Solved selected problems of Real Analysis - Carothers

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Chapter 10 - Sequences of Functions

Proof. **4** Let f be twice continuously differentiable and 2π-periodic, we want to respond why f' and f'' are both 2π-periodic. Since f is 2π-periodic we know that $f(x) = f(x+2\pi n)$ for $n \in \mathbb{N}$ then by differentiating this expression we get that $f'(x) = f'(x+2\pi n)$ and that $f''(x) = f''(x+2\pi n)$ which implies that both f' and f'' are 2π-periodic.

(a) Let us now compute the Fourier coefficient a_n of f using integration by parts where we assume $u(x) = f(x)$ and $v'(x) = \cos(nx)$ hence $v(x) = \sin(nx)/n$ then

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left[\left[f(x)\frac{\sin(nx)}{n}\right]_0^{2\pi} - \int_0^{2\pi} f'(x)\frac{\sin(nx)}{n}dx\right] \\
&= -\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin(nx)dx
\end{aligned}$$

So we have that

$$\begin{aligned}
|a_n| &= \left|\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin(nx)dx\right| \\
&\leq \frac{1}{n\pi}\int_0^{2\pi} |f'(x)\sin(nx)|dx \\
&\leq \frac{1}{n\pi}\int_0^{2\pi} |f'(x)|dx
\end{aligned}$$

Where we used that $|\sin(nx)| \leq 1$. Since f' is 2π-periodic then it is bounded, let us take a bound C' then we have that

$$\frac{1}{\pi}\int_0^{2\pi} |f'(x)|dx \leq 2\pi C' = C$$

which implies that

$$|a_n| \leq C/n$$

In the same way, we compute the Fourier coefficient b_n of f where we assume $u(x) = f(x)$ and $v'(x) = \sin(nx)$ hence $v(x) = -\cos(nx)/n$ then

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \\
&= \frac{1}{\pi} \left[\left[-f(x) \frac{\cos(nx)}{n} \right]_0^{2\pi} + \int_0^{2\pi} f'(x) \frac{\cos(nx)}{n} dx \right] \\
&= \frac{1}{\pi} \left[\left[-\frac{f(2\pi)}{n} + \frac{f(0)}{n} \right] + \int_0^{2\pi} f'(x) \frac{\cos(nx)}{n} dx \right] \\
&= \frac{1}{n\pi} \int_0^{2\pi} f'(x) \cos(nx) dx
\end{aligned}$$

Where we used that f is 2π-periodic and so $f(2\pi) = f(0)$ so we have that

$$\begin{aligned}
|b_n| &= \left| \frac{1}{n\pi} \int_0^{2\pi} f'(x) \cos(nx) dx \right| \\
&\leq \frac{1}{n\pi} \int_0^{2\pi} |f'(x) \cos(nx)| dx \\
&\leq \frac{1}{n\pi} \int_0^{2\pi} |f'(x)| dx
\end{aligned}$$

Where we used again that $|\cos(nx)| \leq 1$ and since f' is 2π-periodic then it is bounded, let us take a bound C' then we have that

$$\frac{1}{\pi} \int_0^{2\pi} |f'(x)| dx \leq 2\pi C' = C$$

which implies that

$$|b_n| \leq C/n$$

Finally, since $1/n \to 0$ as $n \to \infty$ and we know that $0 \leq |a_n| \leq C/n$ and $0 \leq |b_n| \leq C/n$ by the squeeze theorem we have that $|a_n| \to 0$ and $|b_n| \to 0$ as $n \to \infty$.

(b) Let us now integrate by parts again the Fourier coefficient a_n we got where we assume $u(x) = f'(x)$ and $v'(x) = \sin(nx)$ hence $v(x) = -\cos(nx)/n$ then

$$\begin{aligned}
a_n &= -\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin(nx)dx \\
&= -\frac{1}{n\pi}\left[\left[-f'(x)\frac{\cos(nx)}{n}\right]_0^{2\pi} - \int_0^{2\pi} f''(x)\frac{\cos(nx)}{n}dx\right] \\
&= -\frac{1}{n\pi}\left[\left[-\frac{f'(2\pi)}{n} + \frac{f'(0)}{n}\right] - \int_0^{2\pi} f''(x)\frac{\cos(nx)}{n}dx\right] \\
&= \frac{1}{n^2\pi}\int_0^{2\pi} f''(x)\cos(nx)dx
\end{aligned}$$

Where we used that f' is 2π-periodic and so $f'(2\pi) = f'(0)$ so we have that

$$\begin{aligned}
|a_n| &= \left|\frac{1}{n^2\pi}\int_0^{2\pi} f''(x)\cos(nx)dx\right| \\
&\leq \frac{1}{n^2\pi}\int_0^{2\pi} |f''(x)\cos(nx)|dx \\
&\leq \frac{1}{n^2\pi}\int_0^{2\pi} |f''(x)|dx
\end{aligned}$$

Where we used again that $|\cos(nx)| \leq 1$ and since f'' is 2π-periodic then it is bounded, let us take a bound C' then we have that

$$\frac{1}{\pi}\int_0^{2\pi} |f''(x)|dx \leq 2\pi C' = C$$

which implies that

$$|a_n| \leq C/n^2$$

In the same way, we can integrate by parts again the Fourier coefficient b_n we got where we assume $u(x) = f'(x)$ and $v'(x) = \cos(nx)$ hence $v(x) = \sin(nx)/n$ then

$$\begin{aligned} b_n &= \frac{1}{n\pi}\int_0^{2\pi} f'(x)\cos(nx)dx \\ &= \frac{1}{n\pi}\left[\left[f'(x)\frac{\sin(nx)}{n}\right]_0^{2\pi} - \int_0^{2\pi} f''(x)\frac{\sin(nx)}{n}dx\right] \\ &= -\frac{1}{n^2\pi}\int_0^{2\pi} f''(x)\sin(nx)dx \end{aligned}$$

So we have that

$$\begin{aligned} |b_n| &= \left|\frac{1}{n^2\pi}\int_0^{2\pi} f''(x)\sin(nx)dx\right| \\ &\leq \frac{1}{n^2\pi}\int_0^{2\pi} |f''(x)\sin(nx)|dx \\ &\leq \frac{1}{n^2\pi}\int_0^{2\pi} |f'(x)|dx \end{aligned}$$

Where we used again that $|\sin(nx)| \leq 1$ and since f'' is 2π-periodic then it is bounded, let us take a bound C' then we have that

$$\frac{1}{\pi}\int_0^{2\pi} |f''(x)|dx \leq 2\pi C' = C$$

which implies that

$$|b_n| \leq C/n^2$$

Finally, let $x \in \mathbb{R}$ then the Fourier series s for $f(x)$ is given by

$$s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

we see that both the terms $a_n\cos(nx)$ and $b_n\sin(nx)$ tend to 0 as $n \to \infty$ which implies that the series converges and therefore takes a value on $\mathbb{R}$.

□

Proof. **7** Let (f_n) and (g_n) be real-valued function on a set X and suppose that (f_n) and (g_n) converge uniformly on X. We want to show $(f_n + g_n)$ converges uniformly on X.

Since (f_n) converge uniformly then given $\epsilon/2 > 0$ there is $N \geq 1$ (which may depend on ϵ) such that $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in X$ an all $n \geq N$.

In the same way, since (g_n) converge uniformly then there is $N' \geq 1$ (which may depend on ϵ) such that $|g_n(x) - g(x)| < \epsilon/2$ for all $x \in X$ an all $n \geq N'$.

Let us take $M = \max(N, N')$ so we know that for all $x \in X$ and for all $n \geq M$ we have that

$$|f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

and by the triangle inequality, we see that

$$|(f_n(x) + g_n(x)) - (g(x) + f(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon$$

which implies that $(f_n + g_n)$ converges uniformly.

Let us take now $f_n(x) = g_n(x) = x + 1/n$ where we see that they are uniformly convergent to $f(x) = g(x) = x$ on $\mathbb{R}$. So we define $f_n g_n = (x + 1/n)^2$ but we see that

$$\sup_{x\in\mathbb{R}} \left| \left(x + \frac{1}{n} \right)^2 - x^2 \right| = \sup_{x\in\mathbb{R}} \left| \frac{2x}{n} + \frac{1}{n^2} \right| = +\infty$$

Therefore $(f_n g_n)$ is not uniformly convergent. $\square$

Proof. **9**

(a) Let $f_n(x) = x^n$ on $(-1, 1]$. We know that (f_n) converges to 0 if $x \in [0, 1)$ and to 1 if $x = 1$. Let $-1 < x < 0$ then there must be some $a < 0$ such that $x = 1/a$ hence $x^n = 1/a^n$ and we see that $1/a^n \to 0$ as $n \to \infty$ then $x^n \to 0$ as $n \to \infty$. So in summary the pointwise limit for (f_n) is given by

$$f(x) = \begin{cases} 0 & x \in (-1, 1) \\ 1 & x = 1 \end{cases}$$

Let us take now an interval $(a, b) \subset (-1, 1]$ then if $x \in (a, b)$ we have that

$$\sup_{x \in (a,b)} |f_n(x) - f(x)| = \sup_{x \in (a,b)} |x^n - 0| = |b^n|$$

and we see that $|b^n| \to 0$ as $n \to \infty$ since $-1 < b < 1$. Therefore (f_n) is uniformly convergent to 0 in any interval $(a, b) \subset (-1, 1]$ as long as $b < 1$.

Given that $f_n \to f$ pointwise we want to check if $f_n' \to f'$ too. So we have that $f_n'(x) = nx^{n-1}$ if $x \in [0, 1)$ then there is some $a > 1$ such that $x = 1/a$ hence $nx^{n-1} = n/a^{n-1} = an/a^n$ and we know that the polynomial an goes slower to infinity than a^n so we have that $nx^{n-1} \to 0$. The same thing can be shown for $x \in (-1, 0)$. But if $x = 1$ then $f_n'(1) = n$ which goes to ∞ as $n \to \infty$.

Finally, we want to check that if $\int f_n \to \int f$. We see that

$$\begin{aligned} \int_{-1}^{1} f_n(x)dx &= \int_{-1}^{1} x^n dx \\ &= \left[\frac{1^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1}\right] \\ &= \frac{-1^n + 1}{n+1} \end{aligned}$$

and we have that $(-1^n + 1)/(n + 1) \to 0$ as $n \to \infty$.

(b) Let $f_n(x) = n^2x(1-x^2)^n$ on $[0,1]$. Let us take some $x \in (0,1)$ then we see that $0 < 1-x^2 < 1$ hence $(1-x^2)^n \to 0$ as $n \to \infty$ but $xn^2 \to \infty$ as $n \to \infty$ so let us write $\lim_{n\to\infty} f_n(x)$ as

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{\frac{1}{(1-x^2)^n}}$$

So we can apply L'Hôpital rule twice to get

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{2nx}{-\frac{\log(1-x^2)}{(1-x^2)^n}} \\ &= \lim_{n\to\infty} \frac{2x}{\frac{\log^2(1-x^2)}{(1-x^2)^n}} \\ &= 0\end{aligned}$$

Also, if $x = 0$ we get that $f_n(0) = 0$ and if $x = 1$ we have that $f_n(1) = 0$. Therefore f_n converges pointwise to $f(x) = 0$ on $[0,1]$.

Let us take now the interval $[0,1]$ and let us analyze the maximum value of the series by derivating

$$f_n'(x) = -n^2(1-x^2)^{n-1}(-1+(1+2n)x^2)$$

so $f_n(x)$ is a maximum when $x = 1/\sqrt{2n+1}$ hence we have that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |n^2x(1-x^2)^n - 0| = \frac{2^n n^{n+2}}{(2n+1)^{n+1/2}}$$

And we see that $2^n n^{n+2}/(2n+1)^{n+1/2} \to \infty$ as $n \to \infty$. Therefore (f_n) is not uniformly convergent on $[0,1]$.

Let us check now if there is another interval where f_n is uniformly convergent. We see that $1/\sqrt{2n+1} \to 0$ as $n \to \infty$ so the value of x that gives us the maximum will move towards 0 so if we take an interval $(a,b) \subset [0,1]$ where $a > 0$ then the maximum will happen at $x = a$ but as we saw $f_n(a) \to 0$ hence $\sup_{x\in(a,b)} |f_n(x) - 0| = |f_n(a)| \to 0$ so for any interval (a,b) where $a > 0$ the sequence (f_n) is uniformly convergent to 0.

Let us check now if $f_n' \to f'$. We see that

$$f_n'(x) = -n^2(1-x^2)^{n-1}(-1+(1+2n)x^2)$$

By applying multiple times the L'Hôpital rule we get that $f_n' \to 0$ as $n \to \infty$.

Let us check now if $\int f_n \to \int f$. We see that

$$\int_0^1 f_n(x) = \frac{n^2}{2n+2}$$

But in this case, we see that $\int f_n \to \infty$ as $n \to \infty$.

(c) Let $f_n(x) = nx/(1+xn)$ on $[0,\infty)$. We can write $f_n(x)$ as

$$f_n(x) = \frac{x}{1/n + x}$$

So we see that $\lim_{n\to\infty} f_n(x) = 1$ for $x \in (0,\infty)$ and if $x = 0$ we get that $f_n(0) = 0$.

Let us take an interval $(a,b) \subset [0,\infty)$ then we have that

$$\begin{aligned}
\sup_{x\in(a,b)} |f_n(x) - f(x)| &= \sup_{x\in(a,b)} \left| \frac{x}{1/n + x} - 1 \right| \\
&= \sup_{x\in(a,b)} \left| \frac{x - 1/n - x}{1/n + x} \right| \\
&= \sup_{x\in(a,b)} \left| \frac{1}{1 + nx} \right| \\
&= \frac{1}{1 + na}
\end{aligned}$$

Since the supremum for $1/(1+na)$ is given at $x = a$ and we see that $1/(1+na) \to 0$ as $n \to \infty$.

Therefore (f_n) is uniformly convergent to 1 in any interval $(a,b) \subset [0,\infty)$ as long as $a > 0$ otherwise we get that $\sup_{x\in[0,b)} |f_n(x) - f(x)| = 1$ which does not tend to 0 as $n \to \infty$.

Let us check now if $f_n' \to f'$. We see that

$$f_n'(x) = \frac{n}{(nx+1)^2}$$

By applying the L'Hôpital rule we get that $f_n' \to 0$ as $n \to \infty$.

Let us check now if $\int f_n \to \int f$. In this case, the integral $\int_0^\infty f_n(x)dx$ does not converge.

(d) Let $f_n(x) = nx/(1+x^2n^2)$ on $[0,\infty)$. We can write $f_n(x)$ as

$$f_n(x) = \frac{x}{1/n + x^2 n}$$

So we see that $\lim_{n\to\infty} f_n(x) = 0$ for $x \in (0,\infty)$ and if $x = 0$ we get that $f_n(0) = 0$. Hence (f_n) converges pointwise to 0.

Let us take the derivative of $f_n(x)$ to see where the maximum happens

$$f_n'(x) = \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

Then if $f_n'(x) = 0$ we get that $n^3x^2 = n$ which implies that the maximum happens at $x = 1/n$. So we have that

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \frac{x}{1/n + x^2n} - 0 \right| = \left| \frac{1/n}{2/n} \right| = 1/2$$

So we see that $\sup_{x\in[0,\infty)} |f_n(x) - f(x)|$ does not tend to 0 as $n \to \infty$ which implies that (f_n) is not uniformly convergent on $[0,\infty)$ but we see that $1/n \to 0$ as $n \to \infty$ so the value of x that gives us the maximum will move towards 0 thus if we take an interval $(a,b) \subset [0,\infty)$ where $a > 0$ then the maximum will happen at $x = a$ but as we saw $f_n(a) \to 0$ hence $\sup_{x\in(a,b)} |f_n(x) - 0| = |f_n(a)| \to 0$ so for any interval (a,b) where $a > 0$ the sequence (f_n) is uniformly convergent to 0.

Let us check now if $f_n' \to f'$. We saw that

$$f_n'(x) = \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

By applying the L'Hôpital rule multiple times we get that $f_n' \to 0$ as $n \to \infty$.

Let us check now if $\int f_n \to \int f$. We see that

$$\int_0^\infty \frac{x}{1/n + x^2n}\, dx = \left[\frac{\log(1+n^2x^2)}{2n} \right]_0^\infty = \infty$$

So $\int f_n$ does not converge but this was expected since (f_n) is not uniformly convergent on $[0,\infty)$.

(e) Let $f_n(x) = xe^{-nx}$ on $[0, \infty)$. We can write $f_n(x)$ as

$$f_n(x) = \frac{x}{e^{nx}}$$

So we see that $\lim_{n\to\infty} f_n(x) = 0$ for $x \in (0, \infty)$ and if $x = 0$ we get that $f_n(0) = 0$. Hence (f_n) converges pointwise to 0.

Let us take the derivative of $f_n(x)$ to see where the maximum happens

$$f_n'(x) = \frac{1 - nx}{e^{nx}}$$

Then if $f_n'(x) = 0$ we get that $1 - nx = 0$ which implies that the maximum happens at $x = 1/n$. So we have that

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left|\frac{x}{e^{nx}} - 0\right| = \left|\frac{1/n}{e}\right| = \left|\frac{1}{ne}\right|$$

And we see that $|1/ne| \to 0$ as $n \to \infty$ which implies that (f_n) is uniformly convergent on $[0, \infty)$.

Let us check now if $f_n' \to f'$. We saw that

$$f_n'(x) = \frac{1 - nx}{e^{nx}}$$

By applying the L'Hôpital rule we get that $f_n' \to 0$ as $n \to \infty$ which implies that $f_n' \to f'$.

Let us check now if $\int f_n \to \int f$. We see that

$$\int_0^\infty \frac{x}{e^{nx}}\ dx = \left[\frac{nx + 1}{n^2 e^{nx}}\right]_0^\infty = \frac{1}{n^2}$$

So we see that $\int f_n \to 0$ as $n \to \infty$ i.e. $\int f_n \to \int f$ as we wanted.

(f) Let $f_n(x) = nxe^{-nx}$ on $[0, \infty)$. Let $x \in (0, \infty)$ then by applying L'Hôpital rule we get that $x/ne^{nx} \to 0$ as $n \to \infty$ and if $x = 0$ we also have that $f_n(0) = 0$. Hence (f_n) converges pointwise to 0.

Let us take the derivative of $f_n(x)$ to see where the maximum happens

$$f_n'(x) = \frac{(1 - nx)n}{e^{nx}}$$

Then if $f_n'(x) = 0$ we get that $1 - nx = 0$ which implies that the maximum happens at $x = 1/n$. So we have that

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \frac{nx}{e^{nx}} - 0 \right| = \left| \frac{1}{e} \right|$$

So we see that $\sup_{x\in[0,\infty)} |f_n(x) - f(x)|$ does not tend to 0 as $n \to \infty$ which implies that (f_n) is not uniformly convergent on $[0, \infty)$ but we see that $1/n \to 0$ as $n \to \infty$ so the value of x that gives us the maximum will move towards 0 thus if we take an interval $(a, b) \subset [0, \infty)$ where $a > 0$ then the maximum will happen at $x = a$ but as we saw $f_n(a) \to 0$ hence $\sup_{x\in(a,b)} |f_n(x) - 0| = |f_n(a)| \to 0$ so for any interval (a, b) where $a > 0$ the sequence (f_n) is uniformly convergent to 0.

Let us check now if $f_n' \to f'$. We saw that

$$f_n'(x) = \frac{(1 - nx)n}{e^{nx}}$$

Let $x \in (0, \infty)$, by applying the L'Hôpital rule we get that $f_n' \to 0$ as $n \to \infty$ and if $x = 0$ we get that $f_n' \to \infty$ as $n \to \infty$.

Let us check now if $\int f_n \to \int f$. We see that

$$\int_0^\infty \frac{nx}{e^{nx}}\, dx = \left[-\frac{nx + 1}{ne^{nx}} \right]_0^\infty = \frac{1}{n}$$

So we see that $\int f_n \to 0$ as $n \to \infty$ i.e. $\int f_n \to \int f$ as we wanted.

$\square$

Proof. **13** Let $f_n : X \to Y$ be continuous for each n, let (f_n) to be pointwise convergent to f on X and let a sequence $(x_n) \subseteq X$ such that $x_n \to x$ in X but $f_n(x_n) \not\to f(x)$, we want to show that (f_n) does not converge uniformly to f on X.

Let us suppose (f_n) does converge uniformly to f on X, we want to arrive at a contradiction. Let $\epsilon > 0$ then there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\sup_{x \in X} \rho(f_n(x), f(x)) < \epsilon$ this also implies that $\rho(f_n(x), f(x)) < \epsilon$ for all $x \in X$.

On the other hand, since each f_n is continuous given $x_n, x \in X$ and $\epsilon > 0$ we know there is $\delta > 0$ such that whenever $d(x_n, x) < \delta$ we have that $\rho(f_n(x_n), f_n(x)) < \epsilon$. So adding these inequalities and using the triangle inequality we have that

$$\rho(f_n(x_n), f(x)) \leq \rho(f_n(x), f(x)) + \rho(f_n(x_n), f_n(x)) < 2\epsilon$$

Which implies that $f_n(x_n) \to f(x)$ but we said that $f_n(x_n) \not\to f(x)$ hence we have a contradiction. Therefore must be that (f_n) is not uniformly continuous. □

Proof. **14** Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous for each n, and suppose f_n converges uniformly to f on each closed, bounded interval $[a, b]$. We want to show that f is continuous on $\mathbb{R}$.

We know that f is continuous on $[a, b]$ because of Theorem 10.4. Let $x \in \mathbb{R}$ then we can build a closed, bounded interval $[x-1, x+1]$ where f is continuous so f is continuous in x. Therefore f is continuous in $\mathbb{R}$. □

Proof. **15** Let (X, d) and (Y, ρ) be metric spaces and let $f, f_n : X \to Y$ with $f_n \rightrightarrows f$ on X. If each f_n is continuous at $x \in X$, and if $x_n \to x$, we want to prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Let $\epsilon/2 > 0$ then there is $N' \in \mathbb{N}$ such that when $n \geq N'$ we have that $\rho(f_n(y), f(y)) < \epsilon/2$ for all $y \in X$ since (f_n) converges uniformly to f so if in particular we choose $y = x_n$ we get that

$$\rho(f_n(x_n), f(x_n)) < \epsilon/2$$

On the other hand, because of Theorem 10.4, we know that f is continuous so using the same $\epsilon/2 > 0$ there is $M \in \mathbb{N}$ such that when $n \geq M$ we have that

$$\rho(f(x_n), f(x)) < \epsilon/2$$

Finally, let us take $N = \max(N', M)$ so both inequalities are true, then adding both inequalities and using the triangle inequality we get that

$$\rho(f_n(x_n), f(x)) \leq \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f(x)) < \epsilon$$

Which implies that $\lim_{n\to\infty} f_n(x_n) \to f(x)$.

□

Proof. **26** Let $\sum_{n=1}^{\infty} |a_n| < \infty$ we want to prove that $\sum_{n=1}^{\infty} a_n \sin(nx)$ and $\sum_{n=1}^{\infty} a_n \cos(nx)$ are uniformly convergent on $\mathbb{R}$.

Let $f_n(x) = a_n \sin(nx)$ we know that $|\sin(nx)| \leq 1$ then $|a_n \sin(nx)| \leq |a_n|$ also, we have that $|a_n \sin(nx)| \leq \sup_{x\in\mathbb{R}} |a_n \sin(nx)| \leq |a_n|$ so summing over n we get that

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} = \sum_{n=1}^{\infty} \sup_{x\in\mathbb{R}} |a_n \sin(nx)| \leq \sum_{n=1}^{\infty} |a_n| < \infty$$

Then because of the Weierstrass M-test, we have that $\sum_{n=1}^{\infty} a_n \sin(nx)$ is uniformly convergent on $\mathbb{R}$.

Finally, given that $|\cos(nx)| < 1$ all we said is still valid for a sequence of functions $f_n(x) = a_n \cos(nx)$ therefore $\sum_{n=1}^{\infty} a_n \cos(nx)$ is uniformly convergent on $\mathbb{R}$ too. $\square$

Proof. **29**

(a) Let us consider the sequence $f_n(x) = ne^{-nx}$ then we see that if $x > 0$ we have that $\lim_{n\to\infty} ne^{-nx} = 0$ so the series $\sum_{n=1}^{\infty} ne^{-nx}$ converges for $x > 0$.

Now we want to determine in which intervals $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly if we consider the interval $(0, \infty)$ we see that

$$\|ne^{-nx}\|_\infty = \sup_{x\in(0,\infty)} |ne^{-nx}| = n$$

so the series $\sum_{n=1}^{\infty} \|ne^{-nx}\|_\infty$ does not converge hence $\sum_{n=1}^{\infty} ne^{-nx}$ does not converge uniformly. So let us take an interval $[r, \infty)$ for some $r > 0$ then we have that

$$\|ne^{-nx}\|_\infty = \sup_{x\in[r,\infty)} |ne^{-nx}| = ne^{-nr} \to 0$$

as $n \to \infty$. Hence the series $\sum_{n=1}^{\infty} \|ne^{-nx}\|_\infty$ converge and therefore because of the Weierstrass M-test the series $\sum_{n=1}^{\infty} ne^{-nx}$ converge uniformly on $[r, \infty)$ for some $r > 0$.

(b) Let us consider now the series $\sum_{k=1}^{n} e^{-kx}$ then we have that

$$\begin{aligned}
(1 - e^{-x}) \sum_{k=1}^{n} (e^{-x})^k &= (1 - e^{-x})(e^{-x} + (e^{-x})^2 + ... + (e^{-x})^n) \\
&= (e^{-x} + e^{-2x} + ... + e^{-nx} - \\
&\qquad - e^{-2x} - e^{-3x} - ... - e^{-(n+1)x}) \\
&= e^{-x} - e^{-(n+1)x} \\
&= e^{-x}(1 - e^{-nx})
\end{aligned}$$

Then we have that $\sum_{k=1}^{n} e^{-kx} = e^{-x}(1 - e^{-nx})/(1 - e^{-x})$ but also we see that

$$\frac{d}{dx} \sum_{k=1}^{n} e^{-kx} = -\sum_{k=1}^{n} ke^{-kx}$$

So the sequence we are interested in is $f_n(x) = \sum_{k=1}^{n} ke^{-kx}$ hence we have that

$$f_n(x) = \sum_{k=1}^{n} ke^{-kx} = \frac{e^x + ne^{-nx} - (n+1)e^{-x(n+1)}}{(-1 + e^x)^2}$$

So assuming $x > 0$ we saw that $f_n(x)$ is uniformly convergent to $f(x)$ which is given by

$$\begin{aligned}
f(x) &= \lim_{n\to\infty} f_n(x) \\
&= \lim_{n\to\infty} \frac{e^x + ne^{-nx} - (n+1)e^{-x(n+1)}}{(-1 + e^x)^2} \\
&= \frac{e^x}{(e^x - 1)^2}
\end{aligned}$$

Now, using Theorem 10.5 we can take the limit inside the integral where we get that

$$
\begin{aligned}
\lim_{n\to\infty}\int_1^2 f_n(x)dx &= \lim_{n\to\infty}\int_1^2 \sum_{k=1}^{n} ke^{-kx}dx \\
&= \int_1^2 \sum_{k=1}^{\infty} ke^{-kx}dx \\
&= \int_1^2 \frac{e^x}{(e^x-1)^2}dx \\
&= \left[\frac{1}{1-e^2} - \frac{1}{1-e}\right] \\
&= \frac{e}{e^2-1}
\end{aligned}
$$

□