1)$ is convex.

Let $z = \alpha x + (1-\alpha)y$ where $x, y \in \tilde{B}(0,1)$ and $0 \leq \alpha \leq 1$ we want to show that $z \in \tilde{B}(0,1)$. So let us compute the following

$$\begin{aligned}
\|z\| &= \|\alpha x + (1-\alpha)y\| \\
&\leq \|\alpha x\| + \|(1-\alpha)y\| \\
&= |\alpha|\|x\| + |1-\alpha|\|y\| \\
&\leq |\alpha| + |1-\alpha|
\end{aligned}$$

Therefore we see that $\|z\| \leq 1$ hence $z \in \tilde{B}(0,1)$ and in addition $\tilde{B}(0,1)$ is convex. □

2.3 - Further Properties of Normed Spaces

Proof. **1** Let $c \subset l^\infty$, we want to prove it's a vector subspace of l^∞.

Let $x, y \in c$ then they are sequences of the form $x = (\xi_j)$ and $y = (\eta_j)$ where each ξ_j and η_j are complex numbers, let also α, β be scalars. We want to show that $z = \alpha x + \beta y \in c$. We see that $z = (\alpha\xi_j + \beta\eta_j)$ is a sequence of complex numbers so we are left to prove it's also convergent.

Suppose $\xi_j \to \xi$ and $\eta_j \to \eta$ we want to prove z converges to $\alpha\xi + \beta\eta$ hence we compute what follows

$$\begin{aligned}
|(\alpha\xi_j + \beta\eta_j) - (\alpha\xi + \beta\eta)| &\leq |\alpha\xi_j - \alpha\xi| + |\beta\eta_j - \beta\eta| \\
&= |\alpha||\xi_j - \xi| + |\beta||\eta_j - \eta| \\
&\leq |\alpha|\epsilon + |\beta|\epsilon
\end{aligned}$$

Where in the last step we used the fact that both (ξ_j) and (η_j) converge. This implies that $\alpha\xi_j + \beta\eta_j \to \alpha\xi + \beta\eta$ and hence that $z \in c$, therefore c is a vector subspace of l^∞.

Now, let $c_0 \subset l^\infty$ be the space of all sequences of scalars converging to 0, we want to prove that it's a vector subspace of l^∞.

Let $x, y \in c_0$ then they are sequences of the form $x = (\xi_j)$ and $y = (\eta_j)$ such that $\xi_j \to 0$ and $\eta_j \to 0$, let also α, β be scalars. We want to show that $z = \alpha x + \beta y \in c_0$. We see that $z = (\alpha\xi_j + \beta\eta_j)$ is a sequence of scalars so we are left to prove it's also convergent to 0 hence let us compute what follows

$$\begin{aligned}
|(\alpha\xi_j + \beta\eta_j) - 0| &\leq |\alpha\xi_j| + |\beta\eta_j| \\
&= |\alpha||\xi_j - 0| + |\beta||\eta_j - 0| \\
&\leq |\alpha|\epsilon + |\beta|\epsilon
\end{aligned}$$

Where in the last step we used the fact that both (ξ_j) and (η_j) converge to 0. This implies that $\alpha\xi_j + \beta\eta_j \to 0$ and hence that $z \in c_0$, therefore c_0 is a vector subspace of l^∞.

□

Proof. **2** Let $c_0 \subset l^\infty$ be the space of all sequences of scalars converging to 0, we want to prove c_0 is closed. Let us consider $x = (\xi_j) \in \bar{c}_0$ the closure of c_0. By Theorem 1.4-6(a) there are $x_n = (\xi_j^{(n)}) \in c_0$ such that $x_n \to x$. Hence, given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for $n \geq N$ and all j we have that

$$|\xi_j^{(n)} - \xi_j| \leq \sup_j |\xi_j^{(n)} - \xi_j| < \epsilon/2$$

In particular let us take $n = N$ we know that $x_N \in c_0$ and hence x_N is convergent to 0 so there is some $N_1 \in \mathbb{N}$ such that when $j \geq N_1$ we have that

$$|\xi_j^{(N)} - 0| < \epsilon/2$$

Finally, let us apply the triangle inequality to $|\xi_j - 0|$ when $j \geq N_1$ then

$$|\xi_j - 0| = |\xi_j + \xi_j^{(N)} - \xi_j^{(N)}| \leq |\xi_j - \xi_j^{(N)}| + |\xi_j^{(N)} - 0| < \epsilon$$

This implies that x converges to 0 and that $x \in c_0$ but also since $x \in \bar{c}_0$ was arbitrary, this proves that $c_0 \subset l^\infty$ is closed. □

Proof. **3** Let $Y \subset l^\infty$ be the set of all sequences with only finitely many nonzero terms. We want to prove it is a subspace of l^∞ but not a closed subspace.

Let $x, y \in Y$ where x has n nonzero terms and y has m nonzero terms, let also α, β be scalars. We must show that $z = \alpha x + \beta y$ is in Y to show that Y is a subspace of l^∞. We see that αx still has n nonzero terms where each of them is multiplied by the scalar α and the same happens to βy which has m nonzero terms. Finally, $z = \alpha x + \beta y$ will have at most $n + m$ nonzero terms so z is in Y and therefore Y is a subspace of l^∞.

Let us consider the sequence $(x_n) \subset Y$ where

$$\begin{aligned} x_1 &= 1, 0, 0, 0, ... \\ x_2 &= 1, 1/2, 0, 0, ... \\ x_3 &= 1, 1/2, 1/3, 0, ... \\ &\vdots \end{aligned}$$

i.e. the first n terms of each element of the sequence have values given by $1/n$ and the rest of them are zero. We want to prove that (x_n) tends to the sequence $x = (1/n)$. Let $\epsilon > 0$ then we can find $N \in \mathbb{N}$ such that when $n \geq N$ we have that

$$d(x_n, x) = \sup_i |\xi_i^{(n)} - \xi_i| \leq \frac{1}{N+1} < \epsilon$$

Therefore $x_n \to x$ and this implies that $x \in \overline{Y}$ but $x \notin Y$ because it doesn't have finitely many nonzero terms hence Y is not closed. □

Proof. **4** Let X be a normed vector space, we want to prove that vector addition and multiplication by a scalar are continuous operations.

The vector addition is a map $(x, y) \to x + y$ from $X \times X$ to X so let us define a norm for the space $X \times X$ as

$$\|(x, y)\|_{X\times X} = \max(\|x\|_X, \|y\|_X)$$

We want to prove that given $\epsilon > 0$ there is some delta $\delta > 0$ such that when $\|(x', y') - (x, y)\|_{X\times X} = \|(x' - x, y' - y)\|_{X\times X} < \delta$ we have that

$$\|(x' + y') - (x + y)\|_X < \epsilon$$

We see that

$$\begin{aligned}\|(x' + y') - (x + y)\|_X &= \|(x' - x) + (y' - y)\|_X \\ &\leq \|x' - x\|_X + \|y' - y\|_X \\ &\leq 2\max(\|x' - x\|_X, \|y' - y\|_X) \\ &= 2\|(x' - x, y' - y)\|_{X\times X}\end{aligned}$$

Therefore we see that when $\delta = \epsilon/2$ we have that $\|(x' - x, y' - y)\|_{X\times X} < \delta$ implies that $\|(x' + y') - (x + y)\|_X < \epsilon$ i.e. vector addition is a continuous map.

The multiplication by a scalar is a map $(\alpha, x) \to \alpha x$ from $K \times X$ to X so let us define a norm for the space $K \times X$ as

$$\|(\alpha, x)\|_{K\times X} = \max(|\alpha|, \|x\|_X)$$

We want to prove that given $\epsilon > 0$ there is some delta $\delta > 0$ such that when $\|(\beta, x') - (\alpha, x)\|_{K\times X} = \|(\beta - \alpha, x' - x)\|_{K\times X} < \delta$ we have that

$$\|\beta x' - \alpha x\|_X < \epsilon$$

We see that

$$\begin{aligned}\|\beta x' - \alpha x\|_X &= \|\beta x' - \alpha x + \beta x - \beta x\|_X \\ &= \|x(\beta - \alpha) + \beta(x' - x)\|_X \\ &\leq \|x(\beta - \alpha)\|_X + \|\beta(x' - x)\|_X \\ &= |\beta - \alpha|\|x\|_X + |\beta|\|x' - x\|_X\end{aligned}$$

Without loss of generality let us choose $|\beta - \alpha| < 1$ then we see that

$$|\beta| \leq |\beta - \alpha| + |\alpha| < 1 + |\alpha|$$

Hence we can continue the inequality chain as follows

$$\begin{aligned}\|\beta x' - \alpha x\|_X &\leq |\beta - \alpha|\|x\|_X + |\beta|\|x' - x\|_X \\ &\leq |\beta - \alpha|\|x\|_X + (1 + |\alpha|)\|x' - x\|_X \\ &\leq 2\max(1 + |\alpha|, \|x\|_X)\max(|\beta - \alpha|, \|x' - x\|_X) \\ &= 2\max(1 + |\alpha|, \|x\|_X)\|(\beta - \alpha, x' - x)\|_{K\times X}\end{aligned}$$

Therefore we see that when

$$\delta = \frac{\epsilon}{2\max(1+|\alpha|, \|x\|_X)}$$

we have that

$$\|(\beta - \alpha, x' - x)\|_{K\times X} < \delta$$

implies that $\|\beta x' - \alpha x\|_X < \epsilon$ i.e. multiplication by a scalar is a continuous map. ☐

Proof. **5** Let $x_n \to x$ and $y_n \to y$. Let $\epsilon/2 > 0$ then there is $N_x, N_y \in \mathbb{N}$ such that when $n \geq N_x$ we have that $\|x_n - x\| < \epsilon/2$ and when $n \geq N_y$ we have that $\|y_n - y\| < \epsilon/2$. So by using the triangle inequality and by taking $N = \max(N_x, N_y)$ then when $n \geq N$ we have that

$$\|(x_n + y_n) - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore we see that $x_n + y_n \to x + y$.

Now let $x_n \to x$ and $\alpha_n \to \alpha$. This implies that if we let $\epsilon/2\|x\| > 0$ then there is $N_x \in \mathbb{N}$ such that when $n \geq N_x$ we have that $\|x_n - x\| < \epsilon/2\|x\|$. Also, since (α_n) converges to α it is bounded so there is $M > 0$ such that $|\alpha_n| < M$. Then if we let $\epsilon/2M > 0$ when $n \geq N_\alpha$ we have that $\|\alpha_n - \alpha\| < \epsilon/2M$.

On the other hand, by using the triangle inequality on

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha x + \alpha_n x - \alpha_n x\| = \|\alpha_n(x_n - x) + x(\alpha_n - \alpha)\|$$

we see that

$$\begin{aligned}\|\alpha_n(x_n - x) + x(\alpha_n - \alpha)\| &\leq \|\alpha_n(x_n - x)\| + \|x(\alpha_n - \alpha)\| \\ &\leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\|\end{aligned}$$

Hence by taking $N = \max(N_x, N_\alpha)$ we can continue the inequality chain as follows

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &\leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \\ &< M\|x_n - x\| + |\alpha_n - \alpha|\|x\| \\ &< M\frac{\epsilon}{2M} + \frac{\epsilon}{2\|x\|}\|x\| = \epsilon\end{aligned}$$

Therefore this implies that $\alpha_n x_n \to \alpha x$. ☐

Proof. **6** Let $\overline{Y}$ be the closure of a subspace Y of a normed space X we want to show that $\overline{Y}$ is a vector subspace.

Let $x, y \in \overline{Y}$ then there are two sequences $(x_n), (y_n) \subseteq Y$ such that $x_n \to x$ and $y_n \to y$. Hence, let α, β be scalars and let $\epsilon/(|\alpha| + |\beta|) > 0$ then there is $N_x, N_y \in \mathbb{N}$ such that when $n \geq N_x$ we have that $\|x_n - x\| < \epsilon/(|\alpha| + |\beta|)$ and when $n \geq N_y$ we have that $\|y_n - y\| < \epsilon/(|\alpha| + |\beta|)$.

We want to show that $\alpha x + \beta y$ is also in $\overline{Y}$. Since Y is a subspace then the sequence $(\alpha x_n + \beta y_n)$ is in Y, we want to show that $\alpha x_n + \beta y_n \to \alpha x + \beta y$.

Let $N = \max(N_x, N_y)$ then using the triangle inequality we see that

$$\begin{aligned}
\|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)\| &\leq \|\alpha(x_n - x)\| + \|\beta(y_n - y)\| \\
&\leq |\alpha|\|x_n - x\| + |\beta|\|y_n - y\| \\
&< \frac{\epsilon}{(|\alpha| + |\beta|)}(|\alpha| + |\beta|) = \epsilon
\end{aligned}$$

which implies that $\alpha x_n + \beta y_n \to \alpha x + \beta y$. Therefore $\alpha x + \beta y$ is in $\overline{Y}$ and hence $\overline{Y}$ is also a subspace of X. □

Proof. **10** Let X be a normed space which has a Schauder basis (e_n). We want to prove that there is a subset $M \subseteq X$ which is dense in X which would imply that X is separable.

Let us define

$$M = \{q_1 e_1 + ... + q_n e_n : q_k \in \mathbb{Q},\ e_k \in (e_n) \text{ and } n \in \mathbb{N}\}$$

we want to prove M is dense in X.

Let $x \in X$ and let $\epsilon > 0$. Since (e_n) is a Schauder basis for X then there is a sequence $(\alpha_1 e_1 + ... + \alpha_n e_n)$ where the α_k are scalars such that when $n \geq N$ for some $N \in \mathbb{N}$ we have that

$$\|(\alpha_1 e_1 + ... + \alpha_n e_n) - x\| < \epsilon/2$$

Also, let $q_1, ..., q_n \in \mathbb{Q}$ such that $|q_k - \alpha_k|\|e_k\| < \epsilon/2n$ for $1 \leq k \leq n$ then we have that

$$\begin{aligned}
\|(q_1 e_1 + ... + q_n e_n) - (\alpha_1 e_1 + ... + \alpha_n e_n)\| \leq |q_1 - \alpha_1|\|e_1\| + ... + |q_n - \alpha_n|\|e_n\| < \\
< n(\epsilon/2n) = \epsilon/2
\end{aligned}$$

So let us name $y = (\alpha_1 e_1 + ... + \alpha_n e_n)$ then by adding both inequalities and applying the triangle inequality we have that

$$\|(q_1 e_1 + ... + q_n e_n) - x\| \leq \|y - x\| + \|(q_1 e_1 + ... + q_n e_n) - y\| < \epsilon$$

This implies that $(q_1 e_1 + ... + q_n e_n) \to x$ and hence that $x \in \overline{M}$. But x was arbitrary so we have that $X = \overline{M}$ and therefore M is a countable dense set and X is separable.

□

Proof. **11** We want to prove that $(e_n) \subset l^p$ where $e_n = (\delta_{nj})$ is a Schauder basis for l^p.

Let $x \in l^p$ where x can be written as (ξ_j) then we have that

$$\begin{aligned}
&\|(\xi_1, \xi_2, ...) - (\alpha_1 e_1 + ... + \alpha_n e_n)\|_p = \\
&\qquad = \|(\xi_1, \xi_2, ...) - (\alpha_1(1, 0, ...) + ... + \alpha_n(0, ..., 1, 0, ...))\|_p \\
&\qquad = \|(\xi_1, \xi_2, ...) - (\alpha_1, ..., \alpha_n, 0, 0, ...)\|_p \\
&\qquad = \|((\xi_1 - \alpha_1), ..., (\xi_n - \alpha_n), \xi_{n+1}, ...)\|_p
\end{aligned}$$

So if we select $\alpha_i = \xi_i$ for $1 \leq i \leq n$ we get that

$$\|x - (\alpha_1 e_1 + ... + \alpha_n e_n)\|_p = \|(0, ..., 0, \xi_{n+1}, ...)\|_p = \left(\sum_{j=n+1}^{\infty} |\xi_j|^p \right)^{1/p}$$

We know that $\sum_{j=1}^{\infty} |\xi_j|^p < \infty$, let us suppose it converges to some L then

$$L = \sum_{j=1}^{\infty} |\xi_j|^p = \sum_{j=1}^{n} |\xi_j|^p + \sum_{j=n+1}^{\infty} |\xi_j|^p$$

Hence

$$\sum_{j=n+1}^{\infty} |\xi_j|^p = L - \sum_{j=1}^{n} |\xi_j|^p$$

But $\sum_{j=1}^{n} |\xi_j|^p \to L$ as $n \to \infty$ then $\sum_{j=n+1}^{\infty} |\xi_j|^p \to 0$ as $n \to \infty$ therefore

$$\|x - (\alpha_1 e_1 + ... + \alpha_n e_n)\|_p = \|(0, ..., 0, \xi_{n+1}, ...)\|_p \to 0$$

This implies that (e_n) is a Schauder basis for l^p. $\square$

2.4 - Finite Dimensional Normed Spaces and Subspaces

Proof. **1** Let $Y \subset l^\infty$ be the set of all sequences with only finitely many nonzero terms. We want to prove it is a subspace of l^∞ but not a closed subspace.

Let $x, y \in Y$ where x has n nonzero terms and y has m nonzero terms, let also α, β be scalars. We must show that $z = \alpha x + \beta y$ is in Y to show that Y is a subspace of l^∞. We see that αx still has n nonzero terms where each of them is multiplied by the scalar α and the same happens to βy which has m nonzero terms. Finally, $z = \alpha x + \beta y$ will have at most $n + m$ nonzero terms so z is in Y and therefore Y is a subspace of l^∞.

Let us consider the sequence $(x_n) \subset Y$ where

$$\begin{aligned}
x_1 &= 1, 0, 0, 0, ...\\
x_2 &= 1, 1/2, 0, 0, ...\\
x_3 &= 1, 1/2, 1/3, 0, ...\\
&\vdots
\end{aligned}$$

i.e. the first n terms of each element of the sequence have values given by $1/n$ and the rest of them are zero. We want to prove that (x_n) tends to the sequence $x = (1/n)$. Let $\epsilon > 0$ then we can find $N \in \mathbb{N}$ such that when $n \geq N$ we have that

$$d(x_n, x) = \sup_i |\xi_i^{(n)} - \xi_i| \leq \frac{1}{N+1} < \epsilon$$

Therefore $x_n \to x$ and this implies that $x \in \overline{Y}$ but $x \notin Y$ because it doesn't have finitely many nonzero terms hence Y is not closed.

If we now take the same subspace Y but as a subspace of l^2 we can consider the same sequence $(x_n) \subset Y$ and we can show that (x_n) tends to $x = (1/n)$ as $n \to \infty$ as follows.

We know that

$$\begin{aligned}
d(x_n, x) &= \sqrt{\sum_{i=1}^{\infty} |\xi_i^{(n)} - \xi_i|^2}\\
&= \sqrt{\sum_{i=n+1}^{\infty} |\xi_i|^2}
\end{aligned}$$

but also we have by definition that $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$ so let us suppose it converges to some L then

$$L = \sum_{i=1}^{\infty} |\xi_i|^2 = \sum_{i=1}^{n} |\xi_i|^2 + \sum_{i=n+1}^{\infty} |\xi_i|^2$$

Hence

$$\sum_{i=n+1}^{\infty} |\xi_i|^2 = L - \sum_{i=1}^{n} |\xi_i|^2$$

But $\sum_{i=1}^{n} |\xi_i|^2 \to L$ as $n \to \infty$ so $\sum_{i=n+1}^{\infty} |\xi_i|^2 \to 0$ which implies that $d(x_n, x) \to 0$ and that $x_n \to x$. Therefore $x \in \overline{Y}$ but $x \notin Y$ thus $Y \subset l^2$ is not closed in l^2 either. $\square$

Proof. **4** Let $\|\cdot\|$ be a norm on a vector space X such that $\|\cdot\|_0$ is an equivalent norm, hence there are $a, b > 0$ such that $a\|x\| \leq \|x\|_0 \leq b\|x\|$ for all $x \in X$.

Let $\mathcal{T}_{\|\cdot\|}$ be the topology generated by the norm $\|\cdot\|$ over X and let $\mathcal{T}_{\|\cdot\|_0}$ be the topology generated by the norm $\|\cdot\|_0$ over X.

Let U be an open set of $\mathcal{T}_{\|\cdot\|}$ then for every $x_0 \in U$ there is $B_{\|\cdot\|}(x_0; r) \subseteq U$ where $B_{\|\cdot\|}(x_0; r)$ is an open ball centered at x_0.

Then by definition, we have that

$$B_{\|\cdot\|}(x_0; r) = \{x \in X : \|x - x_0\| < r\}$$

Given that $x, x_0 \in X$ then $x - x_0 \in X$ so if $\|x - x_0\|_0 < ar$ then by the equivalence of norms, this implies that

$$\|x - x_0\| \leq \frac{1}{a}\|x - x_0\|_0 < r$$

Hence we see that

$$B_{\|\cdot\|_0}(x_0; ar) = \{x \in X : \|x - x_0\|_0 < ar\} \subseteq B_{\|\cdot\|}(x_0; r) \subseteq U$$

So for every $x_0 \in U$ there is $B_{\|\cdot\|_0}(x_0; ar) \subseteq U$ and hence $U \in \mathcal{T}_{\|\cdot\|_0}$ which implies that $\mathcal{T}_{\|\cdot\|} \subseteq \mathcal{T}_{\|\cdot\|_0}$.

In the same way, if U is an open set of $\mathcal{T}_{\|\cdot\|_0}$ then for every $x_0 \in U$ there is $B_{\|\cdot\|_0}(x_0; r) \subseteq U$. So if $\|x - x_0\| < cr$ then by the equivalence of norms we know that $c\|x\|_0 \leq \|x\| \leq d\|x\|_0$ for some scalars $c, d > 0$ and this implies that

$$\|x - x_0\|_0 \leq \frac{1}{c}\|x - x_0\|_0 < r$$

Hence we see that

$$B_{\|\cdot\|}(x_0; cr) = \{x \in X : \|x - x_0\| < cr\} \subseteq B_{\|\cdot\|_0}(x_0; r) \subseteq U$$

Hence $U \in \mathcal{T}_{\|\cdot\|}$ which implies that $\mathcal{T}_{\|\cdot\|_0} \subseteq \mathcal{T}_{\|\cdot\|}$.

Therefore joining both results we see that $\mathcal{T}_{\|\cdot\|} = \mathcal{T}_{\|\cdot\|_0}$ $\square$

Proof. **5** Let $\|\cdot\|$ be a norm on a vector space X such that $\|\cdot\|_0$ is an equivalent norm, hence there are $a, b > 0$ such that $a\|x\| \leq \|x\|_0 \leq b\|x\|$ for all $x \in X$. In the same way, there are $c, d > 0$ such that $c\|x\|_0 \leq \|x\| \leq d\|x\|_0$ for all $x \in X$.

Let $(x_n) \subseteq (X, \|\cdot\|)$ be a Cauchy sequence then given $\epsilon/b > 0$ there is $N \in \mathbb{N}$ such that when $n, m \geq N$ we have that $\|x_n - x_m\| < \epsilon/b$ but also from the equivalence of norms we have that

$$\|x_n - x_m\|_0 \leq b\|x_n - x_m\| < \epsilon$$

So this implies that (x_n) is also Cauchy in $(X, \|\cdot\|_0)$

In the opposite way if $(x_n) \subseteq (X, \|\cdot\|_0)$ is a Cauchy sequence then given $\epsilon/d > 0$ there is $N \in \mathbb{N}$ such that when $n, m \geq N$ we have that $\|x_n - x_m\|_0 < \epsilon/d$ but also from the equivalence of norms we have that

$$\|x_n - x_m\| \leq d\|x_n - x_m\|_0 < \epsilon$$

Therefore this implies that (x_n) is also Cauchy in $(X, \|\cdot\|)$

Finally, joining both results we see that the Cauchy sequences in $(X, \|\cdot\|)$ and in $(X, \|\cdot\|_0)$ are the same. □

2.5 - Compactness and Finite Dimension

Proof. **1** By Lemma 2.5-2 we know that a subset M of a metric space is closed and bounded so if we see $\mathbb{R}^n$ as a subset of itself we see that $\mathbb{R}^n$ is not bounded so $\mathbb{R}^n$ is not compact. In the same way, $\mathbb{C}^n$ is not bounded so $\mathbb{C}^n$ is not compact either. □

Proof. **2** Let X be a discrete metric space with infinitely many point. Let also $(x_n) \subseteq X$ be a sequence of X such that $d(x_i, x_j) = 1$ for every $i \neq j$. Suppose $(x_{n_k}) \subseteq (x_n)$ is a convergent subsequence, we want to arrive at a contradiction.

Given that (x_{n_k}) is convergent then (x_{n_k}) is also Cauchy. Let $\epsilon = 1/2$ then there must be $N \in \mathbb{N}$ such that when $k, j \geq N$ we get that

$$d(x_{n_k}, x_{n_j}) < \epsilon$$

but this cannot happen since $d(x_{n_k}, x_{n_j}) = 1$ by definition for every k and j, so we arrived at a contradition.

Therefore X is not compact. □

Proof. **3** Let us define a function $f : [0,1] \to \mathbb{R}^2$ as $f(x) = (x, x)$ we see that $[0, 1]$ is closed and bounded so it is compact because of Theorem 2.5-3 in addition f is continuous and hence by Theorem 2.5-6 we have that $f([0, 1])$ is compact.

On the other hand, let us consider the graph of x^2 i.e. $\{(x, x^2) : x \in \mathbb{R}\}$ then we see that x^2 is not bounded so the graph of x^2 is not compact. $\square$

Proof. **5** Let $x \in \mathbb{R}$ then we have the set $[x - 1, x + 1] \subset \mathbb{R}$ which is closed and bounded and thus compact. Therefore since x was arbitrary then $\mathbb{R}$ is locally compact.

Let $z \in \mathbb{C}$ where z has the form $z = a + bi$ for some $a, b \in \mathbb{R}$ then the set

$$A = \{x + yi : a - 1 \leq x \leq a + 1, b - 1 \leq y \leq b + 1\}$$

is closed and bounded and thus compact. Therefore $\mathbb{C}$ is locally compact.

Let $x \in \mathbb{R}^n$ where x has the form $(x_1, ..., x_n)$ then the set

$$\{y \in \mathbb{R}^n : \|x - y\| \leq 1\}$$

which is a closed ball in $\mathbb{R}^n$ is closed and bounded and thus compact. Therefore since x was arbitrary then $\mathbb{R}^n$ is locally compact.

Let $z \in \mathbb{C}^n$ where z has the form $(z_1, ..., z_n)$ then the set

$$\{y \in \mathbb{C}^n : \|z - y\| \leq 1\}$$

which is a closed ball in $\mathbb{C}^n$ is closed and bounded and thus compact. Therefore since z was arbitrary then $\mathbb{C}^n$ is locally compact. $\square$

Proof. **6** Let X be a compact metric space so if we take $x \in X$ we see that X itself is a compact neighborhood of x. Therefore X is locally compact. $\square$

Proof. **9** Let X be a compact metric space and $M \subseteq X$ be a closed set. Let also $(x_n) \subseteq M$ to be a sequence in M then (x_n) has a subsequence (x_{n_k}) that converges to some $x \in X$ since X is compact. But also we know that M is closed so it must happen that $x \in M$. Therefore M is compact. $\square$

Proof. **10** Let X and Y be metric spaces, X compact, and $T : X \to Y$ bijective and continuous. We want to show that T is a homeomorphism.

Let $M \subseteq X$ be a closed set then M is compact because what we proved in Problem 9 then $T(M)$ is also compact and therefore closed. This implies that T^{-1} is continuous.

Finally, since T is bijective and continuous and T^{-1} is continuous then T is a homeomorphism. $\square$

2.6 - Linear Operators

Proof. **2**

- Let T_1 be an operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $(\xi_1, \xi_2) \to (\xi_1, 0)$. We want to prove it is linear.

  (i) The domain $\mathscr{D}(T_1) = \mathbb{R}^2$ is a vector space and the range $\mathscr{R}(T_1) = \mathbb{R} \times \{0\}$ is on the vector space $\mathbb{R}^2$ as well.

  (ii) Let $x, y \in \mathscr{D}(T_1)$ and α, β scalars where $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ then

$$\begin{aligned} T_1(\alpha x + \beta y) &= T_1(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) \\ &= T_1((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2)) \\ &= (\alpha\xi_1 + \beta\eta_1, 0) \\ &= \alpha(\xi_1, 0) + \beta(\eta_1, 0) \\ &= \alpha T_1 x + \beta T_1 y \end{aligned}$$

  Therefore T_1 is a linear operator. Geometrically T_1 projects every point to the x-axis.

- Let T_2 be an operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $(\xi_1, \xi_2) \to (0, \xi_2)$. We want to prove it is linear.

  (i) The domain $\mathscr{D}(T_2) = \mathbb{R}^2$ is a vector space and the range $\mathscr{R}(T_2) = \{0\} \times \mathbb{R}$ is on the vector space $\mathbb{R}^2$ as well.

  (ii) Let $x, y \in \mathscr{D}(T_2)$ and α, β scalars where $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ then

$$\begin{aligned} T_2(\alpha x + \beta y) &= T_2(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) \\ &= T_2((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2)) \\ &= (0, \alpha\xi_2 + \beta\eta_2) \\ &= \alpha(0, \xi_2) + \beta(0, \eta_2) \\ &= \alpha T_2 x + \beta T_2 y \end{aligned}$$

  Therefore T_2 is a linear operator. Geometrically T_2 projects every point to the y-axis.

- Let T_3 be an operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $(\xi_1, \xi_2) \to (\xi_2, \xi_1)$. We want to prove it is linear.

  (i) The domain $\mathscr{D}(T_3) = \mathbb{R}^2$ is a vector space and the range $\mathscr{R}(T_3) = \mathbb{R}^2$ is a vector space as well.

  (ii) Let $x, y \in \mathscr{D}(T_3)$ and α, β scalars where $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ then

$$\begin{aligned} T_3(\alpha x + \beta y) &= T_3(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) \\ &= T_3((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2)) \\ &= (\alpha\xi_2 + \beta\eta_2, \alpha\xi_1 + \beta\eta_1) \\ &= \alpha(\xi_2, \xi_1) + \beta(\eta_2, \eta_1) \\ &= \alpha T_3 x + \beta T_3 y \end{aligned}$$

  Therefore T_3 is a linear operator. Geometrically T_3 is a reflection over the $y = x$ line.

- Let T_4 be an operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $(\xi_1, \xi_2) \to (\gamma\xi_1, \gamma\xi_2)$. We want to prove it is linear.

  (i) The domain $\mathscr{D}(T_4) = \mathbb{R}^2$ is a vector space and the range $\mathscr{R}(T_4) = \mathbb{R}^2$ is a vector space as well.

  (ii) Let $x, y \in \mathscr{D}(T_4)$ and α, β scalars where $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ then

$$\begin{aligned} T_4(\alpha x + \beta y) &= T_4(\alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)) \\ &= T_4((\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2)) \\ &= (\gamma(\alpha\xi_1 + \beta\eta_1), \gamma(\alpha\xi_2 + \beta\eta_2)) \\ &= \gamma\alpha(\xi_1, \xi_2) + \gamma\beta(\eta_1, \eta_2) \\ &= \alpha(\gamma\xi_1, \gamma\xi_2) + \beta(\gamma\eta_1, \gamma\eta_2) \\ &= \alpha T_4 x + \beta T_4 y \end{aligned}$$

  Therefore T_4 is a linear operator. Geometrically T_4 is a dilation.

$\square$

Proof. **3**

- The domain $\mathscr{D}(T_1)$ of T_1 is $\mathbb{R}^2$.
  The range $\mathscr{R}(T_1)$ of T_1 is $\mathbb{R} \times \{0\}$.
  The null space $\mathscr{N}(T_1)$ of T_1 is $\{0\} \times \mathbb{R}$.

- The domain $\mathscr{D}(T_2)$ of T_2 is $\mathbb{R}^2$.
  The range $\mathscr{R}(T_2)$ of T_2 is $\{0\} \times \mathbb{R}$.
  The null space $\mathscr{N}(T_2)$ of T_2 is $\mathbb{R} \times \{0\}$.

- The domain $\mathscr{D}(T_3)$ of T_3 is $\mathbb{R}^2$.
  The range $\mathscr{R}(T_3)$ of T_3 is $\mathbb{R}^2$.
  The null space $\mathscr{N}(T_3)$ of T_3 is $\{(0,0)\}$.

□

Proof. **5** Let $T : X \to Y$ be a linear operator and let $V \subseteq X$ be a subspace. We want to show that $T(V)$ is a vector space.

Let $x, y \in V$ and α, β scalars then $\alpha x + \beta y \in V$ since V is a subspace. We know that $Tx, Ty \in T(V)$ and $T(\alpha x + \beta y) \in T(V)$ as well but also we know that $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ therefore $\alpha Tx + \beta Ty \in T(V)$ which implies that $T(V)$ is a vector space as well.

Now, let $W \subseteq Y$ be a subspace. We want to prove that $T^{-1}(W)$ is a vector space on X. We know that $T^{-1}(W)$ is defined as

$$T^{-1}(W) = \{x \in X : Tx \in W\}$$

Let $Tx, Ty \in W$ for some $x, y \in X$ and α, β scalars then $\alpha Tx + \beta Ty \in W$ since W is a subspace. We know that $x, y \in T^{-1}(W)$ by definition and since $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ which is in W we see that $\alpha x + \beta y \in T^{-1}(W)$. This implies that $T^{-1}(W)$ is a vector space on X. □

Proof. **6** Let $T : X \to Y$ and $S : Y \to Z$ be linear operators and suppose the product of these linear operators $ST : X \to Z$ exists. We want to show that ST is linear.

We know that $\mathscr{D}(ST) = X$ is a vector space and $\mathscr{R}(ST)$ lies on the vector space Z.

Let $x, y \in X$ and α, β scalars then $\alpha x + \beta y \in X$ since X is a vector space. We know that $T(\alpha x + \beta y) = \alpha Tx + \beta Ty \in Y$ since T is a linear operator but also

$$S(T(\alpha x + \beta y)) = S(\alpha Tx + \beta Ty) = \alpha S(Tx) + \beta S(Ty)$$

since S is a linear operator. Therefore

$$ST(\alpha x + \beta y) = \alpha STx + \beta STy$$

which implies that ST is a linear operator as well. □

Proof. **7** We want to check if T_1 and T_3 from Problem 2 commute so let us compute the following

$$T_1(T_3(\xi_1, \xi_2)) = T_1(\xi_2, \xi_1) = (\xi_2, 0)$$

But

$$T_3(T_1(\xi_1, \xi_2)) = T_3(\xi_1, 0) = (0, \xi_1)$$

We see that $(T_1T_3)x \neq (T_3T_1)x$ for any $x \in \mathbb{R}^2$ therefore T_1 T_3 do not commute. □

Proof. **8** We want to write the operators in Problem 2 using 2x2 matrices. Hence we have that

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad T_4 = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$$

□

Proof. **13** Let $T : \mathscr{D}(T) \to Y$ be a linear operator whose inverse exists. Also, let $\{x_1, ..., x_n\}$ be a linearly independent set in $\mathscr{D}(T)$ then we know that

$$\alpha_1 x_1 + ... + \alpha_n x_n = 0$$

only when $\alpha_1 = ... = \alpha_n = 0$. Applying T to this equation we have that

$$\begin{aligned} T(\alpha_1 x_1 + ... + \alpha_n x_n) &= T0 \\ \alpha_1 Tx_1 + ... + \alpha_n Tx_n &= 0 \end{aligned}$$

Where we used that $T0 = 0$ since the inverse of T exists. Let us suppose now that $Tx_1, ..., Tx_n$ are not linearly independent, then the equation is satisfied for $\alpha_1, ..., \alpha_n$ where not all are zeros, we want to arrive at a contradiction. Since T^{-1} exists then applying it to this equation gives us

$$\begin{aligned} T^{-1}(\alpha_1 Tx_1 + ... + \alpha_n Tx_n) &= T^{-1}0 \\ \alpha_1 T^{-1}(Tx_1) + ... + \alpha_n T^{-1}(Tx_n) &= 0 \\ \alpha_1 x_1 + ... + \alpha_n x_n &= 0 \end{aligned}$$

But we know that $\{x_1, ..., x_n\}$ are linearly independent so must be that $\alpha_i = 0$ for every $1 \leq i \leq n$, a contradiction. Therefore $\{Tx_1, ..., Tx_n\}$ are linearly independent. □

Proof. **14** Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$

($\Rightarrow$) By the rank-nullity theorem we know that

$$\dim \mathscr{D}(T) = \dim \mathscr{R}(T) + \dim \mathscr{N}(T)$$

But also we know that $\mathscr{R}(T) = Y$ then we have that

$$\dim X = \dim Y + \dim \mathscr{N}(T)$$

Hence must be that $\dim \mathscr{N}(T) = 0$ since $\dim X = \dim Y = n$ but this implies that the only element of $\mathscr{N}(T)$ is 0 therefore T^{-1} exists.

($\Leftarrow$) Let us suppose T^{-1} exists then for every $y \in Y$ we have a $T^{-1}(y)$ such that

$$T(T^{-1}(y)) = y$$

Therefore T is surjective and hence $\mathscr{R}(T) = Y$.

□

Proof. **15** Let $z(t) \in X$ then $z'(t) \in X$ since by definition $z(t)$ have derivatives of all orders everywhere on $\mathbb{R}$. But $z(t)$ was arbitrary so there is an $x(t)$ for each $x'(t) \in X$ hence $\mathscr{R}(T)$ is all of X for $T : X \to X$.

For T^{-1} to exist it must happen that $x'(t) = Tx(t) = 0$ implies that $x(t) = 0$ but if we take $x(t) = C$ where C is a constant then $x'(t) = 0$ and so an infinite number of functions $x(t) = C$ are sent to $x'(t) = 0$. Therefore T^{-1} doesn't exist. □

Proof. **1** Let $T_1 : X \to Y$ and $T_2 : Y \to Z$ then by applying the definition of bounded operator twice we see that

$$\|T_1(T_2x)\| \leq \|T_1\|\|T_2x\| \leq \|T_1\|\|T_2\|\|x\|$$

Hence dividing by $\|x\|$ and taking the supremum we get that

$$\sup_{\substack{x \in \mathscr{D}(T_1T_2) \\ x \neq 0}} \frac{\|T_1(T_2x)\|}{\|x\|} \leq \|T_1\|\|T_2\|$$

Therefore

$$\|T_1T_2\| \leq \|T_1\|\|T_2\|$$

In the same way, let $T : X \to X$ then we see that

$$\|T^n(x)\| = \|T(T^{n-1}(x))\| \leq \|T\|\|T^{n-1}(x)\|$$

So Applying the bounded operator definition n times we get that

$$\|T^n(x)\| \leq \|T\|\|T\|...\|T\|\|x\| = \|T\|^n\|x\|$$

Hence dividing by $\|x\|$ and taking the supremum we get that

$$\sup_{\substack{x \in \mathscr{D}(T^n) \\ x \neq 0}} \frac{\|T^n(x)\|}{\|x\|} \leq \|T\|^n$$

Therefore

$$\|T^n\| \leq \|T\|^n$$

□

Proof. **2** Let X and Y be normed spaces and let $T : X \to Y$ to be a linear operator.

($\Rightarrow$) Let T be bounded and let $D \subseteq \mathscr{D}(T)$ be a bounded subset in X then there is a scalar d such that for every $x \in D$ we have that $\|x\| \leq d$. Also, we know that there is a scalar c such that $\|Tx\| \leq c\|x\|$ for every x in D since T is bounded. Hence

$$\|Tx\| \leq c\|x\| \leq cd$$

which implies that the subset $M = \{Tx : x \in D\}$ is a bounded subset of Y. Therefore T takes bounded subsets in X to bounded subsets in Y.

($\Leftarrow$) Let T map bounded sets in X into bounded sets in Y. Then let $D \subseteq X$ be a bounded subset such that $\|x\| = 1$ for every $x \in D$ then $T(D)$ is a bounded subset of Y, hence $\|Tx\| \leq c$ for a scalar c and for every $Tx \in T(D)$.

So we have that there is some $\|T\|$ such that

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ \|x\|=1}} \|Tx\|$$

Given that we can find the norm of T and we know by definition that also

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

then

$$\|Tx\| \leq \|T\|\|x\|$$

Which implies that T is bounded.

$\square$

Proof. **5** Let $T : l^\infty \to l^\infty$ such that $Tx = (\xi_j/j)$ where $x = (\xi_j)$. We want to show it's linear and bounded.

Let $z = \alpha x + \beta y \subseteq l^\infty$ be a sequence such that $x = (\xi_j)$ and $y = (\eta_j)$then

$$\begin{aligned} T(\alpha x + \beta y) &= T(\alpha(\xi_j) + \beta(\eta_j)) \\ &= \frac{\alpha(\xi_j) + \beta(\eta_j)}{j} \\ &= \alpha\left(\frac{\xi_j}{j}\right) + \beta\left(\frac{\eta_j}{j}\right) \\ &= \alpha Tx + \beta Ty \end{aligned}$$

Therefore T is linear.

On the other hand, we see that

$$\sup_j \left|\frac{\xi_j}{j}\right| \leq \sup_j |\xi_j|$$

So we have that

$$\|Tx\|_\infty \leq \|x\|_\infty$$

Which implies that T is bounded. $\square$

Proof. **6** Let us take the linear operator $T : l^\infty \to l^\infty$ defined in Problem 5 then the range of T is

$$\mathscr{R}(T) = \{(\xi_j/j) : (\xi_j) \in l^\infty\}$$

Let us take the following sequence of sequences

$$\begin{aligned}
&\left(1, 0, 0, 0, ...\right) \\
&\left(1, \frac{\sqrt{2}}{2}, 0, 0, ...\right) = \left(1, \frac{1}{\sqrt{2}}, 0, 0, ...\right) \\
&\left(1, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{3}, 0, ...\right) = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, 0, ...\right) \\
&\quad\vdots
\end{aligned}$$

We see that all of them are in $\mathscr{R}(T)$ but this sequence of sequences tend to

$$\left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, ...\right) \notin \mathscr{R}(T)$$

Therefore $\mathscr{R}(T)$ is not closed. $\square$

Proof. **7** Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$\|Tx\| \geq b\|x\| \quad \text{for all } x \in X$$

We want to show that $T^{-1} : Y \to X$ exists and is bounded.

Let $Tx = 0$ then

$$0 \geq b\|x\|$$

which implies that $\|x\| = 0$ but then by the norms definition we have that $x = 0$. Therefore T^{-1} exists by Theorem 2.6-10.

Let $y \in Y$ then there is $T^{-1}y \in X$ then we have that

$$\|y\| \geq b\|T^{-1}y\|$$

Hence since $b > 0$ we get that

$$\begin{aligned} b\|T^{-1}y\| &\leq \|y\| \\ \|T^{-1}y\| &\leq \frac{1}{b}\|y\| \end{aligned}$$

Which implies that T^{-1} is bounded. □

Proof. **8** Let $T : l^\infty \to l^\infty$ defined by $Tx = (\xi_j/j)$ where $x = (\xi_j)$. If $Tx = (0)$ then must be that $x = (0)$, hence T^{-1} exists.

Suppose T^{-1} is bounded we want to arrive at a contradiction. Let

$$y = (1, 1, 1, ..., 1, 0, 0, ...)$$

i.e. y has n ones. Then must be that

$$T^{-1}y = (1, 2, 3, ..., n, 0, 0, ...)$$

Hence

$$\begin{aligned} \|y\| &= \sup_j |\xi_j/j| = 1 \\ \|T^{-1}y\| &= \sup_j |\xi_j| = n \end{aligned}$$

So if T^{-1} were bounded we have that

$$\begin{aligned} \|T^{-1}y\| &\leq c\|y\| \\ n &\leq c \end{aligned}$$

But if we pick $c = n+1$ then it's not going to work for a $y = (1, 1, ..., 1, 1, 0, 0, ...)$ with $n + 1$ ones therefore we have a contradiction and T^{-1} cannot be bounded. $\square$

Proof. **9** Let $T : C[0,1] \to C[0,1]$ defined by

$$y(t) = \int_0^t x(\tau)\ d\tau$$

Let $y(t)$ be a differentiable function such that $y(0) = 0$ we want to show there is some $x(t)$ such that

$$y(t) = \int_0^t x(\tau)\ d\tau$$

Let us take $x(t) = y'(t)$ then we have that

$$\int_0^t y'(\tau)\ d\tau = y(t) - y(0) = y(t)$$

Since $y(t)$ is the antiderivative of $x(t)$ by definition. Then $\mathscr{R}(T)$ is every continuous function $f : [0,1] \to \mathbb{R}$ such that f is differentiable and $f(0) = 0$.

The inverse T^{-1} is the map that takes continuous functions $f : [0,1] \to \mathbb{R}$ to their derivatives $f' : [0,1] \to \mathbb{R}$ hence $T^{-1} : \mathscr{R}(T) \to C[0,1]$ is defined as

$$T^{-1}y(t) = y'(t)$$

T^{-1} is the differentiation operator and we saw that this operator is linear but not bounded in Example 2.7-5. □

2.8 - Linear Functionals

Proof. **2** Let the functional

$$f_1(x) = \int_a^b x(t)y_0(t)\ dt$$

Where $y_0 \in C[a,b]$.
Let $x, y \in C[a,b]$ and α, β be scalars then

$$\begin{aligned}
\alpha f_1(x) + \beta f_1(y) &= \alpha \int_a^b x(t)y_0(t)\ dt + \beta \int_a^b y(t)y_0(t)\ dt \\
&= \int_a^b \alpha x(t)y_0(t) + \beta y(t)y_0(t)\ dt \\
&= \int_a^b (\alpha x(t) + \beta y(t))y_0(t)\ dt \\
&= f_1(\alpha x + \beta y)
\end{aligned}$$

Then f_1 is linear.
On the other hand, we see that

$$|f_1(x)| = \left| \int_a^b x(t)y_0(t)\ dt \right| \leq (b-a) \max_{t\in[a,b]} |x(t)| \max_{t\in[a,b]} |y_0(t)| = (b-a)\|x\|\|y_0\|$$

Given that $\|y_0\|$ is a constant then taking $c = (b-a)\|y_0\|$ we get that

$$|f_1(x)| \leq (b-a)\|y_0\|\|x\|$$

Therefore f_1 is bounded.

Let now the functional

$$f_2(x) = \alpha x(a) + \beta x(b)$$

Where α, β are fixed.
Let $x, y \in C[a,b]$ and δ, η be scalars then

$$\begin{aligned}
\delta f_2(x) + \eta f_2(y) &= \delta(\alpha x(a) + \beta x(b)) + \eta(\alpha y(a) + \beta y(b)) \\
&= \alpha(\delta x(a) + \eta y(a)) + \beta(\delta x(b) + \eta y(b)) \\
&= f_2(\delta x + \eta y)
\end{aligned}$$

Then f_2 is linear.
On the other hand, we see that

$$\begin{aligned}
|f_2(x)| = |\alpha x(a) + \beta x(b)| &\leq |\alpha||x(a)| + |\beta||x(b)| \\
&\leq (|\alpha| + |\beta|) \max(|x(a)|, |x(b)|) \\
&\leq (|\alpha| + |\beta|) \max_{t\in C[a,b]} |x(t)| = (|\alpha| + |\beta|)\|x\|
\end{aligned}$$

Therefore taking $c = |\alpha| + |\beta|$ we see that f_2 is bounded. □

Proof. **4** Let the functional

$$f_1(x) = \max_{t \in J} x(t)$$

Where $J = [a, b]$.
Let $a = 0$, $b = 1$ and

$$x(t) = t \text{ and } y(t) = -t$$

Also, let $\alpha = \beta = 1$ then

$$\alpha f_1(x) + \beta f_1(y) = \max_{t \in [0,1]} x(t) + \max_{t \in [0,1]} y(t) = 1 + 0 = 1$$

But

$$f_1(\alpha x + \beta y) = \max_{t \in [0,1]} (t - t) = 0$$

Therefore $\alpha f_1(x) + \beta f_1(y) \neq f_1(\alpha x + \beta y)$ and hence f_1 is not linear.
On the other hand, we see that

$$|f_1(x)| = |\max_{t \in J} x(t)| \leq \max_{t \in J} |x(t)| = \|x(t)\|$$

then taking $c = 1$ we get that

$$|f_1(x)| \leq \|x(t)\|$$

Therefore f_1 is bounded.

Let now the functional

$$f_2(x) = \min_{t \in J} x(t)$$

Where $J = [a, b]$.
Let $a = 0$, $b = 1$ and

$$x(t) = t \text{ and } y(t) = -t$$

Also, let $\alpha = \beta = 1$ then

$$\alpha f_2(x) + \beta f_2(y) = \min_{t \in [0,1]} x(t) + \min_{t \in [0,1]} y(t) = 0 + (-1) = -1$$

But

$$f_2(\alpha x + \beta y) = \min_{t \in [0,1]} (t - t) = 0$$

Therefore $\alpha f_2(x) + \beta f_2(y) \neq f_2(\alpha x + \beta y)$ and hence f_2 is not linear.
On the other hand, we see that

$$|f_2(x)| = |\min_{t \in J} x(t)| \leq \max_{t \in J} |x(t)| = \|x(t)\|$$

then taking $c = 1$ we get that

$$|f_2(x)| \leq \|x(t)\|$$

Therefore f_2 is bounded. $\square$

Proof. **5** Let $f(x) = \xi_n$ (n fixed) where $x = (\xi_j)$, we want to prove that f is a linear functional. Let $x = (\xi_j)$, $y = (\eta_j)$ and α, β scalars then

$$f(\alpha x + \beta y) = \alpha \xi_n + \beta \eta_n = \alpha f(x) + \beta f(y)$$

So f is a linear operator and since the domain of f is the sequence space X and the range of f is in the scalar field K of X then f defines a linear functional.
Let now $X = l^\infty$ then we see that

$$|f(x)| = |\xi_n| \leq \sup_j |\xi_j| = \|x\|_\infty$$

Therefore f is bounded. □

Proof. **11** Let $f_1 \neq 0$ and $f_2 \neq 0$ be two linear functionals, they are defined on a vector space X and they have the same null space.
Let $z \notin \mathscr{N}(f_1)$ then there is some c such that $f_1(z) = cf_2(z)$.
Let also $x \in X$ such that $x \neq z$, first we prove that x can be written as $x = az + w$ where a is some scalar and $w \in \mathscr{N}(f_1)$, note that

$$f_1(x) = f_1(az + w) = af_1(z) + f_1(w) = af_1(z)$$

So it's enough to define $a = f_1(x)/f_1(z)$. On the other hand, let us define $w = x - az$, we want to prove that w is in the null space

$$f_1(w) = f_1(x - az) = f_1(x) - af_1(z) = af_1(z) - af_1(z) = 0$$

then w is in the null space and x can be written as we want. Then we can write that

$$f_1(x) = af_1(z) + f_1(w) = acf_2(z) + f_2(w) = cf_2(az + w) = cf_2(x)$$

Therefore f_1 and f_2 are proportional. □

2.9 - Linear Operators and Functionals on Finite Dimensional Spaces

Proof. **3** The dual basis of $\{(1,0,0),(0,1,0),(0,0,1)\} \in \mathbb{R}^3$ is $F = \{f_1, f_2, f_3\}$ where

$$f_1 = (1,0,0) \quad f_2 = (0,1,0) \quad f_3 = (0,0,1)$$

Since

$$f_k(e_j) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

□

Proof. **4** Let $F = \{f_1, f_2, f_3\}$ be the dual basis of $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ where $e_1 = (1,1,1)$, $e_2 = (1,1,-1)$ and $e_3 = (1,-1,-1)$. For F to be a dual basis must happen that

$$f_k(e_j) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

For f_1 then must happen that

$$\begin{aligned} 1 \cdot \alpha_1 + 1 \cdot \alpha_2 + 1 \cdot \alpha_3 &= 1 \\ 1 \cdot \alpha_1 + 1 \cdot \alpha_2 - 1 \cdot \alpha_3 &= 0 \\ 1 \cdot \alpha_1 - 1 \cdot \alpha_2 - 1 \cdot \alpha_3 &= 0 \end{aligned}$$

The solution to this set of equations imply that $\alpha_1 = 1/2$, $\alpha_2 = 0$ and $\alpha_3 = 1/2$ i.e. $f_1 = (1/2, 0, 1/2)$. In the same way, we can find $f_2 = (0, 1/2, -1/2)$ and $f_3 = (1/2, -1/2, 0)$.
Finally, let $x = (1,0,0)$ then

$$\begin{aligned} f_1(x) &= 1 \cdot 1/2 + 0 \cdot 0 + 0 \cdot 1/2 = 1/2 \\ f_2(x) &= 1 \cdot 0 + 0 \cdot 1/2 - 0 \cdot 1/2 = 0 \\ f_3(x) &= 1 \cdot 1/2 - 0 \cdot 1/2 + 0 \cdot 0 = 1/2 \end{aligned}$$

□

Proof. **5** Let $f : X \to K$ be a linear functional on an n-dimensional vector space X. By the Rank-Nullity Theorem we know that

$$\dim(\mathscr{N}(f)) + \dim(\mathscr{R}(f)) = \dim(\mathscr{D}(f))$$

If f is not the zero functional then $\dim(\mathscr{R}(f)) = 1$ and since we know that $\dim(\mathscr{D}(f)) = n$ then must be that $\dim(\mathscr{N}(f)) = n - 1$.

On the other hand, if f is the zero functional then $\dim(\mathscr{R}(f)) = 0$ hence $\dim(\mathscr{N}(f)) = n$.

Therefore $\dim(\mathscr{N}(f))$ can be of dimension n or $n - 1$. □

Proof. **6** Let $f(x) = \xi_1 + \xi_2 - \xi_3$ we want to find a basis for the null space of f this implies we want to find $\{e_1, e_2\} \subset \mathbb{R}^3$ such that $f(e_1) = f(e_2) = 0$ where e_1 and e_2 are linearly independent.

Let $e_1 = (0, 1, 1)$ and $e_2 = (1, 0, 1)$ we see they are linearly independent and

$$\begin{aligned} f(e_1) &= 0 + 1 - 1 = 0 \\ f(e_2) &= 1 + 0 - 1 = 0 \end{aligned}$$

Therefore they are a basis for the null space of f. □

Proof. **13** Let $B_Z = \{e_1, ..., e_m\} \subset Z$ be a basis for Z where $m < n$ then we know that we can extend B_Z with $\{e_{m+1}, ..., e_n\}$ to get a basis for X.

Then let us define a linear functional $\tilde{f}$ as $\tilde{f}(e_i) = f(e_i)$ when $1 \leq i \leq m$ and $\tilde{f}(e_i) = 0$ when $m + 1 \leq i \leq n$.

Now, let $x \in X$ then x can be written as $x = \sum_{i=1}^{m} \alpha_i e_i + \sum_{j=m+1}^{n} \alpha_j e_j$ and hence

$$\tilde{f}(x) = \sum_{i=1}^{m} \alpha_i \tilde{f}(e_i) + \sum_{j=m+1}^{n} \alpha_j \tilde{f}(e_j) = \sum_{i=1}^{m} \alpha_i f(e_i)$$

And we see that if $x \in Z$ then

$$\tilde{f}(x) = \sum_{i=1}^{m} \alpha_i \tilde{f}(e_i) = \sum_{i=1}^{m} \alpha_i f(e_i)$$

Therefore $\tilde{f}$ and f agree on $x \in Z$. □

Proof. **1** Let $T \in B(X,Y)$ and let us call the zero element of the vector space $B(X,Y)$ as $\theta \in B(X,Y)$ then $\theta : X \to Y$ must take any element of $x \in X$ to the zero element in Y let us call it $0 \in Y$ so we have that

$$(T+\theta)x = Tx + \theta x = Tx + 0 = Tx$$

Hence θ has the property $T + \theta = T$ for any $T \in B(X,Y)$.

On the other hand, let us define the additive inverse of T as $(-T) \in B(X,Y)$ such that it sends an element $x \in X$ to the additive inverse of $Tx \in Y$ then we have that

$$(T+(-T))x = Tx + (-T)x = 0 = \theta x$$

So we can write that $T + (-T) = \theta$. □

Proof. **2** Let f, g be bounded linear functionals and α, β be nonzero scalars. Let $x, y \in \mathscr{D}(f) \cap \mathscr{D}(g)$ and let δ, η be scalars then we see that

$$\begin{aligned} h(\delta x + \eta y) &= \alpha f(\delta x + \eta y) + \beta g(\delta x + \eta y) \\ &= \alpha(\delta f(x) + \eta f(y)) + \beta(\delta g(x) + \eta g(y)) \\ &= \delta(\alpha f(x) + \beta g(x)) + \eta(\alpha f(y) + \beta g(y)) \\ &= \delta h(x) + \eta h(y) \end{aligned}$$

This implies that h is a linear operator. Also, since $\mathscr{D}(h) = \mathscr{D}(f) \cap \mathscr{D}(g) \in X$ and $\mathscr{R}(h)$ is in the scalar field K of X (as $\mathscr{R}(f)$ and $\mathscr{R}(g)$ are) then h is a linear functional.

On the other hand, we know that f and g are bounded so there are two scalars c, d such that

$$|f(x)| \leq c\|x\| \quad \text{and} \quad |g(x)| \leq d\|x\|$$

for all $x \in \mathscr{D}(f) \cap \mathscr{D}(g)$. Also, we have that

$$|h(x)| = |\alpha f(x) + \beta g(x)| \leq |\alpha f(x)| + |\beta g(x)| = |\alpha||f(x)| + |\beta||g(x)|$$

Then

$$|h(x)| \leq |\alpha||f(x)| + |\beta||g(x)| \leq |\alpha|c\|x\| + |\beta|d\|x\| = (|\alpha|c + |\beta|d)\|x\|$$

Therefore taking the constant $|\alpha|c + |\beta|d$ the linear functional h is also bounded over the domain $\mathscr{D}(f) \cap \mathscr{D}(g)$. □

Proof. **3** Let T_1, T_2 be bounded linear operators and α, β be nonzero scalars. Let $x, y \in \mathscr{D}(T_1) \cap \mathscr{D}(T_2)$ and let δ, η be scalars then we see that

$$\begin{aligned} T(\delta x + \eta y) &= \alpha T_1(\delta x + \eta y) + \beta T_2(\delta x + \eta y) \\ &= \alpha(\delta T_1(x) + \eta T_1(y)) + \beta(\delta T_2(x) + \eta T_2(y)) \\ &= \delta(\alpha T_1(x) + \beta T_2(x)) + \eta(\alpha T_1(y) + \beta T_2(y)) \\ &= \delta T(x) + \eta T(y) \end{aligned}$$

This implies that T is a linear operator.

On the other hand, we know that T_1 and T_2 are bounded so there are two scalars c, d such that

$$\|T_1(x)\| \leq c\|x\| \quad \text{and} \quad \|T_2(x)\| \leq d\|x\|$$

for all $x \in \mathscr{D}(T_1) \cap \mathscr{D}(T_2)$. Also, we have that

$$\|T(x)\| = \|\alpha T_1(x) + \beta T_2(x)\| \leq \|\alpha T_1(x)\| + \|\beta T_2(x)\| = |\alpha|\|T_1(x)\| + |\beta|\|T_2(x)\|$$

Then

$$\|T(x)\| \leq |\alpha|\|T_1(x)\| + |\beta|\|T_2(x)\| \leq |\alpha|c\|x\| + |\beta|d\|x\| = (|\alpha|c + |\beta|d)\|x\|$$

Therefore taking the constant $|\alpha|c + |\beta|d$ the linear operator T is also bounded over the domain $\mathscr{D}(T_1) \cap \mathscr{D}(T_2)$. $\square$

Proof. **6** Let X be the space of ordered n-tuples of real numbers with norm $\|x\| = \max_j |\xi_j|$ where $x = (\xi_1, ..., \xi_n)$. Then $f \in X'$ can be written as $f(x) = a_1\xi_1 + ... + a_n\xi_n$ so we have that

$$\begin{aligned}|f(x)| = |a_1\xi_1 + ... + a_n\xi_n| \leq |a_1||\xi_1| + ... + |a_n||\xi_n| \leq \\ \leq |a_1| \max_j |\xi_j| + ... + |a_n| \max_j |\xi_j| = |a_1|\|x\| + ... + |a_n|\|x\|\end{aligned}$$

Then taking the supremum on both sides for any $x \in X$ such that $\|x\| = 1$ we get that

$$\sup_{\substack{x\in X \\ \|x\|=1}} |f(x)| \leq |a_1| + ... + |a_n|$$

But from the first inequality we see that if we take x such that $\xi_j = \pm 1$ such that $a_j\xi_j \geq 0$ we get that

$$|f(x)| = |a_1 + ... + a_n| = |a_1| + ... + |a_n|$$

Therefore we found x such that

$$\sup_{\substack{x\in X \\ \|x\|=1}} |f(x)| = |a_1| + ... + |a_n|$$

□

Proof. **8** A Schauder basis for c_0 is (e_k) where $e_k = (\delta_{kj})$. Let $x \in c_0$ then we can write x as

$$x = \sum_{k=1}^{\infty} \xi_k e_k$$

Also, let $f \in c_0'$ where c_0' is the dual space of c_0. Since f is linear and bounded we have that

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k$$

Where $\gamma_k = f(e_k)$. Now, let $x_n = (\xi_k^{(n)}) \in c_0$ defined as

$$\xi_k^{(n)} = \begin{cases} -1 & \text{if } \gamma_k < 0 \text{ and } k \leq n \\ 1 & \text{if } \gamma_k > 0 \text{ and } k \leq n \\ 0 & \text{if } \gamma_k = 0 \text{ or } k > n \end{cases}$$

Then we have that

$$|f(x_n)| = \left| \sum_{k=1}^{n} \xi_k^{(n)} \gamma_k \right| = \left| \sum_{k=1}^{n} \text{sign}(\gamma_k)\gamma_k \right| = \sum_{k=1}^{n} |\gamma_k| \leq \|f\| \|x_n\| = \|f\|$$

Where we used that $\|x_n\| = \sup_j \xi_j^{(n)} = 1$. Since n is arbitrary, letting $n \to \infty$ we obtain that

$$\sum_{k=1}^{\infty} |\gamma_k| \leq \|f\|$$

This shows that $(\gamma_k) \in l^1$.

Conversely, let $b = (\beta_k) \in l^1$ we can get a corresponding bounded linear functional g on c_0. In fact, we may define g on c_0 by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$$

Where $x = (\xi_k) \in c_0$. Then g is linear, also we see that

$$|g(x)| = \left| \sum_{k=1}^{\infty} \xi_k \beta_k \right| \leq \sum_{k=1}^{\infty} |\xi_k||\beta_k| \leq \sup_j |\xi_j| \sum_{k=1}^{\infty} |\beta_k| \leq \|x\| \sum_{k=1}^{\infty} |\beta_k|$$

But $\sum_{k=1}^{\infty} |\beta_k| < \infty$ then g is bounded and hence $g \in c_0'$.

Finally, we show that the norm of f is the norm on the space l^1. We see that

$$|f(x)| \leq \sum_{k=1}^{\infty} |\xi_k||\gamma_k| \leq \sup_j |\xi_j| \sum_{k=1}^{\infty} |\gamma_k| \leq \|x\| \sum_{k=1}^{\infty} |\gamma_k|$$

Then taking the supremum over all x of norm 1 we obtain

$$\|f\| \leq \sum_{k=1}^{\infty} |\gamma_k|$$

But also we saw that $\sum_{k=1}^{\infty} |\gamma_k| \leq \|f\|$ then must be that

$$\|f\| = \sum_{k=1}^{\infty} |\gamma_k|$$

This can be written as $\|f\| = \|c\|_1$, where $c = (\gamma_k) \in l^1$ and $\gamma_k = f(e_k)$. The mapping of c_0' onto l^1 defined by $f \to c$ is linear and bijective, and it is norm preserving, so it is an isomorphism. $\square$

Proof. **12** We know that if Y is a Banach space, then $B(X, Y)$ is a Banach space. Where $B(X, Y)$ is the set of all bounded linear operators from X into Y.

In the case of bounded linear functionals we have that $Y = \mathbb{R}$ then since $\mathbb{R}$ is a Banach space we have that $B(X, \mathbb{R})$ is a Banach space.

The dual space of $\mathbb{R}^n$ is $B(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^n$ which implies that $\mathbb{R}^n$ is a Banach space and hence complete.

The dual space of l^1 is $B(l^1, \mathbb{R}) = l^\infty$ which implies that l^∞ is a Banach space and hence complete.

Finally, the dual space of l^p is $B(l^p, \mathbb{R}) = l^q$ which implies that l^q is a Banach space and hence complete. $\square$