Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 4 - Index Notation

Solution. **BOX 4.1** We want to check $\delta^\mu{}_\nu A^\nu = A^\mu$ by writing explicitly the summation then for $\mu = t$ we have that

$$\delta^t{}_t A^t + \delta^t{}_x A^x + \delta^t{}_y A^y + \delta^t{}_z A^z = A^t$$

Since $\delta^t{}_x = \delta^t{}_y = \delta^t{}_z = 0$ and $\delta^t{}_t = 1$. In the same way for $\mu = x, y, z$ we have that

$$\begin{aligned}
\delta^x{}_t A^t + \delta^x{}_x A^x + \delta^x{}_y A^y + \delta^x{}_z A^z &= A^x \\
\delta^y{}_t A^t + \delta^y{}_x A^x + \delta^y{}_y A^y + \delta^y{}_z A^z &= A^y \\
\delta^z{}_t A^t + \delta^z{}_x A^x + \delta^z{}_y A^y + \delta^z{}_z A^z &= A^z
\end{aligned}$$

Since $\delta^x{}_t = \delta^x{}_y = \delta^x{}_z = 0$, $\delta^y{}_t = \delta^y{}_x = \delta^y{}_z = 0$ and $\delta^z{}_t = \delta^z{}_x = \delta^z{}_y = 0$ but $\delta^x{}_x = \delta^y{}_y = \delta^z{}_z = 1$. Therefore we get that $\delta^\mu{}_\nu A^\nu = A^\mu$.

Now let us compute $\delta^\mu{}_\nu \eta_{\mu\alpha} = \eta_{\nu\alpha}$ where the summation in this case is over μ then for $\nu = t$ we have that

$$\delta^t{}_t \eta_{t\alpha} + \delta^x{}_t \eta_{x\alpha} + \delta^y{}_t \eta_{y\alpha} + \delta^z{}_t \eta_{z\alpha} = \eta_{t\alpha}$$

In the same way for $\nu = x, y, z$ we have that

$$\begin{aligned}
\delta^t{}_x \eta_{t\alpha} + \delta^x{}_x \eta_{x\alpha} + \delta^y{}_x \eta_{y\alpha} + \delta^z{}_x \eta_{z\alpha} &= \eta_{x\alpha} \\
\delta^t{}_y \eta_{t\alpha} + \delta^x{}_y \eta_{x\alpha} + \delta^y{}_y \eta_{y\alpha} + \delta^z{}_y \eta_{z\alpha} &= \eta_{y\alpha} \\
\delta^t{}_z \eta_{t\alpha} + \delta^x{}_z \eta_{x\alpha} + \delta^y{}_z \eta_{y\alpha} + \delta^z{}_z \eta_{z\alpha} &= \eta_{z\alpha}
\end{aligned}$$

Therefore we get that $\delta^\mu{}_\nu \eta_{\mu\alpha} = \eta_{\nu\alpha}$ as we wanted. □

Solution. **BOX 4.3** - Exercise 4.3.1. We want to write the implied sum of $dp^\mu/d\tau = qF^{\mu\nu}\eta_{\nu\alpha}u^\alpha$ for $\mu = x$ hence

$$\begin{aligned}\frac{dp^x}{d\tau} &= q(F^{xt}\eta_{tt}u^t + F^{xx}\eta_{xt}u^t + F^{xy}\eta_{yt}u^t + F^{xz}\eta_{zt}u^t \\ &\quad + F^{xt}\eta_{tx}u^x + F^{xx}\eta_{xx}u^x + F^{xy}\eta_{yx}u^x + F^{xz}\eta_{zx}u^x \\ &\quad + F^{xt}\eta_{ty}u^y + F^{xx}\eta_{xy}u^y + F^{xy}\eta_{yy}u^y + F^{xz}\eta_{zy}u^y \\ &\quad + F^{xt}\eta_{tz}u^z + F^{xx}\eta_{xz}u^z + F^{xy}\eta_{yz}u^z + F^{xz}\eta_{zz}u^z)\end{aligned}$$

We know that $F^{tt} = F^{xx} = F^{yy} = F^{zz} = 0$ and that except for $\eta^{tt}, \eta^{xx}, \eta^{yy}$ and η^{zz} any other component is equal to 0 so we are left with

$$\begin{aligned}\frac{dp^x}{d\tau} &= q(F^{xt}\eta_{tt}u^t + F^{xy}\eta_{yy}u^y + F^{xz}\eta_{zz}u^z) \\ &= q((-E_x \cdot -1 \cdot 1) + (B_z \cdot 1 \cdot v_y) + (-B_y \cdot 1 \cdot v_z)) \\ &= q(E_x + v_yB_z - v_zB_y)\end{aligned}$$

Where we used that $\eta_{tt} = -1$, $\eta_{yy} = \eta_{zz} = 1$, $F^{xt} = -E_x$, $F^{xy} = B_z$ and $F^{xz} = -B_y$ but also since we are considering low velocities we have that $u^t = 1$, $u^y = v_y$ and $u^z = v_z$. We see that the equation we have is the x component of the Lorentz force law where $(\vec{v} \times \vec{B})\hat{\boldsymbol{x}} = (v_yB_z - v_zB_y)\hat{\boldsymbol{x}}$. □

Solution. **BOX 4.3** - Exercise 4.3.2. First, we want to show that

$$\partial F^{\mu\nu}/\partial x^v = 4\pi k J^\mu$$

becomes Gauss's law when $\mu = t$ so we see that

$$\begin{aligned}\frac{\partial F^{tt}}{\partial t} + \frac{\partial F^{tx}}{\partial x} + \frac{\partial F^{ty}}{\partial y} + \frac{\partial F^{tz}}{\partial z} &= 4\pi k J^t \\ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= \frac{\rho}{\epsilon_0}\end{aligned}$$

Where we used that $4\pi k = 1/\epsilon_0$, $J^t = \rho$, $F^{tt} = 0$, $F^{tx} = E_x$, $F^{ty} = E_y$ and $F^{tz} = E_z$. Therefore, as we wanted, the result is Gauss's law.

Finally, we want to show that

$$\partial F^{\mu\nu}/\partial x^v = 4\pi k J^\mu$$

becomes the x component of Ampere-Maxwell relation when $\mu = x$ so we see that

$$\begin{aligned}\frac{\partial F^{xt}}{\partial t} + \frac{\partial F^{xx}}{\partial x} + \frac{\partial F^{xy}}{\partial y} + \frac{\partial F^{xz}}{\partial z} &= 4\pi k J^x \\ -\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} &= \mu_0 J^x \\ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \mu_0\epsilon_0\frac{\partial E_x}{\partial t} &= \mu_0 J^x\end{aligned}$$

Where we used that $4\pi k = \mu_0$ and that $\epsilon_0\mu_0 = 1$. Therefore, as we wanted, the result is the x component of Ampere-Maxwell relation. □

Solution. **BOX 4.4**

(a) Equation:

$$\eta_{\mu\nu}\frac{du^\mu}{d\tau}u^\nu = 0$$

Has no free indexes, and represents 1 component.

(b) Equation:

$$\eta_{\mu\nu}(\Lambda^{-1})^\nu{}_\alpha = \eta_{\mu\beta}\Lambda^\beta{}_\alpha$$

Has 2 free indexes μ and α, and the equation represents 16 components which are the combination of μ's 4 components α's 4 components.

(c) Equation:

$$\frac{dp^\mu}{d\tau} = 0$$

Has 1 free index μ, and the equation represents the 4 components from μ.

(d) Equation:

$$\eta_{\alpha\mu}\eta_{\beta\nu}\frac{\partial F^{\mu\nu}}{\partial x^\sigma} + \eta_{\sigma\mu}\eta_{\alpha\nu}\frac{\partial F^{\mu\nu}}{\partial x^\beta} + \eta_{\beta\mu}\eta_{\sigma\nu}\frac{\partial F^{\mu\nu}}{\partial x^\alpha} = 0$$

Has 3 free indexes, α, β, and σ, and the equation represents 64 components.

(e) Equation:

$$F^{\mu\nu}\eta_{\mu\alpha}\eta_{\nu\beta}u^\alpha u^\beta = 0$$

Has no free indexes, and the equation represents 1 component.

(f) Equation:

$$\eta_{\mu\nu}\eta^{\mu\nu} = 4$$

Has no free indexes, and the equation represents 1 component.

□

Solution. **BOX 4.5** - Exercise 4.5.1.

(a) Violates Rule 1 since α and β are not in both sides of the equation.

(b) Does not violate Rule 1.

(c) Does not violate Rule 1.

(d) Does not violate Rule 1.

(e) Violates Rule 1 since μ and ν are not in both sides of the equation.

□

Solution. **BOX 4.5** - Exercise 4.5.2.

(a) Violates Rule 2 since now μ appears only once on the RHS.

(b) Violates Rule 2 since μ is already in use.

(c) Does not violate Rule 2.

(d) Does not violate Rule 2.

(e) Does not violate Rule 2.

(f) Violates Rule 2 since μ and ν are already in use in the same term.

□

Solution. **BOX 4.6** Let $\boldsymbol{A}$ be an arbitrary four-vector then we know that $A^2 = \eta_{\mu\nu} A^\mu A^\nu$ so we have that

$$\begin{aligned}
\frac{d}{d\tau}(A^2) &= \frac{d}{d\tau}(\eta_{\mu\nu} A^\mu A^\nu) \\
&= \eta_{\mu\nu}\left[A^\mu \frac{d}{d\tau}(A^\nu) + A^\nu \frac{d}{d\tau}(A^\mu)\right] + A^\mu A^\nu \frac{d}{d\tau}(\eta_{\mu\nu}) \\
&= \eta_{\mu\nu} A^\mu \frac{dA^\nu}{d\tau} + \eta_{\mu\nu} A^\nu \frac{dA^\mu}{d\tau}
\end{aligned}$$

We see that $A^\mu A^\nu \frac{d}{d\tau}(\eta_{\mu\nu}) = 0$ since $\eta_{\mu\nu}$ does not depend on τ. Also, we can invert the superscripts in the last term since the order in which we multiply gives us the same result therefore

$$\begin{aligned}
\frac{d}{d\tau}(A^2) &= \eta_{\mu\nu} A^\mu \frac{dA^\nu}{d\tau} + \eta_{\mu\nu} A^\mu \frac{dA^\nu}{d\tau} \\
&= 2\eta_{\mu\nu} A^\mu \frac{dA^\nu}{d\tau}
\end{aligned}$$

□

Solution. **P4.1**

a. The equation

$$0 = m^2 + (p^\mu)^2$$

is not a valid index equation since m^2 doesn't have the μ free index.

b. The equation

$$dF^{\mu\nu}/d\tau = 0$$

is a valid index equation.

c. The equation

$$dp^\mu/d\tau = g$$

is not a valid index equation since g doesn't have the μ free index.

d. The equation

$$F_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}F^{\mu\sigma}$$

is not a valid index equation since the RHS of the equation has ν and σ as free indexes in addition to α and β but they do not appear in the LHS.

e. The equation

$$A^{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}F^{\mu\nu}$$

is not a valid index equation since the RHS of the equation does not have the same free indexes superscripts as the LHS.

f. The equation

$$A^\mu = \delta^\mu{}_\alpha A^\alpha$$

is a valid index equation.

g. The equation

$$0 = A^\mu + B^\nu$$

is not a valid index equation since A and B have different free indexes superscripts.

h. The equation

$$qF^{\mu\nu} = dp^\mu/d\tau$$

is not a valid index equation since dp doesn't have the same LHS superscripts.

□

Solution. **P4.2**

a.

$$A^2 = \eta_{\alpha\beta}A^\alpha A^\beta \Rightarrow A^2 = \eta_{\mu\nu}A^\alpha A^\beta$$

This isn't a valid renaming since not every occurrence was renamed.

b.

$$0 = \eta_{\alpha\beta}A^\beta + \eta_{\alpha\mu}B^\mu \Rightarrow 0 = \eta_{\alpha\beta}(A^\beta + B^\beta)$$

This is a valid renaming.

c.

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu \Rightarrow \eta_{\mu\nu} = \eta_{\alpha\alpha}\Lambda^\alpha{}_\mu\Lambda^\alpha{}_\nu$$

This isn't a valid renaming since α is already used in the equation.

d.

$$dp^\mu/d\tau = qF^{\mu\nu}\eta_{\nu\alpha}u^\alpha \Rightarrow dp^\mu/d\tau = qF^{\mu\nu}\eta_{\nu\mu}u^\mu$$

This isn't a valid renaming since μ is already used in the equation as a free index.

e.

$$(\Lambda^{-1})^\alpha{}_\mu\eta_{\alpha\nu} = \eta_{\mu\beta}\Lambda^\beta{}_\nu \Rightarrow (\Lambda^{-1})^\beta{}_\mu\eta_{\beta\nu} = \eta_{\mu\alpha}\Lambda^\alpha{}_\nu$$

This is a valid renaming.

□

Solution. **P4.3** We have from equation (4.18) with α and ν as free indexes that

$$\eta_{\alpha\nu} = \eta_{\sigma\beta}\Lambda^{\sigma}{}_{\alpha}\Lambda^{\beta}{}_{\nu}$$

So multiplying both sides by $(\Lambda^{-1})^{\alpha}{}_{\mu}$ we get that

$$\begin{aligned}(\Lambda^{-1})^{\alpha}{}_{\mu}\eta_{\alpha\nu} &= \eta_{\sigma\beta}\Lambda^{\sigma}{}_{\alpha}(\Lambda^{-1})^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\\ (\Lambda^{-1})^{\alpha}{}_{\mu}\eta_{\alpha\nu} &= \eta_{\sigma\beta}\delta^{\sigma}{}_{\mu}\Lambda^{\beta}{}_{\nu}\end{aligned}$$

Where we used that $\Lambda^{\sigma}{}_{\alpha}(\Lambda^{-1})^{\alpha}{}_{\mu} = \delta^{\sigma}{}_{\mu}$ hence since $\eta_{\sigma\beta}\delta^{\sigma}{}_{\mu} = \eta_{\mu\beta}$ we have that

$$(\Lambda^{-1})^{\alpha}{}_{\mu}\eta_{\alpha\nu} = \eta_{\mu\beta}\Lambda^{\beta}{}_{\nu}$$

□

Solution. **P4.4** From the equation we derived on P4.3 but using α and ν as free indexes we have that

$$\eta_{\alpha\sigma}\Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{\alpha}$$

So multiplying both sides by $(\Lambda^{-1})^{\nu}{}_{\beta}$ we get that

$$\begin{aligned}\eta_{\alpha\sigma}\Lambda^{\sigma}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\beta} &= \eta_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}\\ \eta_{\alpha\sigma}\delta^{\sigma}{}_{\beta} &= \eta_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}\\ \eta_{\alpha\beta} &= \eta_{\mu\nu}(\Lambda^{-1})^{\mu}{}_{\alpha}(\Lambda^{-1})^{\nu}{}_{\beta}\end{aligned}$$

Where we used that $\Lambda^{\sigma}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\beta} = \delta^{\sigma}{}_{\beta}$ and that $\eta_{\alpha\sigma}\delta^{\sigma}{}_{\beta} = \eta_{\alpha\beta}$. □

Solution. **P4.5** We want to know what is the value of $\delta^{\mu}{}_{\mu}$ hence we write the implied sum as

$$\sum_{\mu=1}^{4}\delta^{\mu}{}_{\mu} = \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} + \delta^{4}{}_{4} = 1 + 1 + 1 + 1 = 4$$

□

Solution. **P4.6** From Eq. 4.20 we have that

$$\frac{dp^2}{d\tau} = 2\eta_{\alpha\mu}p^{\alpha}\frac{dp^{\mu}}{d\tau}$$

We know that $p^{\alpha} = mu^{\alpha}$ and using Eq. 4.15 we have that

$$\begin{aligned}\frac{dp^2}{d\tau} &= 2\eta_{\mu\alpha}\ mu^{\alpha}qF^{\mu\nu}\eta_{\nu\beta}u^{\beta}\\ &= 2mq\ F^{\mu\nu}\eta_{\mu\alpha}\eta_{\nu\beta}u^{\alpha}u^{\beta}\end{aligned}$$

We changed $\eta_{\alpha\mu}$ to $\eta_{\mu\alpha}$ since the order of multiplication doesn't matter. Finally, we know that $F^{\mu\nu}\eta_{\mu\alpha}\eta_{\nu\beta}u^{\alpha}u^{\beta} = 0$ from Eq. 4.21 therefore $dp^2/d\tau = 0$ i.e. the square magnitude of a charged particle's four-momentum is conserved. □

Solution. **P4.8** Let $F^{\mu\nu}$ be the electromagnetic field tensor then by writing the implied sum of $\eta_{\mu\nu}F^{\mu\nu}$ we have that

$$\begin{aligned}
&\eta_{tt}F^{tt} + \eta_{tx}F^{tx} + \eta_{ty}F^{ty} + \eta_{tz}F^{tz} \\
&\quad + \eta_{xt}F^{xt} + \eta_{xx}F^{xx} + \eta_{xy}F^{xy} + \eta_{xz}F^{xz} \\
&\quad + \eta_{yt}F^{yt} + \eta_{yx}F^{yx} + \eta_{yy}F^{yy} + \eta_{yz}F^{yz} \\
&\quad + \eta_{zt}F^{zt} + \eta_{zx}F^{zx} + \eta_{zy}F^{zy} + \eta_{zz}F^{zz} \\
&= -1\cdot 0 + 0\cdot E_x + 0\cdot E_y + 0\cdot E_z \\
&\quad - 0\cdot E_x + 1\cdot 0 + 0\cdot B_z - 0\cdot B_y \\
&\quad - 0\cdot E_y - 0\cdot B_z + 1\cdot 0 + 0\cdot B_x \\
&\quad - 0\cdot E_z + 0\cdot B_y - 0\cdot B_x + 1\cdot 0 \\
&= 0
\end{aligned}$$

Therefore $\eta_{\mu\nu}F^{\mu\nu} = 0$. □

Solution. **P4.11** By writing $\boldsymbol{u}\cdot\boldsymbol{a}$ on index notation we get that

$$\boldsymbol{u}\cdot\boldsymbol{a} = \eta_{\mu\nu}u^{\mu}\frac{du^{\nu}}{d\tau}$$

but we know that $d(A^2)/d\tau = 2\eta_{\mu\nu}A^{\mu}(dA/d\tau)^{\nu}$ for any four-vector $\boldsymbol{A}$ hence

$$\boldsymbol{u}\cdot\boldsymbol{a} = \frac{1}{2}\frac{d(u^2)}{d\tau} = \frac{1}{2}\frac{d(\boldsymbol{u}\cdot\boldsymbol{u})}{d\tau} = 0$$

where we used that $d(\boldsymbol{u}\cdot\boldsymbol{u})/d\tau = d(-1)/d\tau = 0$. □