Solutions to selected problems on Introduction to Topological Manifolds - John M. Lee.

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Chapter 2 - Topological Spaces

Problems

Proof. **2-1**

(a) We want to show that $\mathcal{T}_1 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is finite}\}$ is a topology on X.

(i) By definition $\emptyset$ is in $\mathcal{T}_1$. If $U = X$ then $X \setminus X = \emptyset$ and $\emptyset$ is finite then $X \in \mathcal{T}_1$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_1$ such that $U_i = \emptyset$ or $X \setminus U_i$ is finite for every i. Also, we see that

$$X \setminus (U_1 \cap ... \cap U_n) = (X \setminus U_1) \cup ... \cup (X \setminus U_n)$$

And the finite union of finite sets is itself a finite set hence $U_1 \cap ... \cap U_n \in \mathcal{T}_1$. We assumed that not all of the elements are empty, but otherwise we already saw that $\emptyset \in \mathcal{T}_1$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_1$ such that $U_i = \emptyset$ or $X \setminus U_i$ is finite for every i. Also, we have that

$$X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} X \setminus U_\alpha$$

So this is the intersection between finite sets then itself it's a finite set hence $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_1$.

Therefore $\mathcal{T}_1$ is a topology on X.

(b) We want to show that $\mathcal{T}_2 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is countable}\}$ is a topology on X.

(i) By definition $\emptyset$ is in $\mathcal{T}_2$. If $U = X$ then $X \setminus X = \emptyset$ and $\emptyset$ is countable then $X \in \mathcal{T}_2$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_2$ such that $U_i = \emptyset$ or $X \setminus U_i$ is countable for every i. Also, we see that

$$X \setminus (U_1 \cap ... \cap U_n) = (X \setminus U_1) \cup ... \cup (X \setminus U_n)$$

And the finite union of countable sets is itself a countable set hence $U_1 \cap ... \cap U_n \in \mathcal{T}_2$. We assumed that not all of the elements are empty, but otherwise we already saw that $\emptyset \in \mathcal{T}_2$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_2$ such that $U_i = \emptyset$ or $X \setminus U_i$ is countable for every i. Also, we have that

$$X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} X \setminus U_\alpha$$

So this is the intersection between countable sets then itself it's a countable set hence $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_2$.

Therefore $\mathcal{T}_2$ is a topology on X.

(c) We want to show that $\mathcal{T}_3 = \{U \subseteq X : U = \emptyset \text{ or } p \in U\}$ is a topology on X.

(i) By definition $\emptyset$ is in $\mathcal{T}_3$. Since $p \in X$ by definition then $X \in \mathcal{T}_3$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_3$ such that $U_i = \emptyset$ or $p \in U_i$ for every i then $U_1 \cap ... \cap U_n$ at least have the element p in common so $U_1 \cap ... \cap U_n \in \mathcal{T}_3$. This result is true assuming not every $U_i = \emptyset$ otherwise $U_1 \cap ... \cap U_n = \emptyset$ and we saw that $\emptyset \in \mathcal{T}_3$ so anyway $U_1 \cap ... \cap U_n \in \mathcal{T}_3$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_3$ such that $U_i = \emptyset$ or $p \in U_i$ for every i. Then assuming not every $U_i = \emptyset$ we have that $p \in \bigcup_{\alpha \in A} U_\alpha$ which implies that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_3$. If every $U_i = \emptyset$ then $\bigcup_{\alpha \in A} U_\alpha = \emptyset$ and we saw that $\emptyset \in \mathcal{T}_3$ so anyway $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_3$.

Therefore $\mathcal{T}_3$ is a topology on X.

(d) We want to show that $\mathcal{T}_4 = \{U \subseteq X : U = X \text{ or } p \notin U\}$ is a topology on X.

(i) By definition X is in $\mathcal{T}_4$. Since $p \notin \emptyset$ then $\emptyset \in \mathcal{T}_4$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_4$ such that $U_i = X$ or $p \notin U_i$ for every i then p is not in $U_1 \cap ... \cap U_n$. This result is true assuming not every $U_i = X$ otherwise $U_1 \cap ... \cap U_n = X$ and we saw that $X \in \mathcal{T}_4$ so anyway $U_1 \cap ... \cap U_n \in \mathcal{T}_4$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_4$ such that $U_i = X$ or $p \notin U_i$ for every i. Then assuming no $U_i = X$ we have that $p \notin \bigcup_{\alpha \in A} U_\alpha$ which implies that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_4$. If some $U_i = X$ then $\bigcup_{\alpha \in A} U_\alpha = X$ and we saw that $X \in \mathcal{T}_4$ so anyway $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_4$.

Therefore $\mathcal{T}_4$ is a topology on X.

(e) We want to determine if $\mathcal{T}_5 = \{U \subseteq X : U = X \text{ or } X \setminus U \text{ is infinite}\}$ is a topology on X. If we let $X = \mathbb{Z}$ then $\mathbb{Z}^+ \in \mathcal{T}_5$ and $\mathbb{Z}^- \in \mathcal{T}_5$ but $\mathbb{Z} \setminus (\mathbb{Z}^+ \cup \mathbb{Z}^-) = \{0\}$ but $\{0\}$ is finite so $(\mathbb{Z}^+ \cup \mathbb{Z}^-) \notin \mathcal{T}_5$. Therefore $\mathcal{T}_5$ is not a topology on X.

$\square$

Proof. **2-3**

(a) Let $x \in \overline{X \setminus B}$ then for every neighborhood U where $x \in U$ contains a point of $X \setminus B$ this implies that no neighborhood that contains x is in $\text{Int}(B)$ hence $x \in X \setminus \text{Int}(B)$ and $\overline{X \setminus B} \subseteq X \setminus \text{Int}(B)$.

Let $x \in X \setminus \text{Int}(B)$ then $x \notin \text{Int}(B)$ so there is no neighborhood of x contained in B then for every neighborhood U that contains x we have that $U \cap X \setminus B \neq \emptyset$ hence $x \in \overline{X \setminus B}$ and $X \setminus \text{Int}(B) \subseteq \overline{X \setminus B}$.

Therefore $\overline{X \setminus B} = X \setminus \text{Int}(B)$.

(b) Let $x \in \text{Int}(X \setminus B)$ then x has a neighborhood contained in $X \setminus B$ but then x is also in $\text{Ext}(B) = X \setminus \overline{B}$ then $\text{Int}(X \setminus B) \subseteq X \setminus \overline{B}$.

In the same way if $x \in X \setminus \overline{B} = \text{Ext}(B)$ then it has a neighborhood in $X \setminus B$ hence $X \setminus B$ is open and hence $x \in \text{Int}(X \setminus B)$ this implies that $X \setminus \overline{B} \subseteq \text{Int}(X \setminus B)$.

Therefore $\text{Int}(X \setminus B) = X \setminus \overline{B}$.

$\square$

Proof. **2-4**

(a) We know that $\bigcap_{A\in\mathcal{A}} \overline{A}$ is a closed set where $\bigcap_{A\in\mathcal{A}} A \subseteq \bigcap_{A\in\mathcal{A}} \overline{A}$ but also we know that the closure of $\bigcap_{A\in\mathcal{A}} A$ is the smallest closed set containing $\bigcap_{A\in\mathcal{A}} A$. Therefore it must happen that $\overline{\bigcap_{A\in\mathcal{A}} A} \subseteq \bigcap_{A\in\mathcal{A}} \overline{A}$.

Let $\mathcal{A} = \{(0,1),(1,2)\}$ where they are intervals of $\mathbb{R}$ then $\bigcap_{A\in\mathcal{A}} \overline{A} = \{1\}$ and $\overline{\bigcap_{A\in\mathcal{A}} A} = \emptyset$ so we see that $\bigcap_{A\in\mathcal{A}} \overline{A} \not\subseteq \overline{\bigcap_{A\in\mathcal{A}} A}$. Therefore when $\mathcal{A}$ is a finite collection the equality is not preserved.

(b) We know that $A \subseteq \bigcup_{A\in\mathcal{A}} A$ then $A \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$ also $\overline{A} \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$ therefore

$$\bigcup_{A\in\mathcal{A}} \overline{A} \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$$

Let us assume now that $\mathcal{A}$ is a finite collection. Then we see that $\bigcup_{A\in\mathcal{A}} \overline{A}$ is a closed set and that $\bigcup_{A\in\mathcal{A}} A \subseteq \bigcup_{A\in\mathcal{A}} \overline{A}$ but also we know that $\overline{\bigcup_{A\in\mathcal{A}} A}$ is the smallest closed set that contains $\bigcup_{A\in\mathcal{A}} A$ therefore it must happen that

$$\overline{\bigcup_{A\in\mathcal{A}} A} \subseteq \bigcup_{A\in\mathcal{A}} \overline{A}$$

Hence the equality holds when $\mathcal{A}$ is a finite collection.

(c) Let $x \in \operatorname{Int}(\bigcap_{A\in\mathcal{A}} A)$ then there is a neighborhood such that $U \subseteq \bigcap_{A\in\mathcal{A}} A$ hence $U \subseteq A$ for every $A \in \bigcap_{A\in\mathcal{A}} A$ hence because of Proposition 2.8 (a) we have that $x \in \operatorname{Int}(A)$ but since this is true for every A then must be that $x \in \bigcap_{A\in\mathcal{A}} \operatorname{Int}(A)$. Therefore

$$\operatorname{Int}\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} \operatorname{Int}(A)$$

Let us assume now that $\mathcal{A}$ is a finite collection. Then we see that $\bigcap_{A\in\mathcal{A}} \operatorname{Int}(A)$ is an open set and $\bigcap_{A\in\mathcal{A}} \operatorname{Int}(A) \subseteq \bigcap_{A\in\mathcal{A}} A$ but also we know that $\operatorname{Int}(\bigcap_{A\in\mathcal{A}} A)$ is the biggest open set contained in $\bigcap_{A\in\mathcal{A}} A$ therefore it must be that

$$\bigcap_{A\in\mathcal{A}} \operatorname{Int}(A) \subseteq \operatorname{Int}\left(\bigcap_{A\in\mathcal{A}} A\right)$$

Hence the equality holds when $\mathcal{A}$ is a finite collection.

(d) We know that $\bigcup_{A\in\mathcal{A}} \text{Int}(A)$ is open since it's an arbitrary union of open sets and also we see that $\bigcup_{A\in\mathcal{A}} \text{Int}(A) \subseteq \bigcup_{A\in\mathcal{A}} A$ but also we know that $\text{Int}\left(\bigcup_{A\in\mathcal{A}} A\right)$ is the biggest open set contained in $\bigcup_{A\in\mathcal{A}} A$ therefore it must happen that

$$\bigcup_{A\in\mathcal{A}} \text{Int}(A) \subseteq \text{Int}\left(\bigcup_{A\in\mathcal{A}} A\right)$$

Let $\mathcal{A} = \{[0,1],[1,2]\}$ where they are intervals of $\mathbb{R}$ then $\bigcup_{A\in\mathcal{A}} \text{Int}(A) = (0,1)\cup(1,2)$ and $\text{Int}(\bigcup_{A\in\mathcal{A}} A) = (0,2)$ so we see that $\text{Int}(\bigcup_{A\in\mathcal{A}} A) \not\subseteq \bigcup_{A\in\mathcal{A}} \text{Int}(A)$. Therefore when $\mathcal{A}$ is a finite collection the equality is not preserved.

$\square$

Proof. **2-5**

(a) Let $X = [0,\infty)\times\{0\} \subset \mathbb{R}^2$ and $Y = [-1,1]\times\{0\} \subset \mathbb{R}^2$ such that $f(x) = \sin(1/x)$ for $x > 0$ and if $x = 0$ we have that $f(0) = 0$.

Let $U \subset X$ be an open set such that U is of the form (a,b) which is a basis and since $U \subset (0,\infty)$ and f is continuous in $(0,\infty)$ because of the intermediate value theorem, we have that $f(U)$ is also open, therefore f is an open map.

Let $E = \{1/n : n \in \mathbb{N}\}\cup\{0\}$ we know that $E \subset X$ and E is a closed set then let us consider $f(E) = \{\sin(n) : n \in \mathbb{N}\}\cup\{0\}$ we see that $f(E)$ is a dense set hence if we let $x \in [-1,1]$ such that $x \notin f(E)$ then every neighborhood of x will have a point of $f(E)$ so x is a limit point of $f(E)$ which is not contained in $f(E)$, therefore $f(E)$ is not closed and f is not a closed map.

Let us consider now a sequence $(x_n) \subseteq X$ defined as $x_n = 1/(\pi n + \pi/2)$ we see that $x_n \to 0$ but $f(x_n) = -1$ if n is odd and $f(x_n) = 1$ if n is even hence f does not converge and f is therefore not continuous.

(b) Let $X = \mathbb{R}\times\{0\}$ and $Y = \mathbb{R}\times\{0\}$ such that $f(x) = 0$ if $x \neq 0$ and $f(x) = 1$ if $x = 0$.

Let $E \subset X$ be a closed set then $f(E) = \{0\}$ or $f(E) = \{1\}$ or $f(E) = \{0,1\}$ in any case they are closed sets since they are singletons hence f is a closed map.

Let $U \subset X$ be an open set then $f(U) \subseteq \{0,1\}$ and we know that neither $\{0,1\}$ nor $\{0\}$ nor $\{1\}$ are open sets hence f is not an open map.

Finally f is not continuous at $y = 0$ since if we take $\epsilon = 1/2$ we have that $|f(x) - f(0)| = |0 - 1| = 1 > 1/2$ no matter which $\delta > 0$ we take.

(c) Let $X = \mathbb{R} \times \{0\}$ and $Y = \mathbb{R} \times \{0\}$ such that $f(x) = 0$ if $x \leq 0$ and $f(x) = \arctan(x)$ if $x > 0$. We see that f is continuous.

Let $U = (-1, 0) \subset \mathbb{R}$ we know U is an open set of X then $f(U) = \{0\}$ but $\{0\}$ is not an open set in $Y = \mathbb{R}$. Therefore f is not an open map.

Let $E = [0, \infty) \subset X$ we know that E is closed in X, also we see that $f(E) = [0, 1)$ but $[0, 1)$ is not an open nor a closed set in $Y = \mathbb{R}$ therefore f is not a closed map.

(d) Let $X = [0, \infty) \times \{0\}$ and $Y = \mathbb{R} \times \{0\}$ such that $f(x) = \arctan(x)$ we see that f is continuous.

Let $U \subset X$ be an open set such that U is of the form (a, b) which is a basis and since $U \subset (0, \infty)$ and f is continuous in $(0, \infty)$ because of the intermediate value theorem, we have that $f(U)$ is also open, therefore f is an open map.

Let $E = [0, \infty) \subset X$ we know that E is closed in X, also we see that $f(E) = [0, 1)$ but $[0, 1)$ is not an open nor a closed set in $Y = \mathbb{R}$ therefore f is not a closed map.

(e) Let $X = \mathbb{R} \times \{0\}$ and $Y = \mathbb{R} \times \{0\}$ such that $f(x) = 0$ we see that f is continuous.

Let $E \subset X$ be a closed set then $f(E) = \{0\}$ which is a closed set in $Y = \mathbb{R}$ since it is a singleton hence f is a closed map.

Let $U \subset X$ be an open set then $f(U) = \{0\}$ and we know that $\{0\}$ is not an open sets in $Y = \mathbb{R}$ hence f is not an open map.

(f) Let $X = \mathbb{R} \times \{0\}$ and $Y = (-\infty, 1] \cup (2, \infty) \times \{0\}$ such that $f(x) = x$ if $x \leq 1$ and $f(x) = 2x$ if $x > 1$.

Let $U \subset X$ be an open set such that U is of the form (a, b) which is a basis hence it's valid for any open set if $(a, b) \subset (-\infty, 1]$ we know that f is continuous in this interval so because of the Intermediate Value Theorem $f(U)$ is an open map and in the same way if $(a, b) \subset (1, \infty)$ then $f(U)$ is open because f is continuous in this interval and the Intermediate Value Theorem. Now suppose (a, b) such that $a < 1 < b$ then $(a, b) = (a, 1] \cup (1, b)$ and we see that $f((a, 1]) = (a, 1]$ which is open in $(-\infty, 1]$ since $(-\infty, 1] \setminus (a, 1] = (-\infty, a]$ which we know is a closed set hence $(a, 1]$ is an open set. Also, we have that $f((1, b)) = (2, 2b)$ which is an open set so in this case $f((a, b))$ is the union of two open sets i.e. it's an open set. Therefore f is an open map.

Let $E \subseteq X$ be a closed set and let us take a sequence $(y_n) \subseteq f(E)$ such that $y_n \to y$ we want to prove that $y \in f(E)$ which would imply that $f(E)$ is closed. By definition there is $x_n \in E$ such that $f(x_n) = y_n$.

But also we know that $x_n = y_n$ or $x_n = y_n/2$ or a combination of both but only for a finite number of points by the definition of f.

In the first case, this implies that $x_n \to x$ where $x = y$ and since E is a closed set then $x \in E$ but also we know that f is continuous in $(-\infty, 1]$ hence $y \in f(E)$.

Lastly if $x_n = y_n/2$ we have that $x_n \to x/2$ where $x/2 = y$ and since E is a closed set then $x \in E$ but also we know that f is continuous in $(1, \infty)$ hence $y \in f(E)$.

Therefore f is a closed map.

$\square$

Proof. **2-6**

(a) ($\Rightarrow$) Let f be continuous and let $A \subseteq X$ also $\overline{f(A)}$ is closed on Y hence $f^{-1}(\overline{f(A)})$ is closed on X because f is continuous but also we know that $A \subseteq f^{-1}(\overline{f(A)})$ and since $f^{-1}(\overline{f(A)})$ is closed it must happen that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ this in turn implies that $f(\overline{A}) \subseteq f(f^{-1}(\overline{f(A)})) \subseteq \overline{f(A)}$.

($\Leftarrow$) Let $B \subseteq Y$ be a closed set we want to show that $A = f^{-1}(B)$ is also closed in X. Given that $f(\overline{A}) \subseteq \overline{f(A)}$ we have that

$$f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} \subseteq \overline{B} = B$$

since B is closed. Then we have that $f(\overline{A}) \subseteq B$ hence

$$\overline{A} \subseteq f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A$$

Therefore A is closed since it contains $\overline{A}$.

(b) ($\Rightarrow$) Let f be a closed map and let A be any set of X we know that $A \subseteq \overline{A}$ then $f(A) \subseteq f(\overline{A})$ but since f is a closed map then $f(\overline{A})$ is closed which implies that the closure of $f(A)$ must be contained in $f(\overline{A})$ i.e. $\overline{f(A)} \subseteq f(\overline{A})$.

($\Leftarrow$) Let A be a closed set of X we want to prove that $f(A)$ is also closed. Since A is closed we have that $A = \overline{A}$ hence $f(A) = f(\overline{A})$ but we know that $\overline{f(A)} \subseteq f(\overline{A}) = f(A)$ thus the closure of $f(A)$ is contained or equal to $f(A)$ therefore $f(A)$ is closed implying that f is a closed map.

(c) ($\Rightarrow$) Let f be continuous and let $B \subseteq Y$ we know that by definition $\text{Int}(B) \subseteq B$ so $f^{-1}(\text{Int}(B)) \subseteq f^{-1}(B)$ we know that $f^{-1}(\text{Int}(B))$ is open since f is continuous and $\text{Int}(B)$ is an open set hence it must also happen that $f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$ since by definition $\text{Int}(f^{-1}(B))$ is the largest open set contained in $f^{-1}(B)$.

($\Leftarrow$) Let $B \subseteq Y$ be an open set we want to prove that $f^{-1}(B)$ is an open set which implies that f is continuous. Since B is open we have that $B = \text{Int}(B)$ but also we know that $f^{-1}(B) = f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$ and by definition we know that $\text{Int}(f^{-1}(B)) \subseteq f^{-1}(B)$ therefore $\text{Int}(f^{-1}(B)) = f^{-1}(B)$ which implies that $f^{-1}(B)$ is an open set.

(d) ($\Rightarrow$) Let f be an open map and let $B \subseteq Y$ also let us name $C = \text{Int}(f^{-1}(B)) \subseteq f^{-1}(B)$ then $f(C) \subseteq f(f^{-1}(B)) \subseteq B$ where $f(C)$ is open since f is an open map hence it must also happen that $f(C) \subseteq \text{Int}(B)$ since $\text{Int}(B)$ is the biggest open set contained in B so we have that $\text{Int}(f^{-1}(B)) = C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(\text{Int}(B))$.

($\Leftarrow$) Let $A \subseteq X$ be an open set we want to prove that $f(A)$ is an open set too which implies that f is open. Since $f(A)$ is a set of Y we know that $\text{Int}(f^{-1}(f(A))) \subseteq f^{-1}(\text{Int}(f(A)))$ also we have that $\text{Int}A \subseteq \text{Int}(f^{-1}(f(A)))$ then joining this two equations and applying f to both sides we have that

$$f(A) = f(\text{Int}(A)) \subseteq f(f^{-1}(\text{Int}(f(A)))) \subseteq \text{Int}(f(A))$$

where we used that $\text{Int}(A) = A$ since A is an open set. Therefore we have that $f(A) \subseteq \text{Int}(f(A))$ but also by definition we know that $\text{Int}(f(A)) \subseteq f(A)$ which implies that $\text{Int}(f(A)) = f(A)$ thus $f(A)$ is an open set and f is an open map.

$\square$

Proof. **2-7** Let X be a Hausdorff space where $A \subseteq X$. Let $p \in X$ be a limit point of A and suppose U is a neighborhood of p that contains finitely many points of A we want to arrive at a contradiction. Suppose $\{p_1, p_2, ..., p_n\} \subset A$ is the set of points that are in U other than p. Also, since X is a Hausdorff space we know that there is a set U' that contains p but does not contain any point from the set $\{p_1, p_2, ..., p_n\}$ hence $U' = \{p\}$ but this is a contradiction since p is a limit point so it must contain at least one point of A. Therefore if p is a limit point of A then every neighborhood of p must contain infinitely many points of A. □

Proof. **2-9**

(a) Let a map $f : D \to A$ where D is a discrete space and A is any arbitrary topological space, we want to prove that f is continuous. Let us take an open set $U \subseteq A$ then $f^{-1}(U)$ must be a set of D but since D is a discrete space (with the discrete topology) it must happen that $f^{-1}(U)$ is an open set of D. Therefore every map from D to A is continuous.

(b) Let a map $f : A \to T$ where T is a space with the trivial topology and A is any arbitrary topological space, we want to prove that f is continuous.
Let us take an open set $U \subseteq T$ then it must happen that $U = T$ or $U = \emptyset$ since T is a space with the trivial topology then $f^{-1}(U) = f^{-1}(T) = A$ or $f^{-1}(U) = f^{-1}(\emptyset) = \emptyset$ where both A and $\emptyset$ are open sets by definition. Therefore every map from A to T is continuous.

(c) Let us suppose that a map $f : T \to H$ where T is a space with the trivial topology and H is a Hausdorff space is continuous and it's not a constant map, we want to arrive at a contradiction.
Let $x, y \in T$ such that $f(x) \neq f(y)$ since H is a Hausdorff space it must happen that there is $f(x) \in U_x \subseteq H$ and $f(y) \in U_y \subseteq H$ such that $U_x \cap U_y = \emptyset$. On the other hand, we know that f is continuous then $f^{-1}(U_x)$ must be an open set where either $f^{-1}(U_x) = T$ or $f^{-1}(U_x) = \emptyset$ and since $x \in f^{-1}(U_x)$ it must happen that $f^{-1}(U_x) = T$ because of the same reason we have that $y \in f^{-1}(U_y) = T$ hence $x \in f^{-1}(U_y)$ but then $f(x) \in U_y$, a contradiction. Therefore the only continuous maps from T to H are the constant maps.

□

Proof. **2-10** Let $f, g : X \to Y$ be two continuous maps and Y a Hausdorff space. We want to prove that $U = \{x \in X : f(x) = g(x)\}$ is closed on X. Let us see that $X \setminus U = \{x \in X : f(x) \neq g(x)\}$ and let us take $x \in X \setminus U$ then since $f(x) \neq g(x)$ and Y is a Hausdorff space there must be $V_{f(x)}, V_{g(x)} \subseteq Y$ such that $V_{f(x)} \cap V_{g(x)} = \emptyset$.
We want to prove now that $f^{-1}(V_{f(x)}) \cap g^{-1}(V_{g(x)}) \subset X \setminus U$ suppose there is $y \in f^{-1}(V_{f(x)}) \cap g^{-1}(V_{g(x)})$ such that $y \notin X \setminus U$ we want to arrive at a contradiction. Since $y \in f^{-1}(V_{f(x)})$ we have that $f(y) \in V_{f(x)}$ but also by the same reasoning we have that $g(y) \in V_{g(x)}$ and since $y \notin X \setminus U$ it must happen that $f(y) = g(y)$ thus $V_{f(x)} \cap V_{g(x)} \neq \emptyset$ which is a contradiction. Therefore it must be that $f^{-1}(V_{f(x)}) \cap g^{-1}(V_{g(x)}) \subset X \setminus U$.
Finally, since this must be true for every $x \in X \setminus U$ then

$$X \setminus U = \bigcup_{x \in X \setminus U} f^{-1}(V_{f(x)}) \cap g^{-1}(V_{g(x)})$$

hence $X \setminus U$ is open since it's an arbitrary union of open sets, which implies that $U = \{x \in X : f(x) = g(x)\}$ is a closed set.

Let now $f, g : X \to Y$ be two continuous maps where $X = Y = \mathbb{R}$ with the trivial topology such that $f(x) = x$ and $g(x) = 0$ for all $x \in X$. We see that $U = \{x \in X : f(x) = g(x)\} = \{0\}$ but U is not closed since the only closed sets are $X = \mathbb{R}$ and $\emptyset$. □

Proof. **2-14**

(a) ($\Rightarrow$) Let $x \in \overline{A}$ since X is a first countable space then there is $\mathcal{B}_x$ a nested neighborhood basis for X at x and hence there is a sequence $(U_i)_{i=1}^{\infty}$ of neighborhoods of x such that $U_{i+1} \subseteq U_i$. On the other hand, since $x \in \overline{A}$ we know that every neighborhood of x contains a point of A then we can build a sequence $(x_i) \subseteq A$ such that $x_i \in U_i$ and $x_i \to x$ as we wanted.

($\Leftarrow$) Let $(x_n) \subset A$ be a sequence such that $x_n \to x$ where $x \in X$ we want to show that $x \in \overline{A}$. By definition of convergence, for every neighborhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$ but we know that $x_n \in A$ therefore $x \in \overline{A}$.

(b) ($\Rightarrow$) Let $x \in \text{Int}A$ and let $(x_n) \subseteq X$ be a sequence that converges to x then for every neighborhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$ if we take $U = \text{Int}A$ then (x_n) is be eventually in A.

($\Leftarrow$) Let $(x_n) \subseteq X$ be a sequence that converges to $x \in X$ and is eventually in A we want to prove that $x \in \text{Int}A$. Let us note that $\text{Int}A = X \setminus \overline{X \setminus A}$ so if x is not in $\text{Int}A$ it must be in $\overline{X \setminus A}$, let us suppose this is the case, we want to arrive at a contradiction. By what we proved in part (a) if $x \in \overline{X \setminus A}$ then it is a limit of a sequence of points in $X \setminus A$ hence we have a sequence that converges to x but it's not eventually in A, a contradiction. Therefore it must be that $x \in \text{Int}A$.

(c) ($\Rightarrow$) Let A be closed in X and let $(x_n) \subseteq A$ be a convergent sequence that converges to $x \in X$. We know that $A = \overline{A}$ and because of part (a) since x is the limit of a sequence of points in A must happen that $x \in \overline{A} = A$. Therefore A contains every limit of every convergent sequence of points in A.

($\Leftarrow$) Let $x \in \overline{A}$ then because of part (a) x is the limit of a sequence $(x_n) \subseteq A$ but we know A contains every limit of every convergent sequence of points in A since (x_n) is convergent then must happen that $x \in A$ and this happens for every point in $\overline{A}$ therefore we have that $\overline{A} \subseteq A$ which implies by definition that $A = \overline{A}$ hence A is closed in X.

(d) ($\Rightarrow$) Let A be open in X and let $(x_n) \subseteq X$ such that $x_n \to x$ where $x \in A$. We know that $A = \text{Int}A$ and because of part (b) since $x \in A = \text{Int}A$ then every sequence converging to it is eventually in A.

($\Leftarrow$) Let $x \in A$ and $(x_n) \subseteq X$ such that $x_n \to x$ we know that (x_n) is eventually in A then by part (b) we know that $x \in \text{Int}A$ so we have that $A \subseteq \text{Int}A$ which implies by definition of interior that $A = \text{Int}A$ hence A is open in X.

$\square$

Proof. **2-15**

(a) Let $f : X \to Y$ be a continuous map and $p_n \to p$ in X. We want to show that $f(p_n) \to f(p)$.

Let us take a neighborhood U of $f(p)$ in Y then since f is continuous we have that $f^{-1}(U) \subseteq X$ is open where $p \in f^{-1}(U)$ but also we know that $p_n \to p$ so by definition there is $N \in \mathbb{N}$ such that for all $i \geq N$ we have that $p_i \in f^{-1}(U)$. Also, in general, we know that $f(f^{-1}(U)) \subseteq U$ hence $f(p_i) \in U$ this implies that for any neighborhood $U \subseteq Y$ of $f(p)$ we can find an $N \in \mathbb{N}$ such that for all $i \geq N$ we have that $f(p_i) \in U$, therefore $f(p_i) \to f(p)$.

(b) Let X be first countable and $f : X \to Y$ be a map such that $p_n \to p$ in X implies that $f(p_n) \to f(p)$ in Y, we want to prove that f is continuous.

Let us suppose f is not continuous we want to arrive at a contradiction. If f is not continuous then there is a neighborhood $U \subseteq Y$ of $f(p)$ such that $f^{-1}(U)$ is not open.

We know that $f(p_n) \to f(p)$ so there is $N \in \mathbb{N}$ such that for all $i \geq N$ we have that $f(p_i) \in U$ this implies that $p \in f^{-1}(U)$ and $p_i \in f^{-1}(U)$ and since $p_n \to p$ we see that (p_n) is eventually in $f^{-1}(U)$ but also we know that X is first countable hence $f^{-1}(U)$ is open, a contradiction. Therefore f must be continuous.

$\square$

Proof. **2-18**

(a) Let us take some $p \in \mathbb{R}$ and let us define the topology of $\mathbb{R}$ as

$$\mathcal{T} = \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } p \in U\}$$

First, we want to prove that $\mathbb{R}$ with this topology is first countable. Let us define a collection $\mathcal{B}$ with only one set $\{p\}$ which is open by definition of the topology. We see that every neighborhood U of p contains p. This implies that $(\mathbb{R}, \mathcal{T})$ is first countable.

Now we want to prove $(\mathbb{R}, \mathcal{T})$ is separable. Let us take the set $\{p\}$. By definition every open set U contains p so $\{p\}$ is dense and countable in $\mathbb{R}$ hence $(\mathbb{R}, \mathcal{T})$ is separable.

Let us suppose now that $(\mathbb{R}, \mathcal{T})$ is second countable we want to arrive at a contradiction. Then $(\mathbb{R}, \mathcal{T})$ admits a countable basis $\{U_n\}$. Also, we know that for each $x \in \mathbb{R}$ the set $\{p, x\}$ is open then there must be some n for which $x \in U_n \subseteq \{p, x\}$ but since this must be true for each $x \in \mathbb{R}$ and $\mathbb{R}$ is not countable then $\{U_n\}$ is uncountable, a contradiction. Therefore $(\mathbb{R}, \mathcal{T})$ is not second countable.

Finally, let us suppose that $(\mathbb{R}, \mathcal{T})$ is Lindelöf we want to arrive at a contradiction. Then let us take $\mathcal{U} = \{\{p, x\} : x \in \mathbb{R}\}$ as an open cover of $\mathbb{R}$ since each set $\{p, x\}$ is open by definition. We know it must have a countable subcover $\{U_n\}$. Let $x \in \mathbb{R}$ then there is U_n such that $x \in U_n$ for some $n \in \mathbb{N}$ but by definition $U_n = \{p, x\}$ and this must be this way for every $x \in \mathbb{R}$ then $\{U_n\}$ is uncountable which is a contradiction. Therefore $(\mathbb{R}, \mathcal{T})$ is not Lindelöf.

(b) Let us take some $p \in \mathbb{R}$ and let us define the topology of $\mathbb{R}$ as

$$\mathcal{T} = \{U \subseteq \mathbb{R} : U = \mathbb{R} \text{ or } p \notin U\}$$

First, we want to prove that $\mathbb{R}$ with this topology is first countable. Let $x \in \mathbb{R}$ such that $x \neq p$ also let us define a collection with one element as $\mathcal{B} = \{\{x\}\}$ where $\{x\}$ is open by definition of the topology. We see that every neighborhood U of x contains x. Now let us consider the case where we take $x = p$ then the only neighborhood for p is $\mathbb{R}$ so if we take $\mathcal{B} = \{\mathbb{R}\}$ we have a countable neighborhood basis for p too. This implies that $(\mathbb{R}, \mathcal{T})$ is first countable.

Now we want to prove $(\mathbb{R}, \mathcal{T})$ is Lindelöf. Let $\mathcal{U}$ be some open cover of $\mathbb{R}$ then $\mathcal{U}$ is a collection of $U \in \mathcal{T}$ but one of them must be $\mathbb{R}$ since otherwise p is not covered by definition. Then we can take a subcover $\mathcal{U}' = \{\mathbb{R}\}$ which is countable and still covers $\mathbb{R}$. Therefore $(\mathbb{R}, \mathcal{T})$ is Lindelöf.

Let us suppose now that $(\mathbb{R}, \mathcal{T})$ is second countable we want to arrive at a contradiction. Then $(\mathbb{R}, \mathcal{T})$ admits a countable basis $\{U_n\}$. Also, we know that for each $x \in \mathbb{R}$ such that $x \neq p$ the set $\{x\}$ is open then there must be some n for which $x \in U_n \subseteq \{x\}$ but since this must be true for each $x \in \mathbb{R}$ where $x \neq p$ and $\mathbb{R} \setminus \{p\}$ is not countable then $\{U_n\}$ must be uncountable too, a contradiction. Therefore $(\mathbb{R}, \mathcal{T})$ is not second countable.

Finally, let us suppose that $(\mathbb{R}, \mathcal{T})$ is separable we want to arrive at a contradiction. Then there must be a set U which is dense and countable in $\mathbb{R}$. Let $x \in \mathbb{R}$ such that $x \neq p$. We know that the set $\{x\}$ is open by definition then U must contain a point of $\{x\}$ hence $\{x\} \subseteq U$ but this must happen for every $x \in \mathbb{R} \setminus \{p\}$ which is uncountable then U must be uncountable which is a contradiction. Therefore $(\mathbb{R}, \mathcal{T})$ is not separable.

(c) Let us define the topology of $\mathbb{R}$ as

$$\mathcal{T} = \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite}\}$$

First, we want to prove that $\mathbb{R}$ with this topology is separable. Let D be an infinite countable set in $\mathbb{R}$ and let $U \in \mathcal{T}$ be an open set such that $\mathbb{R} \setminus U$ is finite. Let us suppose that no element of D is in U we want to arrive at a contradiction. Then it must be in $D \subseteq \mathbb{R} \setminus U$ but $\mathbb{R} \setminus U$ is finite but D is inifinite so we have a contradiction. Therefore since this must be true for any $U \in \mathcal{T}$ then D is dense.

Now we want to prove $(\mathbb{R}, \mathcal{T})$ is Lindelöf. Let $\mathcal{U}$ be some open cover of $\mathbb{R}$ then $\mathcal{U}$ is a collection of sets U from $\mathcal{T}$. Let us take any $U_1 \in \mathcal{U}$ then $\mathbb{R} \setminus U_1$ is a finite set, let us suppose it has n elements so there are n points of $\mathbb{R}$ that are not covered by U_1 so there are at most n sets of $\mathcal{U}$ necessary to cover $\mathbb{R}$ completely, hence we can build in the worst-case $\mathcal{U}' = \{U_1, U_2, ..., U_{n+1}\} \subseteq \mathcal{U}$ which is a countable subcover of $\mathcal{U}$. Therefore $(\mathbb{R}, \mathcal{T})$ is Lindelöf.

Let us suppose now that $(\mathbb{R}, \mathcal{T})$ is first countable, we want to arrive at a contradiction. Then for each $x \in \mathbb{R}$ there is a countable neighborhood basis $\{U_n\}$ hence for every neighborhood U of x there must be some U_n such that $U_n \subseteq U$. Let us we take $U' = \bigcap_{i=1}^{\infty} U_n$ we see that this set must contain $\mathbb{R}$ but countably many points. So there is some $y \in U'$ where $y \neq x$ which we can use to build a neighborhood of x as $\mathbb{R} \setminus \{y\}$. We see that $y \in U_n$ for all n but $y \notin \mathbb{R} \setminus \{y\}$ so there is no U_n such that $U_n \subseteq \mathbb{R} \setminus \{y\}$ which is a contradiction. Therefore $(\mathbb{R}, \mathcal{T})$ is not first countable.

Finally, by Theorem 2.50 $(\mathbb{R}, \mathcal{T})$ can't be second countable since it's not first countable.

□

Proof. **2-20** We want to show that second countability, separability, and Lindelöf properties are all equivalent for metric spaces.

Let (X, d) be some metric space which is second countable then by Theorem 2.50 we know it's Lindelöf and separable.

Let now (X, d) be a separable metric space then it contains a countable dense subset A. We want to prove the following collection is a basis for (X, d)

$$\mathcal{B} = \{B_q(x) : x \in A \text{ and } q \in \mathbb{Q}\}$$

Then $\mathcal{B}$ is the collection of balls centered at every $x \in A$ which is countable with radius $q \in \mathbb{Q}$ which is also countable. Let $U \subseteq (X, d)$ be an open set then since A is dense there is a point $x \in A$ such that $x \in U$ and since the collection of balls (of any radius) is a basis for the metric space then there must be some $B_r(x) \subseteq (X, d)$ such that $B_r(x) \subseteq U$. So we can take some $q \in \mathbb{Q}$ such that $0 < q < r$ and some $y \in B_q(x)$ such that $y \in B_q(x) \subseteq B_r(x) \subseteq U$ which implies that $\mathcal{B}$ is a countable basis for (X, d) and thus (X, d) is second countable.

Let now (X, d) be a Lindelöf metric space then for every open cover of (X, d) there is a countable subcover. Let us define an open cover as $C_q = \{B_q(x) : x \in (X, d)\}$ where $q \in \mathbb{Q}$ then there is a countable subcover $C'_q \subseteq C_q$. We want to prove the collection $\mathcal{B} = \bigcup_{q \in \mathbb{Q}} C'_q$ is a basis for (X, d). Let $U \subseteq (X, d)$ be an open set then given some $x \in U$ there is $B_r(x) \subseteq U$ for some $r \in \mathbb{R}$, let us take some $q \in \mathbb{Q}$ such that $0 < q < r/2$ then there is some $y \in B_{r/2}(x)$ such that $x \in B_q(y) \in \mathcal{B}$. Now let us suppose $z \in B_q(y)$ we see that $d(y, z) < q < r/2$ but also we know that $d(x, y) < r/2$ so by the triangle inequality, we have that $d(x, z) \leq d(x, y) + d(y, z) < r/2 + r/2 = r$ hence $z \in B_r(x)$ and therefore $x \in B_q(y) \subseteq B_r(x) \subseteq U$ which implies that $\mathcal{B}$ is a countable basis for (X, d) and thus (X, d) is second countable. □

Proof. **2-21** We want to show that every locally Euclidean space is first countable.

We know that $\mathbb{R}^n$ is first countable, and that

$$\mathcal{B} = \{B_q(x) : x \in \mathbb{R}^n \text{ and } q \in \mathbb{Q}\}$$

is a countable basis for $\mathbb{R}^n$.

Let M be a locally Euclidean space. Let us take $p \in M$ with a neighborhood $U \subseteq M$ then since M is locally Euclidean there must be some map $\varphi : U \to B_r$ which is a homeomorphism to an open ball $B_r \subseteq \mathbb{R}^n$. Then $B_r(\varphi(p))$ is a neighborhood of $\varphi(p)$ and since $\mathcal{B}$ is a basis for $\mathbb{R}^n$ there is some $q \in \mathbb{Q}$ such that $B_q(\varphi(p)) \subseteq B_r(\varphi(p))$ but since φ is a bijection there is $V_q = \varphi^{-1}(B_q(\varphi(p)))$ such that $V_q \subseteq U$ where V_q is a neighborhood of p.

Since we can do this for any neighborhood U of p then the collection V_q is a countable basis for p and therefore M is first countable. $\square$

Proof. **2-23** Let M be a manifold. We know that for every $p \in M$ there is $\varphi : U_p \to O_p$ which is a homeomorphism from a neighborhood U_p of p to an open set $O_p \in \mathbb{R}^n$ which exists since M is locally Euclidean.

So let us build a collection $\mathcal{B}_p = \{\varphi^{-1}(B_r(\varphi(p))) : r > 0\}$ which is a collection of open sets of U_p which are homeomorphic to an open ball in $\mathbb{R}^n$. Then we can build a collection $\mathcal{B} = \bigcup_{p \in M} \mathcal{B}_p$. We want to prove $\mathcal{B}$ is a basis for M.

So let $V \subseteq M$ be an open set where $p \in V$ and let us consider the set $U = V \cap U_p$ then we see that U is open, $p \in U$ and $\varphi(U)$ is open in $\mathbb{R}^n$ so there is some ball $B_r(\varphi(p)) \subseteq \varphi(U)$ but $\varphi^{-1}(B_r(\varphi(p)))$ is in $\mathcal{B}$ then we have that

$$p \in \varphi^{-1}(B_r(\varphi(p))) \subset U \subset V$$

which implies that $\mathcal{B}$ is a basis for M. Therefore every manifold M has a basis of coordinate balls. $\square$

Proof. **2-25** Let M be an n-dimensional manifold with boundary and let $p \in \mathrm{Int}M$ then there is a domain of an interior chart $U \subseteq M$ of p such that $\varphi(U)$ is an open set in $\mathbb{R}^n$ where φ is a homeomorphism.

We want to prove first that $U \subseteq \mathrm{Int}M$ by contradiction. Let us suppose that there is at least one point $q \in U$ such that $q \notin \mathrm{Int}M$ then $q \in \partial M$ and so $\varphi(q)$ must be in $\partial\mathbb{H}^n$ so U is not the domain of an interior chart which is a contradiction and must be that $U \subseteq \mathrm{Int}M$.

Then $\mathrm{Int}M$ is second countable, a Hausdorff space and locally Euclidean therefore $\mathrm{Int}M$ is an n-manifold without boundary.

On the other hand, for each $p \in \mathrm{Int}M$ we have a domain of an interior chart (or neighborhood) $U_p \subseteq \mathrm{Int}M$ then $\mathrm{Int}M = \bigcup_{p \in \mathrm{Int}M} U_p$ and since each U_p is open then $\mathrm{Int}M$ is open. $\square$