Solutions to selected problems on Introduction to Topological Manifolds - John M. Lee.

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Chapter 3 - New Spaces from Old

Problems

Proof. **3-1** Let M be an n-dimensional manifold with boundary, we want to show that ∂M is an $(n-1)$-manifold (without boundary) when endowed with the subspace topology.

We know that M is a second countable and a Hausdorff space and since ∂M is endowed with the subspace topology then it's a subspace of M hence by Proposition 3.11 (d) and (f) we know that ∂M is also second countable and Hausdorff subspace.

On the other hand, let $p \in \partial M$ then since M is a manifold with boundary there is a neighborhood $U \subseteq M$ of p which is homeomorphic to an open set $V \subseteq \mathbb{H}^n$ i.e. there is a homeomorphism $\varphi : U \to V$. Since $p \in \partial M$ then $\varphi(p) \in V \cap \partial \mathbb{H}^n$ i.e. $\varphi(p) = (x_1, ..., x_n)$ with $x_n = 0$. So let us define $\varphi\big|_{U\cap\partial M} : U \cap \partial M \to V \cap \partial \mathbb{H}^n$ as the restriction of φ to $U \cap \partial M$ we want to show it is a homeomorphism too. We know $\varphi\big|_{U\cap\partial M}$ bijective since φ is bijective and since φ is continuous then the restriction $\varphi\big|_{U\cap\partial M}$ is also continuous. Finally, since φ^{-1} is continuous (given that φ is a homeomorphism) then the restriction $\varphi^{-1}\big|_{U\cap\partial M}$ is also continuous hence $\varphi\big|_{U\cap\partial M}$ is a homeomorphism from $U \cap \partial M$ to $V \cap \partial \mathbb{H}^n$.

This implies that every point of ∂M has a neighborhood $U \cap \partial M$ homeomorphic to an open set $V \cap \partial \mathbb{H}^n$ in $\mathbb{R}^{n-1}$.

Therefore ∂M is an $(n-1)$-manifold without boundary. $\square$

Proof. **3-3** Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and $Y = \mathbb{R}$, and let $\{A_i\} = \{0\} \cup \{\{1/n\} : n \in \mathbb{N}\}$ be an infinite closed cover of X. Also, let us define $f_i : A_i \to \mathbb{R}$ as $f_0(0) = 1$ and and $f_i(1/i) = 0$. We want to prove that each f_i is continuous. Let us take $U = (-1, 1)$ for any f_i such that $i \geq 1$ we see that U is open in $\mathbb{R}$ and $f_i^{-1}(U)$ is open in A_i. Also, if we take $U = (0, 2)$ for f_0 we see that U is open in $\mathbb{R}$ and $f_0^{-1}(U)$ is open in A_0. Therefore each f_i is continuous.

Now, let us take $f : X \to \mathbb{R}$ such that $f|_{A_i} = f_i$, we want to prove f is not continuous. Let $U = (0, 2) \subset \mathbb{R}$ which is open in $\mathbb{R}$ then $f^{-1}(U) = \{0\}$ but $\{0\}$ is not open in X.

Therefore f is not continuous and the Gluing Lemma does not need to hold when we consider an infinite closed cover. □

Proof. **3-6** Let X be a topological space and let $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ be the diagonal of $X \times X$. We want to prove that X is Hausdorff if and only if Δ is closed.

($\Rightarrow$) Let $(x, y) \in X \times X \setminus \Delta$ and let X be Hausdorff then there are open sets $U_x, U_y \subseteq X$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. Then we can build an open rectangle $U_x \times U_y$ such that no point of Δ is in $U_x \times U_y$ therefore for every point of $X \times X \setminus \Delta$ there is a neighborhood which is contained in $X \times X \setminus \Delta$ which implies that Δ is closed.

($\Leftarrow$) Let Δ be a closed set then $X \times X \setminus \Delta$ is open then every point of $X \times X \setminus \Delta$ has a neighborhood contained in $X \times X \setminus \Delta$. Let $(x, y) \in X \times X \setminus \Delta$ then there is an open rectangle $U_x \times U_y \subset X \times X \setminus \Delta$ where $U_x, U_y \subset X$ are open sets such that no point of Δ is in $U_x \times U_y$ so we have two neighborhoods $x \in U_x$ and $y \in U_y$ which are disjoint i.e. $U_x \cap U_y = \emptyset$. Therefore this implies that X is Hausdorff.

□

Proof. **3-10**

(⇒) Let $f : \coprod_{\alpha \in A} X_\alpha \to Y$ be continuous, we want to prove that $f|_{X_\alpha}$ for each $\alpha \in A$ is continuous too.

Let U be an open set of Y we know that $f^{-1}(U)$ is open in $\coprod_{\alpha \in A} X_\alpha$ then by the definition of disjoint union topology $f^{-1}(U) \cap X_\alpha$ is open in X_α for $\alpha \in A$. Also, we know that

$$f|_{X_\alpha}^{-1}(U) = \{x \in X_\alpha : f|_{X_\alpha}(x) \in Y\} = \{x \in X_\alpha : f(x) \in Y\}$$

and that

$$f^{-1}(U) = \{x \in \coprod_{\alpha \in A} X_\alpha : f(x) \in Y\}$$

Therefore we see that $f|_{X_\alpha}^{-1}(U) = f^{-1}(U) \cap X_\alpha$ and thus $f|_{X_\alpha}^{-1}(U)$ is open in X_α which implies that $f|_{X_\alpha}$ is continuous.

(⇐) Let $f|_{X_\alpha} : X_\alpha \to Y$ be continuous for each $\alpha \in A$, we want to prove that $f : \coprod_{\alpha \in A} X_\alpha \to Y$ is continuous too.

Let U be an open set of Y we know that $f|_{X_\alpha}^{-1}(U)$ is open in X_α for every $\alpha \in A$. Also, we know that $f|_{X_\alpha}^{-1}(U) = f^{-1}(U) \cap X_\alpha$ then by the definition of disjoint union topology, we see that $f^{-1}(U)$ must be open in $\coprod_{\alpha \in A} X_\alpha$ and therefore f is continuous.

Suppose now we define $\mathcal{T}$ to be the disjoint union topology and $\mathcal{T}'$ to be another topology with the Characteristic Property.

Invoking the Characteristic Property of $\mathcal{T}'$ with $Y = \coprod_{\alpha \in A} X_\alpha$ in the disjoint union topology $\mathcal{T}$ shows that the identity map

$$i : (\coprod_{\alpha \in A} X_\alpha)_{\mathcal{T}'} \to (\coprod_{\alpha \in A} X_\alpha)_{\mathcal{T}}$$

is continuous and the same happens if we take $Y = \coprod_{\alpha \in A} X_\alpha$ in $\mathcal{T}'$ and we apply the Characteristic Property of $\mathcal{T}$ so the inverse is also continuous. Therefore the two toplogies are equal. □

Proof. **3-12**

(a) Let us define $\mathcal{T}_X$ to be the topology on X, let $\mathcal{T}_S = \{U \cap S : U \in \mathcal{T}_X\}$ be the subspace topology on S and let $\mathcal{T}$ be the coarsest topology on S for which $\iota_S : S \to X$ is continuous then by definition $\mathcal{T}$ is given by $\mathcal{T} = \{\iota_S^{-1}(U) : U \in \mathcal{T}_X\}$.

We want to show that $\mathcal{T} = \mathcal{T}_S$ and this will happen if $\iota_S^{-1}(U) = U \cap S$ for all $U \in \mathcal{T}_X$.

Let $U \in \mathcal{T}_X$ then by definition $\iota_S^{-1}(U) = \{x \in S : \iota_S(x) \in U\}$ hence we have that

$$\iota_S^{-1}(U) = \{x \in S : \iota_S(x) \in U\} = \{x \in S : x \in U\} = U \cap S$$

Therefore $\mathcal{T} = \mathcal{T}_S$ and hence the subspace topology on S is the coarsest topology such that $\iota_S : S \to X$ is continuous.

(b) Let A be finite so we will write $\prod_{i=1}^n X_i$ instead of $\prod_{\alpha \in A} X_\alpha$. Let $\mathcal{T}$ be a topology on $\prod_{i=1}^n X_i$ for which the canonical projection $\pi_i : \prod_{i=1}^n X_i \to X_i$ is continuous. Also, let $\mathcal{T}_p$ be the product topology on $\prod_{i=1}^n X_i$, we want to show that $\mathcal{T}_p \subseteq \mathcal{T}$ which implies that $\mathcal{T}_p$ is the coarsest topology where each π_i is continuous.

By definition $\mathcal{T}$ is given by $\mathcal{T} = \bigcup_{i=1}^n \{\pi_i^{-1}(U_i) : U_i \text{ is open in} X_i\}$ so we see that

$$\begin{aligned}
\pi_i^{-1}(U_i) &= \{x \in \prod_{i=1}^n X_i : \pi_i(x) \in U_i\} \\
&= \{x \in \prod_{i=1}^n X_i : x_i \in U_i\} \\
&= X_1 \times ... \times U_i \times ... \times X_n
\end{aligned}$$

Also, we see that there is a collection of open set $U_i \in X_i$ such that

$$U_1 \times ... \times U_i \times ... \times U_n \subseteq X_1 \times ... \times U_i \times ... \times X_n$$

This implies that $\mathcal{T}_p \subseteq \mathcal{T}$ and therefore that $\mathcal{T}_p$ is the coarsest topology on $\prod_{i=1}^n X_i$ where each $\pi_i : \prod_{i=1}^n X_i \to X_i$ is continuous.

(c) Let $\mathcal{T}$ be a topology on $\coprod_\alpha X_\alpha$ for which every canonical projection $\iota_\alpha : X_\alpha \to \coprod_\alpha X_\alpha$ is continuous. Also, let $\mathcal{T}_d$ be the disjoint union topology on $\coprod_\alpha X_\alpha$, we want to show that $\mathcal{T} \subseteq \mathcal{T}_d$ which implies that $\mathcal{T}_d$ is the finest topology where each ι_α is continuous.

Let U be an open set of $\mathcal{T}$ then by definition each $\iota_\alpha^{-1}(U)$ is open in X_α but we see that

$$\iota_\alpha^{-1}(U) = \{x \in X_\alpha : \iota_\alpha(x) \in U\} = X_\alpha \cap U$$

Then by definition of disjoint union topology, we have that $U \subseteq \mathcal{T}_d$ which implies that $\mathcal{T} \subseteq \mathcal{T}_d$ and therefore $\mathcal{T}_d$ is the finest topology on $\coprod_\alpha X_\alpha$ where each ι_α is continuous.

(d) Let $q : X \to Y$ be a surjective map and let $\mathcal{T}$ be a topology on Y for which q is continuous. Also, let $\mathcal{T}_q$ be the quotient topology on Y, we want to show that $\mathcal{T} \subseteq \mathcal{T}_q$ which implies that $\mathcal{T}_q$ is the finest topology for which q is continuous.

Let U be an open set of $\mathcal{T}$ then by definition since q is continuous in this topololgy $q^{-1}(U)$ is open in X. In the oposite way if $q^{-1}(U)$ is open in X then U must be open in Y since q is continuous.

But by definition of quotient topology this implies that $U \in \mathcal{T}_q$ which implies that $\mathcal{T} \subseteq \mathcal{T}_q$ and therefore $\mathcal{T}_q$ is the finest topology on Y where q is continuous.

$\square$

Proof. **3-13** Let $f : X \to Y$ be a continuous map.

(b) Let f be a map that admits a continuous right inverse g. We want to show that f is a quotient map.

Let U be an open set of Y then $g(U)$ is open in X since f is continuous.

On the other hand, let $V = g(U) \subseteq X$ be an open set for some set U in Y. Then we see that $f(V) = f(g(U)) = U$ since f admits a right inverse g which is continuous.

Therefore f is a quotient map.

(c) Let us consider $f : [0, 1) \to [0, 1]$ such that $f(x) = x$, we see that f is a topological embedding since it's injective, continuous and is a homeomorphism onto its image. Let $g : [0, 1] \to [0, 1)$ be the left inverse of f we want to arrive at a contradiction. We know that $[0, 1]$ is compact so if g is continuous then $g([0, 1])$ must be compact but $g([0, 1]) = [0, 1)$ is not compact, a contradiction. Therefore f has no continuous left inverse.

Let us consider now $q : [0, 1] \to \mathbb{S}^1$ such that $q(s) = e^{2\pi i s}$. In Example 3.66 we showed that q is a quotient map. We want to show that there is no continuous map $r : \mathbb{S}^1 \to [0, 1]$ such that $q(r(z)) = z$ for every $z \in \mathbb{S}^1$. From the definition of q we see that both points $0, 1 \in [0, 1]$ are sent to $(1, 0) \in \mathbb{S}^1$ but $r((1, 0))$ cannot be both $0, 1 \in [0, 1]$ so if we set $r((1, 0)) = 0$ then we have a discontinuity at $1 \in [0, 1]$ and the opposite happens if we set $r((1, 0)) = 1$. Therefore q doesn't have a continuous right inverse.

$\square$

Proof. **3-16** Let X be the subset $(\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2$ and let us define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. We want to show that the quotient space $X \backslash \sim$ is locally Euclidean and second countable, but not Hausdorff.

- Let us also define two maps $f : \mathbb{R} \to \mathbb{R} \times \{0\}$ such that $f(x) = (x, 0)$ and $g : \mathbb{R} \to \mathbb{R} \times \{1\}$ such that $g(x) = (x, 1)$.

  Then if we take a point $[(x, 0)] \in X \backslash \sim$ and a neighborhood $U \subseteq X \backslash \sim$ of $[(x, 0)]$ then by definition $q^{-1}(U)$ is open and $f^{-1}(q^{-1}(U))$ is open since f is a homeomorphism hence $q \circ f$ is continuous.

  On the other hand if we let $x \in \mathbb{R}$ and $U \subseteq \mathbb{R}$ a neighborhood of x then $f(U)$ is open since f is a homeomorphism, also, $q(f(U))$ is also open hence $(q \circ f)^{-1}$ is continuous.

  In the same way, if we take a point $[(x, 1)] \in X \backslash \sim$ and a neighborhood $U \subseteq X \backslash \sim$ of $[(x, 1)]$ then by definition $q^{-1}(U)$ is open and $g^{-1}(q^{-1}(U))$ is open since g is a homeomorphism hence $q \circ g$ is continuous.

  And if we let $x \in \mathbb{R}$ and $U \subseteq \mathbb{R}$ a neighborhood of x then $g(U)$ is open since g is a homeomorphism, also, $q(g(U))$ is also open hence $(q \circ g)^{-1}$ is continuous as well.

  Also, we see that both the map $q \circ f$ and $q \circ g$ are bijective from $\mathbb{R}$ to $X \backslash \sim$.

  Therefore for every neighborhood in $X \backslash \sim$ there is a homeomorphism $q \circ f$ or $q \circ g$ to $\mathbb{R}$ and hence $X \backslash \sim$ is locally Euclidean.

- Given that $X = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\})$ then X is second countable since it is composed of two copies of $\mathbb{R}$ and $\mathbb{R}$ is second countable. In particular the countable basis $\mathcal{B}$ for X has to be the union of the countable basis of $\mathbb{R} \times \{0\}$ and the countable basis of $\mathbb{R} \times \{1\}$ which we can assume are the same.

  Let us suppose we take a ball B from the countable basis of $\mathbb{R} \times \{0\}$ then $q^{-1}(q(B))$ will contain the two balls from $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ and hence it's open which implies that $q(B)$ is open in $X \backslash \sim$ so we can take the set of $\mathcal{B}' = \{q(B) : B \in \mathcal{B}\}$ as a countable basis for $X \backslash \sim$ and therefore $X \backslash \sim$ is second countable.

- Let us take $[(0,0)],[(0,1)] \in X\backslash \sim$ and let us suppose there is some neighborhoods U_0 and U_1 of $[(0,0)]$ and $[(0,1)]$ respectively such that $U_0 \cap U_1 = \emptyset$, we want to arrive at a contradiction.

  By definition $q^{-1}(U_0)$ and $q^{-1}(U_1)$ are open in X but since part of $q^{-1}(U_0)$ is in $\mathbb{R} \times \{0\}$ and part of it is in $\mathbb{R} \times \{1\}$ because of the equivalence relation between them and the same thing happens for $q^{-1}(U_1)$ we have that $q^{-1}(U_0) \cap q^{-1}(U_1) \neq \emptyset$ then there must be a point that $(x,0) \in q^{-1}(U_0)$ and $(x,0) \in q^{-1}(U_1)$ such that $q(x,0) \in U_0$ and $q(x,0) \in U_1$ and hence $U_0 \cap U_1 \neq \emptyset$, a contradiction.

  Therefore $X\backslash \sim$ is not Hausdorff.

$\square$