Solved selected problems of Real Analysis - Carothers

Franco Zacco

Chapter 4 - Open Sets and Closed Sets

Proof. **1** Let $U = (a, b) \times (c, d)$ and let $(x, y) \in U$ we want to check that if $(x', y') \in B_\epsilon((x, y))$ then $(x', y') \in U$. Since $(x', y') \in B_\epsilon((x, y))$ we have that

$$d_\infty((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\} < \epsilon$$

Then if $\max\{d(x, x'), d(y, y')\} = d(x, x')$ this means that $d(y, y') \leq d(x, x') < \epsilon$. Then $x' \in B_\epsilon(x)$ and since (a, b) is an open set in $\mathbb{R}$ this means that $x' \in (a, b)$, the same can be shown for y' such that $y' \in (c, d)$. Therefore $(x', y') \in U$.

Generalizing, let $U = A \times B$ and let $(a, b) \in U$ we want to check that if $(a', b') \in B_\epsilon((a, b))$ then $(a', b') \in U$. So in the same way since $(a', b') \in B_\epsilon((a, b))$ we have that

$$d_\infty((a, b), (a', b')) = \max\{d(a, a'), d(b, b')\} < \epsilon$$

Then if $\max\{d(a, a'), d(b, b')\} = d(a, a')$ this means that $d(b, b') \leq d(a, a') < \epsilon$. Then $a' \in B_\epsilon(a)$ and since A is an open set in $\mathbb{R}$ this means that $a' \in A$, the same can be shown for b' such that $b' \in B$. Therefore $(a', b') \in U$.

Let now $U = A \times B$ where A and B are closed sets in $\mathbb{R}$, we want to prove that U is also closed in $\mathbb{R}^2$. We see that $(\mathbb{R} \setminus A) \times \mathbb{R}$ and $\mathbb{R} \times (\mathbb{R} \setminus B)$ are open sets because $\mathbb{R} \setminus A$, $\mathbb{R} \setminus B$ and $\mathbb{R}$ are open sets. Also, we know that the union of open sets is also an open set so $\mathbb{R} \times \mathbb{R} \setminus A \times B$ is also an open set which means that $A \times B$ must be a closed set. $\square$

Proof. **3**

($\rightarrow$) Let $x \in U$ where U is an open set of (M, d) and let (x_n) be a sequence that converges to x since it is an open set we know that $x_n \in U$ for all but finitely many n, i.e. there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(x_n, x) < \epsilon$ for some $\epsilon > 0$ which means that $d(x_n, x) \rightarrow 0$ but since the metric d and ρ are equivalent then if $d(x_n, x) \rightarrow 0$ we have that $\rho(x_n, x) \rightarrow 0$. Therefore either ρ or d generate the same set U.

($\leftarrow$) Let U be an open set that is generated either by d and by ρ then if $x \in U$ and we have a sequence (x_n) that converge to x we know that $x_n \in U$ for all but finitely many n i.e. there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(x_n, x) < \epsilon$ and $\rho(x_n, x) < \epsilon$ because they both generate U this means that $d(x_n, x) \rightarrow 0$ and that $\rho(x_n, x) \rightarrow 0$ which implies that they are equivalent.

□

Proof. **6** An example of an infinite closed set in $\mathbb{R}$ containing only irrationals is the set of all the square roots of the prime numbers, i.e.

$$F = \{\sqrt{2}, \sqrt{3}, \sqrt{5}, ...\sqrt{p_n}, \sqrt{p_{n+1}}, ...\}$$

So the complement of this set is the set

$$F^c = (-\infty, \sqrt{2}) \ \cup \ (\sqrt{2}, \sqrt{3}) \ \cup \ ... \ \cup \ (\sqrt{p_n}, \sqrt{p_{n+1}}) \ \cup \ ...$$

Where the intervals are open intervals of $\mathbb{R}$ and they are open sets plus the union of open sets is open, therefore F^c is open and F by definition is closed.

Let us suppose that we have a set $F \subset \mathbb{R}$ that is an open set consisting entirely of irrationals we want to arrive at a contradiction. Let us grab an element $x \in F$ where by definition is irrational, then the ball around x is defined as $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$, but we know that $\mathbb{Q}$ is a dense set in $\mathbb{R}$ so there is an element $q \in \mathbb{Q}$ such that $q \in (x - \epsilon, x + \epsilon)$ so we have a contradiction and $B_\epsilon(x) \not\subset F$. Therefore there is no open set consisting entirely of irrationals. □

Proof. **7** Let F be an open set in $\mathbb{R}$ then for each $x \in F$ we have that there is $B_\epsilon(x) = (x-\epsilon, x+\epsilon)$ where $B_\epsilon(x) \subset F$. Also, we know that $\mathbb{Q}$ is dense in $\mathbb{R}$ so there is $a_x, b_x \in \mathbb{Q}$ such that $a_x \in (x-\epsilon, x)$ and $b_x \in (x, x+\epsilon)$ so we have that $x \in (a_x, b_x)$ then we can write that

$$F = \bigcup_{x \in F} (a_x, b_x)$$

Finally, we see that each interval is in $\mathbb{Q} \times \mathbb{Q}$ and we know that $\mathbb{Q} \times \mathbb{Q}$ is equivalent to $\mathbb{N}$, therefore since the intervals involved are a subset of $\mathbb{Q} \times \mathbb{Q}$ they are countable.

From what we proved we see that each open set F is a countable union of intervals with rational endpoints, this suggests an injective function that sends an open set F to $F \cap \mathbb{Q}$ where $F \cap \mathbb{Q} \in \mathcal{P}(\mathbb{Q})$ so we can construct an injective map

$$f : \mathcal{U} \to \mathcal{P}(\mathbb{Q})$$

Also notice that $\mathcal{P}(\mathbb{Q})$ is equivalent to $\mathbb{R}$, so we can construct an injective map that sends $x \in \mathbb{R}$ to $(-\infty, x) \in \mathcal{U}$ i.e. we have a map $g : \mathcal{P}(\mathbb{Q}) \to \mathcal{U}$, therefore using the Bernstein's Theorem we get that there is a bijective map $h : \mathcal{U} \to \mathcal{P}(\mathbb{Q})$ implying that

$$\text{card}(\mathcal{U}) = \text{card}(\mathbb{R})$$

□

Proof. **11** Let (x_n) be a sequence of sequences from $E = \{e^{(k)} : k \geq 1\}$ then $d(x_n, x_m) = 2$ if $x_n \neq x_m$ and $d(x_n, x_m) = 0$ if $x_n = x_m$. This means that (x_n) converges to some $x \in l_1$ if eventually $x = x_n$ but then $x \in E$. Therefore this implies that E is a closed set of l_1 □

Proof. **13** Let $(x^{(n)})$ be a sequence of sequences from c_0 that converge to $x \in l_\infty$ we want to prove that also $x \in c_0$. Since $(x^{(n)})$ converges to $x \in l_\infty$ then for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\|x^{(n)} - x\|_\infty < \epsilon$. Then we have that

$$\|x^{(n)} - x\|_\infty = \sup_{k \in \mathbb{N}} |x_k^{(n)} - x_k| < \epsilon$$

So we get that

$$\begin{aligned}
|x_k| &= |x_k - x_k^{(n)} + x_k^{(n)}| \\
&\leq |x_k - x_k^{(n)}| + |x_k^{(n)}| \\
&\leq \sup_{k \in \mathbb{N}} |x_k^{(n)} - x_k| + |x_k^{(n)}| \\
&< \epsilon + |x_k^{(n)}|
\end{aligned}$$

And since $x^{(n)} \in c_0$ then $|x_k^{(n)}| \to 0$ when $k \to \infty$. Therefore $|x_k| < \epsilon$ implying that $|x_k| \to 0$ and that $x \in c_0$. □

Proof. **15** Let $A = \{y \in M : d(x, y) \leq r\}$ be the closed ball around x, we want to show that $M \setminus A$ is an open set which implies that A is a closed set. If $M \setminus A$ is an open set then for every $z \in M \setminus A$ there is an open ball $B_t(z)$ such that $B_t(z) \subset M \setminus A$.

We have that $d(z, x) > r$ which implies that $d(z, x) - r > 0$ so let us define $t = d(z, x) - r$ then we have found $t > 0$ such that $B_t(z) \subset M \setminus A$ as we wanted. Therefore $B_t(z)$ is an open ball and $M \setminus A$ is an open set, which implies that A is a closed set.

Now let's see that A is not necessarily equal to the closure of the open ball $B_r(x)$. Let's define a metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Then the open ball $B_1(x)$ with this metric is given by

$$B_1(x) = \{y \in M : d(x, y) < 1\} = \{x\}$$

So now we claim that $\mathrm{cl}(B_1(x)) = \{x\}$ we see this is true because let $y \in M$ such that $y \neq x$ so with this metric $d(x, y) = 1$ then there is an open ball $B_{1/2}(y) \subset M \setminus B_1(x)$ implying that $M \setminus B_1(x)$ is open and $\{x\}$ is closed. Which is different from the closed ball

$$A = \{y \in M : d(x, y) \leq 1\} = M$$

□

Proof. **16** Let $A = \{x \in V : \|x\| < 1\}$ and $B = \{x \in V : \|x\| \leq 1\}$. We know $x \in \bar{A}$ if there is a sequence $(x_n) \subset A$ such that $x_n \to x$. Then suppose $x \in V$ such that $\|x\| = 1$ and let us define a sequence (x_n) that converge to x as

$$x_n = \frac{n-1}{n} x$$

We see that $\|\frac{n-1}{n} x\| = |\frac{n-1}{n}| \|x\| = |\frac{n-1}{n}| \cdot 1 < 1$ then $(x_n) \subset A$ and this implies that $x \in \bar{A}$. Therefore B is always the closure of A. □

Proof. **17**

- ($\rightarrow$) If A is an open set, since $\mathring{A}$ is the largest open set contained in A then $\mathring{A} = A$.
- ($\leftarrow$) If $\mathring{A} = A$ since $\mathring{A}$ is an open set then A is open.
- ($\rightarrow$) If A is closed, since $\bar{A}$ is the smallest closed set that contains A then $\bar{A} = A$.
- ($\leftarrow$) If $\bar{A} = A$ since $\bar{A}$ is a closed set then A is closed.

□

Proof. **18** Since E is a nonempty bounded subset of $\mathbb{R}$ then there is a non-decreasing sequence $(x_n) \subset E$ where $\lim_{n\to\infty} x_n = \sup E$ therefore $\sup E \in \bar{E}$. In the same way, we know there is a non-increasing sequence $(x_n) \subset E$ where $\lim_{n\to\infty} x_n = \inf E$ therefore $\inf E \in \bar{E}$. $\square$

Proof. **20** Since $A \subset B$ and $B \subset \bar{B}$ then $A \subset \bar{B}$. Now let

$$C = \{F : F \text{ is closed set and } A \subset F\}$$

We know that

$$\bar{A} = \bigcap \{F : F \text{ is closed set and } A \subset F\} = \bigcap C$$

Then this means that $\bar{B} \in C$. Therefore since $\bar{A}$ is the intersection of C we see that $\bar{A} \subset \bar{B}$.

Let us now see why $\bar{A} \subset \bar{B}$ does not imply that $A \subset B$ by checking the following example. Let us define $A = (0, 1]$ and $B = \mathbb{Q}$ then $\bar{A} = [0, 1]$ and $\bar{B} = \mathbb{R}$ so we see that $\bar{A} \subset \bar{B}$ but $A \not\subset B$. $\square$

Proof. **24** Let $A \subset M$ so $A^c = M \setminus A$. Let us also define

$$U = \bigcup \{F : F \text{ is open and } F \subset M \setminus A\} = \text{int}(A^c)$$

So by definition, U is an open set then $U^c = M \setminus U$ is closed and $A \subset U^c$ because of the definition of U also we see that U^c must be the smallest closed set containing A again because of how we defined U. Therefore

$$\text{cl}(A) = (\text{int}(A^c))^c$$

Let us now define

$$I = \bigcap \{F : F \text{ is closed and } M \setminus A \subset F\} = \text{cl}(A^c)$$

So we see that I is a closed set then $I^c = M \setminus I$ is open and $I^c \subseteq A$ because of the definition of I. Also, we see that I^c must be the largest open set contained in A because of how we defined I. Therefore

$$\text{int}(A) = (\text{cl}(A^c))^c$$

$\square$

Proof. **26**

($\rightarrow$) Let $d(x, A) = 0$ then this means that $\inf\{d(x, a) : a \in A\} = 0$ for which we have two cases. If $x \in A$ then we have that

$$\min\{d(x, a) : a \in A\} = \inf\{d(x, a) : a \in A\} = d(x, x) = 0$$

and we have that $x \in \bar{A}$ since $A \subset \bar{A}$.

If $x \notin A$ and we know that $\inf\{d(x, a) : a \in A\} = 0$ then it is possible to form a sequence $(x_n) \subset A$ such that $x_n \rightarrow x$ i.e. $d(x_n, x) \rightarrow 0$ which implie that $x \in \bar{A}$.

($\leftarrow$) If $x \in \bar{A}$ then there is a sequence $(x_n) \subset A$ such that $x_n \rightarrow x$ which implies that $d(x, x_n) \rightarrow 0$ and since by definition of the metrics $d(x, a) \geq 0$ for any $a \in A$ then $\inf\{d(x, a) : a \in A\} = 0$. Therefore $d(x, A) = 0$.

□

Proof. **28** Let $D = \{x \in M : d(x, A) < \epsilon\}$ and let us define $\epsilon' = \epsilon - d(x, A)$ where we see that $\epsilon' > 0$. We want to prove that $B_{\epsilon'}(x) \subset D$ where we know that $B_{\epsilon'}(x) = \{y \in M : d(y, x) < \epsilon'\}$ then we have that

$$\begin{aligned} d(y, x) &< \epsilon - d(x, A) \\ d(y, A) &\leq d(y, x) + d(x, A) < \epsilon \end{aligned}$$

Then this implies that $B_{\epsilon'}(x) \subset D$ and therefore D is an open set.

Let now $F = \{x \in M : d(x, A) \leq \epsilon\}$ and let us suppose that there is a sequence $(x_n) \subset F$ such that $x_n \rightarrow x$ where $x \in M$ then this implies that there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $d(x_n, x) < \epsilon'$ for some $\epsilon' > 0$. Also from problem 27 we have that

$$|d(x, A) - d(x_n, A)| \leq d(x_n, x) < \epsilon'$$

And from the triangle inequality, we see that

$$\begin{aligned} d(x, A) = &|d(x, A) - d(x_n, A) + d(x_n, A)| \leq \\ &\leq |d(x, A) - d(x_n, A)| + |d(x_n, A)| \end{aligned}$$

Then

$$d(x, A) \leq \epsilon' + \epsilon$$

In particular, let us take $\epsilon' = (d(x, A) - \epsilon)/2$ then we have that

$$\begin{aligned} d(x, A) &\leq \frac{d(x, A)}{2} + \frac{\epsilon}{2} \\ d(x, A) &\leq \epsilon \end{aligned}$$

Therefore $x \in F$ which implies that F is a closed set.

Finally, if $x \in A$ we have that $d(x, A) = d(x, x) = 0 < \epsilon$ which implies that $A \subset D$ and $A \subset F$. □

Proof. **29**

(i) From the hint we have, we see that each set $\{x \in M : d(x, A) < 1/n\}$ is an open set. Let's see that

$$\bigcap_{n=1}^{\infty} \{x \in M : d(x, A) < 1/n\} = \{x \in M : d(x, A) = 0\}$$

So, we need to prove that $d(x, A) = 0$ if and only if for all n it holds that $d(x, A) < 1/n$.

($\rightarrow$) If $d(x, A) = 0$ then $d(x, A) = 0 < 1/n$ for all $n \in \mathbb{N}$.

($\leftarrow$) On the other hand, if $d(x, A) < 1/n$ for all n then let us suppose $d(x, A) > 0$ we want to arrive to a contradiction. We know that there is $n \in \mathbb{N}$ such that $n > 1/d(x, A)$ then $d(x, A) > 1/n$ which is a contradiction. Therefore it must be that $d(x, A) = 0$.

Also we know that $d(x, A) = 0$ if and only if $x \in \bar{A}$ so we have that

$$\{x \in M : d(x, A) = 0\} = \bar{A}$$

Where we know that $\bar{A}$ is closed. Therefore every closed set in M is the intersection of countably many open sets.

(ii) Now let's see that $\{x \in M : d(x, A) \geq 1/n\}$ is the complement of the set $\{x \in M : d(x, A) < 1/n\}$ which implies that $\{x \in M : d(x, A) \geq 1/n\}$ is a closed set. Then because of what we saw in part (i) we have that

$$\bigcup_{n=1}^{\infty} \{x \in M : d(x, A) \geq 1/n\} = \bar{A}^c$$

And we know that $\bar{A}^c$ is open, therefore every open set in M is the intersection of countably many closed sets. □

Proof. **33** Let $(B_\epsilon(x) \setminus \{x\}) \cap A = \{x_1, x_2, ..., x_n\}$ i.e. $B_\epsilon(x) \setminus \{x\}$ has finitely many points of A for all $\epsilon > 0$ we want to arrive to a contradiction.

Let us take $x_m \in \{x_1, x_2, ..., x_n\}$ such that

$$d(x_m, x) = \min\{d(x_1, x), d(x_2, x), ..., d(x_n, x)\}$$

Since $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ for all $\epsilon > 0$ in particular let us take $\epsilon' = d(x, x_m)$ then we see that $(B_{\epsilon'}(x) \setminus \{x\}) \cap A = \emptyset$ which is a contradiction and therefore $(B_\epsilon(x) \setminus \{x\}) \cap A$ hast infinitely many points. □

Proof. **34**

($\rightarrow$) Let x be a limit point of A then let us take a sequence $(x_n) \subset ((B_\epsilon(x) \setminus \{x\}) \cap A)$ which we know it exists because $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ then we see that for each $x_n \in (x_n)$ we have that $d(x_n, x) < \epsilon$ because of the definition of open ball. Therefore $x_n \rightarrow x$ and by definition of (x_n) we know that $(x_n) \subset A$ and $x_n \neq x$ for all n.

($\leftarrow$) Let $(x_n) \subset A$ such that $x_n \rightarrow x$ and $x_n \neq x$ for all n. Then this implies that given some $\epsilon > 0$ for $n \geq N$ we have that $d(x_n, x) < \epsilon$ where $N \in \mathbb{N}$. So we have that $x_n \in ((B_\epsilon(x) \setminus \{x\}) \cap A)$ for all $n \geq N$ by the definition of an open ball. Therefore since ϵ is arbitrary we have that $B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ i.e. x is a limit point of A.

$\square$