Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 2 - Review of Special Relativity

Solution. **BOX 2.2** We want to compute g and G in GR units, hence

$$\begin{aligned} g &= 9.81\ \frac{m}{s^2}\left(\frac{1\ s}{2.99792458 \times 10^8\ m}\right)^2 \\ &= 1.091 \times 10^{-16}\ m^{-1} = \frac{1}{9.17 \times 10^{15}\ m} \end{aligned}$$

and since a light year is $9.4607 \times 10^{15}\ m$ we could say that $g \approx 1\ ly^{-1}$.

Knowing that $N = kg\ m/s^2$ for $G = 6.6743 \times 10^{-11}\ Nm^2/kg^2$ we have that

$$\begin{aligned} G &= 6.6743 \times 10^{-11}\frac{m^3}{s^2 kg}\left(\frac{1\ s}{2.99792458 \times 10^8\ m}\right)^2 \\ &= 7.4261 \times 10^{-28}\frac{m}{kg} \end{aligned}$$

and since 1 solar mass is equivalent to $1.988 \times 10^{30}\ kg$ we have that

$$\begin{aligned} G &= 7.4261 \times 10^{-28}\frac{m}{kg}\left(\frac{1.988 \times 10^{30}\ kg}{1\ \text{solar mass}}\right) \\ &= 1477\ m/\text{solar mass} \end{aligned}$$

$\square$

Solution. **BOX 2.6** Let $\Delta t > 0$ in the frame S and $\Delta s^2 > 0$, the last inequality implies that $\Delta x^2 + \Delta y^2 + \Delta z^2 > \Delta t^2$ and since the events occur in the $+x$ axis we have that $\Delta x^2 > \Delta t^2$ thus $\Delta x > \Delta t > 0$.

Also, let us propose $\beta = \Delta t/\Delta x$ since we know that $0 < \Delta t < \Delta x$ then we have that $0 < \beta = \Delta t/\Delta x < 1$. We want to prove next that it's possible to get $\Delta t' < 0$ for some β such that $0 < \Delta t/\Delta x < \beta < 1$. We see that

$$\begin{aligned} \Delta t &< \beta \Delta x \\ \Delta t - \beta \Delta x &< 0 \\ \gamma \Delta t - \gamma \beta \Delta x &< 0 \end{aligned}$$

But by the Lorentz transformations for a frame S' moving at a speed β with respect to S we know that

$$\Delta t' = \gamma \Delta t - \gamma \beta \Delta x$$

Therefore we have that $\Delta t' < 0$.

If we now suppose $\Delta s^2 < 0$ then this implies that $\Delta x < \Delta t$ and for $\Delta t' < 0$ we know it must happen for β that $\Delta t/\Delta x < \beta < 1$ but this is not possible since as we said $\Delta x < \Delta t$ implying that $1 < \Delta t/\Delta x$. □

Solution. **BOX 2.7** We know that $d\tau = dt'$ since a clock carried by the particle should read the same time as an inertial clock in the same infinitesimal path then in terms of the spacetime interval we have that

$$\begin{aligned} d\tau = dt' &= \sqrt{-ds^2} \\ &= \sqrt{dt^2 - dx^2 - dy^2 - dz^2} \\ &= \sqrt{dt^2(1 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2)} \\ &= dt\sqrt{1 - v^2} \end{aligned}$$

Where we used that $v^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$ □

Solution. **P2.9**

(a) We want to calculate the trip measured by clocks on A. We know that $v = 6/13$ since A travels at a constant speed and in a straight path from Alpha to Beta then using the proper time equation we have that

$$\Delta t_A = \sqrt{1 - v^2} \int_0^{13} dt$$
$$\Delta t_A = 13\sqrt{1 - (6/13)^2}$$
$$\Delta t_A = \sqrt{133} \text{ Tm} = 11.53 \text{ Tm}$$

(b) We want to calculate now the trip measured by clocks on B. In this case, the semi-circular motion gives us a velocity $v = \omega r = (\pi/13) \cdot 3$ hence

$$\Delta t_B = \sqrt{1 - v^2} \int_0^{13} dt$$
$$\Delta t_B = 13\sqrt{1 - (3\pi/13)^2}$$
$$\Delta t_B = 8.95 \text{ Tm}$$

□

Solution. **P2.12**

(a) Let us analyze a light ray that is going upwards in the S' frame then it does not have a velocity component in the x' direction. Since S' is travelling at a speed β, seen from frame S this light ray will have a velocity component in the x direction of magnitude β and since the total velocity of the light ray must be $c = 1$ then the velocity component in the y direction must be $\sqrt{1-\beta^2}$ so the angle with respect to the x axis is given by

$$\sin(\theta) = \frac{\sqrt{1-\beta^2}}{1}$$
$$\theta = \sin^{-1}(\sqrt{1-\beta^2})$$

(b) Replacing the value of $\beta = 0.99$ in the above equation we get that

$$\theta = \sin^{-1}(\sqrt{1-(0.99)^2}) = 0.1415 \text{ rad} = 8.10^\circ$$

□