Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 14 - Event Horizon

Solution. **BOX 14.1** - Exercise 14.1.1.
Let

$$u = \sqrt{1 - \frac{2GM}{r}}$$

Then

$$\begin{aligned} u^2 &= 1 - \frac{2GM}{r} \\ \frac{2GM}{r} &= 1 - u^2 \\ r &= \frac{2GM}{1 - u^2} \end{aligned}$$

□

Solution. **BOX 14.1** - Exercise 14.1.2.
Let

$$r = \frac{2GM}{1 - u^2}$$

Then

$$\frac{\mathrm{d}r}{\mathrm{d}u} = \frac{4GMu}{(1 - u^2)^2} \quad \text{hence} \quad dr = \frac{4GMu}{(1 - u^2)^2}\, du$$

Replacing in (14.7) we have that

$$\Delta s = \int \frac{4GMudu}{u(1 - u^2)^2} = 4GM \int \frac{du}{(1 - u^2)^2}$$

Finally, since we want to integrate between $r = 2GM$ and $r = R$ then u at $2GM$ is 0 so we write

$$\Delta s = 4GM \int_0^{u(R)} \frac{du}{(1 - u^2)^2}$$

□

Solution. **BOX 14.1** - Exercise 14.1.3.
Knowing that

$$\int \frac{du}{(1-u^2)^2} = \frac{u}{2(1-u^2)} + \frac{1}{4}\log\left|\frac{1+u}{1-u}\right|$$

and that

$$\tanh^{-1} u = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|$$

We can solve Δs equation as follows

$$\begin{aligned}
\Delta s &= 4GM \int_0^{u(R)} \frac{du}{(1-u^2)^2} \\
&= 4GM \left[\frac{u}{2(1-u^2)} + \frac{\tanh^{-1} u}{2}\right]_0^{u(R)} \\
&= 4GM \left[\frac{u(R)}{2(1-u(R)^2)} + \frac{\tanh^{-1} u(R)}{2} - 0\right]
\end{aligned}$$

Finally, we replace back $u(R) = \sqrt{1-2GM/R}$

$$\begin{aligned}
\Delta s &= 4GM \left[\frac{\sqrt{1-2GM/R}}{4GM/R} + \frac{\tanh^{-1}\sqrt{1-2GM/R}}{2}\right] \\
&= R\sqrt{1-2GM/R} + 2GM\tanh^{-1}\sqrt{1-2GM/R}
\end{aligned}$$

$\square$

Solution. **BOX 14.1** - Exercise 14.1.4.
Let now $R = 3GM$ then the physical distance from $r = 2GM$ to $R = 3GM$ is

$$\begin{aligned}
\Delta s &= 3GM\sqrt{1-2/3} + 2GM\tanh^{-1}\sqrt{1-2/3} \\
&= 3GM\sqrt{1/3} + 2GM\tanh^{-1}\sqrt{1/3} \\
&= 1.7320\ GM + 1.3169\ GM \\
&= 3.048\ GM
\end{aligned}$$

$\square$

Solution. **BOX 14.2** - Exercise 14.2.1.
Let

$$\Delta\tau = 2\sqrt{\frac{R}{2GM}} \int_0^{u_0} \frac{u^2 du}{\sqrt{u_0^2 - u^2}}$$

Then

$$\begin{aligned}
\Delta\tau &= \sqrt{\frac{R}{2GM}} \left[u_0^2 \arctan\left(\frac{u}{\sqrt{u_0^2 - u^2}} \right) - u\sqrt{u_0^2 - u^2} \right]_0^{u_0} \\
&= \sqrt{\frac{R}{2GM}} \left[u_0^2 \frac{\pi}{2} - 0 \right] \\
&= \frac{\pi}{2} R \sqrt{\frac{R}{2GM}} \\
&= \frac{\pi}{2} \sqrt{\frac{R^3}{2GM}}
\end{aligned}$$

□

Solution. **BOX 14.3** - Exercise 14.3.1.
Let

$$\Delta\tau = \int_0^{2GM} \frac{r^{1/2} dr}{\sqrt{2GM - r}}$$

Then letting $u = \sqrt{r}$ we have that

$$\Delta\tau = 2 \int_0^{\sqrt{2GM}} \frac{u^2 du}{\sqrt{2GM - u^2}}$$

Hence

$$\begin{aligned}
\Delta\tau &= \left[2GM \arctan\left(\frac{u}{\sqrt{2GM - u^2}} \right) - u\sqrt{2GM - u^2} \right]_0^{\sqrt{2GM}} \\
&= \left[2GM \frac{\pi}{2} - 0 \right] \\
&= \pi GM
\end{aligned}$$

□

Solution. **P14.1** From BOX 14.2 we know that the proper time (measured by our own watch) we have to live until we reach $r = 0$ when we started from a radially inward fall from $r = R$ is

$$\Delta\tau = \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}}$$

In our current case $R = 10GM$ hence

$$\begin{aligned}
\Delta\tau &= \frac{\pi}{2}\sqrt{\frac{(10GM)^3}{2GM}} \\
&= 35.124GM \\
&= 35.124 \cdot 1477 \cdot 10^6 \ m \\
&= 51878256788.735 \ m \\
&= 173.042 \ s
\end{aligned}$$

Where we used that 299800 $km = 1$ s. $\square$

Solution. **P14.2**

a. The time the clock registers between its launch at $r = 16GM$ and when it comes to rest at $r = 32GM$ is the same time it registers between falling from rest from $32GM$ to $r = 16GM$ hence, we can compute this time using the equation from BOX 14.2 as follows

$$\begin{aligned}
\Delta\tau &= \sqrt{\frac{32GM}{2GM}}\left[32GM \arctan\left(\frac{u}{\sqrt{32GM - u^2}}\right) - u\sqrt{32GM - u^2}\right]_{\sqrt{16GM}}^{\sqrt{32GM}} \\
&= 4\left[16\pi GM - 8\pi GM + 16GM\right] \\
&= 164.53 \cdot 1477 \cdot 10^6 \\
&= 243010810000.0\ m = 810.57\ s
\end{aligned}$$

b. In the same way, the time the clock registers between being at rest at $r = 32GM$ and falling to $r = 2GM$ (the event horizon) is

$$\begin{aligned}
\Delta\tau &= \sqrt{\frac{32GM}{2GM}}\left[32GM \arctan\left(\frac{u}{\sqrt{32GM - u^2}}\right) - u\sqrt{32GM - u^2}\right]_{\sqrt{2GM}}^{\sqrt{32GM}} \\
&= 4\left[16\pi GM - 8.085GM + 7.746GM\right] \\
&= 199.705 \cdot 1477 \cdot 10^6 \\
&= 294964285000.0\ m = 983.87\ s
\end{aligned}$$

c. Finally, to compute the time the clock registers between crossing the event horizon and its destruction at the origin, we need to compute the time it registers from falling at rest from $r = 32GM$ to $r = 0$ and then subtract the time we computed in part **b.**

$$\begin{aligned}
\Delta\tau &= \sqrt{\frac{32GM}{2GM}}\left[32GM \arctan\left(\frac{u}{\sqrt{32GM - u^2}}\right) - u\sqrt{32GM - u^2}\right]_{0}^{\sqrt{32GM}} \\
&= 4\left[16\pi GM - 0\right] \\
&= 201.061 \cdot 1477 \cdot 10^6 \\
&= 296967097000.0\ m = 990.55\ s
\end{aligned}$$

Therefore

$$\Delta\tau = 990.55\ s - 983.87\ s = 6.67\ s$$

$\square$

Solution. **P14.3** Let us consider an inward-falling object moving in the equatorial plane with arbitrary e and l, then from Table 14.1 we have that $dt/d\tau$, $dr/d\tau$ and $d\phi/d\tau$ for this object are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = e\left(1 - \frac{2GM}{r}\right)^{-1}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{l^2}{r^2}\right)}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{l}{r^2}$$

So the four-velocity vector of the object at $r = R$ is

$$\boldsymbol{u} = \begin{bmatrix} e\left(1 - \frac{2GM}{R}\right)^{-1} \\ \sqrt{e^2 - \left(1 - \frac{2GM}{R}\right)\left(1 + \frac{l^2}{R^2}\right)} \\ 0 \\ \frac{l}{R^2} \end{bmatrix}$$

On the other hand, from equation (12.10) we know that the basis vectors of an observer at rest at $r = R$ are

$$(\boldsymbol{o}_t)^\mu = \begin{bmatrix} \frac{1}{\sqrt{1-2GM/R}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\boldsymbol{o}_x)^\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{R} \end{bmatrix}$$

$$(\boldsymbol{o}_y)^\mu = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{R} \\ 0 \end{bmatrix} \quad (\boldsymbol{o}_z)^\mu = \begin{bmatrix} 0 \\ \sqrt{1 - 2GM/R} \\ 0 \\ 0 \end{bmatrix}$$

Then we can compute the components of the four-velocity in the observer's frame as $u^t_{obs} = -\boldsymbol{o}_t \cdot \boldsymbol{u}$ and $u^\mu_{obs} = \boldsymbol{o}_\mu \cdot \boldsymbol{u}$ for $\mu \in \{x, y, z\}$ hence

$$\begin{aligned} u^t_{obs} &= -g_{tt}(\boldsymbol{o}_t)^t \boldsymbol{u}^t \\ &= \left(1 - \frac{2GM}{R}\right)\left(1 - \frac{2GM}{R}\right)^{-1/2} e\left(1 - \frac{2GM}{R}\right)^{-1} \\ &= e\left(1 - \frac{2GM}{R}\right)^{-1/2} \end{aligned}$$

$$\begin{aligned} u^x_{obs} &= g_{\phi\phi}(\boldsymbol{o}_x)^\phi \boldsymbol{u}^\phi \\ &= R^2 \frac{1}{R}\frac{l}{R^2} \\ &= \frac{l}{R} \end{aligned}$$

$$
\begin{aligned}
u^z_{obs} &= g_{rr}(\boldsymbol{o}_z)^r \boldsymbol{u}^r \\
&= \left(1 - \frac{2GM}{R}\right)^{-1} \sqrt{1 - \frac{2GM}{R}} \sqrt{e^2 - \left(1 - \frac{2GM}{R}\right)\left(1 + \frac{l^2}{R^2}\right)} \\
&= \sqrt{\left(1 - \frac{2GM}{R}\right)^{-1} e^2 - \left(1 + \frac{l^2}{R^2}\right)}
\end{aligned}
$$

Then the speed components of the object are

$$
v_{obs,x} = \frac{u^x_{obs}}{u^t_{obs}} = \frac{l}{eR}\left(1 - \frac{2GM}{R}\right)^{1/2}
$$

And

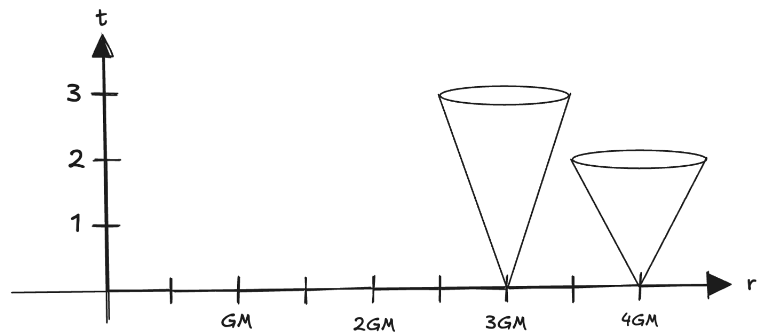
$$
\begin{aligned}
v_{obs,z} = \frac{u^z_{obs}}{u^t_{obs}} &= \frac{\sqrt{e^2\left(1 - \frac{2GM}{R}\right)^{-1} - \left(1 + \frac{l^2}{R^2}\right)}}{e\sqrt{\left(1 - \frac{2GM}{R}\right)^{-1}}} \\
&= \sqrt{\frac{e^2\left(1 - \frac{2GM}{R}\right)^{-1} - \left(1 + \frac{l^2}{R^2}\right)}{e^2\left(1 - \frac{2GM}{R}\right)^{-1}}} \\
&= \sqrt{1 - \frac{1}{e^2}\left(1 + \frac{l^2}{R^2}\right)\left(1 - \frac{2GM}{R}\right)}
\end{aligned}
$$

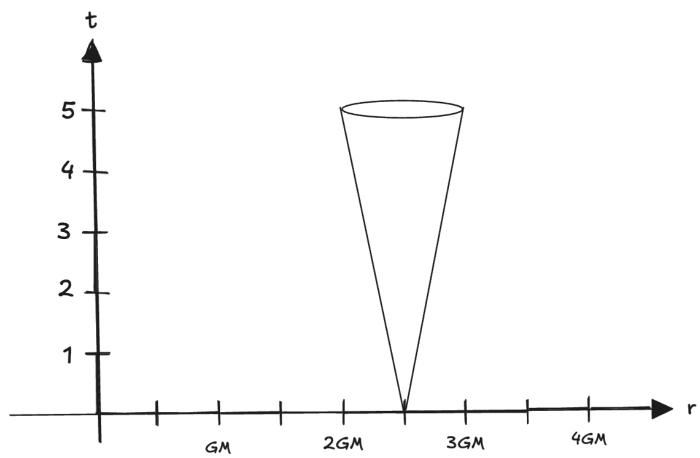
Therefore the squared speed of the object as measured by an observer at rest at $r = R$ is

$$
\begin{aligned}
v^2_{obs} &= v^2_{obs,x} + v^2_{obs,z} \\
&= \frac{l^2}{e^2R^2}\left(1 - \frac{2GM}{R}\right) + 1 - \frac{1}{e^2}\left(1 + \frac{l^2}{R^2}\right)\left(1 - \frac{2GM}{R}\right) \\
&= \frac{l^2}{e^2R^2}\left(1 - \frac{2GM}{R}\right) + 1 - \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right) - \frac{l^2}{e^2R^2}\left(1 - \frac{2GM}{R}\right) \\
&= 1 - \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)
\end{aligned}
$$

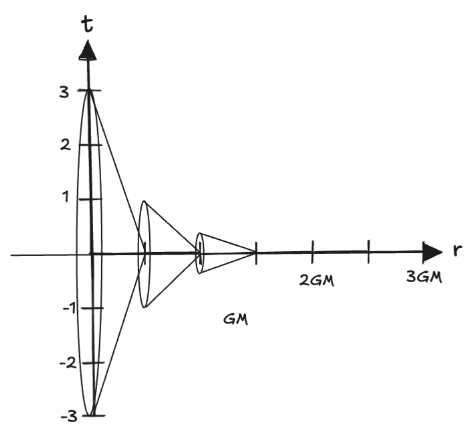
$\square$

Solution. **P14.4** Following the same analysis we see that for $r = 3GM$ the slope must be $dt/dr > \pm 3$ and for $r = \frac{5}{2}GM$ we get that $dt/dr > \pm 5$ so we get the following drawings





Now for $r = \frac{3}{2}GM, GM, \frac{1}{2}GM$ doing the same analysis and considering that forward in proper time corresponds to $dr < 0$ we get that



We see that the light cones in both cases narrow as they approach $r = 2GM$. $\square$

Solution. **P14.5** Let an observer at $R > 2GM$ launch an object with nonzero mass radially outward with initial velocity v_0 as measured by the observer. Given that an object at infinity falling from rest radially has $e = 1$ then the same is true for an object moving radially from a radius R to rest at infinity, then from equations on Table 14.1 the particle's four velocity vector is

$$\boldsymbol{u} = \begin{bmatrix} \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}r}{\mathrm{d}\tau} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{2GM}{R}\right)^{-1} \\ \sqrt{\frac{2GM}{R}} \\ 0 \\ 0 \end{bmatrix}$$

Where we used that for an object travelling radially $l = 0$.
On the other hand, from equation (12.10) we know that the basis vectors of an observer at rest at $r = R$ are

$$(\boldsymbol{o}_t)^\mu = \begin{bmatrix} \frac{1}{\sqrt{1-2GM/R}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\boldsymbol{o}_x)^\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{R} \end{bmatrix}$$

$$(\boldsymbol{o}_y)^\mu = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{R} \\ 0 \end{bmatrix} \quad (\boldsymbol{o}_z)^\mu = \begin{bmatrix} 0 \\ \sqrt{1 - 2GM/R} \\ 0 \\ 0 \end{bmatrix}$$

Then we can compute the components of the four-velocity in the observer's frame as $u^t_{obs} = -\boldsymbol{o}_t \cdot \boldsymbol{u}$ and $u^\mu_{obs} = \boldsymbol{o}_\mu \cdot \boldsymbol{u}$ for $\mu \in \{x, y, z\}$ hence

$$\begin{aligned} u^t_{obs} &= -g_{tt}(\boldsymbol{o}_t)^t \boldsymbol{u}^t \\ &= \left(1 - \frac{2GM}{R}\right)\left(1 - \frac{2GM}{R}\right)^{-1/2}\left(1 - \frac{2GM}{R}\right)^{-1} \\ &= \frac{1}{\sqrt{1 - \frac{2GM}{R}}} \end{aligned}$$

$$\begin{aligned} u^z_{obs} &= g_{rr}(\boldsymbol{o}_z)^r \boldsymbol{u}^r \\ &= \left(1 - \frac{2GM}{R}\right)^{-1}\sqrt{1 - \frac{2GM}{R}}\sqrt{\frac{2GM}{R}} \\ &= \sqrt{\left(1 - \frac{2GM}{R}\right)^{-2}\left(1 - \frac{2GM}{R}\right)}\sqrt{\frac{2GM}{R}} \\ &= \sqrt{\frac{\frac{2GM}{R}}{1 - \frac{2GM}{R}}} \end{aligned}$$

Also, $u^x_{obs} = u^y_{obs} = 0$ because the $\boldsymbol{u}$ components for these velocities are 0. Then the speed of the object as measured by an observer at rest at $r = R$ is

$$v_{obs} = \frac{u^z_{obs}}{u^t_{obs}} = \frac{\sqrt{\frac{\frac{2GM}{R}}{1-\frac{2GM}{R}}}}{\frac{1}{\sqrt{1-\frac{2GM}{R}}}} = \sqrt{\frac{2GM}{R}}$$

But we said that the object has an initial velocity of v_0 as measured by the observer then must be that

$$v_0 = \sqrt{\frac{2GM}{R}}$$

Therefore the escape speed as measured by this observer is the same as in Newtonian mechanics. Also, if we let $R \to 2GM$ then $v_0 = 1$ i.e. the speed of light. $\square$

Solution. **P14.6** From problem P12.7 we know that basis vector $\boldsymbol{o}_t$ is

$$(\boldsymbol{o}_t)^\mu = \begin{bmatrix} (1-2GM/r)^{-1} \\ -\sqrt{2GM/r} \\ 0 \\ 0 \end{bmatrix}$$

Also, from the equations 12.12a and 12.12b (the four-momentum of a photon) we can compute $E_{obs} = -\boldsymbol{o}_t \cdot \boldsymbol{p}$ using the metric given by equation 14.16 as follows

$$\begin{aligned}
E_{obs} &= -\boldsymbol{o}_t \cdot \boldsymbol{p} \\
&= g_{tt}(\boldsymbol{o}_t)^t p^t - g_{rr}(\boldsymbol{o}_t)^r p^r \\
&= -\left(\frac{2GM}{r} - 1\right)\left(1 - \frac{2GM}{r}\right)^{-1} E\left(1 - \frac{2GM}{r}\right)^{-1} \\
&\quad + \left(\frac{2GM}{r} - 1\right)^{-1} \sqrt{\frac{2GM}{r}} E \sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)} \\
&= E\left(1 - \frac{2GM}{r}\right)^{-1}\left(1 - \sqrt{\frac{2GM}{r}}\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}\right)
\end{aligned}$$

But since it's falling radially then $b = 0$ so

$$E_{obs} = E\left(1 - \frac{2GM}{r}\right)^{-1}\left(1 - \sqrt{\frac{2GM}{r}}\right)$$

Finally, the fractional change in wavelength is given by

$$\begin{aligned}
\frac{h/\lambda_E}{h/\lambda_R} &= \frac{E/\sqrt{1 - \frac{2GM}{r_E}}}{E\left(1 - \frac{2GM}{r_R}\right)^{-1}\left(1 - \sqrt{\frac{2GM}{r_R}}\right)} \\
\frac{\lambda_R}{\lambda_E} &= \frac{\left(1 - \frac{2GM}{r_R}\right)}{\sqrt{1 - \frac{2GM}{r_E}}\left(1 - \sqrt{\frac{2GM}{r_R}}\right)}
\end{aligned}$$

But since $r_E = \infty$ i.e. the signal is coming from infinity we can write that

$$\frac{\lambda_R}{\lambda_E} = \frac{\left(1 - \frac{2GM}{r_R}\right)}{\left(1 - \sqrt{\frac{2GM}{r_R}}\right)} = 1 + \sqrt{\frac{2GM}{r_R}}$$

Therefore the expression is correct for $r < 2GM$ as well. $\square$

Solution. **P14.7**

a. We want to show that curves of constant R are hyperbolas so let us compute $t^2 - r^2$, assuming $R > 1/2$ then

$$\begin{aligned} t^2 - r^2 &= b^2(2R-1)\sinh^2 T - b^2(2R-1)\cosh^2 T \\ &= b^2(1-2R)[\cosh^2 T - \sinh^2 T] \\ &= b^2(1-2R) \end{aligned}$$

Where we used that $\cosh^2 q - \sinh^2 q = 1$ for all q.
Therefore $t^2 - r^2 =$ constant and since $b^2(1-2R) < 0$ for $R > 1/2$ then we can write that $r^2 - t^2 = |b^2(1-2R)|$ which implies that the hyperbolas face rightward or leftward.

In the same way, for $R < 1/2$ we have that

$$\begin{aligned} t^2 - r^2 &= b^2(1-2R)\cosh^2 T - b^2(1-2R)\sinh^2 T \\ &= b^2(1-2R)[\cosh^2 T - \sinh^2 T] \\ &= b^2(1-2R) \end{aligned}$$

But, in this case $b^2(1-2R) > 0$ for $R < 1/2$ so the hyperbolas face up and down.

b. We want to show that curves of constant T are straight lines with constant slope, so let us compute t/x for $R > 1/2$ then

$$\frac{t}{x} = \frac{b\sqrt{2R-1}\sinh T}{b\sqrt{2R-1}\cosh T} = \frac{\sinh T}{\cosh T} = \frac{e^T - e^{-T}}{e^T + e^{-T}} = \frac{1 - e^{-2T}}{1 + e^{-2T}}$$

We see that $0 < e^{-2T} \leq 1$ for $T \geq 0$ so the denominator is bigger than the numerator so the slope of the line is less than 1.

In the same way, for $R < 1/2$ we have that

$$\frac{t}{x} = \frac{b\sqrt{1-2R}\cosh T}{b\sqrt{1-2R}\sinh T} = \frac{\cosh T}{\sinh T} = \frac{e^T + e^{-T}}{e^T - e^{-T}} = \frac{1 + e^{-2T}}{1 - e^{-2T}}$$

Therefore in this case the slope of the line is greater than 1.

c. Let the metric for this coordinate system be

$$ds^2 = -(2R-1)b^2 dT^2 + \frac{b^2 dR^2}{2R-1} + dy^2 + dz^2$$

We see that the coordinate R is defined for $R < 1/2$ and $R > 1/2$ and there the metric is well-defined so the metric is well-defined everywhere.

d. We see that the g_{TT} component of the metric is negative and g_{RR} is positive when $R > 1/2$ so R is a spatial coordinate and T is a time coordinate in this case.
But when $R < 1/2$ we have that g_{TT} is positive and g_{RR} is negative so the situation is reversed and R is a time coordinate and T is a spatial coordinate. Then when $R < 1/2$ the future happens as $R \to 0$.

e. Particles can cross from larger Rs to smaller Rs but they cannot go in reverse because when $R < 1/2$ we know that R is a time coordinate so as soon as we cross $R = 1/2$ we cannot reverse cause that would imply going backwards in time.
We consider the line where $R = 1/2$ and $T = \infty$ to be an event horizon in T, R coordinates because the behaviour of coordinates change depending if we are on one side or the other but this doesn't happen in t, x coordinates.

f. Let us consider a photon at $R = 1/2$, we know that for a photon the metric equation looks as follows

$$0 = -(2R-1)b^2 dT^2 + \frac{b^2 dR^2}{2R-1} + dy^2 + dz^2$$

Where we used that for photons $ds^2 = 0$. Then

$$\begin{aligned}\frac{b^2 dR^2}{2R-1} &= (2R-1)b^2 dT^2 - dy^2 - dz^2 \\ \frac{dR^2}{dT^2} &= (2R-1)^2 - \frac{(2R-1)}{b^2}\frac{dy^2}{dT^2} - \frac{(2R-1)}{b^2}\frac{dz^2}{dT^2}\end{aligned}$$

So we see that when $R = 1/2$ we get that $dR/dT = 0$ therefore the photon must be at rest at $R = 1/2$.

On the other hand, since $ds^2 \neq 0$ for a particle of nonzero rest mass then the particle cannot be at rest at $R = 1/2$.

g. The geodesic equation for the T component is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{T\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right) - \frac{1}{2}\partial_T g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{TT}\frac{\mathrm{d}T}{\mathrm{d}\tau}\right) = 0$$

Then $g_{TT}\frac{\mathrm{d}T}{\mathrm{d}\tau}$ is equal to a constant, hence

$$g_{TT}\frac{\mathrm{d}T}{\mathrm{d}\tau} = -e$$
$$-(2R-1)b^2\frac{\mathrm{d}T}{\mathrm{d}\tau} = -e$$
$$\frac{\mathrm{d}T}{\mathrm{d}\tau} = \frac{e}{(2R-1)b^2}$$

Where we named the constant $-e$.
Now, let us consider the metric equation for a particle with no y, z velocity, then

$$-1 = -(2R-1)b^2\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 + \frac{b^2}{2R-1}\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2$$
$$\frac{b^2}{2R-1}\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 = (2R-1)b^2\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 - 1$$
$$\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 = (2R-1)^2\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 - \frac{2R-1}{b^2}$$
$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = \sqrt{\frac{e^2}{b^4} - \frac{2R-1}{b^2}}$$

We combine this result with $dT/d\tau$ to obtain dR/dT as follows

$$\frac{\mathrm{d}R}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}T} = \frac{(2R-1)b^2}{e}\sqrt{\frac{e^2}{b^4} - \frac{2R-1}{b^2}}$$

Hence

$$\frac{\mathrm{d}R}{\mathrm{d}T} = \frac{(2R-1)}{e}\sqrt{e^2 - b^2(2R-1)}$$

Let us consider now a particle released from rest at $R = 1$ and $T = 0$ then we have that

$$0 = \frac{2-1}{e}\sqrt{e^2 - b^2(2-1)}$$
$$e = b$$

So integrating dR/dT we get that

$$\begin{aligned}
\int_0^T dT &= \int_1^{1/2} \frac{b}{(2R-1)} \frac{dR}{b\sqrt{1-(2R-1)}} \\
T &= \Big[-\operatorname{arctanh}(\sqrt{2(1-R)}) \Big]_1^{1/2} \\
T &= \Big[-\operatorname{arctanh}(\sqrt{1}) + \operatorname{arctanh}(0) \Big] \to \infty
\end{aligned}$$

But integrating $dR/d\tau$ we have that

$$\begin{aligned}
\int_0^\tau d\tau &= \int_1^{1/2} \frac{dR}{b\sqrt{1-(2R-1)}} \\
\tau &= \frac{1}{b}\Big[-\sqrt{2(1-R)} \Big]_1^{1/2} \\
\tau &= -\frac{1}{b}
\end{aligned}$$

Therefore this trip requires infinite time T but finite time τ.

$\square$

Solution. **P14.8**

a. We know that $u = 1 - \sqrt{2GM/r}$ then

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{2r}\sqrt{\frac{2GM}{r}}$$

From Table 14.1, we have that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\left(1 - \frac{2GM}{r}\right)\sqrt{1 - \frac{1}{e^2}\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{l^2}{r^2}\right)}$$

We took the negative sign because the laser is going to the blackhole. But since the laser is falling radially from rest we have that $l = 0$ and $e = 1$, hence

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}u}{\mathrm{d}r} &= -\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}}\frac{\mathrm{d}u}{\mathrm{d}r} \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= -\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}}\frac{1}{2r}\sqrt{\frac{2GM}{r}} \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)
\end{aligned}$$

b. Using that

$$1 - \frac{2GM}{r} = \left(1 - \sqrt{\frac{2GM}{r}}\right)\left(1 + \sqrt{\frac{2GM}{r}}\right)$$

We can write that

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= -\frac{GM}{r^2}\left(1 - \sqrt{\frac{2GM}{r}}\right)\left(1 + \sqrt{\frac{2GM}{r}}\right) \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= -u\frac{GM}{r^2}\left(1 + \sqrt{\frac{2GM}{r}}\right)
\end{aligned}$$

Then if $r \to 2GM$ we see that

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{r\to 2GM} = -u\frac{GM}{4(GM)^2}\left(1 + \sqrt{\frac{2GM}{2GM}}\right) = -\frac{u}{2GM}$$

c. The outgoing photon's energy an observer on the laser's frame measures is

$$
\begin{aligned}
E_{obs} &= -\boldsymbol{o}_t \cdot \boldsymbol{p} \\
&= E\left(1-\frac{2GM}{r}\right)^{-1} + \left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}E\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)} \\
&= E\left(1-\frac{2GM}{r}\right)^{-1}\left(1+\sqrt{\frac{2GM}{r}}\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)}\right)
\end{aligned}
$$

Where we used the positive sign for p^r since the photon is outgoing. But since it's falling radially then $b = 0$ so

$$
E_{obs} = E\left(1-\frac{2GM}{r}\right)^{-1}\left(1+\sqrt{\frac{2GM}{r}}\right)
$$

Finally, the fractional change in wavelength is given by

$$
\begin{aligned}
\frac{h/\lambda_R}{h/\lambda_E} &= \frac{E/\sqrt{1-\frac{2GM}{r_R}}}{E\left(1-\frac{2GM}{r_E}\right)^{-1}\left(1+\sqrt{\frac{2GM}{r_E}}\right)} \\
\frac{\lambda_E}{\lambda_R} &= \frac{\left(1-\frac{2GM}{r_E}\right)}{\sqrt{1-\frac{2GM}{r_R}}\left(1+\sqrt{\frac{2GM}{r_E}}\right)}
\end{aligned}
$$

But since $r_R \to \infty$ i.e. the signal is received at infinity we can write that

$$
\frac{\lambda_R}{\lambda_E} = \frac{\left(1-\frac{2GM}{r_R}\right)}{\left(1+\sqrt{\frac{2GM}{r_R}}\right)} = 1-\sqrt{\frac{2GM}{r}}
$$

$\square$