Solved selected problems of General Relativity - Thomas A. Moore

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Chapter 17 - The Absolute Gradient

Solution. **BOX 17.1** - Exercise 17.1.1.
Let $d\boldsymbol{A} = d(A^\mu \boldsymbol{e}_\mu)$ then by the product rule we have that

$$d(A^\mu \boldsymbol{e}_\mu) = \frac{\partial A^\mu}{\partial x^\sigma} dx^\sigma \boldsymbol{e}_\mu + A^\mu \frac{\partial \boldsymbol{e}_\mu}{\partial x^\sigma} dx^\sigma$$

Also, by definition we know that $\partial \boldsymbol{e}_\alpha / \partial x^\mu = \Gamma^\nu_{\mu\alpha} \boldsymbol{e}_\nu$ so

$$\begin{aligned} d(A^\mu \boldsymbol{e}_\mu) &= \frac{\partial A^\mu}{\partial x^\sigma} dx^\sigma \boldsymbol{e}_\mu + A^\nu \frac{\partial \boldsymbol{e}_\nu}{\partial x^\sigma} dx^\sigma \\ &= \frac{\partial A^\mu}{\partial x^\sigma} dx^\sigma \boldsymbol{e}_\mu + A^\nu \Gamma^\mu_{\sigma\nu} \boldsymbol{e}_\mu dx^\sigma \\ &= \left[\frac{\partial A^\mu}{\partial x^\sigma} + A^\nu \Gamma^\mu_{\sigma\nu} \right] \boldsymbol{e}_\mu dx^\sigma \end{aligned}$$

Where in the first step we renamed in the second term μ to ν. □

Solution. **BOX 17.2** - Exercise 17.2.1.
From equation (17.17) we know that

$$\nabla_\alpha (A^\mu B_\mu) = \left[\frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\nu} A^\nu \right] B_\mu + A^\mu (\nabla_\alpha B_\mu)$$

And from equation (17.18) we know that

$$\nabla_\alpha (A^\mu B_\mu) = \frac{\partial A^\mu}{\partial x^\alpha} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha}$$

Then

$$\begin{aligned} \frac{\partial A^\mu}{\partial x^\alpha} B_\mu + \Gamma^\mu_{\alpha\nu} A^\nu B_\mu + A^\mu (\nabla_\alpha B_\mu) &= \frac{\partial A^\mu}{\partial x^\alpha} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha} \\ \Gamma^\mu_{\alpha\nu} A^\nu B_\mu + A^\mu (\nabla_\alpha B_\mu) &= A^\mu \frac{\partial B_\mu}{\partial x^\alpha} \\ -A^\mu \frac{\partial B_\mu}{\partial x^\alpha} + \Gamma^\sigma_{\alpha\mu} A^\mu B_\sigma + A^\mu (\nabla_\alpha B_\mu) &= 0 \\ A^\mu \left[-\frac{\partial B_\mu}{\partial x^\alpha} + \Gamma^\sigma_{\alpha\mu} B_\sigma + \nabla_\alpha B_\mu \right] &= 0 \end{aligned}$$

Where in the third step we renamed the ν index to μ and the μ index to σ. □

Solution. **BOX 17.4** - Exercise 17.4.1.
Equations (17.22), (17.23) and (17.24) state that

$$\Gamma^{\nu}_{\mu\alpha} g_{\nu\rho} + \Gamma^{\nu}_{\rho\alpha} g_{\nu\mu} = \partial_\alpha g_{\mu\rho}$$
$$\Gamma^{\nu}_{\rho\mu} g_{\nu\alpha} + \Gamma^{\nu}_{\alpha\mu} g_{\nu\rho} = \partial_\mu g_{\rho\alpha}$$
$$\Gamma^{\nu}_{\alpha\rho} g_{\nu\mu} + \Gamma^{\nu}_{\mu\rho} g_{\nu\alpha} = \partial_\rho g_{\alpha\mu}$$

Then adding (17.22) to (17.23) and subtracting (17.24) we get that

$$\begin{aligned}
\Gamma^{\nu}_{\mu\alpha} g_{\nu\rho} + \Gamma^{\nu}_{\rho\alpha} g_{\nu\mu} + \Gamma^{\nu}_{\rho\mu} g_{\nu\alpha} + \Gamma^{\nu}_{\alpha\mu} g_{\nu\rho} - \Gamma^{\nu}_{\alpha\rho} g_{\nu\mu} - \Gamma^{\nu}_{\mu\rho} g_{\nu\alpha} &= \partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu} \\
g_{\nu\rho}(\Gamma^{\nu}_{\mu\alpha} + \Gamma^{\nu}_{\alpha\mu}) + g_{\nu\mu}(\Gamma^{\nu}_{\rho\alpha} - \Gamma^{\nu}_{\alpha\rho}) + g_{\nu\alpha}(\Gamma^{\nu}_{\rho\mu} - \Gamma^{\nu}_{\mu\rho}) &= \partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu} \\
g_{\nu\rho}(\Gamma^{\nu}_{\mu\alpha} + \Gamma^{\nu}_{\mu\alpha}) + g_{\nu\mu}(\Gamma^{\nu}_{\rho\alpha} - \Gamma^{\nu}_{\rho\alpha}) + g_{\nu\alpha}(\Gamma^{\nu}_{\rho\mu} - \Gamma^{\nu}_{\rho\mu}) &= \partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu}
\end{aligned}$$

Where we used the symmetry in the lower indices of the Christoffel symbols, then

$$2 g_{\nu\rho} \Gamma^{\nu}_{\mu\alpha} = \partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu}$$

So multiplying this equation by $\frac{1}{2} g^{\sigma\rho}$ and using that $g^{\sigma\rho} g_{\nu\rho} = \delta^{\sigma}_{\nu}$ we get that

$$\begin{aligned}
g^{\sigma\rho} g_{\nu\rho} \Gamma^{\nu}_{\mu\alpha} &= \frac{1}{2} g^{\sigma\rho} [\partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu}] \\
\delta^{\sigma}_{\nu} \Gamma^{\nu}_{\mu\alpha} &= \frac{1}{2} g^{\sigma\rho} [\partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu}] \\
\Gamma^{\sigma}_{\mu\alpha} &= \frac{1}{2} g^{\sigma\rho} [\partial_\alpha g_{\mu\rho} + \partial_\mu g_{\rho\alpha} - \partial_\rho g_{\alpha\mu}]
\end{aligned}$$

□

Solution. **BOX 17.5** - Exercise 17.5.1.
Equation (17.27) states that

$$\frac{1}{2}\partial_\sigma g_{\mu\nu}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}+\frac{1}{2}\partial_\sigma g_{\mu\nu}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}+g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}-\frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=0$$

Renaming $\sigma\to\alpha$, $\nu\to\beta$ in the first term and $\sigma\to\beta,\nu\to\alpha$ in the second term we get that

$$\frac{1}{2}\partial_\alpha g_{\mu\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}+\frac{1}{2}\partial_\beta g_{\mu\alpha}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}+g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}-\frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=0$$

So, multiplying the equation by $g^{\sigma\mu}$ and using the fact that $g^{\sigma\mu}g_{\mu\nu}=\delta^\sigma_\nu$ we see that

$$\frac{1}{2}g^{\sigma\mu}\left[\partial_\alpha g_{\mu\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}+\partial_\beta g_{\mu\alpha}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}-\partial_\mu g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right]+\delta^\sigma_\nu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}=0$$
$$\frac{1}{2}g^{\sigma\mu}\left[\partial_\alpha g_{\mu\beta}+\partial_\beta g_{\mu\alpha}-\partial_\mu g_{\alpha\beta}\right]\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}+\frac{\mathrm{d}^2x^\sigma}{\mathrm{d}\tau^2}=0$$

Finally, using equation (17.10) we have that

$$\Gamma^\sigma_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}+\frac{\mathrm{d}^2x^\sigma}{\mathrm{d}\tau^2}=0$$

Which renaming $\sigma\to\mu$ gives us equation (17.12). $\square$

Solution. **BOX 17.6** - Exercise 17.6.1.
Let $\mu = \theta$ then the geodesic equation becomes

$$\partial_\sigma g_{\theta\nu} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\theta\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} - \frac{1}{2} \partial_\theta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

For the first and second term, since the metric is diagonal the only non-zero term is $g_{\theta\theta}$ and this term only depends on r so the only derivative that is non-zero is when $\sigma = r$.
For the last term, the only component that depends on θ is the $g_{\phi\phi}$ component, then the equation reduces to

$$\begin{aligned}
\partial_r g_{\theta\theta} \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + g_{\theta\theta} \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - \frac{1}{2} \partial_\theta g_{\phi\phi} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= 0 \\
\partial_r (r^2) \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + r^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - \frac{1}{2} \partial_\theta (r^2 \sin^2\theta) \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= 0 \\
2r \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + r^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - r^2 \sin\theta \cos\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= 0 \\
\frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - \sin\theta \cos\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= 0
\end{aligned}$$

Therefore, from the 16 Christoffel symbols $\Gamma^\theta_{\mu\nu}$ that have θ as a superscript the only non-zero terms are

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \quad \text{and} \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$\square$

Solution. **BOX 17.7** - Exercise 17.7.1.
Let us list all 20 valid combinations of the three indices as follows

333	233	322	311	300	012
222	133	122	211	200	123
111	033	022	011	100	230
000					301

□

Solution. **BOX 17.7** - Exercise 17.7.2.
From equation (17.35) we know that

$$\boldsymbol{g}' = [\boldsymbol{a} + \boldsymbol{b}\Delta\boldsymbol{x}' + \boldsymbol{c}(\Delta\boldsymbol{x}')^2 + ...]^2[\boldsymbol{g}_P + \partial\boldsymbol{g}_P\Delta\boldsymbol{x}' + \frac{1}{2}\partial^2\boldsymbol{g}_P(\Delta\boldsymbol{x}')^2 + ...]$$

Then expanding this expression we get that

$$\begin{aligned}\boldsymbol{g}' = {} & [\boldsymbol{a}^2 + \boldsymbol{ab}\Delta\boldsymbol{x}' + \boldsymbol{ac}(\Delta\boldsymbol{x}')^2 + \boldsymbol{ba}\Delta\boldsymbol{x}' + \boldsymbol{b}^2(\Delta\boldsymbol{x}')^2 + \boldsymbol{bc}(\Delta\boldsymbol{x}')^3 + \boldsymbol{ca}(\Delta\boldsymbol{x}')^2 \\ & + \boldsymbol{cb}(\Delta\boldsymbol{x}')^3 + \boldsymbol{c}^2(\Delta\boldsymbol{x}')^4 + ...][\boldsymbol{g}_P + \partial\boldsymbol{g}_P\Delta\boldsymbol{x}' + \frac{1}{2}\partial^2\boldsymbol{g}_P(\Delta\boldsymbol{x}')^2 + ...] \\ = {} & [\boldsymbol{a}^2 + [\boldsymbol{ab} + \boldsymbol{ba}]\Delta\boldsymbol{x}' + [\boldsymbol{ac} + \boldsymbol{b}^2 + \boldsymbol{ca}](\Delta\boldsymbol{x}')^2 + ...][\boldsymbol{g}_P + \partial\boldsymbol{g}_P\Delta\boldsymbol{x}' + \frac{1}{2}\partial^2\boldsymbol{g}_P(\Delta\boldsymbol{x}')^2 + ...] \\ = {} & \boldsymbol{a}^2\boldsymbol{g}_P + \boldsymbol{a}^2\partial\boldsymbol{g}_P\Delta\boldsymbol{x}' + \boldsymbol{a}^2\frac{1}{2}\partial^2\boldsymbol{g}_P(\Delta\boldsymbol{x}')^2 + [\boldsymbol{ab} + \boldsymbol{ba}]\boldsymbol{g}_P\Delta\boldsymbol{x}' + [\boldsymbol{ab} + \boldsymbol{ba}]\partial\boldsymbol{g}_P(\Delta\boldsymbol{x}')^2 \\ & + \boldsymbol{g}_P[\boldsymbol{ac} + \boldsymbol{b}^2 + \boldsymbol{ca}](\Delta\boldsymbol{x}')^2 + ... \\ = {} & \boldsymbol{a}^2\boldsymbol{g}_P + [\boldsymbol{a}^2\partial\boldsymbol{g}_P + [\boldsymbol{ab} + \boldsymbol{ba}]\boldsymbol{g}_P]\Delta\boldsymbol{x}' \\ & + [\boldsymbol{a}^2\frac{1}{2}\partial^2\boldsymbol{g}_P + [\boldsymbol{ab} + \boldsymbol{ba}]\partial\boldsymbol{g}_P + \boldsymbol{g}_P[\boldsymbol{ac} + \boldsymbol{b}^2 + \boldsymbol{ca}]](\Delta\boldsymbol{x}')^2 + ...\end{aligned}$$

Where we removed the higher order terms. Therefore we get that

$$\begin{aligned}\boldsymbol{g}' = {} & \boldsymbol{a}^2\boldsymbol{g}_P + [\boldsymbol{a}^2\partial\boldsymbol{g}_P + \boldsymbol{abg}_P + \boldsymbol{bag}_P]\Delta\boldsymbol{x}' \\ & + [\frac{1}{2}\boldsymbol{a}^2\partial^2\boldsymbol{g}_P + \boldsymbol{ab}\partial\boldsymbol{g}_P + \boldsymbol{ba}\partial\boldsymbol{g}_P + \boldsymbol{acg}_P + \boldsymbol{b}^2\boldsymbol{g}_P + \boldsymbol{cag}_P](\Delta\boldsymbol{x}')^2 + ...\end{aligned}$$

□

Solution. **P17.1**
We know that in polar-coordinates

$$ds^2 = dr^2 + r^2 d\theta^2$$

So the geodesic equation for $\mu = r$ is

$$\partial_\sigma g_{r\nu} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{r\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} - \frac{1}{2} \partial_r g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

For the first and second term, since the metric is diagonal the only non-zero term is g_{rr} and it's constant so every derivative is 0.
For the last term, the only component that depends on r is the $g_{\theta\theta}$ component, then the equation reduces to

$$\begin{aligned} g_{rr} \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \frac{1}{2} \partial_r g_{\theta\theta} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 0 \\ \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - r \frac{\mathrm{d}\theta}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 0 \end{aligned}$$

Comparing this to the 4 Christoffel symbols $\Gamma^r_{\mu\nu}$ that have r as a superscript the only-non-zero term is

$$\Gamma^r_{\theta\theta} = -r$$

Now, if we let $\mu = \theta$ the geodesic equation becomes

$$\partial_\sigma g_{\theta\nu} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\theta\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} - \frac{1}{2} \partial_\theta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

For the first and second term, since the metric is diagonal the only non-zero term is $g_{\theta\theta}$ and it depends only on r so the r-derivative is the only non-zero derivative.
For the last term, no component of the metric depends on θ so all derivatives with respect to θ are zero. Then the equation becomes

$$\begin{aligned} \partial_r g_{\theta\theta} \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + g_{\theta\theta} \frac{\mathrm{d}^2 \theta}{\mathrm{d}\tau^2} &= 0 \\ 2r \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + r^2 \frac{\mathrm{d}^2 \theta}{\mathrm{d}\tau^2} &= 0 \\ \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}\tau} \frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{\mathrm{d}^2 \theta}{\mathrm{d}\tau^2} &= 0 \end{aligned}$$

Comparing this to the 4 Christoffel symbols $\Gamma^\theta_{\mu\nu}$ that have θ as a superscript the only-non-zero terms are

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

Therefore, from the 8 Christoffel symbols for polar coordinates only 3 are non-zero. Finally, we can check that the result we got for $\Gamma^r_{\theta\theta}$ matches to the result given by equation (17.10) by computing it as follows

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{r\sigma} [\partial_\theta g_{\theta\sigma} + \partial_\theta g_{\sigma\theta} - \partial_\sigma g_{\theta\theta}]$$

Given that no metric component depends on θ we can drop the first two terms, and since the metric inverse is also diagonal the only non-zero term is g^{rr}, then we get that

$$\Gamma^r_{\theta\theta} = -\frac{1}{2} g^{rr} \partial_r g_{\theta\theta} = -\frac{1}{2} \frac{1}{g_{rr}} (2r) = -\frac{1}{2} \cdot 1 \cdot (2r) = -r$$

Since the metric is diagonal, we used that $g^{\mu\nu} = 1/g_{\mu\nu}$ and we see that the result matches the previous one. $\square$

Solution. **P17.2**
We computed the Christoffel symbols for when t and θ are superscripts in BOX 17.6, so we have to compute Christoffel symbols for when r and ϕ are superscripts.
Let us start considering the metric equation for $\mu = r$.

$$\partial_\sigma g_{r\nu}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{r\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} - \frac{1}{2}\partial_r g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

For the first and second term, since the metric is diagonal the only non-zero term is g_{rr} and this term only depends on r so the only derivative that is non-zero is when $\sigma = r$.
Then the equation reduces to

$$\begin{aligned}
&\partial_r g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + g_{rr}\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} \\
&\quad - \frac{1}{2}\left[\partial_r g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} + \partial_r g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + \partial_r g_{\theta\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \partial_r g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = 0
\end{aligned}$$

$$\begin{aligned}
&-\frac{2GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + \left(1-\frac{2GM}{r}\right)^{-1}\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} \\
&\quad - \frac{1}{2}\left[-\frac{2GM}{r^2}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} - \frac{2GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + 2r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + 2r\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = 0
\end{aligned}$$

$$\begin{aligned}
&-\frac{2GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} \\
&\quad - \frac{1}{2}\left[-\frac{2GM}{r^2}\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} - \frac{2GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right. \\
&\quad \left. + 2r\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + 2r\sin^2\theta\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = 0
\end{aligned}$$

$$\begin{aligned}
&-\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&\quad - r\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - r\sin^2\theta\left(1-\frac{2GM}{r}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 0
\end{aligned}$$

So from the 16 Christoffel symbols $\Gamma^r_{\mu\nu}$ that have r as a superscript the only non-zero terms are

$$\Gamma^r_{tt} = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) \qquad \Gamma^r_{rr} = -\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}$$

$$\Gamma^r_{\theta\theta} = -r\left(1-\frac{2GM}{r}\right) \qquad \Gamma^r_{\phi\phi} = -r\sin^2\theta\left(1-\frac{2GM}{r}\right)$$

Now, let us consider the metric equation for $\mu = \phi$.

$$\partial_\sigma g_{\phi\nu}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\phi\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} - \frac{1}{2}\partial_\phi g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

For the first and second term, since the metric is diagonal the only non-zero term is $g_{\phi\phi}$ and this term depends only on r and θ so the only derivatives that are non-zero are when $\sigma = r, \theta$.
For the last term, no component of the metric depends on ϕ so all derivatives with respect to ϕ are zero. Then the equation becomes

$$\begin{aligned}\partial_r g_{\phi\phi}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \partial_\theta g_{\phi\phi}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + g_{\phi\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} &= 0\\ 2r\sin^2\theta\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + 2r^2\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + r^2\sin^2\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} &= 0\\ \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + 2\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} &= 0\\ \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} &= 0\end{aligned}$$

Therefore from the 16 Christoffel symbols $\Gamma^{\phi}_{\mu\nu}$ that have ϕ as a superscript the only non-zero terms are

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r} \qquad\qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

Finally, we can check that the result we got for $\Gamma^{\phi}_{r\phi}$ matches to the result given by equation (17.10) by computing it as follows

$$\Gamma^{\phi}_{r\phi} = \frac{1}{2}g^{\phi\sigma}[\partial_r g_{\phi\sigma} + \partial_\phi g_{\sigma r} - \partial_\sigma g_{r\phi}]$$

Given that no metric component depends on ϕ we can drop the second term, the third term is zero as well since the metric is diagonal. Also, since the metric is diagonal the only non-zero terms are when $\sigma = \phi$, then the equation becomes

$$\Gamma^{\phi}_{r\phi} = \frac{1}{2}g^{\phi\phi}[\partial_r g_{\phi\phi}] = \frac{1}{2}\frac{1}{g_{\phi\phi}}[2r\sin^2\theta] = \frac{1}{2}\frac{1}{r^2\sin^2\theta}[2r\sin^2\theta] = \frac{1}{r}$$

Where we used that $g^{\mu\nu} = 1/g_{\mu\nu}$ since the metric is diagonal. $\square$

Solution. **P17.4**
Let us take a LIF centered on a given event, then the absolute gradient reduces to the ordinary gradient and hence we get that

$$\begin{aligned}\nabla_\beta(B^\mu_\nu A^\alpha) &= \partial_\beta(B^\mu_\nu A^\alpha)\\ &= (\partial_\beta B^\mu_\nu)A^\alpha + (\partial_\beta A^\alpha)B^\mu_\nu\\ &= (\nabla_\beta B^\mu_\nu)A^\alpha + (\nabla_\beta A^\alpha)B^\mu_\nu\end{aligned}$$

Where we used the product rule of the ordinary gradient. Since the last equation is a tensor equation, if it holds in any coordinate system, it must hold in all. Therefore the absolute gradient obeys the product rule. □

Solution. **P17.6**

The correct way of computing $\boldsymbol{a}$ is by using equation (17.12) where we compute $d\boldsymbol{v}/dt$ taking into account the change in the basis vectors

$$a^\mu = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}v^\mu}{\mathrm{d}t} + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta$$

Let us consider first the case of cartesian coordinates.
We see that the Christoffel symbols can be calculated using the equation

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}]$$

Since all components of the metric are constant (either 0 or 1) then all derivatives are zero and hence the Christoffel components are all zero, hence

$$a^\mu = \frac{\mathrm{d}v^\mu}{\mathrm{d}t} = 0$$

Where we used that $v^x = v$ and $v^y = 0$ i.e. the object is moving with constant velocity in cartesian coordinates.
So, as expected, the components of $\boldsymbol{a}$ are zero in cartesian coordinates.
Now, let us compute the acceleration in the sinusoidal coordinates u, w where $u = x$ and $w = y - A\sin(bx)$ then from problem P5.5 we know that metric in matrix form is

$$g_{\mu\nu} = \begin{bmatrix} 1 + (Ab)^2\cos^2(bu) & Ab\cos(bu) \\ Ab\cos(bu) & 1 \end{bmatrix}$$

We see that $\det(g_{\mu\nu}) = 1$ so the inverse metric is given by

$$g^{\mu\nu} = \begin{bmatrix} 1 & -Ab\cos(bu) \\ -Ab\cos(bu) & 1 + (Ab)^2\cos^2(bu) \end{bmatrix}$$

For the case $\alpha = u$ we can compute the Christoffel symbols as follows

$$\begin{aligned}
\Gamma^u_{uu} &= \frac{1}{2} g^{uu}[\partial_u g_{uu} + \partial_u g_{uu} - \partial_u g_{uu}] + \frac{1}{2} g^{uw}[\partial_u g_{uw} + \partial_u g_{wu} - \partial_w g_{uu}] \\
&= \frac{1}{2} g^{uu}\partial_u g_{uu} + g^{uw}\partial_u g_{uw} \\
&= -A^2 b^3 \sin(bu)\cos(bu) + A^2 b^3 \cos(bu)\sin(bu) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^u_{uw} &= \frac{1}{2} g^{uu}[\partial_u g_{wu} + \partial_w g_{uu} - \partial_u g_{uw}] + \frac{1}{2} g^{uw}[\partial_u g_{ww} + \partial_w g_{wu} - \partial_w g_{uw}] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^u_{wu} &= \frac{1}{2} g^{uu}[\partial_w g_{uu} + \partial_u g_{uw} - \partial_u g_{wu}] + \frac{1}{2} g^{uw}[\partial_w g_{uw} + \partial_u g_{ww} - \partial_w g_{wu}] \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\Gamma^u_{ww} &= \frac{1}{2}g^{uu}[\partial_w g_{wu} + \partial_w g_{uw} - \partial_u g_{ww}] + \frac{1}{2}g^{uw}[\partial_w g_{ww} + \partial_w g_{ww} - \partial_w g_{ww}] \\
&= 0
\end{aligned}
$$

For the case $\alpha = w$ we can compute the Christoffel symbols as follows

$$
\begin{aligned}
\Gamma^w_{uu} &= \frac{1}{2}g^{wu}[\partial_u g_{uu} + \partial_u g_{uu} - \partial_u g_{uu}] + \frac{1}{2}g^{ww}[\partial_u g_{uw} + \partial_u g_{wu} - \partial_w g_{uu}] \\
&= \frac{1}{2}g^{wu}\partial_u g_{uu} + g^{ww}\partial_u g_{uw} \\
&= Ab\cos(bu)A^2b^3\cos(bu)\sin(bu) - (1 + (Ab)^2\cos^2(bu))Ab^2\sin(bu) \\
&= A^3b^4\cos^2(bu)\sin(bu) - Ab^2\sin(bu) - A^3b^4\cos^2(bu)\sin(bu) \\
&= -Ab^2\sin(bu)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^w_{uw} &= \frac{1}{2}g^{wu}[\partial_u g_{wu} + \partial_w g_u - \partial_u g_{uw}] + \frac{1}{2}g^{ww}[\partial_u g_{ww} + \partial_w g_{wu} - \partial_w g_{uw}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^w_{wu} &= \frac{1}{2}g^{wu}[\partial_w g_u + \partial_u g_{uw} - \partial_u g_{wu}] + \frac{1}{2}g^{ww}[\partial_w g_{uw} + \partial_u g_{ww} - \partial_w g_{wu}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^w_{ww} &= \frac{1}{2}g^{wu}[\partial_w g_{wu} + \partial_w g_{uw} - \partial_u g_{ww}] + \frac{1}{2}g^{ww}[\partial_w g_{ww} + \partial_w g_{ww} - \partial_w g_{ww}] \\
&= 0
\end{aligned}
$$

Therefore the only non-zero Christoffel symbol is Γ^w_{uu}, then

$$
a^u = \frac{\mathrm{d}v^u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v) = 0
$$

And

$$
\begin{aligned}
a^w &= \frac{\mathrm{d}v^w}{\mathrm{d}t} + \Gamma^w_{uu}(v^u)^2 \\
&= Ab^2v^2\sin(bu) - Ab^2\sin(bu)v^2 \\
&= 0
\end{aligned}
$$

Where we used that $v^u = v$ and $v^w = -Abv\cos(bu)$.
Therefore the components of the acceleration in the sinusoidal coordinate system are zero as well. $\square$

Solution. **P17.7**

We know that an observer at rest at r has a four-velocity $\boldsymbol{u} = \boldsymbol{o}_t$ and the only non-zero component of $\boldsymbol{u}$ is the t component given by

$$u^t = \frac{1}{\sqrt{1+2GM/r}}$$

The rest of the component are 0 because the observer is at rest.
On the other hand, we know that the acceleration component are given by

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$$

So, for $\mu = t$ we get that

$$\begin{aligned} a^t &= \frac{\mathrm{d}u^t}{\mathrm{d}\tau} + \Gamma^t_{rt} u^r u^t + \Gamma^t_{tr} u^t u^r \\ &= u^r \frac{\mathrm{d}u^t}{\mathrm{d}r} + 0 \\ &= 0 \end{aligned}$$

Where we used that $u^r = 0$. In the case of $\mu = r$

$$\begin{aligned} a^r &= \frac{\mathrm{d}u^r}{\mathrm{d}\tau} + \Gamma^r_{\alpha\beta} u^\alpha u^\beta \\ a^r &= 0 + \left[\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)\right]\left(\frac{1}{\sqrt{1-2GM/r}}\right)^2 \\ a^r &= \frac{GM}{r^2} \end{aligned}$$

For $\mu = \theta$ we see that $u^\phi = u^r = u^\theta = 0$ and the only non-zero Christoffel symbols are $\Gamma^\theta_{r\theta}, \Gamma^\theta_{\theta r}$ and $\Gamma^\theta_{\phi\phi}$, hence

$$a^\theta = 0$$

And for $\mu = \phi$ we see that the only non-zero Christoffel symbols are $\Gamma^\phi_{r\phi}, \Gamma^\phi_{\phi r}, \Gamma^\phi_{\theta\phi}$ and $\Gamma^\phi_{\phi\theta}$ hence again

$$a^\phi = 0$$

Finally, we compute the magnitude $a = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ as follows

$$\begin{aligned} a &= \sqrt{g_{rr} a^r a^r} \\ &= \sqrt{\left(1 - \frac{2GM}{r}\right)^{-1} \frac{(GM)^2}{r^4}} \\ &= \frac{GM}{r^2}\sqrt{\left(1 - \frac{2GM}{r}\right)^{-1}} \end{aligned}$$

□

Solution. **P17.8**
Let $v^\mu = [(1-2GM/r), 0, 0, 0]$ we want to compute the components of the absolute gradient in the Schwartzschild coordinate basis of v^μ. The absolute gradient components are computed using the following equation

$$\nabla_\alpha v^\mu = \frac{\partial v^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\nu} v^\nu$$

Let $\mu = t$ then the only non-zero Christoffel symbols with t as a superscript are Γ^t_{tr} and Γ^t_{rt} then

$$\nabla_t v^t = \frac{\partial v^t}{\partial t} + \Gamma^t_{tt} v^t + \Gamma^t_{tr} v^r + \Gamma^t_{t\theta} v^\theta + \Gamma^t_{t\phi} v^\phi = 0$$

Where we used that $v^r = 0$ and $\frac{\partial v^t}{\partial t} = 0$. In the same way

$$\begin{aligned}
\nabla_r v^t &= \frac{\partial v^t}{\partial r} + \Gamma^t_{rt} v^t + \Gamma^t_{rr} v^r + \Gamma^t_{r\theta} v^\theta + \Gamma^t_{r\phi} v^\phi \\
&= \frac{\partial v^t}{\partial r} + \Gamma^t_{rt} v^t \\
&= \frac{2GM}{r^2} + \frac{GM}{r^2}\left(1 - \frac{2GM}{r^2}\right)^{-1}\left(1 - \frac{2GM}{r^2}\right) \\
&= \frac{3GM}{r^2}
\end{aligned}$$

Given that $\Gamma^t_{\theta t} = \Gamma^t_{\theta r} = \Gamma^t_{\theta\theta} = \Gamma^t_{\theta\phi} = 0$ and that $\Gamma^t_{\phi t} = \Gamma^t_{\phi r} = \Gamma^t_{\phi\theta} = \Gamma^t_{\phi\phi} = 0$ then

$$\nabla_\theta v^t = 0 \quad \text{and} \quad \nabla_\phi v^t = 0$$

Let now, $\mu = r$ then the only non-zero Christoffel symbols with r as a superscript are $\Gamma^r_{tt}, \Gamma^r_{rr}, \Gamma^r_{\theta\theta}$ and $\Gamma^r_{\phi\phi}$ also $\frac{\partial v^r}{\partial x^\alpha} = 0$ for $\alpha = t, r, \theta, \phi$ so the absolute gradient components of v^r are

$$\begin{aligned}
\nabla_t v^r &= \Gamma^r_{tt} v^t = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^2 \\
\nabla_r v^r &= \Gamma^r_{rr} v^r = 0 \\
\nabla_\theta v^r &= \Gamma^r_{\theta\theta} v^\theta = 0 \\
\nabla_\phi v^r &= \Gamma^r_{\phi\phi} v^\phi = 0
\end{aligned}$$

For $\mu = \theta$ the only non-zero Christoffel symbols with θ as a superscript are $\Gamma^\theta_{r\theta}, \Gamma^\theta_{\theta r}$ and $\Gamma^\theta_{\phi\phi}$ also $\frac{\partial v^\theta}{\partial x^\alpha} = 0$ for $\alpha = t, r, \theta, \phi$ so the absolute gradient components of v^θ are

$$\begin{aligned}
\nabla_t v^\theta &= 0 \\
\nabla_r v^\theta &= \Gamma^\theta_{r\theta} v^\theta = 0 \\
\nabla_\theta v^\theta &= \Gamma^\theta_{\theta r} v^r = 0 \\
\nabla_\phi v^\theta &= \Gamma^\theta_{\phi\phi} v^\phi = 0
\end{aligned}$$

Finally for $\mu = \phi$ the only non-zero Christoffel symbols with ϕ as a superscript are $\Gamma^{\phi}_{r\phi}, \Gamma^{\phi}_{\phi r}, \Gamma^{\phi}_{\theta\phi}$ and $\Gamma^{\phi}_{\phi\theta}$ also $\frac{\partial v^{\phi}}{\partial x^{\alpha}} = 0$ for $\alpha = t, r, \theta, \phi$ so the absolute gradient components of v^{ϕ} are

$$\begin{aligned}
\nabla_t v^{\phi} &= 0 \\
\nabla_r v^{\phi} &= \Gamma^{\phi}_{r\phi} v^{\phi} = 0 \\
\nabla_\theta v^{\phi} &= \Gamma^{\phi}_{\theta\phi} v^{\phi} = 0 \\
\nabla_\phi v^{\phi} &= \Gamma^{\phi}_{\phi r} v^{r} + \Gamma^{\phi}_{\phi\theta} v^{\theta} = 0
\end{aligned}$$

Therefore the only non-zero components of the absolute gradient of v^{μ} are

$$\begin{aligned}
\nabla_r v^{t} &= \frac{3GM}{r^2} \\
\nabla_t v^{r} &= \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^2
\end{aligned}$$

□

Solution. **P17.9**
The absolute gradient of the metric is

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^{\sigma}_{\alpha\mu} g_{\sigma\nu} - \Gamma^{\sigma}_{\alpha\nu} g_{\mu\sigma}$$

From equation (17.10) we know that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}]$$

Then

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \frac{1}{2} g^{\sigma\nu}[\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}] g_{\sigma\nu} - \frac{1}{2} g^{\sigma\mu}[\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}] g_{\mu\sigma} \\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}[\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}] - \frac{1}{2}[\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}] \\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}[\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}] - \frac{1}{2}[\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\nu\alpha}] \\
&= \partial_\alpha g_{\mu\nu} - \partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\mu g_{\nu\alpha} + \frac{1}{2}\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\nu g_{\alpha\mu} + \frac{1}{2}\partial_\mu g_{\nu\alpha} \\
&= 0
\end{aligned}$$

□

Solution. **P17.10**
Let us consider a coordinate transformation such that

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \Gamma^\alpha_{\mu\nu,P}(x'^\nu - x'^\nu_P)$$

Equation (17.31) state that

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

Then plugging the expression we have for $\partial x^\alpha / x'^\mu$ and valuing the expression at P give us

$$\begin{aligned}
g'_{\mu\nu,P} &= [\delta^\alpha_\mu - \Gamma^\alpha_{\mu\sigma,P}(x'^\sigma_P - x'^\sigma_P)][\delta^\beta_\nu - \Gamma^\beta_{\nu\sigma,P}(x'^\sigma_P - x'^\sigma_P)]g_{\alpha\beta,P} \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta,P} \\
&= g_{\mu\nu,P}
\end{aligned}$$

Then the primed metric at P has the same components as the unprimed metric at P.
Let us compute the derivative of equation (17.31) as follows

$$\begin{aligned}
\partial_\rho g'_{\mu\nu} &= \partial_\rho \Big([\delta^\alpha_\mu - \Gamma^\alpha_{\mu\sigma,P}(x'^\sigma - x'^\sigma_P)][\delta^\beta_\nu - \Gamma^\beta_{\nu\sigma,P}(x'^\sigma - x'^\sigma_P)]g_{\alpha\beta} \Big) \\
&= -\Gamma^\alpha_{\mu\rho,P}[\delta^\beta_\nu - \Gamma^\beta_{\nu\sigma,P}(x'^\sigma_P - x'^\sigma_P)]g_{\alpha\beta} \\
&\quad - [\delta^\alpha_\mu - \Gamma^\alpha_{\mu\sigma,P}(x'^\sigma - x'^\sigma_P)]\Gamma^\beta_{\nu\rho,P} g_{\alpha\beta} \\
&\quad + [\delta^\alpha_\mu - \Gamma^\alpha_{\mu\sigma,P}(x'^\sigma - x'^\sigma_P)][\delta^\beta_\nu - \Gamma^\beta_{\nu\sigma,P}(x'^\sigma - x'^\sigma_P)]\partial_\rho g_{\alpha\beta}
\end{aligned}$$

On the other hand, since $\nabla_\rho g_{\alpha\beta} = 0$ from equation (17.40) we get that

$$\begin{aligned}
0 &= \partial_\rho g_{\alpha\beta} - \Gamma^\sigma_{\rho\alpha} g_{\sigma\beta} - \Gamma^\sigma_{\rho\beta} g_{\alpha\sigma} \\
\partial_\rho g_{\alpha\beta} &= \Gamma^\sigma_{\rho\alpha} g_{\sigma\beta} + \Gamma^\sigma_{\rho\beta} g_{\alpha\sigma}
\end{aligned}$$

If we value this expression at P we get that

$$\partial_\rho g_{\alpha\beta}\Big|_P = \Gamma^\sigma_{\rho\alpha,P} g_{\sigma\beta,P} + \Gamma^\sigma_{\rho\beta,P} g_{\alpha\sigma,P}$$

Also, if we value the expression we have for $\partial_\rho g'_{\mu\nu}$ at P we get that

$$\begin{aligned}
\partial_\rho g'_{\mu\nu}\Big|_P &= -\Gamma^\alpha_{\mu\rho,P}\delta^\beta_\nu g_{\alpha\beta,P} - \delta^\alpha_\mu \Gamma^\beta_{\nu\rho,P} g_{\alpha\beta,P} + \delta^\alpha_\mu \delta^\beta_\nu \partial_\rho g_{\alpha\beta}\Big|_P \\
&= -\Gamma^\sigma_{\mu\rho,P} g_{\sigma\nu,P} - \Gamma^\sigma_{\nu\rho,P} g_{\mu\sigma,P} + \delta^\alpha_\mu \delta^\beta_\nu [\Gamma^\sigma_{\rho\alpha,P} g_{\sigma\beta,P} + \Gamma^\sigma_{\rho\beta,P} g_{\alpha\sigma,P}] \\
&= -\Gamma^\sigma_{\mu\rho,P} g_{\sigma\nu,P} - \Gamma^\sigma_{\nu\rho,P} g_{\mu\sigma,P} + \Gamma^\sigma_{\rho\mu,P} g_{\sigma\nu,P} + \Gamma^\sigma_{\rho\nu,P} g_{\mu\sigma,P} \\
&= -\Gamma^\sigma_{\mu\rho,P} g_{\sigma\nu,P} - \Gamma^\sigma_{\nu\rho,P} g_{\mu\sigma,P} + \Gamma^\sigma_{\mu\rho,P} g_{\sigma\nu,P} + \Gamma^\sigma_{\nu\rho,P} g_{\mu\sigma,P} \\
&= 0
\end{aligned}$$

Where we used that $\Gamma^\sigma_{\mu\rho} = \Gamma^\sigma_{\rho\mu}$. $\square$