Solved selected problems of Real Analysis - Carothers

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Chapter 6 - Connectedness

Proof. **1** To finalize the Lemma 6.3 proof we need to prove the Claim which states that $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) = \emptyset$ for every $x \in U$ and $y \in V$.

Let $z \in M$ such that $d(x,z) < \epsilon_x/2$ and $d(y,z) < \delta_y/2$ we want to arrive to a contradiction then

$$d(x,y) \leq d(x,z) + d(z,y) < \epsilon_x/2 + \delta_y/2$$

Since $y \notin B_{\epsilon_x}(x)$ then $d(x,y) > \epsilon_x$ and since $x \notin B_{\delta_y}(y)$ then $d(x,y) > \delta_y$ so we have that $\epsilon_x < \epsilon_x/2 + \delta_y/2$ and $\delta_y < \epsilon_x/2 + \delta_y/2$ and then $\epsilon_x > \delta_y$ and $\epsilon_x < \delta_y$ which is a contradiction. Therefore $B_{\epsilon_x/2}(x) \cap B_{\delta_y/2}(y) = \emptyset$. □

Proof. **3** Let A and B be disjoint open sets in M and $E \subset A \cup B$ be connected in M we want to show that $E \subset A$ or $E \subset B$ by contradiction. We know that if $x \in E$ it must happen that $x \in A$ or $x \in B$ because they are disjoint. But then if we Let $x, y \in E$ such that $x \in A$ and $y \in B$ we have that $A \cap E \neq \emptyset$ and $B \cap E \neq \emptyset$. That, in addition to $E \subset A \cup B$ implies that E is disconnected which is a contradiction, then either $x, y \in A$ or $x, y \in B$. Therefore $E \subset A$ or $E \subset B$. □

Proof. **5** Suppose $E \cup F$ is disconnected we want to arrive at a contradiction. Then there are $A, B \subset E \cup F$ such that A and B are disjoint, nonempty open sets of $E \cup F$ and $A \cup B = E \cup F$. Also, since E is connected and $E \subset A \cup B$ because of problem 3 we have that $E \subset A$ of $E \subset B$ and in the same way $F \subset A$ or $F \subset B$. It must happen that both $E, F \subset A$ or $E, F \subset B$ because otherwise $E \cap F = \emptyset$ which cannot happen because we know that $E \cap F \neq \emptyset$. But then if $E, F \subset A$ then $E \cup F \subset A$ but we know that $E \cup F = A \cup B$ where A and B are disjoint therefore we have a contradiction and $E \cup F$ must be connected. □

Proof. **6** Let $\mathcal{C}$ be a collection of connected subsets of M. Suppose $\bigcup \mathcal{C}$ is disconnected we want to arrive at a contradiction.

Then there are $A, B \subset \bigcup \mathcal{C}$ such that A and B are disjoint nonempty open sets of $\bigcup \mathcal{C}$ and $A \cup B = \bigcup \mathcal{C}$. Let $C, D \in \mathcal{C}$ be connected sets of M then $C \subset A \cup B$ and $D \subset A \cup B$ also, because of proof 3 we have that $C \subset A$ or $C \subset B$ and $D \subset A$ or $D \subset B$. But since C and D share a point it must happen that $C, D \subset A$ or $C, D \subset B$ because A and B are disjoint, let us suppose that $C, D \subset A$ (if $C, D \subset B$ the proof is analogous) then all subsets of $\mathcal{C}$ must be in A because they share a point but $A \cup B = \bigcup \mathcal{C}$ so B must be empty, a contradiction. Therefore $\bigcup \mathcal{C}$ is connected.

Finally, we want to prove that $\mathbb{R}$ is connected. We know that $(-\infty, a]$ and $[a, \infty)$ are connected sets so $(-\infty, a] \cup [a, \infty) = \mathbb{R}$ is connected because of what we proved before. $\square$

Proof. **7** Let $a \in M$ then for every $x \in M$ there is a connected set $E_x \subset M$ such that $a, x \in E_x$ then $M = \bigcup E_x$ and since each connected set E_x share the element a with every other connected set E_x we have that M is connected because of proof 6. $\square$

Proof. **9** Let $A \subset B \subset \bar{A} \subset M$ and let A be connected. Suppose that B is disconnected, we want to arrive at a contradiction. If B is disconnected then there are $C, D \subset B$ such that they are disjoint nonempty open sets of B and $C \cup D = B$ but since A is connected then $A \subset C$ or $A \subset D$. Suppose $A \subset C$ i.e. $A \cap C \neq \emptyset$ and $A \cap D = \emptyset$. Now let us take $x \in B$ such that $x \in D$ since $x \in \bar{A}$ it must happen that $B_\epsilon(x) \cap A \neq \emptyset$ for every $\epsilon > 0$ in particular D is an open neighborhood of x so it must happen that $D \cap A \neq \emptyset$ which is a contradiction. Therefore B is connected.

In the same way, suppose now that $\bar{A}$ is disconnected we want to arrive at a contradiction. If $\bar{A}$ is disconnected then there are $C, D \subset \bar{A}$ such that they are disjoint nonempty open sets and $C \cup D = \bar{A}$ but since A is connected then $A \subset C$ or $A \subset D$. Suppose $A \subset C$ i.e. $A \cap C \neq \emptyset$ and $A \cap D = \emptyset$. Now let us take $x \in \bar{A}$ such that $x \in D$ it must happen that $B_\epsilon(x) \cap A \neq \emptyset$ for every $\epsilon > 0$ in particular D is an open neighborhood of x so it must happen that $D \cap A \neq \emptyset$ which is a contradiction. Therefore $\bar{A}$ is connected. $\square$

Proof. **12** Let $a, b \in M$ and let us define $f(x) = d(x, a)$ we want to show that $f(M) \subset \mathbb{R}$ is an interval. Suppose $f(M)$ is not an interval we want to arrive at a contradiction. Then if $f(a), f(b) \in f(M)$ there exist $c \in M$ such that $f(a) < f(c) < f(b)$ and $f(c) \notin f(M)$. We know that since M is connected then $f(M)$ is connected because d is continuous. So let us take the sets

$$A = \{f(x) \in f(M) : f(x) < f(c)\} \quad \text{and} \quad B = \{f(x) \in f(M) : f(x) > f(c)\}$$

Then we have that $f(M) = A \cup B$ and A and B are disjoint, nonempty, open sets so this would imply that $f(M)$ is disconnected, a contradiction. Therefore $f(M)$ is an interval.

Now let us prove that f is surjective and $f(M)$ is not a singleton. Let us start with surjective-ness for that let $c \in f(M)$ we know that c is of the form $d(e, a)$ for some $e \in M$ then f is surjective. Also, let $a, b \in M$ by definition, then $f(a), f(b) \in f(M)$ and $d(a, a) = 0 \neq d(b, a)$ otherwise $a = b$ so $f(M)$ is not a singleton.

Finally, suppose M is countable we want to arrive at a contradiction. Since f is surjective this would imply that $f(M)$ is countable, which is a contradiction since it is an interval from $\mathbb{R}$. Therefore it must be that M is uncountable. □

Proof. **26** Let $H = \{(x, f(x)) : x \in (0, 1]\}$ and $G = \{(x, f(x)) : x \in [0, 1]\}$. We want to prove that H is connected. Let $h : (0, 1] \to \mathbb{R}^2$ where $h(x) = (x, f(x))$ we know because of Lemma 5.8 that h is continuous because f is continuos in $(0, 1]$. Then since $(0, 1]$ is connected we have that $h((0, 1]) = H$ is also connected.

Let $x_n = 1/n\pi$ and $y_n = \sin(n\pi)$ where $(x_n) \subset (0, 1]$ and $(y_n) \subset (0, 1]$. We know that $x_n \to 0$ and $y_n \to 0$ so it must happen that $(x_n, y_n) \to (0, 0)$ which implies that $(0, 0) \in \mathrm{cl}(H)$ hence $G \subset \mathrm{cl}(H)$ but also we have that $H \subset G \subset \mathrm{cl}(H)$ therefore since H is connected G is also connected i.e. the graph of the function f is connected. □

Proof. **27** We want to prove that $f(t) = x + t(y-x)$ is a homeomorphism from $[0,1]$ to V where $x \neq y$.

First, we want to prove f is continuous. Let $\epsilon > 0$ and let $s, t \in [0,1]$ if $|s-t| < \delta$ where $\delta = \epsilon/\|y-x\|$ (we can do this since $x \neq y$) we have that

$$\begin{aligned} |s-t| &< \frac{\epsilon}{\|y-x\|} \\ \|(s-t)(y-x)\| &< \epsilon \\ \|s(y-x) - t(y-x)\| &< \epsilon \\ \|x + s(y-x) - (x + t(y-x))\| &< \epsilon \\ \|f(s) - f(t)\| &< \epsilon \end{aligned}$$

Therefore f is continuous.

To prove that f is one-to-one suppose $f(t) = f(s)$ then $x + t(y-x) = x + s(y-x)$ which implies that $t = s$ and therefore f is one-to-one.

To prove that f is onto let $w \in V$ then there is $t \in [0,1]$ where $t = (w-x)/(y-x)$ such that $f(t) = x + \frac{(w-x)}{(y-x)}(y-x) = w$ therefore f is onto.

If $f(t_n) \to f(t)$ then given $\epsilon' > 0$ there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\|f(t_n) - f(t)\| < \epsilon'$ so we have that

$$\begin{aligned} \|x + t_n(y-x) - (x + t(y-x))\| &< \epsilon' \\ \|t_n(y-x) - t(y-x))\| &< \epsilon' \\ |t_n - t|\|y-x\| &< \epsilon' \\ |t_n - t| &< \frac{\epsilon'}{\|y-x\|} = \epsilon \end{aligned}$$

Therefore when $f(t_n) \to f(t)$ we have that also must happen that $t_n \to t$ so f^{-1} is also continuous.

Finally, since f and f^{-1} are continuous and f is one-to-one and onto then f is a homeomorphism from $[0,1]$ to V. $\square$

Proof. **28** Let f be a homeomorphism from $[0,1]$ to V where V is a normed vector space and it's defined as $f(t) = x + t(y-x)$ where $x \neq y \in V$.

We know that $[0,1]$ is connected then $f([0,1]) = [x,y]$ is also connected because of Theorem 6.6 and the interval notation is justified because f is a homeomorphism then $E = \bigcup_{x \neq y \in V} [x,y]$ is also connected so every pair of points of V is in E and because of the result of problem 7 we have that V is also connected. $\square$