Solved selected problems of Introductory functional analysis with applications - Erwin Kreyszig

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Chapter 1 - Metric Spaces

1.2 - Further Examples of Metric Spaces

Proof. **4** Let us consider a sequence $x_n = 1/\log(n)$ we see that $\lim_{n\to\infty} 1/\log(n) = 0$ but $\sum_{n=1}^{\infty} |1/\log(n)|^p$ diverges for any $1 \leq p < \infty$ therefore $x_n \notin l^p$. $\square$

Proof. **5** Let us consider a sequence $x_n = 1/n$ we see that $\sum_{n=1}^{\infty} |1/n|$ diverges so $x_n \notin l^1$ but $\sum_{n=1}^{\infty} |1/n|^p$ doesn't diverge, therefore $x_n \in l^p$ for any $1 < p < \infty$. $\square$

1.3 - Open Set, Closed Set, Neighborhood

Proof. **13**

($\Rightarrow$) Let X be a separable metric space then X has a countable dense set Y. Let $x \in X$ we know that for each neighborhood of x no matter how small there is a point $y \in Y$ which is in this neighborhood hence for every $\epsilon > 0$ there is $y \in Y$ such that $d(x, y) < \epsilon$.

($\Leftarrow$) Let $Y \subseteq X$ be a countable subset with the property that for each $\epsilon > 0$ and every $x \in X$ there is $y \in Y$ such that $d(x, y) < \epsilon$. We want to prove that Y is dense in X.

Let $x \in X$ and $\epsilon > 0$ then from the property we have, there is $y \in Y$ such that $d(x, y) < \epsilon$ which implies that $y \in B(x, \epsilon)$ hence x is an accumulation point of Y but since x was arbitrary then every point of X is in the closure of Y i.e. $\overline{Y} = X$ hence Y is dense in X and since it's also countable then X is separable.

$\square$

1.4 - Examples. Completeness Proofs

Proof. **3** Let $M \subset l^\infty$ be the subspace consisting of all sequences $x = (\varepsilon_j)$ with at most finitely many nonzero terms we want to show M is not complete.

Let us take a sequence $(x_n) \subset M$ where each x_n is of the form $x_n = (1, 1/2, ..., 1/n, 0, 0, ...)$ i.e. the first n elements are $1, 1/2, ..., 1/n$ and the rest infinitely many are 0s. Then given $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that when $m, n > N$ we have that $d(x_n, x_m) < \epsilon$ since

$$d(x_n, x_m) = \sup_j |\varepsilon_j^{(m)} - \varepsilon_j^{(n)}| = \left| \frac{1}{m} - 0 \right| < \epsilon$$

assuming $m > n > N$ so (x_n) is a Cauchy sequence on M.

Now we want to prove this sequence converges to $x = (1, 1/2, 1/3, ...) = (1/n)$. Let $\epsilon > 0$ then we can find $N \in \mathbb{N}$ such that when $n \geq N$ we have that

$$d(x_n, x) = \sup_j |\varepsilon_j^{(n)} - \varepsilon_j| = \left| 0 - \frac{1}{n+1} \right| < \epsilon$$

Thus (x_n) converges to x but $x \notin M$ since x has infinitely many nonzero elements. Therefore M is not complete. □

Proof. **4** We saw in the problem 3 that a sequence $(x_n) \subset M$ where each x_n is of the form $x_n = (1, 1/2, ..., 1/n, 0, 0, ...)$ tends to $x = (1, 1/2, 1/3, ...) = (1/n)$. But $x \notin M$ since x has infinitely many nonzero elements then by Theorem 1.4-6 (b) we have that M is not closed, therefore since M is not closed then M is not complete in l^∞ by Theorem 1.4-7. □

Proof. **8** Let $Y \subset C[a, b]$ be the set of $x \in C[a, b]$ such that $x(a) = x(b)$. We want to prove that Y is complete.

Let $(x_n) \subseteq Y$ be a sequence such that $x_n \to x$ we want to prove that $x \in Y$. Let $\epsilon > 0$ then there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that

$$d(x_n, x) = \max_{t \in [a,b]} |x_n(t) - x(t)| < \epsilon$$

Then $|x_n(t) - x(t)| < \epsilon$ for every $t \in [a, b]$. Let us take $n = N$ then by the triangle inequality for numbers we have that

$$\begin{aligned} |x(a) - x(b)| &\leq |x(a) - x_N(a)| + |x_N(a) - x(b)| \\ &\leq |x(a) - x_N(a)| + |x_N(a) - x_N(b)| + |x_N(b) - x(b)| \\ &< 2\epsilon \end{aligned}$$

This implies that $x(a) = x(b)$. Therefore $x \in Y$ which implies that Y is closed in $C[a, b]$ by Theorem 1.4-6(b) and finally by Theorem 1.4-7 we have that Y is complete. □

1.6 - Completion of Metric Spaces

Proof. **10** Let (x_n) and (x'_n) be convergent sequences in a metric space (X, d) where they have the same limit l. We want to prove that

$$\lim_{n\to\infty} d(x_n, x'_n) = 0$$

Let $\epsilon/2 > 0$ then there is $N, N' \in \mathbb{N}$ such that when $n \geq N$ we have that $d(x_n, l) < \epsilon/2$ and when $m \geq N'$ we have that $d(x'_m, l) < \epsilon/2$. Let us take $M = \max(N, N')$ then when $n \geq M$ we have that $d(x_n, l) < \epsilon/2$ and that $d(x'_n, l) < \epsilon/2$. So by applying the triangle inequality to $d(x_n, x'_n)$ we have that

$$d(x_n, x'_n) \leq d(x_n, l) + d(x'_n, l) < \epsilon$$

but also since $d(x_n, x'_n) \geq 0$ we must have that

$$-\epsilon < -(d(x_n, l) + d(x'_n, l)) < d(x_n, x'_n)$$

Adding both results we get that

$$|d(x_n, x'_n) - 0| = |d(x_n, x'_n)| < \epsilon$$

Which implies that $\lim_{n\to\infty} d(x_n, x'_n) = 0$. $\square$

Proof. **11** Let Y be the set of all Cauchy sequences of elements of X, we want to prove that $\lim_{n\to\infty} d(x_n, x'_n) = 0$ defines an equivalence relation on Y.

(a) Let $(x_n) \in Y$ then we see that

$$\lim_{n\to\infty} d(x_n, x_n) = 0$$

which implies that $\lim_{n\to\infty} d(x_n, x'_n) = 0$ is reflexive.

(b) Let $(x_n), (x'_n) \in Y$ such that $\lim_{n\to\infty} d(x_n, x'_n) = 0$ then by the properties of the metric d we have that $\lim_{n\to\infty} d(x'_n, x_n) = 0$ is also true. Then the relation is also symmetric.

(c) Let $(x_n), (y_n), (z_n) \in Y$ such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(y_n, z_n) = 0$$

This implies that given $\epsilon/2 > 0$ there are some $N, N' \in \mathbb{N}$ such that when $n \geq N$ we have that $|d(x_n, y_n)| < \epsilon/2$ and when $n \geq N'$ we have that $|d(y_n, z_n)| < \epsilon/2$. Let us select $M = \max(N, N')$ then when $n \geq M$ we have that $|d(x_n, y_n)| < \epsilon/2$ and $|d(y_n, z_n)| < \epsilon/2$.

But also by the triangle inequality for metrics, we know that

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \epsilon$$

Also, since $d(x_n, z_n) \geq 0$ we must have that

$$-\epsilon < -(d(x_n, y_n) + d(y_n, z_n)) < d(x_n, z_n)$$

Adding both results we get that

$$|d(x_n, z_n) - 0| = |d(x_n, z_n)| < \epsilon$$

Which implies that $\lim_{n\to\infty} d(x_n, z_n) = 0$ and hence the relation is also transitive.

Therefore $\lim_{n\to\infty} d(x_n, x'_n) = 0$ defines an equivalence relation on Y. $\square$

Proof. **12** Let $(x_n) \subseteq (X, d)$ be a Cauchy sequence and let $(x'_n) \subseteq (X, d)$ such that $\lim_{n\to\infty} d(x_n, x'_n) = 0$, we want to show that (x'_n) is also Cauchy. Let $\epsilon/3 > 0$ then since (x_n) is Cauchy we know there is $N \in \mathbb{N}$ such that when $n, m \geq N$ we have that

$$d(x_n, x_m) < \epsilon/3$$

Also, we know that for the same $\epsilon/3 > 0$ there is $N' \in \mathbb{N}$ such that when $n \geq N'$ we have that

$$d(x_n, x'_n) = |d(x_n, x'_n) - 0| < \epsilon/3$$

On the other hand, by the triangle inequality applied twice, we know that

$$\begin{aligned} d(x'_n, x'_m) &\leq d(x'_n, x_n) + d(x_n, x'_m) \\ &\leq d(x'_n, x_n) + d(x_n, x_m) + d(x_m, x'_m) \end{aligned}$$

So if we take $M = \max(N, N')$ when $n, m \geq M$ we have that

$$d(x'_n, x'_m) \leq d(x'_n, x_n) + d(x_n, x_m) + d(x_m, x'_m) < \epsilon$$

This implies that (x'_n) is also Cauchy as we wanted. $\square$