Solved selected problems of Real Analysis - Carothers

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Chapter 4 - Continuous Functions

Proof. **7**

(a) Let $(a, \infty) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(a, \infty)] = \{x : x \in M \text{ and } f(x) > a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv). In the same way, let $(-\infty, a) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(-\infty, a)] = \{x : x \in M \text{ and } f(x) < a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv).

(b) We proved the more general result in part (c) which also applies in this case.

(c) Let V be an open set of $\mathbb{R}$ then since the collection of open intervals with rational endpoints is a base for $\mathbb{R}$ we can write V as

$$V = \bigcup_{\alpha} (p_\alpha, q_\alpha)$$

where $p_\alpha, q_\alpha \in \mathbb{Q}$ so we have that

$$f^{-1}[V] = \bigcup_{\alpha} f^{-1}[(p_\alpha, q_\alpha)]$$

then we can write that

$$f^{-1}[(p_\alpha, q_\alpha)] = f^{-1}[(p_\alpha, \infty)] \cap f^{-1}[(-\infty, q_\alpha)]$$

Also, we know that

$$f^{-1}[(p_\alpha, \infty)] = \{x : f(x) > p_\alpha\} \text{ and } f^{-1}[(-\infty, q_\alpha)] = \{x : f(x) < q_\alpha\}$$

and we know both of them are open sets so $f^{-1}[(p_\alpha, q_\alpha)]$ is the intersection of a finite number of open sets then it is also an open set. Finally, since $f^{-1}[V]$ is the union of open sets it's also an open set. Therefore f is continuous.

□

Proof. **10** Let $\epsilon > 0$ and let us take $\delta = 1$ no matter the value of ϵ then

$$B_\delta(2) = \{x \in A : d(2, x) < 1\} = \{2\}$$

So we have that $f(B_\delta(2)) = \{f(2)\}$ and certainly it must happen that $\{f(2)\} \subset B_\epsilon(f(2))$ because $f(2) \in B_\epsilon(f(2))$. Therefore f is continuous at 2. □

Proof. **11**

(a) Let $x \in A \cup B$, then $x \in A$, $x \in B$ or both of them, also let $\epsilon > 0$ then we know there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ because f is continuous at x by the definition.

(b) Let $A = (0, 1)$ and $B = [1, 2)$ also let $f : A \to \mathbb{R}$ be defined as $f(x) = x$ and $f : B \to \mathbb{R}$ as $f(x) = x + 1$ then we see that $f : A \cup B \to \mathbb{R}$ is not continuous at $x = 1$. Therefore the statement is false.

□