Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 7 - Maxwell's Equations

Solution. **BOX 7.1** - Exercise 7.1.1. In the same way, as we did for the x direction the flux going through the face which has a unit vector $\hat{\boldsymbol{n}}$ pointing to the $-y$ direction is

$$\bar{\boldsymbol{E}}(x, y - \frac{1}{2}\Delta y, z) \cdot \hat{\boldsymbol{n}} \Delta x \Delta z = -E_y(x, y - \frac{1}{2}\Delta y, z)\Delta x \Delta z$$

Similarly, the flux through the face with a unit vector pointing to the $+y$ direction gives us

$$\bar{\boldsymbol{E}}(x, y + \frac{1}{2}\Delta y, z) \cdot \hat{\boldsymbol{n}} \Delta x \Delta z = E_y(x, y + \frac{1}{2}\Delta y, z)\Delta x \Delta z$$

Therefore the net flux through these two faces is

$$\begin{aligned}
[E_y(x, y + \frac{1}{2}\Delta y, z) - E_y(x, y - \frac{1}{2}\Delta y, z)]\Delta x \Delta z = \\
&= \left[\frac{E_y(x, y + \frac{1}{2}\Delta y, z) - E_y(x, y - \frac{1}{2}\Delta y, z)}{\Delta y}\right]\Delta x \Delta y \Delta z \\
&\approx \frac{\partial E_y}{\partial y}\Delta x \Delta y \Delta z
\end{aligned}$$

On the other hand, by the same means, we can get that net flux in the faces perpendicular to the z axis is

$$\begin{aligned}
[E_z(x, y, z + \frac{1}{2}\Delta z) - E_z(x, y, z - \frac{1}{2}\Delta z)]\Delta x \Delta y = \\
&= \left[\frac{E_z(x, y, z + \frac{1}{2}\Delta z) - E_z(x, y, z - \frac{1}{2}\Delta z)}{\Delta z}\right]\Delta x \Delta y \Delta z \\
&\approx \frac{\partial E_z}{\partial z}\Delta x \Delta y \Delta z
\end{aligned}$$

□

Solution. **BOX 7.2** - Exercise 7.2.1. We know that $-m^2 = p_\mu p^\mu = p^\mu \eta_{\mu\nu} p^\nu$ hence by derivating with respect to τ and applying the product rule we get that

$$\begin{aligned}
\frac{d}{d\tau}(-m^2) &= \frac{d}{d\tau}(p^\mu \eta_{\mu\nu} p^\nu) \\
0 &= \eta_{\mu\nu} p^\nu \frac{dp^\mu}{d\tau} + p^\mu \frac{d(\eta_{\mu\nu} p^\nu)}{d\tau} \\
0 &= \eta_{\mu\nu} p^\nu \frac{dp^\mu}{d\tau} + \eta_{\nu\mu} p^\mu \frac{dp^\nu}{d\tau} \\
0 &= 2\eta_{\mu\nu} p^\nu \frac{dp^\mu}{d\tau} \\
0 &= 2p_\mu \frac{dp^\mu}{d\tau}
\end{aligned}$$

Where we used that the metric tensor $\eta_{\mu\nu}$ is symmetric. □

Solution. **BOX 7.3** - Exercise 7.3.1. Let B_μ be an arbitrary covector, then let us raise the index in cartesian coordinates as follows

$$\begin{aligned}
B^t &= \eta^{tt} B_t + \eta^{tx} B_x + \eta^{ty} B_y + \eta^{tz} B_z = (-1) \cdot B_t + 0 + 0 + 0 = -B_t \\
B^x &= \eta^{xt} B_t + \eta^{xx} B_x + \eta^{xy} B_y + \eta^{xz} B_z = 0 + 1 \cdot B_x + 0 + 0 = B_x \\
B^y &= \eta^{yt} B_t + \eta^{yx} B_x + \eta^{yy} B_y + \eta^{yz} B_z = 0 + 0 + 1 \cdot B_y + 0 = B_y \\
B^x &= \eta^{zt} B_t + \eta^{zx} B_x + \eta^{zy} B_y + \eta^{zz} B_z = 0 + 0 + 0 + 1 \cdot B_z = B_z
\end{aligned}$$

□

Solution. **BOX 7.4** - Exercise 7.4.1. In the same way, as we did for the x direction the total amount of charge that moves out of the box through the front face during a time interval Δt is

$$\begin{aligned}\Delta q_{\text{front}} &\approx -\rho(x, y - \frac{1}{2}\Delta y, z)v_y(x, y - \frac{1}{2}\Delta y, z)\Delta t \Delta x \Delta z \\ &\approx -J^y(x, y - \frac{1}{2}\Delta y, z)\Delta t \Delta x \Delta z\end{aligned}$$

In the same way, the amount of charge flowing out of the back face is $\Delta q_{\text{back}} \approx J^y(x, y + \frac{1}{2}\Delta y, z)\Delta t \Delta x \Delta z$ and hence the net amount of charge flowing out of these two faces during Δt will be

$$\begin{aligned}\Delta q_{\text{back}} + \Delta q_{\text{front}} &\approx [J^y(x, y + \frac{1}{2}\Delta y, z) - J^y(x, y - \frac{1}{2}\Delta y, z)]\Delta t \Delta x \Delta z \\ \frac{\Delta q_{\text{back}} + \Delta q_{\text{front}}}{\Delta t \Delta x \Delta y \Delta z} &\approx \frac{J^y(x, y + \frac{1}{2}\Delta y, z) - J^y(x, y - \frac{1}{2}\Delta y, z)}{\Delta y}\end{aligned}$$

Note that this approximations become exact in the limit when $\Delta x, \Delta y, \Delta z$ and Δt goes to zero, hence by the definition of partial derivative we get that

$$\lim_{\Delta y \to 0} \frac{J^y(x, y + \frac{1}{2}\Delta y, z) - J^y(x, y - \frac{1}{2}\Delta y, z)}{\Delta y} = \frac{\partial J^y}{\partial y}$$

On the other hand, by the same means, we can get the net amount of charge flowing out of the top and bottom faces during Δt as

$$\begin{aligned}\Delta q_{\text{top}} + \Delta q_{\text{bottom}} &\approx [J^z(x, y, z + \frac{1}{2}\Delta z) - J^z(x, y, z - \frac{1}{2}\Delta z)]\Delta t \Delta x \Delta y \\ \frac{\Delta q_{\text{top}} + \Delta q_{\text{bottom}}}{\Delta t \Delta x \Delta y \Delta z} &\approx \frac{J^z(x, y, z + \frac{1}{2}\Delta z) - J^z(x, y, z - \frac{1}{2}\Delta z)}{\Delta z}\end{aligned}$$

And as Δz goes to 0 we get that

$$\lim_{\Delta z \to 0} \frac{J^z(x, y, z + \frac{1}{2}\Delta z) - J^y(x, y, z - \frac{1}{2}\Delta z)}{\Delta z} = \frac{\partial J^z}{\partial z}$$

$\square$

Solution. **BOX 7.4** - Exercise 7.4.2. Combining expressions 7.29 and 7.30 we get that

$$
\begin{aligned}
\frac{\Delta q_{\text{left}}+\Delta q_{\text{right}}}{\Delta t\Delta x\Delta y\Delta z}+\frac{\Delta q_{\text{back}}+\Delta q_{\text{front}}}{\Delta t\Delta x\Delta y\Delta z}+\frac{\Delta q_{\text{top}}+\Delta q_{\text{bottom}}}{\Delta t\Delta x\Delta y\Delta z}\approx\\
\approx\frac{J^x(x+\frac{1}{2}\Delta x,y,z)-J^x(x-\frac{1}{2}\Delta x,y,z)}{\Delta x}+\\
+\frac{J^y(x,y+\frac{1}{2}\Delta y,z)-J^y(x,y-\frac{1}{2}\Delta y,z)}{\Delta y}+\\
+\frac{J^z(x,y,z+\frac{1}{2}\Delta z)-J^z(x,y,z-\frac{1}{2}\Delta z)}{\Delta z}
\end{aligned}
$$

Hence by applying the limit to both sides and using equation 7.32 we have that

$$-\frac{\partial\rho}{\partial t}=\frac{\partial J^x}{\partial x}+\frac{\partial J^y}{\partial y}+\frac{\partial J^z}{\partial z}$$

But also we know that $\rho=J^t$ therefore

$$\frac{\partial J^t}{\partial t}+\frac{\partial J^x}{\partial x}+\frac{\partial J^y}{\partial y}+\frac{\partial J^z}{\partial z}=0$$

Or using Einstein notation $\partial_\mu J^\mu=0$. □

Solution. **BOX 7.5** - Exercise 7.5.1. We know that $F^{\mu\nu}=-F^{\nu\mu}$ then we have that

$$\partial_\mu\partial_\nu F^{\mu\nu}=\partial_\mu\partial_\nu(-F^{\nu\mu})=-\partial_\nu\partial_\mu F^{\nu\mu}=-\partial_\mu\partial_\nu F^{\mu\nu}$$

where we used that the order of partial derivatives is irrelevant and in the last equality, we renamed the variables $\nu\to\mu$ and $\mu\to\nu$. This implies that $\partial_\mu\partial_\nu F^{\mu\nu}=0$. □

Solution. **BOX 7.6** - Exercise 7.6.1. Let $\vec{B}=\vec{\nabla}\times\vec{A}$ then by solving the cross product we have that

$$
\begin{aligned}
\vec{B}=\vec{\nabla}\times\vec{A}&=\begin{vmatrix}\hat{\boldsymbol{x}}&\hat{\boldsymbol{y}}&\hat{\boldsymbol{z}}\\\partial_x&\partial_y&\partial_z\\A^x&A^y&A^z\end{vmatrix}\\
&=(\partial_yA^z-\partial_zA^y)\hat{\boldsymbol{x}}+(\partial_zA^x-\partial_xA^z)\hat{\boldsymbol{y}}+(\partial_xA^y-\partial_yA^x)\hat{\boldsymbol{y}}\\
&=\left(\frac{\partial A^z}{\partial y}-\frac{\partial A^y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial A^x}{\partial z}-\frac{\partial A^z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial A^y}{\partial x}-\frac{\partial A^x}{\partial y}\right)\hat{\boldsymbol{y}}
\end{aligned}
$$

□

Solution. **BOX 7.6** - Exercise 7.6.2. Let $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ then for $\mu = t$ and $\nu = x$ we have that

$$\begin{aligned} F^{tx} &= \partial^t A^x - \partial^x A^t \\ E_x &= -\frac{\partial A^x}{\partial t} - \frac{\partial A^t}{\partial x} \end{aligned}$$

which is the x component of equation 7.35. For $\mu = \nu = t$ we have that

$$F^{tt} = \partial^t A^t - \partial^t A^t = 0$$

which is the correct component of the field tensor. □

Solution. **BOX 7.7** - Exercise 7.7.1. Let $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ then by replacing we have that

$$\begin{aligned} \partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha} &= \partial^\alpha(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\nu(\partial^\alpha A^\mu - \partial^\mu A^\alpha) + \\ &\quad + \partial^\mu(\partial^\nu A^\alpha - \partial^\alpha A^\nu) \\ &= \partial^\alpha\partial^\mu A^\nu - \partial^\alpha\partial^\nu A^\mu + \partial^\nu\partial^\alpha A^\mu - \partial^\nu\partial^\mu A^\alpha + \\ &\quad + \partial^\mu\partial^\nu A^\alpha - \partial^\mu\partial^\alpha A^\nu \\ &= (\partial^\alpha\partial^\mu A^\nu - \partial^\alpha\partial^\mu A^\nu) + (\partial^\nu\partial^\alpha A^\mu - \partial^\nu\partial^\alpha A^\mu) \\ &\quad + (\partial^\mu\partial^\nu A^\alpha - \partial^\mu\partial^\nu A^\alpha) \\ &= 0 \end{aligned}$$

Where we used that the order of partial derivatives does not matter. □

Solution. **P7.1** From the general equation for the transformation properties of a tensor we have that

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta}$$

In the special case for the Lorentz transformations the partial derivatives become

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So in particular for $\mu = t$ and $\nu = x$ we get that

$$\begin{aligned}
F'^{tx} = E'_x &= \frac{\partial x'^{t}}{\partial x^{\alpha}} \frac{\partial x'^{x}}{\partial x^{\beta}} F^{\alpha\beta} \\
&= \frac{\partial x'^{t}}{\partial x^{\alpha}} [\frac{\partial x'^{x}}{\partial x^{t}} F^{\alpha t} + \frac{\partial x'^{x}}{\partial x^{x}} F^{\alpha x} + \frac{\partial x'^{x}}{\partial x^{y}} F^{\alpha y} + \frac{\partial x'^{x}}{\partial x^{z}} F^{\alpha z}] \\
&= \frac{\partial x'^{t}}{\partial x^{\alpha}} [-\gamma\beta F^{\alpha t} + \gamma F^{\alpha x}] \\
&= \frac{\partial x'^{t}}{\partial x^{t}} [-\gamma\beta F^{tt} + \gamma F^{tx}] + \frac{\partial x'^{t}}{\partial x^{x}} [-\gamma\beta F^{xt} + \gamma F^{xx}] \\
&= \gamma^2 E_x - \gamma^2\beta^2 E_x \\
&= \gamma^2 E_x (1 - \beta^2) \\
&= E_x
\end{aligned}$$

For $\mu = t$ and $\nu = y$ we get that

$$\begin{aligned}
F'^{ty} = E'_y &= \frac{\partial x'^{t}}{\partial x^{\alpha}} \frac{\partial x'^{y}}{\partial x^{\beta}} F^{\alpha\beta} \\
&= \frac{\partial x'^{t}}{\partial x^{\alpha}} [\frac{\partial x'^{y}}{\partial x^{t}} F^{\alpha t} + \frac{\partial x'^{y}}{\partial x^{x}} F^{\alpha x} + \frac{\partial x'^{y}}{\partial x^{y}} F^{\alpha y} + \frac{\partial x'^{y}}{\partial x^{z}} F^{\alpha z}] \\
&= \frac{\partial x'^{t}}{\partial x^{\alpha}} F^{\alpha y} \\
&= \frac{\partial x'^{t}}{\partial x^{t}} F^{ty} + \frac{\partial x'^{t}}{\partial x^{x}} F^{xy} \\
&= \gamma E_y - \gamma\beta B_z
\end{aligned}$$

And for $\mu = t$ and $\nu = z$ we get that

$$\begin{aligned}
F'^{tz} = E'_z &= \frac{\partial x'^{t}}{\partial x^{\alpha}} \frac{\partial x'^{z}}{\partial x^{\beta}} F^{\alpha\beta} \\
&= \frac{\partial x'^{t}}{\partial x^{\alpha}} [\frac{\partial x'^{z}}{\partial x^{t}} F^{\alpha t} + \frac{\partial x'^{z}}{\partial x^{x}} F^{\alpha x} + \frac{\partial x'^{z}}{\partial x^{y}} F^{\alpha y} + \frac{\partial x'^{z}}{\partial x^{z}} F^{\alpha z}] \\
&= \frac{\partial x'^{t}}{\partial x^{\alpha}} F^{\alpha z} \\
&= \frac{\partial x'^{t}}{\partial x^{t}} F^{tz} + \frac{\partial x'^{t}}{\partial x^{x}} F^{xz} \\
&= \gamma E_z + \gamma\beta B_y
\end{aligned}$$

On the other hand, for the magnetic field, taking $\mu = y$ and $\nu = z$ we get the following

$$\begin{aligned}
F'^{yz} = B'_x &= \frac{\partial x'^y}{\partial x^\alpha}\frac{\partial x'^z}{\partial x^\beta}F^{\alpha\beta} \\
&= \frac{\partial x'^y}{\partial x^\alpha}[\frac{\partial x'^z}{\partial x^t}F^{\alpha t} + \frac{\partial x'^z}{\partial x^x}F^{\alpha x} + \frac{\partial x'^z}{\partial x^y}F^{\alpha y} + \frac{\partial x'^z}{\partial x^z}F^{\alpha z}] \\
&= \frac{\partial x'^y}{\partial x^\alpha}F^{\alpha z} \\
&= \frac{\partial x'^y}{\partial x^y}F^{yz} \\
&= B_x
\end{aligned}$$

For $\mu = z$ and $\nu = x$ we have that

$$\begin{aligned}
F'^{zx} = B'_y &= \frac{\partial x'^z}{\partial x^\alpha}\frac{\partial x'^x}{\partial x^\beta}F^{\alpha\beta} \\
&= \frac{\partial x'^z}{\partial x^\alpha}[\frac{\partial x'^x}{\partial x^t}F^{\alpha t} + \frac{\partial x'^x}{\partial x^x}F^{\alpha x} + \frac{\partial x'^x}{\partial x^y}F^{\alpha y} + \frac{\partial x'^x}{\partial x^z}F^{\alpha z}] \\
&= \frac{\partial x'^z}{\partial x^\alpha}[\frac{\partial x'^x}{\partial x^t}F^{\alpha t} + \frac{\partial x'^x}{\partial x^x}F^{\alpha x}] \\
&= \frac{\partial x'^z}{\partial x^z}[\frac{\partial x'^x}{\partial x^t}F^{zt} + \frac{\partial x'^x}{\partial x^x}F^{zx}] \\
&= \gamma B_y + \gamma\beta E_z
\end{aligned}$$

And finally, for $\mu = x$ and $\nu = y$ we have that

$$\begin{aligned}
F'^{xy} = B'_z &= \frac{\partial x'^x}{\partial x^\alpha}\frac{\partial x'^y}{\partial x^\beta}F^{\alpha\beta} \\
&= \frac{\partial x'^x}{\partial x^\alpha}[\frac{\partial x'^y}{\partial x^t}F^{\alpha t} + \frac{\partial x'^y}{\partial x^x}F^{\alpha x} + \frac{\partial x'^y}{\partial x^y}F^{\alpha y} + \frac{\partial x'^y}{\partial x^z}F^{\alpha z}] \\
&= \frac{\partial x'^z}{\partial x^\alpha}F^{\alpha y} \\
&= \frac{\partial x'^x}{\partial x^t}F^{ty} + \frac{\partial x'^x}{\partial x^x}F^{xy} \\
&= \gamma B_z - \gamma\beta E_y
\end{aligned}$$

$\square$

Solution. **P7.2**

a. The equation 7.20 states that

$$\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha} = 0$$

In the first term, for example, we know that $F^{\mu\nu}$ has 16 components since $F^{\mu\nu}$ is a second rank tensor. Viewing then $\partial^\alpha F^{\mu\nu}$ as a product of tensors of rank 1 and rank 2 the result gives us a tensor of rank $1+2=3$ which by definition has 64 components. The same can be said for the rest of the terms so the equation has 64 components.

b. Let us suppose that $\mu = \nu$ and $\alpha \neq \mu$ then we see that

$$\begin{aligned}
\partial^\alpha F^{\mu\mu} + \partial^\mu F^{\alpha\mu} + \partial^\mu F^{\mu\alpha} &= 0 \\
0 + \partial^\mu F^{\alpha\mu} + \partial^\mu F^{\mu\alpha} &= 0 \\
\partial^\mu F^{\alpha\mu} - \partial^\mu F^{\alpha\mu} &= 0
\end{aligned}$$

Where we used that $F^{\mu\mu} = 0$ and that $F^{\mu\alpha} = -F^{\alpha\mu}$ since $F^{\mu\nu}$ is antisymmetric. So in this case, we see that the equation is identically zero.

In the case where $\mu = \nu = \alpha$ since $F^{\mu\mu} = 0$ then the equation is also identically zero.

Therefore in the only case where the equation is not identically zero is when $\mu \neq \nu \neq \alpha$.

c. Let $\alpha = t$, $\mu = x$ and $\nu = y$ then we have that

$$\begin{aligned}
\partial^t F^{xy} + \partial^y F^{tx} + \partial^x F^{yt} &= 0 \\
\partial^t B_z + \partial^y E_x - \partial^x E_y &= 0 \\
-\frac{\partial B_z}{\partial t} + \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} &= 0 \\
\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0
\end{aligned}$$

Where we see that $\partial^t B_z$ is the negative z component of $\partial \vec{B}/\partial t$.

On the other hand, let us compute $\boldsymbol{\nabla} \times \vec{E}$ as follows

$$\begin{aligned}
\boldsymbol{\nabla} \times \vec{E} &= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \\
&= \frac{\partial E_z}{\partial y}\hat{\boldsymbol{x}} + \frac{\partial E_x}{\partial z}\hat{\boldsymbol{y}} + \frac{\partial E_y}{\partial x}\hat{\boldsymbol{z}} - \frac{\partial E_y}{\partial z}\hat{\boldsymbol{x}} - \frac{\partial E_z}{\partial x}\hat{\boldsymbol{y}} - \frac{\partial E_x}{\partial y}\hat{\boldsymbol{z}} \\
&= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{z}}
\end{aligned}$$

So we see that $\partial^y E_x - \partial^x E_y$ is the negative z component of $\boldsymbol{\nabla} \times \vec{E}$. Therefore with these indexes, we get the z component of Faraday's law as shown.

Let now $\alpha = x$, $\mu = y$ and $\nu = z$

$$\begin{aligned}
\partial^x F^{yz} + \partial^z F^{xy} + \partial^y F^{zx} &= 0 \\
\partial^x B_x + \partial^z B_z + \partial^y B_y &= 0 \\
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0
\end{aligned}$$

Therefore we see that this is $\boldsymbol{\nabla} \cdot \vec{B} = 0$ i.e. Gauss's law for the magnetic field.

Finally for $\alpha = y$, $\mu = z$ and $\nu = t$ we get that

$$\begin{aligned}
\partial^y F^{zt} + \partial^t F^{yz} + \partial^z F^{ty} &= 0 \\
-\partial^y E_z + \partial^t B_x + \partial^z E_y &= 0 \\
-\frac{\partial B_x}{\partial t} + \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} &= 0 \\
\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= 0
\end{aligned}$$

Where we see that $\partial^t B_x$ is the negative x component of $\partial\vec{B}/\partial t$ and $\partial^z E_y - \partial^y E_z$ is the negative x component of $\boldsymbol{\nabla} \times \vec{E}$.

Therefore we get the x component of Faraday's law with these indexes.

$\square$

Solution. **P7.3**

a. Equation (7.5) states the following

$$\partial_\nu F^{\mu\nu} = 4\pi k J^\mu$$

So since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ we have that

$$\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 4\pi k J^\mu$$

b. Let $A^\mu_{\text{new}} = A^\mu + \partial^\mu \Lambda$ be a new four-potential then the equation (7.5) becomes

$$\begin{aligned}
\partial_\nu(\partial^\mu A^\nu_{\text{new}} - \partial^\nu A^\mu_{\text{new}}) &= 4\pi k J^\mu \\
\partial_\nu(\partial^\mu(A^\nu + \partial^\nu \Lambda) - \partial^\nu(A^\mu + \partial^\mu \Lambda)) &= 4\pi k J^\mu \\
\partial_\nu(\partial^\mu A^\nu + \partial^\mu \partial^\nu \Lambda - \partial^\nu A^\mu - \partial^\mu \partial^\nu \Lambda) &= 4\pi k J^\mu \\
\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= 4\pi k J^\mu
\end{aligned}$$

Where we used the fact that the order in which we apply the partial derivatives don't matter. Therefore we get the equation (7.5) in its original.

c. Let A^μ now be a four-potential such that $\partial_\mu A^\mu = 0$ then the equation (7.5) becomes

$$\begin{aligned}
\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= 4\pi k J^\mu \\
\partial^\mu \partial_\nu A^\nu - \partial_\nu \partial^\nu A^\mu &= 4\pi k J^\mu \\
-\partial_\nu \partial^\nu A^\mu &= 4\pi k J^\mu \\
\partial_\nu \partial^\nu A^\mu &= -4\pi k J^\mu
\end{aligned}$$

□

Solution. **P7.5** We know from equation (7.7) that

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu$$

So for example for $\mu = x$ we have that

$$\begin{aligned}
\frac{\mathrm{d}p^x}{\mathrm{d}\tau} &= qF^{x\nu}u_\nu \\
\frac{\mathrm{d}p^x}{\mathrm{d}\tau} &= q(F^{xt}u_t + F^{xx}u_x + F^{xy}u_y + F^{xz}u_z) \\
\frac{\mathrm{d}p^x}{\mathrm{d}\tau} &= q(-E_x u_t + B_z u_y - B_y u_z) \\
\frac{\mathrm{d}p^x}{\mathrm{d}\tau} &= q(E_x u^t + B_z u^y - B_y u^z)
\end{aligned}$$

But since the particle is moving relativistically we have that

$$\frac{\mathrm{d}p^x}{\mathrm{d}\tau} = q\left(\frac{E_x}{\sqrt{1-v^2}} + B_z\frac{v_y}{\sqrt{1-v^2}} - B_y\frac{v_z}{\sqrt{1-v^2}}\right)$$

And mutiplying the equation by $d\tau/dt$ we get that

$$\begin{aligned}
\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}p^x}{\mathrm{d}\tau} &= q\left(\frac{E_x}{\sqrt{1-v^2}} + B_z\frac{v_y}{\sqrt{1-v^2}} - B_y\frac{v_z}{\sqrt{1-v^2}}\right)\frac{\mathrm{d}\tau}{\mathrm{d}t} \\
\frac{\mathrm{d}p^x}{\mathrm{d}t} &= q\left(\frac{E_x}{\sqrt{1-v^2}} + B_z\frac{v_y}{\sqrt{1-v^2}} - B_y\frac{v_z}{\sqrt{1-v^2}}\right)\frac{\mathrm{d}t\sqrt{1-v^2}}{\mathrm{d}t} \\
\frac{\mathrm{d}p^x}{\mathrm{d}t} &= q(E_x + B_z v_y - B_y v_z)
\end{aligned}$$

Which is the x component of the Lorentz force equation in the reference frame where $t, \vec{v}, \vec{E}$ and $\vec{B}$ are measured.

In the same way, for $\mu = y$ we have that

$$\begin{aligned}
\frac{\mathrm{d}p^y}{\mathrm{d}\tau} &= qF^{y\nu}u_\nu \\
\frac{\mathrm{d}p^y}{\mathrm{d}\tau} &= q(F^{yt}u_t + F^{yx}u_x + F^{yy}u_y + F^{yz}u_z) \\
\frac{\mathrm{d}p^y}{\mathrm{d}\tau} &= q(-E_y u_t - B_z u_x + B_x u_z) \\
\frac{\mathrm{d}p^y}{\mathrm{d}\tau} &= q(E_y u^t + B_x u^z - B_z u^x) \\
\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}p^y}{\mathrm{d}\tau} &= q\left(\frac{E_y}{\sqrt{1-v^2}} + B_x\frac{v_z}{\sqrt{1-v^2}} - B_z\frac{v_x}{\sqrt{1-v^2}}\right)\frac{\mathrm{d}\tau}{\mathrm{d}t} \\
\frac{\mathrm{d}p^y}{\mathrm{d}t} &= q(E_y + B_x v_z - B_z v_x)
\end{aligned}$$

And finally, for $\mu = z$ we have that

$$
\begin{aligned}
\frac{\mathrm{d}p^z}{\mathrm{d}\tau} &= qF^{z\nu}u_\nu \\
\frac{\mathrm{d}p^z}{\mathrm{d}\tau} &= q(F^{zt}u_t + F^{zx}u_x + F^{zy}u_y + F^{zz}u_z) \\
\frac{\mathrm{d}p^z}{\mathrm{d}\tau} &= q(-E_z u_t + B_y u_x - B_x u_y) \\
\frac{\mathrm{d}p^z}{\mathrm{d}\tau} &= q(E_z u^t + B_y u^x - B_x u^y) \\
\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}p^z}{\mathrm{d}\tau} &= q\left(\frac{E_z}{\sqrt{1-v^2}} + B_y\frac{v_x}{\sqrt{1-v^2}} - B_x\frac{v_y}{\sqrt{1-v^2}}\right)\frac{\mathrm{d}\tau}{\mathrm{d}t} \\
\frac{\mathrm{d}p^z}{\mathrm{d}t} &= q(E_z + B_y v_x - B_x v_y)
\end{aligned}
$$

Which are the y and z components of the Lorentz force equation. Therefore the Lorentz force equation is correct even in the relativistic limit. $\square$