Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 6 - Tensor Equations

Solution. **BOX 6.1** - Exercise 6.1.1. The required partial derivatives are

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

□

Solution. **BOX 6.1** - Exercise 6.1.2. Let $\Phi = bxy = br^2\cos\theta\sin\theta$ then the components of the gradient are

$$\frac{\partial \Phi}{\partial x} = by \qquad \frac{\partial \Phi}{\partial y} = bx$$

On the other hand, for r and θ we have that

$$\frac{\partial \Phi}{\partial r} = 2br\cos\theta\sin\theta$$
$$\frac{\partial \Phi}{\partial \theta} = br^2(\cos^2\theta - \sin^2\theta)$$

□

Solution. **BOX 6.1** - Exercise 6.1.3. Now, if we treat the gradient as a covector we have that

$$\begin{aligned}\frac{\partial \Phi}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial \Phi}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \Phi}{\partial y}\\ &= by\cos\theta + bx\sin\theta\\ &= br\sin\theta\cos\theta + br\cos\theta\sin\theta\\ &= 2br\sin\theta\cos\theta\end{aligned}$$

And that

$$\begin{aligned}\frac{\partial \Phi}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial \Phi}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \Phi}{\partial y}\\ &= -byr\sin\theta + bxr\cos\theta\\ &= -br^2\sin^2\theta + br^2\cos^2\theta\\ &= br^2(\cos^2\theta - \sin^2\theta)\end{aligned}$$

Which match the equations we got in Exercise 6.1.2. □

Solution. **BOX 6.2** - Exercise 6.2.1. Let $v^x = 1$ and $v^y = 0$ then to lower the indices we compute the following

$$\begin{aligned} v_x &= g_{x\nu} v^\nu = g_{xx} v^x + g_{xy} v^y = 1 \cdot 1 + 0 \cdot 0 = 1 \\ v_y &= g_{y\nu} v^\nu = g_{yx} v^x + g_{yy} v^y = 0 \cdot 1 + 1 \cdot 0 = 0 \end{aligned}$$

where we used that

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Solution. **BOX 6.2** - Exercise 6.2.2. Now, we compute v_r and v_θ by using the covector transformations as follows

$$v_r = \frac{\partial x^\alpha}{\partial r} v_\alpha = \frac{\partial x}{\partial r} v_x + \frac{\partial y}{\partial r} v_y = \cos\theta \cdot 1 + \sin\theta \cdot 0 = \cos\theta$$

And

$$v_\theta = \frac{\partial x^\alpha}{\partial \theta} v_\alpha = \frac{\partial x}{\partial \theta} v_x + \frac{\partial y}{\partial \theta} v_y = -r\sin\theta \cdot 1 + r\cos\theta \cdot 0 = -r\sin\theta$$

□

Solution. **BOX 6.2** - Exercise 6.2.3. Finally, we want to show that $v'^\mu v'_\mu = 1$ hence we have that

$$\begin{aligned} v'^\mu v'_\mu &= v^r v_r + v^\theta v_\theta \\ &= (\cos\theta)(\cos\theta) + \left(-\frac{\sin\theta}{r}\right)(-r\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

This makes sense since the length of the vector is 1 and this generalizes the notion of length. □

Solution. **BOX 6.3** - Exercise 6.3.1. By using equation 6.16 and summing over the resulting Kronecker delta we get that

$$\begin{aligned} g'^{\mu\beta} g'_{\beta\nu} &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\delta}{\partial x'^\nu} g^{\alpha\sigma} g_{\sigma\delta} \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\delta}{\partial x'^\nu} \delta^\alpha_\delta \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} \\ &= \delta^\mu_\nu \end{aligned}$$

□

Solution. **BOX 6.4** - Exercise 6.4.1. By using the fundamental identity we have that

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \delta^{\alpha}_{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu}$$

□

Solution. **BOX 6.5** - Exercise 6.5.1. We want to show that $C_{\mu\nu}{}^{\alpha} = A_{\mu\nu} B^{\alpha}$ satisfies the tensor transformations.

$$\begin{aligned} C'_{\mu\nu}{}^{\alpha} = A'_{\mu\nu} B'^{\alpha} &= \left(\frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} A_{\beta\gamma} \right) \left(\frac{\partial x'^{\alpha}}{\partial x^{\sigma}} B^{\sigma} \right) \\ &= \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial x'^{\alpha}}{\partial x^{\sigma}} \left(A_{\beta\gamma} B^{\sigma} \right) = \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial x'^{\alpha}}{\partial x^{\sigma}} \; C_{\beta\gamma}{}^{\sigma} \end{aligned}$$

□

Solution. **BOX 6.5** - Exercise 6.5.2. As we saw, to raise the first index of $C_{\mu\nu}{}^{\alpha}$ we multiply it by $g^{\mu\sigma}$ then we have that

$$\begin{aligned} C'^{\mu}{}_{\nu}{}^{\alpha} = g'^{\mu\sigma} C'_{\sigma\nu}{}^{\alpha} &= \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} \; g^{\beta\gamma} \right) \left(\frac{\partial x^{\gamma}}{\partial x'^{\sigma}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \frac{\partial x'^{\alpha}}{\partial x^{\phi}} \; C_{\gamma\delta}{}^{\phi} \right) \\ &= \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\gamma}}{\partial x'^{\sigma}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \frac{\partial x'^{\alpha}}{\partial x^{\phi}} \right) \left(g^{\beta\gamma} C_{\gamma\delta}{}^{\phi} \right) \\ &= \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x^{\delta}}{\partial x'^{\nu}} \frac{\partial x'^{\alpha}}{\partial x^{\phi}} \; C^{\beta}{}_{\delta}{}^{\phi} \end{aligned}$$

Therefore $C^{\mu}{}_{\nu}{}^{\alpha}$ transforms like a tensor as we wanted. □

Solution. **BOX 6.5** - Exercise 6.5.3. We saw that $C^{\mu}{}_{\nu}{}^{\alpha}$ transforms like a tensor, hence for $\nu = \mu$ we have that

$$C'^{\mu}{}_{\mu}{}^{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x'^{\alpha}}{\partial x^{\phi}} \; C^{\beta}{}_{\beta}{}^{\phi} = \frac{\partial x'^{\alpha}}{\partial x^{\phi}} \; C^{\beta}{}_{\beta}{}^{\phi}$$

But the four-vector C^{α} transforms as $C'^{\alpha} = (\partial x'^{\alpha} / \partial x^{\phi}) C^{\phi}$.
Therefore $C^{\mu}{}_{\mu}{}^{\alpha}$ transforms as a four-vector. □

Solution. **P6.1** Let us consider the following polar coordinates in 2D flat space

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

a.

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} \end{aligned}$$

b. Using the vector transformations we have that

$$\begin{aligned} v^r &= \frac{\partial r}{\partial x^\alpha} v^\alpha = \frac{\partial r}{\partial x} v^x + \frac{\partial r}{\partial y} v^y = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta \\ v^\theta &= \frac{\partial \theta}{\partial x^\alpha} v^\alpha = \frac{\partial \theta}{\partial x} v^x + \frac{\partial \theta}{\partial y} v^y = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r} \end{aligned}$$

Where in the last step of each equation we used that $x = r\cos\theta$ and $y = r\sin\theta$ respectively.

c. We know that the inverse metric should satisfy

$$g'^{\mu\alpha} g'_{\alpha\nu} = \delta^\mu{}_\nu$$

hence

$$\begin{aligned} g'^{r\alpha} g'_{\alpha r} &= g'^{rr} g'_{rr} + g'^{r\theta} g'_{\theta r} = g'^{rr} \cdot 1 + g'^{r\theta} \cdot 0 = 1 \\ g'^{r\alpha} g'_{\alpha\theta} &= g'^{rr} g'_{r\theta} + g'^{r\theta} g'_{\theta\theta} = g'^{rr} \cdot 0 + g'^{r\theta} \cdot r^2 = 0 \\ g'^{\theta\alpha} g'_{\alpha r} &= g'^{\theta r} g'_{rr} + g'^{\theta\theta} g'_{\theta r} = g'^{\theta r} \cdot 1 + g'^{\theta\theta} \cdot 0 = 0 \\ g'^{\theta\alpha} g'_{\alpha\theta} &= g'^{\theta r} g'_{r\theta} + g'^{\theta\theta} g'_{\theta\theta} = g'^{\theta r} \cdot 0 + g'^{\theta\theta} \cdot r^2 = 1 \end{aligned}$$

So from each equation we see that $g'^{rr} = 1$, $g'^{r\theta} = g'^{\theta r} = 0$ and $g'^{\theta\theta} = 1/r^2$ which implies that

$$g'^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1/r^2 \end{bmatrix}$$

d. To raise the polar-coordinates components of the covector v_μ we compute the following

$$\begin{aligned} v^r &= g'^{r\nu} v_\nu = g'^{rr} v_r + g'^{r\theta} v_\theta = \cos\theta \\ v^\theta &= g'^{\theta\nu} v_\nu = g'^{\theta r} v_r + g'^{\theta\theta} v_\theta = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \end{aligned}$$

□

Solution. **P6.2** Let's compute $v^\mu \partial_\mu \Phi$ in both the cartesian and polar coordinates. From the result of BOX 6.1 we have the following for cartesian coordinates

$$v^\mu \partial_\mu \Phi = v^x \partial_x \Phi + v^y \partial_y \Phi = 1 \cdot by + 0 \cdot bx = by$$

and the following for polar coordinates

$$\begin{aligned}
v^\mu \partial_\mu \Phi &= v^r \partial_r \Phi + v^\theta \partial_\theta \Phi \\
&= (\cos\theta) \cdot (2br \cos\theta \sin\theta) - \left(\frac{\sin\theta}{r}\right) \cdot br^2 (\cos^2\theta - \sin^2\theta) \\
&= br \sin\theta (2\cos^2\theta - (\cos^2\theta - \sin^2\theta)) \\
&= br \sin\theta (\cos^2\theta + \sin^2\theta) \\
&= br \sin\theta = by
\end{aligned}$$

Therefore we see that the value of $v^\mu \partial_\mu \Phi$ is the same in both coordinate systems. □

Solution. **P6.3** Let $F^{\mu\nu}$ be a second-rank tensor and let $F = F^\mu{}_\mu = g_{\mu\nu} F^{\mu\nu}$ be the trace of $F^{\mu\nu}$ we want to prove that it transforms like a scalar invariant. Then by applying the definition in a primed coordinate system we have that

$$\begin{aligned}
F'^\mu{}_\mu &= g'_{\mu\nu} F'^{\mu\nu} \\
&= \left[\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}\right] \left[\frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} F^{\alpha\beta}\right] \\
&= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\beta} g_{\alpha\beta} F^{\alpha\beta} \\
&= g_{\alpha\beta} F^{\alpha\beta} \\
&= F^\alpha{}_\alpha
\end{aligned}$$

Therefore $F^\mu{}_\mu$ transforms like a scalar invariant. □

Solution. **P6.4**

a. No we cannot add the four-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ because they have different units and makes no sense to add quantities with different units.

b. The components of the second-rank tensor product $M^{\mu\nu} = A^\mu B^\nu$ are given by

$$\begin{aligned}
M^{00} &= A^0 B^0 = 1\ m \cdot 3\ s^{-1} = 3\ ms^{-1} \\
M^{10} &= A^1 B^0 = 2\ m \cdot 3\ s^{-1} = 6\ ms^{-1} \\
M^{20} &= A^2 B^0 = -1\ m \cdot 3\ s^{-1} = -3\ ms^{-1} \\
M^{30} &= A^3 B^0 = 0\ m \cdot 3\ s^{-1} = 0\ ms^{-1}
\end{aligned}$$

Continuing this process for the rest of the components we get that

$$M^{\mu\nu} = \begin{bmatrix} 3 & -1 & 0 & -2 \\ 6 & -2 & 0 & -4 \\ -3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} ms^{-1}$$

c. We want to check now if the tensor product is commutative, so we compute the following

$$B^\mu A^\nu = \begin{bmatrix} 3 & 6 & -3 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -4 & 2 & 0 \end{bmatrix} ms^{-1}$$

We see then that the tensor product is not commutative.

d. If the metric for this coordinate system is the flat-space metric $\eta_{\mu\nu}$ then $M^\mu{}_\mu$ is given by

$$\begin{aligned}
M^\mu{}_\mu &= \eta_{\mu\nu} M^{\mu\nu} \\
&= \eta_{0\nu} M^{0\nu} + \eta_{1\nu} M^{1\nu} + \eta_{2\nu} M^{2\nu} + \eta_{3\nu} M^{3\nu} \\
&= -3 - 2 + 0 + 0 \\
&= -5
\end{aligned}$$

The relation of $M^\mu{}_\mu$ to $\boldsymbol{A} \cdot \boldsymbol{B}$ is that $M^\mu{}_\mu$ is the sum of the diagonal with the first component with the opposite sign.

□

Solution. **P6.5** To compute the numerical value of the scalar $g_{\mu\nu}g^{\mu\nu}$ let us recall that $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$ hence we have that

$$\begin{aligned}
g_{\mu\nu}g^{\mu\nu} &= g^{\mu\nu}g_{\mu\nu} \\
&= g^{\mu\nu}g_{\nu\mu} \\
&= \delta^{\mu}{}_{\mu} \\
&= D
\end{aligned}$$

Where we used that $g_{\mu\nu} = g_{\nu\mu}$ i.e. that the metric tensor is symmetric and D is the number of dimensions of $g_{\mu\nu}$. □

Solution. **P6.6**

a. Let us suppose $M^{\mu\nu} = M^{\nu\mu}$ i.e. $\boldsymbol{M}$ is symmetric we want to prove it's also symmetric in a different coordinate system. Since a tensor transforms as

$$M'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} M^{\alpha\beta}$$

Hence we have that

$$M'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} M^{\alpha\beta} = \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} M^{\beta\alpha} = M'^{\nu\mu}$$

Therefore $\boldsymbol{M}$ is also symmetric in a different coordinate system. The same can be proven for antisymmetric tensors.

b. Let $F^{\mu\nu}$ be antisymmetric then $F^{\mu\nu} = -F^{\nu\mu}$. Let us compute the following

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} = g_{\nu\beta} g_{\mu\alpha} (-F^{\beta\alpha}) = -(g_{\nu\beta} g_{\mu\alpha} F^{\beta\alpha}) = -F_{\nu\mu}$$

Therefore $F_{\mu\nu}$ is also antisymmetric.

Now, let $M^{\mu\nu}$ be symmetric then $M^{\mu\nu} = M^{\nu\mu}$. So let us compute the following

$$M_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} M^{\alpha\beta} = g_{\nu\beta} g_{\mu\alpha} M^{\beta\alpha} = M_{\nu\mu}$$

Therefore $M_{\mu\nu}$ is also symmetric.

c. Let $\boldsymbol{M}$ be symmetric and $\boldsymbol{F}$ be antisymmetric then we have that

$$M_{\mu\nu} F^{\mu\nu} = -M_{\nu\mu} F^{\nu\mu}$$

Then it must be that $M_{\mu\nu} F^{\mu\nu} = 0$.

d. Let $\boldsymbol{F}$ be antisymmetric, then the trace $F^{\mu}{}_{\mu}$ is given by

$$F^{\mu}{}_{\mu} = g_{\mu\nu} F^{\mu\nu} = g_{\mu\nu} (-F^{\nu\mu}) = -(g_{\nu\mu} F^{\nu\mu}) = -F^{\nu}{}_{\nu}$$

Where we used that $g_{\mu\nu}$ is symmetric. This implies that $F^{\mu}{}_{\mu} = 0$.

e. A symmetric tensor in $4D$ spacetime has 16 components, but the components above and below the diagonal are equal hence they depend on each other this implies that only 10 components are independent. For an antisymmetric tensor, the components of the diagonal are 0 and the components above and below the diagonal depend on each other therefore only 6 components are independent.

□

Solution. **P6.7** Let $\boldsymbol{A}$ be an arbitrary four-vector that depends on a particle's position in spacetime.

a. We want to show that $dA^\mu/d\tau$ is not a four-vector unless the coordinate transformation partials are independent of position in spacetime.

Given that $\boldsymbol{A}$ is a four-vector then it transforms like it so knowing this and using the product rule we have that

$$\begin{aligned}\frac{\mathrm{d}A'^\mu}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}A^\nu\right)\\ &= A^\nu\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) + \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}A^\nu}{\mathrm{d}\tau}\end{aligned}$$

Then we see that $dA^\mu/d\tau$ does not transform like a four-vector and hence it's not a four-vector. But if the coordinate transformation partials $\partial x'^\mu/\partial x^\nu$ are independent of position in spacetime then the derivative of this quatity with respect to τ is zero and hence we get that

$$\frac{\mathrm{d}A'^\mu}{\mathrm{d}\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}A^\nu}{\mathrm{d}\tau}$$

Which transforms like a four-vector and therefore is a four-vector.

b. Let now $\boldsymbol{\phi}(x^\mu)$ be a scalar that depends on position, we want to prove that $d\boldsymbol{\phi}/d\tau$ along a particle's worldline is a valid scalar. Then by the chain rule we have that

$$\frac{\mathrm{d}\boldsymbol{\phi}(x^\mu)}{\mathrm{d}\tau} = \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

Also, we know that the scalar $\boldsymbol{\phi}$ depends on x^μ and so $d\boldsymbol{\phi}/dx^\mu$ is a scalar value, but also x^μ depends on τ so $dx^\mu/d\tau$ is a scalar value.

Therefore the implied sum gives us a scalar as we wanted.

□