Solved selected problems of Modern Physics for Scientists and Engineers - John Taylor

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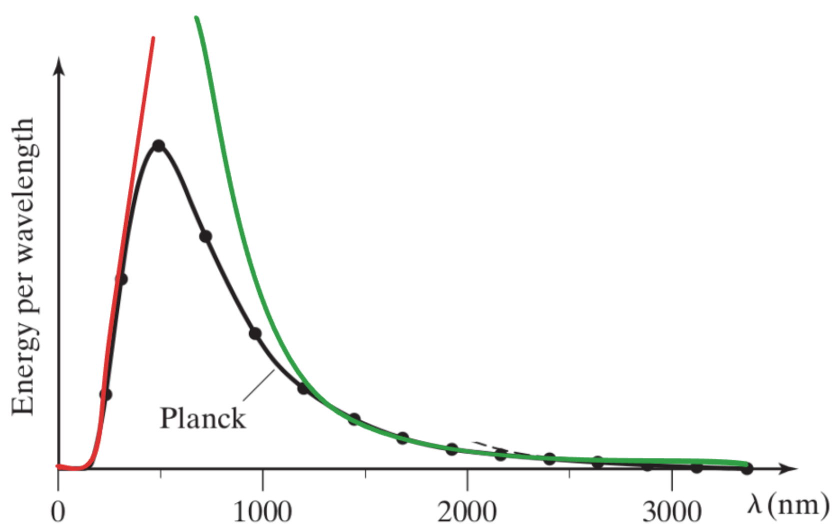
Chapter 4 - Quantization of Light

Solution. 4.1 We know the Planck distribution function for blackbody radiation is

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{(hc/\lambda k_B T)} - 1}$$

If we take the first factor $2\pi hc^2/\lambda^5$ then as $\lambda \to 0$ we see that $2\pi hc^2/\lambda^5 \to \infty$ and as $\lambda \to \infty$ we see that $2\pi hc^2/\lambda^5 \to 0$.
For the second factor we see that as $\lambda \to 0$ we have that $1/(e^{(hc/\lambda k_B T)} - 1) \to 0$ and as $\lambda \to \infty$ we see that $1/(e^{(hc/\lambda k_B T)} - 1) \to \infty$.

So if we sketch both factors we see in red the factor $1/(e^{(hc/\lambda k_B T)} - 1)$ and in green the factor $2\pi hc^2/\lambda^5$.



□

Solution. 4.2 The Planck formula is given by

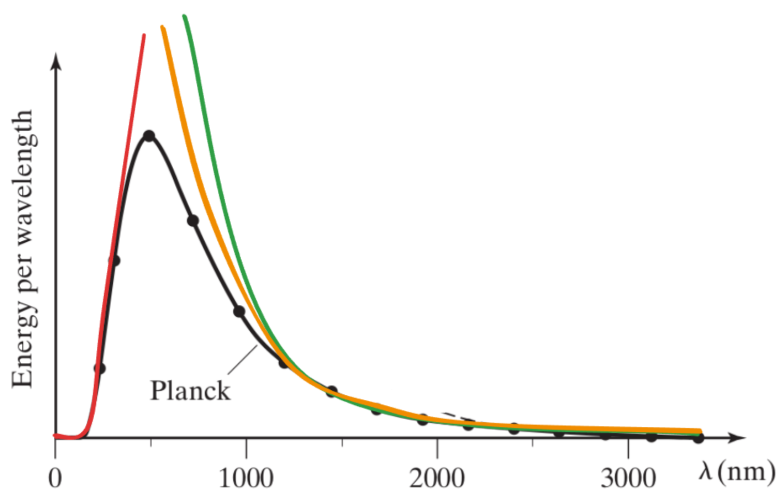
$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{(hc/\lambda k_B T)} - 1}$$

If we approximate $e^{(hc/\lambda k_B T)}$ by a 2-term Taylor series we have that

$$\begin{aligned} I(\lambda, T) &= \frac{2\pi hc^2}{\lambda^5}\frac{1}{1 + \frac{hc}{\lambda k_B T} - 1} \\ &= \frac{2\pi hc^2}{\lambda^5}\frac{\lambda k_B T}{hc} \\ &= \frac{2\pi c k_B T}{\lambda^4} \end{aligned}$$

Which is the Rayleigh-Jeans distribution function.

We already have a sketch for the two factors of Plank's formula from problem 4.1 adding the sketch for the Rayleigh-Jeans distribution function in orange, we see that



Since $2\pi hc^2/\lambda^5$ goes faster to infinity than $2\pi c k_B T/\lambda^4$ as $\lambda \to 0$

We might expect the classical result to be better at long wavelength since the quantum phenomenons are more notable at smaller scales, and they are not so important at long wavelengths. $\square$

Solution. 4.3

(a) We know that the intensity of radiation between λ and $\lambda + d\lambda$ is $I(\lambda, T)d\lambda$. If we consider the frequencies related to the wavelength between λ and $\lambda + d\lambda$ the intensity of radiation between f and $f + df$ must be $I(f, T)df$ and these intensities must match so $I(\lambda, T)d\lambda = I(f, T)df$ therefore must be that

$$I(f, T) = I(\lambda, T)\left|\frac{d\lambda}{df}\right|$$

Where we added the absolute signs to indicate that both distribution functions must be positive.

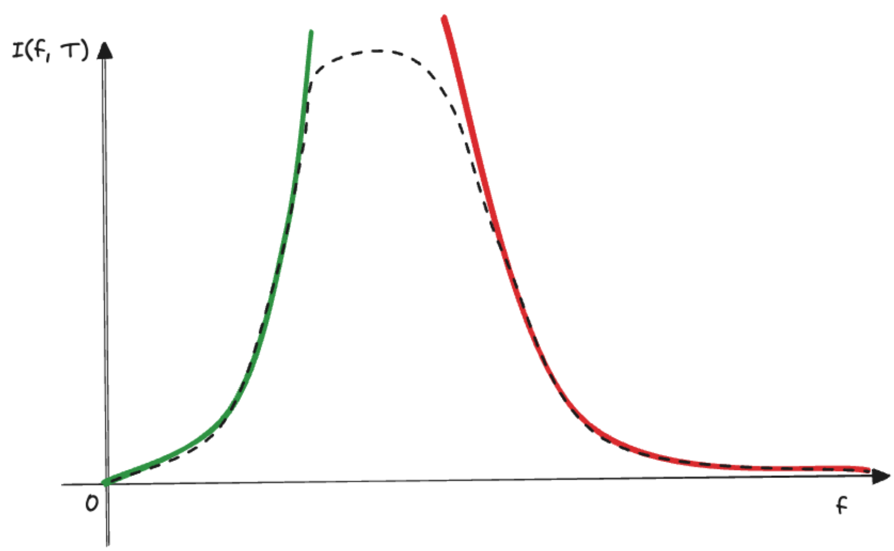
(b) We know that the Planck distribution in terms of the wavelength is given by

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T} - 1}$$

Also, knowing that $\lambda = c/f$ we have that $d\lambda/df = -c/f^2$. Therefore

$$\begin{aligned} I(f, T) &= \frac{2\pi hc^2 f^5}{c^5}\frac{1}{e^{hf/k_B T} - 1}\left|\frac{d\lambda}{df}\right| \\ I(f, T) &= \frac{2\pi hf^5}{c^3}\frac{1}{e^{hf/k_B T} - 1}\frac{c}{f^2} \\ I(f, T) &= \frac{2\pi hf^3}{c^2}\frac{1}{e^{hf/k_B T} - 1} \end{aligned}$$

If we take the first factor $2\pi hf^3/c^2$ then as $f \to 0$ we see that $2\pi hf^3/c^2 \to 0$ and as $f \to \infty$ we see that $2\pi hf^3/c^2 \to \infty$.
For the second factor we see that as $f \to 0$ we have that $1/(e^{(hf/k_B T)} - 1) \to \infty$ and as $f \to \infty$ we see that $1/(e^{(hf/k_B T)} - 1) \to 0$.
Sketching both factors we see in red the factor $1/(e^{(hf/k_B T)} - 1)$ and in green the factor $2\pi hf^3/c^2$ therefore



□

Solution. 4.4

(a) Let $x = hc/\lambda k_B T$ then from $dx/d\lambda$ we can get $d\lambda$ as follows

$$\left|\frac{dx}{d\lambda}\right| = \frac{hc}{k_B T \lambda^2}$$
$$d\lambda = \frac{k_B T \lambda^2}{hc} dx$$

By replacing these values we can get total intensity $I(T)$ radiated by the following integral

$$\begin{aligned}
I(T) &= \int_0^\infty \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}\, d\lambda \\
&= \int_0^\infty \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^x - 1} \frac{k_B T \lambda^2}{hc} dx \\
&= \int_0^\infty \frac{2\pi c k_B T}{\lambda^3} \frac{1}{e^x - 1} \frac{(k_B T)^3}{(k_B T)^3} \frac{(hc)^3}{(hc)^3} dx \\
&= \int_0^\infty \frac{2\pi c (k_B T)^4 x^3}{(hc)^3} \frac{1}{e^x - 1}\, dx \\
&= T^4 \left(\int_0^\infty \frac{2\pi k_B^4 x^3}{h^3 c^2} \frac{1}{e^x - 1}\, dx \right)
\end{aligned}$$

Therefore we see that $I(T)$ has the form $I(T) = \sigma T^4$ where σ is a constant independent of the temperature.

(b) Knowing that $\int_0^\infty x^3\, dx/(e^x - 1) = \pi^4/15$ and the result from part (a) we get that the Stephan - Boltzmann constant is given by

$$\begin{aligned}
\sigma &= \frac{2\pi k_B^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1}\, dx \\
&= \frac{2\pi^5 k_B^4}{15 h^3 c^2}
\end{aligned}$$

(c) Replacing the numerical values of the constants involved in the Stephen-Boltzmann constant we get that

$$\sigma = \frac{2\pi^5 \cdot (1.38 \times 10^{-23})^4}{15 \cdot (6.63 \times 10^{-34})^3 \cdot (2.998 \times 10^8)^2} = 5.649 \times 10^{-8} \frac{W}{m^2 K^4}$$

Then the total power radiated by a steel ball of 1 cm radius at 1000 K is

$$P = I(1000\ K) \cdot A = (5.649 \times 10^{-8})(1000)^4 (4\pi 0.01^2) = 70.987\ W$$

$\square$

Solution. 4.5 Given that the visible light has a wavelength between 400 nm and 700 nm we have that $1/700 < 1/\lambda < 1/400$ hence by multiplying the inequality by $hc = 1240\ eV \cdot nm$ we get that

$$\frac{1240\ eV \cdot nm}{700\ nm} < \frac{hc}{\lambda} < \frac{1240\ eV \cdot nm}{400\ nm}$$

Then the range of energies $E = hc/\lambda$ of visible photons is between

$$1.77\ eV < E < 3.1\ eV$$

Given that UV radiation has shorter wavelength than visible light then UV photons will have bigger energies than visible and opposite to this since IR radiation has longer wavelength than visible then IR photons will have smaller energies than visible. □

Solution. 4.9

(a) For low-intensity microwave radiation with a wavelength of 1cm to break a bond in a biological molecule, the photon coming from the microwave must have an energy of at least 4eV but a microwave photon will have an energy of

$$E = \frac{hc}{\lambda} = \frac{1240\ eV \cdot nm}{1 \times 10^7\ nm} = 0.000124\ eV$$

Therefore a microwave photon cannot break a biological molecule bond.

(b) To break a 4eV bond we need a photon with a wavelength of

$$\lambda = \frac{hc}{E} = \frac{1240\ eV \cdot nm}{4\ eV} = 310\ nm$$

(c) Given that visible light has wavelengths between 400nm and 700nm then 310nm correspond to UV radiation which has a shorter wavelength.

□

Solution. 4.18

(a) Given that the work function for tungsten is $\phi = 4.6\ eV$ the critical frequency below which no electron is emited happens at

$$f_0 = \frac{\phi}{h} = \frac{4.6\ eV}{4.135 \times 10^{-15}\ eV \cdot s} = 1.11 \times 10^{15}\ Hz$$

and this is a critical wavelength of

$$\lambda_0 = \frac{c}{f_0} = \frac{2.998 \times 10^{17}\ nm/s}{1.11 \times 10^{15}\ 1/s} = 270.1\ nm$$

(b) If tungsten is irradiated with a photon of wavelength $\lambda = 200\ nm$ the maximum kinetic energy of the ejected electrons is

$$K_{max} = \frac{hc}{\lambda} - \phi = \frac{1240}{200} - 4.6 = 1.6\ eV$$

(c) Finally, if tungsten is irradiated with a photon of wavelength $\lambda = 300\ nm$ the maximum kinetic energy of the ejected electrons is

$$K_{max} = \frac{hc}{\lambda} - \phi = \frac{1240}{300} - 4.6 = -0.46\ eV$$

The negative result implies that a wavelength of 300nm is not enough to eject an electron as the critical wavelength is about 270nm, so to eject an electron we need a photon with a wavelength less than or equal to this wavelength.

□

Solution. 4.19 Let the crystal planes be separated by $d = 0.31\ nm$ and let the X-rays be of wavelength $\lambda = 0.05\ nm$ then by the Bragg law the glancing angle θ is given by

$$\theta = \arcsin\left(\frac{\lambda}{2d}\right) = \arcsin\left(\frac{0.05}{2 \cdot 0.31}\right) = 0.0807\ rad = 4.67\ deg$$

□

Solution. 4.21 Given that an X-ray of $\lambda = 0.0438\ nm$ gives us three maxima at $\theta = 36.7°, 52.8°$ and $84.5°$ then by the Bragg's law for the first angle we have that

$$\begin{aligned} 2d\sin(36.7°) &= n_1 0.0438 \\ d &= n_1 \frac{0.0438}{2\sin(36.7°)} \\ d &= n_1 0.03664 \end{aligned}$$

And for the second and third angle we have that

$$\begin{aligned} d &= n_2 \frac{0.0438}{2\sin(52.8°)} \\ d &= n_2 0.02749 \end{aligned}$$

$$\begin{aligned} d &= n_3 \frac{0.0438}{2\sin(84.5°)} \\ d &= n_3 0.022 \end{aligned}$$

Then making them equal and solving for n_1/n_2 and n_1/n_3 we get that

$$\begin{aligned} \frac{n_1}{n_2} &= \frac{0.02749}{0.03664} = 0.7502 \\ \frac{n_1}{n_3} &= \frac{0.022}{0.03664} = 0.6004 \end{aligned}$$

In the first case, n_1/n_2 is close to $3/4$ and in the second case n_1/n_3 is close to $3/5$ therefore this implies that $n_1 = 3$, $n_2 = 4$ and $n_3 = 5$.

Finally, using Bragg's law for the first angle we get that the spacing between crystal planes is

$$d = n_1 0.03664 = 3 \cdot 0.03664 = 0.10992\ nm$$

□

Solution. 4.24 We know that crystals of NaCl have a plane spacing of $d = 0.28\ nm$ and a monochromatic beam of X-rays is hitting the crystal with a glancing angle of $\theta = 20°$. By Bragg's law the wavelength of these X-rays is

$$\lambda = 2d\sin\theta = 2 \cdot 0.28 \cdot \sin 20° = 0.191\ nm$$

The Duane-Hunt law requires that the electrons' kinetic energy V_0e in the X-ray tube must be at least equal to the energy hf of the X-ray photons. Therefore we require that

$$V_0e \geq hf = \frac{hc}{\lambda} = \frac{1240\ eV \cdot nm}{0.191\ nm} = 64083.76\ eV$$

or

$$V_0 \geq 64083.76\ V$$

$\square$

Solution. 4.28 Let us consider a collision between a photon (with momentum $\boldsymbol{p_0}$ and energy E_0) and a stationary free electron.

(a) If the photon bounces back with momentum $\boldsymbol{p}$ in the direction $-\boldsymbol{p_0}$ and energy E, by conservation of energy we have that

$$\begin{aligned} E_e + E &= mc^2 + E_0 \\ \sqrt{(p_e c)^2 + (mc^2)^2} &= mc^2 + E_0 - E \end{aligned}$$

Where we used that $E_e = \sqrt{(p_e c)^2 + (mc^2)^2}$. Also, by conservation of momentum

$$\begin{aligned} \boldsymbol{p_e} + \boldsymbol{p} &= \boldsymbol{p_0} \\ \boldsymbol{p_e} &= \boldsymbol{p_0} - \boldsymbol{p} \end{aligned}$$

So

$$p_e^2 = p_0^2 + p^2 + 2p_0 p$$

Hence since $E_0 = p_0 c$ and $E = pc$ we get that

$$\begin{aligned} \sqrt{(p_0^2 + p^2 + 2p_0 p)c^2 + (mc^2)^2} &= mc^2 + p_0 c - pc \\ \sqrt{p_0^2 + p^2 + 2p_0 p + (mc)^2} &= mc + p_0 - p \\ p_0^2 + p^2 + 2p_0 p + (mc)^2 &= (mc)^2 + 2mc(p_0 - p) + p_0^2 + p^2 - 2p_0 p \\ 2p_0 p &= +2mc(p_0 - p) - 2p_0 p \\ 2p_0 p &= +mc(p_0 - p) \\ 2\frac{p_0 p}{p_0 - p} &= mc \\ \frac{p_0 - p}{p_0 p} &= \frac{2}{mc} \\ \frac{1}{p} - \frac{1}{p_0} &= \frac{2}{mc} \end{aligned}$$

(b) If we set $\theta = \pi$ in the Compton's fomula we get that

$$\begin{aligned} \frac{1}{p} - \frac{1}{p_0} &= \frac{1}{mc}(1 - \cos\pi) \\ \frac{1}{p} - \frac{1}{p_0} &= \frac{1}{mc}(1 - (-1)) \\ \frac{1}{p} - \frac{1}{p_0} &= \frac{2}{mc} \end{aligned}$$

Which is the same result we got.

□

Solution. 4.29 Let a 100W beam of light shine for 1000s on a 1g black object initially at rest in a frictionless environment.

(a) Given that the object absorbs all the light produced, the total energy absorbed is the energy generated by the beam hence

$$E = 100\ W \cdot 1000\ s = 100000\ J$$

And therefore the momentum absorbed is

$$p = \frac{E}{c} = \frac{100000\ J}{2.998 \times 10^8\ m/s} = 0.0003335\ kg \cdot m/s$$

(b) From conservation of momentum we have that

$$\begin{aligned} p_{body} &= p_{beam} \\ mv &= p_{beam} \\ v &= \frac{p_{beam}}{m} \end{aligned}$$

Hence

$$v = \frac{0.0003335\ kg \cdot m/s}{0.001\ kg} = 0.3335\ m/s$$

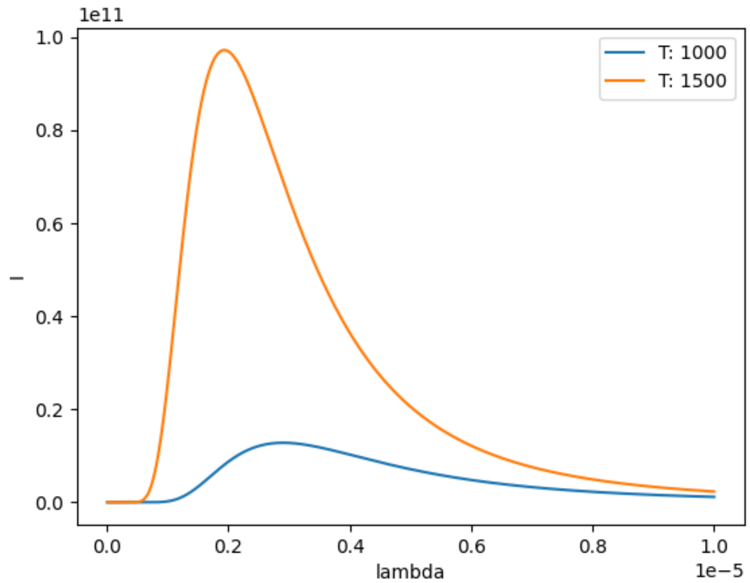
(c) The body's final kinetic energy is then

$$\begin{aligned} K &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} \cdot 0.001\ kg \cdot (0.3335\ m/s)^2 \\ &= 5.5611 \times 10^{-5}\ J \end{aligned}$$

This result is a lot less than the 100000J we started from but this is because the body is not only transforming the energy into kinetic energy but also in kinetic energy on the particles that compose it which translate into heat.

□

Solution. **4.32** Below we show a plot of the Plank distribution function for two temperatures $T = 1000\ K$ and $T = 1500\ K$



The curve for $T = 1000\ K$ reaches its maximum approximately at $\lambda = 2.89\ \mu m$ and the curve for $T = 1500\ K$ reaches its maximum approximately at $\lambda = 1.93\ \mu m$.

The total radiated power is bigger for $T = 1500\ K$ since it is a function of T^4. $\square$

Solution. 4.33 We know the Planck distribution function for blackbody radiation is

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{(hc/\lambda k_B T)} - 1}$$

but if we set $x = \lambda k_B T/hc$ we get that

$$I(x, T) = \frac{2\pi k_B^5 T^5}{h^4 c^3}\frac{1}{x^5}\frac{1}{e^{1/x} - 1}$$

If we derivate this expression using a computer solver we get that

$$\frac{\mathrm{d}I(x,T)}{\mathrm{d}x} = \frac{2\pi k_B^5 T^5}{h^4 c^3}\frac{e^{1/x}(1-5x) + 5x}{(e^{1/x}-1)^2 x^7}$$

and making it equal to 0 we get the x where the maximum happens for $I(x, T)$ hence

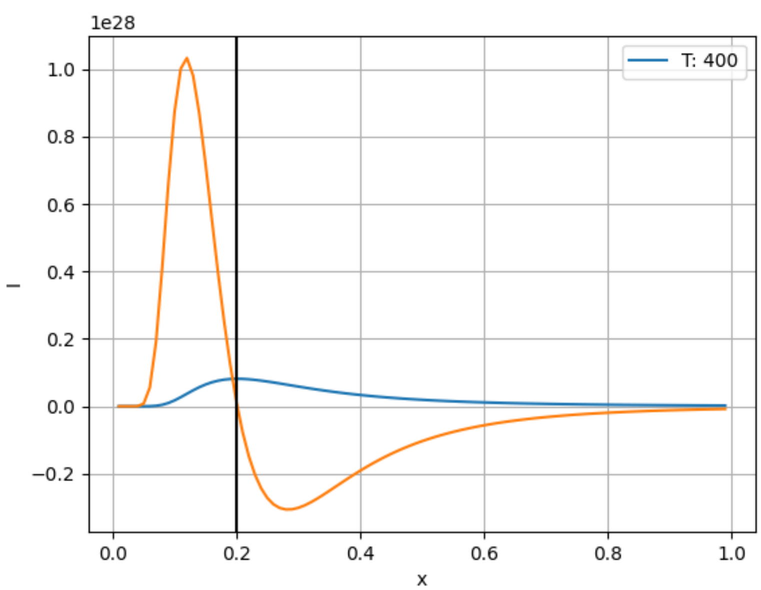
$$e^{1/x}(1-5x) + 5x = 0$$
$$x_{max} = 0.201$$

But x was equal to $\lambda k_B T/hc$ then the value of λ for which $I(\lambda, T)$ is maximum is given by

$$\lambda_{max} = \frac{0.201hc}{k_B T}$$

Where we see that λ_{max} is proportional to $1/T$ as Wien's displacement law states.

Finally, we do the same analysis but graphically. Let us plot $I(x, T)$ and $dI(x, T)/dx$ for $T = 400\ K$ then graphically we see that



Where $dI(x,T)/dx$ is 0 approximately at $x = 0.2$ as we saw in the solution returned by the computer solver. $\square$