Solved selected problems of Real Numbers and Real Analysis - Bloch

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Chapter 1 - Construction of the Real Numbers

Proof. **1.2.1** To prove the uniqueness of $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we suppose that there are two operations $\cdot$ and $\odot$ on $\mathbb{N}$ that satisfy the two properties of the theorem. Let

$$G = \{x \in \mathbb{N} \mid n \cdot x = n \odot x \text{ for all } n \in \mathbb{N}\}$$

we want to prove that $G = \mathbb{N}$, which will imply that $\cdot$ and $\odot$ are the same operation. It is clear that $G \subseteq \mathbb{N}$. By part (a) applied to each of $\cdot$ and $\odot$ we see that $n \cdot 1 = n = n \odot 1$ for all $n \in \mathbb{N}$ and then $1 \in G$.

Now let $q \in G$ so $n \cdot q = n \odot q$ and then from part (b) we have that $n \cdot s(q) = (n \cdot q) + n = (n \odot q) + n = n \odot s(q)$ hence $s(q) \in G$.

Finally, we use part (c) of Peano Postulates to conclude that $G = \mathbb{N}$ $\square$

Proof. **1.2.2**

(2) Let

$$G = \{c \in \mathbb{N} \mid (a + b) + c = a + (b + c) \text{ for all } a, b \in \mathbb{N}\}$$

We will show that $G = \mathbb{N}$ which will imply the desired result. Clearly $G \subseteq \mathbb{N}$. To show that $1 \in G$, let $j, k \in \mathbb{N}$ so $(j + k) + 1 = s(j + k) = j + s(k) = j + (k + 1)$ then $1 \in G$, where we used from Theorem 1.2.5 that $s(n + m) = n + s(m)$.

Now let $r \in G$. Let $j, k \in G$ and suppose $(j+k)+s(r) = j+(k+s(r))$ by Theorem 1.2.5 we know that $(j + k) + s(r) = s((j + k) + r)$ and since $r \in G$ then $s((j + k) + r) = s(j + (k + r))$ then $s(j + (k + r)) = j + s(k + r) = j + (k + s(r))$ therefore $s(r) \in G$ and $G = \mathbb{N}$.

(3) Let

$$H = \{a \in \mathbb{N} \mid a + 1 = 1 + a = s(a)\}$$

We will show that $H = \mathbb{N}$ which will imply the desired result. Clearly $H \subseteq \mathbb{N}$. If $a = 1$ replacing we have that $1 + 1 = 1 + 1 = s(1)$ then $1 \in H$.

Now let $r \in H$ and suppose $s(r) + 1 = 1 + s(r)$ then

$$\begin{aligned}
s(r) + 1 &= (r+1) + 1 && \text{because part (a) of Theorem 1.2.5} \\
&= (1+r) + 1 && \text{because } r \in H \\
&= 1 + (r+1) && \text{because } a + (b+c) = (a+b) + c \\
&= 1 + s(r) && \text{because part (a) of Theorem 1.2.5}
\end{aligned}$$

Therefore $H = \mathbb{N}$.

(4) Let

$$G = \{a \in \mathbb{N} \mid a + b = b + a \text{ for all } b \in \mathbb{N}\}$$

We will show that $G = \mathbb{N}$ which will imply the desired result. It is clear that $G \subseteq \mathbb{N}$. To show that $1 \in G$, let $j \in \mathbb{N}$ so $1 + j = j + 1$ because we proved that in part (3) then $1 \in G$.

Now let $r \in G$ and suppose that $j + s(r) = s(r) + j$ then

$$\begin{aligned}
j + s(r) &= s(j+r) && \text{because part (b) of Theorem 1.2.5} \\
&= s(r+j) && \text{because } r \in G \\
&= r + s(j) && \text{because part (b) of Theorem 1.2.5} \\
&= r + (j+1) && \text{because part (a) of Theorem 1.2.5} \\
&= r + (1+j) && \text{because part (3)} \\
&= (r+1) + j && \text{because associative law for addition} \\
&= s(r) + j && \text{because part (a) of Theorem 1.2.5}
\end{aligned}$$

Therefore $G = \mathbb{N}$.

(7) Let

$$G = \{a \in N \mid 1 \cdot a = a \cdot 1 = a\}$$

We will show that $G = \mathbb{N}$ which will imply the desired result. It is clear that $G \subseteq \mathbb{N}$. If $a = 1$ replacing we have that $1 \cdot 1 = 1 \cdot 1 = 1$ because of part (a) in Theorem 1.2.6 then $1 \in G$.

Now let $r \in G$ and suppose $s(r) \cdot 1 = 1 \cdot s(r) = s(r)$ then

$$\begin{aligned}
s(r) \cdot 1 &= s(r) && \text{because part(a) of Theorem 1.2.6} \\
&= r + 1 && \text{because part(a) of Theorem 1.2.5} \\
&= (r \cdot 1) + 1 && \text{because } r \in G \\
&= (1 \cdot r) + 1 && \text{because } r \in G \\
&= 1 \cdot s(r) && \text{because part (b) of Theorem 1.2.6}
\end{aligned}$$

Therefore $G = \mathbb{N}$.

(8) Let
$$H = \{c \in \mathbb{N} \mid (a+b)c = ac + bc \text{ for all } a, b \in \mathbb{N}\}$$

We want to show that $H = \mathbb{N}$ which will imply the desired result. It is clear that $H \subseteq \mathbb{N}$. To show that $1 \in H$ let $j, k \in \mathbb{N}$ then $(j+k)\cdot 1 = j + k = j \cdot 1 + k \cdot 1$ because what we proved in part (7), then $1 \in H$.

Now let $r \in H$, let $j, k \in \mathbb{N}$ and suppose $(j+k)\cdot s(r) = j \cdot s(r) + k \cdot s(r)$ then

$$\begin{aligned}
(j+k)\cdot s(r) &= ((j+k)\cdot r) + (j+k) && \text{because part (b) of Theorem 1.2.6}\\
&= (j \cdot r + k \cdot r) + (j+k) && \text{because } r \in H\\
&= (j \cdot r + j) + (k \cdot r + k) && \text{because Commutative and Associative law}\\
&= j \cdot s(r) + k \cdot s(r) && \text{because part (b) of Theorem 1.2.6}
\end{aligned}$$

Therefore $s(r) \in H$ and $H = \mathbb{N}$.

(9) Let
$$H = \{a \in \mathbb{N} \mid ab = ba \text{ for all } b \in \mathbb{N}\}$$

We want to show that $H = \mathbb{N}$ which will imply the desired result. It is clear that $H \subseteq \mathbb{N}$. Also $1 \in H$ because what we proved in part (7).

Now let $r \in H$, let $k \in \mathbb{N}$ and suppose $s(r) \cdot k = k \cdot s(r)$ then

$$\begin{aligned}
s(r) \cdot k &= (r+1)\cdot k && \text{because part (a) of Theorem 1.2.5}\\
&= r \cdot k + 1 \cdot k && \text{because Distributive law}\\
&= k \cdot r + k && \text{because } r \in H\\
&= k \cdot s(r) && \text{because part (b) of Theorem 1.2.6}
\end{aligned}$$

Therefore $s(r) \in H$ and $H = \mathbb{N}$.

(10) Let
$$H = \{c \in \mathbb{N} \mid c(a+b) = ca + cb \text{ for all } a, b \in \mathbb{N}\}$$

We want to show that $H = \mathbb{N}$ which will imply the desired result. It is clear that $H \subseteq \mathbb{N}$. Let $j, k \in \mathbb{N}$, if $c = 1$ then $1 \cdot (j+k) = j + k = 1 \cdot j + 1 \cdot k$ so $1 \in H$.

Now let $r \in H$ and lets suppose that $s(r) \cdot (j+k) = s(r) \cdot j + s(r) \cdot k$ then

$$\begin{aligned}
s(r) \cdot (j+k) &= (j+k) \cdot s(r) && \text{because part (9)}\\
&= j \cdot s(r) + k \cdot s(r) && \text{because right-hand side Distributive law}\\
&= s(r) \cdot j + s(r) \cdot k && \text{because part (9)}
\end{aligned}$$

Therefore $s(r) \in H$ and $H = \mathbb{N}$.

(11) Let
$$H = \{c \in \mathbb{N} \mid (ab)c = a(bc) \text{ for all } a, b \in \mathbb{N}\}$$

We want to show that $H = \mathbb{N}$ which will imply the desired result. It is clear that $H \subseteq \mathbb{N}$. Let $j, k \in \mathbb{N}$, if $c = 1$ then $(j \cdot k) \cdot 1 = j \cdot k = j \cdot (k) = j \cdot (k \cdot 1)$ then $1 \in H$.

Now let $r \in H$ and suppose $(j \cdot k) \cdot s(r) = j \cdot (k \cdot s(r))$ then

$$\begin{aligned}
(j \cdot k) \cdot s(r) &= ((j \cdot k) \cdot r) + (j \cdot k) && \text{because part (b) of Theorem 1.2.6} \\
&= (j \cdot (k \cdot r)) + (j \cdot k) && \text{because } r \in H \\
&= j \cdot ((k \cdot r) + k) && \text{because Distributive law} \\
&= j \cdot (k \cdot s(r)) && \text{because part (b) of Theorem 1.2.6}
\end{aligned}$$

Therefore $s(r) \in H$ and $H = \mathbb{N}$.

(13) Let $ab = 1$ there are a set of cases that we should check

- if $a = 1$ and $b \neq 1$ then $1 \cdot b = b$ but we said that $ab = 1$ then b must be equal to 1 which is a contradiction.
- if $a \neq 1$ and $b = 1$ then $a \cdot 1 = a$ but we said that $ab = 1$ then a must be equal to 1 which is a contradiction.
- if $a \neq 1$ and $b \neq 1$ then because of Lemma 1.2.3 there is a unique $t \in \mathbb{N}$ such that $b = s(t)$ so $a \cdot b = a \cdot s(t) = a \cdot t + a = 1$ and because of part (5) this can't be true.

Now let $a = b = 1$ then $a \cdot b = 1 \cdot 1 = 1$ which is what we wanted. □

Proof. **1.2.3** Let $p_1, p_2 \in \mathbb{N}$ where $p_1 \neq p_2$ such that $a + p_1 = b$ and $a + p_2 = b$ then $a + p_1 = a + p_2$ and because of the Cancellation law we have that $p_1 = p_2$ which is a contradiction. Therefore there is a unique $p \in \mathbb{N}$ such that $a + p = b$. □

Proof. **1.2.4**

(1) Since $a = a$ then by definition of the operation $\leq$ is clear that $a \leq a$.

Now let $a < a$ then there is a $p \in \mathbb{N}$ such that $a = a + p$ but this is not possible because of part (6) of Theorem 1.2.7 then it's a contradiction and therefore $a \not< a$.

Let $b = a + 1$ then if $a < b$ there is a $p \in \mathbb{N}$ such that $b = a + p$ but then $a + 1 = a + p$ and then $p = 1$, so we found $p = 1$ for which $a < b = a + 1$.

(3) If $a < b$ and $b < c$ then there is a $p \in \mathbb{N}$ and a $q \in \mathbb{N}$ such that $b = a+p$ and $c = b + q$ so replacing variable b we have that $c = (a + p) + q = a + (p + q)$ now naming $k = p + q$ we have $c = a + k$ and then by definition $a < c$.

If $a \leq b$ and $b < c$ then either $a < b$ or $a = b$, the first case was already proven so we focus on the second one. We also have that $b < c$ then by definition there is a $p \in \mathbb{N}$ such that $c = b+p$ but $a = b$ then replacing $c = a + p$ and by definition $a < c$.

If now, $a < b$ and $b \leq c$ we have that either $b < c$ or $b = c$ the first case was already proven so we focus on the second one. Given that $a < b$ there is a $p \in \mathbb{N}$ such that $b = a + p$ but if $b = c$ then $c = a + p$ which by definition says that $a < c$.

Finally, if $a \leq b$ and $b \leq c$ then either $a < b$ or $a = b$ and $b < c$ or $b = c$, the last combination we have to prove is the case where $a = b$ and $b = c$ then $a = b = c$ so it's clear that $a \leq c$.

(4) Let $a < b$ then by definition there is a $p \in \mathbb{N}$ such that $b = a + p$ then $b + c = (a + p) + c$ because of part (1) of Theorem 1.2.7, and by the Commutative and Associative law we have that $b + c = (a + c) + p$ which by definition says that $a + c < b + c$.

If $a + c < b + c$ then by definition there is a $p \in \mathbb{N}$ such that $b + c = (a + c) + p$ and by the Commutative and Associative law we have that $b + c = (a + p) + c$ and because of part (1) of Theorem 1.2.7 we have that $b = a + p$ which by definition says that $a < b$.

(5) Let $a < b$ then by definition there is a $p \in \mathbb{N}$ such that $b = a + p$ and because of part (12) of Theorem 1.2.7 we have that $bc = (a + p)c = ac + pc$ where we also applied the Distributive law now naming $k = pc$ we have that $bc = ac + k$ which by definition means that $ac < bc$.

Let $ac < bc$ and suppose $a \nless b$ then by the Trichotomy law either $a > b$ or $a = b$. If $a > b$ then because what we proved $ac > bc$ which is a contradiction to $ac < bc$. If $a = b$ then $ac = bc$ because of part (12) of Theorem 1.2.7. and it's another contradiction to the fact that $ac < bc$. Then must happen that $a < b$.

(11) Let $a < b$ and suppose $b < a + 1$ then $a < b < a + 1$ but this cannot happen because of part (9) so it must happen because of the Trichotomy law that $a + 1 \leq b$.

Now let $a + 1 \leq b$ and suppose that $a = b$ then $a + 1 \leq b = a$ which cannot be because of part (1) of this Theorem, so lets suppose that $a > b$ then $a + 1 < a$ because of part (3) of this Theorem but that cannot be true because of part (1) of this Theorem. Therefore it must be that $a < b$.

□

Proof. **1.2.5** Let $a+a=b+b$ then because of part (7) of Theorem 1.2.7 we can write that $1\cdot a+1\cdot a=1\cdot b+1\cdot b$ and because of the Distributive law we have that $a\cdot(1+1)=b\cdot(1+1)$ if we name $c=1+1$ then $ac=bc$ and because of part (12) of Theorem 1.2.7 we have that $a=b$. □

Proof. **1.2.6** Let

$$H=\{n\in\mathbb{N}\mid 1\leq n\leq b\}\cup\{n\in\mathbb{N}\mid b+1\leq n\}$$

We want to show that $H=\mathbb{N}$. It is clear that $H\subseteq\mathbb{N}$. Because $1\in\{n\in\mathbb{N}\mid 1\leq n\leq b\}$ by definition then $1\in H$.
Now let $r\in H$, we want to show that $r+1\in H$, if $r=b$ then

$$r+1=b+1\in\{n\in\mathbb{N}\mid b+1\leq n\}$$

if $r<b$ then because of part (11) of Theorem 1.2.9 $r+1\leq b$ then

$$r+1\in\{n\in\mathbb{N}\mid 1\leq n\leq b\}$$

if $b<r$ then because of part (11) or Theorem 1.2.9 $b+1\leq r$ also because of part (1) of Theorem 1.2.9 $r<r+1$ then because of part (3) of Theorem 1.2.9 we have that $b+1<r+1$ then

$$r+1\in\{n\in\mathbb{N}\mid b+1\leq n\}$$

Therefore $r+1\in H$ and $H=\mathbb{N}$.
Now Let

$$G=\{n\in\mathbb{N}\mid 1\leq n\leq b\}\cap\{n\in\mathbb{N}\mid b+1\leq n\}$$

Suppose there is an $r\in G$. We will derive a contradiction. Then it must happen that $1\leq r\leq b$ and that $b+1\leq r$ but then because of part (3) of Theorem 1.2.9 it must happen that $b+1\leq b$ which is a contradiction to the part (6) of Theorem 1.2.9. Therefore there is no $r\in G$. □

Proof. **1.2.7**

(1) Let

$$H=\{n\in\mathbb{N}\mid a+n\in A\text{ for all }a\in A\}$$

We want to show that $H=\mathbb{N}$ which will imply the desired result. It is clear that $H\subseteq\mathbb{N}$. To show that $1\in H$ let $b\in A$ then $b+1\in A$ by definition of A and then $1\in H$.

Now let $r\in H$ then $b+r\in A$ for some $b\in A$ and by definition of A we have that $(b+r)+1=b+(r+1)\in A$ then $r+1\in H$ and therefore $H=\mathbb{N}$.

(2) Let $H=\{x\in\mathbb{N}\mid x\geq a\}$ and let $r\in H$ then $r\geq a$ so it must happen that $r=a$ or $r>a$ in the first case it is clear that $r=a\in A$ in the second case by definition there is a $p\in\mathbb{N}$ such that $r=a+p$ and we know because of the part (1) that $a+p\in A$. Therefore it must happen that $H\subseteq A$.

$\square$

Proof. **1.2.8** We want to prove that there is an inverse function for f. We have a set $\mathbb{N}'$ with an element $1' \in \mathbb{N}'$ and a function $s' : \mathbb{N}' \to \mathbb{N}'$ that satisfy the Peano Postulates so we could say because Theorem 1.2.4 that there is a function $g : \mathbb{N}' \to \mathbb{N}$ such that $g(1') = 1$ and $g \circ s' = s \circ g$. Now we have to check that g is the inverse of f.
Let

$$G = \{n \in \mathbb{N} \mid g(f(n)) = n\}$$

We want to show that $G = \mathbb{N}$. It is clear that $G \subseteq \mathbb{N}$. To show that $1 \in G$ we do $g(f(1)) = g(1') = 1$ which means that $1 \in G$.
Now Let $r \in G$ we want to show that $r + 1 \in G$ then

$$\begin{aligned} g(f(r+1)) &= g(f(s(r))) && \text{by definition of } r+1 \\ &= g(s'(f(r))) && \text{because } f \circ s = s' \circ f \\ &= s(g(f(r))) && \text{because } g \circ s' = s \circ g \\ &= s(r) = r + 1 && \text{because } r \in G \end{aligned}$$

Therefore $r + 1 \in G$ and $G = \mathbb{N}$. In the same way, let

$$H = \{n \in \mathbb{N}' \mid f(g(n)) = n\}$$

We want to show that $H = \mathbb{N}'$. It is clear that $H \subseteq \mathbb{N}'$. To show that $1' \in H$ we do $f(g(1')) = f(1) = 1'$ which means that $1' \in H$.
Now let $r' \in H$ we want to show that $r' + 1 \in H$ then

$$\begin{aligned} f(g(r'+1)) &= f(g(s'(r'))) && \text{by definition of } r'+1 \\ &= f(s(g(r'))) && \text{because } g \circ s' = s \circ g \\ &= s'(f(g(r'))) && \text{because } f \circ s = s' \circ f \\ &= s'(r') = r' + 1 && \text{because } r' \in H \end{aligned}$$

Therefore $r' + 1 \in H$ and $H = \mathbb{N}'$.
Finally, g is the inverse of f hence f is bijective, which is what we wanted.

$\square$

Proof. **1.3.1**

(1) We want to prove that $\approx$ is an equivalence relation so we must prove that it is reflexive, symmetric and transitive. Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$. Given that $a^2 b = a^2 b$ then $(a, b) \approx (a, b)$. Therefore $\approx$ is reflexive.
Suppose that $(a, b) \approx (c, d)$ then $a^2 d = c^2 b$ but also $c^2 b = a^2 d$ hence $(c, d) \approx (a, b)$ and therefore $\approx$ is symmetric.
Now suppose that also $(e, f) \in \mathbb{N} \times \mathbb{N}$ and $(c, d) \approx (e, f)$ then $c^2 f = e^2 d$ but we know that $a^2 d = c^2 b$ multiplying this last equation on both sides with f and doing a few re-arrangements we have that $a^2 df = c^2 fb = e^2 db$ then $a^2 f = e^2 b$ which means that $(a, b) \approx (e, f)$. Therefore $\approx$ is transitive.
Finally since $\approx$ is reflexive, symmetric and transitive then $\approx$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

(2) The elements of the equivalence class $[(2,3)]$ are

$$[(2,3)] = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid 4y = (x^2)3\}$$

$\square$

Proof. **1.3.2** We want to complete the proof by showing that $\sim$ is transitive. Let $(e,f) \in \mathbb{N} \times \mathbb{N}$ and $(c,d) \sim (e,f)$ then $c+f = d+e$ also we know that $a+d = b+c$ and adding to both sides f we get that $a+d+f = b+c+f = b+d+e$ then $a+f = b+e$ because of Theorem 1.2.7 part (1). Therefore $(a,b) \sim (e,f)$.
Finally since $\sim$ is reflexive, symmetric and transitive then $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. $\square$

Proof. **1.3.3** Let's prove first that $-$ is well-defined for $\mathbb{Z}$. Let $(a,b),(x,y) \in \mathbb{N} \times \mathbb{N}$ and suppose that $[(a,b)] = [(x,y)]$ then $(a,b) \sim (x,y)$ so $a+y = b+x$ but if we write $b+x = a+y$ we have that $(b,a) \sim (y,x)$ therefore $-[(a,b)] = [(b,a)] = [(y,x)] = -[(x,y)]$.
Now let's prove that $\cdot$ is well-defined for $\mathbb{Z}$. Let $(a,b),(c,d),(x,y),(z,w) \in \mathbb{N} \times \mathbb{N}$ and suppose $[(a,b)] = [(x,y)]$ and $[(c,d)] = [(z,w)]$ so by hypothesis $(a,b) \sim (x,y)$ and $(c,d) \sim (z,w)$ then $a+y = b+x$ and $c+w = d+z$. Taking into account this let's do

$$\begin{aligned}
&(ac+bd+xw+yz)+(xc+yc+xd+yd) = \\
&= c(a+y+x)+d(b+x+y)+xw+yz \\
&= c(b+x+x)+d(a+y+y)+xw+yz \\
&= bc+xc+xc+ad+yd+yd+xw+yz \\
&= ad+bc+x(c+c+w)+y(d+d+z) \\
&= ad+bc+x(c+d+z)+y(d+c+w) \\
&= (ad+bc+xz+yw)+(xc+yc+xd+yd)
\end{aligned}$$

which proves that $ac+bd+xw+yz = ad+bc+xz+yw$ and therefore $[(ac+bd, ad+bc)] = [(xz+yw, xw+yz)]$. $\square$

Proof. **1.3.4**

(1) $(\rightarrow)$ If $[(a,b)] = \hat{0}$ then $[(a,b)] = [(1,1)]$ because of the definition of $\hat{0}$ then $(a,b) \sim (1,1)$ so $a+1 = b+1$ because of the Cancellation Law of $\mathbb{N}$ we have that $a = b$.
$(\leftarrow)$ If $a = b$ then adding to both sides 1 we have that $a+1 = b+1$ then by the Definition 1.3.1 $(a,b) \sim (1,1)$ therefore $[(a,b)] = [(1,1)]$.

(2) $(\rightarrow)$ If $[(a,b)] = \hat{1}$ then $[(a,b)] = [(1+1,1)]$ because of the definition of $\hat{1}$ then $(a,b) \sim (1+1,1)$ so $a+1 = (b+1)+1$ because of the Cancellation Law of $\mathbb{N}$ we have that $a = b+1$.
$(\leftarrow)$ If $a = b+1$ then adding to both sides 1 we have that $a+1 = (b+1)+1 = b+(1+1)$ then by the Definition 1.3.1 $(a,b) \sim (1+1,1)$ therefore $[(a,b)] = [(1+1,1)]$.

(3) First let's prove that $[(a,b)] = [(n,1)]$ for some $n \in \mathbb{N}$ such that $n \neq 1$ is and only if $a > b$.
($\rightarrow$) Let $[(a,b)] = [(n,1)]$ for some $n \in \mathbb{N}$ where $n \neq 1$ then $a+1 = b+n$ and given that $n \in \mathbb{N}$ and $n \neq 1$ then n can be written as $n = q+1$ for some $q \in \mathbb{N}$ then $a+1 = b+q+1$ and by the Cancellation law of $\mathbb{N}$ we have that $a = b+q$ which by definition means that $a > b$.
($\leftarrow$) Let $a > b$ by definition this means that $a = b+m$ where $m \in \mathbb{N}$ then if we add 1 to both sides of the equation we have that $a+1 = b+m+1$ then by naming $n = m+1$ where $n \in \mathbb{N}$ we have that $a+1 = b+n$ and therefore $[(a,b)] = [(n,1)]$ where $n \neq 1$.
Now Let's prove that $a > b$ if and only if $[(a,b)] > \hat{0}$
($\rightarrow$) Let $a > b$ if we add on both sides of the equation 1 we get that $a+1 > b+1$ then this means that $[(a,b)] > \hat{0} = [(1,1)]$.
($\leftarrow$) Let $[(a,b)] > \hat{0} = [(1,1)]$ then $a+1 > b+1$ and because of Theorem 1.2.9 part (4) we have that $a > b$.

(4) First let's prove that $[(a,b)] = [(1,m)]$ for some $m \in \mathbb{N}$ such that $m \neq 1$ if and only if $a < b$.
($\rightarrow$) Let $[(a,b)] = [(1,m)]$ for some $m \in \mathbb{N}$ where $m \neq 1$ then $a+m = b+1$ and given that $m \in \mathbb{N}$ and $m \neq 1$ then m can be written as $m = q+1$ for some $q \in \mathbb{N}$ then $a+q+1 = b+1$ and by the Cancellation law of $\mathbb{N}$ we have that $a+q = b$ which by definition means that $a < b$.
($\leftarrow$) Let $a < b$ by definition this means that $b = a+n$ where $n \in \mathbb{N}$ then if we add 1 to both sides of the equation we have that $b+1 = a+n+1$ then by naming $m = n+1$ where $m \in \mathbb{N}$ we have that $a+m = b+1$ and therefore $[(a,b)] = [(1,m)]$ where $m \neq 1$.
Now Let's prove that $a < b$ if and only if $[(a,b)] < \hat{0}$
($\rightarrow$) Let $a < b$ if we add on both sides of the equation 1 we get that $a+1 < b+1$ then this means that $[(a,b)] < \hat{0} = [(1,1)]$.
($\leftarrow$) Let $[(a,b)] < \hat{0} = [(1,1)]$ then $a+1 < b+1$ and because of Theorem 1.2.9 part (4) we have that $a < b$.

□

Proof. **1.3.5**

(1) Using the definition of addition of integers we see that

$$
\begin{aligned}
(x+y)+z &= ([(a,b)] + [(c,d)]) + [(e,f)] \\
&= [(a+c, b+d)] + [(e,f)] \\
&= [((a+c)+e, (b+d)+f)] \\
&= [(a+(c+e), b+(d+f))] \\
&= [(a,b)] + [(c+d, d+f)] \\
&= [(a,b)] + ([(c,d)] + [(e,f)]) \\
&= x+(y+z)
\end{aligned}
$$

where the middle equality holds because of the Associative law of $\mathbb{N}$.

(3) We want to prove that $x+\hat{0}=x$ by arriving to a contradiction. Let us suppose $x+\hat{0}\neq x$ then this means that $[(a,b)]+[(1,1)]=[(c,d)]$ where $[(c,d)]\neq[(a,b)]$ then $[(a+1,b+1)]=[(c,d)]$ so $a+1+d=b+1+c$ and by the Cancellation of $\mathbb{N}$ we have that $a+d=b+c$ then $[(a,b)]=[(c,d)]$ which is a contradiction and therefore $x+\hat{0}=x$.

(4) Let $x=[(a,b)]$ then from the equation $(a+b)+1=(b+a)+1$ we have that $[(a+b,b+a)]=[(1,1)]$ and thus $[(a,b)]+[(b,a)]=[(1,1)]$ therefore $x+(-x)=\hat{0}$.

(5) Let $x=[(a,b)]$, $y=[(c,d)]$ and $z=[(e,f)]$ then

$$\begin{aligned}
(xy)z &= ([(a,b)]\cdot[(c,d)])\cdot[(e,f)]\\
&= [(ac+bd,ad+bc)]\cdot[(e,f)]\\
&= [((ac+bd)e+(ad+bc)f,(ac+bd)f+(ad+bc)e)]\\
&= [(ace+bde+adf+bcf,acf+bdf+ade+bce)]\\
&= [(a(ce+df)+b(de+cf),a(cf+de)+b(df+ce))]\\
&= [(a,b)]\cdot[(ce+df,de+cf)]\\
&= [(a,b)]\cdot([(c,d)]\cdot[(e,f)])=x(yz)
\end{aligned}$$

(6) Let $x=[(a,b)]$ and $y=[(c,d)]$ then

$$\begin{aligned}
xy &= [(a,b)]\cdot[(c,d)]\\
&= [(ac+bd,ad+bc)]\\
&= [(ca+db,cb+da)]\\
&= [(c,d)]\cdot[(a,b)]=yx
\end{aligned}$$

Where the middle equality is true because of the Commutative Law for Addition and Multiplication.

(7) Let $a,b\in\mathbb{N}$ then the equation $a(1+1)+b(1)+b=b(1+1)+a(1)+a$ is true and if we let $x=[(a,b)]$ then the equation means that

$$[(a(1+1)+b(1),b(1+1)+a(1))]=[(a,b)]$$

then because of the definition of the $\cdot$ operation we have that

$$[(a,b)]\cdot[(1+1,1)]=[(a,b)]$$

therefore $x\cdot\hat{1}=x$.

(8) Let $x=[(a,b)]$, $y=[(c,d)]$ and $z=[(e,f)]$ then

$$\begin{aligned}
x(y+z) &= [(a,b)]\cdot([(c,d)]+[(e,f)])\\
&= [(a,b)]\cdot[(c+e,d+f)]\\
&= [(a(c+e)+b(d+f),a(d+f)+b(c+e))]\\
&= [(ac+ae+bd+bf,ad+af+bc+be)]\\
&= [((ac+bd)+(ae+bf),(ad+bc)+(af+be))]\\
&= [(ac+bd,ad+bc)]+[(ae+bf,af+be)]\\
&= [(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(e,f)]=xy+xz
\end{aligned}$$

(10) Let us first prove that there is no way that two of $x < y$, $x = y$ or $x > y$ can be true at the same time.
Let $x = [(a, b)]$ and $y = [(c, d)]$ and let's suppose that $x < y$ and $x = y$ are both true then $x < x$ which means that $[(a, b)] < [(a, b)]$ and thus $a + b < b + a$ by the Cancellation law $a < a$ but that is a contradiction to Theorem 1.2.9 part (1).
In the same way let's suppose that $x > y$ and $x = y$ are both true then $x > x$, which leads to the same result as before which is a contradition.
Finally, let's suppose that $x < y$ and $y < x$ are both true then from the first inequality we have that $[(a, b)] < [(c, d)]$ then $a + d < b + c$, and from the last inequality we have that $[(c, d)] < [(a, b)]$ then $c + b < a + d$ and by applying the Transitive law of $\mathbb{N}$ we have that $a + d < a + d$ thus $a < a$ and we have already proven that this is a contradiction.
Therefore no two of $x < y$, $x = y$ and $x > y$ can be true at the same time.
Now let's prove that one of them is always true.
Suppose $x, y \in \mathbb{Z}$ then $x = [(a, b)]$ and $y = [(c, d)]$ also it must be true that $a + d < b + c$ or $a + d = b + c$ or $a + d > b + c$ because of Trichotomy of $\mathbb{N}$ if $a + d < b + c$ is true then that means that $[(a, b)] < [(c, d)]$, if $a + d = b + c$ then $[(a, b)] = [(c, d)]$ or if $a + d > b + c$ then $[(a, b)] > [(c, d)]$.

(11) Let $x = [(a, b)]$, $y = [(c, d)]$ and $z = [(e, f)]$ then if $x < y$ and $y < z$ that means that $a + d < b + c$ and from the second inequality we have that $c + f < d + e$ then by definition $b + c = (a + d) + p$ and $d + e = (c + f) + q$ where $p, q \in \mathbb{N}$ then summing both equations we have that

$$(b + c) + (d + e) = (a + d) + p + (c + f) + q$$
$$b + e = (a + f) + (p + q)$$

If we name $k = p + q$ then by definition $a + f < b + e$ and therefore $[(a, b)] < [(e, f)]$ thus $x < z$.

(13) Let $x = [(a, b)]$, $y = [(c, d)]$ and $z = [(e, f)]$ then if $x < y$ this means that $a + d < b + c$ and by definition $b + c = (a + d) + p$ where $p \in \mathbb{N}$ also we know that $\hat{0} < z$ so $1 + f < 1 + e$ and also by definition this means that $e = f + q$ where $q \in \mathbb{N}$. Multiplying both sides of $b + c = (a + d) + p$ with e we get that

$$e(b + c) = e(a + d) + ep$$

And doing the same with f we get that

$$f(a + d) + fp = f(b + c)$$

Then summing both equations we get that

$$f(a + d) + e(b + c) + fp = f(b + c) + e(a + d) + ep$$

Replacing $e = f + q$ on the right hand side of the equation we get that

$$f(a+d) + e(b+c) + fp = f(b+c) + e(a+d) + fp + qp$$

and by the Cancellation law we get that

$$f(a+d) + e(b+c) = f(b+c) + e(a+d) + qp$$

Which means that $f(b+c) + e(a+d) < f(a+d) + e(b+c)$ then $ae + bf + cf + de < af + be + ce + df$ and thus $[(ae + bf, af + be)] < [(ce + df, cf + de)]$ therefore $xy < xz$.

(14) Let's suppose that $\hat{0} = \hat{1}$ we want to arrive to a contradiction, then $[(1,1)] = [(1+1,1)]$ so $1+1 = 1+(1+1)$ and by the Cancellation law we have that $1 = (1+1)$ which cannot be because there is no $a, b \in \mathbb{N}$ such that $a + b = 1$. Therefore $\hat{0} \neq \hat{1}$.

□

Proof. **1.3.6** Let us prove the rest of the Theorem 1.3.7

1. The function $i : \mathbb{N} \to \mathbb{Z}$ is injective.
Let $i(n) = i(m)$ then by definition of i we have that $[(n+1,1)] = [(m+1,1)]$ thus $(n+1)+1 = 1+(m+1)$ and by the Cancellation law we have that $n = m$.

3. $i(1) = \hat{1}$
By definition of i we have that $i(1) = [(1+1,1)]$ and therefore $i(1) = \hat{1}$.

4b. $i(ab) = i(a)i(b)$
By definition of i we have that $i(ab) = [(ab+1,1)]$ then

$$\begin{aligned} i(ab) &= [(ab+1,1)] \\ &= [(ab+a+b+1+1, a+b+1+1)] \\ &= [((a+1)(b+1)+1, (a+1)+(b+1))] \\ &= [(a+1,1)] \cdot [(b+1,1)] \\ &= i(a)i(b) \end{aligned}$$

4c. $a < b$ if and only if $i(a) < i(b)$.
($\to$) By definition $a < b$ means that $b = a + p$ where $p \in \mathbb{N}$ then applying the function i to both sides of the equation we have that $i(b) = i(a+p)$ and because of what we have proven in **4a** we have that $i(b) = i(a) + i(p)$ which by definition means $i(a) < i(b)$.
($\leftarrow$) By definition $i(a) < i(b)$ means that $[(a+1,1)] < [(b+1,1)]$ and thus $(a+1)+1 < 1+(b+1)$ and because of the Cancellation law we have that $a < b$ as we wanted.

□

Proof. **1.3.7**

(1) ($\rightarrow$) Let $x < y$ then by the Addition Law for Order we have that $x + (-x) < y + (-x)$ then by the Inverses Law for Addition we have that $0 < y + (-x)$ applying the Addition Law for Order again we have that $0 + (-y) < (y + (-x)) + (-y)$ then by applying the Commutative and Associative Law for Addition we have that $-y < (y + (-y)) + (-x)$ and therefore $-y < -x$.
($\leftarrow$) Let $-y < -x$ then by the Addition Law for Order we have that $x + (-y) < x + (-x)$ then by the Inverses Law for Addition we have that $x + (-y) < 0$ applying the Addition Law for Order again we have that $(x + (-y)) + y < 0 + y$ then by applying the Commutative and Associative Law for Addition we have that $((-y) + y) + x < y$ and therefore $x < y$.

(2) ($\rightarrow$) Let $z < 0$ and $x < y$ then $-z > 0$ because Lemma 1.4.5 part(8) and by the Multiplication Law for Order we have that $x(-z) < y(-z)$ which by the Lemma 1.3.8 part 6 we know that $-xz < -yz$ thus $xz > yz$ because what we saw in part (1) of this problem.
($\leftarrow$) Let $xz > yz$ where $z < 0$ because what we saw in part (1) of this problem this means that $-xz < -yz$ and then by the Multiplication Law for Order we have that $x < y$ because $-z > 0$.

□

Proof. **1.3.8** From Theorem 1.3.9 we know that if $z \in \mathbb{Z}$ there is no $y \in \mathbb{Z}$ such that $z < y < z + 1$ then there is no x such that $0 < x < 1$ so if $x > 0$ it must be that $x \geq 1$.
If $x < 0$ then $-x > 0$ and as we saw this means that $-x \geq 1$ and by what we proved in problem 1.3.7 given that $-1 < 0$ then $-1 \cdot -x < -1 \cdot 1$ therefore $-(1 \cdot (-x)) = -(-x) = x < -1$ where in the equalities we are using the fact that $(-x)y = -xy = x(-y)$. □

Proof. **1.3.9**

(1) From the part (9) of the Lemma 1.4.5 we know that $0 < 1$ then by the Addition Law for Order we have that $0 + 1 < 1 + 1$ by the Identity Law for Addtion we have that $1 < 1 + 1$ and now let us call 2 the following addition $2 = 1 + 1$ therefore $1 < 2$.

(2) Suppose $2x = 1$ where $x \in \mathbb{Z}$ then as we proved in part (1) we have that $2x < 2 = 2 \cdot 1$ then by the Multiplication Law for Order we have that $x < 1$ then $x \leq 0$ if $x = 0$ then $2 \cdot 0 = 0 \neq 1$ so must happen that $x < 0$ then by Lemma 1.4.5 part (11) $2 \cdot x < 0$ but $1 > 0$ which is a contradiction. Therefore $2x \neq 1$.

□

Proof. **1.3.10** Let's define $G' = i^{-1}(G)$ which is the inverse image of G then $G' \subseteq \mathbb{N}$ and because of the Well-Ordering Principle for $\mathbb{N}$ there is $m \in G'$ such that $m \leq g$ for all $g \in G'$. By appying i to both sides of the inequality we have that $i(m) \leq i(g)$ which we can do because of part 4c of Theorem 1.3.7. We know that $G \subseteq \{x \in \mathbb{Z} \mid x > \hat{0}\} = i(\mathbb{N})$ then the elements of G have the form $i(a)$ where $a \in G'$ then $i(m), i(g) \in G$ so we want to check that $i(m)$ is the minimum then let's suppose that $i(m) > i(g)$ we want to arrive to a contradiction then $[(m+1, 1)] > [(g+1, 1)]$ and thus $m+1+1 > g+1+1$ by the Cancellation law we have that $m > g$ which is a contradiction as we wanted because m is the minimum of G'. Therefore $i(m) \leq i(g)$ for all $i(g) \in G$. $\square$

Proof. **1.3.11**

(1) By adding to both sides of the equation $(-z)$ we get that $(x+z)+(-z) = (y+z)+(-z)$ and by the Associative Law for Addition we have that $x+(z+(-z)) = y+(z+(-z))$ then because of the Inverse Law for Addition we have that $x+0 = y+0$ which means that $x = y$ because of the Identity Law for Addition.

(3) By the Inverses Law for Addition we know that $(x+y)+(-(x+y)) = 0$ then adding to both sides of the equation $-x$ and $-y$ we get that $(-x)+(-y)+(x+y)+(-(x+y)) = (-x)+(-y)$ then by using multiple times the Commutative and Associative Law for addition we get that $(x+(-x))+(y+(-y))+(-(x+y)) = (-x)+(-y)$ then $0+0+(-(x+y)) = (-x)+(-y)$ because of the Inverses Law for Addition and therefore $-(x+y) = (-x)+(-y)$ because of the identity law for Addition.

(4) Let us suppose that $x \cdot 0 \neq 0$ we want to arrive to a contradiction then by the Trychotomy Law it must hold that $x \cdot 0 > 0$ or $x \cdot 0 < 0$. Let's suppose that $x \cdot 0 > 0$ holds then by the Addition Law for Order we have that $(x \cdot 0) + x > 0 + x$ and because the Identity Law for Multiplication we have that $(x \cdot 0)+(x \cdot 1) > 0+x$ then by the Identity Law for Addition and the Distributive Law we have that $x \cdot (0+1) > x$ again by the Identity Law for Addition we have that $x \cdot 1 > x$ then $x > x$ because of the Identity Law for Multiplication but this cannot be then it must hold that $x \cdot 0 < 0$ but by the same type of arguments we see that this cannot be either. Therefore it must be that $x \cdot 0 = 0$.

(5) Suppose that $y = x + k$ where $k \in \mathbb{Z}$ then we get that $xz = yz = (x+k)z$ and by adding to both sides of the equation $-xz$ we get that $xz + (-xz) = (xz + kz) + (-xz)$ then by applying the Associative and Commutative Law for Addition we get that $xz + (-xz) = (xz + (-xz)) + kz$ and by the Inveses Law for Addition we get that $0 = 0 + kz = kz$ then either $k = 0$ or $z = 0$ by the Non Zero Divisors Law, but we know that $z \neq 0$ then it must be that $k = 0$ therefore $y = x + 0 = x$.

(7) ($\rightarrow$) If $xy = 1$ we have a few cases we need to address to prove that either $x = y = 1$ or $x = y = -1$. Given that if $x > 0$ and $y > 0$ then $xy > 0$ and if $x > 0$ and $y < 0$ then $xy < 0$ we can rule a lot of cases by taking into account that either both x and y are positive or both are negative. Also the case where one of them or both are 0 is also ruled out because of part (4) of this Lemma.

- Let's check first the case where both x and y are positive. Let then $x > 1$ and $y > 1$ by multiplying the y inequality by x we have that $xy > x \cdot 1$ then because of the Identity Law for Multiplication we have that $xy > x > 1$ thus $xy \neq 1$.
- Now let $x < -1$ and $y < -1$ then $-x > 1 > 0$ because of part (8) of this Lemma so we multiply both sides of the y inequality by $-x$ as $-xy < -1 \cdot 1$ and because of the Identity Law for Multiplication we have that $-xy < -1 < 0$ and then again by the part (8) of this Lemma we have that $xy > 1$ thus $xy \neq 1$.
- Finally, the only option left is that $x = y = 1$ or $x = y = -1$. In the first case by the Identity Law for Multiplication we have that $xy = 1 \cdot 1 = 1$ and in the second case if $x = y = -1$ we have that $xy = (-1) \cdot (-1) = -((-1) \cdot 1)$ because of part (6) of this Lemma and then $xy = -(-1) = 1$ because the Identity Law for Multiplication and part (2) of this Lemma.

Therefore, if $xy = 1$ then $x = y = 1$ or $x = y = -1$.
($\leftarrow$) As shown before if $x = y = 1$ or $x = y = -1$ then $xy = 1$.

(8) ($\rightarrow$) If $x > 0$ then by adding to both sides of the equation $-x$ we have that $x + (-x) > 0 + (-x)$ then because of the Identity Law for Addition we have that $x + (-x) > -x$ and because of the Inverses Law for Addition $0 > -x$.
($\leftarrow$) If $-x < 0$ then by adding to both sides of the equation x we have that $x + (-x) < x + 0$ and because of the Identity Law and the Inverses Law for Addition we have that $0 < x$.
($\rightarrow$) If $x < 0$ then by adding to both sides of the equation $-x$ we have that $x + (-x) < 0 + (-x)$ and because of the Identity Law and the Inverses Law for Addition we have that $0 < -x$.
($\leftarrow$) If $-x > 0$ then by adding to both sides of the equation x we have that $x + (-x) > x + 0$ and because of the Identity Law and the Inverses Law for Addition we have that $0 > x$.

(10) If $x \leq y$ then either $x = y$ or $x < y$ by definition and in the same way if $y \leq x$ then either $y = x$ or $y < x$.
In case $x = y$ and $y = x$ then we are done.
In case $x = y$ and $y < x$ then by replacing y we have that $x < x$ which isn't true and thus $x = y$ must be true. The same can be proven for $y = x$ and $x < y$.
In case $x < y$ and $y < x$ then by the Transitive Law $x < x$ which is not true and then must be that $x = y$.

(11) If $x > 0$ and $y > 0$ then by multipyling the y inequality by x we have that $xy > x \cdot 0$ and we can do that because of the Theorem 1.3.5 part (13) and because of the result we proved in part (4) of this Lemma then $xy > 0$.
If $x > 0$ and $y < 0$ then by multipyling the y inequality by x we have that $xy < x \cdot 0$ and we can do that because of the Theorem 1.3.5 part (13) and because of the result we proved in part (4) of this Lemma then $xy < 0$.

□

Proof. **1.4.2** Let $n \in \mathbb{N}$ also we know that $\mathbb{N}$ is defined as $\mathbb{N} = \{x \in \mathbb{Z} \mid x > 0\}$ then $n \in \mathbb{Z}$ and $n > 0$ by adding $1 > 0$ to both sides of the inequality we have that $n+1 > 0+1 = 1$ where $0+1 = 1$ because of the Identity Law for Addition and as we saw $1 > 0$ then $n+1 > 0$ and also $n+1 \in \mathbb{Z}$ therefore $n+1 \in \mathbb{N}$. □

Proof. **1.4.3**
$(\rightarrow)$ If $x, y \in \mathbb{Z}$ and $x \leq y$ then by definition either $x = y$ or $x < y$ if the last one holds then by adding $-x$ to both sides of the inequality we have that $x + (-x) < y + (-x)$ and because of the Inverses Law for Addition we have that $0 < y + (-x)$ and if we now add $-y$ to both sides of the inequality we have that $(-y) + 0 < (-y) + (y + (-x))$ and because of the Identity Law for Addition and the Associative Law we have that $-y < ((-y) + y) + (-x)$ then again by the Inverses Law for Addition we have that $-y < -x$.
But if $x = y$ holds then applying the exact same steps as before we have that $-y = -x$.
$(\leftarrow)$ If $x, y \in \mathbb{Z}$ and $-y \leq -x$ then by definition either $-y = -x$ or $-y < -x$ if the last one holds then by adding x to both sides of the inequality we have that $x + (-y) < x + (-x)$ and because of the Inverses Law for Addition we have that $x + (-y) < 0$ and if we now add y to both sides of the inequality we have that $(x + (-y)) + y < 0 + y$ and because of the Identity Law for Addition and the Associative Law we have that $x + ((-y) + y) < y$ then again by the Inverses Law for Addition we have that $x < y$.
But if $-y = -x$ holds then applying the exact same steps as before we have that $x = y$. □

Proof. **1.4.4** We defined $\mathbb{N} = \{x \in \mathbb{Z} \mid x > 0\}$ and we know because of Theorem 1.4.6 that if $z \in \mathbb{Z}$ then there is no $y \in \mathbb{Z}$ such that $z < y < z+1$ then there is no $x \in \mathbb{Z}$ such that $0 < x < 1$ then it must be that $\mathbb{N} = \{x \in \mathbb{Z} \mid x \leq 1\}$. □

Proof. **1.4.5** If $a < b$ then by adding 1 to both sides of the equation we have that $a+1 < b+1$ since we saw that there is no $y \in \mathbb{Z}$ such that $b < y < b+1$ then $a+1 = b$ or $a+1 < b$ therefore $a+1 \leq b$. □

Proof. **1.4.6** From problem 1.4.4 we know that $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 1\}$ so if $n \in \mathbb{N}$ and $n \neq 1$ then it must be that $n > 1$ so by definition there is $b \in \mathbb{N}$ such that $n = b+1$. □

Proof. **1.4.8**

(1) Let us write $F = \{x \in G \mid x+(-a)+1\}$ we want to show by induction that $F = \mathbb{N}$ which would be the same as showing that $G = \{x \in \mathbb{Z} \mid x+(-a)+1 \geq 1\}$. Since $a \in G$ then $a+(-a)+1 = 1 \in F$. Now if $g \in G$ then $g+(-a)+1 \in F$ and by definition we know that $g+1 \in G$ then $g+1+(-a)+1 = (g+(-a)+1)+1 \in F$ by using the Associative and Commutative Law for Addition. Therefore $F = \mathbb{N}$.

(2) Let us write $F = \{x \in H \mid a+(-x)+1\}$ we want to show by induction that $F = \mathbb{N}$ which would be the same as showing that $H = \{x \in \mathbb{Z} \mid 1 \leq a+(-x)+1\}$. Since $a \in H$ then $a+(-a)+1 = 1 \in F$. Now if $h \in H$ then $a+(-h)+1 \in F$ and by definition we know that $h+(-1) \in G$ then $a+(-(h+(-1)))+1 = (a+(-h)+1)+1 \in F$ by using the Associative and Comutative Law for Addition and the fact that $-(-1) = 1$. Therefore $F = \mathbb{N}$.

□

Proof. **1.5.1** We want to prove that $\asymp$ is an equivalence relation so we must prove that it is reflexive, symmetric and transitive. Let $(a,b),(c,d) \in \mathbb{Z}\times\mathbb{Z}^*$. We note that $ab = ba$ because of the Commutative Law for Multiplication then $(a,b) \asymp (a,b)$ thus $\asymp$ is reflexive. Now suppose that $(a,b) \asymp (c,d)$ then $ad = bc$ and because of the Commutative Law for Multiplication we have that $cb = da$ then $(c,d) \asymp (a,b)$, therefore $\asymp$ is also symmetric.
We also proved that $\asymp$ is transitive. Therefore $\asymp$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$. □

Proof. **1.5.2** Let's prove that $+$ is well-defined for $\mathbb{Q}$.
Let $(x,y),(z,w),(a,b),(c,d) \in \mathbb{Z}\times\mathbb{Z}^*$ and suppose that $[(x,y)] = [(a,b)]$ and $[(z,w)] = [(c,d)]$ then $(x,y) \asymp (a,b)$ and $(z,w) \asymp (c,d)$ so $xb = ya$ and $zd = wc$ now multiplying both sides of the the first equation by dw we have that $xbdw = yadw$ also we now multiply both sides of the second equation by yb to obtain $zdyb = wcyb$, now we add both equations

$$xbdw + zdyb = yadw + wcyb$$

by the Commutative Law and the Distribute Law we obtain

$$(xw + yz)(bd) = (yw)(ad + cb)$$

this means that $[(xw + yz, yw)] = [(ad + cb, bd)]$ therefore $+$ is well-defined.
Let's now prove that the unary operation $^{-1}$ is well-defined for $\mathbb{Q}$.
From $xb = ya$ we deduce that $bx = ay$ then $[(b,a)] = [(y,x)]$ so $[(a,b)]^{-1} = [(b,a)] = [(y,x)] = [(x,y)]^{-1}$ therefore $^{-1}$ is well-defined.
Finally, let us prove that $<$ is well-defined for $\mathbb{Q}$.
Let $[(x,y)] = [(a,b)]$ and $[(z,w)] = [(c,d)]$ then we have that $xb = ya$ and $zd = wc$ also we have that $[(x,y)] < [(z,w)]$ if $y > 0$ and $w > 0$ or $y < 0$ and $w < 0$ then $xw < yz$. If $bd > 0$ because b and d are either both positive or both negatives then we can multiply the inequality by bd to obtain

$$xwbd < yzbd$$

which by the Commutative and Associative Law we have that

$$(xb)(wd) < (zd)(yb)$$

then by replacing the values of xb and zd we get that

$$(ya)(wd) < (wc)(yb)$$

and again by the Commutative and Associative Law we have that

$$(yw)(ad) < (yw)(bc)$$

and since $yw > 0$ then $ad < bc$.
If $bd < 0$ because either $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$ then $-(bd) > 0$. By multiplying both sides of the inequality with this number we get that

$$-(xw)(bd) < -(yz)(bd)$$

which by the Commutative and Associative Law we have that

$$-(xb)(wd) < -(zd)(yb)$$

then by replacing the values of xb and zd we get that

$$-(ya)(wd) < -(wc)(yb)$$

and again by the Commutative and Associative Law we have that

$$-(yw)(ad) < -(yw)(bc)$$

and since $yw > 0$ then $-ad < -bc$ and thus $ad > bc$.

Now if either $y < 0$ and $w > 0$ or $y > 0$ and $w < 0$ we have that $xw > yz$. If $bd < 0$ because either $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$ then $-(bd) > 0$. By multiplying both sides of the inequality with this number we get that

$$-(bd)(xw) > -(bd)(yz)$$

which by the Commutative and Associative Law we have that

$$-(xb)(wd) > -(zd)(yb)$$

then by replacing the values of xb and zd we get that

$$-(ya)(wd) > -(wc)(yb)$$

and again by the Commutative and Associative Law we have that

$$-(yw)(ad) > -(yw)(bc)$$

and since $yw < 0$ then $-(yw) > 0$ and thus $ad > bc$.
Finally, if $bd > 0$ because b and d are either both positive or both negatives then we can multiply the inequality by bd to obtain

$$xwbd > yzbd$$

which by the Commutative and Associative Law we have that

$$(xb)(wd) > (zd)(yb)$$

then by replacing the values of xb and zd we get that

$$(ya)(wd) > (wc)(yb)$$

and again by the Commutative and Associative Law we have that

$$(yw)(ad) > (yw)(bc)$$

and since $yw < 0$ then $-(yw) > 0$ so we have that

$$-(yw)(ad) < -(yw)(bc)$$

and thus $-ad < -bc$ which means that $ad > bc$.
Therefore the $<$ operation is well-defined. $\square$

Proof. **1.5.3**

(1) ($\rightarrow$) If $[(x,y)] = \bar{0} = [(0,1)]$ then by definition $x \cdot 1 = y \cdot 0$ and because of the Identity Law for Multiplication and the part (4) of the Lemma 1.4.5 we have that $x = 0$.
($\leftarrow$) If $x = 0$ then because of the Identity Law for Multiplication and the part (4) of the Lemma 1.4.5 we can write that $x \cdot 1 = y \cdot 0$ where $y \in \mathbb{Z}^*$ then $[(x,y)] = [(0,1)] = \bar{0}$.

(2) ($\rightarrow$) If $[(x,y)] = \bar{1} = [(1,1)]$ then by definition $x \cdot 1 = y \cdot 1$ and because of the Identity Law for Multiplication we have that $x = y$.
($\leftarrow$) If $x = y$ then because of the Identity Law for Multiplication we can write that $x \cdot 1 = y \cdot 1$ and by definition that means $[(x,y)] = [(1,1)] = \bar{1}$.

(3) ($\rightarrow$) If $\bar{0} = [(0,1)] < [(x,y)]$ then by definition if $y > 0$ this is $0 \cdot y < 1 \cdot x$ then because of the Identity Law for Multiplication and the part (4) of the Lemma 1.4.5 we have that $x > 0$ and therefore $xy > 0$ because both are positive numbers.
If $y < 0$ then by definition $[(0,1)] < [(x,y)]$ means that $0 \cdot y > 1 \cdot x$ then because of the Identity Law for Multiplication and the part (4) of the Lemma 1.4.5 we have that $x < 0$ and therefore $xy > 0$ because both are negative numbers.
($\leftarrow$) If $xy > 0$ then either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. In the first case i.e. $x > 0$ and $y > 0$ we have can write the first inequality as $0 \cdot y < 1 \cdot x$ because of the Identity Law for Multiplication and the part (4) of the Lemma 1.4.5 which means that $[(0,1)] < [(x,y)]$.
If $x < 0$ and $y < 0$ again from the first inequality we can write that $0 \cdot y > 1 \cdot x$ because of the Identity Law for Multiplication and the part (4) of the Lemma 1.4.5 which means that $[(0,1)] < [(x,y)]$.

□

Proof. **1.5.4**

(1) From the definition of the + operation we have that

$$\begin{aligned}(r+s)+t &= ([(x,y)] + [(z,w)]) + [(u,v)]\\ &= [(xw+yz, yw)] + [(u,v)]\\ &= [((xw+yz)v + (yw)u, (yw)v)]\end{aligned}$$

Because of the Distributive, Commutative and Associative Law for Addition and Multiplication of $\mathbb{Z}$ we have that

$$\begin{aligned}[((xw+yz)v + (yw)u, (yw)v)] &= [(x(wv) + y(zv+wu), y(wv))]\\ &= [(x,y)] + [(zv+wu, wv)]\\ &= [(x,y)] + ([(z,w)] + [(u,v)])\\ &= r + (s+t)\end{aligned}$$

(2) From the definition of the + operation we have that

$$\begin{aligned} r + s &= [(x, y)] + [(z, w)] \\ &= [(xw + yz, yw)] \end{aligned}$$

Because of the Commutative law for Multiplication and Addition of $\mathbb{Z}$ we have that

$$[(xw + yz, yw)] = [(zy + wx, wy)]$$

then

$$[(zy + wx, wy)] = [(z, w)] + [(x, y)] = s + r$$

(3) From the definition of the + operation and the $\bar{0}$ element we have that

$$\begin{aligned} r + \bar{0} &= [(x, y)] + [(0, 1)] \\ &= [(x \cdot 1 + y \cdot 0, y \cdot 1)] \\ &= [(x, y)] = r \end{aligned}$$

Where we are using the Identity Law for Multiplication and that $z \cdot 0 = 0$ where $z \in \mathbb{Z}$.

(5) From the definition of the $\cdot$ operation we have that

$$\begin{aligned} (rs)t &= ([(x, y)] \cdot [(z, w)]) \cdot [(u, v)] \\ &= [(xz, yw)] \cdot [(u, v)] \\ &= [((xz)u, (yw)v)] \\ &= [(x(zu), y(wv))] \quad \text{Because of the Associative and} \\ &\qquad\qquad\qquad\qquad\quad \text{Commutative Law for Multiplication} \\ &= [(x, y)] \cdot [(zu, wv)] \\ &= [(x, y)] \cdot ([(z, w)] \cdot [(u, v)]) = r(st) \end{aligned}$$

(6) From the definition of the $\cdot$ operation we have that

$$\begin{aligned} rs &= [(x, y)] \cdot [(z, w)] \\ &= [(xz, yw)] \\ &= [(zx, wy)] \quad \text{Because of the Commutative Law} \\ &= [(z, w)] \cdot [(x, y)] = sr \end{aligned}$$

(8) From the definition of $r^{-1} = [(y, x)]$ we have that

$$\begin{aligned} rr^{-1} &= [(x, y)] \cdot [(y, x)] \\ &= [(xy, yx)] \\ &= [(yx, yx)] \quad \text{because of the Commutative Law} \\ &= \bar{1} \qquad \text{because of Problem 1.5.3 part (2)} \end{aligned}$$

(9) From the definition of $+$ and $\cdot$ operations we have that

$$\begin{aligned}
rs + rt &= [(x,y)] \cdot [(z,w)] + [(x,y)] \cdot [u,v] \\
&= [(xz, yw)] + [(xu, yv)] \\
&= [((xz)(yv) + (yw)(xu), (yw)(yv))] \\
&= [(y(xzv + wxu), y(ywv))] \quad \text{because of the Distributive and Associative Law} \\
&= [(y,y)] \cdot [(xzv + wxu, y(wv))] \\
&= \bar{1} \cdot [(xzv + wxu, y(wv))] \quad \text{because of Problem 1.5.3 part (2)} \\
&= [(xzv + wxu, y(wv))] \quad \text{because of the Identity Law} \\
&= [(x(zv + wu), y(wv))] \quad \text{because of the Distributive and Associative Law} \\
&= [(x,y)] \cdot [(zv + wu, wv)] \\
&= [(x,y)] \cdot ([(z,w)] + [(u,v)]) = r(s+t)
\end{aligned}$$

(11) We know that if $x < y$ if and only if $y - x \in P$ so given that $r < s$ and $s < t$ then $s - r, t - s \in P$ let us name $s - r = u$ and $t - s = v$ then $u + v = (s - r) + (t - s) = t - r$ and we know that if $x, y \in P$ then $x + y \in P$ then $u + v = t - r \in P$ and therefore $r < t$.

(12) Since $r < s$ then we know that $s - r \in P$ now if we compute $(s + t) - (r + t) = s - r$ we have that $(s + t) - (r + t) \in P$ and therefore $r + t < s + t$.

(14) Let us suppose that $\bar{0} = \bar{1}$ we want to arrive to a contradiction then $[(0,1)] = [(1,1)]$ which means that $0 \cdot 1 = 1 \cdot 1$ but this means that $0 = 1$ which is a contradiction because what we proved in Theorem 1.3.5 part (14). Therefore $\bar{0} \neq \bar{1}$.

$\square$

Proof. **1.5.5**

(1) We want to prove that the function $i : \mathbb{Z} \to \mathbb{Q}$ is injective, then we proceed as follows, let $x_1, x_2 \in \mathbb{Z}$ then if $i(x_1) = i(x_2)$ we have that $[(x_1, 1)] = [(x_2, 1)]$ so $x_1 \cdot 1 = 1 \cdot x_2$ and therefore $x_1 = x_2$ and the function is injective.

(2) By definition $i(0) = [(0,1)] = \bar{0}$ and $i(1) = [(1,1)] = \bar{1}$

(3) (a) By definition of i function and the $+$ operation we have that

$$\begin{aligned}
i(x) + i(y) &= [(x,1)] + [(y,1)] \\
&= [(x \cdot 1 + 1 \cdot y, 1 \cdot 1)] \\
&= [(x + y, 1)] \quad \text{because of the Identity Law for Multiplication} \\
&= i(x + y)
\end{aligned}$$

(b) By definition of the i function we have that $i(-x) = [(-x, 1)]$ and because of the defintion of the $-$ unary operation we have that $[(-x, 1)] = -[(x, 1)]$ therefore because of the definition of the function i again we get that $-[(x, 1)] = -i(x)$.

(c) By definition of i function and the $\cdot$ operation we have that

$$\begin{aligned} i(x)i(y) &= [(x, 1)] \cdot [(y, 1)] \\ &= [(xy, 1 \cdot 1)] \\ &= [(xy, 1)] \quad \text{Because of the Identity Law} \\ &= i(xy) \end{aligned}$$

(d) $(\rightarrow)$ If $x < y$ then because of the Identity Law for Multiplication we have that $x \cdot 1 < 1 \cdot y$ and because of the definition of the $<$ operation we can write that $[(x, 1)] < [(y, 1)]$ which means that $i(x) < i(y)$.
$(\leftarrow)$ If $i(x) < i(y)$ then by the definition of the i function we have that $[(x, 1)] < [(y, 1)]$. Because of the definition of the $<$ operation and given that $1 > 0$ as we proved earlier in the Lemma 1.4.5 we have that $x \cdot 1 < 1 \cdot y$ and because of the Identity Law for Multiplication we have that $x < y$.

$\square$

Proof. **1.5.6**

(2) If $r < s$ then because of the Addition Law for Order we have that $r + (-r) < s + (-r)$ which means because of the Inverses Law for Addition that $0 < s + (-r)$ then again by the Addition Law for Order we have that $(-s) + 0 < ((-s) + s) + (-r)$ then it follows because of the Identity Law for Addition and the Inverses Law for Addition that $-s < -r$ as we wanted.

(3) Let us suppose that $r \cdot 0 \neq 0$ we want to arrive to a contradiction then by the Trychotomy Law it must hold that $r \cdot 0 > 0$ or $r \cdot 0 < 0$. Let's suppose that $r \cdot 0 > 0$ holds then by the Addition Law for Order we have that $(r \cdot 0) + r > 0 + r$ and because the Identity Law for Multiplication we have that $(r \cdot 0) + (r \cdot 1) > 0 + r$ then by the Identity Law for Addition and the Distributive Law we have that $r \cdot (0 + 1) > r$ again by the Identity Law for Addition we have that $r \cdot 1 > r$ then $r > r$ because of the Identity Law for Multiplication but this cannot be then it must hold that $r \cdot 0 < 0$ but by the same type of arguments we see that this cannot be either. Therefore it must be that $r \cdot 0 = 0$.

(1) We know from the Theorem 1.5.5 (14) that $0 \neq 1$ and by the Trychotomy Law then either $0 < 1$ or $1 < 0$. Let us suppose that $1 < 0$ and because of part (2) of this problem we have that $-0 = 0 < -1$ then we multiply both sides of the inequality by -1 as $1 \cdot (-1) < 0 \cdot (-1)$ because what we proved in part (3) we have that $1 \cdot (-1) < 0$ and finally because of the Identity Law for Multiplication we have that $-1 < 0$ which is a contradiction to what we showed earlier. Therefore it must be that $0 < 1$.

On the other hand, because of what we proved in part (2) of this problem and starting from the fact that $0 < 1$ we have that $-1 < -0 = 0$ as we wanted. It follows then that $-1 < 0 < 1$.

(4) If $r > 0$ and $s > 0$ then

$$\begin{aligned} r + s &> 0 + s && \text{by the Addition Law for Order} \\ &= s && \text{by the Identity Law for Addition} \\ &> 0 && \text{by the Transitive Law} \end{aligned}$$

Also, using the $r > 0$ inequality we have that

$$\begin{aligned} rs &> 0 \cdot s && \text{by Multiplication Law for Order since } s > 0 \\ &= 0 && \text{because Problem (3)} \end{aligned}$$

Therefore $r + s > 0$ and $rs > 0$ as we wanted.

(5) We know that $rr^{-1} = 1$ then $rr^{-1} > 0$ because $1 > 0$ as we proved. It followss then that $r^{-1} \neq 0$ because what we proved in problem (3). Let us suppose now that $r^{-1} < 0$ then since $r > 0$ we can multiply both sides of the inequality to obtain that $rr^{-1} < r \cdot 0 = 0$ where we used the result of problem (3) and we have a contradiction to the fact that $rr^{-1} > 0$. Therefore must be that $r^{-1} > 0$.

(6) Because of the Identity Law for Multiplication we can write $r < s$ as $1 \cdot r < s \cdot 1$. Because of the Transitive Law since $r > 0$ we have that $s > 0$ too. If follows then because of Lemma 1.5.8 (6) that $\frac{1}{s} < \frac{1}{r}$ as we wanted.

(7) Given that $s > 0$ we can multiply both sides of the inequality $r < p$ by s to get $rs < ps$ also because of the Transitive Law $p > 0$ so we can multiply the inequality $s < q$ by p to get $ps < pq$ and therefore by the Transitive Law we have that $rs < pq$ as we wanted.

□

Proof. **1.5.7**

(1) We know that $0 < 1$ from problem 1.5.6 (1) and because of the Addition Law for Order we can add to both sides 1 to get $0 + 1 < 1 + 1$. Let us call 2 the addition $1 + 1$ therefore applying this definition and by the Identity Law for Addition we have that $1 < 2$.

(2) Since $s, t \in \mathbb{Q}$ then we can write them as $s = \frac{a}{b}$ and $t = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$.
Let us compute $\frac{s+t}{2} = (s+t) \cdot \frac{1}{2}$ as

$$\begin{aligned}(s+t) \cdot \frac{1}{2} &= (\frac{a}{b} + \frac{c}{d}) \cdot \frac{1}{2} \\ &= (\frac{ad+bc}{bd}) \cdot \frac{1}{2} \quad \text{because Lemma 1.5.8 (2)} \\ &= \frac{ad+bc}{2 \cdot (bd)} \quad \text{because Lemma 1.5.8 (4)}\end{aligned}$$

Therefore since $ad + bc \in \mathbb{Z}$ and $2 \cdot (bd) \in \mathbb{Z}$ and $\frac{s+t}{2}$ can be written as a fraction then $\frac{s+t}{2} \in \mathbb{Q}$.
Let us now prove that $s < \frac{s+t}{2} < t$ as follows.

$$\begin{aligned} s &< t \\ s+s &< s+t \quad \text{by the Addition Law for Order} \\ s(1+1) &< s+t \quad \text{by Distributive and Identity Law} \\ s \cdot 2 &< s+t \quad \text{by Definition of 2} \\ s \cdot (2 \cdot 2^{-1}) &< (s+t) \cdot 2^{-1} \quad \begin{array}{l}\text{by Multiplication Law for Order} \\ \text{since } 2 > 0 \text{ then } 2^{-1} > 0\end{array} \\ s &< \frac{s+t}{2} \quad \text{by Inverses Law for Multiplication}\end{aligned}$$

On the other hand, we have that

$$\begin{aligned} s &< t \\ s+t &< t+t \quad \text{by the Addition Law for Order} \\ s+t &< t(1+1) \quad \text{by Distributive and Identity Law} \\ s+t &< t \cdot 2 \quad \text{by Definition of 2} \\ (s+t) \cdot 2^{-1} &< t \cdot (2 \cdot 2^{-1}) \quad \begin{array}{l}\text{by Multiplication Law for Order} \\ \text{since } 2 > 0 \text{ then } 2^{-1} > 0\end{array} \\ \frac{s+t}{2} &< t \quad \text{by Inverses Law for Multiplication}\end{aligned}$$

Therefore by joining both results we have that $s < \frac{s+t}{2} < t$.

□

Proof. **1.5.8**

(1) We know that $r = \frac{a}{b} > \frac{0}{1} = 0$ where $a, b \in \mathbb{Z}$ and $b \neq 0$.
Let us suppose that $b > 0$ then because of the Lemma 1.5.8 (6) we have that $a \cdot 1 > b \cdot 0$. It follows then that $a > 0$ because of the Identity Law for Multiplication and the result of Exercise 1.5.6 (3).
Let us now suppose that $b < 0$ then because of the Lemma 1.5.8 (6) we have that $a \cdot 1 < b \cdot 0$. It follows then that $a < 0$ because of the Identity Law for Multiplication and the result of Exercise 1.5.6 (3).

(2) Since $r \in \mathbb{Q}$ then we can write $r = \frac{a}{b}$. Where $a, b \in \mathbb{Z}$. If $a > 0$ and $b > 0$ then we are done. So let $a < 0$ and $b < 0$ then $-a > 0$ and $-b > 0$. We know also that

$$a \cdot (-b) = (-a) \cdot b$$

because of Lemma 1.4.5 (6) and by writing

$$a \cdot (-b) = b \cdot (-a)$$

then this means that $r = \frac{a}{b} = \frac{-a}{-b}$ because of the Lemma 1.5.8 (1). Therefore we have found $m = -a$ and $n = -b$ such that $m > 0$, $n > 0$ and $r = \frac{m}{n}$.

□

Proof. **1.5.9**

(1) Let us define $\mathbb{N} = \{\frac{a}{1} \mid a > 0\}$.
We need to find $n > \frac{s}{r}$ where $n \in \mathbb{N}$ and since $\frac{s}{r} \in \mathbb{Q}$ is of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ then we need to find $n = \frac{n}{1} > \frac{a}{b}$ but this means because of Lemma 1.5.8 (1) that $nb > 1 \cdot a = a$ and because Exercise 1.5.8 (2) we know there is $b > 0$ and $a > 0$ such that $\frac{a}{b} > 0$ then $a, b \in \mathbb{N}$ and therefore $b \geq 1$. So if $b = 1$ then we take $n = a + 1$ and then $(a + 1)b = a + 1 > a$ which is true, and if $b > 1$ then by multiplying both sides by a we have that $ab > a$ so it's enough to select $n = a$.

(2) We need to find m such that $\frac{1}{m} < r$ or $m > \frac{1}{r}$ if $r > 0$ and $r \in \mathbb{Q}$ then we can write r as $r = \frac{a}{b}$ where there is $a > 0$ and $b > 0$ because of Exercise 1.5.8 (2) and then $a, b \in \mathbb{N}$. It follows then that $\frac{1}{r} = \frac{b}{a}$ so in other words we want to find because of Lemma 1.5.8 (1) some m such that $ma > b$. Since $a \in \mathbb{N}$ then $a \geq 1$. If $a = 1$ then by taking $m = b + 1$ we are good to go since $m = b + 1 > b$, and if $a > 1$ then by multiplying both sides by b we have that $ab > b$ so it's enough to select $m = b$.

(3) We want to find some $k \in \mathbb{N}$ such that

$$\left(r + \frac{1}{k}\right)^2 = r^2 + 2\frac{r}{k} + \frac{1}{k^2} < p$$

but since $\frac{1}{k^2} < \frac{1}{k}$ the above inequality is going to be satisfied if the folllowing one is satisfied too

$$r^2 + 2\frac{r}{k} + \frac{1}{k} < p$$

then

$$\begin{aligned}
&\frac{2r}{k} + \frac{1}{k} < p - r^2 \quad \text{by adding to both sides } -r^2 \\
&\frac{2r+1}{k} < p - r^2 \quad \text{by adding both left terms} \\
&2r + 1 < (p - r^2)k \quad \text{by multiplying both sides by } k > 0 \\
&\frac{2r+1}{p-r^2} < k \quad \begin{array}{l}\text{by multiplying both sides by } (p-r^2)^{-1} > 0 \\ \text{since } p > 0,\ r > 0 \text{ and } r^2 < p\end{array}
\end{aligned}$$

Now given that $r, p \in \mathbb{Q}$ we can write them as fractions and because of the Exercise 1.5.8 (2) we know there is $a > 0$, $b > 0$, $c > 0$ and $d > 0$ i.e $a, b, c, d \in \mathbb{N}$ such that $r = \frac{a}{b}$ and $p = \frac{c}{d}$. Also, since $p - r^2 \in \mathbb{Q}$ and $p - r^2 > 0$ we can write it as $p - r^2 = \frac{e}{f}$ such that $e, f \in \mathbb{N}$. Then we want to find k such that

$$\begin{aligned}
\frac{\frac{2a}{b} + 1}{\frac{e}{f}} &< k \\
\frac{\frac{2a+b}{b}}{\frac{e}{f}} &< k \\
\frac{f(2a+b)}{eb} &< k \\
f(2a+b) &< (eb)k \quad \text{because of Lemma 1.5.8 (6)}
\end{aligned}$$

Since $a, b, e, f \in \mathbb{N}$ and $2 \in \mathbb{N}$ then $eb \in \mathbb{N}$ and $f(2a+b) \in \mathbb{N}$ so eb must be $eb \geq 1$. If $eb = 1$ then by taking $k = f(2a+b) + 1$ is enough given that $f(2a+b) < f(2a+b) + 1 = k$ and if $eb > 1$ then by multiplying by $f(2a+b)$ both sides we have that $f(2a+b) < (eb)(f(2a+b))$ then taking $k = f(2a+b)$ is enough.

$\square$

Proof. **1.6.1** Let us prove by contradiction that $B-A$ has an infinite amount of elements.
The set $B-A$ is defined as $B-A = \{b \in B \mid b < a \text{ for all } a \in A\}$ and let us suppose that $B-A = \{b_1, b_2, b_3, ..., b_n\}$ where $n \in \mathbb{N}$ let's take $z = min(\{b_1, b_2, b_3, ..., b_n\})$ by definition $z \in B$ and $z < a$ and since B is a Dedekin cut then there is some $w \in B$ such that $w < z$ and by the Transitive Law we have that $w < a$, but z was the minimum value, therefore this is a contradiction and it follows that $B-A$ has an infinite amount of elements. □

Proof. **1.6.2**

(1) Let us prove that T is a Dedekin cut.

  (a) Given that T is defined as $T = \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\}$ then $0 \notin T$ but $0 \in \mathbb{Q}$ then $T \neq \mathbb{Q}$. From Exercise 1.5.7 part (1) we know that $1 < 2$ then since $2 > 0$ we can multiply both sides of the inequality by 2 therefore $2 < 2 \cdot 2 = 2^2$ and $2 \in \mathbb{Q}$ so we see that $2 \in T$ it follows then that $T \neq \emptyset$.

  (b) Let $t \in T$ and $y \in \mathbb{Q}$ and let us suppose that $y > t$. Given that $t > 0$ we have that $y > 0$ by the Transitivity Law. Let us now multiply the inequality by t so we have that $yt > t^2$ and let us also multiply by y to obtain $y^2 > yt$ therefore by the Transitivity Law we have that $2 < t^2 < y^2$. It follows then that $y \in T$.

  (c) Let $t \in T$ then $t > 0$ and $t^2 > 2$ we want to find some r such that $0 < r < t$ and $2 < r^2 < t^2$ so let us take $r = t - \frac{1}{k}$ where $k \in \mathbb{N}$. We want that $t - \frac{1}{k} > 0$ then $t > \frac{1}{k}$ and we know such a k exists because of Problem 1.5.9 part (2).
  We also want that $2 < (t - \frac{1}{k'})^2$ where $k' \in \mathbb{N}$ might not be equal to k then we want some k' such that $2 < t^2 - \frac{2t}{k'} + \frac{1}{k'^2}$ but this means that finding a k' such that $2 < t^2 - \frac{2t}{k'}$ is also good to go. It follows that

$$\begin{aligned} 2 - t^2 &< -\frac{2t}{k'} \\ \frac{2-t^2}{2t} &< -\frac{1}{k'} \quad \text{because } t > 0 \\ \frac{t^2-2}{2t} &> \frac{1}{k'} \quad \text{multiplying by } -1 \end{aligned}$$

  Given that $t \in \mathbb{Q}$ we notice that $\frac{t^2-2}{2t} \in \mathbb{Q}$ and since $t^2 > 2$ then $\frac{t^2-2}{2t} > 0$. It follows then that because of the Problem 1.5.9 part (2) we know that such $k' \in \mathbb{N}$ exist. Therefore if we take $k'' = max(k, k')$ then $t - \frac{1}{k''} > 0$ and $2 < (t - \frac{1}{k''})^2$ then $t - \frac{1}{k''} \in T$.

(2) Let $y \in D_r$ then $y > r$ for some $r \in \mathbb{Q}$, clearly $r > 0$ hence $y^2 > r^2$ but also $T = D_r$ so $y \in T$ it follows that $y > 0$ and $y^2 > 2$.
If $y > 0$ and $y^2 > 2$ then $y \in T$ and so $y \in D_r$ so $y > r$. So we see that $y > r$ implies that $y^2 > 2$ and $y^2 > 2$ implies that $y > r$.
Let us now assume that $r^2 > 2$ if we take $y = r$ then $r^2 = y^2 > 2$ and from what we saw this implies that $r = y > r$ which is a contradiction.
Now let $r^2 < 2$ then we know there is some $q \in \mathbb{Q}$ such that $r^2 < q^2 < 2$ but this means that $r < q$ which implies by what we saw that $q^2 > 2$ which is a contradiction. Therefore must be that $r^2 = 2$.

□

Proof. **1.6.3** We want to prove that

$$M = \{r \in \mathbb{Q} \mid r = ab \text{ for some } a \in A \text{ and } b \in B\}$$

is a Dedekin cut, where we suppose that $0 \in \mathbb{Q} - A$ and $0 \in \mathbb{Q} - B$, and A and B are Dedekin cuts.
So we proceed to prove the three parts of the Dedekin cuts definition.

(a) We know that $A \neq \emptyset$ and $A \neq \mathbb{Q}$, also $B \neq \emptyset$ and $B \neq \mathbb{Q}$. Let $x \in A$ and $y \in B$ then $xy \in M$ so $M \neq \emptyset$.
We also know that $0 \notin A$ and $0 \notin B$ then $0 \cdot 0 = 0 \notin M$ but $0 \in \mathbb{Q}$ therefore $M \neq \mathbb{Q}$.

(b) Let $t \in M$ and $y \in \mathbb{Q}$, and suppose $y \geq t$, we know that $t = ab$ for some $a \in A$ and $b \in B$. Then we can write $y = \frac{yb}{b}$ and because $y \geq t$ then $\frac{y}{b} \geq a$ so $\frac{y}{b} \in A$ because A is a Dedekin cut. Therefore $y = (\frac{y}{b}) \cdot b \in M$.

(c) Let $t \in M$ then $t = ab$ for some $a \in A$ and $b \in B$. Given that $0 \in \mathbb{Q} - A$ and $0 \in \mathbb{Q} - B$ then $0 \notin A$ and $0 \notin B$ and therefore $a > 0$ and $b > 0$. It follows from the definition of the Dedekin cuts that there is some $p < a$ and $q < b$ such that $p \in A$ and $q \in B$ so $pq \in M$ and because $0 < p < a$ and $0 < q < b$ we have that $pq < ab$.

Therefore M is a Dedekin cut. □

Proof. **1.6.4**

(1) ($\rightarrow$) If $A \subsetneqq D_r = \{x \in \mathbb{Q} \mid x > r\}$ then there is some $q \in D_r$ such that $q \notin A$ then $q > r$ and $q \in \mathbb{Q}$ by definition of D_r. It follows that $q \in \mathbb{Q} - A$.
($\leftarrow$) If $q \in \mathbb{Q} - A$ and $q > r$ then $q \notin A$ but $q \in \mathbb{Q}$ so by definition $q \in D_r$. It follows that $q \in D_r - A$ but this means that $A \subsetneqq D_r$.

(2) Let's prove first that $A \subseteq D_r$ if and only if $r \in \mathbb{Q} - A$.
($\rightarrow$) Let $A \subseteq D_r$ as we know $r \notin D_r$ so $r \notin A$ but $r \in \mathbb{Q}$ by definition of D_r. It follows that $r \in \mathbb{Q} - A$.
($\leftarrow$) Let $r \in \mathbb{Q} - A$ then $r \notin A$. Since A is a Dedekin cut then there is no $x \in A$ such that $x < r$. So if $x \in A$ then $x > r$ but then $x \in D_r$ and since x was arbitrary therefore $A \subseteq D_r$.
Now let us prove that $r \in \mathbb{Q} - A$ if and only if $r < a$ for all $a \in A$.
($\rightarrow$) If $r \in \mathbb{Q} - A$ then $r \notin A$ and since A is a Dedekin cut, all $a \in A$ must be $r < a$ otherwise we have a contradiction to the fact that A is a Dedekin cut.
($\leftarrow$) If $r < a$ for all $a \in A$ then by definition $r \notin A$ but $r \in \mathbb{Q}$ by definition so this means that $r \in \mathbb{Q} - A$. $\square$

Proof. **1.7.1**

(1) Let $x \in D_{-r}$ then $x \in \mathbb{Q}$ and $-r < x$ so $-x < r$. We know that $-D_r = \{x \in \mathbb{Q} \mid -x < c \text{ for some } c \in \mathbb{Q} - D_r\}$. Also given that $\mathbb{Q} - D_r$ is defined as $\mathbb{Q} - D_r = \{y \in \mathbb{Q} \mid y < x \text{ for all } x \in D_r\}$ and by definition of D_r, $r < x$ where $x \in D_r$ we have that $r \in \mathbb{Q} - D_r$ then $x \in -D_r$. Therefore $D_{-r} \subseteq -D_r$.
Let $x \in -D_r$ then $x \in \mathbb{Q}$ and $-x < c$ for some $c \in \mathbb{Q} - D_r$. As we saw $r \in \mathbb{Q} - D_r$ and by definition of D_r we have that $x > r$ so $-x < -r < r$ but then $x > -r$ so $x \in D_{-r}$ and therefore $-D_r \subseteq D_{-r}$.
It follows that $D_{-r} = -D_r$.

(2) Let $r > 0$ and $D_{r^{-1}} = \{x \in \mathbb{Q} \mid x > \frac{1}{r}\}$ then $[D_r]^{-1}$ is defined as $[D_r]^{-1} = \{x \in \mathbb{Q} \mid x > 0 \text{ and } \frac{1}{x} < c \text{ for some } c \in \mathbb{Q} - D_r\}$

Let $x \in D_{r^{-1}}$ then from the definition we have that $r > \frac{1}{x}$ which is possible since $x > \frac{1}{r} > 0$ and we know that $r \in \mathbb{Q} - D_r$ then $x \in [D_r]^{-1}$ which means that $D_{r^{-1}} \subseteq [D_r]^{-1}$.

Let $x \in [D_r]^{-1}$ then there is some $c \in \mathbb{Q} - D_r$ such that $\frac{1}{x} < c$ and we know $r \in \mathbb{Q} - D_r$ so if we take $c = r$ then $\frac{1}{x} < r$ so $\frac{1}{r} < x$ it follows then that $x \in D_{r^{-1}}$ which means that $[D_r]^{-1} \subseteq D_{r^{-1}}$. Therefore when $r > 0$ we have that $D_{r^{-1}} = [D_r]^{-1}$

Let now $r < 0$ then $D_{r^{-1}} = D_{-(-r)^{-1}}$ and because of part 1 of this Problem we have that $D_{-(-r)^{-1}} = -D_{(-r)^{-1}}$ and now because $-r > 0$ we have that $-D_{(-r)^{-1}} = -[D_{-r}]^{-1} = -[-D_r]^{-1} = [D_r]^{-1}$.

$\square$

Proof. **1.7.2**

(1) Given that $A > D_0$ and $B > D_0$ then AB is defined as

$$AB = \{r \in \mathbb{Q} \mid r = ab \text{ for some } a \in A \text{ and } b \in B\}$$

given that $a > 0$ and $b > 0$ then $r = ab > 0$ so $AB \subsetneqq D_0$ which by definition means that $AB > D_0$.

(2) Given that $A > D_0$ then A^{-1} is defined as

$$A^{-1} = \{r \in \mathbb{Q} \mid r > 0 and \frac{1}{r} < c \text{ for some } c \in \mathbb{Q} - A\}$$

then by definition $r > 0$ which means that $A^{-1} \subsetneqq D_0$ it follows that $A^{-1} > D_0$.

□

Proof. **1.7.7**

(1) Let $i(r_1) = i(r_2)$ then $D_{r_1} = D_{r_2}$ let $x \in D_{r_1}$ then $x \in D_{r_2}$ so $x > r_1$ and $x > r_2$ then if $r_1 > r_2$ if follows that exists a $r_1 > \frac{r_1+r_2}{2} > r_2$ but $\frac{r_1+r_2}{2} \in D_{r_2}$ and $\frac{r_1+r_2}{2} \notin D_{r_1}$ but we know that $D_{r_1} = D_{r_2}$ therefore we have a contradiction. With the same type of arguments we can show that $r_2 > r_1$ cannot be either. So must be $r_1 = r_2$.

(2) By definition $i(0) = D_0$ and $i(1) = D_1$.

(3) (a) Let $t \in D_{r+s}$ so $t > r + s$. Since D_{r+s} is a Dedekin cut then we know there is $a \in D_{r+s}$ such that $t > a > r + s$ then $a - s > r$ so $a - s \in D_r$. Also, we can write that $t = (a - s) + (t - a + s)$, and since $t > a$ then $t - a > 0$ so $t - a + s > s$ which means that $t - a + s \in D_s$. Therefore we see that $t \in D_r + D_s$.

Let $t \in D_r + D_s$ then $t = x + y$ where $x \in D_r$ and $y \in D_s$. Then $x > r$ and $y > s$ and by adding both inequalities we have that $x + y > r + s$ which means that $x + y \in D_{r+s}$. It follows that $D_r + D_s \subseteq D_{r+s}$.

Therefore must be that $D_{r+s} = D_r + D_s$ which means that
$i(r + s) = i(r) + i(s)$.

(b) By definition $i(-r) = D_{-r}$ and from what we proved in Exercise 1.7.1 we have that $D_{-r} = -D_r = -i(r)$.

Therefore $i(-r) = -i(r)$.

(c) Let $r > 0$ and $s > 0$.

If $t \in D_{rs}$ then $t > rs$. Since D_{rs} is a Dedekin cut then we know there is $a \in D_{rs}$ such that $t > a > rs$. Also, we can write $t = \frac{a}{s} \cdot \frac{ts}{a}$ and from the inequality we have that $\frac{a}{s} > r$ so $\frac{a}{s} \in D_r$. On the other hand $t > a$ so $\frac{ts}{a} > s$ which means that $\frac{ts}{a} \in D_s$. Therefore $t \in D_r D_s$ so $D_{rs} \subseteq D_r D_s$.

Let now $t \in D_r D_s$ so $t = xy$ for some $x \in D_r$ and $y \in D_s$. Also, we know that $x > r > 0$ and $y > s > 0$ then by multiplying the inequalities we have that $t = xy > rs$ it follows that $t \in D_{rs}$ and that $D_r D_s \subseteq D_{rs}$. Therefore $D_{rs} = D_r D_s$.

Now let $r > 0$ and $s < 0$ which means that $D_r \geq D_0$ and $D_s < D_0$. Then

$$D_{rs} = D_{r(-(-s))} = -D_{r(-s)} = -[D_r D_{-s}] = -[D_r(-D_s)] = D_r D_s$$

where we used first the Exercise 1.7.1 part (1) then the result we got for the case where $r > 0$ and $s > 0$ and last the definition of multiplication of Dedekin cuts.

In the case of $r < 0$ and $s > 0$ we have in the same way that

$$D_{rs} = D_{(-(-r))s} = -D_{(-r)s} = -[D_{-r} D_s] = -[(-D_r) D_s] = D_r D_s$$

And finally in the case of $r < 0$ and $s < 0$ we have that

$$\begin{aligned} D_{rs} &= D_{(-(-r))(-(-s))} = D_{(-r)(-s)} = D_{-r} D_{-s} = [(-D_r)(-D_s)] \\ &= D_r D_s \end{aligned}$$

Therefore $D_{rs} = i(rs) = i(r)i(s) = D_r D_s$.

(d) By definition $i(r^{-1}) = D_{r^{-1}}$ and by what we proved in Exercise 1.7.1 part (2) we have that $D_{r^{-1}} = [D_r]^{-1}$.

Therefore $i(r^{-1}) = D_{r^{-1}} = [D_r]^{-1} = [i(r)]^{-1}$.

(e) $(\rightarrow)$ Let $x \in D_s$ then by definition $s < x$ but we know that $r < s < x$ so $x \in D_r$ which means that $D_s \subsetneqq D_r$ then $D_r < D_s$ and $i(r) < i(s)$.

$(\leftarrow)$ If $i(r) < i(s)$ then $D_r < D_s$ and $D_s \subsetneqq D_r$. We will show by contradiction that $r < s$. Suppose that $s < r$ then $r \in D_s$ but then $D_r \subsetneqq D_s$ so $D_s < D_r$, but we know that $D_r < D_s$ so we have a contradiction.

Now suppose that $r = s$ then $D_r = D_s$ but know that $D_r < D_s$ so we have another contradiction.

Therefore by Trichotomy must be the case that $r < s$.

□