Solved selected problems of Real Analysis - Carothers

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Chapter 3 - Metrics and Norms

Proof. **2** We know that

$$|d(x,z) - d(y,z)| = \begin{cases} d(x,z) - d(y,z) \text{ if } d(x,z) \geq d(y,z) \\ d(y,z) - d(x,z) \text{ if } d(x,z) < d(y,z) \end{cases}$$

Also, from the triangle inequality we have that

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ d(x,z) - d(y,z) &\leq d(x,y) \end{aligned}$$

and that

$$\begin{aligned} d(y,z) &\leq d(y,x) + d(x,z) \\ d(y,z) - d(x,z) &\leq d(y,x) = d(x,y) \end{aligned}$$

Therefore $|d(x,z) - d(y,z)| \leq d(x,y)$ □

Proof. **3** We know that $d(x,y) \leq d(x,z) + d(y,z)$ so let $z = x$ then we have that

$$d(x,y) \leq d(x,x) + d(y,x) = d(y,x)$$

since $d(x,x) = 0$. But we also know that $d(y,x) \leq d(y,z) + d(x,z)$ and if we let $z = y$ then we have that

$$d(y,x) \leq d(y,y) + d(x,y) = d(x,y)$$

Therefore $d(x,y) = d(y,x)$.
On the other hand, if we grab the triangle inequality $d(x,z) \leq d(x,y) + d(z,y)$ and we let $z = x$ we have that

$$0 = d(x,x) \leq d(x,y) + d(x,y) = 2d(x,y)$$

Therefore $d(x,y) \geq 0$. □

Proof. **6** If $\rho(x,y) = \sqrt{d(x,y)}$ is a metric then it should follow the (i)-(iv) properties defined.

(i) Since $d(x,y)$ is a metric then $0 \leq d(x,y)$ and since the square root is a function strictly increasing we have that $0 \leq \sqrt{d(x,y)}$.

(ii) If $x = y$ then $\sqrt{d(x,y)} = \sqrt{0} = 0$ since $d(x,y)$ is a metric and $d(x,y) = 0$ if $x = y$.

(iii) Since $d(x,y)$ is a metric and $d(x,y) = d(y,x)$ then $\rho(x,y) = \sqrt{d(x,y)} = \sqrt{d(y,x)} = \rho(y,x)$.

(iv) Since $d(x,y)$ is a metric then $d(x,y) \leq d(x,z) + d(z,y)$ and since the square root is a function strictly increasing we have that

$$\sqrt{d(x,y)} \leq \sqrt{d(x,z) + d(z,y)}$$

But also we know that $\sqrt{d(x,z) + d(z,y)} \leq \sqrt{d(x,z)} + \sqrt{d(z,y)}$ therefore

$$\sqrt{d(x,y)} \leq \sqrt{d(x,z)} + \sqrt{d(z,y)}$$

If $\sigma(x,y) = d(x,y)/(1 + d(x,y))$ is a metric then it should follow the (i)-(iv) properties defined. But first we need to prove that the function $F(t) = t/(1+t)$ is increasing for any $t \geq 0$ since $d(x,y) \geq 0$ and that $F(s+t) \leq F(s)+F(t)$ which is going to clear our way to prove that σ follow the properties defined. If $0 \leq s \leq t$ then $1 + s \leq 1 + t$ so we have that

$$\frac{1}{1+t} \leq \frac{1}{1+s}$$

But then

$$\frac{s}{1+s} = 1 - \frac{1}{1+s} \leq 1 - \frac{1}{1+t} = \frac{t}{1+t}$$

Therefore σ is an increasing function.
Let us prove now that $F(s+t) \leq F(s) + F(t)$, we have that

$$F(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t}$$

and since $s, t \geq 0$ then we have that

$$\frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t}$$

So finally we are ready to prove the properties for σ as follows.

(i) Since $d(x,y)$ is a metric then $0 \leq d(x,y)$ and F is an increasing function we have that $0 = F(0) \leq F(d(x,y)) = d(x,y)/(1 + d(x,y)) = \sigma(x,y)$.

(ii) If $x = y$ then $F(d(x,y)) = d(x,y)/(1+d(x,y)) = 0/(1+0) = 0$ since $d(x,y)$ is a metric and $d(x,y) = 0$ if $x = y$.

(iii) Since $d(x,y)$ is a metric and $d(x,y) = d(y,x)$ then

$$\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \sigma(y,x)$$

.

(iv) Since $d(x,y)$ is a metric then $d(x,y) \le d(x,z) + d(z,y)$ and since the function F is a increasing function we have that

$$F(d(x,y)) \le F(d(x,z) + d(z,y))$$

But also we know that $F(d(x,z) + d(z,y)) \le F(d(x,z)) + F(d(z,y))$ because of what we proved before. Therefore

$$\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)} \le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} = \sigma(x,z) + \sigma(z,y)$$

Finally if $\tau(x,y) = \min\{d(x,y),1\}$ is a metric then it should follow the (i)-(iv) properties defined.

(i) Since $d(x,y)$ is a metric then $d(x,y) \ge 0$ but also $1 > 0$ therefore $\tau(x,y) \ge 0$.

(ii) If $x = y$ then $\tau(x,y) = \min\{d(x,y),1\} = \min\{0,1\} = 0$ since $d(x,y)$ is a metric and $d(x,y) = 0$ if $x = y$.

(iii) Since $d(x,y)$ is a metric and $d(x,y) = d(y,x)$ then

$$\tau(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\} = \tau(y,x)$$

(iv) Since $d(x,y)$ is a metric then $d(x,y) \le d(x,z) + d(z,y)$ and applying the minimum function this inequality is conserved, i.e.

$$\min\{d(x,y),1\} \le \min\{d(x,z) + d(z,y),1\}$$

Let us now check that $\min\{d(x,z) + d(z,y),1\} \le \min\{d(x,z),1\} + \min\{d(z,y),1\}$ by cases

- If $d(x,z) > 1$ and $d(z,y) > 1$ then $\min\{d(x,z) + d(z,y),1\} = 1$ and $\min\{d(x,z),1\} + \min\{d(z,y),1\} = 2$ therefore

$$\min\{d(x,z) + d(z,y),1\} < \min\{d(x,z),1\} + \min\{d(z,y),1\}$$

- If $d(x,z) < 1$ and $d(z,y) > 1$ then $\min\{d(x,z) + d(z,y),1\} = 1$ and $\min\{d(x,z),1\} + \min\{d(z,y),1\} = d(x,z) + 1$ therefore

$$\min\{d(x,z) + d(z,y),1\} < \min\{d(x,z),1\} + \min\{d(z,y),1\}$$

– If $d(x,z) > 1$ and $d(z,y) < 1$ then $\min\{d(x,z)+d(z,y),1\} = 1$ and $\min\{d(x,z),1\} + \min\{d(z,y),1\} = 1 + d(z,y)$ therefore

$$\min\{d(x,z)+d(z,y),1\} < \min\{d(x,z),1\} + \min\{d(z,y),1\}$$

– If $d(x,z) < 1$ and $d(z,y) < 1$ then $\min\{d(x,z)+d(z,y),1\} = d(x,z)+d(z,y)$ and $\min\{d(x,z),1\} + \min\{d(z,y),1\} = d(x,z) + d(z,y)$ therefore

$$\min\{d(x,z)+d(z,y),1\} = \min\{d(x,z),1\} + \min\{d(z,y),1\}$$

Finally we see that

$$\min\{d(x,y),1\} \leq \min\{d(x,z),1\} + \min\{d(z,y),1\}$$

$\square$

Proof. **10**

(i) If $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric in H^{∞} then it should follow the (i)-(iv) properties defined for metrics.

(i) Since $2^n \geq 0$ and $|x_n - y_n| \geq 0$ then $\sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| \geq 0$.

(ii) If $x_n = y_n$ then

$$\sum_{n=1}^{\infty} 2^{-n}|x_n - x_n| = \sum_{n=1}^{\infty} 2^{-n} \cdot 0 = 0$$

(iii) Since $|x_n - y_n| = |y_n - x_n|$ then we have that

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| = \sum_{n=1}^{\infty} 2^{-n}|y_n - x_n| = d(y,x)$$

(iv) From the triangle inequality we have that

$$|x_n - y_n| = |(x_n - z_n) + (z_n - y_n)| \leq |x_n - z_n| + |z_n + y_n|$$

Therefore

$$\sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| \leq \sum_{n=1}^{\infty} 2^{-n}(|x_n - z_n| + |z_n - y_n|)$$

(ii) We know that

$$d(x,y) = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n=k+1}^{\infty} 2^{-n}|x_n - y_n|$$

Then since $M_k = \max\{|x_1 - y_1|, ..., |x_k - y_k|\}$ we have that

$$\sum_{n=1}^{k} 2^{-n}|x_n - y_n| \leq \sum_{n=1}^{k} 2^{-n} M_k$$

And since $|x_n - y_n| \leq 2$

$$\sum_{n=k+1}^{\infty} 2^{-n}|x_n - y_n| \leq \sum_{n=k+1}^{\infty} 2^{-n+1} = 2^{1-k}$$

Then

$$\sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n=k+1}^{\infty} 2^{-n}|x_n - y_n| \leq M_k \sum_{n=1}^{k} 2^{-n} + 2^{1-k}$$

But in addition we see that

$$M_k \sum_{n=1}^{k} 2^{-n} + 2^{1-k} = M_k(1 - 2^{-k}) + 2^{1-k} \leq M_k + 2^{1-k}$$

Therefore

$$\sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| \leq M_k + 2^{1-k}$$

On the other hand, we know that $M_k = |x_m - y_m|$ where $m \in \{1, 2, ..., k\}$ so $m \leq k$ then $2^{-k} \leq 2^{-m}$ so $2^{-k} M_k \leq 2^{-m}|x_m - y_m|$ Therefore

$$2^{-k} M_k \leq \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| \leq M_k + 2^{1-k}$$

$\square$

Proof. **12** We want to check that $d(f,g) = \max_{a\leq t\leq b} |f(t) - g(t)|$ is a metric on $C[a,b]$ then we need to prove it follows the (i)-(iv) properties defined

(i) Since $|f(t) - g(t)| \geq 0$ then $d(f,g) = \max_{a\leq t\leq b} |f(t) - g(t)| \geq 0$

(ii) If $f = g$ then
$$d(f,f) = \max_{a\leq t\leq b} |f(t) - f(t)| = 0$$

(iii) Since $|f(t) - g(t)| = |g(t) - f(t)|$ we have that
$$d(f,g) = \max_{a\leq t\leq b} |f(t) - g(t)| = \max_{a\leq t\leq b} |g(t) - f(t)| = d(g,f)$$

(iv) We know that $|f(t) - h(t)| \leq \max_{a\leq t\leq b} |f(t) - h(t)| = d(f,h)$ and that $|h(t) - g(t)| \leq \max_{a\leq t\leq b} |h(t) - g(t)| = d(h,g)$ and also because of the triangle inequality we have that for any t
$$|f(t) - g(t)| = |(f(t) - h(t)) + (h(t) - g(t))| \leq |f(t) - h(t)| + |h(t) - g(t)|$$
Therefore
$$|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)| \leq d(f,h) + d(h,g)$$
The left-hand side of the inequality depends on t but the right-hand side does not, then maximum of the right-hand side is less that the left-hand side.
$$d(f,g) \leq d(f,h) + d(h,g)$$

□

Proof. **14** Let $S = \cup_{i=1}^{n} S_i$ be the finite union of sets of M where all of them are bounded then there is a set of constants $\{C_1, C_2, ..., C_n\}$ such that for the ith element we have that $d(s_i, x_0) \leq C_i$ for all $s_i \in S_i$. Therefore if we take $C = \max\{C_1, C_2, ..., C_n\}$ we see that $d(s, x_0) \leq C$ where $s \in S$. □

Proof. **15**

($\rightarrow$) If A is bounded then there is some constant C and it exist $x_0 \in M$ such that $d(a, x_0) \leq C$ for all $a \in A$. Let us take $a, b \in A$ then $d(a,b) \leq d(a, x_0) + d(b, x_0) \leq 2C$ therefore $\sup\{d(a,b) : a, b \in A\} \leq 2C$.

($\leftarrow$) If the diameter of A is finite then $\sup\{d(a,b) : a, b \in A\}$ exist and because of the definition of supremum we have that $d(a,b) \leq \sup\{d(a,b) : a, b \in A\}$. Let us now take $x_0 \in A$ since $A \subseteq M$ we have that $x_0 \in M$ so if we take $C = \sup\{d(a,b) : a, b \in A\}$ we see that $d(a, x_0) \leq C$. □

Proof. **16** We want to show that $\|x\| = d(x, 0)$ is a norm on V so we need to show that it satisfies the properties of a norm

(i) Since $d(x, y)$ is a metric we know that $0 \leq \|x\| = d(x, 0) < \infty$.

(ii) $(\rightarrow)$ If $\|x\| = 0 = d(x, 0)$ then $x = 0$ because $d(x, y)$ is a metric.

$(\leftarrow)$ If $x = 0$ then $\|x\| = \|0\| = d(0, 0) = 0$ because $d(x, y)$ is a metric and $d(x, y) = 0$ iff $x = y$.

(iii) $\|\alpha x\| = d(\alpha x, 0) = |\alpha| d(x, 0) = |\alpha| \|x\|$ because we know that $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

(iv) Because of the triangle inequality defined for metrics we have that

$$\|x + y\| = d(x + y, 0) \leq d(x + y, y) + d(y, 0)$$

But also we know that $d(x, y) = d(x - y, 0)$ then $d(x + y, y) = d(x, 0)$ therefore

$$\|x + y\| = d(x + y, 0) \leq d(x, 0) + d(y, 0) = \|x\| + \|y\|$$

Finally an example of a metric on $\mathbb{R}$ that fails to be associated with a norm this way is $\rho(x, y) = \sqrt{|x - y|}$ since $\rho(x, y) = \rho(x - y, 0)$ is true but $\rho(\alpha x, \alpha y) \neq |\alpha| \rho(x, y)$. □

Proof. **17** First, we want to show that $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ is a norm so we need to show that it satisfies the properties of a norm

(i) Since every element $0 \leq |x_i| < \infty$ then $0 \leq \|x\|_1 = \sum_{i=1}^{n} |x_i| < \infty$.

(ii) $(\rightarrow)$ If $\|x\|_1 = 0$ and since $|x_i| \geq 0$ it must happen that every element $|x_i| = 0$ therefore $x = 0$.

$(\leftarrow)$ If $x = 0$ this means that every element $|x_i| = 0$ and therefore $\|x\|_1 = \sum_{i=1}^{n} |x_i| = 0$.

(iii) Let α be a scalar then

$$\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum_{i=1}^{n} |\alpha||x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha| \|x\|_1$$

(iv) We have that $\|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$ and because of the triangle inequality for real numbers we know that $|x_i + y_i| \leq |x_i| + |y_i|$ then

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

And therefore

$$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$$

Now we want to show that $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ is a norm then

(i) Since every element $0 \le |x_i| < \infty$ then $0 \le \max_{1\le i\le n} |x_i| < \infty$.

(ii) $(\rightarrow)$ If $\|x\|_\infty = 0 = \max_{1\le i\le n} |x_i|$ then $|x_i| = 0$ for any i, therefore $x = 0$.

$(\leftarrow)$ If $x = 0$ this means that for any i we have that $|x_i| = 0$ and therefore $\|x\|_\infty = \max_{1\le i\le n} |x_i| = \max_{1\le i\le n} |0| = 0$.

(iii) Let α be a scalar then

$$\|\alpha x\|_\infty = \max_{1\le i\le n} |\alpha x_i| = |\alpha| \max_{1\le i\le n} |x_i| = |\alpha| \|x\|_\infty$$

because we are multiplying every element to the same scalar α the maximum function can be applied to the elements of x only.

(iv) We know that $|x_i| \le \max_{1\le i\le n} |x_i|$ and that $|y_i| \le \max_{1\le i\le n} |y_i|$ then because of the triangle inequality we have that

$$|x_i + y_i| \le |x_i| + |y_i| \le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i|$$

And since the right-hand side of the equation does not depend on i we can apply the maximum to the left-hand side of the equation as

$$\max_{1\le i\le n} |x_i + y_i| \le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i|$$

Therefore

$$\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty$$

□

Proof. **18** First, let us note that

$$(\sum_{i=1}^{n}|x_i|)\cdot(\sum_{i=1}^{n}|x_i|)=\sum_{i=1}^{n}|x_i|^2+2\sum_{i<j}|x_i||x_j|$$

Also, we know that $\|x\|_2=\sqrt{\sum_{i=1}^{n}x_i^2}$ and that $\|x\|_1=\sum_{i=1}^{n}|x_i|$ Then we see that

$$\|x\|_2^2=\sum_{i=1}^{n}|x_i|^2\leq\sum_{i=1}^{n}|x_i|^2+2\sum_{i<j}|x_i||x_j|=\|x\|_1^2$$

Which implies that $\|x\|_2\leq\|x\|_1$.

On the other hand, we see that

$$\|x\|_\infty^2=(\max_{1\leq i\leq n}|x_i|)^2\leq\sum_{i=1}^{n}|x_i|^2=\|x\|_2^2$$

Therefore $\|x\|_\infty\leq\|x\|_2\leq\|x\|_1$ as we wanted.

Finally, we have that

$$\|x\|_1=\sum_{i=1}^{n}|x_i|\leq n\cdot(\max_{1\leq i\leq n}|x_i|)=n\|x\|_\infty$$

And using Cauchy-Schwartz inequality we have that

$$\|x\|_1=\sum_{i=1}^{n}|x_i\cdot 1|\leq\sqrt{\sum_{i=1}^{n}1}\sqrt{\sum_{i=1}^{n}|x_i|^2}=\sqrt{n}\|x\|_2$$

□

Proof. **19**

($\rightarrow$) From the Cauchy-Schwarz inequality proof we see that if

$$\sum_{i=1}^{n} |x_i y_i| = \|x\|_2 \|y\|_2 \tag{1}$$

then the discriminant of the quadratic equation $\|x\|_2 + 2t\langle x, y\rangle + t^2\|y\|_2$ is 0 where $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ therefore we have only one solution for t i.e

$$\begin{aligned} t &= \frac{-2\langle x, y\rangle}{2\|y\|_2^2} \\ |\langle x, y\rangle| &= |t|\|y\|_2^2 \end{aligned}$$

But since the inequality we took should work also for the vectors $(|x_i|)$ and $(|y_i|)$ then we can do the replacement in equation (1) as follows

$$\begin{aligned} \|x\|_2\|y\|_2 &= |t|\|y\|_2^2 \\ \|x\|_2 &= |t|\|y\|_2 = \|ty\|_2 \end{aligned}$$

Therefore this means that x and y are proportional on some value $|t| \geq 0$.

($\leftarrow$) If $y = \alpha x$ for some scalar $\alpha \geq 0$ then

$$\begin{aligned} \|x\|_2\|y\|_2 &= \|x\|_2\|\alpha x\|_2 = |\alpha|\|x\|_2^2 = \\ &= |\alpha| \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} |\alpha x_i|^2 = \sum_{i=1}^{n} |x_i y_i| \end{aligned}$$

Where we used that if $y = \alpha x$ then for every i we have that $y_i = \alpha x_i$.
Therefore $\|x\|_2\|y\|_2 = \sum_{i=1}^{n} |x_i y_i|$ $\square$

Proof. **21** To show that l_1 is a normed vector space we need to show that $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ is a norm on l_1 therefore we need to show that it satisfies the properties of a norm

(i) Since for any i we have that $0 \leq |x_i| < \infty$ then $0 \leq \|x\|_1 < \infty$.

(ii) ($\rightarrow$) If $\|x\|_1 = 0$ then it must be the case that for any i we have $|x_i| = 0$ therefore $x = 0$.

($\leftarrow$) If $x = 0$ then this means that any i we have that $|x_i| = 0$ therefore $\|x\|_1 = 0$.

(iii) In this case we have that

$$\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha| \|x\|_1$$

(iv) For the triangle inequality in this case we have that

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

Therefore

$$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$$

Let us now do the same for l_∞ with the norm $\|x\|_\infty = \sup_{n\geq 1} |x_n|$ then

(i) For any n we have that $0 \leq |x_n| < \infty$ because l_∞ is the set of all bounded sequences, therefore $0 \leq \|x\|_\infty < \infty$.

(ii) ($\rightarrow$) If $\|x\|_\infty = \sup_{n\geq 1} |x_n| = 0$ then by the definition of the supremum we have that $|x_n| \leq 0$ which means that $x = 0$.

($\leftarrow$) If $x = 0$ then this means that for any n we have that $|x_n| = 0$ therefore $\|x\|_\infty = \sup_{n\geq 1} |x_n| = 0$.

(iii) Since α is a scalar and l_∞ is the set of all bounded sequences we have that

$$\sup_{n\geq 1} |\alpha x_n| = |\alpha| \sup_{n\geq 1} |x_n|$$

Therefore $\|\alpha x\|_\infty = |\alpha| \|x\|_\infty$.

(iv) From the triangle inequality property we have that for any n

$$|x_n + y_n| \leq |x_n| + |y_n|$$

Then applying the supremum to both sides of the equation we have that

$$\sup_{n\geq 1} |x_n + y_n| \leq \sup_{n\geq 1} |x_n| + |y_n| \leq \sup_{n\geq 1} |x_n| + \sup_{n\geq 1} |y_n|$$

Therefore

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$$

$\square$

Proof. **22** Since $x \in l_2$ then we can compute $\|x\|_2^2 = \sum_{n=1}^{\infty} |x_n|^2$ by definition of the norm $\|x\|_2$ also we can grab the supremum of the absolute values of the sequence because we know that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ therefore

$$\|x\|_\infty^2 = (\sup_n |x_n|)^2 \leq \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_2^2$$

Now given $x \in l_1$ we can compute

$$\|x\|_1^2 = (\sum_{n=1}^{\infty} |x_n|) \cdot (\sum_{n=1}^{\infty} |x_n|) = \sum_{n=1}^{\infty} |x_n|^2 + 2 \sum_{\substack{n=1 \\ m=2 \\ n<m}}^{\infty} |x_n||x_m|$$

Therefore

$$\|x\|_2^2 = \sum_{n=1}^{\infty} |x_n|^2 \leq \sum_{n=1}^{\infty} |x_n|^2 + 2 \sum_{\substack{n=1 \\ m=2 \\ n<m}}^{\infty} |x_n||x_m| = \|x\|_1^2$$

Which implies that $\|x\|_2 \leq \|x\|_1$ $\square$

Proof. **23** Let x be a sequence from l_1 where

$$\sum_{n=1}^{\infty} |x_n| < \infty$$

Since we know that

$$\sum_{n=1}^{\infty} |x_n|^2 \leq \sum_{n=1}^{\infty} |x_n|^2 + 2 \sum_{\substack{n=1 \\ m=2 \\ n<m}}^{\infty} |x_n||x_m|$$

Therefore

$$\|x\|_2^2 \leq \|x\|_1^2 < \infty$$

i.e. $\|x\|_2$ also converges. So we have that $l_1 \subset l_2$. Also, we know that if a series converge then the sequence of it's elements converge to 0 so

$$l_1 \subset l_2 \subset c_0$$

And finally becuase of how we defined c_0 we have that

$$l_1 \subset l_2 \subset c_0 \subset l_\infty$$

$\square$

Proof. **24** We want to show that the conclusion of Lemma 3.7 also holds for $p = 1$ and $q = \infty$ which means that we want to prove that

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_1 \|y\|_\infty$$

holds. We know that

$$|x_i y_i| = |x_i| \cdot |y_i| \leq |x_i| \cdot \sup_i |y_i| = |x_i| \cdot \|y\|_\infty$$

Then adding the inequalities for every i we have that

$$\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} |x_i| \cdot \|y\|_\infty = \|y\|_\infty \sum_{i=1}^{n} |x_i| = \|x\|_1 \|y\|_\infty$$

□

Proof. **25** We want to prove the analogous to Holder's Inequality so let $1 < p < \infty$ and let q be defined by $1/p + 1/q = 1$ given $f, g \in C[0,1]$ we want to prove that

$$\int_0^1 |f(t)g(t)|dt \leq (\int_0^1 |f(t)|^p dt)^{1/p} \cdot (\int_0^1 |g(t)|^q dt)^{1/q} = \|f\|_p \|g\|_q$$

Let us suppose $\|f\|_p > 0$ and $\|g\|_q > 0$ then from Young's Inequality we have that

$$\left| \frac{f(t)g(t)}{\|f\|_p \|g\|_q} \right| \leq \frac{1}{p} \left| \frac{f(t)}{\|f\|_p} \right|^p + \frac{1}{q} \left| \frac{g(t)}{\|g\|_q} \right|^q \leq \frac{1}{p} + \frac{1}{q} = 1$$

Then since this should work for any t we can integrate the expression

$$\int_0^1 \left| \frac{f(t)g(t)}{\|f\|_p \|g\|_q} \right| dt \leq \frac{1}{p} \int_0^1 \left| \frac{f(t)}{\|f\|_p} \right|^p dt + \frac{1}{q} \int_0^1 \left| \frac{g(t)}{\|g\|_q} \right|^q dt \leq 1$$

Therefore

$$\int_0^1 |f(t)g(t)|dt \leq \|f\|_p \|g\|_q$$

Now we want to prove the analogous to Minkowski's Inequality i.e the triangle inequality, in the same way let $1 < p < \infty$ if $f, g \in C[0,1]$ we want to prove that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
Before we continue with the proof we show the following equality. Since $(p-1)q = p$ we have that

$$\begin{aligned} \||f|^{p-1}\|_q &= \left(\int_0^1 ||f(t)|^{p-1}|^q dt \right)^{1/q} \\ &= \left(\int_0^1 |f(t)|^p dt \right)^{1/q} = \|f\|_p^{p/q} = \|f\|_p^{p-1} \end{aligned}$$

On the other hand we use Holder's Inequality as follows

$$
\begin{aligned}
\int_0^1 |f(t)+g(t)|^p dt &= \int_0^1 |f(t)+g(t)|\cdot|f(t)+g(t)|^{p-1}dt\\
&\leq \int_0^1 |f(t)|\cdot|f(t)+g(t)|^{p-1}dt + \int_0^1 |g(t)|\cdot|f(t)+g(t)|^{p-1}dt\\
&\leq \|f\|_p\cdot\||f+g|^{p-1}\|_q + \|g\|_p\cdot\||f+g|^{p-1}\|_q =\\
&= \|f\|_p\|f+g\|_p^{p-1} + \|g\|_p\|f+g\|_p^{p-1} =\\
&= \|f+g\|_p^{p-1}(\|g\|_p+\|f\|_p)
\end{aligned}
$$

Therefore

$$\|f+g\|_p \leq \|g\|_p + \|f\|_p$$

□

Proof. **26** Let $a<b$ and let us define $r=a/b$ then

$$
\begin{aligned}
(a^p+b^p)^{1/p} &= \left(b^p(1+\frac{a^p}{b^p})\right)^{1/p}\\
&= b(1+r^p)^{1/p}
\end{aligned}
$$

Then applying logarithm to both sides of the equation we have that

$$
\begin{aligned}
\log(a^p+b^p)^{1/p} &= \log b(1+r^p)^{1/p}\\
\frac{1}{p}\log(a^p+b^p) &= \log b + \frac{1}{p}\log(1+r^p)
\end{aligned}
$$

Now let us apply the limit to infinity to the right-side of the equation, then

$$
\begin{aligned}
\lim_{p\to\infty} \log b + \frac{\log(1+r^p)}{p} &= \log b + \lim_{p\to\infty}\frac{\log(1+r^p)}{p}\\
&= \log b + 0 = \log b
\end{aligned}
$$

Then we see that

$$\lim_{p\to\infty}(a^p+b^p)^{1/p} = \lim_{p\to\infty} e^{\log b+\frac{1}{p}\log(1+r^p)} = b$$

The same procedure is valid for when $b<a$ defining $r=b/a$ and we get that

$$\lim_{p\to\infty}(a^p+b^p)^{1/p} = a$$

So we have the last case to prove which happens when $b=a$ then we have that

$$\lim_{p\to\infty}(a^p+b^p)^{1/p} = \lim_{p\to\infty}(2b^p)^{1/p} = \lim_{p\to\infty}2^{1/p}b = b$$

Therefore we see that

$$\lim_{p\to\infty}(a^p+b^p)^{1/p} = \max\{a,b\}$$

□

Proof. **29**

($\rightarrow$) If A is bounded we saw that for any $x \in M$ we have that $\sup_{a\in A} d(x,a) < \infty$ but since $A \subset M$ then we can take $b \in A$ so $\sup_{a\in A} d(b,a) < \infty$ therefore

$$diam(A) = \sup\{d(a,b) : a, b \in A\} < \infty$$

($\leftarrow$) If $diam(A) < \infty$ let $a, b \in A$ this means that $\sup\{d(a,b) : a, b \in A\} < \infty$ but then we have that $d(a,b) < r$ for some r and since $A \subset M$ we see that

$$A \subset \{x, y \in M : d(x,y) < r\} = B_r(x)$$

□

Proof. **30** Let $a_1, a_2 \in A$ since $A \subset B$ then it must be true that $d(a_1, a_2) \leq \text{diam}(B)$ then since the Supremum is the least upper bound we see that $\text{diam}(A) \leq \text{diam}(B)$. □

Proof. **32** We know that the usual metric on V is $d(x,y) = \|x - y\|$ then $B_r(x) = \{z \in V : \|x - z\| < r\}$ and since V is a normed vector space we can write z as $z = x + y$ where $x, y \in V$ therefore

$$\begin{aligned} B_r(x) &= \{x + y \in V : \|x - x + y\| < r\} \\ &= \{x + y \in V : \|y\| < r\} \\ &= x + \{y \in V : \|y\| < r\} = x + B_r(0) \end{aligned}$$

On the other hand we know that $B_r(0) = \{y \in V : \|y\| < r\}$ and we can write that $y = rx$ where $x \in V$ then we have that

$$\begin{aligned} B_r(0) &= \{rx \in V : \|rx\| < r\} \\ &= \{rx \in V : |r|\|x\| < r\} \\ &= \{rx \in V : \|x\| < 1\} = rB_1(0) \end{aligned}$$

□

Proof. **33** Let us suppose that (x_n) converges both to x and y then this means that there is $N \geq 1$ and $N' \geq 1$ for which $d(x_n, x) < \epsilon$ when $n \geq N$ and $d(x_n, y) < \epsilon$ when $n \geq N'$ we want to arrive to a contradiction. This should be valid for any $\epsilon > 0$, so let us grab

$$\epsilon = \frac{d(x,y)}{2}$$

Then we have that

$$d(x_n, x) + d(x_n, y) < 2\epsilon$$

But because of the triagle inequality we have that

$$d(x,y) \leq d(x_n, x) + d(x_n, y)$$

Therefore

$$d(x, y) < 2\epsilon = d(x, y)$$

which is a contradiction, and must be the case that $x = y$. $\square$

Proof. **34** Given that (x_n) converges to x we know that given some $\epsilon > 0$ there is an integer $N \geq 1$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$. Then from the result we got from problem 2 we know that

$$|d(x_n, y) - d(x, y)| \leq d(x_n, x)$$

So

$$|d(x_n, y) - d(x, y)| < \epsilon$$

Which implies that the difference $|d(x_n, y) - d(x, y)| \to 0$.
Therefore $d(x_n, y) \to d(x, y)$

On the other hand, let us see the following inequality

$$d(x, y) \leq d(x, x_n) + d(y, y_n) \leq d(x, x_n) + d(x_n, y_n) + d(y, y_n)$$

Then

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$$

Now we know that $y_n \to y$ so let us take $\epsilon/2 > 0$ so we have that there is $N \geq 1$ such that $d(x_n, x) < \epsilon/2$ whenever $n \geq N$ and that there is $N' \geq 1$ such that $d(y_n, y) < \epsilon/2$ whenever $n \geq N'$ therefore

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) < \epsilon$$

Which implies that the difference $|d(x_n, y_n) - d(x, y)| \to 0$.
Therefore $d(x_n, y_n) \to d(x, y)$ $\square$

Proof. **35** If $x_n \to x$ then given $\epsilon > 0$ there is an integer $N \geq 1$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$. Then we can grab $K \geq 1$ such that $n_K \geq n \geq N$ and since $n_1 < n_2 < ...$ if we take $k \geq K$ we have that $d(x_{n_k}, x) < \epsilon$. Therefore $x_{n_k} \to x$. $\square$

Proof. **36** Since (x_n) is bounded we know that given $\epsilon > 0$ there is an integer $N \geq 1$ such that $\{x_n : n \geq N\} \subset B_\epsilon(x)$ also we have that

$$\text{diam}(\{x_n : n \geq N\}) = \sup\{d(x_n, x_m) : n, m \geq N\}$$

But since $\{x_n : n \geq N\} \subset B_\epsilon(x)$ then it must happen that

$$\text{diam}(\{x_n : n \geq N\}) = \sup\{d(x_n, x_m) : n, m \geq N\} \leq \epsilon$$

Therefore (x_n) is Cauchy.

On the other hand, if (x_n) is Cauchy we know that given $\epsilon > 0$ there is an integer $N \geq 1$ such that $\sup\{d(x_n, x_m) : n, m \geq N\} \leq \epsilon$ then this means that

$$d(x_n, x_m) \leq \epsilon$$

for any $n, m \geq N$ so $\{x_n : n \geq N\} \subset B_\epsilon(x)$ where $B_\epsilon(x)$ is a closed ball. Now we have to prove that the elements of the sequence $\{x_1, x_2, ..., x_{n-1}\}$ are also bounded so let us call M the maximum of the distances between them

$$M = \max\{d(x_n, x_m) : n, m \in \{1, 2, ..., n-1\}\}$$

then we have that

$$\{x_1, x_2, ..., x_{n-1}\} \subset B_M(x)$$

Where $B_M(x)$ is a closed ball, which means that this sequence is also bounded. Therefore (x_n) is bounded. $\square$

Proof. **40** We want to prove that $x^{(k)} \to x$ where $x \in l_1$ and $x^{(k)} \in l_1$ then this means to prove that given an $\epsilon > 0$ there is an integer $K \geq 1$ such that $d(x^{(k)}, x) < \epsilon$ whenever $k \geq K$. We have then that

$$d(x^{(k)}, x) = \|x^{(k)} - x\|_1 = \sum_{n=1}^{\infty} |x_n^{(k)} - x_n| = \sum_{n=k}^{\infty} |x_n|$$

And when $k \to \infty$ since x is a convergent series we have that

$$\lim_{k\to\infty} \sum_{n=k}^{\infty} |x_n| = 0 < \epsilon$$

Therefore $x^{(k)} \to x$.

In the case of $x^{(k)}, x \in l_2$ we have that

$$d(x^{(k)}, x) = \|x^{(k)} - x\|_2 = \sqrt{\sum_{n=1}^{\infty} |x_n^{(k)} - x_n|^2} = \sqrt{\sum_{n=k}^{\infty} |x_n|^2}$$

And when $k \to \infty$ since x is a convergent series we have that

$$\lim_{k\to\infty} \sqrt{\sum_{n=k}^{\infty} |x_n|^2} = 0 < \epsilon$$

Therefore $x^{(k)} \to x$.

Lastly, let $\epsilon = x_1 = \inf_{n\geq 2} x_n$ we have that

$$d(x^{(k)}, x) = \|x_n^k - x_n\|_\infty = \sup_{n\geq 1} |x_n^k - x_n| \not< x_1$$

i.e. there is no $K \geq 1$ such that when $k \geq K$ we get that $d(x^{(k)}, x) < \epsilon$. Therefore $x^{(k)} \not\to x$. □

Proof. **41** We want to prove that $\langle x^{(k)}, y^{(k)}\rangle \to \langle x, y\rangle$ where $x, y \in l_2$ and $x^{(k)}, y^{(k)} \in l_2$ then this means to prove that given an $\epsilon > 0$ there is an integer $K \geq 1$ such that $d(\langle x^{(k)}, y^{(k)}\rangle, \langle x, y\rangle) < \epsilon$ whenever $k \geq K$. We have then

$$\begin{aligned} d(\langle x^{(k)}, y^{(k)}\rangle, \langle x, y\rangle) &= |\langle x^{(k)}, y^{(k)}\rangle - \langle x, y\rangle| \\ &= |\sum_{i=1}^{\infty} x_i^{(k)} y_i^{(k)} - \sum_{i=1}^{\infty} x_i y_i| \\ &= |\sum_{i=k+1}^{\infty} x_i y_i| \end{aligned}$$

Now we have to check that when $k \to \infty$ the expression $|\sum_{i=k+1}^{\infty} x_i y_i|$ converges to 0. From Holder's inequality and since we know that $x^{(k)} \to x$ and $y^{(k)} \to y$ we have that

$$\sum_{i=1}^{\infty} |(x_i^{(k)} - x_i)(y_i^{(k)} - y_i)| \leq \|x^{(k)} - x\|_2 \|y^{(k)} - y\|_2 < 2\epsilon$$

Then

$$\begin{aligned} \sum_{i=1}^{\infty} |(x_i^{(k)} - x_i)(y_i^{(k)} - y_i)| &= \sum_{i=1}^{\infty} |x_i^{(k)} y_i^{(k)} - x_i^{(k)} y_i - x_i y_i^{(k)} + y_i x_i| \\ &= \sum_{i=1}^{\infty} |x_i y_i - x_i y_i^{(k)}| < 2\epsilon \end{aligned}$$

Where we used that the terms $x_i^{(k)} y_i^{(k)} - x_i^{(k)} y_i$ cancel each other. So when $k \to \infty$ since x and y are convergent sequence we have that

$$\lim_{k\to\infty} |\sum_{i=k+1}^{\infty} x_i y_i| = 0 < \epsilon$$

Therefore $\langle x^{(k)}, y^{(k)}\rangle \to \langle x, y\rangle$. $\square$

Proof. **42**

($\rightarrow$) Given that $d(x_n, x) \rightarrow 0$

(i) Then $\rho(x_n, x) = \sqrt{d(x_n, x)} \rightarrow 0$.

(ii) Since

$$\sigma(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} = \frac{1}{1/d(x_n, x) + 1}$$

Therefore when $\sigma(x_n, x) \rightarrow 0$

(iii) Lastly, we see that $\tau(x_n, x) = \min\{d(x_n, x), 1\} \rightarrow 0$ whenever $d(x_n, x) \rightarrow 0$.

($\leftarrow$)

(i) Given that $\rho(x_n, x) \rightarrow 0$ then we know that $x_n \rightarrow x$ so given $\sqrt{\epsilon} > 0$ there is $N \geq 1$ such that $\rho(x_n, x) = \sqrt{d(x_n, x)} < \sqrt{\epsilon}$ whenever $n \geq N$ therefore $d(x_n, x) < \epsilon$ and $d(x_n, x) \rightarrow 0$.

(ii) Given that $\sigma(x_n, x) \rightarrow 0$ then we know that $x_n \rightarrow x$ so given $\epsilon' = 1/(1/\epsilon + 1) > 0$ there is $N \geq 1$ such that

$$\sigma(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} = \frac{1}{1/d(x_n, x) + 1} < \epsilon' = \frac{1}{1/\epsilon + 1}$$

whenever $n \geq N$. Therefore

$$\begin{gathered} \frac{1}{1/d(x_n, x) + 1} < \frac{1}{1/\epsilon + 1} \\ 1/d(x_n, x) + 1 > 1/\epsilon + 1 \\ 1/d(x_n, x) > 1/\epsilon \\ d(x_n, x) < \epsilon \end{gathered}$$

and $d(x_n, x) \rightarrow 0$.

(iii) Finally, given that $\tau(x_n, x) = \min\{d(x_n, x), 1\} \rightarrow 0$ then this must means that $d(x_n, x) \rightarrow 0$.

$\square$

Proof. **43** We define $d'(x_n, x)$ as the discrete metric.

(i) ($\rightarrow$) Let us suppose that the usual metric on $\mathbb{N}$ i.e. $d(x_n, x) = |x_n - x| \rightarrow 0$ where x_n is a sequence in $\mathbb{N}$. Then by definition given some $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$. Let us take $\epsilon = 1$ and let us have N' that satisfy the definition so $|x_n - x| < 1$ whenever $n \geq N'$ but this must mean that $x_n = x$ because x_n is a sequence on $\mathbb{N}$ then x_n is eventually constant so the discrete metric $d'(x_n, x) = 0$. Therefore given some $\epsilon > 0$ we can use the N' we selected before such that $d'(x_n, x) < \epsilon$ whenever $n \geq N'$ no matter which ϵ we are given and because of that $d'(x_n, x) \rightarrow 0$.

($\leftarrow$) Now let us suppose that $d'(x_n, x) \rightarrow 0$ then this means that eventualy $x_n = x$ for $n \geq N$ then $d(x_n, x) = |x_n - x| = 0$ when $n \geq N$. Therefore also $d(x_n, x) \rightarrow 0$.

(ii) ($\rightarrow$) Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set and let the metric of this set be $d_A(a_n, a) \rightarrow 0$ then this mean that for some $N \in \mathbb{N}$ we have that $d_A(a_n, a) = 0$ when $n \geq N$ but as before this means that A eventually becomes constant and $a_n = a$ therefore $d'(a_n, a) \rightarrow 0$.

($\leftarrow$) Let now $d'(a_n, a) \rightarrow 0$ then this means that for some $N \in \mathbb{N}$ we have that $a_n = a$ for any $n \geq N$. But then if we take this N we see that $d_A(a_n, a) \rightarrow 0$ when $n \geq N$.

□

Proof. **44**

If $\|x_m - x\|_1 \rightarrow 0$ then this means that given $\epsilon > 0$ we have some $M \geq 1$ such that $\|x_m - x\|_1 < \epsilon$ whenever $m \geq M$ and since we know that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ then

$$\|x_m - x\|_\infty \leq \|x_m - x\|_2 \leq \|x_m - x\|_1 < \epsilon$$

Therefore $\|x_m - x\|_2 \rightarrow 0$ and $\|x_m - x\|_\infty \rightarrow 0$.

If $\|x_m - x\|_2 \rightarrow 0$ then in the same way we see that

$$\|x_m - x\|_\infty \leq \|x_m - x\|_1 \leq \sqrt{n}\|x_m - x\|_2 < \sqrt{n}\epsilon \leq \epsilon$$

Therefore $\|x_m - x\|_1 \rightarrow 0$ and $\|x_m - x\|_\infty \rightarrow 0$.

If $\|x_m - x\|_\infty \rightarrow 0$ then in the same way we see that

$$\|x_m - x\|_2 \leq \|x_m - x\|_1 \leq n\|x_m - x\|_\infty < n\epsilon \leq \epsilon$$

Therefore $\|x_m - x\|_1 \rightarrow 0$ and $\|x_m - x\|_2 \rightarrow 0$. □

Proof. **45** If the two induced metrics are equivalent then $\|x_n - x\| \to 0$ if and only if $|||x_n - x||| \to 0$ and if both sequences tend to 0 then we have that $\|x_n\| \to 0$ if and only if $|||x_n||| \to 0$ $\square$

Proof. **46** If the sequence (x_n) converge then $\rho(x_n, x) \to 0$ and if the sequence (a_n) converge then $d(a_n, a) \to 0$. Then

$$d_1((a_n, x_n), (a, x)) = d(a_n, a) + \rho(x_n, x) \to 0$$

$$d_2((a_n, x_n), (a, x)) = (d(a_n, a)^2 + \rho(x_n, x)^2)^{1/2} \to 0$$

And also

$$d_\infty((a_n, x_n), (a, x)) = \max\{d(a_n, a), \rho(x_n, x)\} \to 0$$

Lastly, since for this metrics to tend to 0 we need that $d(a_n, a) \to 0$ and that $\rho(x_n, x) \to 0$ then if $d_1((a_n, x_n), (a, x)) \to 0$ this is posible because $d(a_n, a) \to 0$ and $\rho(x_n, x) \to 0$ as we said. Therefore $d_2((a_n, x_n), (a, x)) \to 0$ and $d_\infty((a_n, x_n), (a, x)) \to 0$ and the same can be proved the other way around. $\square$