Solved selected problems of Real Analysis - Carothers

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Chapter 5 - Continuous Functions

Proof. **7**

(a) Let $(a, \infty) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(a, \infty)] = \{x : x \in M \text{ and } f(x) > a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv).

In the same way, let $(-\infty, a) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(-\infty, a)] = \{x : x \in M \text{ and } f(x) < a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv).

(b) We proved the more general result in part (c) which also applies in this case.

(c) Let V be an open set of $\mathbb{R}$ then since the collection of open intervals with rational endpoints is a base for $\mathbb{R}$ we can write V as

$$V = \bigcup_{\alpha} (p_\alpha, q_\alpha)$$

where $p_\alpha, q_\alpha \in \mathbb{Q}$ so we have that

$$f^{-1}[V] = \bigcup_{\alpha} f^{-1}[(p_\alpha, q_\alpha)]$$

then we can write that

$$f^{-1}[(p_\alpha, q_\alpha)] = f^{-1}[(p_\alpha, \infty)] \cap f^{-1}[(-\infty, q_\alpha)]$$

Also, we know that

$$f^{-1}[(p_\alpha, \infty)] = \{x : f(x) > p_\alpha\} \text{ and } f^{-1}[(-\infty, q_\alpha)] = \{x : f(x) < q_\alpha\}$$

and we know both of them are open sets so $f^{-1}[(p_\alpha, q_\alpha)]$ is the intersection of a finite number of open sets then it is also an open set. Finally, since $f^{-1}[V]$ is the union of open sets it's also an open set. Therefore f is continuous.

$\square$

Proof. **10** Let $\epsilon > 0$ and let us take $\delta = 1$ no matter the value of ϵ then

$$B_\delta(2) = \{x \in A : d(2, x) < 1\} = \{2\}$$

So we have that $f(B_\delta(2)) = \{f(2)\}$ and certainly it must happen that $\{f(2)\} \subset B_\epsilon(f(2))$ because $f(2) \in B_\epsilon(f(2))$. Therefore f is continuous at 2. $\square$

Proof. **11**

(a) Let $x \in A \cup B$, then $x \in A$, $x \in B$ or both of them, also let $\epsilon > 0$ then we know there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ because f is continuous at x by the definition.

(b) Let $A = (0, 1)$ and $B = [1, 2)$ also let $f : A \to \mathbb{R}$ be defined as $f(x) = x$ and $f : B \to \mathbb{R}$ as $f(x) = x + 1$ then we see that $f : A \cup B \to \mathbb{R}$ is not continuous at $x = 1$. Therefore the statement is false.

$\square$

Proof. **14** Given that a continuous function on $\mathbb{R}$ is completely determined by its values on $\mathbb{Q}$. For each $q \in \mathbb{Q}$ we have that $f(q)$ has a cardinality of $\mathfrak{c}$ since for each real number $x \in \mathbb{R}$ we can find an f such that $x = f(q)$. So the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ has a cardinality of $\mathfrak{c}^{|\mathbb{Q}|}$ and doing some cardinality algebra we get that

$$\mathfrak{c}^{|\mathbb{Q}|} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$$

Therefore there are $\mathfrak{c}$ continuous function $f : \mathbb{R} \to \mathbb{R}$. $\square$

Proof. **17** Let $x \in M$, and let us also define $(x_n) \subset D$ such that $x_n \to x$ which we know it exists because D is dense. Then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ also we know that $f(x_n) = g(x_n)$ for every $x_n \in (x_n)$. Finally, since sequences have unique limits it must happen that $f(x) = g(x)$ as we wanted to show.

In the same way, suppose we define $(x_n) \subset D$ such that $x_n \to x$ where $x \in M$. Then $f(x_n) \to f(x)$ also $(f(x_n)) \subset f(D)$. So we have a sequence $(f(x_n))$ for any $f(x)$ and we know that every $y \in N$ has the form $y = f(x)$ because f is onto. Therefore $f(D)$ is dense in N. $\square$

Proof. **22** Let $n, m \in \mathbb{N}$ we want to show that $d(E(n), E(m)) = d(n, m)$ this means that $\|E(n) - E(m)\|_1 = |n - m|$. Let us suppose that $n > m$ then $n = m + b$ where $b \in \mathbb{N}$ and so $|n - m| = b$. Then we have that

$$\|E(n) - E(m)\|_1 = \sum_{i=1}^{\infty} |E_i(n) - E_i(m)|$$

where $E_i(n)$ is the value of the ith element in the sequence, the same for $E_i(m)$. If $i \in \{1, 2, ..., m\}$ we have that $|E_i(n) - E_i(m)| = |1 - 1| = 0$ and for $i \in \{n + 1, n + 2, ...\}$ we have that $|E_i(n) - E_i(m)| = |0 - 0| = 0$ so we can write the following

$$\|E(n) - E(m)\|_1 = \sum_{i=m}^{n} |E_i(n) - 0| = \sum_{i=m}^{n} 1 = n - m = b$$

Therefore $\|E(n) - E(m)\|_1 = |n - m|$ as we wanted, in the case of $m \geq n$ the proof is analogous because we are taking the absolute value inside the sum. □

Proof. **23** Let $S : c_0 \to c_0$ be defined as $S(x_0, x_1, ...) = (0, x_0, x_1, ...)$ such that S shifts the entries forward and puts 0 in the empty slot. We want to prove that $d(S(x), S(y)) = d(x, y)$ where $x = (x_0, x_1, ...)$ and $y = (y_0, y_1, ...)$ then

$$\|x - y\|_\infty = \sup\{|x_0 - y_0|, |x_1 - y_1|, ...\}$$

And $\sup\{|x_0 - y_0|, |x_1 - y_1|, ...\} \geq 0$ because both x and y are sequences that tend to 0. Let us suppose that $\sup_n |x_n - y_n| = 0$ then we see that

$$\|S(x) - S(y)\|_\infty = \sup\{|0 - 0|, |x_0 - y_0|, |x_1 - y_1|, ...\} = 0$$

Now let us suppose that $\|x - y\|_\infty > 0$ then we have that

$$\sup\{|x_0 - y_0|, |x_1 - y_1|, ...\} = \sup\{|0 - 0|, |x_0 - y_0|, |x_1 - y_1|, ...\}$$

because $\|S(x) - S(y)\|_\infty$ cannot be 0. Therefore we have that

$$\|S(x) - S(y)\|_\infty = \|x - y\|_\infty$$

as we wanted.

□

Proof. **24** Let $f : \mathbb{R} \to V$ such that for each $\alpha \in \mathbb{R}$ we map it to $\alpha y \in V$ where $y \in V$. Let $\alpha, \beta \in \mathbb{R}$ we want to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(\alpha) - f(\beta)\| < \epsilon$ whenever $|\alpha - \beta| < \delta$. So let us define $\delta = \epsilon/\|y\|$ if $\|y\| \neq 0$ then we have that when $|\alpha - \beta| < \delta$ we get that

$$\begin{aligned} |\alpha - \beta| &< \frac{\epsilon}{\|y\|} \\ |\alpha - \beta|\|y\| &< \epsilon \\ \|\alpha y - \beta y\| &< \epsilon \\ \|f(\alpha) - f(\beta)\| &< \epsilon \end{aligned}$$

Therefore f is continuous.

Let now $f : V \to V$ such that for each $x \in V$ we map it to $x + y \in V$ where $y \in V$. Let $x, x' \in V$ we want to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(x) - f(x')\| < \epsilon$ whenever $\|x - x'\| < \delta$. So let us define $\delta = \epsilon$ then we have that when $\|x - x'\| < \delta$ we get that

$$\|x - x'\| = \|(x + y) - (x' + y)\| = \|f(x) - f(x')\| < \epsilon$$

Therefore f is continuous. □

Proof. **25** Let $f : (M, d) \to (N, \rho)$ be a Lipschitz mapping so there is $K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$ also let's observe that $K \geq 0$ since both $\rho(f(x), f(y)) \geq 0$ and $d(x, y) \geq 0$

Let now $\epsilon > 0$ and $\delta > 0$, we want to prove that if $d(x, y) < \delta$ then $\rho(f(x), f(y)) < \epsilon$ i.e. f is continuous. So let $\delta = \epsilon/K$ with $K > 0$ then we have that if $d(x, y) < \epsilon/K$ then $Kd(x, y) < \epsilon$ and since $\rho(f(x), f(y)) \leq Kd(x, y)$ we get that $\rho(f(x), f(y)) < \epsilon$ as we wanted. If $K = 0$ then it must happen that $\rho(f(x), f(y)) = 0$ and then $\rho(f(x), f(y)) = 0 < \epsilon$ which by definition is true. Therefore f is continuous. □

Proof. **27** Let $k \geq 1$ and $f : l_\infty \to \mathbb{R}$ defined as $f(x) = x_k$ we want to prove that f is continuous but we will prove that f is Lipschitz which implies that f is continuous.

Let $x, y \in l_\infty$ then we have that for some fixed $k \geq 1$ the following is always true because of the definition of supremum

$$|f(x) - f(y)| = |x_k - y_k| \leq \sup_n |x_n - y_n| = \|x - y\|_\infty$$

then for $K = 1$ we see that

$$|f(x) - f(y)| \leq K\|x - y\|_\infty$$

Therefore f is Lipschitz and hence continuous. □

Proof. **31**

(a) Let $x \in M$ since $M = \cup_{n=1}^{\infty} U_n$ then x is in some U_n we want to show that for every sequence $(x_n) \subset M$ such that $x_n \to x$ we have that $f(x_n) \to f(x)$. For every $(x_n) \subset U_n$ such that $x_n \to x$ since f is continuous in U_n then we have that $f(x_n) \to f(x)$. But if (x_n) is not completely in U_n then since U_n is an open set it must happen that eventually for some n onwards $x_n \in U_n$ and since f is continuous in U_n we have that $f(x_n) \to f(x)$. Therefore f is also continuous in M.

(b) Let $F \in N$ be a closed set then we have that

$$f^{-1}(F) \cap M = f^{-1}(F) \cap \bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} f^{-1}(F) \cap E_n$$

We know that each $f^{-1}(F) \cap E_n$ is closed in E_n because f is continuous in E_n and we know that the finite union of closed sets is closed then $f^{-1}(F) \cap M$ is closed and therefore M is continuous.

(c) Let $E_n = [\frac{1}{n}, 1]$ we see that $\cup_{n=1}^{\infty} E_n = (0, 1]$ which is not a closed set. Therefore even though f is continuous in every E_n we based part of our proof on the fact that the finite union of closed sets is closed which is not the case here.

□

Proof. **34** Let $(M \times M, \rho)$ be a metric space where ρ is defined as

$$\rho((a, b), (c, d)) = d(a, c) + d(b, d)$$

and d is a metric on M. We want to prove that if $(x_n, y_n) \to (x, y)$ then $d(x_n, y_n) \to d(x, y)$ i.e. that d is continuous.

If $(x_n, y_n) \to (x, y)$ then this means that $x_n \to x$ and $y_n \to y$ then $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$ which implies that

$$\rho((x_n, y_n), (x, y)) = d(x_n, x) + d(y_n, y) < \epsilon$$

for some $\epsilon > 0$. Also, we have that

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \quad \text{and} \quad d(x, y_n) \leq d(x, y) + d(y, y_n)$$

so joining these inequalities we get that

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y) < \epsilon$$

On the other hand, we also have that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \quad \text{and} \quad d(x_n, y) \leq d(x_n, y_n) + d(y_n, y)$$

then

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y) < \epsilon$$

hence $|d(x, y) - d(x_n, y_n)| < \epsilon$. Therefore $d(x_n, y_n) \to d(x, y)$ as we wanted.

□

Proof. **43**

($\rightarrow$) We want to prove that $i : (M, d) \rightarrow (M, \rho)$ (the identity map) is a homeomorphism from (M, d) to (M, ρ).

Let us check first that i is a one-to-one map. Let $i(a), i(b) \in (M, \rho)$ such that $i(a) = i(b)$ then by definition $a = i(a) = i(b) = b$. Then i is a one-to-one map.

Let us also check that i is an onto map too. Let us take $a \in (M, \rho)$ then by definition we have $a \in (M, d)$ such that $i(a) = a \in (M, \rho)$. Then i is an onto map.

Also, we want to prove that i is continuous. Let $x \in (M, d)$ then if $d(x_n, x) \rightarrow 0$ (i.e. $x_n \rightarrow x$) for any $(x_n) \subset M$ we have that $\rho(x_n, x) \rightarrow 0$ because d and ρ are equivalent metrics. Therefore since i maps both x_n and x to themselves this implies that $i(x_n) \rightarrow i(x)$ where $i(x_n), i(x) \in (M, \rho)$.

Finally, we want to prove that also i^{-1} is continuous. In the same way let $x \in (M, \rho)$ if $\rho(x_n, x) \rightarrow 0$ (i.e. $x_n \rightarrow x$) for any $(x_n) \subset (M, \rho)$ we have that $d(x_n, x) \rightarrow 0$ because d and ρ are equivalent metrics. Therefore since i^{-1} maps both x_n and x to themselves this implies that $i^{-1}(x_n) \rightarrow i^{-1}(x)$ where $i^{-1}(x_n), i^{-1}(x) \in (M, d)$.

Therefore i is an homeomorphism from (M, d) to (M, ρ).

($\leftarrow$) We want to prove that d and ρ are equivalent metrics on M knowing that i (the identity map) is a homeomorphism from (M, d) to (M, ρ).

Let $x \in (M, d)$ and $(x_n) \subset (M, d)$ then if $d(x_n, x) \rightarrow 0$ we have that $\rho(i(x_n), i(x)) \rightarrow 0$ because i is continuous, but by definition, this also implies that $\rho(x_n, x) \rightarrow 0$.

In the same way, let $x \in (M, \rho)$ and $(x_n) \subset (M, \rho)$ then if $\rho(x_n, x) \rightarrow 0$ we have that $d(i^{-1}(x_n), i^{-1}(x)) \rightarrow 0$ because i^{-1} is continuous, but by definition, this also implies that $d(x_n, x) \rightarrow 0$.

Therefore d and ρ are equivalent metrics. $\square$

Proof. **44** We want to prove that "is homeomorphic to" is an equivalence relation, so we will prove it is a reflexive, symmetric and transitive relation.

Reflexivity: Let (M, d) be a metric space, we want to prove that (M, d) is homeomorphic to itself. Let i be the identity map, we saw in problem 43 that i is a homeomorphism from (M, d) to (M, d), then (M, d) is homeomorphic to itself.

Symmetry: Let (M, d) be homeomorphic to (N, ρ) then there is a relation $f : M \to N$ which is one-to-one and onto such that f and f^{-1} are continuous. Then we can define $g : N \to M$ such that $g = f^{-1}$ which is one-to-one and onto because f is bijective. Also, g is continuous because f^{-1} is continuous and $g^{-1} = (f^{-1})^{-1} = f$ is continuous because f is continuous. Therefore (N, ρ) is homeomorphic to (M, d).

Transitivity: Let (M, d) be homeomorphic to (N, ρ) and let (N, ρ) be homeomorphic to (L, τ). We can define $h : M \to L$ as $h = g \circ f$ where $f : M \to N$ and $g : N \to L$ and they are the respective homeomorphisms. Since f and g are one-to-one and onto then h is also one-to-one and onto. Also, we see that $h^{-1} = (g \circ f)^{-1} = g^{-1} \circ f^{-1}$ and by the properties of the composition of functions we see that if f, f^{-1}, g and g^{-1} are continuous then both $h = g \circ f$ and $h^{-1} = g^{-1} \circ f^{-1}$ are also continuous. Therefore (M, d) is homeomorphic to (L, τ).

Finally, since the relation "is homeomorphic to" is reflective, symmetric and transitive then it is an equivalence relation between metric spaces. □

Proof. **45** Let $M = \{1/n : n \geq 1\}$ and $f : \mathbb{N} \to M$ such that $f(n) = 1/n$.

Let us prove first that f is one-to-one then let $n, m \in \mathbb{N}$ such that $f(n) = f(m)$ then $1/n = 1/m$ i.e. $n = m$ so f is one-to-one.

Let us prove now that f is an onto map. Let us take $a \in M$ then a has the form of $a = 1/b$ where $b \in \mathbb{N}$ then there is always a $b \in \mathbb{N}$ such that $f(b) = 1/b = a$ i.e. f is an onto map.

Now we want to prove that f is continuous. Let $f^{-1} : M \to \mathbb{N}$ defined as $f^{-1}(1/n) = n$. Since $\mathbb{N}$ and M are discrete then a subset $V \subset M$ is open and because f^{-1} is bijective then $f^{-1}(V) \subset \mathbb{N}$ is also open. Therefore f is continuous.

In the same way, we want to prove that f^{-1} defined as we said is also continuous. Since $\mathbb{N}$ and M are discrete then a subset $V' \subset \mathbb{N}$ is open and because f is bijective $(f^{-1})^{-1}(V') = f(V') \subset M$ is also open. Therefore f^{-1} is continuous.

So taking into account all these results we conclude that $\mathbb{N}$ is homeomorphic to M. □

Proof. **48** We want to prove first that $\mathbb{R}$ is homeomorphic to $(0,1)$ this can be accomplished if we define $f : \mathbb{R} \to (0,1)$ such that $f(x) = \arctan(x)/\pi + 1/2$ since this map is bijective and continuous also $f^{-1}(x) = \tan(\pi(x - 1/2))$ is continuous over $(0,1)$.

Now we want to prove that $(0,1)$ is homeomorphic to $(0,\infty)$ but we will prove first that $\mathbb{R}$ is homeomorphic to $(0,\infty)$ so we define $f : \mathbb{R} \to (0,\infty)$ such that $f(x) = e^x$ we see that f is bijective and continuous also $f^{-1}(x) = \log(x)$ is continuous over $(0,\infty)$. So by composing these results, we have that a map $g : (0,1) \to (0,\infty)$ such that $g(x) = e^{\tan(\pi(x-1/2))}$ is an homeomorphism between $(0,1)$ and $(0,\infty)$ as we wanted.

Let $x = 0$ and $y = 2$ then $|0 - 2| = 2$ but $|f(0) - f(2)|$ is at most close to 1. Therefore $\mathbb{R}$ is not isometric to $(0,1)$.

Let $f : \mathbb{R} \to (0,\infty)$ be an isometry from $\mathbb{R}$ to $(0,\infty)$ we want to arrive at a contradiction. Let $x \in \mathbb{R}$ and suppose $|x - 0| = |x| = |f(x) - f(0)|$ then we have that $f(x) = f(0) + x$ or $f(x) = f(0) - x$. Let us take $a \notin (0,\infty)$ then $f(a - f(0)) = f(0) + a - f(0) = a$ or $f(a - f(0)) = f(0) - a + f(0) = 2f(0) - a$ where we see that the first one cannot happen (otherwise $a \in (0,\infty)$). Also, let us consider that $f(f(0) - a) = 2f(0) - a$ or $f(f(0) - a) = a$ must be true where again the last case cannot happen, but also since f is injective we have that $f(0) - a = a - f(0)$ which implies that $f(0) = a$ hence $a \in (0,\infty)$ a contradiction. Therefore $\mathbb{R}$ is not isometric to $(0,\infty)$. $\square$

Proof. **49** Let $y \in V$ and $f : V \to V$ which is defined as $f(x) = x + y$ we want to show that f is an isometry on V. Let $x, z \in V$ since V is a normed vector space then we have that

$$\|f(x) - f(z)\| = \|x + y - (z + y)\| = \|x - z\|$$

Therefore f is an isometry on V.

Now let $\alpha \in \mathbb{R}$ and $g : V \to V$ which is defined as $g(x) = \alpha x$ we want to prove that g is a homeomorphism on V.

Let $x, z \in V$ and suppose $g(x) = g(z)$ then $x = z$ hence g is a one-to-one map. Also, let $v \in V$ such that v is in the image of g then by definition there must be an $x \in V$ such that $v = \alpha x$ so g is an onto map too.

Let us show now that g is continuous. Let $\epsilon > 0$ and $x, z \in V$ then let us take $\delta = \epsilon/|\alpha|$ so when $\|x - z\| < \delta$ we get that

$$\begin{aligned} \|x - z\| &< \frac{\epsilon}{|\alpha|} \\ |\alpha|\|x - z\| &< \epsilon \\ \|\alpha x - \alpha z\| &< \epsilon \\ \|g(x) - g(z)\| &< \epsilon \end{aligned}$$

Therefore g is continuous.

Finally, we want to show that $g^{-1}(x) = x/\alpha$ (which we can define this way because α is nonzero) is continuous. Let $\epsilon > 0$ and let $x, z \in V$ (the image) then let us take $\delta = \epsilon|\alpha|$ so when $\|x - z\| < \delta$ we have that

$$\begin{aligned} \|x - z\| &< \epsilon|\alpha| \\ \left|\frac{1}{\alpha}\right| \|x - z\| &< \epsilon \\ \left\|\frac{x}{\alpha} - \frac{z}{\alpha}\right\| &< \epsilon \\ \|g^{-1}(x) - g^{-1}(z)\| &< \epsilon \end{aligned}$$

Therefore g^{-1} is continuous.

Joining all these results we see that g is a homeomorphism on V. □

Proof. **50** Let us define a map $f : (M, d) \to (M, \rho)$ such that

$$f(m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 1 \\ 1/n & \text{if } m = 1/n \text{ where } n \geq 2 \end{cases}$$

Let $a, b \in (M, d)$ then

(i) If $a = b = 0$ we have that

$$d(a, b) = |a - b| = 0 = \rho(1, 1) = \rho(f(a), f(b))$$

(ii) If $a = b = 1$ we have that

$$d(a, b) = |a - b| = 0 = \rho(0, 0) = \rho(f(a), f(b))$$

(iii) If $a = 1/n$ and $b = 1/n'$ where $n, n' \geq 2$ we have that

$$d(a, b) = |1/n - 1/n'| = \rho(1/n, 1/n') = \rho(f(a), f(b))$$

(iv) If $a = 0$ and $b = 1$ (or $a = 1$ and $b = 0$) we have that

$$d(a, b) = |a - b| = 1 = \rho(1, 0) = \rho(f(a), f(b))$$

(v) If $a = 0$ and $b = 1/n$ (or $a = 1/n$ and $b = 0$) where $n \geq 2$ we have that

$$d(a, b) = |0 - 1/n| = 1/n = \rho(1, 1/n) = \rho(f(a), f(b))$$

(vi) If $a = 1$ and $b = 1/n$ (or $a = 1/n$ and $b = 1$) where $n \geq 2$ we have that

$$d(a, b) = |1 - 1/n| = 1 - 1/n = \rho(0, 1/n) = \rho(f(a), f(b))$$

Therefore f is an isometry which implies it's also a homeomorphism.

Finally, we want to prove that $i : (M, d) \to (M, \rho)$ the identity map is not continuous. We see that $\{0\} \subset (M, d)$ is not an open set since there is no $\epsilon > 0$ such that $B^d_\epsilon(0) \subset (M, d)$. But $\{0\} \subset (M, \rho)$ is an open set since $\rho(0, 1/n) = 1 - 1/n \geq 1/2$ and $\rho(0, 1) = 1$ then there is $\epsilon = 1/2$ such that $B^\rho_\epsilon(0) = \{0\} \subset (M, \rho)$. So if we take $V = \{0\}$ an open set in (M, ρ) we have that $i^{-1}(V) = i^{-1}(\{0\}) = \{0\} \subset (M, d)$ is not open and therefore i is not continuous. $\square$

Proof. **52** We want to probe Theorem 55. Let $f : (M,d) \to (N,\rho)$ be one-to-one and onto.

$(i) \Rightarrow (ii)$ Suppose f is a homeomorphisms and there is $(x_n) \subset (M,d)$ such that $x_n \to x$ then since f is continuous we have that $f(x_n) \to f(x)$. Now let us suppose that $f(x_n) \to f(x)$ then since f is an homeomorphism there is f^{-1} which is also continuous then we have that $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ hence $x_n \to x$.

$(ii) \Rightarrow (iii)$ Let G be an open set in M and let a sequence $(x_n) \subset M$ such that $x_n \to x$ where $x \in G$ then we have that $x_n \in G$ for all but finitely many n, but also we have that $f(x_n) \to f(x)$ so since $f(x) \in f(G)$ and f is bijective it must happen that all but finitely many $f(x_n) \in f(G)$ therefore $f(G) \subset N$ is an open set.

Now let us suppose $f(G) \subset N$ is open, also, let $f(x) \in f(G)$ such that $f(x_n) \to f(x)$ since $f(G)$ is open then we have that $f(x_n) \in f(G)$ for all but finitely many n. But also we know that f is bijective then $x \in G$ and if $x_n \to x$ it must happen that $x_n \in G$ for all but finitely many n. Therefore $G \subset M$ is an open set.

$(iii) \Rightarrow (iv)$ Let $E \subset M$ be a closed set then $M \setminus E$ is open in M then $f(M \setminus E)$ is an open set in N therefore $N \setminus f(M \setminus E)$ is a closed set in N and since f is bijective it must happen that $f(E) = N \setminus f(M \setminus E)$.

Let now $f(E)$ be a closed set in N then $N \setminus f(E)$ is an open set in N then $f^{-1}(N \setminus f(E))$ is an open set in M therefore $M \setminus f^{-1}(N \setminus f(E))$ is closed and since f^{-1} is bijective (as well as f) we have that $E = M \setminus f^{-1}(N \setminus f(E))$.

$(i) \Leftrightarrow (v)$ Let f be a homeomorphisms and $(x_n) \subset (M,d)$ be a sequence that tends to $x \in M$. Since f is continuous if $x_n \to x$ then $f(x_n) \to f(x)$ then $d(x_n, x) \to 0$ and $\hat{d}(x_n, x) = \rho(f(x_n), f(x)) \to 0$ hence d is equivalent to $\hat{d}$.

Let $\hat{d}(x,y) = \rho(f(x), f(y))$ be equivalent to the metric $d(x,y)$ on M. We want to prove that f and f^{-1} are continuous, i.e. f is a homeomorphism (we already know f is bijective). Let $(x_n) \subset M$ be a sequence that tends to $x \in M$, then $d(x_n, x) \to 0$ but since $\hat{d}$ is also equivalent to d this implies that $\rho(f(x_n), f(x)) \to 0$ hence f is continuous. Now let $(f(x_n)) \subset N$ be a sequence that tends to $f(x) \in N$ then $\hat{d}(x_n, x) = \rho(f(x_n), f(x)) \to 0$ but since $\hat{d}$ is equivalent to d this implies that $d(x_n, x) \to 0$, hence f^{-1} is continuous. Therefore f is a homeomorphism.

$(iv) \Rightarrow (i)$ We want to prove that f and f^{-1} are continuous (we already know that f is bijective). Let E be closed in N, then because of (iv) we have that $f^{-1}(E)$ is closed in M hence f is continuous. In the same way, if E is closed in M then $f(E)$ is closed in N so f^{-1} is also continuous. Therefore f is a homeomorphism.

□

Proof. **55**

($\rightarrow$) Let M be separable and $f : (M, d) \rightarrow (N, \rho)$ a homeomorphism. Also, let $A \subset M$ be a countable dense set in M then $f(A)$ is countable since f is bijective but also we know that for every $x \in M$ there is a sequence $(x_n) \subset A$ such that $x_n \rightarrow x$. Since f is continuous we have $(f(x_n)) \subset f(A)$ such that $f(x_n) \rightarrow f(x)$ where $f(x) \in N$. Hence $f(A)$ is a countable dense subset of N and therefore N is separable.

($\leftarrow$) In the same way, if N is separable let us define a countable dense subset $B \subset N$ then $f^{-1}(B) \subset M$ is countable since f^{-1} is bijective (as well as f). But also we know that for every $y \in N$ there is a sequence $(y_n) \subset B$ such that $y_n \rightarrow y$. Since f^{-1} is continuous we have $(f^{-1}(y_n)) \subset f^{-1}(B)$ such that $f^{-1}(y_n) \rightarrow f^{-1}(y)$ where $f^{-1}(y) \in M$. Hence $f^{-1}(B)$ is a countable dense subset of M and therefore M is separable. □

Proof. **57** Let $f : (M, d) \rightarrow (N, \rho)$ be one-to-one and onto.

$(i) \Rightarrow (ii)$ Suppose f is open, so if $U \subset M$ is open then $f(U) \subset N$ is open. From Theorem 5.5. we know that "U is open if and only if $f(U)$ is open" is equivalent to "E is closed if and only if $f(E)$ is closed" when f is bijective (like in this case). Therefore f is closed

$(ii) \Rightarrow (iii)$ We want to prove that f^{-1} is continuous. We know that if E is closed in M then $f(E)$ is closed in N therefore f^{-1} is continuous.

$(iii) \Rightarrow (i)$ From Theorem 5.1. we know that if f^{-1} is continuous and if U is open in M then $f(U)$ is open in N. Therefore f is open.

□

Proof. **58**

($\Rightarrow$) Let f be a homeomorphism and A a subset of M. Also let $x \in \overline{A}$ then there is $(x_n) \in A$ such that $x_n \to x$. Since f is a homeomorphism we have that also $f(x_n) \to f(x)$ where $f(x_n) \in f(A)$ and $f(x) \in f(\overline{A})$ because f is bijective. But also $f(x) \in \overline{f(A)}$ because of Corollary 4.11. Then this implies that $f(\overline{A}) \subseteq \overline{f(A)}$.

In the same way, let $f(x) \in \overline{f(A)}$ then there is $(f(x_n)) \subset f(A)$ such that $f(x_n) \to f(x)$. Since f is a homeomorphism we also have that $x_n \to x$ where $x_n \in A$ and $x \in \overline{A}$ because f is bijective. But also again since f is bijective we have that $f(x) \in f(\overline{A})$. Then this implies that $\overline{f(A)} \subseteq f(\overline{A})$. Therefore $\overline{f(A)} = f(\overline{A})$.

($\Leftarrow$) Let E be a closed set in M so $E = \overline{E}$ then $f(E) = f(\overline{E}) = \overline{f(E)}$ therefore f is closed.

On the other hand, let B be closed set in N since f is bijective there must be A such that $f(A) = B$ but also since B is closed we have that $B = \overline{B} = \overline{f(A)}$. We also have that $f(\overline{A}) = \overline{f(A)}$ so $B = f(\overline{A})$ then $f^{-1}(B) = \overline{A}$ which is closed. Therefore f^{-1} is closed.

Finally, since f and f^{-1} are both closed then f is a homeomorphism. $\square$

Proof. **63**

(i) We want to show that $\sigma(t) = a + t(b-a)$ is a homeomorphism. We will assume $b \neq a$.

Let us prove first it is a one-to-one function. Suppose $\sigma(t) = \sigma(t')$ then $a + t(b-a) = a + t'(b-a)$ which implies that $t = t'$ since $b \neq a$.

Now we want to prove it is an onto function. Let $c \in [a,b]$ then there is $t_c = \frac{c-a}{b-a}$ such that $\sigma(t_c) = c$ i.e. σ is onto.

Next, we want to prove that σ is continuous. Let $\epsilon > 0$ we want to show that $|\sigma(x) - \sigma(y)| < \epsilon$ whenever $|x-y| < \delta$. Let $\delta = \epsilon/|b-a|$ then if $|x-y| < \delta$ we have that

$$\begin{aligned} |x-y| &< \frac{\epsilon}{|b-a|} \\ |x-y||b-a| &< \epsilon \\ |a + x(b-a) - a - y(b-a)| &< \epsilon \\ |\sigma(x) - \sigma(y)| &< \epsilon \end{aligned}$$

Finally, we want to prove that $\sigma^{-1}(t) = \frac{t-a}{b-a}$ is also continuous. Let $\epsilon > 0$ we want to show that $|\sigma^{-1}(x) - \sigma^{-1}(y)| < \epsilon$ whenever $|x-y| < \delta$. Let $\delta = \epsilon|b-a|$ then if $|x-y| < \delta$ we have that

$$\begin{aligned} |x-y| &< \epsilon|b-a| \\ \frac{|x-y|}{|b-a|} &< \epsilon \\ \frac{|(x-a) - (y-a)|}{|b-a|} &< \epsilon \\ \left|\frac{x-a}{b-a} - \frac{y-a}{b-a}\right| &< \epsilon \\ |\sigma^{-1}(x) - \sigma^{-1}(y)| &< \epsilon \end{aligned}$$

(ii) ($\Rightarrow$) Let $f \in C[a,b]$. Since σ is a homeomorphism it is also continuous and because of the Lemma 5.7 we have that $f \circ \sigma$ is also continuous therefore $f \circ \sigma$ is a continuous map from $[0,1]$ to $\mathbb{R}$ i.e. $f \circ \sigma \in C[0,1]$.

($\Leftarrow$) Let $f \circ \sigma \in C[0,1]$, since σ is a homeomorphism then σ^{-1} is continuous and because of the Lemma 5.7 we have that $(f \circ \sigma) \circ \sigma^{-1}$ is also continuous but $(f \circ \sigma) \circ \sigma^{-1} = f$ therefore $f \in C[a,b]$.

(iii) Let $f, g \in C[a,b]$ we want to prove that the map $f \to f \circ \sigma$ is an isometry from $C[a,b]$ to $C[0,1]$ then we have on one hand that

$$d(f,g) = \|f-g\|_\infty = \max_{a\leq t\leq b} |f(t)-g(t)|$$

And on the other hand, we have that

$$\begin{aligned}\rho(f\circ\sigma, g\circ\sigma) &= \|f\circ\sigma - g\circ\sigma\|_\infty \\ &= \max_{0\leq t\leq 1} |(f\circ\sigma)(t)-(g\circ\sigma)(t)| \\ &= \max_{0\leq t\leq 1} |f(a+t(b-a))-g(a+t(b-a))| \\ &= \max_{a\leq t\leq b} |f(t)-g(t)|\end{aligned}$$

Therefore the map $f \to f\circ\sigma$ is an isometry.

(iv) Let $\alpha, \beta \in \mathbb{R}$ then we have that

$$\begin{aligned}T((\alpha f+\beta g)(t)) &= ((\alpha f+\beta g)\circ\sigma)(t) \\ &= (\alpha f+\beta g)(a+t(b-a)) \\ &= \alpha f(a+t(b-a)) + \beta g(a+t(b-a)) \\ &= \alpha(f\circ\sigma)(t)+\beta(g\circ\sigma)(t) \\ &= \alpha T(f(t))+\beta T(g(t))\end{aligned}$$

(v) In this case, we have that

$$\begin{aligned}T((fg)(t)) &= ((fg)\circ\sigma)(t) \\ &= (fg)(a+t(b-a)) \\ &= f(a+t(b-a))g(a+t((b-a))) \\ &= (f\circ\sigma)(t)(g\circ\sigma)(t) \\ &= T(f(t))T(g(t))\end{aligned}$$

(vi) $(\Rightarrow)$ If $T(f(t)) \leq T(g(t))$ for any $t \in [0,1]$ then we have that

$$\begin{gathered}f\circ\sigma(t) \leq g\circ\sigma(t) \\ f(a+t(b-a)) \leq g(a+t(b-a)) \\ f(t') \leq g(t')\end{gathered}$$

This implies that $f \leq g$ for any $t' \in [a,b]$.

$(\Leftarrow)$ If $f(t) \leq g(t)$ for any $t \in [a,b]$ then we have that $f(a+t'(b-a)) \leq g(a+t'(b-a))$ for any $t' \in [0,1]$ so $f\circ\sigma(t') \leq g\circ\sigma(t')$. Therefore $T(f(t)) \leq T(g(t))$.

□

Proof. **64** Given $f, g \in C(\mathbb{R})$, we want to prove that

$$d(f, g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f, g)}{(1 + d_n(f, g))}$$

defines a metric on $C(\mathbb{R})$ then

(i) Since $d_n(f, g) \geq 0$ for any $n \in \mathbb{N}$ and any $f, g \in C(\mathbb{R})$ then $d(f, g) \geq 0$. Also, given that $d_n(f, g) < 1 + d_n(f, g)$ we have that

$$0 \leq \frac{2^{-n} d_n(f, g)}{(1 + d_n(f, g))} \leq 2^{-n}$$

and since $\sum_{n=1}^{\infty} 2^{-n}$ converges we have that $d(f, g)$ also converges i.e. $d(f, g) < \infty$.

(ii) ($\Rightarrow$) Suppose $d(f, g) = 0$ then

$$\sum_{n=1}^{\infty} \frac{2^{-n} d_n(f, g)}{(1 + d_n(f, g))} = 0$$

So it must happen that $d_n(f, g) = 0$ for every $n \in \mathbb{N}$ this implies that $\max_{|t| \leq n} |f(t) - g(t)| = 0$ for every $n \in \mathbb{N}$ hence it must happen that $f(t) = g(t)$ for all $|t| \leq n$. Since this is true for every $n \in \mathbb{N}$ then $f = g$.

($\Leftarrow$) Suppose $f = g$ then it must happen that $d_n(f, g) = 0$ for every $n \in \mathbb{N}$ since it is a pseudometric. Therefore $d(f, g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{(1 + d_n(f,g))} = 0$.

(iii) Let $f, g \in C(\mathbb{R})$ then

$$d(f, g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f, g)}{(1 + d_n(f, g))} = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(g, f)}{(1 + d_n(g, f))} = d(g, f)$$

Where we used that $d_n(f, g) = d_n(g, f)$ for every $n \in \mathbb{N}$ since it is a pseudometric and it has the symmetry property.

(iv) Let $f, g, h \in C(\mathbb{R})$. Since d_n is a pseudometric we have that

$$d_n(f, g) \leq d_n(f, h) + d_n(h, g)$$

Also, from problem 5 (of Chapter 3) we know that the function $F(t) = t/(1 + t)$ is increasing so we have that

$$\begin{aligned} F(d_n(f, g)) &\leq F(d_n(f, h) + d_n(h, g)) \\ \frac{d_n(f, g)}{1 + d_n(f, g)} &\leq \frac{d_n(f, h) + d_n(h, g)}{1 + d_n(f, h) + d_n(h, g)} \end{aligned}$$

But also F satisfies $F(s+t) \leq F(s) + F(t)$ for $s, t \geq 0$ then joining these inequalities we have that

$$\frac{d_n(f,g)}{1+d_n(f,g)} \leq \frac{d_n(f,h)+d_n(h,g)}{1+d_n(f,h)+d_n(h,g)} \leq \frac{d_n(f,h)}{1+d_n(f,h)} + \frac{d_n(h,g)}{1+d_n(h,g)}$$

This implies that

$$\sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{1+d_n(f,g)} \leq \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,h)}{1+d_n(f,h)} + \frac{2^{-n} d_n(h,g)}{1+d_n(h,g)}$$

Therefore $d(f,g) \leq d(f,h) + d(h,g)$ as we wanted.

Finally, since the metric d satisfies (i) to (iv) it defines a metric on $C(\mathbb{R})$. $\square$