Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 5 - Arbitrary Coordinates

Solution. **BOX 5.1** Let us define a coordinate basis where $\boldsymbol{e}_r$ has magnitude 1 and $\boldsymbol{e}_\theta$ has magnitude $r = 1$ then we can write $\Delta\boldsymbol{s}$ as

$$\Delta\boldsymbol{s} = \boldsymbol{e}_r + 2\boldsymbol{e}_\theta$$

but if we use polar coordinates then $\Delta\boldsymbol{s}$ is given by

$$\Delta\boldsymbol{s} = \Delta r\boldsymbol{e}_{\hat{r}} + r\Delta\theta\boldsymbol{e}_{\hat{\theta}} = \boldsymbol{e}_{\hat{r}} + \frac{\pi}{2}\boldsymbol{e}_{\hat{\theta}}$$

Which is not the same vector. □

Solution. **BOX 5.2** We know that

$$g'_{\mu\nu}dx'^{\mu}dx'^{\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}dx'^{\mu}dx'^{\nu}$$

but we can write it as

$$g'_{\mu\nu}dx'^{\mu}dx'^{\nu} - g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}dx'^{\mu}dx'^{\nu} = 0$$
$$dx'^{\mu}dx'^{\nu}\left(g'_{\mu\nu} - g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\right) = 0$$

And since this must be true for any displacement dx'^{μ}, dx'^{ν} then it must be that

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}$$

□

Solution. **BOX 5.3** - Exercise 5.3.1. We want to check equations (5.24) are the correct inverses of the equations (5.23) so suppose we start from a set of cartesian coordinates x and y then we transform them to $p = x$ and $q = y - cx^2$ so if equations (5.24) are the correct inverses from the p and q we got we should get again our x and y coordinates so we see that

$$\begin{aligned} x(p,q) &= p = x \\ y(p,q) &= cp^2 + q = cx^2 + y - cx^2 = y \end{aligned}$$

Hence they are the correct inverses. □

Solution. **BOX 5.3** - Exercise 5.3.2. Assuming p and q are represented by the unprimed coordinates x^μ and x and y are represented by the primed coordinates x'^μ we have that

$$\frac{\partial x(p,q)}{\partial p} = 1 \quad \frac{\partial x(p,q)}{\partial q} = 0 \quad \frac{\partial y(p,q)}{\partial p} = 2cp \quad \frac{\partial y(p,q)}{\partial q} = 1$$

$$\frac{\partial p(x,y)}{\partial x} = 1 \quad \frac{\partial p(x,y)}{\partial y} = 0 \quad \frac{\partial q(x,y)}{\partial x} = -2cp \quad \frac{\partial q(x,y)}{\partial y} = 1$$

□

Solution. **BOX 5.3** - Exercise 5.3.3. If $\mu = \nu = p$ we have that

$$\begin{aligned} g_{pp} &= \frac{\partial x^\alpha}{\partial p}\frac{\partial x^\beta}{\partial p} g_{\alpha\beta} \\ &= \frac{\partial x}{\partial p}\frac{\partial x}{\partial p} g_{xx} + \frac{\partial x}{\partial p}\frac{\partial y}{\partial p} g_{xy} + \frac{\partial y}{\partial p}\frac{\partial x}{\partial p} g_{yx} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial p} g_{yy} \\ &= 1 \cdot 1 \cdot 1 + 1 \cdot 2cp \cdot 0 + 2cp \cdot 1 \cdot 0 + 2cp \cdot 2cp \cdot 1 \\ &= 1 + 4c^2p^2 \end{aligned}$$

If $\mu = q$ and $\nu = p$ we have that

$$\begin{aligned} g_{qp} &= \frac{\partial x^\alpha}{\partial q}\frac{\partial x^\beta}{\partial p} g_{\alpha\beta} \\ &= \frac{\partial x}{\partial q}\frac{\partial x}{\partial p} g_{xx} + \frac{\partial x}{\partial q}\frac{\partial y}{\partial p} g_{xy} + \frac{\partial y}{\partial q}\frac{\partial x}{\partial p} g_{yx} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial p} g_{yy} \\ &= 0 \cdot 1 \cdot 1 + 0 \cdot 2cp \cdot 0 + 1 \cdot 1 \cdot 0 + 1 \cdot 2cp \cdot 1 \\ &= 2cp \end{aligned}$$

Finally, if $\mu = \nu = q$ we have that

$$\begin{aligned} g_{qq} &= \frac{\partial x^\alpha}{\partial q}\frac{\partial x^\beta}{\partial q} g_{\alpha\beta} \\ &= \frac{\partial x}{\partial q}\frac{\partial x}{\partial q} g_{xx} + \frac{\partial x}{\partial q}\frac{\partial y}{\partial q} g_{xy} + \frac{\partial y}{\partial q}\frac{\partial x}{\partial q} g_{yx} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial q} g_{yy} \\ &= 0 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 0 + 1 \cdot 0 \cdot 0 + 1 \cdot 1 \cdot 1 \\ &= 1 \end{aligned}$$

The off-diagonal of the metric tensor makes sense because the basis vectors are not orthogonal i.e. $\boldsymbol{e}_p \cdot \boldsymbol{e}_q \neq 0$. $\square$

Solution. **BOX 5.3** - Exercise 5.3.4. Let $\boldsymbol{A}$ be a vector with p, q components $A^p = 1$ and $A^q = 0$

a) From equation (5.7) we know that

$$A^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu$$

Hence

$$\begin{aligned} A^x &= \frac{\partial x}{\partial p} A^p + \frac{\partial x}{\partial q} A^q = 1 \cdot 1 + 0 \cdot 0 = 1 \\ A^y &= \frac{\partial y}{\partial p} A^p + \frac{\partial y}{\partial q} A^q = 2cp \cdot 1 + 1 \cdot 0 = 2cp \end{aligned}$$

b) Yes the components make sense since both of them gives us the same vector.

c) We want to show $A^2 = \boldsymbol{A} \cdot \boldsymbol{A}$ has the same value in both systems then we see that

$$\begin{aligned} A^2 &= (A^p \boldsymbol{e}_p + A^q \boldsymbol{e}_q) \cdot (A^p \boldsymbol{e}_p + A^q \boldsymbol{e}_q) \\ &= (A^p)^2 (\boldsymbol{e}_p \cdot \boldsymbol{e}_p) + (A^p A^q)(\boldsymbol{e}_p \cdot \boldsymbol{e}_q) \\ &\quad + (A^q A^p)(\boldsymbol{e}_q \cdot \boldsymbol{e}_p) + (A^q)^2 (\boldsymbol{e}_q \cdot \boldsymbol{e}_q) \\ &= 1 \cdot (1 + 4c^2 p^2) + 0 \cdot 2cp + 0 \cdot 2cp + 0 \cdot 1 \\ &= 1 + 4c^2 p^2 \end{aligned}$$

and also

$$\begin{aligned} A^2 &= (A^x \boldsymbol{e}_x + A^y \boldsymbol{e}_y) \cdot (A^x \boldsymbol{e}_x + A^y \boldsymbol{e}_y) \\ &= (A^x)^2 (\boldsymbol{e}_x \cdot \boldsymbol{e}_x) + (A^x A^y)(\boldsymbol{e}_x \cdot \boldsymbol{e}_y) \\ &\quad + (A^y A^x)(\boldsymbol{e}_y \cdot \boldsymbol{e}_x) + (A^y)^2 (\boldsymbol{e}_y \cdot \boldsymbol{e}_y) \\ &= 1 \cdot 1 + 2cp \cdot 0 + 2cp \cdot 0 + 4c^2 p^2 \cdot 1 \\ &= 1 + 4c^2 p^2 \end{aligned}$$

$\square$

Solution. **BOX 5.4** - Exercise 5.4.1. Let $\mu = x$ and $\nu = t$ then we have that

$$\frac{\partial x'^x}{\partial x^t} = \frac{\partial x'}{\partial t} = \frac{\partial}{\partial t} \gamma(x - \beta t) = -\gamma\beta = \Lambda^x{}_t$$

And for $\mu = \nu = y$ we have that

$$\frac{\partial x'^y}{\partial x^y} = \frac{\partial y'}{\partial y} = \frac{\partial}{\partial y} y = 1 = \Lambda^y{}_y$$

$\square$

Solution. **BOX 5.5** - Exercise 5.5.1. Let $\alpha = t$ and $\beta = x$ then we have that

$$\begin{aligned}
\eta'_{tx} &= \eta_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{t}(\Lambda^{-1})^{\nu}{}_{x} \\
&= \eta_{t\nu}(\Lambda^{-1})^{t}{}_{t}(\Lambda^{-1})^{\nu}{}_{x} + \eta_{x\nu}(\Lambda^{-1})^{x}{}_{t}(\Lambda^{-1})^{\nu}{}_{x} \\
&\quad + \eta_{y\nu}(\Lambda^{-1})^{y}{}_{t}(\Lambda^{-1})^{\nu}{}_{x} + \eta_{z\nu}(\Lambda^{-1})^{z}{}_{t}(\Lambda^{-1})^{\nu}{}_{x}
\end{aligned}$$

Now $\eta_{t\nu}$ is only nonzero when $\nu = t$, $\eta_{x\nu}$ only when $\nu = x$ and so on. Moreover $(\Lambda^{-1})^{y}{}_{t} = (\Lambda^{-1})^{z}{}_{t} = 0$ then

$$\begin{aligned}
\eta'_{tx} &= \eta_{tt}(\Lambda^{-1})^{t}{}_{t}(\Lambda^{-1})^{t}{}_{x} + \eta_{xx}(\Lambda^{-1})^{x}{}_{t}(\Lambda^{-1})^{x}{}_{x} \\
&\quad + \eta_{yy}(\Lambda^{-1})^{y}{}_{t}(\Lambda^{-1})^{y}{}_{x} + \eta_{zz}(\Lambda^{-1})^{z}{}_{t}(\Lambda^{-1})^{z}{}_{x} \\
&= -1 \cdot \gamma \cdot (\gamma\beta) + 1 \cdot (\gamma\beta) \cdot \gamma + 0 + 0 \\
&= 0 = \eta_{tx}
\end{aligned}$$

Now let $\alpha = \beta = x$ then we have that

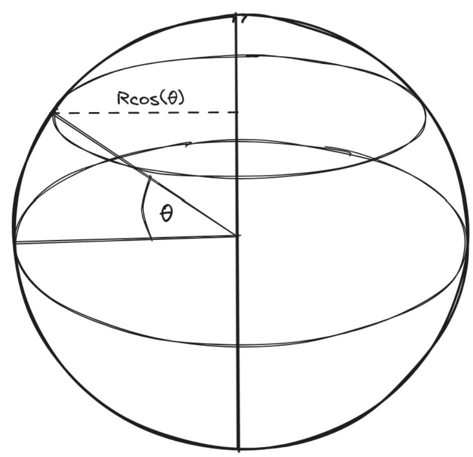
$$\begin{aligned}
\eta'_{xx} &= \eta_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{x}(\Lambda^{-1})^{\nu}{}_{x} \\
&= \eta_{t\nu}(\Lambda^{-1})^{t}{}_{x}(\Lambda^{-1})^{\nu}{}_{x} + \eta_{x\nu}(\Lambda^{-1})^{x}{}_{x}(\Lambda^{-1})^{\nu}{}_{x} \\
&\quad + \eta_{y\nu}(\Lambda^{-1})^{y}{}_{x}(\Lambda^{-1})^{\nu}{}_{x} + \eta_{z\nu}(\Lambda^{-1})^{z}{}_{x}(\Lambda^{-1})^{\nu}{}_{x}
\end{aligned}$$

Using the same reasoning as before we have that

$$\begin{aligned}
\eta'_{xx} &= \eta_{tt}(\Lambda^{-1})^{t}{}_{x}(\Lambda^{-1})^{t}{}_{x} + \eta_{xx}(\Lambda^{-1})^{x}{}_{x}(\Lambda^{-1})^{x}{}_{x} \\
&\quad + \eta_{yy}(\Lambda^{-1})^{y}{}_{x}(\Lambda^{-1})^{y}{}_{x} + \eta_{zz}(\Lambda^{-1})^{z}{}_{x}(\Lambda^{-1})^{z}{}_{x} \\
&= -1 \cdot (\gamma\beta) \cdot (\gamma\beta) + 1 \cdot \gamma \cdot \gamma + 0 + 0 \\
&= \gamma^2(1 - \beta^2) = 1 = \eta_{xx}
\end{aligned}$$

□

Solution. **BOX 5.6** - Exercise 5.6.1. If we measure θ up from the equator then we would have that a curve of constant latitude θ would be at a distance from the vertical line crossing north-south of $R\cos\theta$ instead of $R\sin\theta$ (see figure below) and hence the length of the infinitesimal displacement corresponding to an infinitesimal change $d\phi$ along a circle of constant latitude must have a length $R\cos\theta d\phi$ therefore the metric component would be $g_{\phi\phi} = R^2\cos^2\theta$.



□

Solution. **P5.1**

a. The transformation equations are given by

$$x(r,\theta) = r\cos\theta \qquad y(r,\theta) = r\sin\theta$$

and oppositely they are

$$r(x,y) = \sqrt{x^2+y^2} \qquad \theta(x,y) = \arctan\left(\frac{y}{x}\right)$$

b. The required partial derivatives are

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

and oppositely

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

c. The metric tensor for the cartesian coordinates is given by

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we can get the metric tensor for the polar coordinates by applying the following equation

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

Thus

$$\begin{aligned} g_{rr} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} g_{xx} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial r} g_{xy} + \frac{\partial y}{\partial r}\frac{\partial x}{\partial r} g_{yx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} g_{yy} \\ &= \cos^2\theta + 0 + 0 + \sin^2\theta \\ &= 1 \end{aligned}$$

$$\begin{aligned} g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} g_{xy} + \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta} g_{yx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} g_{yy} \\ &= -r\cos\theta\sin\theta + 0 + 0 + r\sin\theta\cos\theta \\ &= 0 \end{aligned}$$

$$\begin{aligned} g_{\theta r} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} g_{xx} + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} g_{xy} + \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial r} g_{yx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r} g_{yy} \\ &= -r\sin\theta\cos\theta + 0 + 0 + r\cos\theta\sin\theta \\ &= 0 \end{aligned}$$

$$\begin{aligned} g_{\theta\theta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta} g_{xy} + \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta} g_{yx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} g_{yy} \\ &= r^2\sin^2\theta + 0 + 0 + r^2\cos^2\theta \\ &= r^2 \end{aligned}$$

Therefore the metric tensor for polar coordinates is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

Which is consistent with equation 5.19.

$\square$

Solution. **P5.2**

a. We know from the polar metric equation that

$$ds^2 = dr^2 + r^2 d\theta^2$$

so dividing by dt^2 we get that

$$\begin{aligned}\left(\frac{ds}{dt}\right)^2 &= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \\ v^2 &= (v^r)^2 + r^2(v^\theta)^2 \\ v^\theta &= \pm\frac{v}{r}\end{aligned}$$

Where we used that $v^r = 0$ for an object in a uniform circular motion.

b. We know from problem ***P*5.1** that

$$x = r\cos\theta \qquad y = r\sin\theta$$

so by derivating these expressions, we get that

$$\begin{aligned}v^x = \frac{dx}{dt} &= \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta \\ &= v^r\cos\theta - rv^\theta\sin\theta \\ &= (\pm v)(-\sin\theta)\end{aligned}$$

and in the same way

$$\begin{aligned}v^y = \frac{dy}{dt} &= \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta \\ &= v^r\sin\theta + rv^\theta\cos\theta \\ &= \pm v\cos\theta\end{aligned}$$

So we can write the velocity vector as $\boldsymbol{v} = \pm v(-\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}})$ which is going to be tangent to a circle.

□

Solution. **P5.3**

a. We know from problem ***P5.1*** that

$$r(x,y) = \sqrt{x^2+y^2} \qquad \theta(x,y) = \arctan\left(\frac{y}{x}\right)$$

so by derivating these expressions, we get that

$$\begin{aligned} v^r = \frac{dr}{dt} &= \frac{x(dx/dt)+y(dy/dt)}{\sqrt{x^2+y^2}} \\ &= \frac{xv^x+yv^y}{\sqrt{x^2+y^2}} \\ &= \frac{yv}{\sqrt{x^2+y^2}} \\ &= v\sin\theta \end{aligned}$$

and in the same way

$$\begin{aligned} v^\theta = \frac{d\theta}{dt} &= \frac{x(dy/dt)-y(dx/dt)}{x^2+y^2} \\ &= \frac{xv^y-yv^x}{x^2+y^2} \\ &= \frac{xv}{x^2+y^2} \\ &= \frac{v\cos\theta}{r} \end{aligned}$$

b. By replacing the values for $t>0$ we get that

$$r = \sqrt{b^2+(vt)^2} \qquad \theta = \arctan\left(\frac{vt}{b}\right)$$

And for the velocities, we have that

$$\begin{aligned} v^r &= \frac{v^2t}{\sqrt{b^2+(vt)^2}} = \frac{v^2t}{r} \\ v^\theta &= \frac{bv}{b^2+(vt)^2} = \frac{vb}{r^2} \end{aligned}$$

We see that when $t < b/vr$ then

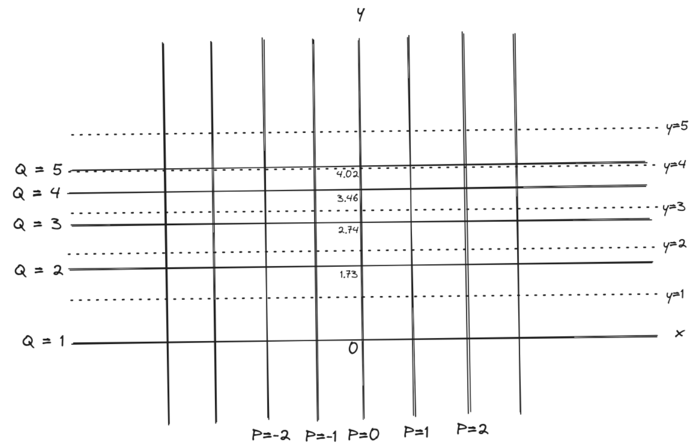
$$v^r = \frac{v^2t}{r} < \frac{vb}{r^2} = v^\theta$$

Hence there is a predominance of v^θ over v^r and when $t > b/vr$ we get that v^r is predominant over v^θ.

$\square$

Solution. **P5.4**

a. The following is a sketch of the "curves" of constant p and q when $b = 0.4\ cm^{-1}$



b. Since any vector must transform as

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

and the acceleration **a** is a vector then we have that

$$a^p = \frac{\partial p}{\partial x} a^x + \frac{\partial p}{\partial y} a^y = a^x = 0.2\ cm/s^2$$
$$a^q = \frac{\partial q}{\partial x} a^x + \frac{\partial q}{\partial y} a^y = b e^{by} a^y = -0.445\ 1/s^2$$

c. The metric tensor for the cartesian coordinates is given by

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we can get the metric tensor for the semilog coordinate system by applying the following equation

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

using that $x = p$ and that $y = \log(q)/b$ thus

$$\begin{aligned} g_{pp} &= \frac{\partial x}{\partial p}\frac{\partial x}{\partial p} g_{xx} + \frac{\partial x}{\partial p}\frac{\partial y}{\partial p} g_{xy} + \frac{\partial y}{\partial p}\frac{\partial x}{\partial p} g_{yx} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial p} g_{yy} \\ &= 1 + 0 + 0 + 0 \\ &= 1 \end{aligned}$$

$$
\begin{aligned}
g_{pq} &= \frac{\partial x}{\partial p}\frac{\partial x}{\partial q}g_{xx} + \frac{\partial x}{\partial p}\frac{\partial y}{\partial q}g_{xy} + \frac{\partial y}{\partial p}\frac{\partial x}{\partial q}g_{yx} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q}g_{yy} \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{qp} &= \frac{\partial x}{\partial q}\frac{\partial x}{\partial p}g_{xx} + \frac{\partial x}{\partial q}\frac{\partial y}{\partial p}g_{xy} + \frac{\partial y}{\partial q}\frac{\partial x}{\partial p}g_{yx} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial p}g_{yy} \\
&= 0 + 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{qq} &= \frac{\partial x}{\partial q}\frac{\partial x}{\partial q}g_{xx} + \frac{\partial x}{\partial q}\frac{\partial y}{\partial q}g_{xy} + \frac{\partial y}{\partial q}\frac{\partial x}{\partial q}g_{yx} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial q}g_{yy} \\
&= 0 + 0 + 0 + \frac{1}{bq}\frac{1}{bq} \\
&= \frac{1}{(bq)^2}
\end{aligned}
$$

Therefore the metric tensor for the semilog coordinate system is

$$
g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1/(bq)^2 \end{bmatrix}
$$

As we see the metric is diagonal which makes sense since the basis vectors are orthogonal.

d. The magnitude of $\boldsymbol{a}$ in cartesian coordinates is given by

$$
\begin{aligned}
|\boldsymbol{a}| &= \sqrt{(a^x)^2(\boldsymbol{e}_x \cdot \boldsymbol{e}_x) + (a^y)^2(\boldsymbol{e}_y \cdot \boldsymbol{e}_y)} \\
&= \sqrt{(0.2)^2 + (-0.5)^2} \\
&= 0.538
\end{aligned}
$$

And in the semilog coordinate system is given by

$$
\begin{aligned}
|\boldsymbol{a}| &= \sqrt{(a^p)^2(\boldsymbol{e}_p \cdot \boldsymbol{e}_p) + (a^q)^2(\boldsymbol{e}_q \cdot \boldsymbol{e}_q)} \\
&= \sqrt{(0.2)^2 + (-0.445)^2\left(\frac{1}{(0.4 \cdot 2.225)^2}\right)} \\
&= 0.538
\end{aligned}
$$

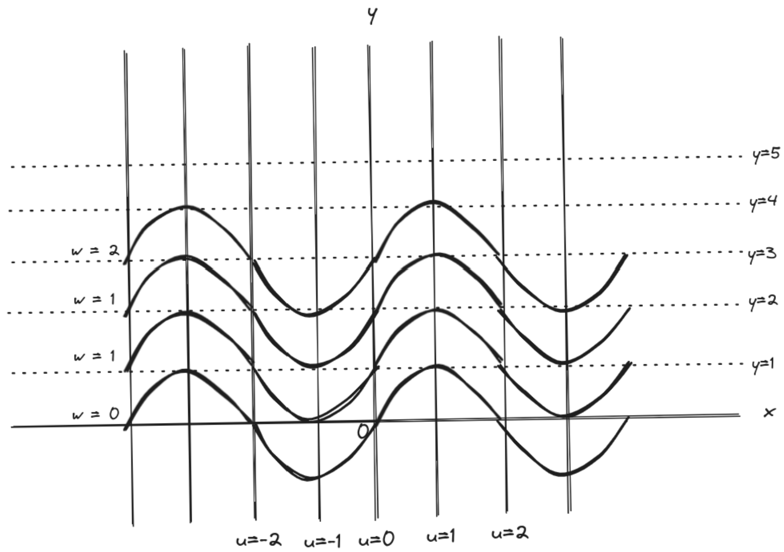
Where we used that $b = 0.4\ cm^{-1}$ and $q = 2.225$.

e. The length of the basis vector $\boldsymbol{e}_q$ is $|\boldsymbol{e}_q| = 1/bq$ as we can see from the metric we computed.

□

Solution. **P5.5**

a. The following is a sketch of the curves of constant u and w when $A = 1.0\ cm^{-1}$ and $b = \pi/2\ cm^{-1}$



b. The metric tensor for the cartesian coordinates is given by

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we can get the metric tensor for the sinusoidal coordinate system by applying the following equation

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

using that $x = u$ and that $y = w + A\sin(bu)$ thus

$$\begin{aligned} g_{uu} &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial u} g_{xx} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial u} g_{xy} + \frac{\partial y}{\partial u}\frac{\partial x}{\partial u} g_{yx} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial u} g_{yy} \\ &= 1 + 0 + 0 + (Ab)^2\cos^2(bu) \\ &= 1 + (Ab)^2\cos^2(bu) \end{aligned}$$

$$\begin{aligned} g_{uw} &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial w} g_{xx} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial w} g_{xy} + \frac{\partial y}{\partial u}\frac{\partial x}{\partial w} g_{yx} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial w} g_{yy} \\ &= 0 + 0 + 0 + (Ab)^2\cos^2(bu) \\ &= Ab\cos(bu) \end{aligned}$$

$$\begin{aligned} g_{wu} &= \frac{\partial x}{\partial w}\frac{\partial x}{\partial u} g_{xx} + \frac{\partial x}{\partial w}\frac{\partial y}{\partial u} g_{xy} + \frac{\partial y}{\partial w}\frac{\partial x}{\partial u} g_{yx} + \frac{\partial y}{\partial w}\frac{\partial y}{\partial u} g_{yy} \\ &= 0 + 0 + 0 + Ab\cos(bu) \\ &= Ab\cos(bu) \end{aligned}$$

$$
\begin{aligned}
g_{ww} &= \frac{\partial x}{\partial w}\frac{\partial x}{\partial w}g_{xx} + \frac{\partial x}{\partial w}\frac{\partial y}{\partial w}g_{xy} + \frac{\partial y}{\partial w}\frac{\partial x}{\partial w}g_{yx} + \frac{\partial y}{\partial w}\frac{\partial y}{\partial w}g_{yy} \\
&= 0 + 0 + 0 + 1 \\
&= 1
\end{aligned}
$$

Therefore the metric tensor for the sinusoidal coordinate system is

$$
g_{\mu\nu} = \begin{bmatrix} 1 + (Ab)^2\cos^2(bu) & Ab\cos(bu) \\ Ab\cos(bu) & 1 \end{bmatrix}
$$

As we see the metric is not diagonal which makes sense since the basis vectors are not always orthogonal.

c. By derivating the expressions of $x = u$ and $y = w + A\sin(bx)$ with respect to t we have that

$$
v^x = \frac{dx}{dt} = \frac{du}{dt} = v^u
$$

and that

$$
\begin{aligned}
v^y = \frac{dy}{dt} &= \frac{dw}{dt} + Ab\frac{dx}{dt}\cos(bx) \\
&= v^w + Abv^x\cos(bx)
\end{aligned}
$$

Hence

$$
v^u = v \qquad v^w = -Abv\cos(bvt)
$$

Where we used that $v^y = 0$ and that $x = vt$.

d. Let us compute $\boldsymbol{v}^2$ in this coordinate system as follows

$$
\begin{aligned}
\boldsymbol{v}^2 &= (v^u\boldsymbol{e}_u + v^w\boldsymbol{e}_w)\cdot(v^u\boldsymbol{e}_u + v^w\boldsymbol{e}_w) \\
&= (v^u)^2(\boldsymbol{e}_u\cdot\boldsymbol{e}_u) + v^uv^w(\boldsymbol{e}_u\cdot\boldsymbol{e}_w) \\
&\quad + v^wv^u(\boldsymbol{e}_w\cdot\boldsymbol{e}_u) + (v^w)^2(\boldsymbol{e}_w\cdot\boldsymbol{e}_w) \\
&= v^2(1 + (Ab)^2\cos^2(bvt)) - 2(Ab)^2v^2\cos(bvt) \\
&\quad + (Ab)^2v^2\cos^2(bvt) \\
&= v^2
\end{aligned}
$$

We used that $u = x = vt$. The component v^w is not constant because $\boldsymbol{e}_w$ changes directions in this coordinate system. So to maintain $\boldsymbol{v}$'s direction and magnitude the component v^w must change according to the changes in the direction product of the coordinate system.

e. Given that v^w changes with time we want to prove that $dv^w/dt \neq a^w$ where a^w is the component of the object's accelertaion vector $\boldsymbol{a}$. So by derivating v^w we get that

$$\frac{\mathrm{d}v^w}{\mathrm{d}t} = -Ab^2v^2\sin(bvt)$$

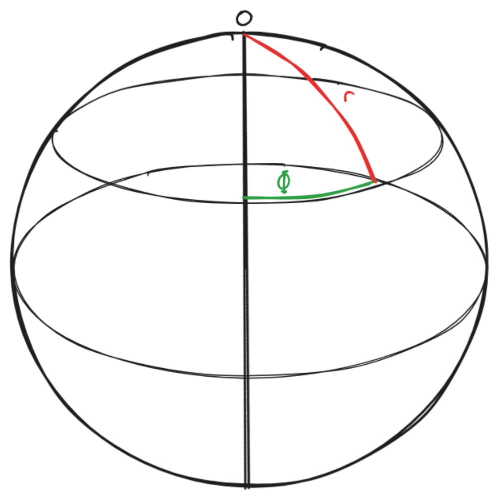
But on the other hand, the acceleration $\boldsymbol{a}$ is a vector so a^w must transform as follows

$$a^w = \frac{\partial w}{\partial x}a^x + \frac{\partial w}{\partial y}a^y$$

but we know that $a^x = a^y = 0$ hence $a^w = 0$ which implies that $dv^w/dt \neq a^w$.

$\square$

Solution. **P5.6** We are considering a system where the coordinates look like the following



Let us consider an infinitesimal displacement in the r direction. The basis vector $\boldsymbol{e}_r$ has a magnitude of 1 so

$$d\boldsymbol{s} = dr\boldsymbol{e}_r$$

Now if we consider an infinitesimal displacement in the ϕ direction we see that the basis vector has a magnitude $R\sin(r/R)$ since the angle between the vertical and the line connecting the center of the sphere and the coordinate point r is $\theta = r/R$ hence we have that

$$d\boldsymbol{s} = d\phi\boldsymbol{e}_\phi$$

So an arbitrary infinitesimal displacement in any direction can be written as

$$d\boldsymbol{s} = dr\boldsymbol{e}_r + d\phi\boldsymbol{e}_\phi$$

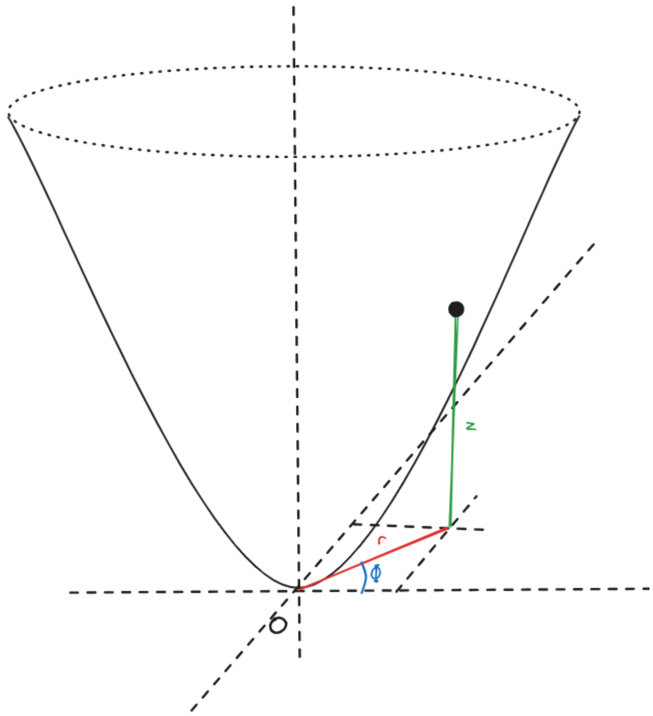
Then the metric in this case is given by

$$g_{\mu\nu} = \begin{bmatrix} \boldsymbol{e}_r \cdot \boldsymbol{e}_r & \boldsymbol{e}_r \cdot \boldsymbol{e}_\phi \\ \boldsymbol{e}_\phi \cdot \boldsymbol{e}_r & \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^2\sin^2(r/R) \end{bmatrix}$$

□

Solution. **P5.7**

a. We are considering a system where the coordinates look like the following



b. Let us consider an infinitesimal distance $d\boldsymbol{s}$ along the surface in the r direction with ϕ fixed. This displacement involves not only a displacement dr in the r direction but also a displacement dz in the z direction. Knowing that $z = br^2$ we can compute $dz/dr = 2br$ and hence $dz = 2brdr$.

Now if we consider an infinitesimal distance $d\boldsymbol{s}$ along the surface in the ϕ direction (with r fixed), then the infinitesimal displacement of an infinitesimal angle $d\phi$ is $rd\phi$.

Finally, given that the displacements are perpendicular we can use the Pythagorean theorem to compute ds^2 for an arbitrary displacement as follows

$$\begin{aligned} ds^2 &= dr^2 + (rd\phi)^2 + dz^2 \\ &= dr^2 + r^2 d\phi^2 + 4b^2 r^2 dr^2 \\ &= (1 + 4br^2) dr^2 + r^2 d\phi^2 \end{aligned}$$

Now, comparing this to the abstract form of the metric equation $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ we see that $g_{rr} = 1 + 4br^2$ and $g_{\phi\phi} = r^2$. Therefore the metric tensor is given by

$$g_{\mu\nu} = \begin{bmatrix} 1 + 4br^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

□