Solved selected problems of Real Analysis - Carothers

Franco Zacco

Chapter 1 - Calculus Review

Proof. **1** Let us take the set $-A = \{-a \mid a \in A\}$ since A is bounded below then $-A$ is bounded above and because of the The Least Upper Bound Axiom we know there is $u = \sup(-A)$ such that if $x \in -A$ then $x \leq \sup(-A)$ so $-x \geq -\sup(-A)$ and $-x \in A$ then $-\sup(-A)$ is an lower bound of A.

Now let us suppose that l is a Lower Bound of A such that $-x \geq l > -\sup(-A)$ we want to arrive to a contradiction to show there is no such an l. Then $x \leq -l < \sup(-A)$ but $\sup(-A)$ is the Least Upper Bound of $-A$ and this cannot be so we have a contradiction and must be the case that if l is a Lower Bound then $l \leq -\sup(-A)$.

Therefore $-\sup(-A)$ is the Greatest Lower Bound of A. $\square$

Proof. **3** Supremum characterization.
$(\rightarrow)$

(i) If $s = \sup(A)$ then s is the Least Upper Bound for A so by definition s is an upper bound for A.

(ii) Let $\epsilon > 0$. Since s is the Least Upper Bound of A, then $s - \epsilon < s$ and $s - \epsilon$ cannot be an upper bound of A thus there exists $a \in A$ such that $s \geq a > s - \epsilon$.

$(\leftarrow)$ Now we want to show by contradiction that s is the Least Upper Bound for A. Suppose $u \neq s$ is the Least Upper Bound for A so $u = \sup(A)$ which means that if $a \in A$ then $a \leq u$ and since s is an upper bound for A then $u < s$. We also have that $a > s - \epsilon$ for every $\epsilon > 0$ so let us take $\epsilon = s - u$ then we have that $a > s - (s - u) = u$ but we said that $a \leq u$ which means that we have a contradiction. Therefore s must be the Least Upper Bound for A i.e. $s = \sup(A)$.

Infimum characterization. Let A be a nonempty set of $\mathbb{R}$ that is bounded below. We want to prove that $i = \inf(A)$ if and only if (i) i is a lower bound for A, and (ii) for every $\epsilon > 0$ there is an $a \in A$ such that $a < i + \epsilon$.
$(\rightarrow)$

(i) If $i = \inf(A)$ then i is the Greatest Lower Bound for A so by definition i is a lower bound for A.

(ii) Let $\epsilon > 0$. Since i is the Greatest Lower Bound for A, then $i + \epsilon > i$ and $i + \epsilon$ cannot be a lower bound of A thus there exists $a \in A$ such that $i \leq a < i + \epsilon$.

$(\leftarrow)$ Now we want to show by contradiction that i is the Greatest Lower Bound for A. Suppose $l \neq i$ is the Greatest Lower Bound for A so $l = \inf(A)$ which means that if $a \in A$ then $a \geq l$ and since i is an lower bound for A then $i < l$. We also have that $a < i + \epsilon$ for every $\epsilon > 0$ so let us take $\epsilon = l - i$ then we have that $a < i + (l - i) = l$ but we said that $a \geq l$ which means that we have a contradiction. Therefore i must be the Least Upper Bound for A i.e. $i = \inf(A)$. □

Proof. **6** Let the sequence (a_n) to be convergent to $a \in \mathbb{R}$, so for every positive $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Also let us notice that

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < \epsilon + |a|$$

so in summary $|a_n| < |a| + \epsilon$. Let us take then

$$M = max\{|a_1|, |a_2|, ..., |a_n|, |a| + \epsilon\}$$

so we see that $|a_n| < M$ and therefore (a_n) is bounded.

Given that (a_n) is bounded below and above then because of The Least Upper Bound Axiom and The Greatest Lower Bound Axiom we know that (a_n) has a Supremum and an Infimum.

Now we want to show by contradiction that $a \leq \sup(a_n)$. Let us suppose that $\sup(a_n) < a$ then if we take $\epsilon = a - \sup(a_n) > 0$ we have that there must be some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < a - \sup(a_n)$ so this means that $-a + \sup(a_n) < a_n - a < a - \sup(a_n)$ then $\sup(a_n) < a_n < 2a - \sup(a_n)$ but $\sup(a_n)$ is the supremum of a_n so we have a contradiction. Therefore must be the case that $a \leq \sup(a_n)$.

In the same way we want to show by contradiction that $\inf(a_n) \leq a$. Let us suppose now that $\inf(a_n) > a$ then if we take $\epsilon = \inf(a_n) - a > 0$ we have that there must be some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < \inf(a_n) - a$ so this means that $a - \inf(a_n) < a_n - a < \inf(a_n) - a$ then $2a - \inf(a_n) < a_n < \inf(a_n)$ but $\inf(a_n)$ is the Infimum of a_n so we have a contradiction. Therefore must be the case that $\inf(a_n) \leq a$. $\square$

Proof. **7** Since $b - a > 0$ we can apply Lemma 1.2 to get a positive integer q' such that $q'(b - a) > 1$ we also know that $\sqrt{2} > 1$ so $\sqrt{2}q'(b - a) > 1$ then we see that $\sqrt{2}q'b$ is bigger than $\sqrt{2}q'a$ by a value bigger than 1 so this means that there is some $p \in \mathbb{Z}$ between them, thus $\sqrt{2}q'b > p > \sqrt{2}q'a$ then it follows that $a < \sqrt{2}p/q < b$ where we used that $2q' = q$ and that $\sqrt{2} \cdot \sqrt{2} = 2$. Therefore there is some irrational number of the form $\sqrt{2}p/q$ between a and b. $\square$

Proof. **13**

($\rightarrow$) We know that (s_n) converges so let $\epsilon > 0$ it follows then that there is some $s \in \mathbb{R}$ such that when $n \geq N$ then $|s_n - s| < \epsilon$ which means that $|s_n| < |s| + \epsilon$. Let us take then $M = \max\{|s_1|, |s_2|, ..., |s_n|, |s| + \epsilon\}$, so we see that $|s_n| < M$ and since $a_n \geq 0$ then $s_n = \sum_{i=1}^{n} a_i \geq 0$ which means that $s_n < M$. Therefore (s_n) is bounded.

($\leftarrow$) Since we know now that (s_n) is bounded we want to prove by induction that it's a monotone (increasing) sequence. First we see that $a_1 \geq 0$ and $a_2 \geq 0$ then $s_1 = a_1 \leq a_1 + a_2 = s_2$.
Now let us suppose that the following expression is true

$$s_{n-1} = \sum_{i=1}^{n-1} a_i \leq \sum_{i=1}^{n} a_i = s_n$$

then since $a_{n+1} \geq 0$ we have that

$$s_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a_i + a_{n+1} = s_{n+1}$$

Therefore we showed that (s_n) is bounded and monotone it follows then that it is convergent. $\square$

Proof. **22** Let us prove first by contradiction that $\inf_n a_n \leq \liminf_{n\to\infty} a_n$. Suppose $\inf_n a_n > \liminf_{n\to\infty} a_n = \sup t_n$ then $\inf_n a_n > \sup t_n \geq t_n$ but we know that $\inf_n a_n \leq t_n$ so we have a contradiction then it must be the case that $\inf_n a_n \leq \liminf_{n\to\infty} a_n$.

Now let us prove by contradiction that $\limsup_{n\to\infty} a_n \leq \sup_n a_n$. Suppose $\inf T_n = \limsup_{n\to\infty} > \sup_n a_n$ then $T_n \geq \inf T_n > \sup_n a_n$ but we know that $T_n \leq \sup_n a_n$ so we have a contradiction then it must be the case that $\limsup_{n\to\infty} a_n \leq \sup_n a_n$.

Finally we want to prove that

$$\sup t_n = \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n = \inf T_n$$

We know that $t_n \leq T_n$ so if we take limits in both sides we have that $\lim_{n\to\infty} t_n \leq \lim_{n\to\infty} T_n$ and since a_n is bounded then we have that

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} t_n \leq \lim_{n\to\infty} T_n = \limsup_{n\to\infty} a_n$$

as we wanted.

Therefore joining the results we have that

$$\inf_n a_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup_n a_n$$

$\square$

Proof. **23** We know that (a_n) converges to some $a \in \mathbb{R}$ so let $\epsilon > 0$ then $|a_n - a| < \epsilon$ when $n \geq N$ then we have that

$$a - \epsilon < a_n < a + \epsilon$$

but this also means that

$$a - \epsilon \leq t_n \leq T_n \leq a + \epsilon$$

and therefore their limits should be between that interval too, then

$$a - \epsilon \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq a + \epsilon$$

Therefore this means that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ both converge to a or what it's the same

$$\lim \inf_{n\to\infty} a_n = \lim \sup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

□

Proof. **24** We know that $\limsup\limits_{n\to\infty} a_n = \inf\{\sup\{a_n, a_{n+1}, ...\}\}$ so this means that

$$\limsup_{n\to\infty} -a_n = \inf\{\sup\{-a_n, -a_{n+1}, ...\}\}$$

But since $\sup -A = -\inf A$ we have that

$$\limsup_{n\to\infty} -a_n = \inf\{-\inf\{a_n, a_{n+1}, ...\}\}$$

We also know that $\inf -A = -\sup A$ therefore

$$\limsup_{n\to\infty} -a_n = -\sup\{\inf\{a_n, a_{n+1}, ...\}\}$$

It follows then by definition that

$$\limsup_{n\to\infty} -a_n = -\liminf_{n\to\infty} a_n$$

□

Proof. **25** We know that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k = -\infty$ so this means that if we have an $M < 0$ then we can find an $N \in \mathbb{N}$ such that if $k \geq N$ then $\sup_{k\geq N} a_k < M$ and in particular $a_k < M$ for any $k \geq N$. Therefore $\lim_{n\to\infty} a_n = -\infty$.

Now we have that $\limsup_{n\to\infty} a_n = +\infty$ which means that $\lim_{n\to\infty} \sup_{k\geq n} a_k = +\infty$ so we have that for every n that $\sup_{k\geq n} a_k = +\infty$ in particular

$$\sup_{k\geq 1} a_k > 1$$

This means that exists $k_1 \geq 1$ such that $a_{k_1} > 1$. Now using again this fact we have that

$$\sup_{k\geq k_1+1} a_k > 2$$

Then we can find k_2 such that $k_2 > k_1$ and $a_{k_2} > 2$. Continuing this procedure we may find a set of increasing indices

$$k_1 < k_2 < ... < k_n < k_{n+1} < ...$$

such that $a_{k_n} > n$ therefore we have an unbounded increasing subsequence (a_{k_n}) that diverge to $+\infty$.

Now let's see what happens when $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq N} a_n = +\infty$ this means that if we have an $M > 0$ then we can find an $N \in \mathbb{N}$ such that if $k \geq N$ then $\inf_{k\geq N} a_n > M$ and in particular $a_k > M$ for any $k \geq N$. Therefore $\lim_{n\to\infty} a_n = +\infty$.

In the same way as above if $\liminf_{n\to\infty} a_n = -\infty$ this means that $\lim_{n\to\infty} \inf_{k\geq n} a_k = -\infty$ so we have that for every n that $\inf_{k\geq n} a_k = -\infty$ in particular

$$\inf_{k\geq 1} a_k < -1$$

This means that exists $k_1 \geq 1$ such that $a_{k_1} < -1$. Now using again this fact we have that

$$\inf_{k\geq k_1+1} a_k < -2$$

Then we can find k_2 such that $k_2 > k_1$ and $a_{k_2} < -2$. Continuing this procedure we may find a set of increasing indices

$$k_1 < k_2 < ... < k_n < k_{n+1} < ...$$

such that $a_{k_n} < -n$ therefore we have an unbounded decreasing subsequence (a_{k_n}) that diverge to $-\infty$.

$\square$

Proof. **26** ($\rightarrow$) We know that $M = \limsup_{n\to\infty} a_n$ which means that for some $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|\sup_{k\geq n} a_k - M| < \epsilon$$

or in other words

$$M - \epsilon < \sup_{k\geq n} a_k < M + \epsilon$$

We also know that $a_n \leq \sup_{k\geq n} a_k < M + \epsilon$ so $a_n < M + \epsilon$ but this only works for $n \geq N$ then this is only valid for all but finitely many $n < N$.

Now let us suppose that $M - \epsilon > a_n$ we want to show by contradiction that this cannot be true for all $n \geq N$ then in that case the $\sup_{k\geq n} a_k \leq M - \epsilon$ but we saw by the definition of the lim sup that $\sup_{k\geq n} a_k > M - \epsilon$ so we have a contradiction and if $M - \epsilon > a_n$ happen then it cannot happen for all $n \geq N$ therefore $M - \epsilon < a_n$ happen for infinitely many n.

($\leftarrow$) Now we know that M satisfies that for every $\epsilon > 0$, we have $a_n < M + \epsilon$ for all but finitely many n, and $M - \epsilon < a_n$ for infinitely many n. Since (a_n) is bounded $M + \epsilon$ is an upper bound for (a_n) and since this is true for all but finitely many n, let us take $N \in \mathbb{N}$ such that if $n \geq N$ we have that $a_n < M + \epsilon$ then given that (a_n) is bounded (a_n) has a supremum and $\sup_{k\geq n} a_k \leq M + \epsilon$.

Also since $M - \epsilon < a_n$ for infinitely many n, then $M - \epsilon < \sup_{k\geq n} a_k$ is true. Therefore $M = \limsup_{n\to\infty} a_n$

Let us now caracterize $m = \liminf_{n\to\infty} a_n$ as

$$\begin{cases} \text{for every } \epsilon > 0 \text{ we have } a_n < m + \epsilon \text{ for infinitely many } n \\ \text{and } m - \epsilon < a_n \text{ for all but finitely many } n \end{cases}$$

($\rightarrow$) We know that $m = \liminf_{n\to\infty} a_n$ which means that for some $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|\inf_{k\geq n} a_k - m| < \epsilon$$

or in other words

$$m - \epsilon < \inf_{k\geq n} a_k < m + \epsilon$$

We also know that $m - \epsilon < \inf_{k\geq n} a_k \leq a_n$ so $m - \epsilon < a_n$ but this only works for $n \geq N$ then this is only valid for all but finitely many $n < N$.

Now let us suppose that $a_n > m+\epsilon$ we want to show by contradiction that this cannot be true for all $n \geq N$ then if that were true the $\inf_{k\geq n} a_k \geq m+\epsilon$ but we saw by the definition of the lim inf that $\inf_{k\geq n} a_k < m+\epsilon$ so we have a contradiction and if $a_n > m + \epsilon$ happen then it cannot happen for all $n \geq N$ therefore $a_n < m + \epsilon$ happen for infinitely many n.

($\leftarrow$) Now we know that m satisfies that for every $\epsilon > 0$, we have $a_n < m + \epsilon$ for infinitely many n, and $m - \epsilon < a_n$ for all but finitely many n.

Since (a_n) is bounded $m - \epsilon$ is a lower bound for (a_n) and since this is true for all but finitely many n, let us take $N \in \mathbb{N}$ such that if $n \geq N$ we have that $m - \epsilon < a_n$ then given that (a_n) is bounded (a_n) has a infimum and $m - \epsilon \leq \inf_{k \geq n} a_k$.

Also since $a_n < m + \epsilon$ for infinitely many n, then $\inf_{k \geq n} a_k < m - \epsilon$ is true. Therefore $m = \liminf_{n \to \infty} a_n$

□

Proof. **27** The case where $M = \limsup_{n \to \infty} a_n = \pm\infty$ was handled in Excercise 25 so we will focus on the case where $M = \limsup_{n \to \infty} a_n \neq \pm\infty$.

Let $\epsilon = 1$. Since $M = \limsup_{n \to \infty} a_n$ is caracterized by $(*)$ then there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $a_n < M + 1$ but we also have that $M - 1 < a_n$ for infinitely many n, so we can choose $n_1 \in \mathbb{N}$ such that $n_1 \geq N$ where both inequalities are satisfied.

Similarly we can choose $n_2 > n_1 \geq N$ such that $a_{n_2} < M + \frac{1}{2}$ and $M + \frac{1}{2} < a_{n_2}$ for inifinitely many n. Then following this procedure we can find a subsequence (a_{n_k}) such that $|a_{n_k} - M| < \frac{1}{k}$ which implies that (a_{n_k}) converge to M.

Let us now show that there is also a subsequence that converge to $\liminf_{n \to \infty} a_n$. We saw that is true in Excercise 25 for the case where $\liminf_{n \to \infty} a_n = \pm\infty$.

In the case that $m = \liminf_{n \to \infty} a_n \neq \pm\infty$ since $\liminf$ is caracterized by an analogous statement given for $\limsup$ then the proof follows the same structure as above. □

Proof. **30** If $a_n \leq b_n$ then $\inf a_n \leq \inf b_n$ and so

$$\inf_{k \geq n} a_k \leq \inf_{k \geq n} b_k$$

by applying the limit then we have that

$$\lim_{n \to \infty} \inf_{k \geq n} a_n \leq \lim_{n \to \infty} \inf_{k \geq n} b_k$$

but we know that (a_n) converge then because of Excercise 23 we have that $\lim_{n \to \infty} a_n = \liminf_{n \to \infty} a_n$, therefore

$$\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} \inf_{k \geq n} b_k$$

□

Proof. **32** Let us prove first the case where $\limsup_{n\to\infty} a_n = +\infty$ then because of Exercise 25 we have that (a_n) has a subsequence (a_{k_n}) that diverge to $+\infty$ then

$$\sup S = \lim_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_n = +\infty$$

Now in the case where $\limsup_{n\to\infty} a_n = -\infty$ because of Exercise 25 we have that (a_n) diverge to $-\infty$ too so every subsequence must diverge to $-\infty$ then

$$\sup S = \limsup_{n\to\infty} a_n = -\infty$$

Finally, suppose $\limsup_{n\to\infty} a_n \neq \pm\infty$ if we take any subsequence (a_{k_n}) we know that $a_{k_n} \leq \sup a_{k_n} \leq \sup a_n$ and then

$$a_{k_n} \leq \sup_{m\geq n} a_m$$

so if we apply the limit we have that

$$\lim_{n\to\infty} a_{k_n} \leq \lim_{n\to\infty} \sup_{m\geq n} a_n$$

Which implies that the limit of any subsequence of (a_n) is less or equal to $\limsup_{n\to\infty} a_n$ therefore $\sup S = \limsup_{n\to\infty} a_n$.

Let's see what happens with the infimum. Let us prove first the case where $\liminf_{n\to\infty} a_n = -\infty$ then because of Exercise 25 we have that (a_n) has a subsequence (a_{k_n}) that diverge to $-\infty$ then

$$\inf S = \lim_{n\to\infty} a_{k_n} = \liminf_{n\to\infty} a_n = -\infty$$

Now in the case where $\liminf_{n\to\infty} a_n = +\infty$ because of Exercise 25 we have that (a_n) diverge to $+\infty$ too so every subsequence must diverge to $+\infty$ then

$$\inf S = \liminf_{n\to\infty} a_n = +\infty$$

Finally, suppose $\liminf_{n\to\infty} a_n \neq \pm\infty$ if we take any subsequence (a_{k_n}) we know that $\inf a_n \leq \inf a_{k_n} \leq a_{k_n}$ and then

$$\inf_{m\geq n} a_m \leq a_{k_n}$$

so if we apply the limit we have that

$$\lim_{n\to\infty} \inf_{m\geq n} a_m \leq \lim_{n\to\infty} a_{k_n}$$

Which implies that the limit of any subsequence of (a_n) is bigger or equal to $\liminf_{n\to\infty} a_n$ therefore $\inf S = \liminf_{n\to\infty} a_n$. $\square$

Proof. **40**

- $(i) \to (ii)$ We know that $\lim_{x\to a} f(x) = L$ so by the $\epsilon - \delta$ definition we have that for every $\epsilon > 0$ there is some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever x satisfies $0 < |x - a| < \delta$.

  But also we know that $\lim_{n\to\infty} x_n = a$ then by the definition of the sequence limits we know that we can find some $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - a| < \epsilon'$ for any $\epsilon' > 0$ so if we take $\epsilon' = \delta$ then we have that $0 < |x_n - a| < \delta$ because we also know that $x_n \neq a$. Therefore it must happen that $|f(x_n) - L| < \epsilon$ for every $\epsilon > 0$.

- $(ii) \to (iii)$ Let $y_n = f(x_n)$. From part $(i) \to (ii)$ we know we can find $N \in \mathbb{N}$ so if $n \geq N$ then $0 < |x_n - a| < \delta$ and because of that $|y_n - L| < \epsilon$ therefore y_n converges to some limit L.

- $(iii) \to (ii)$ If $(f(x_n))$ converges to something when $x_n \to a$ then there is some $N \in \mathbb{N}$ and when $n \geq N$ we have that $0 < |x_n - a| < \epsilon'$ but this also means that $|f(x_n) - L| < \epsilon$ when $n \geq N$ because $(f(x_n))$ converges to something, but this means that there exists some number L such that $f(x_n) \to L$ whenever $x_n \to a$ where $x_n \neq a$.

  So we know that $f(x_n) \to L$ when $x_n \to a$ but we want to check that this is true for any sequence we choose. Suppose we take another sequence (y_n) and then we build (z_n) as $z_n = x_1, y_1, x_2, y_2, ...$ Now let us suppose that $f(z_n) \to M$ then $f(y_n)$ and $f(x_n)$ are subsequences of $f(z_n)$ but we know that if a sequence converge to some number then all its subsequences must converge to the same number so $f(x_n) \to M$ and $f(y_n) \to M$ as we wanted.

- $(ii) \to (i)$ Let us suppose that $\lim_{x\to a} f(x) \neq L$ we want to arrive to a contradiction. Then for some $\epsilon > 0$ there isn't a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever x satisfies $0 < |x - a| < \delta$.

  But from part $(iii) \to (ii)$ we know that when $0 < |x_n - a| < \epsilon' = \delta$ then $f(x_n) \to L$ so $|f(x_n) - L| < \epsilon$ therefore we found a set of values of x such that if $0 < |x_n - a| < \epsilon' = \delta$ then $|f(x_n) - L| < \epsilon$ so we have a contradiction and $\lim_{x\to a} f(x) = L$ must be true.

$\square$

Proof. **44** Since f is increasing and bounded then we can construct a sequence $f(n)$ where $n \in \mathbb{N}$ such that $(f(n))$ is convergent because it's increasing and bounded. This means that there is $N \in \mathbb{N}$ such that when $n \geq N$ then $|f(n) - L| < \epsilon$ moreover since $(f(n))$ is bounded and increasing we can write that $f(n) \leq L$. Let $x \geq N$ then since f is increasing we can write that $f(n) \leq f(x)$ therefore $|f(x) - L| < \epsilon$ for any $x \geq N$.

In the case of $lim_{x \to -\infty} f(x)$ we can take a sequence $(f(-n))$ which is going to be a decreasing and bounded sequence and therefore convergent. This means that there is $N \in \mathbb{N}$ such that when $n \geq N$ then $|f(-n) - L'| < \epsilon$ moreover since $(f(-n))$ is bounded and decreasing we can write that $f(-n) \geq L'$. Let $x \leq -N$ since f is increasing we can write that $f(-n) \geq f(x)$ therefore $|f(x) - L| < \epsilon$ for any $x \leq -N$. □

Proof. **45** Suppose $c \in [a, b]$ and $c \notin \mathbb{Q}$ such that $f(c) \neq 0$. We want to arrive to a contradiction where we show that $f(c)$ must be equal to 0. Since f is continuous at c let $\epsilon = |f(c)|/2$ then there is some $\delta > 0$ such that when $|x - c| < \delta$ we have that $|f(x) - f(c)| < \epsilon$ so if x is rational the expression becomes $| - f(c)| < \epsilon$ but we said that $\epsilon = |f(c)|/2$ and $| - f(c)| \not< |f(c)|/2$ therefore we have a contradiction and must be true that $f(c) = 0$. □

Proof. **46**

(a) Since $f(0) > 0$ we can find an $\epsilon > 0$ such that $f(0) - \epsilon > 0$ (for example $\epsilon = \frac{f(0)}{2}$). Now from the definition of continuity let us choose $\epsilon > 0$ such that $f(0) - \epsilon > 0$ then there is some $\delta > 0$ such that when $|x - 0| < \delta$ we have that $|f(x) - f(0)| < \epsilon$ then $f(0) - \epsilon < f(x) < f(0) + \epsilon$ so $0 < f(0) - \epsilon < f(x)$.

(b) Let $a \in \mathbb{R}$ and $a \notin \mathbb{Q}$ and let us suppose that $f(a) < 0$ we want to arrive at a contradiction where we show this cannot be true. We know from the analogous of part (a) that there is some interval, let's say $(a - \delta, a + \delta)$ where if $x \in (a - \delta, a + \delta)$ then $f(x) < 0$ but also in this interval there is some rational b for which $f(b) \geq 0$ then we have a contradiction and therefore must be true that $f(a) \geq 0$.

This result doesn't hold if we change from ≥ 0 to > 0. Take for example $f(x) = |\pi - x|$ then $f(x) > 0$ for all rationals but if $x = \pi$ then $f(\pi) = 0$.

□