Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 3 - Four Vectors

Solution. **BOX 3.1** Let us compute the A'^t and A'^x components of $\boldsymbol{A'}\cdot\boldsymbol{B'}$ using the Lorentz transformations as follows

$$\begin{aligned}
-A'^tB'^t + A'^xB'^x &= -(\gamma A^t - \gamma\beta A^x)(\gamma B^t - \gamma\beta B^x)\\
&\quad + (\gamma A^x - \gamma\beta A^t)(\gamma B^x - \gamma\beta B^t)\\
&= (\gamma^2\beta A^xB^t - \gamma^2\beta^2 A^xB^x - \gamma^2A^tB^t + \gamma^2\beta A^tB^x)\\
&\quad + (\gamma^2A^xB^x - \gamma^2\beta B^tA^x - \gamma^2\beta A^tB^x + \gamma^2\beta^2A^tB^t)\\
&= \gamma^2A^xB^x(1-\beta^2) - \gamma^2A^tB^t(1-\beta^2)\\
&= -A^tB^t + A^xB^x
\end{aligned}$$

Also, we have that $A'^yB'^y + A'^zB'^z = A^yB^y + A^zB^z$ therefore

$$-A'^tB'^t + A'^xB'^x + A'^yB'^y + A'^zB'^z = -A^tB^t + A^xB^x + A^yB^y + A^zB^z$$

□

Solution. **BOX 3.2** We know that

$$d\boldsymbol{s}\cdot d\boldsymbol{s} = ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and that $d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$ so let us compute $\boldsymbol{u}\cdot\boldsymbol{u}$ as follows

$$\begin{aligned}
\boldsymbol{u}\cdot\boldsymbol{u} &= \frac{ds^2}{d\tau^2}\\
&= \frac{-dt^2 + dx^2 + dy^2 + dz^2}{d\tau^2}\\
&= \frac{-(dt^2 - dx^2 - dy^2 - dz^2)}{dt^2 - dx^2 - dy^2 - dz^2}\\
&= -1
\end{aligned}$$

□

Solution. **BOX 3.5** We want to verify equation 3.44. We know that

$$\begin{aligned} p_p^2 + 2\boldsymbol{p}_p\boldsymbol{p}_\gamma &= -m_p^2 - 2m_p m_\pi - m_\pi^2 \\ 2\boldsymbol{p}_p\boldsymbol{p}_\gamma &= -2m_p m_\pi - m_\pi^2 \end{aligned}$$

Where we used that $p_p^2 = -m_p^2$ and that $p_\gamma^2 = -m_\gamma^2 = 0$. Since we are considering an approximation of $p_{px} \approx E$ we get that

$$\boldsymbol{p}_p = \begin{bmatrix} E_p \\ E_p \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_\gamma = \begin{bmatrix} E \\ -E \\ 0 \\ 0 \end{bmatrix}$$

Then

$$\begin{aligned} -2(E_p E + E_p E) &= -2m_p m_\pi - m_\pi^2 \\ 4E_p E &= 2m_p m_\pi + m_\pi^2 \\ E_p &= \frac{m_\pi(2m_p + m_\pi)}{4E} \end{aligned}$$

Finally, we know that $m_p = 938\ MeV$, $m_\pi = 135\ MeV$ and $E = 6.4 \times 10^{-4}\ eV$ hence E_p is

$$\begin{aligned} E_p &= \frac{135 \times 10^6\ eV(2(938 \times 10^6\ eV) + 135 \times 10^6\ eV)}{4(6.4 \times 10^{-4}\ eV)} \\ E_p &= 1.06 \times 10^{20}\ eV \end{aligned}$$

$\square$

Solution. **P3.1**

a. We know that $u^x = dx/d\tau$ hence $u^x = \sinh(g\tau)$.

b. We know that

$$\boldsymbol{u} \cdot \boldsymbol{u} = -(u^t)^2 + (u^x)^2 = -1$$

Hence

$$\begin{aligned} -(u^t)^2 + \sinh^2(g\tau) &= -1 \\ u^t &= \sqrt{1 + \sinh^2(g\tau)} \\ u^t &= \cosh(g\tau) \end{aligned}$$

c. Given that we know u^t and u^x we can compute $v = v_x$ as

$$v = u^x/u^t = \tanh(g\tau)$$

and since tanh is asymptotic to 1 then v cannot be greater than 1.

d. We know that $dt/d\tau = u^t = \cosh(g\tau)$ then

$$\begin{aligned} \int dt &= \int \cosh(g\tau)\ d\tau \\ t &= \frac{\sinh(g\tau)}{g} + C \\ gt &= \sinh(g\tau) \end{aligned}$$

Where we used that $t = 0$ when $\tau = 0$ and hence $C = 0$.

e. From the previous part we have that $g\tau = \sinh^{-1}(gt)$ hence

$$u^x = \sinh(\sinh^{-1}(gt)) = gt$$

Also, we get that

$$\begin{aligned} u^t &= \cosh(\sinh^{-1}(gt)) \\ &= \sqrt{1 + (gt)^2} \end{aligned}$$

And finally that

$$\begin{aligned} v &= \tanh(\sinh^{-1}(gt)) \\ &= \frac{\sinh(\sinh^{-1}(gt))}{\cosh(\sinh^{-1}(gt))} \\ &= \frac{gt}{\sqrt{1 + (gt)^2}} \end{aligned}$$

□

Solution. **P3.2**

a. We know that $u^t = 1/\sqrt{1-v^2}$ so

$$\begin{aligned} u^t &= \frac{1}{\sqrt{1-v^2}} \\ &= \frac{1}{\sqrt{1-(1-1/(gt+1)^2)}} \\ &= \frac{1}{\sqrt{\frac{1}{(gt+1)^2}}} \\ &= gt+1 \end{aligned}$$

b. We know that $dt/d\tau = u^t = gt+1$ then

$$\begin{aligned} \int_0^t \frac{dt}{gt+1} &= \int_0^\tau d\tau \\ \frac{\log(gt+1)}{g} - 0 &= \tau - 0 \\ \log(gt+1) &= g\tau \end{aligned}$$

c. Using the equations we got in part **a** and **b** we get that

$$\begin{aligned} \log(u^t) &= \log(gt+1) \\ \log(u^t) &= g\tau \\ u^t &= e^{g\tau} \end{aligned}$$

d. We know that $u^x = u^t v$ so we have that

$$\begin{aligned} u^x &= (gt+1)\sqrt{1-\frac{1}{(gt+1)^2}} \\ &= (gt+1)\sqrt{\frac{(gt+1)^2-1}{(gt+1)^2}} \\ &= \sqrt{(gt+1)^2-1} \end{aligned}$$

Also we know that $gt+1 = e^{g\tau}$ so we can write

$$u^x = \sqrt{e^{2g\tau}-1}$$

e. We know that $u^x = dx/d\tau$ so we can integrate u^x to obtain $x(\tau)$ as follows

$$\begin{aligned} \int_0^{x(\tau)} dx &= \int_0^\tau \sqrt{e^{2g\tau}-1}\, d\tau \\ x(\tau) - 0 &= \frac{\sqrt{e^{2g\tau}-1} - \arctan(\sqrt{e^{2g\tau}-1})}{g} - 0 \\ x(\tau) &= \frac{\sqrt{e^{2g\tau}-1} - \arctan(\sqrt{e^{2g\tau}-1})}{g} \end{aligned}$$

Finally, we know that $u^t = dt/d\tau$ so we can integrate u^t to obtain $t(\tau)$ as follows

$$\begin{aligned} \int_0^{t(\tau)} dt &= \int_0^{\tau} e^{g\tau}\ d\tau \\ t(\tau) - 0 &= \frac{e^{g\tau}}{g} - \frac{1}{g} \\ t(\tau) &= \frac{e^{g\tau} - 1}{g} \end{aligned}$$

$\square$

Solution. **P3.7**

a. In the source frame we have that $p^t = E$ and that $p^x = Ev_x = E$ since $v_x = 1, v_y = 0, v_z = 0$ so using the Lorentz transformations we get that

$$
\begin{aligned}
p'^t &= \gamma E - \gamma v E \\
E' &= E\frac{(1-v)}{\sqrt{1-v^2}} \\
E' &= E\frac{(1-v)\sqrt{1-v}}{\sqrt{(1+v)}\sqrt{(1-v)^2}} \\
E' &= \frac{h}{\lambda_0}\frac{\sqrt{1-v}}{\sqrt{1+v}}
\end{aligned}
$$

Where we also used that $p'^t = E'$. Hence the wavelength λ' in the source frame is given by

$$
\begin{aligned}
\frac{h}{\lambda'} &= \frac{h}{\lambda_0}\frac{\sqrt{1-v}}{\sqrt{1+v}} \\
\lambda' &= \lambda_0\frac{\sqrt{1+v}}{\sqrt{1-v}}
\end{aligned}
$$

b. Let now $\boldsymbol{u}_{obs}$ be the velocity of the observer seen from the source frame then we have that $u^t_{obs} = \gamma$, $u^x_{obs} = \gamma v$ and $u^y_{obs} = u^z_{obs} = 0$. Also, let $\boldsymbol{p}$ be the 4-momentum of the photon in the source frame where $p^t = E$ and that $p^x = Ev_x = E$ since $v_x = 1$ and $v_y = v_z = 0$.

On the other hand, we know that $E_{obs} = -\boldsymbol{p}\boldsymbol{u}_{obs}$ so

$$
\begin{aligned}
E_{obs} &= -(-p^t u^t_{obs} + p^x u^x_{obs} + p^y u^y_{obs} + p^z u^z_{obs}) \\
E_{obs} &= -(-E\gamma + E\gamma v + 0 + 0) \\
E_{obs} &= E\gamma(1-v)
\end{aligned}
$$

We have the same equation we got in part **a** so we can follow the same steps to show that

$$
\begin{aligned}
E_{obs} &= \frac{h}{\lambda_0}\frac{\sqrt{1-v}}{\sqrt{1+v}} \\
\lambda_{obs} &= \lambda_0\frac{\sqrt{1+v}}{\sqrt{1-v}}
\end{aligned}
$$

□