Solved selected problems of Real Analysis - Carothers

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Chapter 4 - Continuous Functions

Proof. **7**

(a) Let $(a, \infty) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(a, \infty)] = \{x : x \in M \text{ and } f(x) > a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv).

In the same way, let $(-\infty, a) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(-\infty, a)] = \{x : x \in M \text{ and } f(x) < a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv).

(b) We proved the more general result in part (c) which also applies in this case.

(c) Let V be an open set of $\mathbb{R}$ then since the collection of open intervals with rational endpoints is a base for $\mathbb{R}$ we can write V as

$$V = \bigcup_{\alpha} (p_\alpha, q_\alpha)$$

where $p_\alpha, q_\alpha \in \mathbb{Q}$ so we have that

$$f^{-1}[V] = \bigcup_{\alpha} f^{-1}[(p_\alpha, q_\alpha)]$$

then we can write that

$$f^{-1}[(p_\alpha, q_\alpha)] = f^{-1}[(p_\alpha, \infty)] \cap f^{-1}[(-\infty, q_\alpha)]$$

Also, we know that

$$f^{-1}[(p_\alpha, \infty)] = \{x : f(x) > p_\alpha\} \text{ and } f^{-1}[(-\infty, q_\alpha)] = \{x : f(x) < q_\alpha\}$$

and we know both of them are open sets so $f^{-1}[(p_\alpha, q_\alpha)]$ is the intersection of a finite number of open sets then it is also an open set. Finally, since $f^{-1}[V]$ is the union of open sets it's also an open set. Therefore f is continuous.

$\square$

Proof. **10** Let $\epsilon > 0$ and let us take $\delta = 1$ no matter the value of ϵ then

$$B_\delta(2) = \{x \in A : d(2, x) < 1\} = \{2\}$$

So we have that $f(B_\delta(2)) = \{f(2)\}$ and certainly it must happen that $\{f(2)\} \subset B_\epsilon(f(2))$ because $f(2) \in B_\epsilon(f(2))$. Therefore f is continuous at 2. $\square$

Proof. **11**

(a) Let $x \in A \cup B$, then $x \in A$, $x \in B$ or both of them, also let $\epsilon > 0$ then we know there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ because f is continuous at x by the definition.

(b) Let $A = (0, 1)$ and $B = [1, 2)$ also let $f : A \to \mathbb{R}$ be defined as $f(x) = x$ and $f : B \to \mathbb{R}$ as $f(x) = x + 1$ then we see that $f : A \cup B \to \mathbb{R}$ is not continuous at $x = 1$. Therefore the statement is false.

$\square$

Proof. **14** Given that a continuous function on $\mathbb{R}$ is completely determined by its values on $\mathbb{Q}$. For each $q \in \mathbb{Q}$ we have that $f(q)$ has a cardinality of $\mathfrak{c}$ since for each real number $x \in \mathbb{R}$ we can find an f such that $x = f(q)$. So the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ has a cardinality of $\mathfrak{c}^{|\mathbb{Q}|}$ and doing some cardinality algebra we get that

$$\mathfrak{c}^{|\mathbb{Q}|} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$$

Therefore there are $\mathfrak{c}$ continuous function $f : \mathbb{R} \to \mathbb{R}$. $\square$

Proof. **17** Let $x \in M$, and let us also define $(x_n) \subset D$ such that $x_n \to x$ which we know it exists because D is dense. Then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ also we know that $f(x_n) = g(x_n)$ for every $x_n \in (x_n)$. Finally, since sequences have unique limits it must happen that $f(x) = g(x)$ as we wanted to show.

In the same way, suppose we define $(x_n) \subset D$ such that $x_n \to x$ where $x \in M$. Then $f(x_n) \to f(x)$ also $(f(x_n)) \subset f(D)$. So we have a sequence $(f(x_n))$ for any $f(x)$ and we know that every $y \in N$ has the form $y = f(x)$ because f is onto. Therefore $f(D)$ is dense in N. $\square$

Proof. **22** Let $n, m \in \mathbb{N}$ we want to show that $d(E(n), E(m)) = d(n, m)$ this means that $\|E(n) - E(m)\|_1 = |n - m|$. Let us suppose that $n > m$ then $n = m + b$ where $b \in \mathbb{N}$ and so $|n - m| = b$. Then we have that

$$\|E(n) - E(m)\|_1 = \sum_{i=1}^{\infty} |E_i(n) - E_i(m)|$$

where $E_i(n)$ is the value of the ith element in the sequence, the same for $E_i(m)$. If $i \in \{1, 2, ..., m\}$ we have that $|E_i(n) - E_i(m)| = |1 - 1| = 0$ and for $i \in \{n + 1, n + 2, ...\}$ we have that $|E_i(n) - E_i(m)| = |0 - 0| = 0$ so we can write the following

$$\|E(n) - E(m)\|_1 = \sum_{i=m}^{n} |E_i(n) - 0| = \sum_{i=m}^{n} 1 = n - m = b$$

Therefore $\|E(n) - E(m)\|_1 = |n - m|$ as we wanted, in the case of $m \geq n$ the proof is analogous because we are taking the absolute value inside the sum. $\square$

Proof. **23** Let $S : c_0 \to c_0$ be defined as $S(x_0, x_1, ...) = (0, x_0, x_1, ...)$ such that S shifts the entries forward and puts 0 in the empty slot. We want to prove that $d(S(x), S(y)) = d(x, y)$ where $x = (x_0, x_1, ...)$ and $y = (y_0, y_1, ...)$ then

$$\|x - y\|_\infty = \sup\{|x_0 - y_0|, |x_1 - y_1|, ...\}$$

And $\sup\{|x_0 - y_0|, |x_1 - y_1|, ...\} \geq 0$ because both x and y are sequences that tend to 0. Let us suppose that $\sup_n |x_n - y_n| = 0$ then we see that

$$\|S(x) - S(y)\|_\infty = \sup\{|0 - 0|, |x_0 - y_0|, |x_1 - y_1|, ...\} = 0$$

Now let us suppose that $\|x - y\|_\infty > 0$ then we have that

$$\sup\{|x_0 - y_0|, |x_1 - y_1|, ...\} = \sup\{|0 - 0|, |x_0 - y_0|, |x_1 - y_1|, ...\}$$

because $\|S(x) - S(y)\|_\infty$ cannot be 0. Therefore we have that

$$\|S(x) - S(y)\|_\infty = \|x - y\|_\infty$$

as we wanted.

$\square$

Proof. **24** Let $f : \mathbb{R} \to V$ such that for each $\alpha \in \mathbb{R}$ we map it to $\alpha y \in V$ where $y \in V$. Let $\alpha, \beta \in \mathbb{R}$ we want to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(\alpha) - f(\beta)\| < \epsilon$ whenever $|\alpha - \beta| < \delta$. So let us define $\delta = \epsilon/\|y\|$ if $\|y\| \neq 0$ then we have that when $|\alpha - \beta| < \delta$ we get that

$$\begin{aligned} |\alpha - \beta| &< \frac{\epsilon}{\|y\|} \\ |\alpha - \beta|\|y\| &< \epsilon \\ \|\alpha y - \beta y\| &< \epsilon \\ \|f(\alpha) - f(\beta)\| &< \epsilon \end{aligned}$$

Therefore f is continuous.

Let now $f : V \to V$ such that for each $x \in V$ we map it to $x + y \in V$ where $y \in V$. Let $x, x' \in V$ we want to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(x) - f(x')\| < \epsilon$ whenever $\|x - x'\| < \delta$. So let us define $\delta = \epsilon$ then we have that when $\|x - x'\| < \delta$ we get that

$$\|x - x'\| = \|(x + y) - (x' + y)\| = \|f(x) - f(x')\| < \epsilon$$

Therefore f is continuous. $\square$

Proof. **25** Let $f : (M, d) \to (N, \rho)$ be a Lipschitz mapping so there is $K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$ also let's observe that $K \geq 0$ since both $\rho(f(x), f(y)) \geq 0$ and $d(x, y) \geq 0$

Let now $\epsilon > 0$ and $\delta > 0$, we want to prove that if $d(x, y) < \delta$ then $\rho(f(x), f(y)) < \epsilon$ i.e. f is continuous. So let $\delta = \epsilon/K$ with $K > 0$ then we have that if $d(x, y) < \epsilon/K$ then $Kd(x, y) < \epsilon$ and since $\rho(f(x), f(y)) \leq Kd(x, y)$ we get that $\rho(f(x), f(y)) < \epsilon$ as we wanted. If $K = 0$ then it must happen that $\rho(f(x), f(y)) = 0$ and then $\rho(f(x), f(y)) = 0 < \epsilon$ which by definition is true. Therefore f is continuous. $\square$

Proof. **27** Let $k \geq 1$ and $f : l_\infty \to \mathbb{R}$ defined as $f(x) = x_k$ we want to prove that f is continuous but we will prove that f is Lipschitz which implies that f is continuous.

Let $x, y \in l_\infty$ then we have that for some fixed $k \geq 1$ the following is always true because of the definition of supremum

$$|f(x) - f(y)| = |x_k - y_k| \leq \sup_n |x_n - y_n| = \|x - y\|_\infty$$

then for $K = 1$ we see that

$$|f(x) - f(y)| \leq K\|x - y\|_\infty$$

Therefore f is Lipschitz and hence continuous. $\square$

Proof. **31**

(a) Let $x \in M$ since $M = \cup_{n=1}^{\infty} U_n$ then x is in some U_n we want to show that for every sequence $(x_n) \subset M$ such that $x_n \to x$ we have that $f(x_n) \to f(x)$. For every $(x_n) \subset U_n$ such that $x_n \to x$ since f is continuous in U_n then we have that $f(x_n) \to f(x)$. But if (x_n) is not completely in U_n then since U_n is an open set it must happen that eventually for some n onwards $x_n \in U_n$ and since f is continuous in U_n we have that $f(x_n) \to f(x)$. Therefore f is also continuous in M.

(b) Let $F \in N$ be a closed set then we have that

$$f^{-1}(F) \cap M = f^{-1}(F) \cap \bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} f^{-1}(F) \cap E_n$$

We know that each $f^{-1}(F) \cap E_n$ is closed in E_n because f is continuous in E_n and we know that the finite union of closed sets is closed then $f^{-1}(F) \cap M$ is closed and therefore M is continuous.

(c) Let $E_n = [\frac{1}{n}, 1]$ we see that $\cup_{n=1}^{\infty} E_n = (0, 1]$ which is not a closed set. Therefore even though f is continuous in every E_n we based part of our proof on the fact that the finite union of closed sets is closed which is not the case here.

□

Proof. **34** Let $(M \times M, \rho)$ be a metric space where ρ is defined as

$$\rho((a, b), (c, d)) = d(a, c) + d(b, d)$$

and d is a metric on M. We want to prove that if $(x_n, y_n) \to (x, y)$ then $d(x_n, y_n) \to d(x, y)$ i.e. that d is continuous.

If $(x_n, y_n) \to (x, y)$ then this means that $x_n \to x$ and $y_n \to y$ then $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$ which implies that

$$\rho((x_n, y_n), (x, y)) = d(x_n, x) + d(y_n, y) < \epsilon$$

for some $\epsilon > 0$. Also, we have that

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \quad \text{and} \quad d(x, y_n) \leq d(x, y) + d(y, y_n)$$

so joining these inequalities we get that

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y) < \epsilon$$

On the other hand, we also have that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \quad \text{and} \quad d(x_n, y) \leq d(x_n, y_n) + d(y_n, y)$$

then

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y) < \epsilon$$

hence $|d(x, y) - d(x_n, y_n)| < \epsilon$. Therefore $d(x_n, y_n) \to d(x, y)$ as we wanted.

□

Proof. **43**

($\rightarrow$) We want to prove that $i : (M, d) \rightarrow (M, \rho)$ (the identity map) is a homeomorphism from (M, d) to (M, ρ).

Let us check first that i is a one-to-one map. Let $i(a), i(b) \in (M, \rho)$ such that $i(a) = i(b)$ then by definition $a = i(a) = i(b) = b$. Then i is a one-to-one map.

Let us also check that i is an onto map too. Let us take $a \in (M, \rho)$ then by definition we have $a \in (M, d)$ such that $i(a) = a \in (M, \rho)$. Then i is an onto map.

Also, we want to prove that i is continuous. Let $x \in (M, d)$ then if $d(x_n, x) \rightarrow 0$ (i.e. $x_n \rightarrow x$) for any $(x_n) \subset M$ we have that $\rho(x_n, x) \rightarrow 0$ because d and ρ are equivalent metrics. Therefore since i maps both x_n and x to themselves this implies that $i(x_n) \rightarrow i(x)$ where $i(x_n), i(x) \in (M, \rho)$.

Finally, we want to prove that also i^{-1} is continuous. In the same way let $x \in (M, \rho)$ if $\rho(x_n, x) \rightarrow 0$ (i.e. $x_n \rightarrow x$) for any $(x_n) \subset (M, \rho)$ we have that $d(x_n, x) \rightarrow 0$ because d and ρ are equivalent metrics. Therefore since i^{-1} maps both x_n and x to themselves this implies that $i^{-1}(x_n) \rightarrow i^{-1}(x)$ where $i^{-1}(x_n), i^{-1}(x) \in (M, d)$.

Therefore i is an homeomorphism from (M, d) to (M, ρ).

($\leftarrow$) We want to prove that d and ρ are equivalent metrics on M knowing that i (the identity map) is a homeomorphism from (M, d) to (M, ρ).

Let $x \in (M, d)$ and $(x_n) \subset (M, d)$ then if $d(x_n, x) \rightarrow 0$ we have that $\rho(i(x_n), i(x)) \rightarrow 0$ because i is continuous, but by definition, this also implies that $\rho(x_n, x) \rightarrow 0$.

In the same way, let $x \in (M, \rho)$ and $(x_n) \subset (M, \rho)$ then if $\rho(x_n, x) \rightarrow 0$ we have that $d(i^{-1}(x_n), i^{-1}(x)) \rightarrow 0$ because i^{-1} is continuous, but by definition, this also implies that $d(x_n, x) \rightarrow 0$.

Therefore d and ρ are equivalent metrics. $\square$

Proof. **44** We want to prove that "is homeomorphic to" is an equivalence relation, so we will prove it is a reflexive, symmetric and transitive relation.

Reflexivity: Let (M,d) be a metric space, we want to prove that (M,d) is homeomorphic to itself. Let i be the identity map, we saw in problem 43 that i is a homeomorphism from (M,d) to (M,d), then (M,d) is homeomorphic to itself.

Symmetry: Let (M,d) be homeomorphic to (N,ρ) then there is a relation $f : M \to N$ which is one-to-one and onto such that f and f^{-1} are continuous. Then we can define $g : N \to M$ such that $g = f^{-1}$ which is one-to-one and onto because f is bijective. Also, g is continuous because f^{-1} is continuous and $g^{-1} = (f^{-1})^{-1} = f$ is continuous because f is continuous. Therefore (N,ρ) is homeomorphic to (M,d).

Transitivity: Let (M,d) be homeomorphic to (N,ρ) and let (N,ρ) be homeomorphic to (L,τ). We can define $h : M \to L$ as $h = g \circ f$ where $f : M \to N$ and $g : N \to L$ and they are the respective homeomorphisms. Since f and g are one-to-one and onto then h is also one-to-one and onto. Also, we see that $h^{-1} = (g \circ f)^{-1} = g^{-1} \circ f^{-1}$ and by the properties of the composition of functions we see that if f, f^{-1}, g and g^{-1} are continuous then both $h = g \circ f$ and $h^{-1} = g^{-1} \circ f^{-1}$ are also continuous. Therefore (M,d) is homeomorphic to (L,τ).

Finally, since the relation "is homeomorphic to" is reflective, symmetric and transitive then it is an equivalence relation between metric spaces. □

Proof. **45** Let $M = \{1/n : n \geq 1\}$ and $f : \mathbb{N} \to M$ such that $f(n) = 1/n$.

Let us prove first that f is one-to-one then let $n, m \in \mathbb{N}$ such that $f(n) = f(m)$ then $1/n = 1/m$ i.e. $n = m$ so f is one-to-one.

Let us prove now that f is an onto map. Let us take $a \in M$ then a has the form of $a = 1/b$ where $b \in \mathbb{N}$ then there is always a $b \in \mathbb{N}$ such that $f(b) = 1/b = a$ i.e. f is an onto map.

Now we want to prove that f is continuous. Let $f^{-1} : M \to \mathbb{N}$ defined as $f^{-1}(1/n) = n$. Since $\mathbb{N}$ and M are discrete then a subset $V \subset M$ is open and because f^{-1} is bijective then $f^{-1}(V) \subset \mathbb{N}$ is also open. Therefore f is continuous.

In the same way, we want to prove that f^{-1} defined as we said is also continuous. Since $\mathbb{N}$ and M are discrete then a subset $V' \subset \mathbb{N}$ is open and because f is bijective $(f^{-1})^{-1}(V') = f(V') \subset M$ is also open. Therefore f^{-1} is continuous.

So taking into account all these results we conclude that $\mathbb{N}$ is homeomorphic to M. □

Proof. **48** We want to prove first that $\mathbb{R}$ is homeomorphic to $(0,1)$ this can be accomplished if we define $f : \mathbb{R} \to (0,1)$ such that $f(x) = \arctan(x)/\pi + 1/2$ since this map is bijective and continuous also $f^{-1}(x) = \tan(\pi(x - 1/2))$ is continuous over $(0,1)$.

Now we want to prove that $(0,1)$ is homeomorphic to $(0,\infty)$ but we will prove first that $\mathbb{R}$ is homeomorphic to $(0,\infty)$ so we define $f : \mathbb{R} \to (0,\infty)$ such that $f(x) = e^x$ we see that f is bijective and continuous also $f^{-1}(x) = \log(x)$ is continuous over $(0,\infty)$. So by composing these results, we have that a map $g : (0,1) \to (0,\infty)$ such that $g(x) = e^{\tan(\pi(x-1/2))}$ is an homeomorphism between $(0,1)$ and $(0,\infty)$ as we wanted.

Let $x = 0$ and $y = 2$ then $|0 - 2| = 2$ but $|f(0) - f(2)|$ is at most close to 1. Therefore $\mathbb{R}$ is not isometric to $(0,1)$.

Let $f : \mathbb{R} \to (0,\infty)$ be an isometry from $\mathbb{R}$ to $(0,\infty)$ we want to arrive at a contradiction. Let $x \in \mathbb{R}$ and suppose $|x - 0| = |x| = |f(x) - f(0)|$ then we have that $f(x) = f(0) + x$ or $f(x) = f(0) - x$. Let us take $a \notin (0,\infty)$ then $f(a - f(0)) = f(0) + a - f(0) = a$ or $f(a - f(0)) = f(0) - a + f(0) = 2f(0) - a$ where we see that the first one cannot happen (otherwise $a \in (0,\infty)$). Also, let us consider that $f(f(0) - a) = 2f(0) - a$ or $f(f(0) - a) = a$ must be true where again the last case cannot happen, but also since f is injective we have that $f(0) - a = a - f(0)$ which implies that $f(0) = a$ hence $a \in (0,\infty)$ a contradiction. Therefore $\mathbb{R}$ is not isometric to $(0,\infty)$. □

Proof. **49** Let $y \in V$ and $f : V \to V$ which is defined as $f(x) = x + y$ we want to show that f is an isometry on V. Let $x, z \in V$ since V is a normed vector space then we have that

$$\|f(x) - f(z)\| = \|x + y - (z + y)\| = \|x - z\|$$

Therefore f is an isometry on V.

Now let $\alpha \in \mathbb{R}$ and $g : V \to V$ which is defined as $g(x) = \alpha x$ we want to prove that g is a homeomorphism on V.

Let $x, z \in V$ and suppose $g(x) = g(z)$ then $x = z$ hence g is a one-to-one map. Also, let $v \in V$ such that v is in the image of g then by definition there must be an $x \in V$ such that $v = \alpha x$ so g is an onto map too.

Let us show now that g is continuous. Let $\epsilon > 0$ and $x, z \in V$ then let us take $\delta = \epsilon/|\alpha|$ so when $\|x - z\| < \delta$ we get that

$$\begin{aligned} \|x - z\| &< \frac{\epsilon}{|\alpha|} \\ |\alpha|\|x - z\| &< \epsilon \\ \|\alpha x - \alpha z\| &< \epsilon \\ \|g(x) - g(z)\| &< \epsilon \end{aligned}$$

Therefore g is continuous.

Finally, we want to show that $g^{-1}(x) = x/\alpha$ (which we can define this way because α is nonzero) is continuous. Let $\epsilon > 0$ and let $x, z \in V$ (the image) then let us take $\delta = \epsilon|\alpha|$ so when $\|x - z\| < \delta$ we have that

$$\begin{aligned} \|x - z\| &< \epsilon|\alpha| \\ \left|\frac{1}{\alpha}\right| \|x - z\| &< \epsilon \\ \left\|\frac{x}{\alpha} - \frac{z}{\alpha}\right\| &< \epsilon \\ \|g^{-1}(x) - g^{-1}(z)\| &< \epsilon \end{aligned}$$

Therefore g^{-1} is continuous.

Joining all these results we see that g is a homeomorphism on V. $\square$

Proof. **50** Let us define a map $f : (M,d) \to (M,\rho)$ such that

$$f(m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 1 \\ 1/n & \text{if } m = 1/n \text{ where } n \geq 2 \end{cases}$$

Let $a, b \in (M,d)$ then

(i) If $a = b = 0$ we have that

$$d(a,b) = |a-b| = 0 = \rho(1,1) = \rho(f(a), f(b))$$

(ii) If $a = b = 1$ we have that

$$d(a,b) = |a-b| = 0 = \rho(0,0) = \rho(f(a), f(b))$$

(iii) If $a = 1/n$ and $b = 1/n'$ where $n, n' \geq 2$ we have that

$$d(a,b) = |1/n - 1/n'| = \rho(1/n, 1/n') = \rho(f(a), f(b))$$

(iv) If $a = 0$ and $b = 1$ (or $a = 1$ and $b = 0$) we have that

$$d(a,b) = |a-b| = 1 = \rho(1,0) = \rho(f(a), f(b))$$

(v) If $a = 0$ and $b = 1/n$ (or $a = 1/n$ and $b = 0$) where $n \geq 2$ we have that

$$d(a,b) = |0 - 1/n| = 1/n = \rho(1, 1/n) = \rho(f(a), f(b))$$

(vi) If $a = 1$ and $b = 1/n$ (or $a = 1/n$ and $b = 1$) where $n \geq 2$ we have that

$$d(a,b) = |1 - 1/n| = 1 - 1/n = \rho(0, 1/n) = \rho(f(a), f(b))$$

Therefore f is an isometry which implies it's also a homeomorphism.

Finally, we want to prove that $i : (M,d) \to (M,\rho)$ the identity map is not continuous. We see that $\{0\} \subset (M,d)$ is not an open set since there is no $\epsilon > 0$ such that $B_\epsilon^d(0) \subset (M,d)$. But $\{0\} \subset (M,\rho)$ is an open set since $\rho(0,1/n) = 1 - 1/n \geq 1/2$ and $\rho(0,1) = 1$ then there is $\epsilon = 1/2$ such that $B_\epsilon^\rho(0) = \{0\} \subset (M,\rho)$. So if we take $V = \{0\}$ an open set in (M,ρ) we have that $i^{-1}(V) = i^{-1}(\{0\}) = \{0\} \subset (M,d)$ is not open and therefore i is not continuous. $\square$

Proof. **52** We want to probe Theorem 55. Let $f : (M, d) \to (N, \rho)$ be one-to-one and onto.

$(i) \Rightarrow (ii)$ Suppose f is a homeomorphisms and there is $(x_n) \subset (M, d)$ such that $x_n \to x$ then since f is continuous we have that $f(x_n) \to f(x)$. Now let us suppose that $f(x_n) \to f(x)$ then since f is an homeomorphism there is f^{-1} which is also continuous then we have that $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ hence $x_n \to x$.

$(ii) \Rightarrow (iii)$ Let G be an open set in M and let a sequence $(x_n) \subset M$ such that $x_n \to x$ where $x \in G$ then we have that $x_n \in G$ for all but finitely many n, but also we have that $f(x_n) \to f(x)$ so since $f(x) \in f(G)$ and f is bijective it must happen that all but finitely many $f(x_n) \in f(G)$ therefore $f(G) \subset N$ is an open set.

Now let us suppose $f(G) \subset N$ is open, also, let $f(x) \in f(G)$ such that $f(x_n) \to f(x)$ since $f(G)$ is open then we have that $f(x_n) \in f(G)$ for all but finitely many n. But also we know that f is bijective then $x \in G$ and if $x_n \to x$ it must happen that $x_n \in G$ for all but finitely many n. Therefore $G \subset M$ is an open set.

$(iii) \Rightarrow (iv)$ Let $E \subset M$ be a closed set then $M \setminus E$ is open in M then $f(M \setminus E)$ is an open set in N therefore $N \setminus f(M \setminus E)$ is a closed set in N and since f is bijective it must happen that $f(E) = N \setminus f(M \setminus E)$.

Let now $f(E)$ be a closed set in N then $N \setminus f(E)$ is an open set in N then $f^{-1}(N \setminus f(E))$ is an open set in M therefore $M \setminus f^{-1}(N \setminus f(E))$ is closed and since f^{-1} is bijective (as well as f) we have that $E = M \setminus f^{-1}(N \setminus f(E))$.

$(i) \Leftrightarrow (v)$ Let f be a homeomorphisms and $(x_n) \subset (M, d)$ be a sequence that tends to $x \in M$. Since f is continuous if $x_n \to x$ then $f(x_n) \to f(x)$ then $d(x_n, x) \to 0$ and $\hat{d}(x_n, x) = \rho(f(x_n), f(x)) \to 0$ hence d is equivalent to $\hat{d}$.

Let $\hat{d}(x, y) = \rho(f(x), f(y))$ be equivalent to the metric $d(x, y)$ on M. We want to prove that f and f^{-1} are continuous, i.e. f is a homeomorphism (we already know f is bijective). Let $(x_n) \subset M$ be a sequence that tends to $x \in M$, then $d(x_n, x) \to 0$ but since $\hat{d}$ is also equivalent to d this implies that $\rho(f(x_n), f(x)) \to 0$ hence f is continuous. Now let $(f(x_n)) \subset N$ be a sequence that tends to $f(x) \in N$ then $\hat{d}(x_n, x) = \rho(f(x_n), f(x)) \to 0$ but since $\hat{d}$ is equivalent to d this implies that $d(x_n, x) \to 0$, hence f^{-1} is continuous. Therefore f is a homeomorphism.

$(iv) \Rightarrow (i)$ We want to prove that f and f^{-1} are continuous (we already know that f is bijective). Let E be closed in N, then because of (iv) we have that $f^{-1}(E)$ is closed in M hence f is continuous. In the same way, if E is closed in M then $f(E)$ is closed in N so f^{-1} is also continuous. Therefore f is a homeomorphism.

□

Proof. **55**

($\rightarrow$) Let M be separable and $f : (M, d) \rightarrow (N, \rho)$ a homeomorphism. Also, let $A \subset M$ be a countable dense set in M then $f(A)$ is countable since f is bijective but also we know that for every $x \in M$ there is a sequence $(x_n) \subset A$ such that $x_n \rightarrow x$. Since f is continuous we have $(f(x_n)) \subset f(A)$ such that $f(x_n) \rightarrow f(x)$ where $f(x) \in N$. Hence $f(A)$ is a countable dense subset of N and therefore N is separable.

($\leftarrow$) In the same way, if N is separable let us define a countable dense subset $B \subset N$ then $f^{-1}(B) \subset M$ is countable since f^{-1} is bijective (as well as f). But also we know that for every $y \in N$ there is a sequence $(y_n) \subset B$ such that $y_n \rightarrow y$. Since f^{-1} is continuous we have $(f^{-1}(y_n)) \subset f^{-1}(B)$ such that $f^{-1}(y_n) \rightarrow f^{-1}(y)$ where $f^{-1}(y) \in M$. Hence $f^{-1}(B)$ is a countable dense subset of M and therefore M is separable. □

Proof. **57** Let $f : (M, d) \rightarrow (N, \rho)$ be one-to-one and onto.

$(i) \Rightarrow (ii)$ Suppose f is open, so if $U \subset M$ is open then $f(U) \subset N$ is open. From Theorem 5.5. we know that "U is open if and only if $f(U)$ is open" is equivalent to "E is closed if and only if $f(E)$ is closed" when f is bijective (like in this case). Therefore f is closed

$(ii) \Rightarrow (iii)$ We want to prove that f^{-1} is continuous. We know that if E is closed in M then $f(E)$ is closed in N therefore f^{-1} is continuous.

$(iii) \Rightarrow (i)$ From Theorem 5.1. we know that if f^{-1} is continuous and if U is open in M then $f(U)$ is open in N. Therefore f is open.

□

Proof. **58**

($\Rightarrow$) Let f be a homeomorphism and A a subset of M. Also let $x \in \overline{A}$ then there is $(x_n) \in A$ such that $x_n \to x$. Since f is a homeomorphism we have that also $f(x_n) \to f(x)$ where $f(x_n) \in f(A)$ and $f(x) \in f(\overline{A})$ because f is bijective. But also $f(x) \in \overline{f(A)}$ because of Corollary 4.11. Then this implies that $f(\overline{A}) \subseteq \overline{f(A)}$.

In the same way, let $f(x) \in \overline{f(A)}$ then there is $(f(x_n)) \subset f(A)$ such that $f(x_n) \to f(x)$. Since f is a homeomorphism we also have that $x_n \to x$ where $x_n \in A$ and $x \in \overline{A}$ because f is bijective. But also again since f is bijective we have that $f(x) \in f(\overline{A})$. Then this implies that $\overline{f(A)} \subseteq f(\overline{A})$. Therefore $\overline{f(A)} = f(\overline{A})$.

($\Leftarrow$) Let E be a closed set in M so $E = \overline{E}$ then $f(E) = f(\overline{E}) = \overline{f(E)}$ therefore f is closed.

On the other hand, let B be closed set in N since f is bijective there must be A such that $f(A) = B$ but also since B is closed we have that $B = \overline{B} = \overline{f(A)}$. We also have that $f(\overline{A}) = \overline{f(A)}$ so $B = f(\overline{A})$ then $f^{-1}(B) = \overline{A}$ which is closed. Therefore f^{-1} is closed.

Finally, since f and f^{-1} are both closed then f is a homeomorphism. $\square$

Proof. **63**

(i) We want to show that $\sigma(t) = a + t(b-a)$ is a homeomorphism. We will assume $b \neq a$.

Let us prove first it is a one-to-one function. Suppose $\sigma(t) = \sigma(t')$ then $a + t(b-a) = a + t'(b-a)$ which implies that $t = t'$ since $b \neq a$.

Now we want to prove it is an onto function. Let $c \in [a,b]$ then there is $t_c = \frac{c-a}{b-a}$ such that $\sigma(t_c) = c$ i.e. σ is onto.

Next, we want to prove that σ is continuous. Let $\epsilon > 0$ we want to show that $|\sigma(x) - \sigma(y)| < \epsilon$ whenever $|x-y| < \delta$. Let $\delta = \epsilon/|b-a|$ then if $|x-y| < \delta$ we have that

$$\begin{aligned} |x-y| &< \frac{\epsilon}{|b-a|} \\ |x-y||b-a| &< \epsilon \\ |a + x(b-a) - a - y(b-a)| &< \epsilon \\ |\sigma(x) - \sigma(y)| &< \epsilon \end{aligned}$$

Finally, we want to prove that $\sigma^{-1}(t) = \frac{t-a}{b-a}$ is also continuous. Let $\epsilon > 0$ we want to show that $|\sigma^{-1}(x) - \sigma^{-1}(y)| < \epsilon$ whenever $|x-y| < \delta$. Let $\delta = \epsilon|b-a|$ then if $|x-y| < \delta$ we have that

$$\begin{aligned} |x-y| &< \epsilon|b-a| \\ \frac{|x-y|}{|b-a|} &< \epsilon \\ \frac{|(x-a) - (y-a)|}{|b-a|} &< \epsilon \\ \left| \frac{x-a}{b-a} - \frac{y-a}{b-a} \right| &< \epsilon \\ |\sigma^{-1}(x) - \sigma^{-1}(y)| &< \epsilon \end{aligned}$$

(ii) ($\Rightarrow$) Let $f \in C[a,b]$. Since σ is a homeomorphism it is also continuous and because of the Lemma 5.7 we have that $f \circ \sigma$ is also continuous therefore $f \circ \sigma$ is a continuous map from $[0,1]$ to $\mathbb{R}$ i.e. $f \circ \sigma \in C[0,1]$.

($\Leftarrow$) Let $f \circ \sigma \in C[0,1]$, since σ is a homeomorphism then σ^{-1} is continuous and because of the Lemma 5.7 we have that $(f \circ \sigma) \circ \sigma^{-1}$ is also continuous but $(f \circ \sigma) \circ \sigma^{-1} = f$ therefore $f \in C[a,b]$.

(iii) Let $f, g \in C[a, b]$ we want to prove that the map $f \to f \circ \sigma$ is an isometry from $C[a, b]$ to $C[0, 1]$ then we have on one hand that

$$d(f, g) = \|f - g\|_\infty = \max_{a \le t \le b} |f(t) - g(t)|$$

And on the other hand, we have that

$$\begin{aligned}
\rho(f \circ \sigma, g \circ \sigma) &= \|f \circ \sigma - g \circ \sigma\|_\infty \\
&= \max_{0 \le t \le 1} |(f \circ \sigma)(t) - (g \circ \sigma)(t)| \\
&= \max_{0 \le t \le 1} |f(a + t(b - a)) - g(a + t(b - a))| \\
&= \max_{a \le t \le b} |f(t) - g(t)|
\end{aligned}$$

Therefore the map $f \to f \circ \sigma$ is an isometry.

(iv) Let $\alpha, \beta \in \mathbb{R}$ then we have that

$$\begin{aligned}
T((\alpha f + \beta g)(t)) &= ((\alpha f + \beta g) \circ \sigma)(t) \\
&= (\alpha f + \beta g)(a + t(b - a)) \\
&= \alpha f(a + t(b - a)) + \beta g(a + t(b - a)) \\
&= \alpha (f \circ \sigma)(t) + \beta (g \circ \sigma)(t) \\
&= \alpha T(f(t)) + \beta T(g(t))
\end{aligned}$$

(v) In this case, we have that

$$\begin{aligned}
T((fg)(t)) &= ((fg) \circ \sigma)(t) \\
&= (fg)(a + t(b - a)) \\
&= f(a + t(b - a))g(a + t((b - a))) \\
&= (f \circ \sigma)(t)(g \circ \sigma)(t) \\
&= T(f(t))T(g(t))
\end{aligned}$$

(vi) $(\Rightarrow)$ If $T(f(t)) \le T(g(t))$ for any $t \in [0, 1]$ then we have that

$$\begin{gathered}
f \circ \sigma(t) \le g \circ \sigma(t) \\
f(a + t(b - a)) \le g(a + t(b - a)) \\
f(t') \le g(t')
\end{gathered}$$

This implies that $f \le g$ for any $t' \in [a, b]$.

$(\Leftarrow)$ If $f(t) \le g(t)$ for any $t \in [a, b]$ then we have that $f(a + t'(b - a)) \le g(a + t'(b - a))$ for any $t' \in [0, 1]$ so $f \circ \sigma(t') \le g \circ \sigma(t')$. Therefore $T(f(t)) \le T(g(t))$.

$\square$

Proof. **64** Given $f, g \in C(\mathbb{R})$, we want to prove that

$$d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{(1 + d_n(f,g))}$$

defines a metric on $C(\mathbb{R})$ then

(i) Since $d_n(f,g) \geq 0$ for any $n \in \mathbb{N}$ and any $f, g \in C(\mathbb{R})$ then $d(f,g) \geq 0$. Also, given that $d_n(f,g) < 1 + d_n(f,g)$ we have that

$$0 \leq \frac{2^{-n} d_n(f,g)}{(1 + d_n(f,g))} \leq 2^{-n}$$

and since $\sum_{n=1}^{\infty} 2^{-n}$ converges we have that $d(f,g)$ also converges i.e. $d(f,g) < \infty$.

(ii) ($\Rightarrow$) Suppose $d(f,g) = 0$ then

$$\sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{(1 + d_n(f,g))} = 0$$

So it must happen that $d_n(f,g) = 0$ for every $n \in \mathbb{N}$ this implies that $\max_{|t| \leq n} |f(t) - g(t)| = 0$ for every $n \in \mathbb{N}$ hence it must happen that $f(t) = g(t)$ for all $|t| \leq n$. Since this is true for every $n \in \mathbb{N}$ then $f = g$.

($\Leftarrow$) Suppose $f = g$ then it must happen that $d_n(f,g) = 0$ for every $n \in \mathbb{N}$ since it is a pseudometric. Therefore $d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{(1+d_n(f,g))} = 0$.

(iii) Let $f, g \in C(\mathbb{R})$ then

$$d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(f,g)}{(1 + d_n(f,g))} = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(g,f)}{(1 + d_n(g,f))} = d(g,f)$$

Where we used that $d_n(f,g) = d_n(g,f)$ for every $n \in \mathbb{N}$ since it is a pseudometric and it has the symmetry property.

(iv) Let $f, g, h \in C(\mathbb{R})$. Since d_n is a pseudometric we have that

$$d_n(f,g) \leq d_n(f,h) + d_n(h,g)$$

Also, from problem 5 (of Chapter 3) we know that the function $F(t) = t/(1+t)$ is increasing so we have that

$$F(d_n(f,g)) \leq F(d_n(f,h) + d_n(h,g))$$
$$\frac{d_n(f,g)}{1 + d_n(f,g)} \leq \frac{d_n(f,h) + d_n(h,g)}{1 + d_n(f,h) + d_n(h,g)}$$

But also F satisfies $F(s+t) \leq F(s) + F(t)$ for $s, t \geq 0$ then joining these inequalities we have that

$$\frac{d_n(f,g)}{1+d_n(f,g)} \leq \frac{d_n(f,h)+d_n(h,g)}{1+d_n(f,h)+d_n(h,g)} \leq \frac{d_n(f,h)}{1+d_n(f,h)} + \frac{d_n(h,g)}{1+d_n(h,g)}$$

This implies that

$$\sum_{n=1}^{\infty} \frac{2^{-n}d_n(f,g)}{1+d_n(f,g)} \leq \sum_{n=1}^{\infty} \frac{2^{-n}d_n(f,h)}{1+d_n(f,h)} + \frac{2^{-n}d_n(h,g)}{1+d_n(h,g)}$$

Therefore $d(f,g) \leq d(f,h) + d(h,g)$ as we wanted.

Finally, since the metric d satisfies (i) to (iv) it defines a metric on $C(\mathbb{R})$. $\square$