Solved selected problems of General Relativity - Thomas A. Moore

Franco Zacco

Chapter 9 - The Schwarzschild Metric

Solution. **BOX 9.1** - Exercise 9.1.1. Applying the binomial approximation to the integral

$$\Delta s = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - 2GM/r}}$$

We get that

$$\Delta s \approx \int_{r_A}^{r_B} (1 + GM/r)\ dr$$

Therefore

$$\Delta s \approx \Big[r + GM\ln(r)\Big]_{r_A}^{r_B} = (r_B - r_A) + GM\ln\left(\frac{r_B}{r_A}\right)$$

if $2GM \ll r$. $\square$

Solution. **BOX 9.1** - Exercise 9.1.2. For every point above the surface of the earth we get that $r \approx 6380\ km$ and $GM = 4.44\ mm$ then we have that $2GM \ll r$ so we can apply the approximation. Therefore the extra distance beyond 100 km is

$$\begin{aligned}\Delta s &= (6480 - 6380) + 4.44 \times 10^{-6}\ln\left(\frac{6480}{6380}\right)\\ &= 100 + ((4.44 \times 10^{-6}) \cdot 0.015552)\\ &= 100.00000006905272\ km\end{aligned}$$

$\square$

Solution. **BOX 9.2** - Exercise 9.2.1. From the tensor equation $\boldsymbol{u} \cdot \boldsymbol{u} = u^\mu g_{\mu\nu} u^\nu = -1$ assuming that the spatial components of u^μ are all zero we have that

$$\begin{aligned} u^t g_{tt} u^t &= -1 \\ \left(1 - \frac{r_s}{r}\right)(u^t)^2 &= 1 \\ (u^t)^2 &= \frac{1}{1 - 2GM/r} \\ u^t &= \sqrt{\frac{1}{1 - 2GM/r}} \\ u^t &= \left(1 - \frac{2GM}{r}\right)^{-1/2} \end{aligned}$$

□

Solution. **BOX 9.2** - Exercise 9.2.2. We know that the geodesic equation for $\gamma = r$ is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -g^{r\alpha}(\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\alpha g_{\mu\nu}) u^\mu u^\nu$$

but we know that the only nonzero terms in the implicit sums for μ and ν are the t components so

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -g^{r\alpha}(\partial_t g_{\alpha t} - \frac{1}{2}\partial_\alpha g_{tt})(u^t)^2$$

Also, no element of the metric depends on t so the derivatives with respect to t are zero

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{1}{2} g^{r\alpha}\partial_\alpha g_{tt}(u^t)^2$$

Finally, the g_{tt} component only depends on the r component so the derivatives with respect to θ, ϕ and t are zero, hence

$$\begin{aligned} \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} &= \frac{1}{2} g^{rr}\partial_r g_{tt}(u^t)^2 \\ &= \frac{1}{2}\frac{1}{g_{rr}}\partial_r g_{tt}(u^t)^2 \\ &= \frac{1}{2}\frac{1}{(1 - r_s/r)^{-1}}\left(-\frac{r_s}{r^2}\right)(1 - r_s/r)^{-1} \\ &= -\frac{1}{2}\frac{2GM}{r^2} \end{aligned}$$

Where we used that $r_s = 2GM$ and $(u^t)^2 = (1 - r_s/r)^{-1}$. Therefore

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{GM}{r^2}$$

□

Solution. **BOX 9.3** - Exercise 9.3.1.
Given that $G = 7.426 \times 10^{-28}\ m/kg$ we have for the earth that

$$GM_{earth} = 7.426 \times 10^{-28}\ m/kg \cdot 5.97 \times 10^{24}\ kg = 0.00443\ m = 4.43\ mm$$

and for the sun

$$GM_{sun} = 7.426 \times 10^{-28}\ m/kg \cdot 1.988 \times 10^{30}\ kg = 1476.28\ m$$

□

Solution. **BOX 9.4** - Exercise 9.4.1.
After the approximations we know that

$$\frac{\lambda_R}{\lambda_E} = 1 + \frac{GM}{r_E}\left(1 - \frac{1}{1 + h/r_E}\right) = 1 + \frac{GM}{r_E}\left(1 - \left(1 + \frac{h}{r_E}\right)^{-1}\right)$$

If we assume that $h \ll r_E$ we can apply the binomial approximation as follows

$$\begin{aligned}\frac{\lambda_R}{\lambda_E} &= 1 + \frac{GM}{r_E}\left(1 - \left(1 - \frac{h}{r_E}\right)\right) \\ &= 1 + \frac{GM}{r_E}\frac{h}{r_E} \\ &= 1 + \frac{GM}{r_E^2}h \\ &= 1 + gh\end{aligned}$$

Where $g = GM/r_E^2$. □

Solution. **BOX 9.4** - Exercise 9.4.2.
Knowing that $u_R^t \approx 1$, $u_R^x = u_R^y = 0$ and $u_R^z \approx v \approx gh$ but also that $p^t = E_E$, $p^x = p^y = 0$ and $p^z = E_E$, we get by using the equation $E_R = -\boldsymbol{p} \cdot \boldsymbol{u}_R$ that

$$\begin{aligned}E_R = -\boldsymbol{p} \cdot \boldsymbol{u}_R &= -(-E_E \cdot 1 + 0 + 0 + E_E \cdot gh) \\ &= E_E - E_E \cdot gh \\ &= E_E(1 - gh)\end{aligned}$$

Hence

$$\frac{\lambda_R}{\lambda_E} = \frac{E_E}{E_R} = \frac{1}{1 - gh} = (1 - gh)^{-1}$$

And by the binomial approximation we get that

$$\frac{\lambda_R}{\lambda_E} = 1 + gh$$

□

Solution. **P9.1**

a. In this case, we have that

$$g = \frac{GM}{R_S^2} = \frac{7.426 \times 10^{-31} \cdot 3 \times 10^{30}}{12^2} = 0.01547\ km^{-1}$$

Then the fractional redshift using the approximate method is

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = \frac{\lambda_R}{\lambda_E} - 1 = gh$$

Where $h = R_R - R_S = 5\ km$ then

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = 0.01547\ km^{-1} \cdot 5\ km = 0.07735$$

If we evaluate g halfway between the surface and the spaceship we get that

$$g = \frac{7.426 \times 10^{-28} \cdot 3 \times 10^{30}}{14.5^2} = 0.01059\ km^{-1}$$

And hence

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = 0.01059\ km^{-1} \cdot 5\ km = 0.05295$$

b. The fractional redshift using the exact formula gives us

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = \frac{\lambda_R}{\lambda_E} - 1 = \sqrt{\frac{1 - 2GM/R_R}{1 - 2GM/R_S}} - 1$$

Hence

$$\frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - (2 \cdot 2.2278/17)}{1 - (2 \cdot 2.2278/12)}} - 1 = 0.08337$$

$\square$

Solution. **P9.3** Let us consider a sphere of radius R then the spherical metric states that

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2$$

Let us consider a point on the sphere of fixed latitude angle θ if we use latitude-longitude coordinates for the sphere. Then the length of a differential step from the north pole in the direction of θ following the surface of the sphere ($d\phi = 0$) is

$$ds = R d\theta$$

and hence by integration between 0 and θ we get that the radial distance along the surface of a sphere is $R\theta$. □

Solution. **P9.4** We know the total radial distance between two points with r coordinates r_A and r_B is

$$\Delta s = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - 2GM/r}}$$

Let $u = 2GM/r$ then $du = -2GMdr/r^2$ and hence $dr = -2GMdu/u^2$ the integral then becomes

$$\begin{aligned}
\Delta s &= -2GM \int_{u_A}^{u_B} \frac{du}{u^2\sqrt{1-u}} \\
&= 2GM \left[\frac{\sqrt{1-u}}{u} + \arctan\bigl(\sqrt{1-u}\bigr) \right]_{u_A}^{u_B} \\
&= 2GM \left[\frac{\sqrt{1-2GM/r}}{2GM/r} + \arctan\left(\sqrt{1 - \frac{2GM}{r}} \right) \right]_{r_A}^{r_B} \\
&= \left[r\sqrt{1 - \frac{2GM}{r}} + 2GM \arctan\left(\sqrt{1 - \frac{2GM}{r}} \right) \right]_{r_A}^{r_B}
\end{aligned}$$

$\square$

Solution. **P9.5** Given that the inner shell has a circumference of $6\pi GM$ and the outer shell has a circumference of $20\pi GM$ then the radial coordinate for each of them are $r_i = 6\pi GM/2\pi = 3GM$ and $r_o = 20\pi GM/2\pi = 10GM$.

Also, we know from problem P9.4 that the physical distance between the two shells with r coordinates r_i and r_o is

$$\Delta s = \left[r\sqrt{1 - \frac{2GM}{r}} + 2GM \arctan\left(\sqrt{1 - \frac{2GM}{r}}\right)\right]_{r_i}^{r_o}$$

Hence

$$\begin{aligned}
\Delta s &= \left[10GM\sqrt{1 - \frac{2}{10}} + 2GM \arctan\left(\sqrt{1 - \frac{2}{10}}\right) \right. \\
&\qquad \left. - 3GM\sqrt{1 - \frac{2}{3}} - 2GM \arctan\left(\sqrt{1 - \frac{2}{3}}\right)\right] \\
&= \left[\frac{20GM}{\sqrt{5}} + 2GM \arctan\left(\frac{2}{\sqrt{5}}\right) - \frac{3GM}{\sqrt{3}} - 2GM\frac{\pi}{6}\right] \\
&= GM\left[4\sqrt{5} + 2\arctan\left(\frac{2}{\sqrt{5}}\right) - \sqrt{3} - \frac{\pi}{3}\right] \\
&= 7.624\ GM
\end{aligned}$$

$\square$

Solution. **P9.6** From equation 9.20 which applies to the observer we know that

$$u^t = \left(1 - \frac{2GM}{R}\right)^{-1/2}$$

And since the observer is stationary we know that $u^r = u^\theta = u^\phi = 0$. Also, we know that $-\boldsymbol{p}\cdot\boldsymbol{u}_{obs} = E$ where E is the energy measured in the observer's frame, hence

$$\begin{aligned}
-(g_{tt}u^t p^t + g_{rr}\cdot 0\cdot p^r + g_{\theta\theta}\cdot 0\cdot p^\theta + g_{\phi\phi}\cdot 0\cdot p^\phi) &= E \\
\left(1 - \frac{2GM}{R}\right)u^t p^t &= E \\
\left(1 - \frac{2GM}{R}\right)\left(1 - \frac{2GM}{R}\right)^{-1/2} p^t &= E \\
p^t &= E\left(1 - \frac{2GM}{R}\right)^{-1/2}
\end{aligned}$$

On the other hand, for a photon we have that $\boldsymbol{p}\cdot\boldsymbol{p} = 0$ and in this case $p^\theta = p^\phi = 0$ because the photon moves radially then

$$\begin{aligned}
-\left(1 - \frac{2GM}{R}\right)(p^t)^2 + \left(1 - \frac{2GM}{R}\right)^{-1}(p^r)^2 + r^2(p^\theta)^2 + r^2\sin^2\theta(p^\phi)^2 &= 0 \\
-E^2\left(1 - \frac{2GM}{R}\right)\left(1 - \frac{2GM}{R}\right) + \left(1 - \frac{2GM}{R}\right)^{-1}(p^r)^2 &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
\left(1 - \frac{2GM}{R}\right)^{-1}(p^r)^2 &= E^2\left(1 - \frac{2GM}{R}\right)^2 \\
p^r &= E\left(1 - \frac{2GM}{R}\right)^{1/2}
\end{aligned}$$

Therefore the four momentum for the photon is

$$\boldsymbol{p} = \begin{bmatrix} E\left(1 - \frac{2GM}{R}\right)^{-1/2} \\ E\left(1 - \frac{2GM}{R}\right)^{1/2} \\ 0 \\ 0 \end{bmatrix}$$

□

Solution. **P9.7**

a. Consider a clock at rest at a given r coordinate ($dr = d\theta = d\phi = 0$). According to the metric the proper time $\Delta\tau$ that this clock measures between two events at its location is related to the coordinate difference Δt between those events as follows

$$\Delta\tau = \int d\tau = \int \sqrt{-ds^2} = \int \sqrt{dt^2 + 0 + 0 + 0} = \Delta t$$

We see that the clock's proper time between the events agrees with the t coordinate difference between those events. So the t coordinate registers the same time a clock at rest registers at any point.

b. Yes, this metric describes a spherically symmetric spacetime since the spatial part of the metric only depends on the coordinate r, not on θ or ϕ.

c. Let us consider a radial line i.e. a line made up of steps where $dt = d\theta = d\phi = 0$ then we get that

$$\begin{aligned} ds^2 &= 0 + dr^2 + 0 + 0 \\ ds &= dr \end{aligned}$$

So by integration we see that in this case the r coordinate is equivalent to the radial distance to the origin and hence the r coordinate is a radial coordinate.

Let us consider now a circle of constant r (meaning that $dr = 0$ for all steps around the circle) in the equatorial plane ($\theta = \pi/2$, so $\sin\theta = 1$, and $d\theta = 0$) at an instant of time (meaning that $dt = 0$ for all events on the circle). The physical distance around this circle is given by integration of

$$\begin{aligned} ds^2 &= 0 + 0 + 0 + R^2 \sinh^2(r/R) d\phi^2 \\ ds &= R \sinh(r/R) d\phi \end{aligned}$$

Hence

$$C = \int ds = R\sinh(r/R) \int_0^{2\pi} d\phi = 2\pi R \sinh(r/R)$$

Where C is the circumference of this circle.

Therefore the circumference will be bigger than $2\pi r$ since $R\sinh(r/R)$ is bigger than r.

d. This coordinate system doesn't have off-diagonal terms in metric tensor this imply that the basis vectors are orthogonal.

□