Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 16 - Blackhole Thermodynamics

Solution. **BOX 16.1** - Exercise 16.1.1.
From BOX 14.2 we know a free-falling object falling from $2GM + \varepsilon$ has

$$e^2 = 1 - \frac{2GM}{2GM + \varepsilon}$$

Then replacing into the equation (16.12) where $l = 0$ we get that

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{\left(1 - \frac{2GM}{2GM + \varepsilon}\right) - \left(1 - \frac{2GM}{r}\right)}$$
$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{\frac{2GM}{r} - \frac{2GM}{2GM + \varepsilon}}$$

Now, integrating between $2GM + \varepsilon$ and $2GM$ we see that

$$\Delta\tau = -\int_{2GM+\varepsilon}^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r} - \frac{2GM}{2GM+\varepsilon}}}$$

Let $\rho = r - 2GM$ then $d\rho = dr$ and inverting the integral limits we have that

$$\Delta\tau = +\int_{0}^{\varepsilon} \frac{d\rho}{\sqrt{\frac{2GM}{2GM+\rho} - \frac{2GM}{2GM+\varepsilon}}}$$

□

Solution. **BOX 16.1** - Exercise 16.1.2.
Let us write equation (16.13) as follows

$$
\begin{aligned}
\Delta\tau &= \int_0^{\varepsilon} \frac{d\rho}{\sqrt{\frac{2GM}{2GM+\rho} - \frac{2GM}{2GM+\varepsilon}}} \\
&= \int_0^{\varepsilon} \frac{d\rho}{\sqrt{\frac{1}{1+\rho/2GM} - \frac{1}{1+\varepsilon/2GM}}} \\
&= \int_0^{\varepsilon} \frac{d\rho}{\sqrt{(1+\rho/2GM)^{-1} - (1+\varepsilon/2GM)^{-1}}}
\end{aligned}
$$

Then if we let $\varepsilon \ll 2GM$ applying the binomial approximation gives us

$$
\begin{aligned}
\Delta\tau &\approx \int_0^{\varepsilon} \frac{d\rho}{\sqrt{(1-\rho/2GM) - (1-\varepsilon/2GM)}} \\
&\approx \sqrt{2GM} \int_0^{\varepsilon} \frac{d\rho}{\sqrt{\varepsilon-\rho}} \\
&\approx \sqrt{2GM} \Big[-2\sqrt{\varepsilon-\rho} \Big]_0^{\varepsilon} \\
&\approx 2\sqrt{2GM\varepsilon}
\end{aligned}
$$

$\square$

Solution. **BOX 16.2** - Exercise 16.2.1.
Given that the spatial components of the frame's four-velocity will be zero at the time the particle-antiparticle pair is formed, then the only nonzero component is u^t then

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{u} = g_{\mu\nu} u^\mu u^\nu &= -1 \\ -\left(1 - \frac{2GM}{r}\right)(u^t)^2 &= -1 \\ u^t &= \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \end{aligned}$$

□

Solution. **BOX 16.2** - Exercise 16.2.2.
We compute $E = -\boldsymbol{o}_t \cdot \boldsymbol{p} = -\boldsymbol{u} \cdot \boldsymbol{p}$ as follows

$$E = -g_{\mu\nu} u^\mu p^\nu = \left(1 - \frac{2GM}{r}\right) \frac{1}{\sqrt{1 - \frac{2GM}{r}}} m \frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1 - \frac{2GM}{r}} m \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

Where we used again that $u^r = u^\theta = u^\phi = 0$ initially. □

Solution. **BOX 16.2** - Exercise 16.2.3.
We know that

$$e = \left(1 - \frac{2GM}{r}\right) \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

Then mutiplying the equation by m we get that

$$\begin{aligned} me &= \left(1 - \frac{2GM}{r}\right) m \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ E_\infty &= \left(1 - \frac{2GM}{r}\right) \frac{E}{\sqrt{1 - \frac{2GM}{r}}} \\ E_\infty &= \sqrt{1 - \frac{2GM}{r}} E \end{aligned}$$

Hence

$$E = \frac{E_\infty}{\sqrt{1 - \frac{2GM}{r}}}$$

Where E_∞ is the particle's energy at infinity.

□

Solution. **BOX 16.3** - Exercise 16.3.1.
We know that in SI $k_B = 1.3807 \times 10^{-23} J/K$ and $\hbar = 1.0546 \times 10^{-34} Js$ then

$$\frac{\hbar}{8\pi k_B G M_\odot} = \frac{1.0546 \times 10^{-34}\ Js \cdot (299800000\ m/1\ s)}{8\pi \cdot 1.3807 \times 10^{-23} J/K \cdot 1477\ m} = 6.168 \times 10^{-8} K$$

□

Solution. **BOX 16.4** - Exercise 16.4.1.
We know that a black hole's mass-energy M is radiated away following the Stefan-Boltzmann formula

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A\sigma T^4$$

Hence, we need to integrate as follows

$$\int_0^\tau dt = \frac{1}{\sigma}\int_0^M \frac{dM}{AT^4}$$

So, using that $T = \hbar/8\pi k_B G M$ and $A = 4\pi(2GM)^2$ we get that

$$\tau = \frac{(8\pi k_B G)^4}{16\pi G^2 \sigma \hbar^4}\int_0^M M^2 dM = \frac{8^4 \pi^3 k_B^4 G^2}{16\sigma\hbar^4}\frac{M^3}{3} = \frac{256\pi^3 k_B^4}{3G\sigma\hbar^4}(GM)^3$$

□

Solution. **BOX 16.4** - Exercise 16.4.2.
Using that $k_B = 1.536 \times 10^{-40}\ kg/K$, $\hbar = 3.518 \times 10^{-43}\ kgm$, and $\sigma = 2.105 \times 10^{-33}\ kgm^{-3}K^{-4}$ we have that

$$\begin{aligned}\tau &= \frac{256\pi^3(1.536 \times 10^{-40})^4}{3 \cdot (7.426 \times 10^{-28})(2.105 \times 10^{-33})(3.518 \times 10^{-43})^4}(1477)^3\left(\frac{M}{M_\odot}\right)^3 \\ &= 1.982 \times 10^{83}\ m \cdot \frac{1\ y}{9.461 \times 10^{15}\ m} \cdot \left(\frac{M}{M_\odot}\right)^3 \\ &= (2.095 \times 10^{67}\ y)\left(\frac{M}{M_\odot}\right)^3\end{aligned}$$

□

Solution. **P16.1** From equation 16.6 we know that the relativistic energy at infinity is

$$E_\infty = \frac{\hbar}{4GM}$$

If M is a solar mass then we get that

$$E_\infty = \frac{3.518 \times 10^{-43}\ kg \cdot m}{4 \cdot 1477\ m} = 5.95 \times 10^{-47}\ kg \approx 6 \times 10^{-47}\ kg$$

Or in eV

$$E_\infty = 5.95 \times 10^{-47}\ kg \cdot \frac{1\ eV}{1.782 \times 10^{-36}\ kg} = 3.33 \times 10^{-11}\ eV$$

□

Solution. **P16.2**

a. Using equation 16.9 we know that a blackhole survives for

$$\tau_{\text{life}} = (2.095 \times 10^{67}\ y)\left(\frac{M}{M_\odot}\right)^3$$

So if a balckhole is just evaporating today, then we can compute how massive it was in the beginning as follows

$$\begin{aligned}(2.095 \times 10^{67}\ y)\left(\frac{M}{M_\odot}\right)^3 &= 13.7 \times 10^9\ y \\ \frac{M}{M_\odot} &= \sqrt[3]{\frac{13.7 \times 10^9}{2.095 \times 10^{67}}} \\ M &= 8.679 \times 10^{-20} M_\odot\end{aligned}$$

So the black hole was 8.679×10^{-20} solar masses or $173.58 \times 10^9\ kg$.

b. We want to know now how much energy a black hole releases in the last second of its life. So, from the Stefan-Boltzmann formula we know that

$$-\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}E_{rad}}{\mathrm{d}t} = A\sigma T^4$$

We need to integrate it to get the energy radiated but we integrate with respect to the mass. So we get again that

$$1\ s = (2.095 \times 10^{67}\ y)\left(\frac{M}{M_\odot}\right)^3$$

Where we inegrated t from 0 to 1 second. Then solving for $M = E_{rad}$ we see that

$$\begin{aligned}E_{rad} = M &= \sqrt[3]{\frac{3.17098 \times 10^{-8}\ y}{2.095 \times 10^{67}\ y}} M_\odot \\ &= 229632.39\ kg \cdot \frac{1\ J}{1.1126 \times 10^{-17}\ kg} \\ &= 2.06 \times 10^{22}\ J\end{aligned}$$

This is 51.5×10^6 times an atomic bomb.

$\square$

Solution. **P16.3** A black hole of 1 TeV mass or $1.782 \times 10^{-24}\ kg$ will have a lifetime of

$$\begin{aligned}
\tau &= (2.095 \times 10^{67}\ y)\left(\frac{1.782 \times 10^{-24}\ kg}{2 \times 10^{30}\ kg}\right)^3 \\
&= 2.095 \times 10^{67} \cdot 7.073 \times 10^{-163}\ y \\
&= 1.481 \times 10^{-95}\ y = 4.67 \times 10^{-88}\ s
\end{aligned}$$

□

Solution. **P16.4**

a. From equation 14.3 we know that the physical distance from $r = R$ to $r = 2GM$ is

$$\Delta s = R\sqrt{1-\frac{2GM}{r}} + 2GM\tanh^{-1}\sqrt{1-\frac{2GM}{r}}$$

We know that the box can be lowered down at most a physical distance from the event horizon of $\Delta s = \lambda/2$, also, applying the approximation $\tanh^{-1} x \approx x$ and assuming that $R \approx 2GM$ we get that

$$\begin{aligned}\frac{\lambda}{2} &\approx 2GM\sqrt{1-\frac{2GM}{r}} + 2GM\sqrt{1-\frac{2GM}{r}}\\ \frac{\lambda}{2} &\approx 4GM\sqrt{1-\frac{2GM}{r}}\\ \left(\frac{\lambda}{2}\right)^2 &\approx (4GM)^2\left(1-\frac{2GM}{r}\right)\end{aligned}$$

b. From equation 10.8 we know that

$$\frac{1}{2}(e^2-1) = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}$$

Using that $dr/d\tau = 0$ and $l = 0$ since the object is at rest we get that

$$e = \sqrt{1-\frac{2GM}{r}}$$

Also, from part (a) we see that

$$\begin{aligned}\left(\frac{\lambda}{8GM}\right)^2 &\approx \left(1-\frac{2GM}{r}\right)\\ \frac{\lambda}{8GM} &\approx \sqrt{1-\frac{2GM}{r}}\end{aligned}$$

Therefore

$$e \approx \frac{\lambda}{8GM}$$

c. If we consider that at the beginning the energy per unit of mass of the box is $e = 1$, close to the event horizon is e and that the difference $1 - e$ has become work done on the winch, then the efficiency must be $1 - e$ since all the energy has become work.

Then must be that $e = T_C/T_H$ but we know that $\lambda \approx \hbar/k_B T_H$ so

$$e = \frac{\lambda}{8GM} = \frac{\lambda k_B T_C}{\hbar} = \frac{T_C}{T_H}$$

Therefore T_C must be $T_C = \hbar/8GMk_B$.

□

Solution. **P16.5** According to equation (12.15) observers at $r = R$ will see photons with energy E_{obs}

$$E_{obs} = \frac{E_\infty}{\sqrt{1 - 2GM/R}}$$

Where E_∞ is the photon's energy at infinity. So an observer at $r = R$ will consider the temperature of the black hole to be higher by a factor of $1/\sqrt{1 - 2GM/R}$ than that of an observer at infinity.
As $R \to 2GM$ the temperature tends to infinity.
Let us note that

$$k_B T = E_{obs} = \frac{E_\infty}{\sqrt{1 - 2GM/R}} = \frac{k_B T_\infty}{\sqrt{1 - 2GM/R}}$$

Then

$$T = \frac{T_\infty}{\sqrt{1 - 2GM/R}}$$

So since we are considering a solar-mass black hole we have that $T_\infty = 6.17 \times 10^{-8}\ K$. Therefore to observe a temperature of $T = 300\ K$ we need to be at the following distance (in R) from the event horizon

$$\begin{aligned} R - 2GM &= \frac{2GM}{1 - (T_\infty/T)^2} - 2GM \\ &= 2 \cdot 1477 \left(\frac{1}{1 - (6.17 \times 10^{-8}/300)^2} - 1 \right) \\ &= 1.249 \times 10^{-16}\ m \end{aligned}$$

□

Solution. **P16.6** We know from equation (16.18) that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -A\sigma T^4 = -\frac{4\pi(4G^2M^2)\sigma\hbar^4}{4096\pi^4 k_B^4 G^4 M^4} = -\frac{\sigma\hbar^4}{256\pi^3 k_B^4 G^2 M^2}$$

Where we replaced the values we have for A and T. Then by integration we get that

$$\begin{aligned}
\int_{M_0}^{M} M^2\ dM &= -\frac{\sigma\hbar^4}{256\pi^3 k_B^4 G^2}\int_0^t\ dt \\
\frac{M^3}{3} - \frac{M_0^3}{3} &= -\frac{\sigma\hbar^4}{256\pi^3 k_B^4 G^2} t \\
M^3 &= -\frac{3\sigma\hbar^4}{256\pi^3 k_B^4 G^2} t + M_0^3
\end{aligned}$$

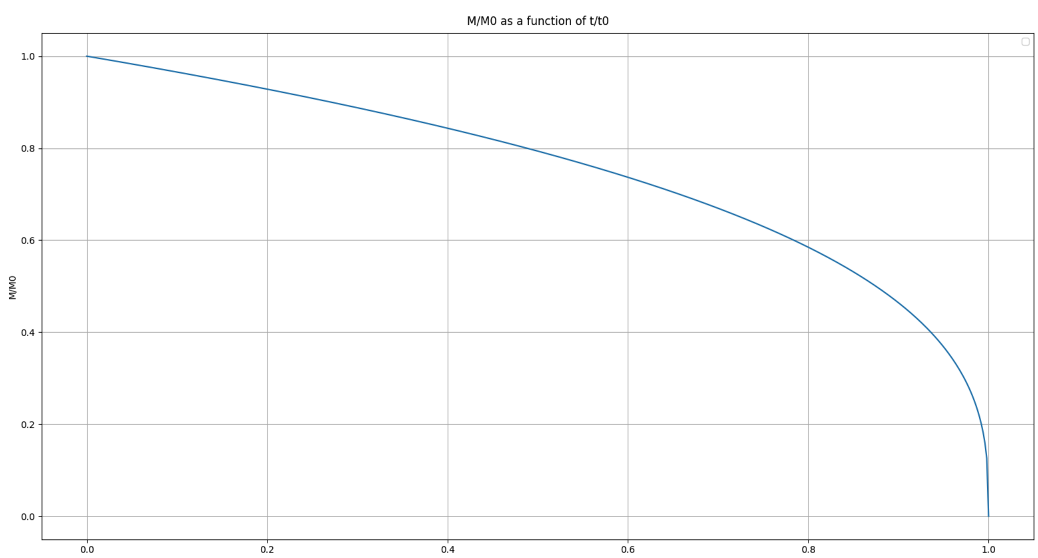
But we know that the lifetime of a black hole is

$$t_0 = \frac{256\pi^3 k_B^4 G^2}{3\sigma\hbar^4} M_0^3$$

Therefore

$$\begin{aligned}
M^3 &= -\frac{M_0^3}{t_0} t + M_0^3 \\
M^3 &= M_0^3\left(1 - \frac{t}{t_0}\right) \\
M &= M_0\sqrt[3]{1 - \frac{t}{t_0}}
\end{aligned}$$

Below we leave a plot of M/M_0 as a function of t/t_0



□

Solution. **P16.7** Let two black holes have masses M_1 and M_2 then the entropy of each is

$$S_1 = \frac{4\pi k_B G}{\hbar} M_1^2 \quad \text{and} \quad S_2 = \frac{4\pi k_B G}{\hbar} M_2^2$$

So the total entropy is

$$S_1 + S_2 = \frac{4\pi k_B G}{\hbar} M_1^2 + \frac{4\pi k_B G}{\hbar} M_2^2 = \frac{4\pi k_B G}{\hbar}(M_1^2 + M_2^2)$$

Now, if we consider they merge toghether to a new black hole of mass $M_1 + M_2$ then the entropy of this new black hole would be

$$S_M = \frac{4\pi k_B G}{\hbar}(M_1 + M_2)^2$$

And since $M_1^2 + M_2^2 < M_1^2 + 2M_1M_2 + M_2^2 = (M_1 + M_2)^2$ we see that

$$S_1 + S_2 < S_M$$

□

Solution. **P16.8** Let us consider a system composed of a thermal reservoir and a black hole, then the entropy of the reservoir is

$$S_R = \frac{U_{tot} - M}{T_R} + C$$

Where U_{tot} is the total conserved energy of the system, M is the mass of the black hole, T_R is the constant temperature of the reservoir and C is a constant.
Also, the entropy of the black hole we know it is

$$S_{BH} = \frac{4\pi k_B G}{\hbar} M^2$$

So the total entropy of the system is

$$S_{tot} = S_{BH} + S_R = \frac{4\pi k_B G}{\hbar} M^2 - \frac{M}{T_R} + \left(\frac{U_{tot}}{T_R} + C \right)$$

We know that a stable equilibrium is when energy is distributed in such a way that the total entropy of the system is a local maximum.
In this case, S_{tot} is a quadratic equation of M and since the term involving M^2 is positive then the branches point upward, hence the only extremum point point is a minimum.
Also, S_{tot} cannot take values such that $M < 0$ and $M > U_{tot}$ so the only local maximum of the quadratic equation can happen for $M = 0$ and $M = U_{tot}$ i.e. the extremes. $\square$

Solution. **P16.9** Let us compute the entropy S of a solar-mass black hole as follows

$$S = \frac{4\pi k_B G M^2}{\hbar} = \frac{4\pi \cdot 1.536 \times 10^{-40} \cdot 1477 \cdot 2 \times 10^{30}}{3.518 \times 10^{-43}} = 1.621 \times 10^{37}\ kg/K$$

On the other hand, if we consider that the black hole was formed by pure ionized hydrogen (protons), then the number of particles forming it is

$$N = \frac{M}{m_p} = \frac{2 \times 10^{30}}{1.672 \times 10^{-27}} = 1.196 \times 10^{57}$$

Therefore the ratio S/Nk_B for a solar-mass blackhole is

$$\frac{S}{Nk_B} = \frac{1.621 \times 10^{37}}{1.196 \times 10^{57} \cdot 1.536 \times 10^{-40}} = 8.823 \times 10^{19}$$

In comparison to the normal matter, where the ratio is $S/Nk_B = 1$, we see that it is 19 orders of magnitude bigger. $\square$