Solved selected problems of General Relativity - Thomas A. Moore

Franco Zacco

Chapter 18 - Geodesic Deviation

Solution. **BOX 18.1** - Exercise 18.1.1.
We know that $\Phi = -GM/r$ then we can compute $\partial_j \Phi$ using the chain rule as follows

$$\partial_j \Phi = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x^j} = \frac{GM}{r^2}\frac{\eta_{jn}x^n}{r} = \frac{GM}{r^3}\eta_{jn}x^n$$

Where we used that $\partial_j r = \eta_{jn}x^n/r$.
Now, we can compute $\partial_k\partial_j\Phi$ using the product rule as follows

$$\begin{aligned}
\partial_k\partial_j\Phi &= \partial_k\left(\frac{GM}{r^3}\eta_{jn}x^n\right)\\
&= \partial_k\left(\frac{GM}{r^3}\right)\eta_{jn}x^n + \frac{GM}{r^3}\partial_k\left(\eta_{jn}x^n\right)\\
&= \partial_k\left(\frac{GM}{r^3}\right)\eta_{jn}x^n + \frac{GM}{r^3}\partial_k\left(\eta_{jn}x^n\right)\\
&= GM\eta_{jn}x^n\left(\partial_k\left(\frac{1}{r}\right)\frac{1}{r^2} + \partial_k\left(\frac{1}{r^2}\right)\frac{1}{r}\right) + \frac{GM}{r^3}\eta_{jk}x^k\\
&= GM\eta_{jn}x^n\left(-\frac{1}{r^3}\frac{1}{r^2}\eta_{km}x^m + \partial_k\left(\frac{1}{r}\right)\frac{1}{r^2} + \partial_k\left(\frac{1}{r}\right)\frac{1}{r^2}\right) + \frac{GM}{r^3}\eta_{jk}x^k\\
&= GM\eta_{jn}x^n\left(-\frac{1}{r^3}\frac{1}{r^2}\eta_{km}x^m - \frac{1}{r^3}\frac{1}{r^2}\eta_{km}x^m - \frac{1}{r^3}\frac{1}{r^2}\eta_{km}x^m\right) + \frac{GM}{r^3}\eta_{jk}x^k\\
&= -\frac{GM}{r^5}\eta_{km}\eta_{jn}x^m x^n + \frac{GM}{r^3}\eta_{jk}x^k
\end{aligned}$$

□

Solution. **BOX 18.1** - Exercise 18.1.2.
The Newtonian tidal deviation equation is

$$\begin{aligned}\frac{\partial^2 n^i}{\partial t^2} &= \frac{3GM}{r^5}\eta^{ij}\eta_{km}\eta_{jn}x^m x^n n^k - \frac{GM}{r^3}\eta^{ij}\eta_{kj}n^k \\ &= \frac{3GM}{r^5}\eta_{km}\delta^i_n x^m x^n n^k - \frac{GM}{r^3}\delta^i_k n^k\end{aligned}$$

Let $i = x$ then

$$\begin{aligned}\frac{\partial^2 n^x}{\partial t^2} &= \frac{3GM}{r^5}\eta_{km}\delta^x_n x^m x^n n^k - \frac{GM}{r^3}\delta^x_k n^k \\ &= \frac{3GM}{r^5}\eta_{km}x^m x^x n^k - \frac{GM}{r^3}n^x \\ &= -\frac{GM}{r^3}n^x\end{aligned}$$

Where we used that $x^x = 0$ (this is just $x = 0$).
Let $i = z$ then

$$\begin{aligned}\frac{\partial^2 n^z}{\partial t^2} &= \frac{3GM}{r^5}\eta_{km}\delta^z_n x^m x^n n^k - \frac{GM}{r^3}\delta^z_k n^k \\ &= \frac{3GM}{r^5}\eta_{km}x^m x^z n^k - \frac{GM}{r^3}n^z \\ &= \frac{3GM}{r^5}\eta_{zz}x^z x^z n^z - \frac{GM}{r^3}n^z \\ &= \frac{3GM}{r^5}r^2 n^z - \frac{GM}{r^3}n^z \\ &= \frac{3GM}{r^3}n^z - \frac{GM}{r^3}n^z \\ &= \frac{2GM}{r^3}n^z\end{aligned}$$

Where we used that $x^x = x^y = 0$ and hence $r^2 = \eta_{mn}x^m x^n = \eta_{zz}x^z x^z$. $\square$

Solution. **BOX 18.2** - Exercise 18.2.1.
Dropping second or higher order terms of n^α equation (18.8) becomes

$$0 = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}n^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) + n^\sigma(\partial_\sigma \Gamma^\alpha_{\mu\nu})\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)$$

Subtracting the geodesic equation we get that

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}n^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) + n^\sigma(\partial_\sigma \Gamma^\alpha_{\mu\nu})\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) \\
&\quad - \left(\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) \\
0 &= \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}n^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) + n^\sigma(\partial_\sigma \Gamma^\alpha_{\mu\nu})\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) \\
0 &= \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + \Gamma^\alpha_{\nu\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}n^\mu}{\mathrm{d}\tau} + n^\sigma(\partial_\sigma \Gamma^\alpha_{\mu\nu})\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) \\
0 &= \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + 2\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + n^\sigma(\partial_\sigma \Gamma^\alpha_{\mu\nu})\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) \\
0 &= \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + 2\Gamma^\alpha_{\mu\nu}u^\mu\frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + n^\sigma(\partial_\sigma \Gamma^\alpha_{\mu\nu})u^\mu u^\nu
\end{aligned}$$

Where we used that $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. □

Solution. **BOX 18.3** - Exercise 18.3.1.
From equation (8.22) we have that

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}\tau} = \frac{\mathrm{d}n^\alpha}{\mathrm{d}\tau}\boldsymbol{e}_\alpha + n^\alpha \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial \boldsymbol{e}_\alpha}{\partial x^\mu}$$

Using that $\partial \boldsymbol{e}_\beta / \partial x^\mu = \Gamma^\alpha_{\beta\mu}\boldsymbol{e}_\alpha$ we get that

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}\tau} = \frac{\mathrm{d}n^\alpha}{\mathrm{d}\tau}\boldsymbol{e}_\alpha + n^\sigma \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\Gamma^\alpha_{\sigma\mu}\boldsymbol{e}_\alpha = \left(\frac{\mathrm{d}n^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\sigma\mu}n^\sigma u^\mu\right)\boldsymbol{e}_\alpha$$

Where we changed the summation over α to be over σ in the second term.

$\square$

Solution. **BOX 18.4** - Exercise 18.4.1.
Equation (18.13) states that

$$\left(\frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}\tau^2}\right)^\alpha = \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + (\partial_\sigma \Gamma^\alpha_{\mu\nu}) u^\sigma u^\mu n^\nu + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} n^\nu + 2\Gamma^\alpha_{\mu\nu} u^\mu \frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + \Gamma^\alpha_{\sigma\nu}\Gamma^\nu_{\beta\gamma} u^\sigma u^\beta n^\gamma$$

Plugging in equations (18.24) we get that

$$\begin{aligned}
\left(\frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}\tau^2}\right)^\alpha &= -2\Gamma^\alpha_{\mu\nu} u^\mu \frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} - (\partial_\sigma \Gamma^\alpha_{\mu\nu}) u^\mu u^\nu n^\sigma + (\partial_\sigma \Gamma^\alpha_{\mu\nu}) u^\sigma u^\mu n^\nu \\
&\quad - \Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\beta} u^\sigma u^\beta n^\nu + 2\Gamma^\alpha_{\mu\nu} u^\mu \frac{\mathrm{d}n^\nu}{\mathrm{d}\tau} + \Gamma^\alpha_{\sigma\nu}\Gamma^\nu_{\beta\gamma} u^\sigma u^\beta n^\gamma \\
&= -(\partial_\sigma \Gamma^\alpha_{\mu\nu}) u^\mu u^\nu n^\sigma + (\partial_\sigma \Gamma^\alpha_{\mu\nu}) u^\sigma u^\mu n^\nu - \Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\beta} u^\sigma u^\beta n^\nu \\
&\quad + \Gamma^\alpha_{\sigma\nu}\Gamma^\nu_{\beta\gamma} u^\sigma u^\beta n^\gamma \\
&= -(\partial_\nu \Gamma^\alpha_{\mu\sigma}) u^\mu u^\sigma n^\nu + (\partial_\sigma \Gamma^\alpha_{\mu\nu}) u^\sigma u^\mu n^\nu - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\sigma\mu} u^\sigma u^\mu n^\nu \\
&\quad + \Gamma^\alpha_{\sigma\gamma}\Gamma^\gamma_{\mu\nu} u^\sigma u^\mu n^\nu \\
&= (\partial_\sigma \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\gamma_{\mu\nu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}) u^\sigma u^\mu n^\nu
\end{aligned}$$

Where in the third equality, we renamed some of the indices in each term.

$\square$

Solution. **BOX 18.5** - Exercise 18.5.1.
We compute the metric equation for the case $\mu = w$ as follows

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{w\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_w g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sin^2(u)\frac{\mathrm{d}w}{\mathrm{d}\tau}\right) - 0 \\
0 &= 2\sin(u)\cos(u)\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}w}{\mathrm{d}\tau} + \sin^2(u)\frac{\mathrm{d}^2w}{\mathrm{d}\tau^2} \\
0 &= \frac{\mathrm{d}^2w}{\mathrm{d}\tau^2} + 2\frac{\cos(u)}{\sin(u)}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}w}{\mathrm{d}\tau} \\
0 &= \frac{\mathrm{d}^2w}{\mathrm{d}\tau^2} + 2\cot(u)\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}w}{\mathrm{d}\tau}
\end{aligned}
$$

Therefore comparing this with equation (18.27) we see that

$$\Gamma^w_{uw} = \Gamma^w_{wu} = \cot(u)$$

and for the rest of the components, we get that $\Gamma^w_{\alpha\beta} = 0$. □

Solution. **BOX 18.5** - Exercise 18.5.2.
Let us evaluate the component R^u_{wuw} of the Riemann tensor as follows

$$
\begin{aligned}
R^u_{wuw} &= \partial_u\Gamma^u_{ww} - \Gamma^u_{w\sigma}\Gamma^\sigma_{wu} \\
&= \partial_u\Gamma^u_{ww} - \Gamma^u_{wu}\Gamma^u_{wu} - \Gamma^u_{ww}\Gamma^w_{wu} \\
&= \frac{1}{a^2}(\sin^2(u) - \cos^2(u)) - 0 + \frac{1}{a^2}\sin(u)\cos(u)\cot(u) \\
&= \frac{1}{a^2}(\sin^2(u) - \cos^2(u)) + \frac{1}{a^2}\cos^2(u) \\
&= \frac{\sin^2(u)}{a^2}
\end{aligned}
$$

□

Solution. **P18.1**

From the Newtonian tidal deviation equation, we know that if we choose the z axis such that both particles are on it, then the separation vector $\vec{n}(t)$ has only one non-zero component n^z.

Also, if we assume that the separation vector has a small magnitude compared to earth's radius we can use the approximated equation (18.4), which states the following

$$\frac{\mathrm{d}^2 n^z}{\mathrm{d}t^2} = \frac{2GM}{R^3} n^z$$

We know that the solution to this second order differential equation is

$$n^z = C_1 \cosh\left(\sqrt{\frac{2GM}{R^3}} t\right) + C_2 \sinh\left(\sqrt{\frac{2GM}{R^3}} t\right)$$

Where C_1 and C_2 are constants. Also, we have that

$$\frac{\mathrm{d}n^z}{\mathrm{d}t} = \sqrt{\frac{2GM}{R^3}} \left(C_1 \sinh\left(\sqrt{\frac{2GM}{R^3}} t\right) + C_2 \cosh\left(\sqrt{\frac{2GM}{R^3}} t\right)\right)$$

Since the balls start from rest then $C_2 = 0$. Also, the balls, start 1 meter apart so from the equation for $n^z(0)$ we get that $C_1 = 1$, then the equation becomes

$$n^z = \cosh\left(\sqrt{\frac{2GM}{R^3}} t\right)$$

So the relative acceleration is given by

$$\frac{\mathrm{d}^2 n^z}{\mathrm{d}t^2} = \frac{2GM}{R^3} \cosh\left(\sqrt{\frac{2GM}{R^3}} t\right)$$

On the other hand, to determine how long it takes for the separation of the balls to grow $1nm$ we compute the following

$$\begin{aligned} t &= \frac{\cosh^{-1}(n^z)}{\sqrt{\frac{2GM}{R^3}}} \\ &= \frac{\cosh^{-1}(1 + 10^{-9})}{\sqrt{\frac{8.87\times 10^{-3}}{(6378\times 10^3)^3}}} \\ &= \frac{4.472 \times 10^{-5}}{5.843 \times 10^{-12}} \\ &= 7.653 \times 10^6 \ m \approx 25.527 \ ms \end{aligned}$$

□

Solution. **P18.2**
Let an infinite flat plate of uniformly distributed mass where $\vec{g}$ is constant in magnitude and direction. Then the potential must be of the form $\Phi = Cz$ where z is the perpendicular distance from the plate to the particle so

$$\vec{g} = -\boldsymbol{\nabla}\Phi = (0, 0, -C)$$

Then the tidal deviation equation become

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}t^2} = -\eta^{ij}\frac{\partial^2\Phi}{\partial x^k \partial x^j}n^k = 0$$

Since every second derivative of Φ vanishes. This implies that there is no relative acceleration between balls freely falling on a reference frame near the plate.
If this result holds in general relativity, then $d^2\boldsymbol{n}/d\tau^2 = 0$ and this implies that we are in a flat spacetime i.e. in an empty spacetime which is not true. Therefore, this field cannot be real in general relativity. $\square$

Solution. **P18.3**

a. From the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

We see that if $\mu = t, y$ or z then since no $g_{\mu\nu}$ depends on t, y or z then the second term of the geodesic equation is 0 (all derivatives or $g_{\mu\nu}$ are zero) implying that all Christoffel symbols where the super index is t, y or z are zero.
For the case where $\mu = x$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{xx}\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_x g_{xx}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= 0\\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(f(x)\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_x f(x)\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= 0\\
\frac{\mathrm{d}f(x)}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} + f(x)\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} - \frac{1}{2}\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= 0\\
\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} + f(x)\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} - \frac{1}{2}\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= 0\\
\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + \frac{1}{2f(x)}\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= 0
\end{aligned}$$

Then the only non-zero Christoffel symbol is

$$\Gamma^x_{xx} = \frac{1}{2f(x)}\frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

b. The Riemann tensor is defined as

$$R^\alpha_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu}$$

Given that all derivatives with respect to t, y or z are zero and that all Christoffel symbols where the super index is t, y or z are all zero as well, then μ,ν and α cannot take these values. Then the only possible non-zero component of the Riemann tensor might be

$$R^x_{\beta xx} = \partial_x\Gamma^x_{\beta x} - \partial_x\Gamma^x_{\beta x} + \Gamma^x_{x\gamma}\Gamma^\gamma_{\beta x} - \Gamma^x_{x\sigma}\Gamma^\sigma_{\beta x}$$

But, since the only non-zero Christoffel symbol is Γ^x_{xx} then must be that $\beta = x$ and the summations of γ and σ have only one term i.e.

$$R^x_{xxx} = \partial_x\Gamma^x_{xx} - \partial_x\Gamma^x_{xx} + \Gamma^x_{xx}\Gamma^x_{xx} - \Gamma^x_{xx}\Gamma^x_{xx} = 0$$

Therefore all components of the Riemann tensor are zero, implying a flat spacetime.

□

Solution. **P18.4** Let the metric

$$ds^2 = dr^2 + r^2 d\theta^2$$

Then from the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

When $\mu = \theta$ we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) &= 0 \\
2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} &= 0 \\
\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 0
\end{aligned}$$

Where we used that all derivatives of $g_{\mu\nu}$ with respect to θ are zero. Then the only non-zero Christoffel symbols when θ is a super index are

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

When $\mu = r$ we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_r g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 0 \\
\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 0
\end{aligned}$$

So we see that the only non-zero Christoffel symbol is

$$\Gamma^r_{\theta\theta} = -r$$

Then, the component $R^r_{\theta r\theta}$ of the Riemann tensor is

$$R^r_{\theta r\theta} = \partial_r\Gamma^r_{\theta\theta} - \partial_\theta\Gamma^r_{\theta r} + \Gamma^r_{r\gamma}\Gamma^\gamma_{\theta r} - \Gamma^r_{\theta\sigma}\Gamma^\sigma_{\theta r}$$

We see that $\partial_\theta\Gamma^r_{\theta r} = 0$ since $\Gamma^r_{\theta r}$ does not depend on θ, also, $\Gamma^r_{r\gamma} = 0$ for any value of γ and must be that $\sigma = \theta$ then

$$R^r_{\theta r\theta} = \partial_r\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} = \partial_r(-r) - (-r)\frac{1}{r} = -1 + 1 = 0$$

□

Solution. **P18.5** Let the metric

$$ds^2 = \frac{dp^2}{1-kp^2} + p^2 dq^2$$

Then when $\mu = p$ the geodesic equation becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{pp}\frac{\mathrm{d}p}{\mathrm{d}\tau}\right) - \frac{1}{2}\left(\partial_p g_{pp}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} + \partial_p g_{qq}\frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{1-kp^2}\frac{\mathrm{d}p}{\mathrm{d}\tau}\right) - \frac{1}{2}\left(\frac{2kp}{(1-kp^2)^2}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} + 2p\frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{1-kp^2}\right)\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} + \frac{1}{1-kp^2}\frac{\mathrm{d}^2p}{\mathrm{d}\tau^2} - \frac{kp}{(1-kp^2)^2}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} - p\frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau} &= 0 \\
\frac{2kp}{(1-kp^2)^2}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} + \frac{1}{1-kp^2}\frac{\mathrm{d}^2p}{\mathrm{d}\tau^2} - \frac{kp}{(1-kp^2)^2}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} - p\frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau} &= 0 \\
\frac{\mathrm{d}^2p}{\mathrm{d}\tau^2} + \frac{kp}{1-kp^2}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} - (1-kp^2)p\frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau} &= 0
\end{aligned}$$

Then the Christoffel symbols when p is a superscript are

$$\Gamma^p_{pp} = \frac{kp}{1-kp^2} \qquad \Gamma^p_{qq} = (kp^2-1)p$$

When $\mu = q$ we get that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{qq}\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) - \frac{1}{2}\left(\partial_q g_{pp}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}p}{\mathrm{d}\tau} + \partial_q g_{qq}\frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(p^2\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) - 0 &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}(p^2)\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau} + p^2\frac{\mathrm{d}^2q}{\mathrm{d}\tau^2} &= 0 \\
\frac{\mathrm{d}^2q}{\mathrm{d}\tau^2} + \frac{2}{p}\frac{\mathrm{d}p}{\mathrm{d}\tau}\frac{\mathrm{d}q}{\mathrm{d}\tau} &= 0
\end{aligned}$$

Then the Christoffel symbols when q is a superscript are

$$\Gamma^q_{pq} = \Gamma^q_{qp} = \frac{1}{p}$$

Since in a two-dimensional space the components of the Riemann tensor are either identically zero or equal to $\pm R^p_{qpq}$, then we need to compute only R^p_{qpq}

$$\begin{aligned}
R^p_{qpq} &= \partial_p\Gamma^p_{qq} - \partial_q\Gamma^p_{qp} + \Gamma^p_{p\gamma}\Gamma^\gamma_{qq} - \Gamma^p_{q\sigma}\Gamma^\sigma_{qp} \\
&= \partial_p\Gamma^p_{qq} + \Gamma^p_{pp}\Gamma^p_{qq} - \Gamma^p_{qq}\Gamma^q_{qp} \\
&= 3kp^2 - 1 + \frac{kp}{1-kp^2}(kp^2-1)p - \frac{(kp^2-1)p}{p} \\
&= 3kp^2 - 1 - kp^2 - kp^2 + 1 \\
&= kp^2
\end{aligned}$$

Therefore this metric describes a curved space. $\square$

Solution. **P18.6**

a. From BOX 17.6 we know that

$$0 = \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}$$

Then the only non-zero Christoffel symbols that have t as a superscript are

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}$$

b. We want to compute now $R^t{}_{rtr}$ then

$$\begin{aligned} R^t{}_{rtr} &= \partial_t \Gamma^t_{rr} - \partial_r \Gamma^t_{rt} + \Gamma^t_{t\gamma}\Gamma^\gamma_{rr} - \Gamma^t_{r\sigma}\Gamma^\sigma_{rt} \\ &= -\partial_r \Gamma^t_{rt} + \Gamma^t_{tr}\Gamma^r_{rr} - \Gamma^t_{rt}\Gamma^t_{rt} \end{aligned}$$

Then using that $\Gamma^r_{rr} = -\Gamma^t_{rt}$ we get that

$$\begin{aligned} R^t{}_{rtr} &= -\partial_r \Gamma^t_{rt} - \Gamma^t_{rt}\Gamma^t_{rt} - \Gamma^t_{rt}\Gamma^t_{rt} \\ &= -\partial_r \Gamma^t_{rt} - 2(\Gamma^t_{rt})^2 \\ &= -\left[-\frac{2GM}{r^3}\left(1 - \frac{2GM}{r}\right)^{-1} - \frac{GM}{r^2}\frac{2GM}{(r-2GM)^2}\right] - \frac{2(GM)^2}{r^4}\left(1 - \frac{2GM}{r}\right)^{-2} \\ &= -\left[-\frac{2GM}{r^3}\left(1 - \frac{2GM}{r}\right)^{-1} - \frac{2(GM)^2}{r^4}\left(1 - \frac{2GM}{r}\right)^{-2}\right] - \frac{2(GM)^2}{r^4}\left(1 - \frac{2GM}{r}\right)^{-2} \\ &= \frac{2GM}{r^3}\left(1 - \frac{2GM}{r}\right)^{-1} \end{aligned}$$

c. Given that at least one component of the Riemann tensor is not zero then the spacetime must be curved.
But if we take a big r we can use the binomial approximation to get that

$$R^t{}_{rtr} \approx \frac{2GM}{r^3}\left(1 + \frac{2GM}{r}\right) = \frac{2GM}{r^3} + \frac{4(GM)^2}{r^4}$$

Then as $r \to \infty$ we get that $R^t{}_{rtr} \to 0$, so as r increases the spacetime becomes flatter.

□

Solution. **P18.7** The equation of geodesic deviation states that

$$\left(\frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}\tau^2}\right)^\alpha = -R^\alpha_{\mu\beta\sigma}u^\sigma u^\mu n^\beta$$

Since we are considering a LIF we get that $\partial_\alpha g_{\mu\nu} = 0$ so all Christoffel symbols are zero, but the second derivatives of $g_{\mu\nu}$ might not be zero so the derivatives of the Christoffel symbols might not be zero, so from equation (18.13) we get that

$$\left(\frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}\tau^2}\right)^\alpha = \frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + (\partial_\sigma\Gamma^\alpha_{\mu\nu})u^\sigma u^\mu n^\nu$$

And replacing this in the equation of geodesic deviation we get that

$$\frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} + (\partial_\sigma\Gamma^\alpha_{\mu\nu})u^\sigma u^\mu n^\nu = -R^\alpha_{\mu\beta\sigma}u^\sigma u^\mu n^\beta$$

Also, replacing the value of the Riemann tensor we get that

$$\begin{aligned}
\frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} &= -(\partial_\beta\Gamma^\alpha_{\mu\sigma} - \partial_\sigma\Gamma^\alpha_{\mu\beta})u^\sigma u^\mu n^\beta - \partial_\sigma\Gamma^\alpha_{\mu\nu}u^\sigma u^\mu n^\nu \\
&= -\partial_\beta\Gamma^\alpha_{\mu\sigma}u^\sigma u^\mu n^\beta + \partial_\sigma\Gamma^\alpha_{\mu\beta}u^\sigma u^\mu n^\beta - \partial_\sigma\Gamma^\alpha_{\mu\nu}u^\sigma u^\mu n^\nu \\
&= -\partial_\beta\Gamma^\alpha_{\mu\nu}u^\nu u^\mu n^\beta + \partial_\sigma\Gamma^\alpha_{\mu\nu}u^\sigma u^\mu n^\nu - \partial_\sigma\Gamma^\alpha_{\mu\nu}u^\sigma u^\mu n^\nu \\
&= -\partial_\beta\Gamma^\alpha_{\mu\nu}u^\nu u^\mu n^\beta
\end{aligned}$$

Where in the third step we changed the indices $\beta \to \nu$ in the second term. Since we are considering a freely falling frame then $u^t = 1$ and 0 for the spacial components, so we get that

$$\frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} = -\partial_\beta\Gamma^\alpha_{tt}n^\beta$$

Finally, if we consider the Riemann tensor component $R^\alpha_{t\beta t}$ we see that

$$R^\alpha_{t\beta t} = \partial_\beta\Gamma^\alpha_{tt} - \partial_t\Gamma^\alpha_{t\beta}$$

But argueing that the time derivative of the Christoffel symbols are 0, we get that

$$R^\alpha_{t\beta t} = \partial_\beta\Gamma^\alpha_{tt}$$

And hence

$$\frac{\mathrm{d}^2 n^\alpha}{\mathrm{d}\tau^2} = -R^\alpha{}_{t\beta t}n^\beta$$

□

Solution. **P18.8**

a. Let us consider $\nabla_\nu a^\alpha$ as a tensor with one lower index and one upper index, then the absolute gradient of it is

$$\nabla_\mu(\nabla_\nu a^\alpha) = \partial_\mu(\nabla_\nu a^\alpha) + \Gamma^\alpha_{\mu\delta}(\nabla_\nu a^\delta) - \Gamma^\sigma_{\mu\nu}(\nabla_\sigma a^\alpha)$$

Now using that

$$\nabla_\nu a^\alpha = \partial_\nu a^\alpha + \Gamma^\alpha_{\nu\beta} a^\beta$$

We get that

$$\begin{aligned}
\nabla_\mu(\nabla_\nu a^\alpha) &= \partial_\mu(\partial_\nu a^\alpha + \Gamma^\alpha_{\nu\beta} a^\beta) + \Gamma^\alpha_{\mu\delta}(\partial_\nu a^\delta + \Gamma^\delta_{\nu\lambda} a^\lambda) - \Gamma^\sigma_{\mu\nu}(\partial_\sigma a^\alpha + \Gamma^\alpha_{\sigma\gamma} a^\gamma) \\
&= \partial_\mu\partial_\nu a^\alpha + (\partial_\mu \Gamma^\alpha_{\nu\beta} a^\beta) + \Gamma^\alpha_{\mu\delta}\partial_\nu a^\delta + \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\nu\lambda} a^\lambda \\
&\quad - \Gamma^\sigma_{\mu\nu}\partial_\sigma a^\alpha - \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\gamma} a^\gamma \\
&= \partial_\mu\partial_\nu a^\alpha + (\partial_\mu \Gamma^\alpha_{\nu\beta}) a^\beta + \Gamma^\alpha_{\nu\beta}\partial_\mu a^\beta + \Gamma^\alpha_{\mu\delta}\partial_\nu a^\delta + \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\nu\lambda} a^\lambda \\
&\quad - \Gamma^\sigma_{\mu\nu}\partial_\sigma a^\alpha - \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\gamma} a^\gamma \\
&= \partial_\mu\partial_\nu a^\alpha + (\partial_\mu \Gamma^\alpha_{\beta\nu}) a^\beta + \Gamma^\alpha_{\beta\nu}\partial_\mu a^\beta + \Gamma^\alpha_{\delta\mu}\partial_\nu a^\delta + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\lambda\nu} a^\lambda \\
&\quad - \Gamma^\sigma_{\nu\mu}\partial_\sigma a^\alpha - \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\gamma\sigma} a^\gamma
\end{aligned}$$

Where we used the product rule and in the last step we used the symmetry of the Christoffel symbol to swap the index order.

b. We see that

$$\begin{aligned}
\nabla_\nu(\nabla_\mu a^\alpha) &= \partial_\nu\partial_\mu a^\alpha + (\partial_\nu \Gamma^\alpha_{\beta\mu}) a^\beta + \Gamma^\alpha_{\beta\mu}\partial_\nu a^\beta + \Gamma^\alpha_{\delta\nu}\partial_\mu a^\delta + \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\lambda\mu} a^\lambda \\
&\quad - \Gamma^\sigma_{\mu\nu}\partial_\sigma a^\alpha - \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\gamma\sigma} a^\gamma
\end{aligned}$$

Then $\nabla_\mu(\nabla_\nu a^\alpha) - \nabla_\nu(\nabla_\mu a^\alpha)$ gives us

$$\begin{aligned}
&\nabla_\mu(\nabla_\nu a^\alpha) - \nabla_\nu(\nabla_\mu a^\alpha) = \\
&\quad = \partial_\mu\partial_\nu a^\alpha + (\partial_\mu \Gamma^\alpha_{\beta\nu}) a^\beta + \Gamma^\alpha_{\beta\nu}\partial_\mu a^\beta + \Gamma^\alpha_{\delta\mu}\partial_\nu a^\delta + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\lambda\nu} a^\lambda \\
&\qquad - \Gamma^\sigma_{\nu\mu}\partial_\sigma a^\alpha - \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\gamma\sigma} a^\gamma - \partial_\nu\partial_\mu a^\alpha - (\partial_\nu \Gamma^\alpha_{\beta\mu}) a^\beta - \Gamma^\alpha_{\beta\mu}\partial_\nu a^\beta \\
&\qquad - \Gamma^\alpha_{\delta\nu}\partial_\mu a^\delta - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\lambda\mu} a^\lambda + \Gamma^\sigma_{\mu\nu}\partial_\sigma a^\alpha + \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\gamma\sigma} a^\gamma \\
&\quad = (\partial_\mu \Gamma^\alpha_{\beta\nu}) a^\beta - (\partial_\nu \Gamma^\alpha_{\beta\mu}) a^\beta + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\lambda\nu} a^\lambda - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\lambda\mu} a^\lambda \\
&\quad = (\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\beta\nu} - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\beta\mu}) a^\beta \\
&\quad = R^\alpha_{\beta\mu\nu} a^\beta
\end{aligned}$$

□