Solved selected problems of General Relativity - Thomas A. Moore

Franco Zacco

Chapter 6 - Tensor Equations

Solution. **BOX 6.1** - Exercise 6.1.1. The required partial derivatives are

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

□

Solution. **BOX 6.1** - Exercise 6.1.2. Let $\Phi = bxy = br^2\cos\theta\sin\theta$ then the components of the gradient are

$$\frac{\partial \Phi}{\partial x} = by \qquad \frac{\partial \Phi}{\partial y} = bx$$

On the other hand, for r and θ we have that

$$\frac{\partial \Phi}{\partial r} = 2br\cos\theta\sin\theta$$
$$\frac{\partial \Phi}{\partial \theta} = br^2(\cos^2\theta - \sin^2\theta)$$

□

Solution. **BOX 6.1** - Exercise 6.1.3. Now, if we treat the gradient as a covector we have that

$$\begin{aligned}\frac{\partial \Phi}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial \Phi}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \Phi}{\partial y}\\ &= by\cos\theta + bx\sin\theta\\ &= br\sin\theta\cos\theta + br\cos\theta\sin\theta\\ &= 2br\sin\theta\cos\theta\end{aligned}$$

And that

$$\begin{aligned}\frac{\partial \Phi}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial \Phi}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \Phi}{\partial y}\\ &= -byr\sin\theta + bxr\cos\theta\\ &= -br^2\sin^2\theta + br^2\cos^2\theta\\ &= br^2(\cos^2\theta - \sin^2\theta)\end{aligned}$$

Which match the equations we got in Exercise 6.1.2. □

Solution. **BOX 6.2** - Exercise 6.2.1. Let $v^x = 1$ and $v^y = 0$ then to lower the indices we compute the following

$$v_x = g_{x\nu}v^\nu = g_{xx}v^x + g_{xy}v^y = 1 \cdot 1 + 0 \cdot 0 = 1$$
$$v_y = g_{y\nu}v^\nu = g_{yx}v^x + g_{yy}v^y = 0 \cdot 1 + 1 \cdot 0 = 0$$

where we used that

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Solution. **BOX 6.2** - Exercise 6.2.2. Now, we compute v_r and v_θ by using the covector transformations as follows

$$v_r = \frac{\partial x^\alpha}{\partial r} v_\alpha = \frac{\partial x}{\partial r} v_x + \frac{\partial y}{\partial r} v_y = \cos\theta \cdot 1 + \sin\theta \cdot 0 = \cos\theta$$

And

$$v_\theta = \frac{\partial x^\alpha}{\partial \theta} v_\alpha = \frac{\partial x}{\partial \theta} v_x + \frac{\partial y}{\partial \theta} v_y = -r\sin\theta \cdot 1 + r\cos\theta \cdot 0 = -r\sin\theta$$

□

Solution. **BOX 6.2** - Exercise 6.2.3. Finally, we want to show that $v'^\mu v'_\mu = 1$ hence we have that

$$\begin{aligned} v'^\mu v'_\mu &= v^r v_r + v^\theta v_\theta \\ &= (\cos\theta)(\cos\theta) + \left(-\frac{\sin\theta}{r}\right)(-r\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

This makes sense since the length of the vector is 1 and this generalizes the notion of length. □

Solution. **BOX 6.3** - Exercise 6.3.1. By using equation 6.16 and summing over the resulting Kronecker delta we get that

$$\begin{aligned} g'^{\mu\beta} g'_{\beta\nu} &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\delta}{\partial x'^\nu} g^{\alpha\sigma} g_{\sigma\delta} \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\delta}{\partial x'^\nu} \delta^\alpha_\delta \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} \\ &= \delta^\mu_\nu \end{aligned}$$

□

Solution. **BOX 6.4** - Exercise 6.4.1. By using the fundamental identity we have that

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\delta^{\alpha}_{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu}$$

□

Solution. **BOX 6.5** - Exercise 6.5.1. We want to show that ${C_{\mu\nu}}^{\alpha} = A_{\mu\nu}B^{\alpha}$ satisfies the tensor transformations.

$$\begin{aligned}{C'_{\mu\nu}}^{\alpha} = A'_{\mu\nu}B'^{\alpha} &= \left(\frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\nu}}A_{\beta\gamma}\right)\left(\frac{\partial x'^{\alpha}}{\partial x^{\sigma}}B^{\sigma}\right)\\ &= \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\nu}}\frac{\partial x'^{\alpha}}{\partial x^{\sigma}}\left(A_{\beta\gamma}B^{\sigma}\right) = \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\nu}}\frac{\partial x'^{\alpha}}{\partial x^{\sigma}}\ {C_{\beta\gamma}}^{\sigma}\end{aligned}$$

□

Solution. **BOX 6.5** - Exercise 6.5.2. As we saw, to raise the first index of ${C_{\mu\nu}}^{\alpha}$ we multiply it by $g^{\mu\sigma}$ then we have that

$$\begin{aligned}{C'^{\mu}}_{\nu}{}^{\alpha} = g'^{\mu\sigma}{C'_{\sigma\nu}}^{\alpha} &= \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial x'^{\sigma}}{\partial x^{\gamma}}\ g^{\beta\gamma}\right)\left(\frac{\partial x^{\gamma}}{\partial x'^{\sigma}}\frac{\partial x^{\delta}}{\partial x'^{\nu}}\frac{\partial x'^{\alpha}}{\partial x^{\phi}}\ {C_{\gamma\delta}}^{\phi}\right)\\ &= \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial x'^{\sigma}}{\partial x^{\gamma}}\frac{\partial x^{\gamma}}{\partial x'^{\sigma}}\frac{\partial x^{\delta}}{\partial x'^{\nu}}\frac{\partial x'^{\alpha}}{\partial x^{\phi}}\right)\left(g^{\beta\gamma}{C_{\gamma\delta}}^{\phi}\right)\\ &= \frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial x^{\delta}}{\partial x'^{\nu}}\frac{\partial x'^{\alpha}}{\partial x^{\phi}}\ {C^{\beta}}_{\delta}{}^{\phi}\end{aligned}$$

Therefore ${C^{\mu}}_{\nu}{}^{\alpha}$ transforms like a tensor as we wanted. □

Solution. **BOX 6.5** - Exercise 6.5.3. We saw that ${C^{\mu}}_{\nu}{}^{\alpha}$ transforms like a tensor, hence for $\nu = \mu$ we have that

$${C'^{\mu}}_{\mu}{}^{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x'^{\alpha}}{\partial x^{\phi}}\ {C^{\beta}}_{\beta}{}^{\phi} = \frac{\partial x'^{\alpha}}{\partial x^{\phi}}\ {C^{\beta}}_{\beta}{}^{\phi}$$

But the four-vector C^{α} transforms as $C'^{\alpha} = (\partial x'^{\alpha}/\partial x^{\phi})C^{\phi}$.
Therefore ${C^{\mu}}_{\mu}{}^{\alpha}$ transforms as a four-vector.

□