Solved selected problems of General Relativity - Thomas A. Moore

Franco Zacco

Chapter 12 - Photon Orbits

Solution. **BOX 12.2** - Exercise 12.2.1.
Equation (12.2) states that

$$0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

Then dividing by $(1 - 2GM/r)dt^2$ we get that

$$0 = -1 + \left(1 - \frac{2GM}{r}\right)^{-2} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2 \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2$$

Finally, replacing equation (12.1) squared we get that

$$1 = \left(1 - \frac{2GM}{r}\right)^{-2} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{b^2}{r^2} \left(1 - \frac{2GM}{r}\right)$$

□

Solution. **BOX 12.3** - Exercise 12.3.1.
The effective potential energy is

$$\tilde{V}(r) = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

Then the extremum of $\tilde{V}(r)$ happens at $d\tilde{V}/dr = 0$ hence we compute the derivative and we equate it to zero as follows

$$\begin{aligned}
-\frac{2}{r^3}\left(1 - \frac{2GM}{r}\right) + \frac{1}{r^2}\frac{2GM}{r^2} &= 0 \\
-\frac{2}{r^3} + \frac{4GM}{r^4} + \frac{2GM}{r^4} &= 0 \\
\frac{6GM}{r} &= 2 \\
r &= 3GM
\end{aligned}$$

Finally we compute $\tilde{V}(3GM)$ as shown below

$$\begin{aligned}
\tilde{V}(3GM) &= \frac{1}{9(GM)^2}\left(1 - \frac{2GM}{3GM}\right) \\
&= \frac{1}{9(GM)^2}\frac{1}{3} \\
&= \frac{1}{27(GM)^2}
\end{aligned}$$

□

Solution. **BOX 12.4** - Exercise 12.4.1.
In flat spacetime we know that

$$
\begin{aligned}
b &= \frac{l}{e} = \frac{r^2 d\phi/d\tau}{dt/d\tau} = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \\
0 &= -dt^2 + dr^2 + r^2 d\phi^2
\end{aligned}
$$

Then dividing the second equation by dt^2 and replacing $b^2/r^2 = r^2(d\phi/dt)^2$ we get that

$$
\begin{aligned}
0 &= -1 + \left[\frac{\mathrm{d}r}{\mathrm{d}t}\right]^2 + \frac{b^2}{r^2} \\
\frac{1}{b^2} &= \left[\frac{1}{b^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right]^2 + \frac{1}{r^2}
\end{aligned}
$$

Where in the last step we divided by b^2. $\square$

Solution. **BOX 12.5** - Exercise 12.5.1.
In the observer's coordinate system the four-vector $\boldsymbol{A}$ has components

$$A^{\mu}_{obs} = \begin{bmatrix} A^t_{obs} \\ A^x_{obs} \\ A^y_{obs} \\ A^z_{obs} \end{bmatrix}$$

Also, $\boldsymbol{o}_x$ in the observer's reference frame is defined as

$$(\boldsymbol{o}_x)^{\mu}_{obs} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the inner product between these vectors in the observer's frame is

$$\boldsymbol{o}_x \cdot \boldsymbol{A} = \eta_{\mu\nu}(\boldsymbol{o}_x)^{\mu}_{obs} A^{\nu}_{obs} = \eta_{x\nu}(\boldsymbol{o}_x)^{x}_{obs} A^{\nu}_{obs} = \eta_{x\nu}(1) A^{\nu}_{obs} = \eta_{xx} A^{x}_{obs} = A^{x}_{obs}$$

□

Solution. **BOX 12.6** - Exercise 12.6.1.
Let us compute $(\boldsymbol{o}_x)^{\mu}$, we know that $\boldsymbol{o}_x \cdot \boldsymbol{o}_x = \eta_{xx} = 1$ and since we align $\boldsymbol{o}_x$ with ϕ Schwarzschild coordinate the rest of the components must be zero and hence

$$1 = \boldsymbol{o}_x \cdot \boldsymbol{o}_x = g_{\mu\nu}(\boldsymbol{o}_x)^{\mu}(\boldsymbol{o}_x)^{\nu} = g_{\phi\phi}(\boldsymbol{o}_x)^{\phi}(\boldsymbol{o}_x)^{\phi} = r^2 \sin\theta^2((\boldsymbol{o}_x)^{\phi})^2$$

Therefore

$$(\boldsymbol{o}_x)^{\phi} = \frac{1}{r\sin\theta}$$

In the same way, for $(\boldsymbol{o}_y)^{\mu}$ and $(\boldsymbol{o}_z)^{\mu}$ where we align them with $-\theta$ and r respectively, we have that

$$1 = \boldsymbol{o}_y \cdot \boldsymbol{o}_y = g_{\mu\nu}(\boldsymbol{o}_y)^{\mu}(\boldsymbol{o}_y)^{\nu} = g_{\theta\theta}(\boldsymbol{o}_y)^{\theta}(\boldsymbol{o}_y)^{\theta} = r^2((\boldsymbol{o}_y)^{\theta})^2$$

Therefore $(\boldsymbol{o}_y)^{\theta} = 1/r$ but since it's aligned to $-\theta$ must be that

$$(\boldsymbol{o}_y)^{\theta} = -\frac{1}{r}$$

Finally, for $(\boldsymbol{o}_z)^{\mu}$ we have that

$$1 = \boldsymbol{o}_z \cdot \boldsymbol{o}_z = g_{\mu\nu}(\boldsymbol{o}_z)^{\mu}(\boldsymbol{o}_z)^{\nu} = g_{rr}(\boldsymbol{o}_z)^{r}(\boldsymbol{o}_z)^{r} = \left(1 - \frac{2GM}{r}\right)^{-1}((\boldsymbol{o}_z)^{r})^2$$

Hence

$$(\boldsymbol{o}_z)^{r} = \sqrt{1 - \frac{2GM}{r}}$$

□

Solution. **BOX 12.7** We know that the angle ψ an emitted photon's path makes with the outward direction is such that

$$\sin\psi = \frac{v_{x,obs}}{1} = \frac{p^x_{obs}}{p^t_{obs}} = \frac{\boldsymbol{o}_x \cdot \boldsymbol{p}}{-\boldsymbol{o}_t \cdot \boldsymbol{p}}$$

Then from equations 12.10 and 12.12 we have that

$$\begin{aligned}
\sin\psi &= \frac{\boldsymbol{o}_x \cdot \boldsymbol{p}}{-\boldsymbol{o}_t \cdot \boldsymbol{p}} \\
&= \frac{g_{\mu\nu}(\boldsymbol{o}_x)^\mu p^\nu}{-g_{\mu\nu}(\boldsymbol{o}_t)^\mu p^\nu} \\
&= \frac{r^2(1/r)(Eb/r^2)}{(1-2GM/r)(1/\sqrt{1-2GM/r})(E/(1-2GM/r))} \\
&= \frac{b/r}{1/\sqrt{1-2GM/r}} \\
&= \frac{b}{r}\sqrt{1-\frac{2GM}{r}}
\end{aligned}$$

Finally, the critical angle correspond to when $b = GM\sqrt{27}$ hence

$$\sin\psi_c = \frac{GM\sqrt{27}}{r}\sqrt{1-\frac{2GM}{r}}$$

□

Solution. **P12.1** We know that if $b > \sqrt{27}GM$ then a photon coming in from infinity will rebound to infinity.
So if $b = 6GM$ we see that $6GM > \sqrt{27}GM \approx 5.19GM$ then the photon will not be absorbed by the black hole but it will rebound to infinity. □

Solution. **P12.2** We saw that if a photon has $b > \sqrt{27}GM$ then will rebound to infinity then the cylindrical beam of photons must have at most a radius $R = \sqrt{27}GM$, assuming the center of the beam is aligned to the center of the object. □

Solution. **P12.3** In BOX 12.7 we computed the $\sin\psi$ which is essentially v_x since $\sin\psi = v_x/1$ hence

$$v_x = \frac{b}{r}\sqrt{1 - \frac{2GM}{r}} = \sqrt{\frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}$$

In the same way, we compute v_z as follows

$$\begin{aligned}
v_z &= \frac{\boldsymbol{o}_z \cdot \boldsymbol{p}}{-\boldsymbol{o}_t \cdot \boldsymbol{p}} \\
&= \frac{g_{\mu\nu}(\boldsymbol{o}_z)^\mu p^\nu}{-g_{\mu\nu}(\boldsymbol{o}_t)^\mu p^\nu} \\
&= \frac{(1 - 2GM/r)^{-1}\sqrt{1 - 2GM/r}E\sqrt{1 - b^2/r^2(1 - 2GM/r)}}{(1 - 2GM/r)(1/\sqrt{1 - 2GM/r})(E/(1 - 2GM/r))} \\
&= \frac{(1 - 2GM/r)E\sqrt{1 - b^2/r^2(1 - 2GM/r)}}{(1 - 2GM/r)^2(E/(1 - 2GM/r))} \\
&= \sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}
\end{aligned}$$

Therefore the speed the speed of light moving in the equatorial plane as measured by the observer is

$$v = \sqrt{v_x^2 + v_z^2} = \sqrt{\frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right) + 1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)} = 1$$

□

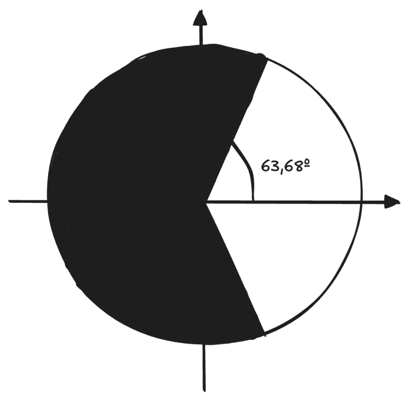
Solution. **P12.4**

a. When an observer is at rest at $r = 6GM$ the critical angle is given by

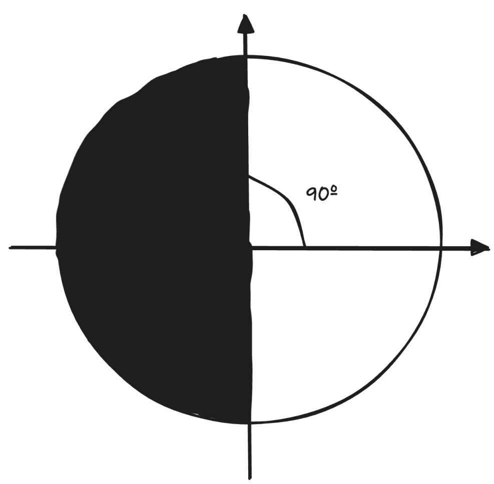
$$\psi_c = \arcsin\left(\frac{\sqrt{27}GM}{6GM}\sqrt{1-\frac{2GM}{6GM}}\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4} = 135^\circ$$

Then any light emited beyond 135° will be captured by the blackhole implying that the blackhole occupy a region between the angles 135° and 225° (measured from the outward direction) or a region of 45° around the inward direction.

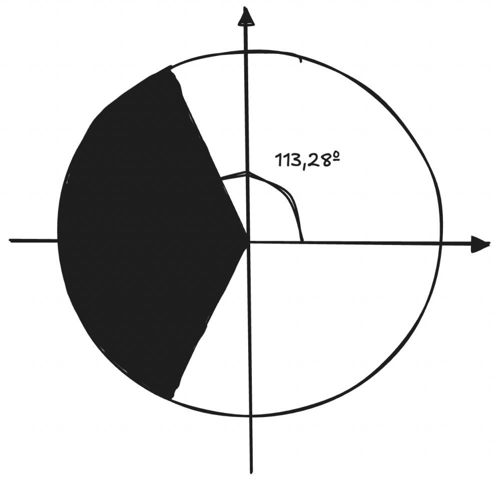
b.
- When $r = 2.5GM$ then $\psi_c = 68.36^\circ$



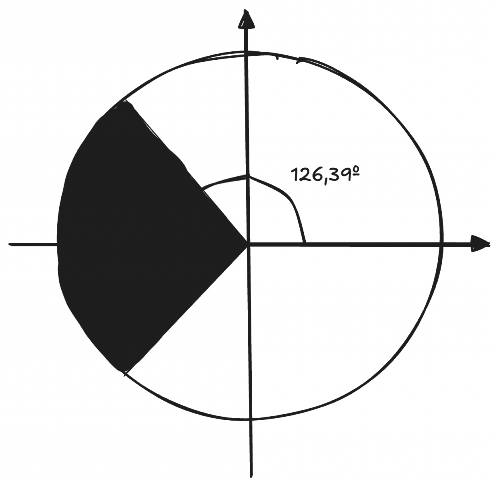
- When $r = 3GM$ then $\psi_c = 90^\circ$



– When $r = 4GM$ then $\psi_c = 66.72° = 113.28°$



– When $r = 5GM$ then $\psi_c = 53.61° = 126.39°$



$\square$

Solution. **P12.5** Equation 9.12 states that

$$\frac{\lambda_R}{\lambda_E} = \frac{\sqrt{1 - 2GM/r_R}}{\sqrt{1 - 2GM/r_E}}$$

If we consider that $E = h/\lambda$ for a photon, equation 12.15 for an observer at r_R becomes

$$E_{obs} = \frac{h}{\lambda_R} = \frac{E}{\sqrt{1 - 2GM/r_R}}$$

And for an observer at r_E we have that

$$\frac{h}{\lambda_E} = \frac{E}{\sqrt{1 - 2GM/r_E}}$$

Hence

$$\begin{aligned} \frac{h/\lambda_E}{h/\lambda_R} &= \frac{E/\sqrt{1 - 2GM/r_E}}{E/\sqrt{1 - 2GM/r_R}} \\ \frac{\lambda_R}{\lambda_E} &= \frac{\sqrt{1 - 2GM/r_R}}{\sqrt{1 - 2GM/r_E}} \end{aligned}$$

Therefore equation 12.15 is consistent with equation 9.12. $\square$

Solution. **P12.6**

a. From equation (12.5) we get that

$$\frac{1}{b^2} = \left[\frac{1}{b}\frac{\mathrm{d}r}{\mathrm{d}t}\right]^2 + \frac{1}{r^2}$$

And from equation (12.19) we get that

$$b = r^2\frac{\mathrm{d}\phi}{\mathrm{d}t} \qquad \text{or} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{b}{r^2}$$

Also, we can combine them as follows

$$\left[\frac{1}{b}\frac{\mathrm{d}r}{\mathrm{d}t}\right]^2 = \frac{1}{b^2} - \frac{1}{r^2}$$

$$\left[\frac{\mathrm{d}r}{\mathrm{d}t}\right]^2 = b^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right)$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{1 - \frac{b^2}{r^2}}$$

b. Let us divide the equations as follows

$$\frac{d\phi/dt}{dr/dt} = \frac{b/r^2}{\sqrt{1-\frac{b^2}{r^2}}}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{b}{r^2\sqrt{1-\frac{b^2}{r^2}}}$$

Now, integrating leaves us with

$$\int d\phi = \int \frac{b}{r^2\sqrt{1-\frac{b^2}{r^2}}}\, dr$$

$$\phi = \operatorname{arccot}\left(\frac{b}{\sqrt{r^2-b^2}}\right) + \alpha$$

$$\cos(\phi) = \cos\left(\operatorname{arccot}\left(\frac{b}{\sqrt{r^2-b^2}}\right)\right) + \alpha$$

$$\cos(\phi) = \frac{b/\sqrt{r^2-b^2}}{\sqrt{1+(b^2/(r^2-b^2))}} + \alpha$$

$$\cos(\phi) = \frac{b/\sqrt{r^2-b^2}}{\sqrt{(b^2/(r^2-b^2))(1+(r^2-b^2)/b^2)}} + \alpha$$

$$\cos(\phi) = \frac{1}{\sqrt{1+(r^2-b^2)/b^2}} + \alpha$$

$$\cos(\phi) = \frac{1}{\sqrt{1+r^2/b^2-1}} + \alpha$$

$$\cos(\phi) = \frac{b}{r} + \alpha$$

$$\phi = \arccos\left(\frac{b}{r}\right) + \alpha$$

c. Let b be the smallest distance between the line and the origin.

From the equation we derived in part **b.** we have that

$$\begin{aligned} \phi - \alpha &= \arccos\left(\frac{b}{r}\right) \\ r\cos(\phi - \alpha) &= b \\ r(\cos\phi\cos\alpha + \sin\phi\sin\alpha) &= b \\ x\cos\alpha + y\sin\alpha &= b \end{aligned}$$

We used that $x = r\cos\phi$ and $y = r\sin\phi$. Therefore we arrived at the normal equation of a line in rectangular coordinates.

$\square$

Solution. **P12.7**

a. Since the observer is falling from rest then $e = 1$ hence

$$\left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1$$

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2GM}{r}\right)^{-1}$$

Also, $l = 0$ since it's falling radially, so $d\phi/d\tau = 0$ so the equation of motion becomes

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} = 0$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sqrt{\frac{2GM}{r}}$$

Finally, if we assume this to happen on the equatorial plane must be that $d\theta/d\tau = 0$.

b. Let us compute $\boldsymbol{o}_x \cdot \boldsymbol{o}_t$ knowing that $\boldsymbol{o}_x$ is aligned to ϕ as follows

$$0 = \boldsymbol{o}_x \cdot \boldsymbol{o}_t = g_{\mu\nu}(\boldsymbol{o}_x)^\mu(\boldsymbol{o}_t)^\nu = g_{tt}(\boldsymbol{o}_x)^t(\boldsymbol{o}_t)^t + g_{\phi\phi}(\boldsymbol{o}_x)^\phi(\boldsymbol{o}_t)^\phi$$

Since $(\boldsymbol{o}_t)^\phi = 0$ then must be that $(\boldsymbol{o}_x)^t = 0$. Also, since $\boldsymbol{o}_x$ has no components in the r and θ direction then computing $\boldsymbol{o}_x \cdot \boldsymbol{o}_x$ gives us

$$1 = \boldsymbol{o}_x \cdot \boldsymbol{o}_x = g_{\mu\nu}(\boldsymbol{o}_x)^\mu(\boldsymbol{o}_x)^\nu = g_{\phi\phi}(\boldsymbol{o}_x)^\phi(\boldsymbol{o}_x)^\phi = r^2\sin^2\theta((\boldsymbol{o}_x)^\phi)^2$$

So as in equation (12.10), $\boldsymbol{o}_x$ is given by

$$(\boldsymbol{o}_x)^\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r\sin\theta} \end{bmatrix}$$

Now, let us compute $\boldsymbol{o}_y \cdot \boldsymbol{o}_t$ so in the same way, we have that

$$0 = \boldsymbol{o}_y \cdot \boldsymbol{o}_t = g_{\mu\nu}(\boldsymbol{o}_y)^\mu(\boldsymbol{o}_t)^\nu = g_{tt}(\boldsymbol{o}_y)^t(\boldsymbol{o}_t)^t + g_{\phi\phi}(\boldsymbol{o}_y)^\theta(\boldsymbol{o}_t)^\theta$$

Which implies that $(\boldsymbol{o}_y)^t = 0$ since $(\boldsymbol{o}_t)^\theta = 0$ and hence $(\boldsymbol{o}_y)^\mu$ is

$$(\boldsymbol{o}_y)^\mu = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{r} \\ 0 \end{bmatrix}$$

Where we used that $1 = r^2((\boldsymbol{o}_y)^\theta)^2$.

Finally, let us compute $\boldsymbol{o}_z \cdot \boldsymbol{o}_t$ then we have that

$$0 = \boldsymbol{o}_z \cdot \boldsymbol{o}_t = g_{\mu\nu}(\boldsymbol{o}_z)^\mu(\boldsymbol{o}_t)^\nu = g_{tt}(\boldsymbol{o}_z)^t(\boldsymbol{o}_t)^t + g_{rr}(\boldsymbol{o}_z)^r(\boldsymbol{o}_t)^r$$

Then

$$0 = -\left(1 - \frac{2GM}{r}\right)(\boldsymbol{o}_z)^t\left(1 - \frac{2GM}{r}\right)^{-1} - \left(1 - \frac{2GM}{r}\right)^{-1}(\boldsymbol{o}_z)^r\sqrt{\frac{2GM}{r}}$$

$$(\boldsymbol{o}_z)^t = -\left(1 - \frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}(\boldsymbol{o}_z)^r$$

But also if we compute $\boldsymbol{o}_z \cdot \boldsymbol{o}_z$ we have that

$$\begin{aligned}
\boldsymbol{o}_z \cdot \boldsymbol{o}_z &= 1 \\
g_{\mu\nu}(\boldsymbol{o}_z)^\mu(\boldsymbol{o}_z)^\nu &= 1 \\
g_{tt}((\boldsymbol{o}_z)^t)^2 + g_{rr}((\boldsymbol{o}_z)^r)^2 &= 1 \\
-\left(1 - \frac{2GM}{r}\right)((\boldsymbol{o}_z)^t)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}((\boldsymbol{o}_z)^r)^2 &= 1 \\
-\left(1 - \frac{2GM}{r}\right)^{-1}\frac{2GM}{r}((\boldsymbol{o}_z)^r)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}((\boldsymbol{o}_z)^r)^2 &= 1 \\
\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1}((\boldsymbol{o}_z)^r)^2 &= 1 \\
(\boldsymbol{o}_z)^r &= 1
\end{aligned}$$

Where we replaced the value fore $(\boldsymbol{o}_z)^t$. Therefore $(\boldsymbol{o}_z)^\mu$ is

$$(\boldsymbol{o}_z)^\mu = \begin{bmatrix} \frac{-\sqrt{2GM/r}}{1-2GM/r} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

c. Let us compute the critical angle using equation (12.13) as follows

$$\begin{aligned}
\sin\psi &= \frac{\boldsymbol{o}_x \cdot \boldsymbol{p}}{-\boldsymbol{o}_t \cdot \boldsymbol{p}} \\
&= \frac{g_{\mu\nu}(\boldsymbol{o}_x)^\mu(\boldsymbol{p})^\nu}{-g_{\mu\nu}(\boldsymbol{o}_t)^\mu(\boldsymbol{p})^\nu} \\
&= \frac{r^2\frac{1}{r}\frac{Eb}{r^2}}{E(1-\frac{2GM}{r})^{-1} \pm (1-\frac{2GM}{r})^{-1}\sqrt{\frac{2GM}{r}}E\sqrt{1-\frac{b^2}{r^2}(1-\frac{2GM}{r})}} \\
&= \frac{\frac{b}{r}(1-\frac{2GM}{r})}{1 \pm \sqrt{\frac{2GM}{r}}\sqrt{1-\frac{b^2}{r^2}(1-\frac{2GM}{r})}}
\end{aligned}$$

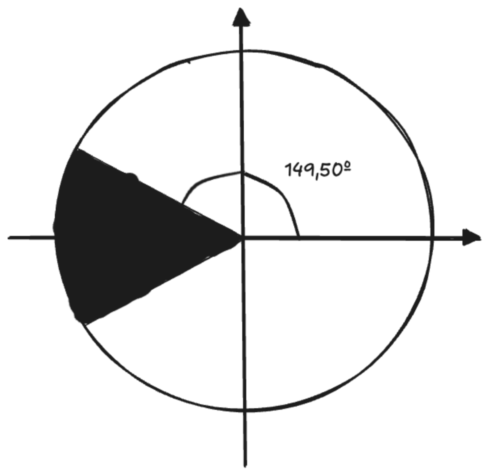
Then setting $b = \sqrt{27}GM$ to get the critical angle we have that

$$\sin\psi_c = \frac{\frac{\sqrt{27}GM}{r}(1-\frac{2GM}{r})}{1 \pm \sqrt{\frac{2GM}{r}}\sqrt{1-\frac{27(GM)^2}{r^2}(1-\frac{2GM}{r})}}$$

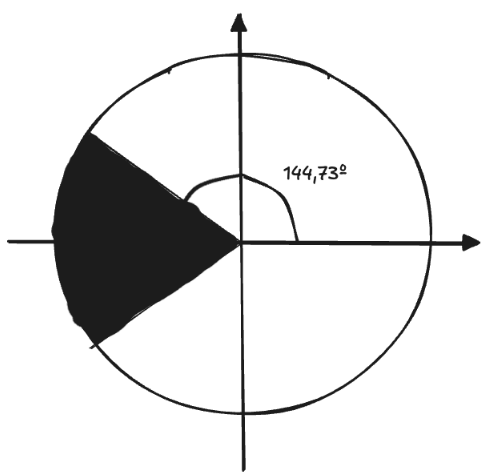
Where the plus sign in the denominator implies an outgoing photon and the minus sign implies an ingoing photon. We see that for $r > 3GM$ the photon must be outgoing and for $r < 3GM$ the photon is incoming. Then if $r = 4GM$ we have that

$$\sin\psi_c = \frac{\frac{\sqrt{27}}{4}(1-\frac{1}{2})}{1+\sqrt{\frac{1}{2}}\sqrt{1-\frac{27}{16}(1-\frac{1}{2})}} = 0.5076$$

Hence $\psi_c = 30.506° = 149.506°$



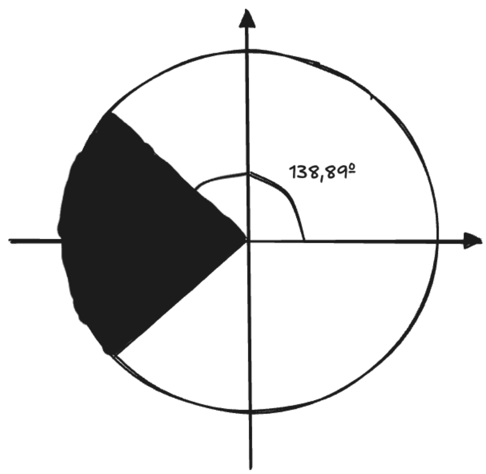
If $r = 3GM$ then $\psi_c = 35.264° = 144.736°$



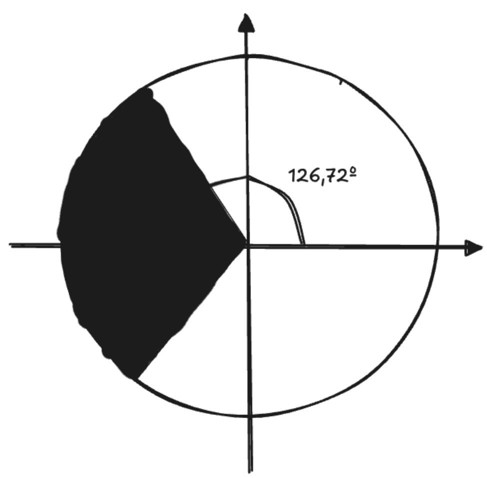
If $r = 2GM$ when using the minus sign in the equation we get a 0/0 indeterminate form so we use l'Hopital rule to compute the limit as follows

$$
\begin{aligned}
\sin\psi_c &= \lim_{r\to 2GM} \frac{\frac{\sqrt{27}GM}{r}(1-\frac{2GM}{r})}{1-\sqrt{\frac{2GM}{r}}\sqrt{1-\frac{27(GM)^2}{r^2}(1-\frac{2GM}{r})}} \\
&= \lim_{r\to 2GM} \frac{\frac{3\sqrt{3GM}}{r^3}(4GM-r)}{\frac{\sqrt{GM/r}(216(GM)^3-81(GM)^2r+r^3)}{\sqrt{2}r^4\sqrt{(r-3GM)^2(6GM+r)/r^3}}} \\
&= \frac{12\sqrt{3}}{31}
\end{aligned}
$$

then $\psi_c = 42.103^\circ = 138.897^\circ$



Finally for $r = GM$ we have that $\psi_c = 53.27^\circ = 126.72^\circ$



d. The incoming photon's energy a falling observer measures is

$$
\begin{aligned}
E_{obs} &= -\boldsymbol{o}_t \cdot \boldsymbol{p} \\
&= E\left(1-\frac{2GM}{r}\right)^{-1} - \left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}E\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)} \\
&= E\left(1-\frac{2GM}{r}\right)^{-1}\left(1-\sqrt{\frac{2GM}{r}}\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)}\right)
\end{aligned}
$$

But since it's falling radially then $b = 0$ so

$$
E_{obs} = E\left(1-\frac{2GM}{r}\right)^{-1}\left(1-\sqrt{\frac{2GM}{r}}\right)
$$

Then we see that $E_{obs} < E$ so the observer receives signals red-shifted.

Finally, the fractional change in wavelength is given by

$$
\begin{aligned}
\frac{h/\lambda_E}{h/\lambda_R} &= \frac{E/\sqrt{1-\frac{2GM}{r_E}}}{E\left(1-\frac{2GM}{r_R}\right)^{-1}\left(1-\sqrt{\frac{2GM}{r_R}}\right)} \\
\frac{\lambda_R}{\lambda_E} &= \frac{\left(1-\frac{2GM}{r_R}\right)}{\sqrt{1-\frac{2GM}{r_E}}\left(1-\sqrt{\frac{2GM}{r_R}}\right)}
\end{aligned}
$$

But since $r_E = \infty$ i.e. the signal is coming from infinity we can write that

$$
\frac{\lambda_R}{\lambda_E} = \frac{\left(1-\frac{2GM}{r_R}\right)}{\left(1-\sqrt{\frac{2GM}{r_R}}\right)}
$$

$\square$

Solution. **P12.8** In the falling observer's frame the velocity components of the object in circular orbit are $u^t_{obs} = -\boldsymbol{o}_t \cdot \boldsymbol{u}$ and $u^\mu_{obs} = \boldsymbol{o}_\mu \cdot \boldsymbol{u}$ for the spatial components. Then

$$\begin{aligned}
u^t_{obs} &= -\boldsymbol{o}_t \cdot \boldsymbol{u} \\
&= -g_{\mu\nu}(\boldsymbol{o}_t)^\mu (\boldsymbol{u})^\nu \\
&= -(g_{tt}(\boldsymbol{o}_t)^t u^t + g_{rr}(\boldsymbol{o}_t)^r u^r) \\
&= \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1} u^t + \left(1 - \frac{2GM}{r}\right)^{-1} \sqrt{\frac{2GM}{r}} u^r \\
&= u^t
\end{aligned}$$

Where we used that $u^r = 0$ for an object in a circular orbit. Also, we see that

$$u^x_{obs} = \boldsymbol{o}_x \cdot \boldsymbol{u} = g_{\mu\nu}(\boldsymbol{o}_x)^\mu (\boldsymbol{u})^\nu = g_{\phi\phi}(\boldsymbol{o}_x)^\phi (\boldsymbol{u})^\phi = r^2 \frac{1}{r} u^\phi = r u^\phi$$

$$u^z_{obs} = \boldsymbol{o}_z \cdot \boldsymbol{u} = g_{\mu\nu}(\boldsymbol{o}_z)^\mu (\boldsymbol{u})^\nu = g_{tt}(\boldsymbol{o}_z)^t (\boldsymbol{u})^t + g_{rr}(\boldsymbol{o}_z)^r (\boldsymbol{u})^r = \sqrt{\frac{2GM}{r}} u^t$$

On the other hand, we know that for an object in circular orbit at $r = 6GM$ we have that $l = \sqrt{12}GM$ and $e = \sqrt{8/9}$ hence

$$\begin{aligned}
u^t_{obs} &= u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = e\left(1 - \frac{2GM}{r}\right)^{-1} = \sqrt{\frac{8}{9}}\left(\frac{2}{3}\right)^{-1} = \sqrt{2} \\
u^x_{obs} &= r u^\phi = r\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{l}{r} = \frac{\sqrt{12}GM}{6GM} = \frac{\sqrt{3}}{3} \\
u^z_{obs} &= \sqrt{\frac{2GM}{r}} u^t = e\sqrt{\frac{2GM}{r}}\left(1 - \frac{2GM}{r}\right)^{-1} = \sqrt{\frac{8}{9}}\sqrt{\frac{1}{3}}\left(\frac{2}{3}\right)^{-1} = \frac{\sqrt{6}}{3}
\end{aligned}$$

Finally, the speed in the observer's frame is

$$\begin{aligned}
v &= \sqrt{v_x^2 + v_z^2} \\
&= \sqrt{\left(\frac{u^x_{obs}}{u^t_{obs}}\right)^2 + \left(\frac{u^z_{obs}}{u^t_{obs}}\right)^2} \\
&= \sqrt{\left(\frac{\sqrt{3}}{3\sqrt{2}}\right)^2 + \left(\frac{\sqrt{6}}{3\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{1}{6} + \frac{1}{3}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

□

Solution. **P12.9** For an observer in a circular orbit of radius r we have that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = e\left(1 - \frac{2GM}{r}\right)^{-1} \quad \frac{\mathrm{d}r}{\mathrm{d}\tau} = 0 \quad \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0 \quad \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{l}{r^2}$$

But also using the equation for the radius of circular orbits we get that

$$l = \frac{\sqrt{GM}r}{\sqrt{r - 3GM}}$$

Also, we know that the equation of e for a circular orbit ($\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0$) becomes

$$\begin{aligned}
e^2 &= -\frac{2GM}{r} + \frac{l^2}{r^2} - \frac{2GMl^2}{r^3} + 1 \\
e &= \sqrt{-\frac{2GM}{r} + \frac{GM}{r - 3GM} - \frac{2(GM)^2}{r(r - 3GM)} + 1} \\
e &= \sqrt{\frac{-2GM(r - 3GM) + GMr - 2(GM)^2 + r^2 - 3GMr}{r(r - 3GM)}} \\
e &= \sqrt{\frac{4(GM)^2 - 4GMr + r^2}{r(r - 3GM)}} \\
e &= (r - 2GM)\sqrt{\frac{1}{r(r - 3GM)}}
\end{aligned}$$

Hence

$$(\boldsymbol{o}_t)^\mu = \begin{bmatrix} \sqrt{\frac{r}{r-3GM}} \\ 0 \\ 0 \\ \frac{\sqrt{GM}}{r\sqrt{r-3GM}} \end{bmatrix}$$

Where we used that $(r - 2GM)\sqrt{\frac{1}{r(r-3GM)}}\left(1 - \frac{2GM}{r}\right)^{-1} = \sqrt{\frac{r}{r-3GM}}$.
Now, let us compute $\boldsymbol{o}_x \cdot \boldsymbol{o}_t$ knowing that $\boldsymbol{o}_x$ is aligned to ϕ as follows

$$\begin{aligned}
0 = \boldsymbol{o}_x \cdot \boldsymbol{o}_t = g_{\mu\nu}(\boldsymbol{o}_x)^\mu(\boldsymbol{o}_t)^\nu &= g_{tt}(\boldsymbol{o}_x)^t(\boldsymbol{o}_t)^t + g_{\phi\phi}(\boldsymbol{o}_x)^\phi(\boldsymbol{o}_t)^\phi = \\
&= -\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{r}{r - 3GM}}(\boldsymbol{o}_x)^t + r^2\frac{\sqrt{GM}}{r\sqrt{r - 3GM}}(\boldsymbol{o}_x)^\phi
\end{aligned}$$

Then

$$\begin{aligned}
\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{r}{r - 3GM}}(\boldsymbol{o}_x)^t &= \frac{r\sqrt{GM}}{r\sqrt{r - 3GM}}(\boldsymbol{o}_x)^\phi \\
(\boldsymbol{o}_x)^t &= \left(1 - \frac{2GM}{r}\right)^{-1}\sqrt{\frac{r - 3GM}{r}}\frac{r\sqrt{GM}}{\sqrt{r - 3GM}}(\boldsymbol{o}_x)^\phi \\
(\boldsymbol{o}_x)^t &= \sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1}(\boldsymbol{o}_x)^\phi
\end{aligned}$$

Also, for $\boldsymbol{o}_x \cdot \boldsymbol{o}_x$ we have that

$$\begin{aligned} 1 = \boldsymbol{o}_x \cdot \boldsymbol{o}_x = g_{\mu\nu}(\boldsymbol{o}_x)^\mu(\boldsymbol{o}_x)^\nu &= g_{tt}((\boldsymbol{o}_x)^t)^2 + g_{\phi\phi}((\boldsymbol{o}_x)^\phi)^2 \\ &= -\left(1 - \frac{2GM}{r}\right)((\boldsymbol{o}_x)^t)^2 + r^2((\boldsymbol{o}_x)^\phi)^2 \end{aligned}$$

So replacing $(\boldsymbol{o}_x)^t$ we get that

$$\begin{aligned} 1 &= -GMr\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-2}((\boldsymbol{o}_x)^\phi)^2 + r^2((\boldsymbol{o}_x)^\phi)^2 \\ 1 &= ((\boldsymbol{o}_x)^\phi)^2\left(r^2 - GMr\left(1 - \frac{2GM}{r}\right)^{-1}\right) \\ 1 &= ((\boldsymbol{o}_x)^\phi)^2\left(r^2 - \frac{GMr^2}{r - 2GM}\right) \\ 1 &= ((\boldsymbol{o}_x)^\phi)^2 r^2\left(\frac{r - 3GM}{r - 2GM}\right) \\ ((\boldsymbol{o}_x)^\phi)^2 &= \frac{1}{r^2}\left(\frac{r - 2GM}{r - 3GM}\right) \\ (\boldsymbol{o}_x)^\phi &= \frac{1}{r}\sqrt{\frac{r - 2GM}{r - 3GM}} \end{aligned}$$

Then $(\boldsymbol{o}_x)^t$ is

$$\begin{aligned} (\boldsymbol{o}_x)^t &= \sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1}\frac{1}{r}\sqrt{\frac{r - 2GM}{r - 3GM}} \\ &= \frac{\sqrt{GMr}}{r - 2GM}\sqrt{\frac{r - 2GM}{r - 3GM}} \\ &= \sqrt{\frac{GMr}{(r - 2GM)(r - 3GM)}} \end{aligned}$$

Let us compute now $\boldsymbol{o}_y \cdot \boldsymbol{o}_t$ as follows

$$0 = \boldsymbol{o}_y \cdot \boldsymbol{o}_t = g_{\mu\nu}(\boldsymbol{o}_y)^\mu(\boldsymbol{o}_t)^\nu = g_{tt}(\boldsymbol{o}_y)^t(\boldsymbol{o}_t)^t + g_{\theta\theta}(\boldsymbol{o}_y)^\theta(\boldsymbol{o}_t)^\theta$$

Since $(\boldsymbol{o}_t)^\theta = 0$ then must be that $(\boldsymbol{o}_y)^t = 0$. So from $\boldsymbol{o}_y \cdot \boldsymbol{o}_y$ we get $(\boldsymbol{o}_y)^\theta$ as follows

$$1 = \boldsymbol{o}_y \cdot \boldsymbol{o}_y = g_{\mu\nu}(\boldsymbol{o}_y)^\mu(\boldsymbol{o}_y)^\nu = g_{\theta\theta}((\boldsymbol{o}_y)^\theta)^2 = r^2((\boldsymbol{o}_y)^\theta)^2$$

Then $(\boldsymbol{o}_y)^\theta = -1/r$ since it's aligned to $-\theta$. In the same way, for $(\boldsymbol{o}_z)^\mu$ we have that $(\boldsymbol{o}_z)^t = 0$ since $(\boldsymbol{o}_t)^r = 0$ and hence the only non-zero component is

$$1 = \boldsymbol{o}_z \cdot \boldsymbol{o}_z = g_{\mu\nu}(\boldsymbol{o}_z)^\mu(\boldsymbol{o}_z)^\nu = g_{rr}((\boldsymbol{o}_z)^r)^2 = \left(1 - \frac{2GM}{r}\right)^{-1}((\boldsymbol{o}_z)^r)^2$$

Then

$$(\boldsymbol{o}_z)^r = \sqrt{1 - \frac{2GM}{r}}$$

Therefore the set of orthonormal basis vectors is

$$(\boldsymbol{o}_t)^\mu = \begin{bmatrix} \sqrt{\frac{r}{r-3GM}} \\ 0 \\ 0 \\ \frac{\sqrt{GM}}{r\sqrt{r-3GM}} \end{bmatrix} \quad (\boldsymbol{o}_x)^\mu = \begin{bmatrix} \sqrt{\frac{GMr}{(r-2GM)(r-3GM)}} \\ 0 \\ 0 \\ \frac{1}{r}\sqrt{\frac{r-2GM}{r-3GM}} \end{bmatrix}$$

$$(\boldsymbol{o}_y)^\mu = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{r} \\ 0 \end{bmatrix} \quad (\boldsymbol{o}_z)^\mu = \begin{bmatrix} 0 \\ \sqrt{1 - \frac{2GM}{r}} \\ 0 \\ 0 \end{bmatrix}$$

If $r \leq 3GM$ then l^2 will be negative hence no circular orbits can exist for $r \leq 3GM$. $\square$