Solutions to selected problems on Introduction to Topological Manifolds - John M. Lee.

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Chapter 3 - New Spaces from Old

Exercises

Proof. **Exercise 3.1.** Let $S \subseteq X$ be any subset of the topological space X. We want to show $\mathcal{T}_S$ is a topology on S.

(i) Given that X is an open set then $S \cap X = S$ is in $\mathcal{T}_S$ also we know that $\emptyset$ is in the topology of X so $\emptyset$ is open in X hence $S \cap \emptyset = \emptyset$ is in $\mathcal{T}_S$.

(ii) Let $U_1, U_2 \in \mathcal{T}_S$ then we know there are open sets $V_1, V_2 \subseteq X$ such that $U_1 = S \cap V_1$ and $U_2 = S \cap V_2$ so we have that

$$\begin{aligned} U_1 \cap U_2 &= (S \cap V_1) \cap (S \cap V_2) \\ &= S \cap (V_1 \cap V_2) \end{aligned}$$

but since $V_1 \cap V_2$ is open in X we have that $U_1 \cap U_2$ is in $\mathcal{T}_S$. We can repeat this process for finitely many $U_i \subseteq S$ since each U_i is related to some open set $V_i \subseteq X$ and we know that the finite intersection of open sets in X is open. Therefore $U_1 \cap ... \cap U_n$ is in $\mathcal{T}_S$.

(iii) Let $U, U' \in \mathcal{T}_S$ then there are open sets $V, V' \subseteq X$ such that $U = S \cap V$ and $U' = S \cap V'$ so we can write that

$$\begin{aligned} U \cup U' &= (S \cap V) \cup (S \cap V') \\ &= ((S \cap V) \cup S) \cap ((S \cap V) \cup V') \\ &= S \cap ((V' \cup S) \cap (V' \cup V)) \\ &= (S \cap (V' \cup S)) \cap (V' \cup V) \\ &= S \cap (V' \cup V) \end{aligned}$$

And we know that $V' \cup V$ is open in X so $U \cup U'$ is in $\mathcal{T}_S$. We can continue this process for an arbitrary union $\bigcup_\alpha U_\alpha$ since we can write in the same way we did that

$$\bigcup_\alpha U_\alpha = S \cap \bigcup_\alpha V_\alpha$$

But $\bigcup_\alpha V_\alpha$ is open in X since X is a topological space. Therefore $\bigcup_\alpha U_\alpha$ in in $\mathcal{T}_S$.

All we proved implies that $\mathcal{T}_S$ is a topology on S. □

Proof. **Exercise 3.2.** Let S be a subspace of X.

($\Rightarrow$) Let $B \subseteq S$ be closed set in S then $S \backslash B$ is an open set in S so $S \backslash B \in \mathcal{T}_S$ which implies that there is an open set $V \in X$ such that $S \setminus B = S \cap V$ then $X \setminus V$ is a closed set in X and $B = X \setminus V \cap S$. Therefore there is a closed set $X \setminus V \subseteq X$ such that B is equal to the intersection of S with $X \setminus V$.

($\Leftarrow$) Let now $B = C \cap S$ where $C \subseteq X$ is a closed subset of X. We want to prove that B is closed in S.

We know that $X \setminus C$ is open in X. Let us define $U = S \setminus B$ then we see that $U = S \setminus B \subseteq X \setminus C$ so there is an open subset $X \setminus C \subseteq X$ such that $U = S \cap X \setminus C$ hence U is open in S. Therefore must be that B must be closed in S.

□

Proof. **Exercise 3.3.** Let $x \in S$ and $r > 0$ we want to prove first that $B_r^S(x) = B_r^M(x) \cap S$.

Let $y \in B_r^S(x)$ meaning that $y \in S$ and $d(x, y) < r$. By definition $B_r^M(x) = \{a \in M : d(x, a) < r\}$ since $S \subseteq M$ and we know that $d(x, y) < r$ then $y \in B_r^M(x)$ this implies that $B_r^S(x) \subseteq B_r^M(x) \cap S$.

Let now $y \in B_r^M(x)$ and $y \in S$ then by definition $B_r^S(x) = \{a \in S : d(x, a) < r\}$ then $y \in B_r^S(x)$ which implies that $B_r^M \cap S \subseteq B_r^S(x)$. Therefore we get that $B_r^S(x) = B_r^M(x) \cap S$.

Let now $U \in \mathcal{T}_S$ so $U = V \cap S$ for some open set $V \subseteq M$. We want to show that U is also in the metric topology for S let us call it $\mathcal{T}_d$.

Let $x \in U$ then $x \in V$ too and since M is a metric space there is a ball $B_r^M(x) \subseteq V$. Let now $y \in B_r^S(x)$ then must be that $y \in U$ because otherwise $y \notin V$ which cannot happen hence $B_r^S(x) \subseteq U$. Since this must happen for any $x \in U$ we have that $U \in \mathcal{T}_d$ since U can be built as a union of open balls. Therefore $\mathcal{T}_S \subseteq \mathcal{T}_d$.

Let now $U \in \mathcal{T}_d$ i.e. U can be written as the union of open balls so if $x \in U$ we have a ball $B_r^S(x) \subseteq U$. Also, let $V = \bigcup_{x \in U} B_{r_x}^M(x)$ be an open set in M. So if $y \in B_r^S(x)$ then $y \in B_r^M(x)$ because of what we proved earlier and $B_r^M(x) \subseteq V$. This implies that if $y \in U$ then $y \in V$ hence $U = V \cap S$. Therefore $\mathcal{T}_d \subseteq \mathcal{T}_S$. Finally, we get that $\mathcal{T}_d = \mathcal{T}_S$ as we wanted. □

Proof. **Exercise 3.6.**

(a) Let $U \subseteq S \subseteq X$ where U is open in S and S is open in X we want to show that U is also open in X. Since U is open in S then there is an open set $V \subseteq X$ such that $U = V \cap S$ but since V and S are open in X and a finite intersection of open sets is open in X then U is open in X.

Now let again $U \subseteq S \subseteq X$ where U is closed in S and S is closed in X we want to prove that U is also closed in X. Because of Exercise 3.2. we know that if U is closed in S then there exists a closed set $B \subseteq X$ such that $U = B \cap S$ but since B and S are closed in X and an intersection of arbitrary many closed subsets is closed in X then U is closed in X.

(b) Let U be a subset of S that is open in X we want to prove it is also open in S. We can write U as $U = U \subset S$ and since U is open in X then U is open in S by definition of the subspace topology.

Now let U be a subset of S that is closed in X we want to prove it is also closed in S. As before we can write U as $U = U \cap S$ since U is closed in X then by Exercise 3.2.we know that U is closed in S.

□

Proof. **Exercise 3.7.**

(a) Let U be a set of S then the closure of U in S that we name $\overline{U}_S$ is by definition the smallest closed set on S that contains U. On the other hand, because of Exercise 3.2. we know that $\overline{U} \cap S$ is a closed set on S that contains U then at least must happen that $\overline{U}_S \subseteq \overline{U} \cap S$.

Now let $x \in \overline{U} \cap S$ we want to show that also $x \in \overline{U}_S$. Let us take an open neighborhood V of x in S so V is of the form $V = G \cap S$ where G is an open set in X. We can say that G is a neighborhood of x in X and since $x \in \overline{U}$ then $G \cap U \neq \emptyset$ also, by definition $U \subseteq S$ so we see that $(S \cap G) \cap U = V \cap U \neq \emptyset$ but this implies that x is in the closure of U in S i.e. $x \in \overline{U}_S$. Finally, this implies that $\overline{U}_S = \overline{U} \cap S$.

(b) Let $U \subseteq S$ and let us name $\text{Int}U_S$ the interior of U in S we want to prove that $\text{Int}U \cap S \subseteq \text{Int}U_S$. Let $x \in \text{Int}U \cap S$ so there is a neighborhood V of x that is contained in $U \subseteq S$ hence, this implies too that $x \in \text{Int}U_S$. Therefore $\text{Int}U \cap S \subseteq \text{Int}U_S$.

Finally, suppose now that $X = \mathbb{R}$, $S = [0,1] \cup (2,3)$ and let us take $U = [0,1]$, we see that U is open in S since $U = (-1,2) \cap [0,1]$ hence $\text{Int}U_S = [0,1]$ but $\text{Int}U \cap S = (0,1)$ so we see they are not equal.

□

Proof. **Exercise 3.11.**

(c) ($\Rightarrow$) Let (p_i) be a sequence of S and $p \in S$ such that $p_i \to p$ in S we want to show that $p_i \to p$ in X. Since $p_i \to p$ in S then for every neighborhood $U \subseteq S$ of p there is $N \in \mathbb{N}$ such that $p_i \in U$ for all $i \geq N$. Also, we know that U is of the form $U = V \cap S$ so there is a neighborhood $V \subseteq X$ of p such that $p_i \in V$ for all $i \geq N$ hence $p_i \to p$ in X as well.

($\Leftarrow$) Let (p_i) be a sequence of S and $p \in S$ such that $p_i \to p$ in X we want to show that $p_i \to p$ in S. Since $p_i \to p$ in X then for every neighborhood $V \subseteq X$ of p there is $N \in \mathbb{N}$ such that $p_i \in V$ for all $i \geq N$. Also, we see that $V \cap S \neq \emptyset$ since $p_i, p \in S$ so we can define $U = V \cap S$ which is a neighborhood of p in S for which $p_i \in U$ when $i \geq N$ hence $p_i \to p$ in S as well.

(d) Let X be a Hausdorff space and let S be a subspace of X we want to prove that S is also Hausdorff.

Let $p_1, p_2 \in S$ since p_1 and p_2 are also in X then there exist two neighborhoods $V_1 \subseteq X$ and $V_2 \subseteq X$ for p_1 and p_2 respectively such that $V_1 \cap V_2 = \emptyset$.

On the other hand, we can define two neighborhoods $U_1 = V_1 \cap S$ and $U_2 = V_2 \cap S$ in S for p_1 and p_2 respectively such that $U_1 \cap U_2 = \emptyset$ since we said that $V_1 \cap V_2 = \emptyset$. Finally, this implies that S is a Hausdorff subspace.

(e) Let X be first countable and let S be a subspace of X we want to prove that S is also first countable.

Let $p \in S$ and let us define the following collection

$$\mathcal{B}_p^S = \{B \cap S : B \in \mathcal{B}_p\}$$

where $\mathcal{B}_p$ is the countable neighborhood basis for X at p.

Let now $U \subseteq S$ be a neighborhood of p then U is of the form $U = V \cap S$ for an open set $V \subseteq X$. Since X is first countable there is $B \in \mathcal{B}_p$ such that $p \in B \subseteq V$ but then we have that $p \in B \cap S \subseteq V \cap S = U$ hence this implies that $\mathcal{B}_p^S$ is a neighborhood basis for S at p which is also countable by definition. Therefore S is also first countable.

(f) Let X be second countable and let S be a subspace of X we want to prove that S is also second countable.

Let us define the following collection

$$\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$$

where $\mathcal{B}$ is a countable basis of X. We want to show that $\mathcal{B}_S$ is a countable basis for S.

Let $U \subseteq S$ be an open set then U is of the form $U = V \cap S$ for an open set $V \subseteq X$ and since X is second countable we can write $V = \bigcup_\alpha B_\alpha$ where $B_\alpha \in \mathcal{B}$ then we have that $U = (\bigcup_\alpha B_\alpha) \cap S$ hence $U = \bigcup_\alpha (B_\alpha \cap S)$ so U can be written as the union of some collection of elements of $\mathcal{B}_S$ and we know by definition that $\mathcal{B}_S$ is countable so $\mathcal{B}_S$ is a countable basis for S and therefore S is second countable.

□

Proof. **Exercise 3.13.** Let $S \subseteq X$ be a subspace of a topological space X we want to show $\iota_S : S \to X$ is a topological embedding.

The inclusion map is injective since $\iota_S(x) = \iota_S(y)$ implies that $x = y$ where $x, y \in S$.

Also, let $U \subseteq X$ be an open subset of X then we have that

$$\iota_S^{-1}(U) = \{x \in S : \iota_S(x) \in U\} = \{x \in S : x \in U\} = U \cap S$$

Since S is a subspace of X then $U \cap S$ is open in S and therefore ι_S is continuous

Now we want to prove $\iota_S^{-1}|_{\iota(S)} : \iota_S(S) \to S$ is continuous, let $U \subseteq S$ be an open set. We want to show that $\iota_S(U)$ is open in $\iota_S(S)$. By definition, $\iota_S(S) = S$ and $\iota_S(U) = U$ then since U is open in S we have that $\iota_S(U)$ is open in $\iota_S(S)$. Therefore $\iota_S^{-1}|_{\iota(S)}$ is continuous.

Finally, we want to prove $\iota_S : S \to \iota(S)$ is surjective. Let $y \in \iota_S(S)$ then there is $\iota_S^{-1}|_{\iota(S)}(y) = y$ such that $\iota_S(y) = y$ hence ι_S as defined is surjective.

Adding all we have proven we see that $\iota_S : S \to X$ is a topological embedding. □

Proof. **Exercise 3.17.** Let $[0, 1) \subset \mathbb{R}$ and let $\iota : [0, 1) \to \mathbb{R}$ be the inclusion map then ι is a topological embedding because of Excercise 3.13. Since ι is defined from $[0, 1)$ to $\mathbb{R}$ then $[0, 1)$ is open and closed in $[0, 1)$ but $\iota([0, 1)) = [0, 1)$ is not open nor closed in $\mathbb{R}$ and therefore ι is not open nor closed. □

Proof. **Exercise 3.19.** Let $f : A \to X$ be a surjective topological embedding then f is a homeomorphism onto its image i.e. $f' : A \to f(A)$ is a homeomorphism but we know that also f is bijective (injective by the topological embedding definition and surjective by definition) so $f(A) = X$. Therefore f is a homeomorphism. □

Proof. **Exercise 3.25.** Let

$$\mathcal{B} = \{U_1 \times ... \times U_n : U_i \text{ is an open subset of } X_i, i = 1, ..., n\}$$

We want to prove that $\mathcal{B}$ is a basis for a topology then

(i) First, we want to show that $\bigcup_{B \in \mathcal{B}} B = X$ where $X = X_1 \times ... \times X_n$. But we know that $X \in \mathcal{B}$ since each X_i is open in X_i therefore it must be that

$$\bigcup_{B \in \mathcal{B}} B = X$$

(ii) Let $B_1, B_2 \in \mathcal{B}$ where $B_1 = U_1 \times ... \times U_n$ and $B_2 = V_1 \times ... \times V_n$ then we have that

$$(U_1 \times ... \times U_n) \cap (V_1 \times ... \times V_n) = (U_1 \cap V_1) \times ... \times (U_n \cap V_n)$$

But since U_i, V_i are open in X_i then $U_i \cap V_i$ is also open in X_i. This implies that $B_1 \cap B_2 \in \mathcal{B}$.

Therefore $\mathcal{B}$ is a basis for a topology. $\square$

Proof. **Exercise 3.26.** Let $\mathcal{T}_\rho$ be the max-metric topology on $\mathbb{R}^n$ and $\mathcal{T}_p$ be the product topology on $\mathbb{R}^n$ generated by the following basis

$$\mathcal{B} = \{U_1 \times ... \times U_n : U_i \text{ is an open subset of } \mathbb{R}, i = 1, ..., n\}$$

Also, let U be an open set from the basis $\mathcal{B}$ such that

$$U = (a_1, b_1) \times ... \times (a_n, b_n)$$

Where $a_i, b_i \in \mathbb{R}$. Let $x = (x_1, ..., x_n) \in U$ then for each coordinate i there is ϵ_i such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq (a_i, b_i)$. Let us take $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\}$. Then let us consider a ball $B_\epsilon^\rho(x) \in \mathbb{R}^n$ where the metric ρ is defined as $\rho(x, y) = \max|x_i - y_i|$ so we have that

$$B_\epsilon^\rho(x) \subseteq (a_1, b_1) \times ... \times (a_n, b_n) = U$$

This implies that $\mathcal{T}_p \subseteq \mathcal{T}_\rho$.

Conversely, let $B_\epsilon^\rho(x) \in \mathbb{R}^n$ and let $y \in B_\epsilon^\rho(x)$ we want to find an open set $V \in \mathcal{B}$ where $y \in V$ such that $V \subseteq B_\epsilon^\rho(x)$. But the metric ρ implies that

$$B_\epsilon^\rho(x) = (x_1 - \epsilon, x_1 + \epsilon) \times ... \times (x_n - \epsilon, x_n + \epsilon)$$

We see that if we take $V = B_\epsilon^\rho(x)$ then $V \in \mathcal{B}$. This implies that $\mathcal{T}_\rho \subseteq \mathcal{T}_p$ and therefore $\mathcal{T}_\rho = \mathcal{T}_p$.

Finally, since the max metric and the Euclidean metric are equivalent, i.e. they generate the same open sets we can say that also $\mathcal{T}_{d_2} = \mathcal{T}_p$ where d_2 is the Euclidean metric. $\square$

Proof. **Exercise 3.29.** Let us consider the following diagram

$$\begin{tikzcd}
 & X_1 \times ... \times X_n \arrow[d, "\pi_i"] \\
X_1 \times ... \times X_n \arrow[ur, "i"] \arrow[r, "\pi_i"] & X_i
\end{tikzcd}$$

This is analogous to the Characteristic Property diagram where we set $Y = X_1 \times ... \times X_n$ and we have replaced f to i and f_i to π_i. We define $i : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ as the identity function. Since i is continuous then by the Characteristic Property we have that $\pi_i = \pi_i \circ i$ is also continuous as we wanted. $\square$

Proof. **Exercise 3.32.**

(a) Let us consider three topologies on the set $X_1 \times X_2 \times X_3$ obtained by thinking of it as $X_1 \times X_2 \times X_3$, $(X_1 \times X_2) \times X_3$ and $X_1 \times (X_2 \times X_3)$, we want to show they are equal.

Let us consider the following bases

$$\begin{aligned}
\mathcal{B}_1 &= \{U_1 \times U_2 \times U_3 : U_i \text{ is an open subset of } X_i, i = 1, 2, 3\} \\
\mathcal{B}_2' &= \{U_1 \times U_2 : U_i \text{ is an open subset of } X_i, i = 1, 2\} \\
\mathcal{B}_2 &= \{U \times U_3 : U_3 \text{ is an open subset of } X_3 \\
&\qquad \text{and } U \text{ is an open subset of } \mathcal{B}_2'\} \\
\mathcal{B}_3' &= \{U_2 \times U_3 : U_i \text{ is an open subset of } X_i, i = 2, 3\} \\
\mathcal{B}_3 &= \{U_1 \times U : U_1 \text{ is an open subset of } X_1 \\
&\qquad \text{and } U \text{ is an open subset of } \mathcal{B}_3'\}
\end{aligned}$$

Where $\mathcal{B}_1$ generates the topology of $X_1 \times X_2 \times X_3$, $\mathcal{B}_2$ generates the topology of $(X_1 \times X_2) \times X_3$ and $\mathcal{B}_3$ generates the topology of $X_1 \times (X_2 \times X_3)$.

Let $V_1 \times V_2 \times V_3 \in \mathcal{B}_1$ then V_i is open in X_i for $i = 1, 2, 3$ then $V_1 \times V_2 \in \mathcal{B}_2'$ and $V_2 \times V_3 \in \mathcal{B}_3'$ and then $(V_1 \times V_2) \times V_3 \in \mathcal{B}_2$ and $V_1 \times (V_2 \times V_3) \in \mathcal{B}_3$.

Let $(V_1 \times V_2) \times V_3 \in \mathcal{B}_2$ then $V_1 \times V_2 \in \mathcal{B}_2'$ and hence V_1, V_2 are open in X_1, X_2 respectively but also we know that V_3 is open in X_3 then $V_1 \times V_2 \times V_3 \in \mathcal{B}_1$ but also since $V_2 \times V_3 \in \mathcal{B}_3'$ we have that $V_1 \times (V_2 \times V_3) \in B_3$.

Finally, let $V_1 \times (V_2 \times V_3) \in \mathcal{B}_3$ then in the same way we can show that $V_1 \times V_2 \times V_3 \in \mathcal{B}_1$ and that $(V_1 \times V_2) \times V_3 \in B_2$.

This implies that $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$ and therefore they all generate the same topology.

(b) Let $f : X_i \to X_1 \times ... \times X_n$ be a map given by

$$f(x) = (x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$$

we want to show that f is a topological embedding of X_i into the product space.

Let us apply the Characteristic Property of the Product Topology when $Y = X_i$ i.e.

$$\begin{tikzcd} & X_1 \times ... \times X_n \arrow[d, "\pi_j"] \\ X_i \arrow[ur, "f"] \arrow[r, "f_j"] & X_j \end{tikzcd}$$

then we want to show that each $f_j = \pi_j \circ f$ is continuous for $j = 1, ..., n$ so then f is continuous.

Let $j = i$, given that f sends $x \in X_i$ to $(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ then applying the canonical projection π_i we send $(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ to x therefore f_i is the identity map which is continuous.

If $j \neq i$ we have that f sends $x \in X_i$ to $(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ then applying the canonical projection π_j we send $(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ to x_j which is a constant. Hence f_j sends x to some constant x_j which is continuous.

Therefore for each j, we have that f_j is continuous and hence f is continuous.

Also, if $f(x) = f(y)$ where $x, y \in X_i$ then

$$(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n) = (x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n)$$

hence $x = y$, therefore f is also injective.

Now we have to prove that f is a homeomorphism to its image. Let U be an open set in X_i then

$$f(U) = \{x_1\} \times ... \times \{x_{i-1}\} \times U \times \{x_{i+1}\} \times ... \times \{x_n\}$$

which is open in

$$f(X_i) = \{x_1\} \times ... \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times ... \times \{x_n\}$$

Since each $\{x_j\}$ is open in $\{x_j\}$ for $j \neq i$ and U is open in X_i by definition. Therefore f^{-1} is continuous.

We already know that f is continuous and bijective to its image hence f is a homeomorphism from X_i to $f(X_i)$.

Joining what we have proven we see that f is a topological embedding of X_i into the product space as we wanted.

(c) Let π_i be the canonical projection we want to show that it's an open map.

Let V be an open subset of $X_1 \times ... \times X_n$ then V can be written as a union of elements of the basis $\mathcal{B}$. Let $V = \cup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{B}$ then $\pi_i(V) = \pi_i(\cup_\alpha U_\alpha) = \cup_\alpha \pi_i(U_\alpha)$ hence it's enough to prove that $\pi_i(U_\alpha)$ is open.

Let $U_1 \times ... \times U_n \in \mathcal{B}$ which is an open subset of $X_1 \times ... \times X_n$ then $\pi_i(U_1 \times ... \times U_n) = U_i$ and U_i by definition is open. Therefore π_i is an open map.

(d) Let $\mathcal{B}_i$ be a basis for the topology of X_i we want to prove that

$$\mathcal{B} = \{B_1 \times ... \times B_n : B_i \in \mathcal{B}_i\}$$

is a basis for the product topology on $X_1 \times ... \times X_n$.

Let $U \in X_1 \times ... \times X_n$ be an open set in the product topology and let $x \in U$. Then there is an open subset $U_1 \times ... \times U_n$ where each U_i is an open subset of X_i for $i = 1, .., n$ where $x \in U_1 \times ... \times U_n \subseteq U$. But then each U_i can be written as $U_i = \bigcup_\alpha B_{i\alpha}$ hence $x \in \bigcup_\alpha B_{1\alpha} \times ... \times \bigcup_\beta B_{n\beta}$ which implies that $x \in B_{1\alpha} \times ... \times B_{n\beta}$ where each $B_{i\mu} \in \mathcal{B}_i$ therefore

$$x \in B_1 \times ... \times B_n \subset U_1 \times ... \times U_n$$

where $B_1 \times ... \times B_n \in \mathcal{B}$. Finally, this implies that $\mathcal{B}$ is a basis for the product topology.

(e) Let S_i be a subspace of X_i for $i = 1, ..., n$ we want to prove that the product topology and the subspace topology on $S_1 \times ... \times S_n \subseteq X_1 \times ... \times X_n$ are equal.

Let $\mathcal{T}_p$ and $\mathcal{T}_s$ be the product topology and the subspace topology on $S_1 \times ... \times S_n$ respectively.

Let $U \in \mathcal{T}_p$ be an open set then we can write U as

$$U = \bigcup_\alpha U_{1\alpha} \times ... \times U_{n\alpha}$$

where each $U_{i\alpha}$ is an open subset of S_i then since each S_i is a subspace of X_i each $U_{i\alpha}$ can be written as $U_{i\alpha} = S_i \cap V_{i\alpha}$ for some open subset $V_{i\alpha} \subseteq X_i$, therefore we can write that

$$\begin{aligned} U &= \bigcup_\alpha \left[(V_{1\alpha} \cap S_1) \times ... \times (V_{n\alpha} \cap S_n)\right] \\ &= \bigcup_\alpha \left[(V_{1\alpha} \times ... \times V_{n\alpha}) \cap (S_1 \times ... \times S_n)\right] \end{aligned}$$

Then by the definition of subspace topology on $S_1 \times ... \times S_n$ this implies that $U \in \mathcal{T}_s$ and since U is arbitrary we have that $\mathcal{T}_p \subseteq \mathcal{T}_s$.

Let $U \in \mathcal{T}_s$ be an open set then by definition of subpace topology on $S_1 \times ... \times S_n$, U can be written as

$$\begin{aligned} U &= (S_1 \times ... \times S_n) \cap \bigcup_\alpha (V_{1\alpha} \times ... \times V_{n\alpha}) \\ &= \bigcup_\alpha [(S_1 \times ... \times S_n) \cap (V_{1\alpha} \times ... \times V_{n\alpha})] \\ &= \bigcup_\alpha [(V_{1\alpha} \cap S_1) \times ... \times (V_{n\alpha} \cap S_n)] \end{aligned}$$

where we used that $V = \bigcup_\alpha (V_{1\alpha} \times ... \times V_{n\alpha})$ is an open set of $X_1 \times ... \times X_n$. Also, each $V_{i\alpha}$ is an open subset of X_i and since each S_i is a subspace of X_i each $U_{i\alpha} = V_{i\alpha} \cap S_i$ is an open subset of S_i then U can be written as

$$U = \bigcup_\alpha U_{1\alpha} \times ... \times U_{n\alpha}$$

Then by definition of product topology on $S_1 \times ... \times S_n$ this implies that $U \in \mathcal{T}_p$ and since U is arbitrary we have that $\mathcal{T}_s \subseteq \mathcal{T}_p$.

Finally, joining what we proved above we have that $\mathcal{T}_s = \mathcal{T}_p$.

(f) Let $p, q \in X_1 \times ... \times X_n$ be two points then they can be written as $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ where each p_i, q_i is in X_i also, let us assume $p \neq q$.

Given that $p \neq q$ then there is at least a p_i and a q_i such that $p_i \neq q_i$ for those p_i, q_i which are different let us take two open sets U_i and V_i such that $p_i \in U_i$ and $q_i \in V_i$ but also $U_i \cap V_i = \emptyset$ which we know we can select in this way since each X_i is Hausdorff.

For those p_i, q_i where $p_i = q_i$ then we select X_i which is open.

Let us assume without loss of generality that only $p_1 \neq q_1$ then we can build two open sets according to the product topology $U = U_1 \times X_2 \times ... \times X_n$ and $V = V_1 \times X_2 \times ... \times X_n$ such that

$$\begin{aligned} U \cap V &= (U_1 \times X_2 \times ... \times X_n) \cap (V_1 \times X_2 \times ... \times X_n) \\ &= (U_1 \cap V_1) \times (X_2 \cap X_2) \times ... \times (X_n \cap X_n) \\ &= \emptyset \times X_2 \times ... \times X_n \\ &= \emptyset \end{aligned}$$

Therefore in any case this implies that $X_1 \times ... \times X_n$ is Hausdorff as well.

(g) (Without loss of generality we set $n = 2$) Let $p = (x_1, x_2) \in X_1 \times X_2$ and let us consider a collection of neighborhoods of p

$$\mathcal{B}_p = \{B_1 \times B_2 : B_1 \in \mathcal{B}_{x_1} \text{ and } B_2 \in \mathcal{B}_{x_2}\}$$

Where $\mathcal{B}_{x_1}$ and $\mathcal{B}_{x_2}$ are the countable neighborhood bases of $x_1 \in X_1$ and $x_2 \in X_2$ respectively since X_1 and X_2 are first countable. We want to show that $\mathcal{B}_p$ is a countable neighborhood basis of p.

Let $U_p \subseteq X_1 \times X_2$ be a neighborhood of p then by the product topology we know that there is a basis subset $U_1 \times U_2$ such that $p \in U_1 \times U_2 \subseteq U_p$ where U_1, U_2 are open sets of X_1 and X_2 respectively. Then we have that $x_1 \in U_1$ and $x_2 \in U_2$ and hence there is $B_1 \in \mathcal{B}_{x_1}$ and $B_2 \in \mathcal{B}_{x_2}$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$ so by the cartesian product properties we have that

$$p \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U_p$$

Therefore given that the cartesian product of countable sets is countable then $\mathcal{B}_p$ is a countable neighborhood basis of p and by definition of first countability $X_1 \times X_2$ is also first countable.

(h) (Without loss of generality we set $n = 2$) We know that both X_1 and X_2 are second countable so they admit a countable basis $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Let us consider the collection

$$\mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\}$$

Then by part (d) we have that $\mathcal{B}$ is a basis for the product topology on $X_1 \times X_2$.

Also, the cartesian product of countable sets is countable then $X_1 \times X_2$ admits a countable basis and therefore it is second countable.

$\square$

Proof. **Exercise 3.34.** Let $f_1, f_2 : X \to \mathbb{R}$ be two continuous functions. Their pointwise sum and product are defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $(f_1 f_2)(x) = f_1(x) f_2(x)$.

Let us define $f : X \to \mathbb{R}^2$ as $f(x) = (f_1(x), f_2(x))$ so we can build a diagram as follows

$$\begin{tikzcd}
& \mathbb{R}^2 \arrow[d, "\pi_1"] \arrow[dr, bend left, "\pi_2"] & \\
X \arrow[ur, "f"] \arrow[r, "f_1"] \arrow[rr, bend right, "f_2"'] & \mathbb{R} & \mathbb{R}
\end{tikzcd}$$

So because of the Characteristic Property and since f_1 and f_2 are continuous then f is continuous.

Finally since the sum function $+ : \mathbb{R}^2 \to \mathbb{R}$ and the product function $\cdot : \mathbb{R}^2 \to \mathbb{R}$ are continuous as well then the compositions $+(f(x)) = f_1(x) + f_2(x)$ and $\cdot(f(x)) = f_1(x) \cdot f_2(x)$ are also continuous. $\square$

Proof. **Exercise 3.40.** We want to prove that the disjoint union topology on $\coprod_{\alpha \in A} X_\alpha$ is indeed a topology. We defined a subset to be open if and only if its intersection with each set X_α is open in X_α. Hence

(i) We first check that $\coprod_{\alpha \in A} X_\alpha$ and $\emptyset$ are in the topology.

We know that $\coprod_{\alpha \in A} X_\alpha \cap X_\alpha = X_\alpha$ and X_α is open in X_α for any $\alpha \in A$ then $\coprod_{\alpha \in A} X_\alpha$ is in the disjoint union topology.

On the other hand, we know that $\emptyset \cap X_\alpha = \emptyset$ and $\emptyset$ is open in X_α for any $\alpha \in A$ then $\emptyset$ is in the disjoint union topology.

(ii) Let U, V be open sets of the disjoint union topology we want to prove that $U \cap V$ is also in the disjoint union topology.

We know that $U \cap X_\alpha$ and $V \cap X_\alpha$ are open in X_α for any $\alpha \in A$. Hence $(U \cap X_\alpha) \cap (V \cap X_\alpha)$ is also open in X_α because the intersection of finitely many open sets is open in X_α. But also we see that

$$(U \cap X_\alpha) \cap (V \cap X_\alpha) = (U \cap V) \cap (X_\alpha \cap X_\alpha) = (U \cap V) \cap X_\alpha$$

Therefore we see that $U \cap V$ is open in X_α and hence $U \cap V$ is in the disjoint topology.

Finally, in the same way, if we consider finitely many open sets of the disjoint union topology we see that $\bigcap_{i=1}^{n} (U_1 \cap X_\alpha)$ where each U_i is an open set of the disjoint union topology is open in X_α.

(iii) Let U, V be open sets of the disjoint union topology we want to prove that $U \cup V$ is also in the disjoint union topology.

We know that $U \cap X_\alpha$ and $V \cap X_\alpha$ are open in X_α for any $\alpha \in A$. Hence $(U \cap X_\alpha) \cup (V \cap X_\alpha)$ is also open in X_α because the union of arbitrarily many open sets is open in X_α. But also we see that

$$(U \cap X_\alpha) \cup (V \cap X_\alpha) = X_\alpha \cap (U \cup V)$$

Therefore we see that $U \cup V$ is open in X_α and hence $U \cup V$ is in the disjoint topology.

Finally, in the same way, if we consider arbitrarily many open sets of the disjoint union topology we see that $\bigcup_{\beta \in B} (U_\beta \cap X_\alpha)$ where each U_β is an open set of the disjoint union topology is open in X_α.

Adding all we have proven we see that the disjoint union topology is indeed a topology on $\coprod_{\alpha \in A} X_\alpha$. $\square$

Proof. **Exercise 3.43.** Let $(X_\alpha)_{\alpha\in A}$ be an indexed family of topological spaces

(a) ($\Rightarrow$) Let $U \subseteq \coprod_{\alpha\in A} X_\alpha$ be a closed subset then $\coprod_{\alpha\in A} X_\alpha \setminus U$ is an open subset, therefore

$$(\coprod_{\alpha\in A} X_\alpha \setminus U) \cap X_\alpha = (\coprod_{\alpha\in A} X_\alpha \cap X_\alpha) \setminus U = X_\alpha \setminus U$$

is open in X_α for every $\alpha \in A$ hence

$$X_\alpha \setminus (X_\alpha \setminus U) = (X_\alpha \cap U) \cup (X_\alpha \setminus X_\alpha) = X_\alpha \cap U$$

is closed in X_α for every $\alpha \in A$

($\Leftarrow$) Let $U \subseteq \coprod_{\alpha\in A} X_\alpha$ be a subset such that $X_\alpha \cap U$ is closed in X_α for every $\alpha \in A$ then

$$X_\alpha \setminus (X_\alpha \cap U) = (X_\alpha \setminus X_\alpha) \cup (X_\alpha \setminus U) = X_\alpha \setminus U$$

is open in X_α but also we see that $X_\alpha \setminus U = (X_\alpha \setminus U) \cap X_\alpha$ which implies that $X_\alpha \setminus U$ is open in $\coprod_{\alpha\in A} X_\alpha$ by the definition of the disjoint union topology, therefore

$$\coprod_{\alpha\in A} X_\alpha \setminus (X_\alpha \setminus U) = (\coprod_{\alpha\in A} X_\alpha \cap U) \cup (\coprod_{\alpha\in A} X_\alpha \setminus X_\alpha) = U$$

is closed in $\coprod_{\alpha\in A} X_\alpha$. Where we used that this must be true for every $\alpha \in A$ and hence $\coprod_{\alpha\in A} X_\alpha \setminus X_\alpha = \emptyset$.

(b) Let $\iota_\alpha : X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ be the canonical injection map for every $\alpha \in A$. We want to prove it's a topological embedding and an open and closed map.

Let $\iota_\alpha(x) = \iota_\alpha(y)$ for some $x, y \in X_\alpha$ then by definition $\iota_\alpha(x) = x \in X_\alpha$ and $\iota_\alpha(y) = y \in X_\alpha$ hence if $\iota_\alpha(x) = \iota_\alpha(y)$ we have that $x = y$. Therefore ι_α is injective.

Let $U \subseteq \coprod_{\alpha\in A} X_\alpha$ be an open subset then by definition $X_\alpha \cap U$ is open in X_α but also

$$\iota_\alpha^{-1}(U) = \iota_\alpha^{-1}(X_\alpha \cap U) = X_\alpha \cap U$$

Therefore since we saw that $X_\alpha \cap U$ is an open set in X_α then ι_α is continuous.

Let us consider now the inverse map restricted to the image of ι_α i.e.

$$\iota_\alpha^{-1}\big|_{\iota_\alpha(X_\alpha)} : \iota_\alpha(X_\alpha) \to X_\alpha$$

we know that $\iota_\alpha(X_\alpha) = X_\alpha$ hence $\iota_\alpha^{-1}\big|_{\iota_\alpha(X_\alpha)}$ is the identity map which is continuous.

Therefore ι_α is a topological embedding.

Let now $U \subseteq X_\alpha$ be an open set then $\iota_\alpha(U) = U = U \cap X_\alpha$ is open by definition but for any other $\beta \in A$ where $\beta \neq \alpha$ and $X_\alpha \neq X_\beta$ we have that $U \cap X_\beta = \emptyset$ which is also open. Therefore U is open in $\coprod_{\alpha \in A} X_\alpha$ and ι_α is an open map.

Let now $V \subseteq X_\alpha$ be a closed set then $\iota_\alpha(V) = V = V \cap X_\alpha$ is closed by what we proved in part (a) but for any other $\beta \in A$ where $\beta \neq \alpha$ and $X_\alpha \neq X_\beta$ we have that $V \cap X_\beta = \emptyset$ which is also closed. Therefore V is closed in $\coprod_{\alpha \in A} X_\alpha$ and ι_α is a closed map too.

(c) Let each X_α be Hausdorff then for each pair of points $p_1, p_2 \in X_\alpha$ there exist neighborhoods U_1 of p_1 and U_2 of p_2 where $U_1 \cap U_2 = \emptyset$.

If we let $p_1, p_2 \in \coprod_{\alpha \in A} X_\alpha$ then if p_1 and p_2 belong to the same X_α we are done. But if $p_1 \in X_\alpha$ and $p_2 \in X_\beta$ for some $\alpha, \beta \in A$ such that $\alpha \neq \beta$ and $X_\alpha \neq X_\beta$ then we can define $U_1 = X_\alpha$ and $U_2 = X_\beta$ as the neighborhoods of p_1 and p_2 respectively where we have that $U_1 \cap U_2 = \emptyset$ since every X_α is disjoint from each other.

Therefore $\coprod_{\alpha \in A} X_\alpha$ is Hausdorff as well.

(d) Let each X_α to be first countable then for each $p \in X_\alpha$ there is a countable collection of neighborhoods $\mathcal{B}_p^\alpha$ such that any neighborhood of p contains some $B \in \mathcal{B}_p^\alpha$.

Let now $p \in \coprod_{\alpha \in A} X_\alpha$ and a neighborhood $U \subseteq \coprod_{\alpha \in A} X_\alpha$ of p. Then $p \in X_\beta$ for some $\beta \in A$.

Also, by the definition of disjoint union topology, we know that $U \cap X_\beta$ is open in X_β then $U \cap X_\beta$ is a neighborhood of p and since X_β is first countable there is some $B \in \mathcal{B}_p^\beta$ such that $B \subseteq U \cap X_\alpha \subseteq U$. Therefore for each $p \in \coprod_{\alpha \in A} X_\alpha$ we have a countable neighborhood basis i.e. $\coprod_{\alpha \in A} X_\alpha$ is first countable.

(e) Let each X_α be second countable and A the index set be countable. We want to prove that $\coprod_{\alpha \in A} X_\alpha$ is second countable.

Let us define a collection $\coprod_{\alpha \in A} \mathcal{B}_\alpha$ where each $\mathcal{B}_\alpha$ is the countable basis of X_α then this collection is countable since the union of countable sets is countable.

Let now $U \subseteq \coprod_{\alpha \in A} X_\alpha$ be an open set then by the definition of disjoint union topology, we know that $U \cap X_\alpha$ is open in each X_α and hence there is $B \in \mathcal{B}_\alpha$ such that $B \subseteq U \cap X_\alpha$ but this implies that there is $B \in \coprod_{\alpha \in A} \mathcal{B}_\alpha$ such that $B \subseteq U \cap X_\alpha \subseteq U$. Therefore $\coprod_{\alpha \in A} X_\alpha$ admits a countable basis i.e. $\coprod_{\alpha \in A} X_\alpha$ is second countable.

$\square$

Proof. **Exercise 3.44.** Let $(X_\alpha)_{\alpha\in A}$ be an indexed family of nonempty n-manifolds.

($\Rightarrow$) Let $\coprod_{\alpha\in A} X_\alpha$ be a n-manifold then $\coprod_{\alpha\in A} X_\alpha$ is second countable. Let us suppose that A is uncountable we want to arrive at a contradiction.

Let us take a collection of open sets $U_\alpha \subseteq X_\alpha$ for every $\alpha \in A$ then U_α is also open in $\coprod_{\alpha\in A} X_\alpha$. Since $\coprod_{\alpha\in A} X_\alpha$ is second countable then it admits a countable basis $\mathcal{B}$ then there is some $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subseteq U_\alpha$. But this must be true for every open set U_α in the collection hence there are uncountably many B_α in $\mathcal{B}$ which is a contradiction. Therefore it must happen that A is countable.

($\Leftarrow$) Let A be countable.

Since every X_α for $\alpha \in A$ is an n-manifold then each one of them is Hausdorff so by what we showed in Proposition 3.42 (c) we have that $\coprod_{\alpha\in A} X_\alpha$ is Hausdorff.

Since every X_α for $\alpha \in A$ is an n-manifold then each one of them is second countable. Also, A is countable so by what we showed in Proposition 3.42 (e) we have that $\coprod_{\alpha\in A} X_\alpha$ is second countable.

Finally, let $p \in \coprod_{\alpha\in A} X_\alpha$ then $p \in X_\alpha$ for some $\alpha \in A$. Since X_α is an n-manifold then there is a neighborhood of p in X_α which is homeomorphic to an open ball in $\mathbb{R}^n$. But p was arbitrary so this must happen for every $p \in \coprod_{\alpha\in A} X_\alpha$. Therefore $\coprod_{\alpha\in A} X_\alpha$ is Locally Euclidean.

Joining the above results we see that $\coprod_{\alpha\in A} X_\alpha$ is an n-manifold.

$\square$

Proof. **Exercise 3.45.** Let X be any space and let Y be a discrete space. Then by definition $X \times Y$ is the collection of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. But also we know by the definition of disjoint union that $\coprod_{y \in Y} X_y$ is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. Therefore $X \times Y = \coprod_{y \in Y} X_y$.

Let $\mathcal{T}_p$ be the product topology generated by the following basis

$$\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Let also $B \in \mathcal{B}$ then $B = U \times V$ for some $U \subset X$ and $V \subset Y$ then B is the collection of ordered pairs (x, y) such that $x \in U$ and $y \in V$.

Now we want to prove that $B \in \mathcal{T}_d$ where $\mathcal{T}_d$ is the disjoint union topology. We can write B as $B = U \times V = \coprod_{y \in V} U \times \{y\}$ then we see that

$$\begin{aligned}
\left(\coprod_{y \in V} U \times \{y\}\right) \cap X_y &= U \times \{y\} \cap X_y \\
&= U \times \{y\} \cap X \times \{y\} \\
&= (U \cap X) \times \{y\}
\end{aligned}$$

where $U \cap X = U$ which is open by definition and $\{y\}$ is open for any $y \in V$ because Y is a discrete space. Therefore $U \times \{y\}$ is open in X_y and hence $B \in \mathcal{T}_d$ but since every open set of $\mathcal{T}_p$ is the union of basis elements then every open subset of $\mathcal{T}_p$ can be constructed with elements of $\mathcal{T}_d$ i.e. $\mathcal{T}_p \subseteq \mathcal{T}_d$.

Let now $U \in \mathcal{T}_d$ then $U \cap X_y$ is open for every X_y. Then we can write U as

$$U = \bigcup_{y \in Y} U \cap X_y = \bigcup_{y \in Y} U \cap (X \times \{y\}) = \bigcup_{y \in Y} (U \cap X) \times \{y\}$$

Therefore we can write U as the union of a set of cartesian products $(U \cap X) \times \{y\}$ where $U \cap X$ is open in X since $U \in \mathcal{T}_d$ and $\{y\}$ is open in Y since Y is a discrete space. Therefore $U \in \mathcal{T}_p$ which implies that $\mathcal{T}_d \subseteq \mathcal{T}_p$.

Finally, by joining the results we got we have that $\mathcal{T}_d = \mathcal{T}_p$. $\square$

Proof. **Exercise 3.46.** We want to prove that the quotient topology on a set Y is indeed a topology. We defined a subset $U \subseteq Y$ to be open if and only if $q^{-1}(U)$ is open in X for a surjective map $q : X \to Y$ and a topological space X.

(i) We first check that Y and $\emptyset$ are in the topology.

We know that $q^{-1}(Y) = X$ and X is open in X so Y is in the quotient topology.

On the other hand, we know that $q^{-1}(\emptyset) = \emptyset$ which is open in X therefore $\emptyset$ is open in Y and hence it is in the quotient topology

(ii) Let U, V be open sets of the quotient topology on Y we want to prove that $U \cap V$ is also in the quotient topology.

Suppose $U \cap V \neq \emptyset$ otherwise $U \cap V$ is in the quotient topology. We know that $q^{-1}(U)$ and $q^{-1}(V)$ are open in X. Hence $q^{-1}(U) \cap q^{-1}(V)$ is open in X but also we have that

$$\begin{aligned} q^{-1}(U) \cap q^{-1}(V) &= \{x \in X : q(x) \in U\} \cap \{x \in X : q(x) \in V\} \\ &= \{x \in X : q(x) \in U \cap V\} \\ &= q^{-1}(U \cap V) \end{aligned}$$

This implies that $U \cap V$ is open in Y and hence that $U \cap V$ is in the quotient topology.

(iii) Let U, V be open sets of the quotient topology we want to prove that $U \cup V$ is also in the quotient topology.

We know that $q^{-1}(U)$ and $q^{-1}(V)$ are open in X. Hence $q^{-1}(U) \cup q^{-1}(V)$ is open in X but also we have that

$$\begin{aligned} q^{-1}(U) \cup q^{-1}(V) &= \{x \in X : q(x) \in U\} \cup \{x \in X : q(x) \in V\} \\ &= \{x \in X : q(x) \in U \cup V\} \\ &= q^{-1}(U \cup V) \end{aligned}$$

This implies that $U \cup V$ is open in Y and hence that $U \cup V$ is in the quotient topology.

Finally, if we consider arbitrarily many open sets U_α of the quotient topology we see that $\bigcup_{\alpha \in A} q^{-1}(U_\alpha) = q^{-1}(\bigcup_{\alpha \in A} U_\alpha)$ which is open in X and therefore $\bigcup_{\alpha \in A} U_\alpha$ is in the quotient topology.

Adding all we have proven we see that the quotient topology is indeed a topology on Y. $\square$

Proof. **Exercise 3.55.** Let $(X_\alpha)_{\alpha\in A}$ be a set of nonempty Hausdorff spaces and let $\bigvee_{\alpha\in A} X_\alpha$ be the wedge sum of the spaces. We want to prove that $\bigvee_{\alpha\in A} X_\alpha$ is Hausdorff.

Associated with this wedge sum there is a set of base points for each X_α denoted as $\{p_\alpha\}_{\alpha\in A}$ which are equivalent in $\bigvee_{\alpha\in A} X_\alpha$.

Let us take some point of the base points set $p_1 \in \{p_\alpha\}_{\alpha\in A}$ and some other $p_2 \in X_\alpha$ such that $p_1 \neq p_2$ then p_1 by definition is in some X_β where α may be equal to β or not. But since we know that $\coprod_{\alpha\in A} X_\alpha$ is Haussdorff if every X_α is Haussdorff then there is U_1 of p_1 and U_2 of p_2 such that $U_1 \cap U_2 = \emptyset$ for this pair of points.

Let us suppose now we take $p_1, p_2 \in \{p_\alpha\}_{\alpha\in A}$ by definition $p_1 \in X_\alpha$ and $p_2 \in X_\beta$ where $\alpha \neq \beta$. Then as before since $\coprod_{\alpha\in A} X_\alpha$ is Haussdorff if every X_α is Haussdorff then there is U_1 of p_1 and U_2 of p_2 such that $U_1 \cap U_2 = \emptyset$ for this pair of points.

Finally, if we take $p_1, p_2 \in \coprod_{\alpha\in A} X_\alpha$ such that $p_1, p_2 \notin \{p_\alpha\}_{\alpha\in A}$ we already know that $\coprod_{\alpha\in A} X_\alpha$ is Haussdorff if every X_α is Haussdorff.

Therefore joining these cases we see that $\bigvee_{\alpha\in A} X_\alpha$ is Hausdorff. $\square$

Proof. **Exercise 3.59.** Let $q : X \to Y$ be any map.

(a) $\to$ (b) Let $U \subseteq X$ be saturated with respect to q, then $U = q^{-1}(V)$ for some subset $V \subseteq Y$. Then we have that

$$q^{-1}(q(U)) = q^{-1}(q(q^{-1}(V))) = q^{-1}(V) = U$$

Where we used that $q(q^{-1}(V)) = V$ since q is a surjective map.

(b) $\to$ (c) Let $V = q(U)$ be a subset of Y then we can write $U = q^{-1}(V)$. Let $y \in V \subseteq Y$ then $q^{-1}(y) \in q^{-1}(V) = U$ and we can see the same for every $y \in V$ then U is the union of every $q^{-1}(y)$ fiber.

(c) $\to$ (d) Let U be a union of fibers. Let $x \in U$ and $x' \in X$ such that $q(x) = q(x')$ we want to show that also $x' \in U$.

Since U is a union of fiber must happen that $q^{-1}(y) = x$ for some $y \in Y$ but y must be $q(x)$ since $q^{-1}(q(x)) = \{x, x'\}$ because q is surjective and $q(x) = q(x')$ then x' must be in U otherwise x will not be a fiber.

(d) $\to$ (b) If $x \in U$ then every $x' \in X$ such that $q(x) = q(x')$ is also in U. We want to prove that $U = q^{-1}(q(U))$.

Let $x \in U$ such that $q(x) \neq q(x')$ for every other $x' \in X$. then since q is surjective we have that $q^{-1}(q(x)) = x$.

If we let $x, x' \in U$ such that $q(x) = q(x')$ then also since q is surjective we see that $q^{-1}(q(x)) = q^{-1}(q(x')) = x' = x$ which implies that in any case if $x \in U$ we can write it as $q^{-1}(q(x))$ therefore must be that $U = q^{-1}(q(U))$.

$\square$

Proof. **Exercise 3.61.**

($\Rightarrow$) Let $q : X \to Y$ be a surjective map which is also a quotient map. Let $U \subseteq X$ be an open saturated set then there is a subset $V \subseteq Y$ such that $U = q^{-1}(V)$ so we see that $q(U) = q(q^{-1}(V)) = V$ since q is surjective, but also since q is a quotient map V is open in Y because $q^{-1}(V) = U$ is open in X by definition. Therefore q takes saturated open subsets to open subsets.

Now let $U \subseteq X$ be a closed saturated set then there is a subset $V \subseteq Y$ such that $U = q^{-1}(V)$ so we see that $q(U) = q(q^{-1}(V)) = V$ since q is surjective. We want to prove that V is closed. Since q is a quotient map, $Y \setminus V$ is open if $q^{-1}(Y \setminus V)$ is open in X but we see that

$$q^{-1}(Y \setminus V) = X \setminus q^{-1}(V) = X \setminus U$$

And since we know that U is closed then $X \setminus U$ is open which implies that $Y \setminus V$ is open. Therefore V is closed and q takes closed saturated sets to closed sets.

($\Leftarrow$) Let $q : X \to Y$ be a surjective map which takes open saturated sets to open sets. We want to prove that q is a quotient map. Let $V \subseteq Y$ and suppose V is open in Y then there is an open saturated set $U \subseteq X$ such that $q(U) = V$ but since U is saturated we have that $U = q^{-1}(V)$ which is open in X by definition.

On the other hand, suppose $q^{-1}(V) = U$ is open in X then U by definition is saturated but also since q is surjective and q takes open saturated sets to open sets we have that $q(U) = q(q^{-1}(V)) = V$ is open in Y.

Therefore q is a quotient map.

Let $q : X \to Y$ be a surjective map which takes closed saturated sets to closed sets. We want to prove that q is a quotient map. Let $V \subseteq Y$ and suppose V is open in Y then $Y \setminus V$ is closed in Y so there is a closed saturated set $U \subseteq X$ such that $q(U) = Y \setminus V$ but since U is saturated we have that $U = q^{-1}(Y \setminus V)$ which is closed in X by definition. Then we see that $q^{-1}(Y \setminus V) = X \setminus q^{-1}(V)$ is closed which implies that $q^{-1}(V)$ is open in X.

On the other hand, suppose $q^{-1}(V)$ is open in X then $X \setminus q^{-1}(V)$ is closed in X but $X \setminus q^{-1}(V) = q^{-1}(Y \setminus V)$ since q is surjective which implies that $X \setminus q^{-1}(V)$ is saturated by definition. So $q(q^{-1}(Y \setminus V)) = Y \setminus V$ is closed in Y since q takes saturated closed sets to closed sets. Therefore V is open in Y and q is a quotient map.

$\square$

Proof. **Exercise 3.62.**

(a) Let $f : X \to Y$ and $g : Y \to Z$ be quotient maps, we want to prove $h = g \circ f$ is a quotient map.

Suppose $U \subseteq Z$ is open then because g is a quotient map we have that $g^{-1}(U)$ is open in Y but also since f is a quotient map we have that $f^{-1}(g^{-1}(U))$ is open in X. Therefore $h^{-1}(U) = f^{-1}(g^{-1}(U))$ is open.

Now suppose $h^{-1}(U)$ is open in X for some $U \subseteq Z$. We want to prove that U is open in Z. Since $h^{-1}(U)$ is open then $f^{-1}(g^{-1}(U))$ is open but since f is a quotient map this implies that $g^{-1}(U)$ is open in Y but since g is also a quotient map must be that also U is open in Z.

Therefore $h : X \to Z$ is a quotient map.

(b) Let $q : X \to Y$ be an injective quotient map, we want to prove it is a homeomorphism.

Given that q is both surjective and injective then q is a bijection.

Given that q is a quotient map then if U is open in Y we have that $q^{-1}(U)$ is open in X and hence q is continuous.

Suppose $U \subseteq X$ is open we want to prove that $q(U)$ is open in Y. Given that q is bijective we have that $q^{-1}(q(U)) = U$ but this implies that U is saturated, so if a set is open in X must be saturated with respect to q but we know that quotient maps send open saturated sets to open sets therefore $q(U)$ is open in Y as we wanted and thus q^{-1} is continuous.

Joining above results we see that q is a homeomorphism.

(c) Let $q : X \to Y$ be a quotient map.

($\Rightarrow$) Let $K \subseteq Y$ be a closed subset. Then $Y \setminus K$ is open in Y so $q^{-1}(Y \setminus K)$ is open because q is a quotient map. But also we see that $q^{-1}(Y \setminus K) = X \setminus q^{-1}(K)$ so $X \setminus q^{-1}(K)$ is open in X which implies that $q^{-1}(K)$ is closed.

($\Leftarrow$) Let $q^{-1}(K)$ be closed in X for some set $K \subseteq Y$ then $X \setminus q^{-1}(K)$ is open in X but we see that $X \setminus q^{-1}(K) = q^{-1}(Y \setminus K)$ this implies that $Y \setminus K$ is open in Y because q is a quotient map. Finally, we get because of this that K is closed in Y.

(d) Let $q : X \to Y$ be a quotient map and $U \subseteq X$ be a saturated open or closed subset. We want to prove that $q|_U : U \to q(U)$ is a quotient map.

Suppose U is a saturated open subset then it is a union of fibers so every open subset of U is a saturated open subset then $q|_U$ sends saturated open subsets to open subsets because q does. Therefore $q|_U$ is a quoriente map.

The same can be shown if U is a saturated closed subset because q sends saturated closed subsets to closed subsets.

(e) Let $\{q_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ be an indexed family of quotient maps. Let also $q : \coprod_\alpha X_\alpha \to \coprod_\alpha Y_\alpha$ where the restriction of q to each X_α is equal to q_α. We want to prove that q is a quotient map.

Suppose $U \subseteq \coprod_\alpha Y_\alpha$ is an open subset, we want to prove that $q^{-1}(U)$ is open in $\coprod_\alpha X_\alpha$. For each $\alpha \in A$ we have that $Y_\alpha \cap U$ is open in Y_α but also we know that $q^{-1}(Y_\alpha \cap U) = q|_{X_\alpha}^{-1}(Y_\alpha \cap U) = q_\alpha^{-1}(Y_\alpha \cap U)$ which is open in X_α since q_α is a quotient map. Also, we know that $q^{-1}(Y_\alpha \cap U) = X_\alpha \cap q^{-1}(U)$. Therefore by the disjoint union topology definition we get that $q^{-1}(U)$ is open in $\coprod_\alpha X_\alpha$.

Suppose now that $q^{-1}(U)$ is open in $\coprod_\alpha X_\alpha$ for some set $U \subseteq \coprod_\alpha Y_\alpha$. We want to prove that U is open in $\coprod_\alpha Y_\alpha$. Since $q^{-1}(U)$ is open then $q^{-1}(U) \cap X_\alpha$ is open in X_α for every $\alpha \in A$ but we know that $q^{-1}(U) \cap X_\alpha = q^{-1}(Y_\alpha \cap U)$ and $q^{-1}(Y_\alpha \cap U) = q|_{X_\alpha}^{-1}(Y_\alpha \cap U) = q_\alpha^{-1}(Y_\alpha \cap U)$. Also, we know that every q_α is a quotient map so if $q_\alpha^{-1}(Y_\alpha \cap U)$ is open this implies that $Y_\alpha \cap U$ is open in Y_α. Therefore by the disjoint union topology definition we get that U is open in $\coprod_\alpha Y_\alpha$.

Finally, joining above results we get that $q : \coprod_\alpha X_\alpha \to \coprod_\alpha Y_\alpha$ is a quotient map.

$\square$

Proof. **Exercise 3.72.** Let X be a topological space, Y a set and $q : X \to Y$ a surjective map. Let us suppose we have a topology $\mathcal{T}$ and the quotient topology $\mathcal{T}_q$ on Y and for both the characteristic property holds. We want to show that $(Y, \mathcal{T})$ and $(Y, \mathcal{T}_q)$ are homeomorphic which implies that $\mathcal{T} = \mathcal{T}_q$.

Applying the characteristic property of $(Y, \mathcal{T})$ on $Z = (Y, \mathcal{T}_q)$ and taking $f = \iota$ where ι is the identity map we have that

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "\iota \circ q"] & \\ (Y, \mathcal{T}) \arrow[r, "\iota"] & (Y, \mathcal{T}_q) \end{tikzcd}$$

But we know that $\iota \circ q = q$ is continuous so ι is continuous.

On the other hand, applying the characteristic property of $(Y, \mathcal{T}_q)$ on $Z = (Y, \mathcal{T})$ and taking $f = \iota$ where ι is the identity map we have that

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "\iota \circ q"] & \\ (Y, \mathcal{T}_q) \arrow[r, "\iota"] & (Y, \mathcal{T}) \end{tikzcd}$$

Again we know that $\iota \circ q = q$ is continuous so ι is continuous.

Therefore implies that ι is a homeomorphism between $\mathcal{T}$ and $\mathcal{T}_q$. $\square$