Solved selected problems of Real Analysis - Carothers

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Chapter 7 - Completeness

Proof. **1** If B is totally bounded then given $\epsilon > 0$ there are finitely many sets $B_1, ..., B_n \subset B$ with $\text{diam}(B_i) < \epsilon$ for all i such that $B \subset \bigcup_{i=1}^n B_i$. Then if $A \subset B$ we see that $A \subset B \subset \bigcup_{i=1}^n B_i$ then A is also totally bounded. □

Proof. **2**

($\Rightarrow$) Let $A \subset \mathbb{R}$ such that A is totally bounded then given $\epsilon > 0$ there are finitely many points $x_1, ..., x_n \in \mathbb{R}$ such that $A \subset \bigcup_{i=1}^n B_\epsilon(x_i)$. Now let us grab $x_M = \max_{1\leq i\leq n}(x_i)$ so we have that $a \leq x_M + \epsilon$ for every $a \in A$ hence A has an upper bound, if we now grab $x_m = \min_{1\leq i\leq n}(x_i)$ then we have that $x_m - \epsilon \leq a$ so A has a lower bound and therefore A is bounded.

($\Leftarrow$) Let $A \subset \mathbb{R}$ such that A is bounded then there are $x_m, x_M \in \mathbb{R}$ such that $x_m \leq a \leq x_M$ for every $a \in A$. Given some $\epsilon > 0$ we select $x_1 = x_m + \epsilon$ then $x_2 = x_1 + 2\epsilon$ and so on such that $x_{i+1} = x_i + 2\epsilon$ until we arrive to some finite n where $x_n \geq x_M - \epsilon$. Then with this set we conclude that $A \in \bigcup_{i=1}^n B_\epsilon(x_i)$. Therefore A is totally bounded.

Finally let I be a closed, bounded, interval in $\mathbb{R}$ and $\epsilon > 0$. Then there are $x_m, x_M \in I$ such that $x_m \leq y \leq x_M$ for all $y \in I$. Let us select $x_1 \in I$ such that $x_1 = x_m + \epsilon/2$ then the ball $J_1 = B_{\epsilon/2}(x_1)$ covers the interval $[x_m, x_m + \epsilon]$ the following $x_2 \in I$ let us select it as $x_2 = x_1 + \epsilon$ so it covers $J_2 = [x_m + \epsilon, x_m + 2\epsilon]$ if we continue this way we can select finitely many $x_{i+1} = x_i + \epsilon$ such that $\bigcup_{i=1}^n J_i$ covers I as we wanted. □

Proof. **3** Given that $(0,1) \subset \mathbb{R}$ and that $(0,1)$ is bounded then $(0,1)$ is totally bounded because of the result we got in problem 2. But $\mathbb{R}$ is not bounded therefore it's not totally bounded. Hence totally boundedness is not preserved by homeomorphisms. □

Proof. **4**

($\Rightarrow$) Let A be a totally bounded set then given $\epsilon/2 > 0$ there exists finitely many points $x_1, ..., x_n \in M$ such that $A \subset \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$. In particular, if we take the closed balls $B'_{\epsilon/2}(x_i) = \{y \in M : d(x_i, y) \leq \epsilon/2\}$ we see that $A \subset \bigcup_{i=1}^{n} B'_{\epsilon/2}(x_i)$ is still true therefore A can be covered by finitely many closed sets of diameter at most ϵ.

($\Leftarrow$) If A can be covered by finitely many closed sets of diameter at most ϵ then A can be also covered by a finite set of closed balls i.e. there are $x_1, ..., x_n$ such that $A \subset \bigcup_{i=1}^{n} B'_{\epsilon/2}(x_i)$ where $B'_{\epsilon/2}(x_i) = \{y \in M : d(x_i, y) \leq \epsilon/2\}$. Now let us take the set of open balls $B_{\epsilon}(x_i)$ where we know that $B'_{\epsilon/2}(x_i) \subset B_{\epsilon}(x_i)$ so A can be covered by this set too i.e. $A \subset \bigcup_{i=1}^{n} B_{\epsilon}(x_i)$. Therefore A is totally bounded. □

Proof. **5**

($\Rightarrow$) Let A be a totally bounded set and $\epsilon/2 > 0$ then there exist finitely many points $x_1, ..., x_n \in M$ such that $A \subset \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$ for each ball we have that $\overline{B_{\epsilon/2}(x_i)} \subseteq B_{\epsilon}(x_i)$. Also, since the closure is the smallest closed set that contains A. It must happen that

$$\overline{A} \subseteq \bigcup_{i=1}^{n} \overline{B_{\epsilon/2}(x_i)} \subseteq \bigcup_{i=1}^{n} B_{\epsilon}(x_i)$$

Hence, $\overline{A}$ is totally bounded too.

($\Leftarrow$) Let $\overline{A}$ be totally bounded and let $\epsilon > 0$ then there exist finitely many points $x_1, ..., x_n \in M$ such that $\overline{A} \subset \bigcup_{i=1}^{n} B_{\epsilon}(x_i)$ but since $A \subseteq \overline{A}$ we have that $A \subset \bigcup_{i=1}^{n} B_{\epsilon}(x_i)$. Therefore A is totally bounded. □

Proof. **10** Let M be a totally bounded metric space then for each $1/n > 0$ there is a set D_n with m finitely many points such that $M \subset \bigcup_{i=1}^{m} B_{1/n}(x_i)$. Now let us define $D = \bigcup_{n=1}^{\infty} D_n$ since the union of countable sets is still countable then D is a countable set too. Also, for each $x \in M$ there is some $x_j \in D$ and some $1/n > 0$ such that $x \in B_{1/n}(x_j)$ i.e. $B_{1/n}(x_j) \cap M \neq \emptyset$ which implies that for every open set U formed by an arbitrary union of open balls we have that $U \cap M \neq \emptyset$. Therefore D is a countable dense set which implies that M is separable. □

Proof. **12** Suppose (A, d) is a complete subset of (M, d) and let (x_n) be a sequence in A that converges to some $x \in M$ then (x_n) is Cauchy in (A, d) hence it converges to some point in A this implies that $x \in A$. Therefore (A, d) is closed in (M, d). □

Proof. **15** Let $f : \mathbb{R} \to (0, 1)$ such that $f(x) = \arctan(x)/\pi + 1/2$ we know that f is continuous and $\mathbb{R}$ is complete but $f(\mathbb{R}) = (0, 1)$ is not complete since we have a sequence $x_n = 1/n$ which is Cauchy and converges to $0 \notin (0, 1)$. Therefore we disproved the statement. □

Proof. **16** Let us assume $\mathbb{R}^n$ is complete under $\|\cdot\|_1$ we want to prove that $\mathbb{R}^n$ is also complete under $\|\cdot\|_\infty$ then let (x_m) be a Cauchy sequence on $\mathbb{R}^n$ under the norm $\|\cdot\|_\infty$ then we know that for some $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that when $i, j > N$ we have that $\|x_i - x_j\|_\infty < \epsilon$ also, let $x \in \mathbb{R}^n$ be the limit of (x_m) under $\|\cdot\|_1$ which we know converges since (x_m) is also a Cauchy sequence under $\|\cdot\|_1$ and $\mathbb{R}^n$ is complete under $\|\cdot\|_1$ then we have that

$$\|x_i - x_j\|_\infty = \|x_i - x + x - x_j\|_\infty \leq \|x_i - x\|_\infty + \|x - x_j\|_\infty$$

Also, we know that there is $M \in \mathbb{N}$ such that when $m > M$ we get that $\|x - x_m\|_1 < \epsilon$ but in addition we know that $\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \|\cdot\|_1 \leq n\|\cdot\|_\infty$ then we get that

$$\|x_i - x\|_\infty + \|x - x_j\|_\infty \leq \|x - x_m\|_1 + \|x - x_m\|_1 < 2\epsilon$$

Therefore since (x_m) also converges under $\|\cdot\|_\infty$ we get that $\mathbb{R}^n$ is complete under $\|\cdot\|_\infty$

In the same way, using the inequality we have, we can prove that assuming $\mathbb{R}^n$ is complete in any metric it is also complete in any of the other metrics. □

Proof. **17**

($\Rightarrow$) Let $(x_n) \subseteq M$ and $(y_n) \subseteq N$ be Cauchy sequences we want to prove that $x_n \to x$ and $y_n \to y$ for some $x \in M$ and $y \in N$. We know that $((x_n, y_n)) \subseteq M \times N$ is a Cauchy sequence from $M \times N$ and since $M \times N$ is complete this implies that $(x_n, y_n) \to (x, y)$ for some $(x, y) \in M \times N$ but this implies that $x_n \to x$ and $y_n \to y$ for $x \in M$ and $y \in N$. Therefore M and N are both complete.

($\Leftarrow$) Let $((x_n, y_n)) \subseteq M \times N$ be a Cauchy sequence we want to prove that $(x_n, y_n) \to (x, y)$ where $(x, y) \in M \times N$. We know that $(x_n) \subseteq M$ and $(y_n) \subseteq N$ are Cauchy sequences such that $x_n \to x$ and $y_n \to y$ because both M and N are complete but this implies that $(x_n, y_n) \to (x, y)$. Therefore $M \times N$ is also complete. □

Proof. **20** If (x_n) and (y_n) are Cauchy in (M,d) then for some $\epsilon/2 > 0$ we know that there is $N_x \in \mathbb{N}$ and $N_y \in \mathbb{N}$ such that when $n,m \geq N_x$ and $n',m' \geq N_y$ we have that $d(x_n,x_m) < \epsilon/2$ and $d(y_{n'},y_{m'}) < \epsilon/2$ so let us define $N = \max(N_x,N_y)$ such that when $n,m \geq N$ we have that $d(x_n,x_m) < \epsilon/2$ and $d(y_n,y_m) < \epsilon/2$.

On the other hand, we have that

$$d(x_n,y_n) \leq d(x_n,x_m) + d(x_m,y_n) \leq d(x_n,x_m) + d(x_m,y_m) + d(y_m,y_n)$$

and that

$$d(x_m,y_m) \leq d(x_m,x_n) + d(x_n,y_m) \leq d(x_m,x_n) + d(x_n,y_n) + d(y_n,y_m)$$

Hence

$$\begin{aligned} d(x_n,y_n) - d(x_m,y_m) &\leq d(x_n,x_m) + d(y_n,y_m) < \epsilon/2 + \epsilon/2 = \epsilon \\ d(x_m,y_m) - d(x_n,y_n) &\leq d(x_n,x_m) + d(y_n,y_m) < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Therefore $|d(x_n,y_n) - d(x_m,y_m)| < \epsilon$ which implies that $(d(x_n,y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. □

Proof. **21**

($\Rightarrow$) Let (x_n) and (y_n) be two Cauchy sequences with the same limit $m \in (M,d)$ then $d(x_n,m) \to 0$ and $d(y_n,m) \to 0$ also we know that

$$0 \leq d(x_n,y_n) \leq d(x_n,m) + d(y_n,m) \to 0$$

Therefore it must also happen that $d(x_n,y_n) \to 0$.

($\Leftarrow$) Let $d(x_n,y_n) \to 0$ also we know that (x_n) and (y_n) are Cauchy in (M,d) which is complete hence they converge to some $x \in M$ and $y \in M$ respectively then $d(x_n,x) \to 0$ and $d(y_n,y) \to 0$. Also, let us notice that

$$0 \leq d(x,y) \leq d(x_n,x) + d(x_n,y) \leq d(x_n,x) + d(x_n,y_n) + d(y_n,y) \to 0$$

Where $d(x_n,x) + d(x_n,y_n) + d(y_n,y) \to 0$ since every term tend to 0. Therefore $d(x,y) \to 0$ and $x = y$ which implies that (x_n) and (y_n) have the same limit. □

Proof. **31** Let $\sum_{n=1}^{\infty} x_n$ be a convergent series in a normed vector space X. Let us some $x_n \in X$ then they must preserve the triangle inequality, hence

$$\|x_n + x_{n+1}\| \leq \|x_n\| + \|x_{n+1}\|$$

this implies that

$$\left\| \sum_{n=1}^{N} x_n \right\| \leq \sum_{n=1}^{N} \|x_n\|$$

for some $N \in \mathbb{N}$. Now if we take the limit of the series on both sides as $N \to \infty$ we get that

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\|$$

$\square$

Proof. **36** Let $f(x) = x^2$ and let $0 < \delta < 1$ also suppose that $|x - p_0| < \delta$ then since $p_0 = 0$ we have that $|x| < 1$ so if we multiply both sides of the inequality by $|x|$ we get that $|x|^2 < |x|$ hence $|x^2 - 0| < |x - 0|$ and therefore $|f(x) - p_0| < |x - p_0|$.

Now we want to conclude that $f^n(x) \to p_0$ whenever $|x - p_0| < \delta$. So given $\epsilon > 0$ we take some δ such that $\delta < \epsilon$ and $0 < \delta < 1$. We know that $|f^n(x) - p_0| = |f(f^{n-1}(x)) - p_0|$ and we saw that $|f(f^{n-1}(x)) - p_0| < |f^{n-1}(x) - p_0|$ so we can continue this process n times to see that $|f^n(x) - p_0| < |x - p_0| < \delta < \epsilon$ which implies that $f^n(x) \to p_0$.

On the other hand, let $\delta = 1/2$ then if $|x - 1| < 1/2$ we get that $1/2 < x < 3/2$ hence $x + 1 > 3/2$ but also $|x + 1| > 3/2$ so we see that

$$|x^2 - 1| = |x + 1||x - 1| > \frac{3}{2}|x - 1| > |x - 1|$$

Therefore since $f(x) = x^2$ and $p_1 = 1$ we get that $|f(x) - p_1| > |x - p_1|$. $\square$

Proof. **37** Let $f : (a, b) \to (a, b)$ with a fixed point $p \in (a, b)$ where f is differentiable. If $|f'(p)| < 1$ then from the definition of $f'(p)$ we have that

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p} \frac{f(x) - p}{x - p}$$

and we know that $\left| \lim_{x \to p} \frac{f(x)-p}{x-p} \right| < 1$. Then by using the limits definition, let $\epsilon < 1 - |f'(p)|$ we know there is some $\delta > 0$ such that when $|x - p| < \delta$ we have that

$$\left| \left| \frac{f(x) - p}{x - p} \right| - |f'(p)| \right| \leq \left| \frac{f(x) - p}{x - p} - f'(p) \right| < \epsilon < 1 - |f'(p)|$$

Then we have that

$$\left| \frac{f(x) - p}{x - p} \right| \leq \left| \left| \frac{f(x) - p}{x - p} \right| - |f'(p)| \right| + |f'(p)| < 1$$

Therefore we get that $|f(x) - p| < |x - p|$ which implies that p is an attracting fixed point for f.

In the same way if $|f'(p)| > 1$ we get that $\left| \lim_{x \to p} \frac{f(x)-p}{x-p} \right| > 1$. Then by using the limits definition, let $\epsilon < |f'(p)| - 1$ we know there is some $\delta > 0$ such that when $|x - p| < \delta$ we have that

$$\left| \left| \frac{f(x) - p}{x - p} \right| - |f'(p)| \right| \leq \left| \frac{f(x) - p}{x - p} - f'(p) \right| < \epsilon < |f'(p)| - 1$$

Then we have that

$$\left| \left| \frac{f(x) - p}{x - p} \right| - |f'(p)| \right| - |f'(p)| < -1$$

so multiplying by -1 and applying to both sides of the inequality the absolute value we get that

$$1 < \left| |f'(p)| - \left| \left| \frac{f(x) - p}{x - p} \right| - |f'(p)| \right| \right|$$

Hence

$$1 < \left| |f'(p)| - \left| \left| \frac{f(x) - p}{x - p} \right| - |f'(p)| \right| \right| \leq \left| \frac{f(x) - p}{x - p} \right|$$

Therefore we get that $|f(x) - p| > |x - p|$ which implies that p is a repelling fixed point for f. $\square$

Proof. **38**

(a) Let $f(x) = \arctan x$ we know that $f'(x) = 1/(x^2+1)$ then if $x = 0$ we get that $f'(0) = 1$ also we know that

$$|f(x) - 0| = |\arctan x| < |x| = |x - 0|$$

Therefore from problem 36, we can say that 0 is an attracting fixed point for f.

(b) Let $g(x) = x^3 + x$ we know that $g'(x) = 3x^2 + 1$ then if $x = 0$ we get that $g'(0) = 1$. Now we want to prove that

$$|x^3 + x| = |g(x) - 0| > |x - 0| = |x|$$

For $x \geq 0$ we see that

$$x^3 \geq 0$$
$$x^3 + x \geq x = |x|$$

and if $x < 0$ we see that

$$x^3 < 0$$
$$x^3 + x < x = -|x|$$

Therefore $|x^3 + x| > |x|$ which implies that 0 is a repelling fixed point for g according to problem 36.

(c) Let $h(x) = x^2 + 1/4$ we know that $h'(x) = 2x$ then if $x = 1/2$ we get that $h'(1/2) = 1$. If $x \geq 1/2$ we see that

$$x^2 - 1/4 \geq x - 1/2$$

hence $|x^2 - 1/4| \geq |x - 1/2|$. On the other hand, we are interested in knowing if this is also true for $x < 1/2$. Suppose $x \in (0, 1/2)$ then we have that $|x - 1/2| = -x + 1/2$ but also in this interval we have that $|x^2 - 1/4| = -x^2 + 1/4$ and we see that

$$-x^2 + 1/4 < -x + 1/2$$

Then $|x^2 - 1/4| < |x - 1/2|$ when $x \in (0, 1/2)$. Therefore $h(x)$ is neither an attracting nor a repelling fixed point because $|h(x) - 1/2| \not< |x - 1/2|$ nor $|h(x) - 1/2| \not> |x - 1/2|$ for every $x \in \mathbb{R}$.

□

Proof. **46** Let $\hat{M}$ be the completion of M then M is dense in $\hat{M}$ which impliest that $\overline{M} = \hat{M}$. But also we know that $\overline{A} = M$ because A is dense in M hence $\overline{M} = \overline{\overline{A}} = \overline{A} = \hat{M}$ which implies that A is also dense in $\hat{M}$. Therefore since (A, d) is an isometry to A and A is dense in $\hat{M}$ we see that $\hat{M}$ is a completion to A too. □