Solutions to selected problems on Introduction to Topological Manifolds - John M. Lee.

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Chapter 2 - Topological Spaces

Exercises

Proof. **Exercise 2.2.**

(a) Let X be any set such that every subset of X is open.
Let also $\mathcal{T} = \mathcal{P}(X)$ we want to prove $\mathcal{T}$ is a topology on X.

  (i) Given that $\mathcal{T} = \mathcal{P}(X)$ is the power set of X, by definition $X \in \mathcal{P}(X)$ and $\emptyset \in \mathcal{P}(X)$.

  (ii) Let $U_1, ..., U_n$ be a finite set of elements of $\mathcal{T} = \mathcal{P}(X)$ then by definition $U_1 \cap ... \cap U_n$ is a set of elements of X and hence they are also in $\mathcal{P}(X) = \mathcal{T}$.

  (iii) Let $(U_\alpha)_{\alpha \in A}$ be any (finite or infinite) family of elements of $\mathcal{T} = \mathcal{P}(X)$ then their union $\bigcup_{\alpha \in A} U_\alpha$ is the union of sets from X hence it's a subset of X then they are also in $\mathcal{P}(X) = \mathcal{T}$.

(b) Let Y be any set and $\mathcal{T} = \{Y, \emptyset\}$ we want to prove that $\mathcal{T}$ is a topology on Y.

  (i) By definition Y and the $\emptyset$ are in $\mathcal{T}$.

  (ii) Any intersection between elements of $\mathcal{T}$ is either Y or $\emptyset$ and both of them are in $\mathcal{T}$.

  (iii) Any union between elements of $\mathcal{T}$ is either Y or $\emptyset$ and both of them are in $\mathcal{T}$.

(c) Let $Z = \{1, 2, 3\}$ and $\mathcal{T} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \emptyset\}$ we want to prove that $\mathcal{T}$ is a topology on Z.

  (i) By definition Z and the $\emptyset$ are in $\mathcal{T}$.

(ii)

$$
\begin{aligned}
\{1\} \cap \{1,2\} \cap \{1,2,3\} \cap \emptyset &= \emptyset \\
\{1,2\} \cap \{1,2,3\} \cap \emptyset &= \emptyset \\
\{1\} \cap \{1,2,3\} \cap \emptyset &= \emptyset \\
\{1\} \cap \{1,2\} \cap \emptyset &= \emptyset \\
\{1\} \cap \{1,2\} \cap \{1,2,3\} &= \{1\} \\
\{1\} \cap \{1,2\} &= \{1\} \\
\{1\} \cap \{1,2,3\} &= \{1\} \\
\{1\} \cap \emptyset &= \emptyset \\
\{1,2\} \cap \{1,2,3\} &= \{1,2\} \\
\{1,2\} \cap \emptyset &= \emptyset \\
\{1,2,3\} \cap \emptyset &= \emptyset
\end{aligned}
$$

Therefore every finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

(iii)

$$
\begin{aligned}
\{1\} \cup \{1,2\} \cup \{1,2,3\} \cup \emptyset &= \{1,2,3\} \\
\{1,2\} \cup \{1,2,3\} \cup \emptyset &= \{1,2,3\} \\
\{1\} \cup \{1,2,3\} \cup \emptyset &= \{1,2,3\} \\
\{1\} \cup \{1,2\} \cup \emptyset &= \{1,2\} \\
\{1\} \cup \{1,2\} \cup \{1,2,3\} &= \{1,2,3\} \\
\{1\} \cup \{1,2\} &= \{1,2\} \\
\{1\} \cup \{1,2,3\} &= \{1,2,3\} \\
\{1\} \cup \emptyset &= \{1\} \\
\{1,2\} \cup \{1,2,3\} &= \{1,2,3\} \\
\{1,2\} \cup \emptyset &= \{1,2\} \\
\{1,2,3\} \cup \emptyset &= \{1,2,3\}
\end{aligned}
$$

Therefore every union of elements of $\mathcal{T}$ is in $\mathcal{T}$.

$\square$

Proof. **Exercise 2.4.**

(a) ($\Rightarrow$) Suppose d and d' generate the same topology on M then the topologies $\mathcal{T}$ and $\mathcal{T}'$ generated by d and d' respectively have the same open sets. Let $x \in M$ and $r > 0$ then $B_r^{(d)}(x) \in \mathcal{T}$ and $B_r^{(d)}(x) \in \mathcal{T}'$ because $B_r^{(d)}(x)$ is an open set then there is $r_1 > 0$ such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$. In the same way $B_r^{(d')}(x) \in \mathcal{T}$ because it's an open set and hence then there is $r_2 > 0$ such that $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$.

($\Leftarrow$) Let $U \subset (M, d)$ be an open set. Let also $\mathcal{T}$ and $\mathcal{T}'$ be the topologies generated by d and d'. It happens that $U \in \mathcal{T}$ since U is open then there is some $r > 0$ such that $B_r^{(d)}(x) \subseteq U$ but also we know that there is some $r_1 > 0$ such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x) \subseteq U$ which implies that there is a ball around x for (M, d') and hence U is also an open set in (M, d') therefore $U \in \mathcal{T}'$. If we take now $U \subset (M, d')$ such that $U \in \mathcal{T}'$ we can show in the same way that $U \in \mathcal{T}$ which implies that d and d' generate the same topology on M.

(b) Let $x \in M$ and $r > 0$. By definition $B_r^{(d)}(x) = \{y \in M : d(x, y) < r\}$ then let $r_1 = rc > 0$ so

$$B_{r_1}^{(d')}(x) = \{y \in M : cd(x, y) < rc\} = \{y \in M : d(x, y) < r\}$$

Hence $B_r^{(d)}(x) = B_{r_1}^{(d')}(x)$.

Now let $B_r^{(d')}(x) = \{y \in M : d'(x, y) < r\}$ if $r_2 = r/c > 0$ we get that

$$B_{r_2}^{(d)}(x) = \{y \in M : d(x, y) < r/c\} = \{y \in M : cd(x, y) < r\}$$

Hence $B_r^{(d')}(x) = B_{r_2}^{(d)}(x)$.

Therefore because of what we proved in (a) we see that d and d' generate the same topology.

(c) Let

$$d(x, y) = |x - y| = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

and

$$d'(x, y) = \max\{|x_1 - y_1|, ..., |x_n - y_n|\}$$

Let also $r > 0$ and $y \in B_r^d(x)$ then $d(x, y) < r$ but we know from problem B.1 that $d'(x, y) \leq d(x, y) \leq \sqrt{n}d'(x, y)$ then we see that $y \in B_r^{d'}(x)$ because $d'(x, y) \leq d(x, y) < r$ which imply that $B_r^d(x) \subseteq B_r^{d'}(x)$.

On the other hand, let $y \in B_{r_1}^{d'}(x)$ where $r_1 = r/\sqrt{n}$ then$\sqrt{n}d'(x, y) < r$ and from problem B.1 we have that $d(x, y) \leq \sqrt{n}d'(x, y) < r$ which implies that $y \in B_r^d(x)$ and hence $B_{r_1}^{d'}(x) \subseteq B_r^d(x)$. With this and the result we got from problem (a) we get that d and d' generate the same topology.

(d) Let $x \in X$ and $0 < r \leq 1$ then $B_r^d(x) = \{x\}$ so $\{x\}$ is an open set. Now let $S \subset X$ be any set of X we see that it can be written as

$$S = \bigcup_{x \in X} B_r^d(x)$$

Which is also open. Hence the topology induced by d is a topology where every set of X is open then the topology is the discrete topology.

(e) We know that the discrete metric on $\mathbb{Z}$ generates the discrete topology on $\mathbb{Z}$ so we want to show that the Euclidean metric generates it as well. It is sufficient to prove that the singletons are open sets in $\mathbb{Z}$ with the Euclidean metric from there we can generate any set as we did for the discrete metric hence they generate the same topology. Let $x \in \mathbb{Z}$ then $\{x\}$ is open since there is $0 \leq r \leq 1$ such that $B_r^d(x) \subseteq \{x\}$ where d is the Euclidean metric.

□

Proof. **Exercise 2.5.** We want to show that

$$\mathcal{T} = \{Z : Z \subset Y \text{ and } Z \text{ is open on } X\}$$

is a topology on Y.

(i) Given that X is a topological space then $\emptyset \subset X$. And since $\emptyset$ is open it is also a subset of Y so $\emptyset \subset \mathcal{T}$. Also, by definition Y is open in X then $Y \subset \mathcal{T}$.

(ii) Given that X is a topological space then any intersection of finitely many open subsets of X is an open subset of X. Then since $Y \subset X$ then any finite intersection of open subsets of Y is an open subset of Y.

(iii) Given that X is a topological space then any union of arbitrarily many open subsets of X is an open subset of X. In particular, since $Y \subset X$ then any union of arbitrarily many open subsets of Y is an open subset of Y.

Therefore $\mathcal{T}$ is a topology on Y. □

Proof. **Exercise 2.6.** Let $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ be a collection of topologies on X. We want to prove that $\mathcal{T} = \bigcap_{\alpha \in A} \mathcal{T}_\alpha$ is also a topology on X.

(i) Given that every $\mathcal{T}_\alpha$ is a topology on X then $\emptyset \in \mathcal{T}_\alpha$ and $X \in \mathcal{T}_\alpha$ for every $\alpha \in A$ therefore $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(ii) Let $U_1, ..., U_n$ be a finite set of elements from $\mathcal{T}$ then each U_i is an open set because it belongs to $\mathcal{T}_\alpha$ for every $\alpha \in A$ also $U_1 \cap ... \cap U_n$ is open and is in every $\mathcal{T}_\alpha$ because they are topologies on X. Therefore $U_1 \cap ... \cap U_n \in \mathcal{T}$.

(iii) Let $(U_\beta)_{\beta \in A}$ be any family of elements of $\mathcal{T}$ as before each U_β is in every $\mathcal{T}_\alpha$ by the definition of $\mathcal{T}$ and since $\mathcal{T}_\alpha$ are topologies on X it must happen that $\bigcup_{\beta \in A} U_\beta \in \mathcal{T}_\alpha$ hence $\bigcup_{\beta \in A} U_\beta \in \mathcal{T}$.

Therefore $\mathcal{T}$ is a topology on X. □

Proof. **Exercise 2.9.** We will prove Proposition 2.8.

(a) ($\Rightarrow$) Let $x \in \operatorname{Int}(A)$ since $\operatorname{Int}(A)$ is open and $\operatorname{Int}(A) \subseteq A$ then x has a neighborhood contained in A.

($\Leftarrow$) Let $U \subseteq A$ be an open neighborhood that contains a point $x \in A$ then since $\operatorname{Int}(A)$ is the largest open subset contained in A it must happen that $U \subseteq \operatorname{Int}(A)$. Therefore $x \in \operatorname{Int}(A)$.

(b) ($\Rightarrow$) Let $x \in \operatorname{Ext}(A)$ then by definition $x \in X \setminus \overline{A}$ which is an open set and we see that $X \setminus \overline{A} \subseteq X \setminus A$ hence x has a neightborhood contained in $X \setminus A$.

($\Leftarrow$) Let $U \subseteq X \setminus A$ be an open neighborhood that contains a point $x \in X \setminus A$ then since $\operatorname{Int}(X \setminus A)$ is the largest open subset contained in $X \setminus A$ it must happen that $U \subseteq \operatorname{Int}(X \setminus A)$ and also we know that $\operatorname{Int}(X \setminus A) = X \setminus \overline{A}$ hence $x \in X \setminus \overline{A} = \operatorname{Ext}(A)$.

(c) ($\Rightarrow$) Let $x \in \partial A$ and U be a neighborhood of x since $x \notin \operatorname{Int}(A)$ then $U \not\subseteq A$ so there is a point $y \in U$ such that $y \in X \setminus A$. On the other hand, since $x \notin \operatorname{Ext}(A)$ then $U \not\subseteq X \setminus A$ so there is a point $z \in U$ such that $z \in A$.

($\Leftarrow$) Let U be a neighborhood of some point x we know there is a point $y \in U$ such that $y \in X \setminus A$ then $U \not\subseteq A$ hence $x \notin \operatorname{Int}(A)$. Also, we know that there is a point $z \in U$ such that $z \in A$ then $U \not\subseteq X \setminus A$ hence $x \notin \operatorname{Ext}(A)$. Therefore this implies that $x \in \partial A$.

(d) ($\Rightarrow$) Let $x \in \overline{A}$ and U be a neighborhood of x since $x \notin \operatorname{Ext}(A) = X \setminus \overline{A}$ then $U \not\subseteq X \setminus A$ so there is a point $y \in U$ such that $y \in A$.

($\Leftarrow$) Let $x \in U$ be a neighborhood of $x \in A$ then $U \not\subseteq X \setminus A$ hence $x \notin \operatorname{Ext}(A) = X \setminus \overline{A}$. Therefore $x \in \overline{A}$.

(e) Let $x \in \overline{A}$ given that $A \subseteq \overline{A}$ then x might be in A as well. Suppose $x \notin A$ hence $x \notin \text{Int}(A)$ but also we know by definition that $x \notin \text{Ext}(A) = X \setminus \overline{A}$ therefore x must be in ∂A. Hence $\overline{A} = A \cup \partial A$.

In the same way, let $x \in \overline{A}$ given that $\text{Int}(A) \subset \overline{A}$ then x might be in $\text{Int}(A)$ as well but let us suppose that $x \notin \text{Int}(A)$ but also we know by definition that $x \notin \text{Ext}(A) = X \setminus \overline{A}$ therefore x must be in ∂A. Hence we also have that $\overline{A} = \text{Int}(A) \cup \partial A$.

(f) By definition $\text{Int}(A)$ is the largest open subset contained in A hence it's open in X. Also, we know that $\overline{A}$ is closed then $X \setminus \overline{A} = \text{Ext}(A)$ is open in X.

By definition $\overline{A}$ is the smallest closed subset containing A hence it's closed in X. Also, we know that $\text{Int}(A) \cup \text{Ext}(A)$ is open because they are both open as we proved before then $\partial A = X \setminus (\text{Int}(A) \cup \text{Ext}(A))$ is closed in X.

(g) Suppose A is open in X then A is the largest open subset contained in A hence $A = \text{Int}(A)$.

If $A = \text{Int}(A)$ then no element of A is in $\text{Ext}(A)$ or in ∂A hence A contains none of its boundary points.

If A contains none of its boundary points then no neighborhood contains a point of $X \setminus A$ hence every point of A has a neighborhood contained in A.

Finally, if every point of A has a neighborhood contained in A then it must happen that $A = \text{Int}(A)$ which we know is open. Therefore A is open in X.

(h) Suppose A is closed in X then A is the smallest closed subset that contains A hence $A = \overline{A}$.

If $A = \overline{A}$ then $A = \text{Int}A \cup \partial A$ hence A contains all of its boundary points.

Let A contain all of its boundary points. Let us note that $\overline{A} = \text{Int}(A) \cup \partial A$ since $\partial A \subseteq A$ and $\text{Int}(A) \subseteq A$ by definition then we have that $\overline{A} \subseteq A$ and hence A is closed which implies that $X \setminus A$ is open and therefore every point of $X \setminus A$ has a neighborhood contained in $X \setminus A$.

Finally, if every point of $X \setminus A$ has a neighborhood contained in $X \setminus A$ then $X \setminus A$ is open and hence A is closed.

$\square$

Proof. **Exercise 2.10.**

($\Rightarrow$) Let $A \subseteq X$ be a closed subset in a topological space X also let $p \in X$ be a limit point of A we want to prove that also $p \in A$. Since every neighborhood U of p contains a point of A then $U \cap A \neq \emptyset$ hence $U \nsubseteq X \setminus A$ and since A is closed then every point of $X \setminus A$ must contain a neighborhood contained in $X \setminus A$ hence $p \notin X \setminus A$ which implies that $p \in A$. Therefore A contains all of its limit points.

($\Leftarrow$) Let $A \subseteq X$ be a subset in a topological space X that contains all of its limit points, we want to prove that A is closed. Suppose that $X \setminus A$ is not open we want to arrive at a contradiction. Let $p \in X \setminus A$ and let U be a neighborhood of p since $X \setminus A$ is not open then for every U of p we have that $U \cap A \neq \emptyset$ then p is a limit point of A but A contains all of its limits point which is a contradiction. Therefore $X \setminus A$ is open which implies that A is closed.

□

Proof. **Exercise 2.11.**

($\Rightarrow$) Let $x \in \overline{A} = X$ then every neighborhood U where $x \in U$ contains a point of A because of Proposition 2.8.(d) therefore every non-empty open subset of X contains a point of A.

($\Leftarrow$) Let us suppose $\overline{A} \neq X$ we want to arrive at a contradiction. Let $x \in X$ such that $x \notin \overline{A}$ but we know that for every neighborhood U where $x \in U$ there is a point of A in it but then $x \in \overline{A}$ because of Proposition 2.8.(d) so we have a contradiction and therefore must be that $\overline{A} = X$. □

Proof. **Exercise 2.12.** Let X be a topological metric space then from the topological convergence definition if $(x_i)_{i=1}^{\infty}$ is a sequence that converges to $x \in X$ we know that for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq N$ but since we are in a metric space we have that for every neighborhood there is a ball $B_\epsilon(x) \subseteq U$ for some $\epsilon > 0$ which implies that there is $N' \in \mathbb{N}$ with $N' \geq N$ such that when $i \geq N'$ we have that $d(x_i, x) < \epsilon$. Therefore when X is a topological metric space the two definitions are equivalent. □

Proof. **Exercise 2.13.** Let (x_i) be a convergent sequence in the discrete topological space X. Hence there is some $x \in X$ such that for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq N$. Since X is a discrete topological space this implies that the set $\{x\}$ is an open set and also a neighborhood around x so there exists some $N \in \mathbb{N}$ such that $x_i \in \{x\}$ for all $i \geq N$ but this implies that $x_i = x$ for every $i \geq N$. Therefore convergent sequences in discrete topological spaces are eventually constant. □

Proof. **Exercise 2.14.** Let $A \subseteq X$ and $(x_i) \subseteq A$ such that $x_i \to x$ where $x \in X$ then by the definition of a convergent sequence we have that for every neighborhood U of x there is some $N \in \mathbb{N}$ such that when $i \geq N$ we have that $x_i \in U$ this implies that every neighborhood of x has at least a point of A therefore $x \in \overline{A}$. $\square$

Proof. **Exercise 2.16.**

($\Rightarrow$) Let $f : X \to Y$ be a continuous map and let $V \subseteq Y$ be closed subset then $Y \setminus V$ is open and since f is continuous we have that $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is also open then

$$X \setminus (X \setminus f^{-1}(V)) = (f^{-1}(V) \cap X) \cup (X \setminus X) = f^{-1}(V)$$

is closed.

($\Leftarrow$) Let $U \subseteq Y$ be an open subset then $Y \setminus U$ is closed so we have that $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is also closed hence

$$X \setminus (X \setminus f^{-1}(U)) = (f^{-1}(U) \cap X) \cup (X \setminus X) = f^{-1}(U)$$

is open which implies that f is continuous. $\square$

Proof. **Exercise 2.18.**

(a) Let $f : X \to Y$ be a constant map where every $x \in X$ is sent to a constant $y \in Y$ and let $U \subseteq Y$ be an open set then if $y \notin U$ we have that $f^{-1}(U) = \emptyset$ which is an open set if $y \in U$ then $f^{-1}(U) = X$ which is also an open set, therefore f is continuous.

(b) Let $Id_X : X \to X$ be the identity map and let $U \subset X$ be an open set hence $Id_X^{-1}(U) = U$ since Id_X is the identity map, hence $Id_X^{-1}(U)$ is also an open set which implies that Id_X is continuous.

(c) Let $f : X \to Y$ be a continuous function and let $f_U : U \to Y$ be a restriction of f to an open subset $U \subset X$ also let $V \subset Y$ be an open set from Y then we see that $f_U^{-1}(V) = f^{-1}(V) \cap U$ and we know that $f^{-1}(V)$ is open since f is continuous and U is open by definition then $f_U^{-1}(V)$ is also an open set. Therefore f_U the restriction of f to an open set $U \subset X$ is continuous.

$\square$

Proof. **Exercise 2.20.** We want to prove that "homeomorphic" is an equivalence relation on the class of all topological spaces then

(a) Let X be a topological space then there is the identity map $Id_X : X \to X$ which is continuous as we saw in Proposition 2.17(b) and bijective by definition. Also, we have that $Id_X = Id_X^{-1}$ hence Id_X^{-1} is also continuous. Therefore X is homeomorphic to X i.e. $X \approx X$.

(b) Let $X \approx Y$ i.e. X is homeomorphic to Y then there is $f : X \to Y$ such that f is bijective and continuous and also f^{-1} is continuos. Now let us define $g = f^{-1}$ hence $g : Y \to X$ where g is bijective and continuous by definition and $g^{-1} = (f^{-1})^{-1} = f$ is also continuos. Therefore $Y \approx X$.

(c) Let $f : X \to Y$ and $g : Y \to Z$ be homeomorphisms (i.e. $X \approx Y$ and $Y \approx Z$) then there is $h = g \circ f$ such that $h : X \to Z$ which we know is continuous because of Proposition 2.17(d) and is bijective since f and g are bijective. Also, if we define $h^{-1} = f^{-1} \circ g^{-1}$ we have that $h^{-1} : Z \to X$ and h^{-1} is continuos since f^{-1} and g^{-1} are continuous and their composition is continuous. Therefore if $X \approx Y$ and $Y \approx Z$ then $X \approx Z$.

Finally since "homeomorphic" is a reflexive, symmetric and transitive relation then "homeomorphic" is an equivalence relation on the class of all topological spaces. $\square$

Proof. **Exercise 2.21.** Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and $f : X_1 \to X_2$ a bijective map.

($\Rightarrow$) Let f be a homeomorphism, we want to prove that $f(\mathcal{T}_1) = \mathcal{T}_2$. Let $U \in \mathcal{T}_1$ i.e. U is an open set from X_1 since f is a homeomorphism then f^{-1} is continuous hence $(f^{-1})^{-1}(U)$ is an open set of X_2 so $(f^{-1})^{-1}(U) \in \mathcal{T}_2$ but $(f^{-1})^{-1}(U) = f(U) \in \mathcal{T}_2$ because f is bijective.

On the other hand, let us name $V = f(U) \in \mathcal{T}_2$ since f is a homeomorphism then f is continuous so $f^{-1}(V) = f^{-1}(f(U)) = U$ is an open set from X_1 hence $U \in \mathcal{T}_1$.

Therefore $U \in \mathcal{T}_1$ if and only if $f(U) \in \mathcal{T}_2$ i.e. $f(\mathcal{T}_1) = \mathcal{T}_2$.

($\Leftarrow$) Let $U \in \mathcal{T}_1$ we know that $f(U) \in \mathcal{T}_2$. Let us name $V = f(U)$ then $f^{-1}(V) = U$ because f is bijective and $U \in \mathcal{T}_1$ hence $f^{-1}(V)$ is an open set of X_1 which implies that f is continuous.

Let $U \in \mathcal{T}_1$ then $(f^{-1})^{-1}(U) = f(U)$ because f is bijective but we know that $f(U) \in \mathcal{T}_2$ hence $(f^{-1})^{-1}(U)$ is an open set of X_2 which implies that f^{-1} is continuous.

Finally, since we know that f is bijective we get that f is a homeomorphism. $\square$

Proof. **Exercise 2.22.** Let $f : X \to Y$ be a homeomorphism and let $U \subseteq X$ be an open subset. Since f is a homeomorphism then f^{-1} is continuous hence $(f^{-1})^{-1}(U)$ is an open set of Y but $(f^{-1})^{-1}(U) = f(U)$ because f is bijective therefore $f(U)$ is an open set of Y.

Let $f|_U : U \to f(U)$ be the restriction of f to U. Since f is a homeomorphism then f is continuous and we know from Exercise 2.18 (c) that then $f|_U$ is continuous. Let also $f|_U^{-1} : f(U) \to U$ we know f^{-1} is continuous since f is a homeomorphism then because of Exercise 2.18 (c) we have that $f|_U^{-1}$ is also continuous. Finally, since f is biyective then $f|_U$ is also biyective. Therefore $f|_U$ is also a homeomorphism. □

Proof. **Exercise 2.23.**

($\Rightarrow$) Let $Id_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be a continuous identity map then let $U \subseteq (X, \mathcal{T}_2)$ since Id_X is continuous we have that $Id_X^{-1}(U) \subseteq (X, \mathcal{T}_1)$ but $Id_X^{-1}(U) = U$ which implies that $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

($\Leftarrow$) Let $\mathcal{T}_1$ be finer than $\mathcal{T}_2$ i.e. $\mathcal{T}_2 \subseteq \mathcal{T}_1$ now let $U \subseteq (X, \mathcal{T}_2)$ then $U \subseteq (X, \mathcal{T}_1)$ but we know that $U = Id_X^{-1}(U)$ hence $Id_X^{-1}(U) \subseteq (X, \mathcal{T}_1)$. Therefore Id_X is continuous.

($\Rightarrow$) Let $Id_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be a homeomorphism then Id_X is continuous which implies from what we just proved that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, on the other hand we know that Id_X^{-1} is also continuous hence we also get that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ which implies that $\mathcal{T}_1 = \mathcal{T}_2$.

($\Leftarrow$) Let $\mathcal{T}_1 = \mathcal{T}_2$ hence we can write that $\mathcal{T}_2 \subseteq \mathcal{T}_1$ which implies that Id_X is continuous from what we just proved, also, if we write $\mathcal{T}_1 \subseteq \mathcal{T}_2$ this implies that Id_X^{-1} is continuous. Finally, we know that Id_X is bijective by definition, therefore Id_X is a homeomorphism. □

Proof. **Exercise 2.27.** Let us first check that

$$\varphi^{-1}(x,y,z) = \frac{(x,y,z)}{\max\{|x|,|y|,|z|\}}$$

is the inverse of φ as defined in example 2.26. so we compute the following

$$\begin{aligned}
\varphi(\varphi^{-1}(x,y,z)) &= \frac{\left(\frac{x}{\max\{|x|,|y|,|z|\}}, \frac{y}{\max\{|x|,|y|,|z|\}}, \frac{z}{\max\{|x|,|y|,|z|\}}\right)}{\sqrt{\frac{x^2}{\max\{|x|,|y|,|z|\}^2} + \frac{y^2}{\max\{|x|,|y|,|z|\}^2} + \frac{z^2}{\max\{|x|,|y|,|z|\}^2}}} \\
&= \frac{\frac{(x,y,z)}{\max\{|x|,|y|,|z|\}}}{\frac{\sqrt{x^2+y^2+z^2}}{\max\{|x|,|y|,|z|\}}} \\
&= \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}
\end{aligned}$$

But since (x,y,z) represents the unit sphere $\mathbb{S}^2$ then $\sqrt{x^2+y^2+z^2} = 1$ hence $\varphi(\varphi^{-1}(x,y,z)) = (x,y,z)$. Now we prove the same for the opposite composition

$$\begin{aligned}
\varphi^{-1}(\varphi(x,y,z)) &= \frac{\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)}{\max\left\{\left|\frac{x}{\sqrt{x^2+y^2+z^2}}\right|, \left|\frac{y}{\sqrt{x^2+y^2+z^2}}\right|, \left|\frac{z}{\sqrt{x^2+y^2+z^2}}\right|, \right\}} \\
&= \frac{\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}}{\frac{\max\{|x|,|y|,|z|\}}{\sqrt{x^2+y^2+z^2}}} \\
&= \frac{(x,y,z)}{\max\{|x|,|y|,|z|\}}
\end{aligned}$$

In the same way here (x,y,z) represents the unit cube $\mathbf{C}$ hence $\max\{|x|,|y|,|z|\} = 1$ therefore $\varphi^{-1}(\varphi(x,y,z)) = (x,y,z)$ as we wanted.

Finally, we want to prove that $\varphi^{-1} : \mathbb{S}^2 \to \mathbf{C}$ is a continuous function. Since $\varphi^{-1}(x,y,z) = (x,y,z)/\max\{|x|,|y|,|z|\}$ then φ^{-1} is the division between two functions the identity function $Id : (x,y,z) \to (x,y,z)$ and the infinity norm $\|(x,y,z)\|_\infty : (x,y,z) \to \max\{|x|,|y|,|z|\}$ where both of them are continuous functions therefore φ^{-1} is continuous. $\square$

Proof. **Exercise 2.28.** Let $a(s) = e^{2\pi i s} = \cos(2\pi s) + i\sin(2\pi s)$ we want to show first that a is continuous then let $\epsilon > 0$ we want to show that $\|a(s) - a(t)\|_2 < \epsilon$ whenever $\|s - t\|_2 < \delta$. Let us suppose $\|a(s) - a(t)\|_2 < \epsilon$ is true then we have that

$$\begin{aligned}
\sqrt{(\cos(2\pi s) - \cos(2\pi t))^2 + (\sin(2\pi s) - \sin(2\pi t))^2} &< \epsilon \\
\sqrt{2 - 2\cos(2\pi s)\cos(2\pi t) - 2\sin(2\pi s)\sin(2\pi t)} &< \epsilon \\
\sqrt{2 - 2\cos(2\pi(s - t))} &< \epsilon \\
1 + \cos(\pi - 2\pi(s - t)) &< \frac{\epsilon^2}{2} \\
-2\pi(s - t) &< \arccos\left(\frac{\epsilon^2}{2} - 1\right) - \pi \\
-(s - t) &< \frac{1}{2\pi}\arccos\left(\frac{\epsilon^2}{2} - 1\right) - \frac{1}{2}
\end{aligned}$$

Therefore if we take $\delta = \frac{1}{2\pi}\arccos\left(\frac{\epsilon^2}{2} - 1\right) - \frac{1}{2}$ whenever

$$\|s - t\|_2 = \sqrt{(s - t)^2} < \delta$$

we get that $\|a(s) - a(t)\|_2 < \epsilon$ which implies that a is continuous.

Let us prove now that a is one-to-one so let us suppose $a(s) = a(t)$ then $e^{2\pi i s} = e^{2\pi i t}$ which implies that $s = t$ so a is one-to-one.

Now we want to prove a is onto, let $v \in \mathbb{S}^1$ we want to show that there is some $s \in [0, 1)$ such that $v = a(s)$ let us take $s = \frac{\log(v)}{2\pi i}$ then

$$a(s) = e^{2\pi i(\frac{\log(v)}{2\pi i})} = e^{\log(v)} = v$$

Therefore for every $v \in \mathbb{S}^1$ there is some $s \in [0, 1)$ such that $a(s) = v$ which implies that a is onto.

Finally, we want to show that a^{-1} is not continuous. Let $x_n = \cos(2\pi(1+1/n)) + i\sin(2\pi(1 + 1/n))$ be a sequence in $\mathbb{S}^1$ which tends to $1 \in \mathbb{S}^1$ and we see that $(1 + 1/n) \to 1$ but $1 \notin [0, 1)$ therefore a^{-1} is not continuous. $\square$

Proof. **Exercise 2.29.**

$(a) \Rightarrow (b)$ Let f be a homeomorphism then if $U \subseteq X$ is an open set we know that $f(U) = (f^{-1})^{-1}(U)$ is open since f^{-1} is continuous.

$(b) \Rightarrow (c)$ Let f be open. If $E \subseteq X$ is a closed set then $X \setminus E$ is open and since f is open then $f(X \setminus E) = Y \setminus f(E)$ is open hence $Y \setminus (Y \setminus f(E))$ is closed and we see that

$$Y \setminus (Y \setminus f(E)) = (f(E) \cap Y) \cup (Y \setminus Y) = f(E) \cup \emptyset = f(E)$$

therefore f is closed.

$(c) \Rightarrow (a)$ Let f be closed. Let $U \subseteq X$ be a open set then $X \setminus U$ is closed hence $f(X \setminus U) = Y \setminus f(U)$ is also closed since f is closed hence $Y \setminus (Y \setminus f(U)) = f(U)$ is an open set. But also we know that $(f^{-1})^{-1}(U) = f(U)$ since f is bijective. Therefore f^{-1} is also continuous which implies that f is a homeomorphism.

□

Proof. **Exercise 2.32.**

(a) Let $f : X \to Y$ be a homeomorphism and let $x \in X$ with a neighborhood $U \subseteq X$ which is open by definition. Since f is a homeomorphism then f^{-1} is continuous hence $(f^{-1})^{-1}(U)$ is an open set of Y but $(f^{-1})^{-1}(U) = f(U)$ because f is bijective therefore $f(U)$ is an open set of Y.

Let $f|_U : U \to f(U)$ be the restriction of f to U. Since f is a homeomorphism then f is continuous and we know from Exercise 2.18 (c) that then $f|_U$ is continuous. Let also $f|_U^{-1} : f(U) \to U$ we know f^{-1} is continuous since f is a homeomorphism then because of Exercise 2.18 (c) we have that $f|_U^{-1}$ is also continuous. Finally, since f is biyective then $f|_U$ is also biyective. Therefore $f|_U$ is also a homeomorphism and f is a local homeomorphism.

(b) Let f be a local homeomorphism and let $U \subseteq X$ be an open subset then for every $x \in U$ there is a neighborhood $V_x \subseteq X$ such that $f|_{V_x}$ is a homeomorphism hence $(f|_{V_x})^{-1}$ is continuous which implies that $f|_{V_x}(U)$ is an open set in $f(V_x)$ and by definition we have that

$$f|_{V_x}(U) = f(V_x) \cap f(U)$$

Hence $f|_{V_x}(U)$ is contained in $f(U)$ therefore every element of $f(U)$ has a neighborhood $f|_{V_x}(U)$ which is contained in $f(U)$ which implies that $f(U)$ is open and f is an open map.

Let $B \subseteq Y$ be an open set then for every $x \in f^{-1}(B) \subseteq X$ there is a neighborhood $U_x \subseteq X$ such that $f|_{U_x}$ is a homeomorphism hence $f|_{U_x}$ is continuous then $(f|_{U_x})^{-1}(B)$ must be open in U_x so by definition we have that

$$(f|_{U_x})^{-1}(B) = U_x \cap f^{-1}(B)$$

Hence $(f|_{U_x})^{-1}(B)$ is contained in $f^{-1}(B)$ therefore every element of $f^{-1}(B)$ has a neighborhood $(f|_{U_x})^{-1}(B)$ which is contained in $f^{-1}(B)$ which implies that $f^{-1}(B)$ is open and f is continuous.

(c) Let f be a bijective local homeomorphism then f is continuous and open from (b). Let $U \subseteq X$ be an open set then $f(U) = (f^{-1})^{-1}(U)$ since f is bijective and it's open since f is an open map. Therefore f^{-1} is also continuous which implies that f is a homeomorphism.

□

Proof. **Exercise 2.33.** Let $(y_i) \subseteq Y$ be a sequence that converges to $y \in Y$ this implies that for every neighborhood U of y there exists $N \in \mathbb{N}$ such that $y_i \in U$ for all $i \geq N$ but we are considering a trivial topology of Y then the only possible neighborhood for y is Y hence there is always $1 \in \mathbb{N}$ such that for all $i \geq 1$ we have that $y_i \in Y$. Therefore since y is arbitrary every sequence (y_i) converges to every element in Y. □

Proof. **Exercise 2.35.** Let $p, q \in X$ then there is $f : X \to \mathbb{R}$ such that $f(p) = 0$ but then $f(q) \neq 0$ and let us call $f(q) = r \in \mathbb{R}$. We can take an open set $U = (-r/2, r/2)$ if $r > 0$ or $U = (r/2, -r/2)$ otherwise such that $0 \in U$ also, we can take a different set $V = (r/2, \infty)$ if $r > 0$ and $V = (-\infty, r/2)$ otherwise such that $r \in V$. We see that U and V are both open sets in $\mathbb{R}$ hence since f is continuous we have that $f^{-1}(U)$ and $f^{-1}(V)$ are open disjoint sets where $p \in f^{-1}(U)$ and $q \in f^{-1}(V)$ therefore X must be a Hausdorff space. □

Proof. **Exercise 2.38.** Let us define $\mathcal{T}$ to be a topology on a finite set X such that $(X, \mathcal{T})$ is a Hausdorff space, we want to show that $\mathcal{T} = \mathcal{P}(X)$ i.e. $\mathcal{T}$ is the discrete topology. Let $x, y \in X$ then there is a neighborhood $U_y \in \mathcal{T}$ of x such that $y \notin U_y$ so we can build $U = \bigcap_{y \neq x} U_y$ where U is a finite intersection of open neighborhoods hence it's an open set where $x \in U$ but $y \notin U$ for every $y \in X$ therefore the singletons $\{x\}$ must be open sets of $\mathcal{T}$. Also, we know that any union of arbitrarily many open subsets of X is an open subset of X hence they are in the topology $\mathcal{T}$ so this implies that it must happen that $\mathcal{T} = \mathcal{P}(X)$ therefore $\mathcal{P}(X)$ is the only topology on a finite set X such that $(X, \mathcal{T})$ is a Hausdorff space. □

Proof. **Exercise 2.40.**

($\Rightarrow$) Let $U \subseteq X$ be an open subset since $\mathcal{B}$ is a basis for X's topology then U can be written as $U = \bigcup_\alpha B_\alpha$ where each $B_\alpha \in \mathcal{B}$ hence for each $p \in U$ must happen that $p \in \bigcup_\alpha B_\alpha$ i.e. p is at least in one $B_\alpha \subseteq U$.

($\Leftarrow$) If for each $p \in U$ there exist $B \in \mathcal{B}$ such that $p \in B \subseteq U$ it must happen that $U = \bigcup_\alpha B_\alpha$ where $B_\alpha \in \mathcal{B}$ which implies that U is the union of some collection of elements of $\mathcal{B}$ therefore U must be an open set.

□

Proof. **Exercise 2.42.**

(a) Let us consider a topology with basis $\mathcal{B}_\infty$ for $\mathbb{R}^n$ where each $B \in \mathcal{B}_\infty$ is defined as $B_{s/2}(x) = \{y \in \mathbb{R} : d_\infty(x, y) < s/2\}$ where $d_\infty(x, y) = \max_{1\leq i\leq n} |x_i - y_i|$ we see that each $B_{s/2}(x)$ is an open cube in $\mathbb{R}^n$ of side length s. On the other hand, we know because of Exercise 2.4(c) that d_∞ generate the same topology as the Euclidean metric i.e. we can generate every open set of the Euclidean topology with an arbitrary union of elements of $\mathcal{B}_\infty$. Hence, since $\mathcal{B}_1$ is also the collection of open cubes of side length s in $\mathbb{R}^n$ it must also be a basis for $\mathbb{R}^n$.

(b) Let $B_\epsilon(t)$ be a ball around $t \in \mathbb{R}^n$. Let us take now some $p \in B_\epsilon(t)$, since $\mathbb{Q}$ is dense on $\mathbb{R}$ then for every coordinate p_i we have that there is $x_i, r_i \in \mathbb{Q}$ such that

$$t_i - \epsilon < r_i < x_i < p_i < x_i + r_i < t_i + \epsilon$$

This implies that for every $p \in B_\epsilon(t)$ there is a ball $B_r(x)$ such that $p \in B_r(x) \subseteq B_\epsilon(t)$. Therefore every open set in $\mathbb{R}^n$ can be built as an arbitrary union of elements (balls) from $\mathcal{B}_2$ hence $\mathcal{B}_2$ is a basis for the Euclidean topology on $\mathbb{R}^n$.

□

Proof. **Exercise 2.45.** Let $\mathcal{B}$ be a basis for some topology on X we want to prove that the basis $\mathcal{B}$ satisfies the two properties (i) and (ii).

(i) By definition of basis we know that every open subset of X is the union of some collection of elements of $\mathcal{B}$. Also, we know that X is open so it must happen that $\bigcup_{B \in \mathcal{B}} B = X$.

(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ we want to prove there is an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Let us take $U = B_1 \cap B_2$ we see that U must be an open set since B_1 and B_2 are also open, then since $\mathcal{B}$ is a basis this implies that U must be an arbitrary union of elements of $\mathcal{B}$ hence there must be at least one set $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Therefore we have that $x \in B \subseteq U \subseteq B_1 \cap B_2$ if we take $B_3 = B$ we are done.

$\square$

Proof. **Exercise 2.51.**

(b) Let $\mathcal{B}$ be a countable basis and let us take an element $x_1 \in B_1 \in \mathcal{B}$ then we can build a set $A = \{x_i : x_i \in B_i \in \mathcal{B} \text{ for } i \in \mathbb{N}\}$ we see that A is countable. We want to prove that A is dense in X.

Let us take an open set $U \subseteq X$ since we have a basis $\mathcal{B}$ we know that $U = \bigcup_\alpha B_\alpha$ then at least one element of A is in U. Therefore A is dense in X.

$\square$

Proof. **Exercise 2.54.** Let M be a 0-manifold, let $p \in M$ and let $U \subseteq M$ be a neighborhood of p such that it is homeomorphic to an open ball in $\mathbb{R}^0$ but an open ball in $\mathbb{R}^0$ is the only point in $\mathbb{R}^0$ and since homeomorphism means bijection it must happen that $U = \{p\}$. Also, since M is second countable we know there is a countable basis $\mathcal{B}$ and hence there is $B \in \mathcal{B}$ such that $p \in B \subseteq \{p\}$ so must be that $B = \{p\}$ and we know that $M = \bigcup_{B \in \mathcal{B}} B$ therefore M is a countable discrete space.

Let now M be a countable discrete space we want to prove it's a 0-manifold.

Let $p, q \in M$ then there are two neighborhoods $\{p\}$ and $\{q\}$ such that $\{p\} \cap \{q\} = \emptyset$. Therefore M is Hausdorff.

Let $\mathcal{B} = \{\{p\} : p \in M\}$ then any open subset of M is the union of some collection of elements of $\mathcal{B}$. Therefore M is second countable.

Let $p \in M$, let $0 \in \mathbb{R}^0$ be the only point in $\mathbb{R}^0$, and let us define a map $T : \{p\} \to \{0\}$. We see that T is a bijection. The only topology for $\mathbb{R}^0$ is $\mathcal{T} = \{\emptyset, \{0\}\}$ and we see that $T^{-1}(\{0\}) = \{p\}$ which is open in M so T is continuous. Since $T(\{p\}) = \{0\}$ we have that T^{-1} is also continuous. Therefore every point of M has a neighborhood homeomorphic to an open ball in $\mathbb{R}^0$ and to $\mathbb{R}^0$ itself. This implies that M is locally Euclidean of dimension 0.

Finally, since M is Hausdorff, second countable and locally Euclidean of dimension 0 we have that M is a 0-manifold. □