Solved selected problems of Lagrangian and Hamiltonian by Mann

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Solution. **4.2** Let

$$x(t) = a\cos(\omega t + \phi) \qquad \dot{x}(t) = -a\omega\sin(\omega t + \phi)$$

To compute the average kinetic energy over a cycle let us assume that $\phi = 0$ or what is the same that we measure the time of the cycle from ϕ to $\omega T + \phi$, hence

$$\begin{aligned}
\langle \frac{1}{2} m\dot{x}^2 \rangle &= \frac{1}{T}\int_0^T \frac{1}{2} m a^2\omega^2 \sin^2(\omega t)\ dt \\
&= \frac{ma^2\omega^2}{2T}\int_0^T \sin^2(\omega t)\ dt \\
&= \frac{ma^2\omega^2}{2T}\left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}\right]_0^T \\
&= \frac{ma^2\omega^2}{2T}\left[\frac{T}{2} - \frac{\sin(2\omega T)}{4\omega}\right] \\
&= \frac{\pi^2 m a^2}{T}
\end{aligned}$$

Where we used that $\omega = 2\pi/T$ and hence $\sin(4\pi) = 0$.
The average potential energy over a cycle is given by

$$\begin{aligned}
\langle \frac{1}{2} kx^2 \rangle &= \frac{1}{T}\int_0^T \frac{1}{2} k a^2 \cos^2(\omega t)\ dt \\
&= \frac{ka^2}{2T}\int_0^T \cos^2(\omega t)\ dt \\
&= \frac{m\omega^2 a^2}{2T}\left[\frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}\right]_0^T \\
&= \frac{ma^2\omega^2}{2T}\left[\frac{T}{2} + \frac{\sin(2\omega T)}{4\omega}\right] \\
&= \frac{\pi^2 m a^2}{T}
\end{aligned}$$

Where we used that $k = m\omega^2$.

Finally, we compute the average energy as follows

$$\begin{aligned}
\langle \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \rangle &= \frac{1}{T}\int_0^T \frac{1}{2}ma^2\omega^2 \sin^2(\omega t) + \frac{1}{2}ka^2 \sin^2(\omega t)\ dt \\
&= \frac{m\omega^2 a^2}{2T}\int_0^T \sin^2(\omega t) + \cos^2(\omega t)\ dt \\
&= \frac{m\omega^2 a^2}{2T}\left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} + \frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}\right]_0^T \\
&= \frac{m\omega^2 a^2}{2T}\cdot T \\
&= \frac{2\pi^2 m a^2}{T}
\end{aligned}$$

Therefore we see that

$$\langle \frac{1}{2}m\dot{x}^2 \rangle = \langle \frac{1}{2}kx^2 \rangle = \langle \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \rangle$$

$\square$