# Solutions to selected problems on Introduction to Topological Manifolds - John M. Lee.

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## Chapter 2 - Topological Spaces

### Problems

*Proof.* **2-1**

(a) We want to show that $\mathcal{T}_1 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is finite}\}$ is a topology on $X$.

(i) By definition $\emptyset$ is in $\mathcal{T}_1$. If $U = X$ then $X \setminus X = \emptyset$ and $\emptyset$ is finite then $X \in \mathcal{T}_1$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_1$ such that $U_i = \emptyset$ or $X \setminus U_i$ is finite for every $i$. Also, we see that

$$X \setminus (U_1 \cap ... \cap U_n) = (X \setminus U_1) \cup ... \cup (X \setminus U_n)$$

And the finite union of finite sets is itself a finite set hence $U_1 \cap ... \cap U_n \in \mathcal{T}_1$. We assumed that not all of the elements are empty, but otherwise we already saw that $\emptyset \in \mathcal{T}_1$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_1$ such that $U_i = \emptyset$ or $X \setminus U_i$ is finite for every $i$. Also, we have that

$$X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} X \setminus U_\alpha$$

So this is the intersection between finite sets then itself it's a finite set hence $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_1$.

Therefore $\mathcal{T}_1$ is a topology on $X$.

(b) We want to show that $\mathcal{T}_2 = \{U \subseteq X : U = \emptyset \text{ or } X \setminus U \text{ is countable}\}$ is a topology on $X$.

(i) By definition $\emptyset$ is in $\mathcal{T}_2$. If $U = X$ then $X \setminus X = \emptyset$ and $\emptyset$ is countable then $X \in \mathcal{T}_2$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_2$ such that $U_i = \emptyset$ or $X \setminus U_i$ is countable for every $i$. Also, we see that

$$X \setminus (U_1 \cap ... \cap U_n) = (X \setminus U_1) \cup ... \cup (X \setminus U_n)$$

And the finite union of countable sets is itself a countable set hence $U_1 \cap ... \cap U_n \in \mathcal{T}_2$. We assumed that not all of the elements are empty, but otherwise we already saw that $\emptyset \in \mathcal{T}_2$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_2$ such that $U_i = \emptyset$ or $X \setminus U_i$ is countable for every $i$. Also, we have that

$$X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} X \setminus U_\alpha$$

So this is the intersection between countable sets then itself it's a countable set hence $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_2$.

Therefore $\mathcal{T}_2$ is a topology on $X$.

(c) We want to show that $\mathcal{T}_3 = \{U \subseteq X : U = \emptyset \text{ or } p \in U\}$ is a topology on $X$.

(i) By definition $\emptyset$ is in $\mathcal{T}_3$. Since $p \in X$ by definition then $X \in \mathcal{T}_3$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_3$ such that $U_i = \emptyset$ or $p \in U_i$ for every $i$ then $U_1 \cap ... \cap U_n$ at least have the element $p$ in common so $U_1 \cap ... \cap U_n \in \mathcal{T}_3$. This result is true assuming not every $U_i = \emptyset$ otherwise $U_1 \cap ... \cap U_n = \emptyset$ and we saw that $\emptyset \in \mathcal{T}_3$ so anyway $U_1 \cap ... \cap U_n \in \mathcal{T}_3$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_3$ such that $U_i = \emptyset$ or $p \in U_i$ for every $i$. Then assuming not every $U_i = \emptyset$ we have that $p \in \bigcup_{\alpha \in A} U_\alpha$ which implies that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_3$. If every $U_i = \emptyset$ then $\bigcup_{\alpha \in A} U_\alpha = \emptyset$ and we saw that $\emptyset \in \mathcal{T}_3$ so anyway $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_3$.

Therefore $\mathcal{T}_3$ is a topology on $X$.

(d) We want to show that $\mathcal{T}_4 = \{U \subseteq X : U = X \text{ or } p \notin U\}$ is a topology on $X$.

(i) By definition $X$ is in $\mathcal{T}_4$. Since $p \notin \emptyset$ then $\emptyset \in \mathcal{T}_4$.

(ii) Let $U_1, ..., U_n$ be elements of $\mathcal{T}_4$ such that $U_i = X$ or $p \notin U_i$ for every $i$ then $p$ is not in $U_1 \cap ... \cap U_n$. This result is true assuming not every $U_i = X$ otherwise $U_1 \cap ... \cap U_n = X$ and we saw that $X \in \mathcal{T}_4$ so anyway $U_1 \cap ... \cap U_n \in \mathcal{T}_4$.

(iii) Let $(U_\alpha)_{\alpha \in A}$ be a family of elements of $\mathcal{T}_4$ such that $U_i = X$ or $p \notin U_i$ for every $i$. Then assuming no $U_i = X$ we have that $p \notin \bigcup_{\alpha \in A} U_\alpha$ which implies that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_4$. If some $U_i = X$ then $\bigcup_{\alpha \in A} U_\alpha = X$ and we saw that $X \in \mathcal{T}_4$ so anyway $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_4$.

Therefore $\mathcal{T}_4$ is a topology on $X$.

(e) We want to determine if $\mathcal{T}_5 = \{U \subseteq X : U = X \text{ or } X \setminus U \text{ is infinite}\}$ is a topology on $X$. If we let $X = \mathbb{Z}$ then $\mathbb{Z}^+ \in \mathcal{T}_5$ and $\mathbb{Z}^- \in \mathcal{T}_5$ but $\mathbb{Z} \setminus (\mathbb{Z}^+ \cup \mathbb{Z}^-) = \{0\}$ but $\{0\}$ is finite so $(\mathbb{Z}^+ \cup \mathbb{Z}^-) \notin \mathcal{T}_5$. Therefore $\mathcal{T}_5$ is not a topology on $X$.

$\square$

*Proof.* **2-3**

(a) Let $x \in \overline{X \setminus B}$ then for every neighborhood $U$ where $x \in U$ contains a point of $X \setminus B$ this implies that no neighborhood that contains $x$ is in $\text{Int}(B)$ hence $x \in X \setminus \text{Int}(B)$ and $\overline{X \setminus B} \subseteq X \setminus \text{Int}(B)$.

Let $x \in X \setminus \text{Int}(B)$ then $x \notin \text{Int}(B)$ so there is no neighborhood of $x$ contained in $B$ then for every neighborhood $U$ that contains $x$ we have that $U \cap X \setminus B \neq \emptyset$ hence $x \in \overline{X \setminus B}$ and $X \setminus \text{Int}(B) \subseteq \overline{X \setminus B}$.

Therefore $\overline{X \setminus B} = X \setminus \text{Int}(B)$.

(b) Let $x \in \text{Int}(X \setminus B)$ then $x$ has a neighborhood contained in $X \setminus B$ but then $x$ is also in $\text{Ext}(B) = X \setminus \overline{B}$ then $\text{Int}(X \setminus B) \subseteq X \setminus \overline{B}$.

In the same way if $x \in X \setminus \overline{B} = \text{Ext}(B)$ then it has a neighborhood in $X \setminus B$ hence $X \setminus B$ is open and hence $x \in \text{Int}(X \setminus B)$ this implies that $X \setminus \overline{B} \subseteq \text{Int}(X \setminus B)$.

Therefore $\text{Int}(X \setminus B) = X \setminus \overline{B}$.

$\square$

*Proof.* **2-4**

(a) We know that $\bigcap_{A\in\mathcal{A}} \overline{A}$ is a closed set where $\bigcap_{A\in\mathcal{A}} A \subseteq \bigcap_{A\in\mathcal{A}} \overline{A}$ but also we know that the closure of $\bigcap_{A\in\mathcal{A}} A$ is the smallest closed set containing $\bigcap_{A\in\mathcal{A}} A$. Therefore it must happen that $\overline{\bigcap_{A\in\mathcal{A}} A} \subseteq \bigcap_{A\in\mathcal{A}} \overline{A}$.

Let $\mathcal{A} = \{(0,1),(1,2)\}$ where they are intervals of $\mathbb{R}$ then $\bigcap_{A\in\mathcal{A}} \overline{A} = \{1\}$ and $\overline{\bigcap_{A\in\mathcal{A}} A} = \emptyset$ so we see that $\bigcap_{A\in\mathcal{A}} \overline{A} \not\subseteq \overline{\bigcap_{A\in\mathcal{A}} A}$. Therefore when $\mathcal{A}$ is a finite collection the equality is not preserved.

(b) We know that $A \subseteq \bigcup_{A\in\mathcal{A}} A$ then $A \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$ also $\overline{A} \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$ therefore

$$\bigcup_{A\in\mathcal{A}} \overline{A} \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$$

Let us assume now that $\mathcal{A}$ is a finite collection. Then we see that $\bigcup_{A\in\mathcal{A}} \overline{A}$ is a closed set and that $\bigcup_{A\in\mathcal{A}} A \subseteq \bigcup_{A\in\mathcal{A}} \overline{A}$ but also we know that $\overline{\bigcup_{A\in\mathcal{A}} A}$ is the smallest closed set that contains $\bigcup_{A\in\mathcal{A}} A$ therefore it must happen that

$$\overline{\bigcup_{A\in\mathcal{A}} A} \subseteq \bigcup_{A\in\mathcal{A}} \overline{A}$$

Hence the equality holds when $\mathcal{A}$ is a finite collection.

(c) Let $x \in \mathrm{Int}(\bigcap_{A\in\mathcal{A}} A)$ then there is a neighborhood such that $U \subseteq \bigcap_{A\in\mathcal{A}} A$ hence $U \subseteq A$ for every $A \in \bigcap_{A\in\mathcal{A}} A$ hence because of Proposition 2.8 (a) we have that $x \in \mathrm{Int}(A)$ but since this is true for every $A$ then must be that $x \in \bigcap_{A\in\mathcal{A}} \mathrm{Int}(A)$. Therefore

$$\mathrm{Int}\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} \mathrm{Int}(A)$$

Let us assume now that $\mathcal{A}$ is a finite collection. Then we see that $\bigcap_{A\in\mathcal{A}} \mathrm{Int}(A)$ is an open set and $\bigcap_{A\in\mathcal{A}} \mathrm{Int}(A) \subseteq \bigcap_{A\in\mathcal{A}} A$ but also we know that $\mathrm{Int}(\bigcap_{A\in\mathcal{A}} A)$ is the biggest open set contained in $\bigcap_{A\in\mathcal{A}} A$ therefore it must be that

$$\bigcap_{A\in\mathcal{A}} \mathrm{Int}(A) \subseteq \mathrm{Int}\left(\bigcap_{A\in\mathcal{A}} A\right)$$

Hence the equality holds when $\mathcal{A}$ is a finite collection.

(d) We know that $\bigcup_{A\in\mathcal{A}} \mathrm{Int}(A)$ is open since it's an arbitrary union of open sets and also we see that $\bigcup_{A\in\mathcal{A}} \mathrm{Int}(A) \subseteq \bigcup_{A\in\mathcal{A}} A$ but also we know that $\mathrm{Int}\left(\bigcup_{A\in\mathcal{A}} A\right)$ is the biggest open set contained in $\bigcup_{A\in\mathcal{A}} A$ therefore it must happen that

$$\bigcup_{A\in\mathcal{A}} \mathrm{Int}(A) \subseteq \mathrm{Int}\left(\bigcup_{A\in\mathcal{A}} A\right)$$

Let $\mathcal{A} = \{[0,1],[1,2]\}$ where they are intervals of $\mathbb{R}$ then $\bigcup_{A\in\mathcal{A}} \mathrm{Int}(A) = (0,1)\cup(1,2)$ and $\mathrm{Int}(\bigcup_{A\in\mathcal{A}} A) = (0,2)$ so we see that $\mathrm{Int}(\bigcup_{A\in\mathcal{A}} A) \not\subseteq \bigcup_{A\in\mathcal{A}} \mathrm{Int}(A)$. Therefore when $\mathcal{A}$ is a finite collection the equality is not preserved.

□

*Proof.* **2-5**

(a) Let $X = [0,\infty)\times\{0\} \subset \mathbb{R}^2$ and $Y = [-1,1]\times\{0\} \subset \mathbb{R}^2$ such that $f(x) = \sin(1/x)$ for $x > 0$ and if $x = 0$ we have that $f(0) = 0$.

Let $U \subset X$ be an open set such that $U$ is of the form $(a,b)$ which is a basis and since $U \subset (0,\infty)$ and $f$ is continuous in $(0,\infty)$ because of the intermediate value theorem, we have that $f(U)$ is also open, therefore $f$ is an open map.

Let $E = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ we know that $E \subset X$ and $E$ is a closed set then let us consider $f(E) = \{\sin(n) : n \in \mathbb{N}\} \cup \{0\}$ we see that $f(E)$ is a dense set hence if we let $x \in [-1,1]$ such that $x \notin f(E)$ then every neighborhood of $x$ will have a point of $f(E)$ so $x$ is a limit point of $f(E)$ which is not contained in $f(E)$, therefore $f(E)$ is not closed and $f$ is not a closed map.

Let us consider now a sequence $(x_n) \subseteq X$ defined as $x_n = 1/(\pi n + \pi/2)$ we see that $x_n \to 0$ but $f(x_n) = -1$ if $n$ is odd and $f(x_n) = 1$ if $n$ is even hence $f$ does not converge and $f$ is therefore not continuous.

(b) Let $X = \mathbb{R}\times\{0\}$ and $Y = \mathbb{R}\times\{0\}$ such that $f(x) = 0$ if $x \neq 0$ and $f(x) = 1$ if $x = 0$.

Let $E \subset X$ be a closed set then $f(E) = \{0\}$ or $f(E) = \{1\}$ or $f(E) = \{0,1\}$ in any case they are closed sets since they are singletons hence $f$ is a closed map.

Let $U \subset X$ be an open set then $f(U) \subseteq \{0,1\}$ and we know that neither $\{0,1\}$ nor $\{0\}$ nor $\{1\}$ are open sets hence $f$ is not an open map.

Finally $f$ is not continuous at $y = 0$ since if we take $\epsilon = 1/2$ we have that $|f(x) - f(0)| = |0 - 1| = 1 > 1/2$ no matter which $\delta > 0$ we take.

(c) Let $X = \mathbb{R} \times \{0\}$ and $Y = \mathbb{R} \times \{0\}$ such that $f(x) = 0$ if $x \leq 0$ and $f(x) = \arctan(x)$ if $x > 0$. We see that $f$ is continuous.

Let $U = (-1, 0) \subset \mathbb{R}$ we know $U$ is an open set of $X$ then $f(U) = \{0\}$ but $\{0\}$ is not an open set in $Y = \mathbb{R}$. Therefore $f$ is not an open map.

Let $E = [0, \infty) \subset X$ we know that $E$ is closed in $X$, also we see that $f(E) = [0, 1)$ but $[0, 1)$ is not an open nor a closed set in $Y = \mathbb{R}$ therefore $f$ is not a closed map.

(d) Let $X = [0, \infty) \times \{0\}$ and $Y = \mathbb{R} \times \{0\}$ such that $f(x) = \arctan(x)$ we see that $f$ is continuous.

Let $U \subset X$ be an open set such that $U$ is of the form $(a, b)$ which is a basis and since $U \subset (0, \infty)$ and $f$ is continuous in $(0, \infty)$ because of the intermediate value theorem, we have that $f(U)$ is also open, therefore $f$ is an open map.

Let $E = [0, \infty) \subset X$ we know that $E$ is closed in $X$, also we see that $f(E) = [0, 1)$ but $[0, 1)$ is not an open nor a closed set in $Y = \mathbb{R}$ therefore $f$ is not a closed map.

(e) Let $X = \mathbb{R} \times \{0\}$ and $Y = \mathbb{R} \times \{0\}$ such that $f(x) = 0$ we see that $f$ is continuous.

Let $E \subset X$ be a closed set then $f(E) = \{0\}$ which is a closed set in $Y = \mathbb{R}$ since it is a singleton hence $f$ is a closed map.

Let $U \subset X$ be an open set then $f(U) = \{0\}$ and we know that $\{0\}$ is not an open sets in $Y = \mathbb{R}$ hence $f$ is not an open map.

(f) Let $X = \mathbb{R} \times \{0\}$ and $Y = (-\infty, 1] \cup (2, \infty) \times \{0\}$ such that $f(x) = x$ if $x \leq 1$ and $f(x) = 2x$ if $x > 1$.

Let $U \subset X$ be an open set such that $U$ is of the form $(a, b)$ which is a basis hence it's valid for any open set if $(a, b) \subset (-\infty, 1]$ we know that $f$ is continuous in this interval so because of the Intermediate Value Theorem $f(U)$ is an open map and in the same way if $(a, b) \subset (1, \infty)$ then $f(U)$ is open because $f$ is continuous in this interval and the Intermediate Value Theorem. Now suppose $(a, b)$ such that $a < 1 < b$ then $(a, b) = (a, 1] \cup (1, b)$ and we see that $f((a, 1]) = (a, 1]$ which is open in $(-\infty, 1]$ since $(-\infty, 1] \setminus (a, 1] = (-\infty, a]$ which we know is a closed set hence $(a, 1]$ is an open set. Also, we have that $f((1, b)) = (2, 2b)$ which is an open set so in this case $f((a, b))$ is the union of two open sets i.e. it's an open set. Therefore $f$ is an open map.

Let $E \subseteq X$ be a closed set and let us take a sequence $(y_n) \subseteq f(E)$ such that $y_n \to y$ we want to prove that $y \in f(E)$ which would imply that $f(E)$ is closed. By definition there is $x_n \in E$ such that $f(x_n) = y_n$.

But also we know that $x_n = y_n$ or $x_n = y_n/2$ or a combination of both but only for a finite number of points by the definition of $f$.

In the first case, this implies that $x_n \to x$ where $x = y$ and since $E$ is a closed set then $x \in E$ but also we know that $f$ is continuous in $(-\infty, 1]$ hence $y \in f(E)$.

Lastly if $x_n = y_n/2$ we have that $x_n \to x/2$ where $x/2 = y$ and since $E$ is a closed set then $x \in E$ but also we know that $f$ is continuous in $(1, \infty)$ hence $y \in f(E)$.

Therefore $f$ is a closed map.

$\square$

*Proof.* **2-6**

(a) $(\Rightarrow)$ Let $f$ be continuous and let $A \subseteq X$ also $\overline{f(A)}$ is closed on $Y$ hence $f^{-1}(\overline{f(A)})$ is closed on $X$ because $f$ is continuous but also we know that $A \subseteq f^{-1}(\overline{f(A)})$ and since $f^{-1}(\overline{f(A)})$ is closed it must happen that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ this in turn implies that $f(\overline{A}) \subseteq f(f^{-1}(\overline{f(A)})) \subseteq \overline{f(A)}$.

$(\Leftarrow)$ Let $B \subseteq Y$ be a closed set we want to show that $A = f^{-1}(B)$ is also closed in $X$. Given that $f(\overline{A}) \subseteq \overline{f(A)}$ we have that

$$f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} \subseteq \overline{B} = B$$

since $B$ is closed. Then we have that $f(\overline{A}) \subseteq B$ hence

$$\overline{A} \subseteq f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A$$

Therefore $A$ is closed since it contains $\overline{A}$.

(b) ($\Rightarrow$) Let $f$ be a closed map and let $A$ be any set of $X$ we know that $A \subseteq \overline{A}$ then $f(A) \subseteq f(\overline{A})$ but since $f$ is a closed map then $f(\overline{A})$ is closed which implies that the closure of $f(A)$ must be contained in $f(\overline{A})$ i.e. $\overline{f(A)} \subseteq f(\overline{A})$.

($\Leftarrow$) Let $A$ be a closed set of $X$ we want to prove that $f(A)$ is also closed. Since $A$ is closed we have that $A = \overline{A}$ hence $f(A) = f(\overline{A})$ but we know that $\overline{f(A)} \subseteq f(\overline{A}) = f(A)$ thus the closure of $f(A)$ is contained or equal to $f(A)$ therefore $f(A)$ is closed implying that $f$ is a closed map.

(c) ($\Rightarrow$) Let $f$ be continuous and let $B \subseteq Y$ we know that by definition $\mathrm{Int}(B) \subseteq B$ so $f^{-1}(\mathrm{Int}(B)) \subseteq f^{-1}(B)$ we know that $f^{-1}(\mathrm{Int}(B))$ is open since $f$ is continuous and $\mathrm{Int}(B)$ is an open set hence it must also happen that $f^{-1}(\mathrm{Int}(B)) \subseteq \mathrm{Int}(f^{-1}(B))$ since by definition $\mathrm{Int}(f^{-1}(B))$ is the largest open set contained in $f^{-1}(B)$.

($\Leftarrow$) Let $B \subseteq Y$ be an open set we want to prove that $f^{-1}(B)$ is an open set which implies that $f$ is continuous. Since $B$ is open we have that $B = \mathrm{Int}(B)$ but also we know that $f^{-1}(B) = f^{-1}(\mathrm{Int}(B)) \subseteq \mathrm{Int}(f^{-1}(B))$ and by definition we know that $\mathrm{Int}(f^{-1}(B)) \subseteq f^{-1}(B)$ therefore $\mathrm{Int}(f^{-1}(B)) = f^{-1}(B)$ which implies that $f^{-1}(B)$ is an open set.

(d) ($\Rightarrow$) Let $f$ be an open map and let $B \subseteq Y$ also let us name $C = \mathrm{Int}(f^{-1}(B)) \subseteq f^{-1}(B)$ then $f(C) \subseteq f(f^{-1}(B)) \subseteq B$ where $f(C)$ is open since $f$ is an open map hence it must also happen that $f(C) \subseteq \mathrm{Int}(B)$ since $\mathrm{Int}(B)$ is the biggest open set contained in $B$ so we have that $\mathrm{Int}(f^{-1}(B)) = C \subseteq f^{-1}(f(C)) \subseteq f^{-1}(\mathrm{Int}(B))$.

($\Leftarrow$) Let $A \subseteq X$ be an open set we want to prove that $f(A)$ is an open set too which implies that $f$ is open. Since $f(A)$ is a set of $Y$ we know that $\mathrm{Int}(f^{-1}(f(A))) \subseteq f^{-1}(\mathrm{Int}(f(A)))$ also we have that $\mathrm{Int}A \subseteq \mathrm{Int}(f^{-1}(f(A)))$ then joining this two equations and applying $f$ to both sides we have that

$$f(A) = f(\mathrm{Int}(A)) \subseteq f(f^{-1}(\mathrm{Int}(f(A)))) \subseteq \mathrm{Int}(f(A))$$

where we used that $\mathrm{Int}(A) = A$ since $A$ is an open set. Therefore we have that $f(A) \subseteq \mathrm{Int}(f(A))$ but also by definition we know that $\mathrm{Int}(f(A)) \subseteq f(A)$ which implies that $\mathrm{Int}(f(A)) = f(A)$ thus $f(A)$ is an open set and $f$ is an open map.

$\square$