Solved selected problems of Real Analysis - Carothers

Franco Zacco

Chapter 1 - Calculus Review

Proof. **1** Let us take the set $-A = \{-a \mid a \in A\}$ since A is bounded below then $-A$ is bounded above and because of the The Least Upper Bound Axiom we know there is $u = \sup(-A)$ such that if $x \in -A$ then $x \leq \sup(-A)$ so $-x \geq -\sup(-A)$ and $-x \in A$ then $-\sup(-A)$ is an lower bound of A.

Now let us suppose that l is a Lower Bound of A such that $-x \geq l > -\sup(-A)$ we want to arrive to a contradiction to show there is no such an l. Then $x \leq -l < \sup(-A)$ but $\sup(-A)$ is the Least Upper Bound of $-A$ and this cannot be so we have a contradiction and must be the case that if l is a Lower Bound then $l \leq -\sup(-A)$.

Therefore $-\sup(-A)$ is the Greatest Lower Bound of A. $\square$

Proof. **3** Supremum characterization.
$(\rightarrow)$

(i) If $s = \sup(A)$ then s is the Least Upper Bound for A so by definition s is an upper bound for A.

(ii) Let $\epsilon > 0$. Since s is the Least Upper Bound of A, then $s - \epsilon < s$ and $s - \epsilon$ cannot be an upper bound of A thus there exists $a \in A$ such that $s \geq a > s - \epsilon$.

$(\leftarrow)$ Now we want to show by contradiction that s is the Least Upper Bound for A. Suppose $u \neq s$ is the Least Upper Bound for A so $u = \sup(A)$ which means that if $a \in A$ then $a \leq u$ and since s is an upper bound for A then $u < s$. We also have that $a > s - \epsilon$ for every $\epsilon > 0$ so let us take $\epsilon = s - u$ then we have that $a > s - (s - u) = u$ but we said that $a \leq u$ which means that we have a contradiction. Therefore s must be the Least Upper Bound for A i.e. $s = \sup(A)$.

Infimum characterization. Let A be a nonempty set of $\mathbb{R}$ that is bounded below. We want to prove that $i = \inf(A)$ if and only if (i) i is a lower bound for A, and (ii) for every $\epsilon > 0$ there is an $a \in A$ such that $a < i + \epsilon$.
$(\rightarrow)$

(i) If $i = \inf(A)$ then i is the Greatest Lower Bound for A so by definition i is a lower bound for A.

(ii) Let $\epsilon > 0$. Since i is the Greatest Lower Bound for A, then $i + \epsilon > i$ and $i + \epsilon$ cannot be a lower bound of A thus there exists $a \in A$ such that $i \leq a < i + \epsilon$.

$(\leftarrow)$ Now we want to show by contradiction that i is the Greatest Lower Bound for A. Suppose $l \neq i$ is the Greatest Lower Bound for A so $l = \inf(A)$ which means that if $a \in A$ then $a \geq l$ and since i is an lower bound for A then $i < l$. We also have that $a < i + \epsilon$ for every $\epsilon > 0$ so let us take $\epsilon = l - i$ then we have that $a < i + (l - i) = l$ but we said that $a \geq l$ which means that we have a contradiction. Therefore i must be the Least Upper Bound for A i.e. $i = \inf(A)$. □

Proof. **6** Let the sequence (a_n) to be convergent to $a \in \mathbb{R}$, so for every positive $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Also let us notice that

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < \epsilon + |a|$$

so in summary $|a_n| < |a| + \epsilon$. Let us take then

$$M = max\{|a_1|, |a_2|, ..., |a_n|, |a| + \epsilon\}$$

so we see that $|a_n| < M$ and therefore (a_n) is bounded.

Given that (a_n) is bounded below and above then because of The Least Upper Bound Axiom and The Greatest Lower Bound Axiom we know that (a_n) has a Supremum and an Infimum.

Now we want to show by contradiction that $a \leq \sup(a_n)$. Let us suppose that $\sup(a_n) < a$ then if we take $\epsilon = a - \sup(a_n) > 0$ we have that there must be some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < a - \sup(a_n)$ so this means that $-a + \sup(a_n) < a_n - a < a - \sup(a_n)$ then $\sup(a_n) < a_n < 2a - \sup(a_n)$ but $\sup(a_n)$ is the supremum of a_n so we have a contradiction. Therefore must be the case that $a \leq \sup(a_n)$.

In the same way we want to show by contradiction that $\inf(a_n) \leq a$. Let us suppose now that $\inf(a_n) > a$ then if we take $\epsilon = \inf(a_n) - a > 0$ we have that there must be some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < \inf(a_n) - a$ so this means that $a - \inf(a_n) < a_n - a < \inf(a_n) - a$ then $2a - \inf(a_n) < a_n < \inf(a_n)$ but $\inf(a_n)$ is the Infimum of a_n so we have a contradiction. Therefore must be the case that $\inf(a_n) \leq a$. $\square$

Proof. **7** Since $b - a > 0$ we can apply Lemma 1.2 to get a positive integer q' such that $q'(b - a) > 1$ we also know that $\sqrt{2} > 1$ so $\sqrt{2}q'(b - a) > 1$ then we see that $\sqrt{2}q'b$ is bigger than $\sqrt{2}q'a$ by a value bigger than 1 so this means that there is some $p \in \mathbb{Z}$ between them, thus $\sqrt{2}q'b > p > \sqrt{2}q'a$ then it follows that $a < \sqrt{2}p/q < b$ where we used that $2q' = q$ and that $\sqrt{2} \cdot \sqrt{2} = 2$. Therefore there is some irrational number of the form $\sqrt{2}p/q$ between a and b. $\square$

Proof. **13**

($\rightarrow$) We know that (s_n) converges so let $\epsilon > 0$ it follows then that there is some $s \in \mathbb{R}$ such that when $n \geq N$ then $|s_n - s| < \epsilon$ which means that $|s_n| < |s| + \epsilon$. Let us take then $M = \max\{|s_1|, |s_2|, ..., |s_n|, |s| + \epsilon\}$, so we see that $|s_n| < M$ and since $a_n \geq 0$ then $s_n = \sum_{i=1}^{n} a_i \geq 0$ which means that $s_n < M$. Therefore (s_n) is bounded.

($\leftarrow$) Since we know now that (s_n) is bounded we want to prove by induction that it's a monotone (increasing) sequence. First we see that $a_1 \geq 0$ and $a_2 \geq 0$ then $s_1 = a_1 \leq a_1 + a_2 = s_2$.
Now let us suppose that the following expression is true

$$s_{n-1} = \sum_{i=1}^{n-1} a_i \leq \sum_{i=1}^{n} a_i = s_n$$

then since $a_{n+1} \geq 0$ we have that

$$s_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a_i + a_{n+1} = s_{n+1}$$

Therefore we showed that (s_n) is bounded and monotone it follows then that it is convergent. $\square$

Proof. **22** Let us prove first by contradiction that $\inf_n a_n \leq \liminf_{n\rightarrow\infty} a_n$. Suppose $\inf_n a_n > \liminf_{n\rightarrow\infty} a_n = \sup t_n$ then $\inf_n a_n > \sup t_n \geq t_n$ but we know that $\inf_n a_n \leq t_n$ so we have a contradiction then it must be the case that $\inf_n a_n \leq \liminf_{n\rightarrow\infty} a_n$.

Now let us prove by contradiction that $\limsup_{n\rightarrow\infty} a_n \leq \sup_n a_n$. Suppose $\inf T_n = \limsup_{n\rightarrow\infty} > \sup_n a_n$ then $T_n \geq \inf T_n > \sup_n a_n$ but we know that $T_n \leq \sup_n a_n$ so we have a contradiction then it must be the case that $\limsup_{n\rightarrow\infty} a_n \leq \sup_n a_n$.

Finally we want to prove that

$$\sup t_n = \lim \inf_{n\rightarrow\infty} a_n \leq \lim \sup_{n\rightarrow\infty} a_n = \inf T_n$$

We know that $t_n \leq T_n$ so if we take limits in both sides we have that $\lim_{n\rightarrow\infty} t_n \leq \lim_{n\rightarrow\infty} T_n$ and since a_n is bounded then we have that

$$\lim \inf_{n\rightarrow\infty} a_n = \lim_{n\rightarrow\infty} t_n \leq \lim_{n\rightarrow\infty} T_n = \lim \sup_{n\rightarrow\infty} a_n$$

as we wanted.

Therefore joining the results we have that

$$\inf_n a_n \leq \lim \inf_{n\rightarrow\infty} a_n \leq \lim \sup_{n\rightarrow\infty} a_n \leq \sup_n a_n$$

$\square$

Proof. **23** We know that (a_n) converges to some $a \in \mathbb{R}$ so let $\epsilon > 0$ then $|a_n - a| < \epsilon$ when $n \geq N$ then we have that

$$a - \epsilon < a_n < a + \epsilon$$

but this also means that

$$a - \epsilon \leq t_n \leq T_n \leq a + \epsilon$$

and therefore their limits should be between that interval too, then

$$a - \epsilon \leq \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n \leq a + \epsilon$$

Therefore this means that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ both converge to a or what it's the same

$$\lim \inf_{n\to\infty} a_n = \lim \sup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

□

Proof. **24** We know that $\limsup\limits_{n\to\infty} a_n = \inf\{\sup\{a_n, a_{n+1}, ...\}\}$ so this means that

$$\limsup_{n\to\infty} -a_n = \inf\{\sup\{-a_n, -a_{n+1}, ...\}\}$$

But since $\sup -A = -\inf A$ we have that

$$\limsup_{n\to\infty} -a_n = \inf\{-\inf\{a_n, a_{n+1}, ...\}\}$$

We also know that $\inf -A = -\sup A$ therefore

$$\limsup_{n\to\infty} -a_n = -\sup\{\inf\{a_n, a_{n+1}, ...\}\}$$

It follows then by definition that

$$\limsup_{n\to\infty} -a_n = -\liminf_{n\to\infty} a_n$$

□

Proof. **25** TODO

We know that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k = -\infty$ so this means that if we have an $M < 0$ then we can find an $N \in \mathbb{N}$ such that if $k \geq N$ then $\sup_{k\geq N} a_k < M$ and in particular $a_k < M$ for any $k \geq N$. Therefore $\lim_{n\to\infty} a_n = -\infty$.

Now we have that $\limsup_{n\to\infty} a_n = +\infty$ so this means that if we have an $M > 0$ then we can find an $N \in \mathbb{N}$ such that if $k \geq N$ then $\sup_{k\geq N} a_k > M$. Let us now take a subsequence b_n such that $b_k = a_k$ if $a_k > a_{k-1}$ but if $a_k \leq a_{k-1}$ then we take $b_k = a_{k-1}$ then b_k is an increasing subsequence which is not bounded, therefore it must diverge to $+\infty$.

For $\liminf_{n\to\infty} a_n = \pm\infty$ the procedure is analogous. □