Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 11 - Precession of the perihelion

Solution. **BOX 11.1** - Exercise 11.1.1.
From equation (11.5) we know that

$$u = \frac{1}{r} \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\phi}$$

Replacing these values in equation (11.6) we get that

$$\tilde{E} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 \frac{l^2}{r^4} - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}$$
$$2\tilde{E} = \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 l^2 - 2GMu + u^2l^2 - 2GMl^2u^3$$

Now taking the derivative with respect to ϕ we have that

$$0 = 2l^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - 2GM\frac{\mathrm{d}u}{\mathrm{d}\phi} + 2l^2u\frac{\mathrm{d}u}{\mathrm{d}\phi} - 6GMl^2u^2\frac{\mathrm{d}u}{\mathrm{d}\phi}$$
$$0 = 2l^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - 2GM + 2l^2u - 6GMl^2u^2$$
$$0 = \frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - \frac{GM}{l^2} + u - 3GMu^2$$

Therefore

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{l^2} + 3GMu^2$$

□

Solution. **BOX 11.2** - Exercise 11.2.1.
First let us note that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{l}{r^2}$$

where we used that $d\phi/dt = l/r^2$ from (11.20). In the same way as in (11.5) we can replace r with u using the following equations

$$u = \frac{1}{r} \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\phi}$$

Then equation (11.20) becomes

$$\begin{aligned} \tilde{E}_N &= \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 \frac{l^2}{r^4} - \frac{GM}{r} + \frac{l^2}{2r^2} \\ 2\tilde{E}_N &= \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 l^2 - 2GMu + l^2u^2 \end{aligned}$$

Now taking the derivative with respect to ϕ we have that

$$\begin{aligned} 0 &= 2l^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - 2GM\frac{\mathrm{d}u}{\mathrm{d}\phi} + 2l^2u\frac{\mathrm{d}u}{\mathrm{d}\phi} \\ 0 &= 2l^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - 2GM + 2l^2u \\ 0 &= \frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - \frac{GM}{l^2} + u \end{aligned}$$

Therefore

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{l^2}$$

□

Solution. **BOX 11.3** - Exercise 11.3.1.
Equation (11.6) states that

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{l^2} + 3GMu^2$$

But if we consider an almost circular orbit then r is going to be constant and hence u too so $du/d\phi \approx 0$ hence we can write that

$$u_c = \frac{GM}{l^2} + 3GMu_c^2$$

Now, if we consider a perturbation of this orbit $u(\phi) = u_c + u_c w(\phi)$ and we replace this value in equation (11.6) we get that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}(u_c + u_c w) + u_c + u_c w &= \frac{GM}{l^2} + 3GM(u_c + u_c w)^2 \\ \frac{\mathrm{d}^2 u_c}{\mathrm{d}\phi^2} + u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + 2\frac{\mathrm{d}u_c}{\mathrm{d}\phi}\frac{\mathrm{d}w}{\mathrm{d}\phi} + w\frac{\mathrm{d}^2 u_c}{\mathrm{d}\phi^2} + u_c + u_c w &= \frac{GM}{l^2} + 3GM(u_c + u_c w)^2 \\ u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + u_c + u_c w &= \frac{GM}{l^2} + 3GM(u_c + u_c w)^2\end{aligned}$$

Where we used again that $du_c/d\phi \approx 0$. Expanding the equation and replacing the value we have for u_c we get that

$$\begin{aligned}u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \frac{GM}{l^2} + 3GMu_c^2 + u_c w &= \frac{GM}{l^2} + 3GMu_c^2 + 6GMu_c^2 w + 3GMu_c^2 w^2 \\ u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + u_c w &= 6GMu_c^2 w + 3GMu_c^2 w^2 \\ \frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + w &= 6GMu_c w + 3GMu_c w^2\end{aligned}$$

□

Solution. **BOX 11.4** - Exercise 11.4.1.
The predicted perihelion shift is given by $6\pi GM/r_c$. Hence replacing values we have that

$$\frac{6\pi \cdot 1.477}{57.9 \times 10^6} = 4.808 \times 10^{-7} \text{ radians}$$

This is the shift predicted for each period of 0.241 years so in 100 years will be

$$4.808 \times 10^{-7} \cdot \frac{100}{0.241} = 0.0001995 \text{ radians/century}$$

Equivalently in arc-second this is

$$0.0001995 \cdot \frac{180}{\pi} \cdot 3600 = 41.15 \text{ arc-second/century}$$

□

Solution. **BOX 11.5** - Exercise 11.5.1.
From equation (11.23) we have that

$$\begin{aligned}1+\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 &= \frac{1}{1-2GM/r}\\ \frac{\mathrm{d}z}{\mathrm{d}r} &= \sqrt{\frac{1}{1-2GM/r}-1}\\ \frac{\mathrm{d}z}{\mathrm{d}r} &= \sqrt{\frac{2GM}{r-2GM}}\end{aligned}$$

Then by integrating we get that

$$\begin{aligned}\int dz &= \int \sqrt{\frac{2GM}{r-2GM}}dr\\ z &= \sqrt{8GM(r-2GM)}+C\end{aligned}$$

□

Solution. **BOX 11.6** - Exercise 11.6.1.
Let α be small then we can approximate $\tan\alpha \approx \alpha$ and $\cos\alpha \approx 1-\frac{1}{2}\alpha^2$ then equation 11.16 becomes

$$\begin{aligned}2\pi-\delta &= 2\pi\left(1-\frac{1}{2}\alpha^2\right)\\ \delta &= \pi\alpha^2\end{aligned}$$

Also, by this approximation we know that $\alpha = \frac{\mathrm{d}z}{\mathrm{d}r}$ and by differentiation of $z(r)$ we get that

$$\alpha = \frac{\mathrm{d}z}{\mathrm{d}r} = \sqrt{\frac{2GM}{r-2GM}}$$

But since we are assuming that $r \gg 2GM$ we can approximate α as

$$\alpha = \sqrt{\frac{2GM}{r}}$$

Therefore replacing the value of α we get that

$$\delta = \frac{2\pi GM}{r}$$

□

Solution. **BOX 11.7** - Exercise 11.7.1.
From equations (11.29a) and (11.29b) lets compute $\frac{1}{2}(r_{n+1}+r_n)$ by adding both equation and multiplying by 1/2 then

$$\begin{aligned}
r_{n+1}+r_n &= \left[r_{n+1/2} + \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}r}{\mathrm{d}\tau}\right]} \frac{1}{2}\Delta\tau + \frac{1}{2} \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\right]} (\frac{1}{2}\Delta\tau)^2 + O(\Delta\tau^3) \right] \\
&\quad + \left[r_{n+1/2} - \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}r}{\mathrm{d}\tau}\right]} \frac{1}{2}\Delta\tau + \frac{1}{2} \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\right]} (\frac{1}{2}\Delta\tau)^2 - O(\Delta\tau^3) \right] \\
r_{n+1}+r_n &= 2r_{n+1/2} + \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\right]} (\frac{1}{2}\Delta\tau)^2 \\
\frac{1}{2}(r_{n+1}+r_n) &= r_{n+1/2} + \frac{1}{2} \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}\right]} \frac{1}{4}\Delta\tau^2
\end{aligned}$$

This implies that $\frac{1}{2}(r_{n+1}+r_n)$ is a second-order-accurate approximation for $r_{n+1/2}$.

Now, let us compute $\phi_{n+1}-\phi_n$ using equations (11.29c) and (11.29d) then

$$\begin{aligned}
\phi_{n+1}-\phi_n &= \left[\phi_{n+1/2} + \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]} \frac{1}{2}\Delta\tau + \frac{1}{2} \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}^2 \phi}{\mathrm{d}\tau^2}\right]} (\frac{1}{2}\Delta\tau)^2 + O(\Delta\tau^3) \right] \\
&\quad - \left[\phi_{n+1/2} - \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]} \frac{1}{2}\Delta\tau + \frac{1}{2} \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}^2 \phi}{\mathrm{d}\tau^2}\right]} (\frac{1}{2}\Delta\tau)^2 - O(\Delta\tau^3) \right] \\
\phi_{n+1}-\phi_n &= \underset{\text{at } \tau_{n+1/2}}{\left[\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]} \Delta\tau + 2O(\Delta\tau^3)
\end{aligned}$$

Therefore $\phi_{n+1}-\phi_n$ is a second-order-accurate approximation for $[d\phi/d\tau]\Delta\tau$ evaluated at $\tau_{n+1/2}$. □

Solution. **BOX 11.7** - Exercise 11.7.2.
Let us take equation (11.24) when $d^2 r_c/d\tau^2 = 0$ then solving for l gives us

$$\begin{aligned}
0 &= -\frac{GM}{r_c^2} + \frac{l_c^2}{r_c^3} - \frac{3GMl_c^2}{r_c^4} \\
l_c^2 \left[\frac{1}{r_c^3} - \frac{3GM}{r_c^4}\right] &= \frac{GM}{r_c^2} \\
l_c^2 &= \frac{GM}{r_c^2}\left[\frac{r_c^4}{r_c - 3GM}\right] \\
l_c^2 &= \frac{r_c^2}{r_c/GM - 3} \\
\frac{l_c}{GM} &= \frac{r_c/GM}{\sqrt{r_c/GM - 3}}
\end{aligned}$$

□

Solution. **P11.1** We know that the equation to compute the perihelion shift is

$$\frac{6\pi GM}{r_c}$$

So this implies a perihelion shift for Venus of

$$\Delta\phi_V = \frac{6\pi \cdot 1.477}{108.2 \times 10^6} = 2.573 \times 10^{-7} \text{ radians}$$

This is the shift predicted for each period of 0.615 years so in 100 years will be

$$2.573 \times 10^{-7} \cdot \frac{100}{0.615} \cdot \frac{180}{\pi} \cdot 3600 = 8.62 \text{ arcseconds/century}$$

In the same way for the Earth and Mars we have that

$$\begin{aligned}
\Delta\phi_E &= \frac{6\pi \cdot 1.477}{149.6 \times 10^6} = 1.861 \times 10^{-7} \text{ radians} \\
\Delta\phi_M &= \frac{6\pi \cdot 1.477}{227.9 \times 10^6} = 1.221 \times 10^{-7} \text{ radians}
\end{aligned}$$

And hence

$$\begin{aligned}
1.861 \times 10^{-7} \cdot \frac{100}{1} \cdot \frac{180}{\pi} \cdot 3600 &= 3.838 \text{ arcseconds/century} \\
1.221 \times 10^{-7} \cdot \frac{100}{1.881} \cdot \frac{180}{\pi} \cdot 3600 &= 1.339 \text{ arcseconds/century}
\end{aligned}$$

□

Solution. **P11.3** We know that the equation to compute the periastron shift is

$$\Delta\phi = \frac{6\pi GM}{r_c}$$

So in this case we have that

$$\Delta\phi = \frac{6\pi \cdot 2.0}{400} = 0.09424 \text{ radians} = 5.3995^\circ$$

For someone at infinity the period of the orbit is given by Kepler's law derived in the previous chapter as

$$T = \frac{2\pi}{\sqrt{GM}}\sqrt{r^3} = \frac{2\pi}{\sqrt{2}}\ \sqrt{400^3} = 35543.0635\ km = 0.1185\ s$$

Therefore the precession rate is

$$\frac{\Delta\phi}{T} = \frac{5.3995}{0.1185} = 45.56\ ^\circ/s$$

□

Solution. **P11.5** Let a spacetime metric given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

a. Let $\mu = t$ then since the metric is both diagonal and time-independent then equation 10.1 becomes

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) + 0$$

Then $-g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau} = e$ where e is a constant, hence

$$\left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = e$$

On the other hand, if $\mu = \phi$ as in Schwarzchild metric, our metric is both diagonal and independent of ϕ so we have that $g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = l$ where l is a constant then

$$r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = l$$

Which implies that for the equatorial plane $\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = l/r^2$.

b. Let us compute $\boldsymbol{u}\cdot\boldsymbol{u} = -1$ as follows

$$\begin{aligned}
-1 &= \boldsymbol{u}\cdot\boldsymbol{u} \\
-1 &= -\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \\
-1 &= -\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \\
1 &= \left(1 - \frac{2GM}{r}\right)^{-1} e^2 - \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{l^2}{r^2}
\end{aligned}$$

Where we used that the plane is equatorial i.e. $\theta = \pi/2$ but also $d\theta/d\tau = 0$ and we replaced the constants we got from **a.**

c. First, let us note that

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{l}{r^2}$$

Then the equation we determined in **b.** becomes

$$1 = \left(1 - \frac{2GM}{r}\right)^{-1} e^2 - \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 \frac{l^2}{r^4} - \frac{l^2}{r^2}$$

Now replacing $u = 1/r$ and $dr/d\phi = -r^2 du/d\phi = -(1/u^2)du/d\phi$ we get that

$$1 = \frac{e^2}{1 - 2GMu} - \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 l^2 - u^2 l^2$$

Taking the derivative with respect to ϕ we see that

$$\begin{aligned}
0 &= \frac{2e^2GM}{(1-2GMu)^2}\frac{\mathrm{d}u}{\mathrm{d}\phi} - 2l^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - 2l^2u\frac{\mathrm{d}u}{\mathrm{d}\phi} \\
0 &= \frac{2e^2GM}{(1-2GMu)^2} - 2l^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - 2l^2u \\
\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u &= \frac{e^2GM}{l^2(1-2GMu)^2}
\end{aligned}$$

d. Applying the binomial approximation on $(1-2GMu)^{-2}$ we get that

$$\begin{aligned}
\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u &= \frac{e^2GM}{l^2}(1+4GMu) \\
\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u &= \frac{e^2GM}{l^2} + 4\left(\frac{eGM}{l}\right)^2 u \\
\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + \left[1 - 4\left(\frac{eGM}{l}\right)^2\right] u &= \frac{e^2GM}{l^2}
\end{aligned}$$

e. Taking $u = u_c =$ constant equation 11.37 becomes

$$\left[1 - 4\left(\frac{eGM}{l}\right)^2\right] u_c = \frac{e^2GM}{l^2}$$

Solving this equation for $(\frac{eGM}{l})^2$ we get that

$$\begin{aligned}
u_c - 4u_c\left(\frac{eGM}{l}\right)^2 &= \left(\frac{eGM}{l}\right)^2\frac{1}{GM} \\
4u_c\left(\frac{eGM}{l}\right)^2 + \left(\frac{eGM}{l}\right)^2\frac{1}{GM} &= u_c \\
\left(\frac{eGM}{l}\right)^2 &= u_c\left(4u_c + \frac{1}{GM}\right)^{-1} \\
\left(\frac{eGM}{l}\right)^2 &= \frac{u_cGM}{4u_cGM+1}
\end{aligned}$$

f. Let us plug in the equation $u(\phi) = u_c + u_c w(\phi)$ in the equation 11.37 then

$$\begin{aligned}
u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \left[1 - 4\left(\frac{eGM}{l}\right)^2\right](u_c + u_c w) &= \frac{e^2 GM}{l^2} \\
u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \left[1 - \frac{4u_c GM}{4u_c GM + 1}\right](u_c + u_c w) &= \frac{u_c GM}{4u_c GM + 1}\frac{1}{GM} \\
u_c\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \left[\frac{4u_c GM + 1 - 4u_c GM}{4u_c GM + 1}\right](u_c + u_c w) &= \frac{u_c}{4u_c GM + 1} \\
\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \left[\frac{1}{4u_c GM + 1}\right](1 + w) &= \frac{1}{4u_c GM + 1} \\
\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \frac{w}{4u_c GM + 1} &= 0
\end{aligned}$$

Or

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} + \frac{w}{4GM/r_c + 1} = 0$$

This is formally the same as the harmonic oscillator equation and we know that the solution to the harmonic oscillator equation is

$$w(\phi) = A\cos(\omega\phi + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{1}{4GM/r_c + 1}}$$

The closest points of approach occur when the argument of the function $A\cos(\omega\phi + \phi)$ changes by 2π i.e. when

$$\omega\Delta\phi = \sqrt{\frac{1}{4GM/r_c + 1}}\Delta\phi = 2\pi$$

Hence applying the binomial approximation again assuming r_c is large enough we get that

$$\begin{aligned}
\Delta\phi &= 2\pi\left(\frac{1}{4GM/r_c + 1}\right)^{-1/2} \\
&= 2\pi\left(1 + \frac{4GM}{r_c}\right)^{1/2} \\
&\approx 2\pi + \frac{4\pi GM}{r_c}
\end{aligned}$$

Therefore the perihelion shift in this case is $4\pi GM/r_c$ i.e. $2/3$ of the perihelion shift predicted by the Schwarzchild metric $(6\pi GM/r_c)$.

$\square$

Solution. **P11.6**

a. Let us define $r = R\sin\theta$ then

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = R\cos\theta$$
$$d\theta = \frac{dr}{R\cos\theta}$$

And by replacing r and $d\theta$ in equation (10.39) we get that

$$ds^2 = \frac{dr^2}{\cos^2\theta} + r^2 d\phi^2$$
$$ds^2 = \frac{dr^2}{1 - \frac{R^2}{R^2}\sin^2\theta} + r^2 d\phi^2$$
$$ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\phi^2$$

b. Let us consider a three-dimensional Euclidean space with the following metric

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

Then we can elimintate dz from the equation by writing that

$$ds^2 = \left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right] dr^2 + r^2 d\phi^2$$

So we want to find $z(r)$ such that both metrics match hence we need that

$$1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 = \frac{1}{1 - \frac{r^2}{R^2}}$$
$$\frac{\mathrm{d}z}{\mathrm{d}r} = \sqrt{\frac{R^2}{R^2 - r^2} - 1}$$
$$\frac{\mathrm{d}z}{\mathrm{d}r} = \sqrt{\frac{r^2}{R^2 - r^2}}$$

Then by integration we get that

$$\int dz = \int \sqrt{\frac{r^2}{R^2 - r^2}}\, dr$$
$$z(r) = \sqrt{R^2 - r^2}$$

c. We see that $z(r)$ is the equation of a circle of radius R but $z(r)$ is independent of ϕ then it's rotationally invariant and therefore $z(r)$ and ϕ describe the surface of a sphere.

□

Solution. **P11.10**

a. Let us consider a circular orbit i.e. $dr = 0$ then from the metric equation for the surface of Flamm's paraboloid we have that

$$
\begin{aligned}
ds^2 &= 0 + r^2 d\phi^2 \\
ds &= r d\phi \\
\frac{\mathrm{d}\phi}{\mathrm{d}s} &= \frac{1}{r}
\end{aligned}
$$

b. The geodesic equation for this metric when $\alpha = r$ gives us

$$
0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(g_{rr}\frac{\mathrm{d}r}{\mathrm{d}s}\right) - \frac{1}{2}\left[\frac{\partial g_{rr}}{\partial r}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2\right]
$$

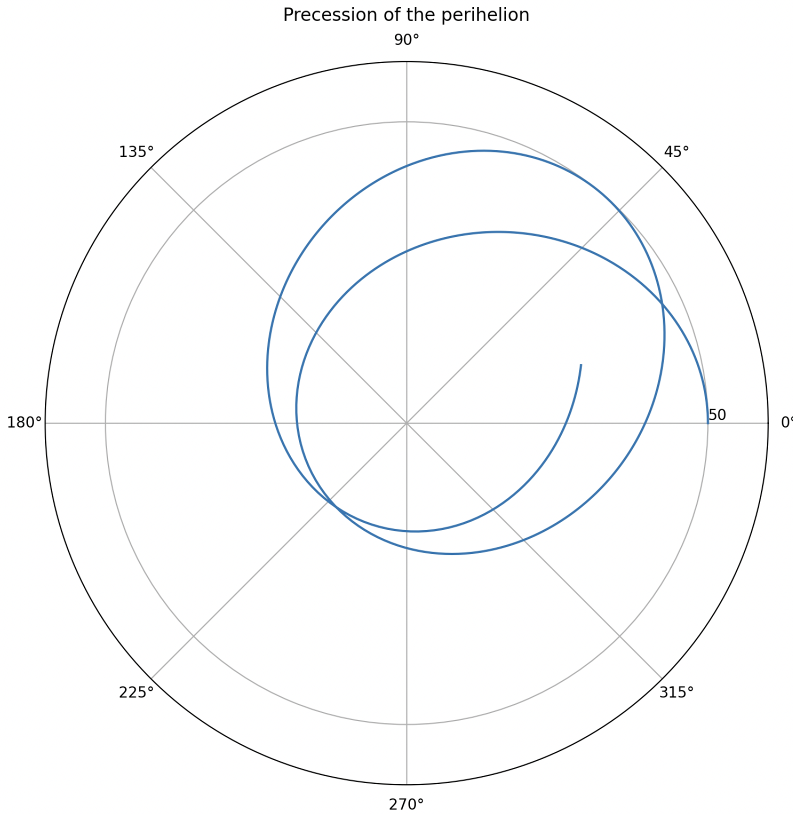
But since we are considering a circular orbit for which $dr/ds = 0$ and where $d\phi/ds = 1/r$ we get that

$$
\begin{aligned}
0 &= 0 - \frac{1}{2}\left[0 + 2r\left(\frac{1}{r}\right)^2\right] \\
0 &= -\frac{1}{r}
\end{aligned}
$$

Therefore we see that unless $r \to \infty$ the geodesic equation is not satisfied for the r component and hence circular orbits are not geodesics on Flamm's paraboloid.

$\square$

Solution. **P11.11** Computer Model for Schwarzchild Orbits.



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