Solved selected problems of Real Analysis - Carothers

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Chapter 8 - Compactness

Proof. **1** If K is a non-empty compact subset of $\mathbb{R}$ then K is bounded and closed therefore the $\sup K \in K$ and $\inf K \in K$. □

Proof. **2** Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ then the complement on $\mathbb{Q}$ is

$$\begin{aligned} E^c =& \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\} \cup \\ & \{x \in \mathbb{Q} : x < 0 \text{ and } x^2 > 3\} \cup \\ & \{x \in \mathbb{Q} : x^2 < 2\} \end{aligned}$$

We see that $\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\} = (\sqrt{3}, \infty) \cap \mathbb{Q}$ where $(\sqrt{3}, \infty)$ and $\mathbb{Q}$ are open sets hence $\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\}$ is open. Also, we see that $\{x \in \mathbb{Q} : x < 0 \text{ and } x^2 > 3\} = (-\infty, -\sqrt{3}) \cap \mathbb{Q}$ and that $\{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ so both $\{x \in \mathbb{Q} : x < 0 \text{ and } x^2 > 3\}$ and $\{x \in \mathbb{Q} : x^2 < 2\}$ are open sets. Therefore since E^c is the union of open sets it's also an open set hence E is closed.

On the other hand, if $x \in E$ then $x \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ or $x \in (-\sqrt{2}, -\sqrt{3}) \cap \mathbb{Q}$ hence $-2 < x < 2$ which implies that E is bounded.

Let us call the $\sup E$ (that we know exists) as $\sqrt{3}$ we want to prove that there is a sequence in E that tends to it. Let us form a sequence (x_n) where each element $x_n \in B_{1/n}(\sqrt{3}) = (\sqrt{3} - 1/n, \sqrt{3} + 1/n)$ then we see that $\sqrt{3} - 1/n < x_n < \sqrt{3}$ for every $n \in \mathbb{N}$ which implies that $x_n \to \sqrt{3}$ therefore we have a Cauchy sequence that converges to a point that is not in E hence E is neither complete nor compact. □

Proof. **3** Let A be compact in M then A is totally bounded so given $\epsilon > 0$ there are finitely many sets $A_1, ..., A_n \subset A$ with $\text{diam}(A_i) < \epsilon$ such that $A \subset \bigcup_{i=1}^n A_i$ so let $B = \bigcup_{i=1}^n A_i$ we see that $\text{diam}(B) < \infty$ since every set is of diameter at most ϵ also we have that $\text{diam}(A) \leq \text{diam}(B) < \infty$ which implies that $\text{diam}(A)$ is finite.

On the other hand, we know that $\text{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. Let us define $(x_n) \subseteq A$ and $(y_n) \subseteq A$ where each x_n and y_n is defined such that $\text{diam}(A) - 1/n < d(x_n, y_n) \leq \text{diam}(A)$ which we know it exists because otherwise $\text{diam}(A) - 1/n$ would be an upper bound which is smaller than $\text{diam}(A) = \sup\{d(a,b) : a, b \in A\}$, implying a contradiction. This in turn implies that $d(x_n, y_n) \to \text{diam}(A)$.
Since A is compact from Theorem 8.2 we have that every sequence in A has a subsequence that converges to a point in A hence there is a subsequence $(x_{n_k}) \subset A$ from (x_n) such that $x_{n_k} \to x$ where $x \in A$ also from (y_n) we can select a subsequence $(y_{n_k}) \subset A$ where we took the n_k's from the (x_{n_k}) subsequence this implies that (y_{n_k}) might not converge but we know there is a subsequence $(y_{n_{k_t}})$ that converges to a point $y \in A$ hence we can take $(x_{n_{k_t}})$ from (x_{n_k}) that also converges to $x \in A$. This implies that $d(x_{n_{k_t}}, y_{n_{k_t}}) \to d(x, y)$. Finally, since every subsequence must converge to the same limit as the main sequence therefore we have that $d(x, y) = \text{diam}(A)$. □

Proof. **4** Let A and B be compact in M, we want to show that $A \cup B$ is compact. Let $(x_n) \subseteq A \cup B$ be a sequence then either $(x_n) \subset A$ or $(x_n) \subset B$ or in both for infinitely many points in any case we can take a subsequence (x_{n_k}) that converges to a point in A and/or in B since they are compact. Therefore since (x_n) has a convergent subsequence $(x_{n_k}) \subset A \cup B$ we get from Theorem 8.2 that $A \cup B$ is compact. □

Proof. **6** Let $(a_n) \subset A$ and $(b_n) \subset B$ be sequences, since A is compact then there is $(a_{n_k}) \subset A$ such that it converges to $a \in A$. We can also take a sequence $(b_{n_k}) \subset (b_n) \subset B$ which has a convergent subsequence $(b_{n_{k_t}}) \subset B$ that converges to $b \in B$ since B is compact, hence we can also take $(a_{n_{k_t}}) \subset A$ which still converges to $a \in A$.

On the other hand, let us also define a sequence $(a_n, b_n) \subset A \times B$. We know because of problem 3.46 that the subsequence $(a_{n_{k_t}}, b_{n_{k_t}}) \subset A \times B$ also converges in $A \times B$ because each subsequence converges separately in A and B. Therefore $A \times B$ is compact as well. □

Proof. **8** Let $K = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ since K is a subset of $\mathbb{R}^n$ to show K is compact in $\mathbb{R}^n$ under the Euclidean norm we need to show that K is closed and bounded under the Euclidean norm.

Let $x \in K$ we know that $0 \leq \|x\|_2 \leq \|x\|_1 = 1$ hence K is bounded under the Euclidean norm.

Now let us define $f(x) = \|x\|_1$ we see that $K = f^{-1}(\{1\})$ since $\{1\}$ is a closed set and f is cotinuous in $\mathbb{R}^n$ under the 1-norm we see that K must be closed under the 1-norm. This implies that for some $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\|x_n - x\|_1 < \epsilon$ but also we know that $\|x_n - x\|_2 \leq \|x_n - x\|_1 < \epsilon$ hence K is also closed under the Euclidean norm.

Therefore K is compact in $\mathbb{R}^n$ under the Euclidean norm. $\square$

Proof. **21** Let $f : [a, b] \to \mathbb{R}$ be a continuous function, since $[a, b]$ is a closed and bounded subset of $\mathbb{R}$ we know that $[a, b]$ is compact hence $f([a, b]) \in \mathbb{R}$ is compact because of Theorem 8.4. then $f([a, b])$ is bounded and closed so there is $c, d \in \mathbb{R}$ such that $c \leq f(x) \leq d$ for every $f(x) \in f([a, b])$ or $f([a, b]) \subset [c, d]$ moreover there is $x_1, x_2 \in [a, b]$ such that $f(x_1) = c$ and $f(x_2) = d$.

Let us take $J = [x_1, x_2]$ if $x_1 \leq x_2$ or $J = [x_2, x_1]$ if $x_1 > x_2$ where $J \subset [a, b]$. Since f is continuous and because of the Intermediate Value Theorem we know that f takes any value between $f(x_1)$ and $f(x_2)$ which implies that $[f(x_1), f(x_2)] = [c, d] \subset f([a, b])$. Therefore $f([a, b]) = [c, d]$.

$\square$

Proof. **22** Let $E \subseteq M$ be a closed set (hence compact because of Corollary 8.3) and let us take a convergent sequence $(y_n) \subseteq f(E)$ such that it converges to $y \in N$ we want to prove that also $y \in f(E)$ which would imply that $f(E)$ is a closed set.

By definition, there is $x_n \in E$ such that $f(x_n) = y_n$ hence we can form a sequence $(x_n) \subseteq E$, but E is compact so there is $(x_{n_k}) \subseteq E$ such that $x_{n_k} \to x$ where $x \in E$. Also, f is continuous so $f(x_{n_k}) \to f(x)$ or $y_{n_k} \to f(x)$ but we knew that $y_n \to y$ so by unicity of limits we have that $y = f(x) \in f(E)$. Therefore $f(E)$ is closed and f is a closed map. $\square$

Proof. **23** Let E be a closed set from M since M is compact and $f : M \to N$ is continuous then from proof 22 we know that f is a closed map hence $f(E)$ is closed in N but also we know that $f(E) = (f^{-1})^{-1}(E)$ since f is bijective therefore f^{-1} is continuous and f is a homeomorphism. $\square$

Proof. **25** Let V be a normed vector space and let a function $f : [0,1] \to V$ defined as $f(t) = x + t(y - x)$ where $x \neq y \in V$.

First, we want to prove that f is continuous. Let $\epsilon > 0$ and let $s, t \in [0,1]$ if $|s - t| < \delta$ where $\delta = \epsilon / \|y - x\|$ (we can do this since $x \neq y$) we have that

$$\begin{aligned}
|s - t| &< \frac{\epsilon}{\|y - x\|} \\
\|(s - t)(y - x)\| &< \epsilon \\
\|s(y - x) - t(y - x)\| &< \epsilon \\
\|x + s(y - x) - (x + t(y - x))\| &< \epsilon \\
\|f(s) - f(t)\| &< \epsilon
\end{aligned}$$

Therefore f is continuous.

Now we want to prove that f is one-to-one and onto (i.e. bijective). Suppose $f(t) = f(s)$ for some $t, s \in [0,1]$ hence

$$\begin{aligned}
x + t(y - x) &= x + s(y - x) \\
t(y - x) &= s(y - x) \\
t &= s
\end{aligned}$$

Therefore f is one-to-one.

To prove that f is onto suppose $z \in V$ we want to prove that there is $t \in [0,1]$ such that $f(t) = z$ let us take $t = (z - x)/(y - x)$ hence

$$f(t) = x + \frac{z - x}{y - x}(y - x) = z$$

Therefore f is onto as we wanted.

Finally, since $[0,1]$ is compact in $\mathbb{R}$ because it's closed and bounded and f is continuous and bijective from proof 23 we have that f is a homeomorphism from $[0,1]$ to V. $\square$

Proof. **30** We want to prove first that (a) is equivalent to (b). Let $\mathcal{F}$ be a collection of closed sets in M such that $\bigcap_{i=1}^n F_i \neq \emptyset$ for all choices of finitely many sets $F_1, ..., F_n$ let us suppose $\bigcap\{F : F \in \mathcal{F}\} = \emptyset$ we want to arrive at a contradiction.

Now let us define $\mathcal{G} = \{F^c : F \in \mathcal{F}\}$ we see that $(\bigcap\{F : F \in \mathcal{F}\})^c = M$ also from De Morgan's law, we have that $(\bigcap\{F : F \in \mathcal{F}\})^c = \bigcup\{F^c : F \in \mathcal{F}\}$ hence $M \subseteq \bigcup\{G : G \in \mathcal{G}\}$ then from (a) we have that there are finitely many sets $G_1, ..., G_n \in \mathcal{G}$ such that $M \subseteq \bigcup_{i=1}^n G_i$ where $G_i = (F_i)^c$ then $(\bigcup_{i=1}^n (F_i)^c)^c = \emptyset$ but we know that $(\bigcup_{i=1}^n (F_i)^c)^c = \bigcap_{i=1}^n ((F_i)^c)^c = \bigcap_{i=1}^n F_i$ hence $\bigcap_{i=1}^n F_i = \emptyset$ but we know that $\bigcap_{i=1}^n F_i \neq \emptyset$ then we have a contradiction therefore it must be that $\bigcap\{F : F \in \mathcal{F}\} \neq \emptyset$.

Finally, we want to prove that (b) is equivalent to (a). Let $\mathcal{G}$ be a collection of open sets in M such that $M \subseteq \bigcup\{G : G \in \mathcal{G}\}$ and let us suppose that for every combination of finitely many sets $G_1, ..., G_n \in \mathcal{G}$ we have that $M \not\subseteq \bigcup_{i=1}^n G_i$ we want to arrive at a contradiction.

Let us define $\mathcal{F} = \{(G_i)^c : G_i \in \mathcal{G}\}$ for $1 \leq i \leq n$ such that $\bigcap_{i=1}^n (G_i)^c \neq \emptyset$ which we know it exists because if $\bigcap_{i=1}^n (G_i)^c = \emptyset$ then $\bigcap_{i=1}^n (G_i)^c = (\bigcup_{i=1}^n G_i)^c = \emptyset$ which implies that $\bigcup_{i=1}^n G_i = M$ but we said that $M \not\subseteq \bigcup_{i=1}^n G_i$. Then because of (b) we have that $\bigcap\{(G)^c : G \in \mathcal{G}\} \neq \emptyset$ but also from De Morgan's law, we have that $(\bigcap\{(G)^c : G \in \mathcal{G}\})^c = \bigcup\{G : G \in \mathcal{G}\}$ so $M \subseteq (\bigcap\{(G)^c : G \in \mathcal{G}\})^c$ hence it must happen that $\bigcap\{(G)^c : G \in \mathcal{G}\} = \emptyset$ which is a contradiction to what we've got from (b), therefore it must happen that there are finitely many sets $G_1, ..., G_n \in \mathcal{G}$ such that $M \subseteq \bigcup_{i=1}^n G_i$. $\square$

Proof. **36** Let us suppose that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} = 0$ we want to arrive at a contradiction. Let us take $(x_n) \subseteq F$ and $(y_n) \subseteq K$ such that $d(x_n, y_n) \to 0$. Since K is compact then (y_n) has a subsequence such that $y_{n_k} \to y$ where $y \in K$. Also, let us take a subsequence $(x_{n_k}) \subseteq (x_n)$ so we have that

$$0 \leq d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y)$$

We see that $d(x_{n_k}, y_{n_k}) \to 0$ since it is a subsequence of $d(x_n, y_n)$ hence both have the same limit and $d(y_{n_k}, y) \to 0$ because K is compact as we just saw therefore $x_{n_k} \to y$ but we know F is closed then $y \in F$ but also $K \cap F = \emptyset$ hence we have a contradiction and must be that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} > 0$.

Finally, let $F = \{(x, y) : y = 0\}$ and $K = \{(x, y) : y = 1/x\}$ we see that both F and K are closed sets and disjoint but $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} = 0$. $\square$

Proof. **44** Let $f : (M, d) \to (N, \rho)$ be a Lipschitz map then there is $K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$ hence given $\epsilon > 0$ there is $\delta = \epsilon/K$ such that when $d(x, y) < \delta = \epsilon/K$ we have that

$$\rho(f(x), f(y)) \leq Kd(x, y) < \epsilon$$

Therefore f is uniformly continuous.

Let us suppose now that f is isometric then we know that $\rho(f(x), f(y)) = d(x, y)$ hence given $\epsilon > 0$ if we take $\delta = \epsilon$ we have that whenever $d(x, y) < \delta = \epsilon$ we get that $\rho(f(x), f(y)) = d(x, y) < \epsilon$. Therefore an isometry is also uniformly continuous. □

Proof. **45** Let $f : \mathbb{N} \to \mathbb{R}$ and if we take $\delta = 1/2$ we have that $|n - m| < 1/2$ for every $n, m \in \mathbb{N}$ hence $n = m$ so $|f(n) - f(m)| < \epsilon$ no mater which $\epsilon > 0$ we take since $f(n) = f(m)$. Therefore f is uniformly continuous. □

Proof. **46** First, we want to prove that $|d(x, z) - d(y, z)| \leq d(x, y)$. From the triangle inequality we know that

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ d(x, z) - d(y, z) &\leq d(x, y) \end{aligned}$$

and that

$$\begin{aligned} d(y, z) &\leq d(y, x) + d(x, z) \\ d(y, z) - d(x, z) &\leq d(x, y) \\ -d(x, y) &\leq d(x, z) - d(y, z) \end{aligned}$$

Hence this implies that $|d(x, z) - d(y, z)| \leq d(x, y)$ as we wanted to show.

Now we will prove that the map $x \to d(x, z)$ for some fixed $z \in M$ is a uniformly continuous map in M. Given some $\epsilon > 0$, let us take $\delta = \epsilon$ then when $d(x, y) < \delta = \epsilon$ from what we proved earlier we have that

$$|d(x, z) - d(y, z)| \leq d(x, y) < \delta = \epsilon$$

Therefore the map $x \to d(x, z)$ is uniformly continuous. □

Proof. **47** First, we want to prove that $|d(x,A) - d(y,A)| \leq d(x,y)$. From the triangle inequality for any $a \in A$ we know that

$$\begin{aligned} d(x,A) = \inf\{d(x,a) : a \in A\} \leq d(x,a) \leq d(x,y) + d(y,a) \\ d(x,A) - d(x,y) \leq d(y,a) \end{aligned}$$

So we see that $d(x,A) - d(x,y)$ is a lower bound for $d(y,a)$ hence we have that

$$\begin{aligned} d(x,A) - d(x,y) &\leq \inf\{d(y,a) : a \in A\} = d(y,A) \\ d(x,A) - d(y,A) &\leq d(x,y) \end{aligned}$$

Similarly, we have that

$$\begin{aligned} d(y,A) = \inf\{d(y,a) : a \in A\} \leq d(y,a) \leq d(y,x) + d(x,a) \\ d(y,A) - d(x,y) \leq d(x,a) \end{aligned}$$

So we see that $d(y,A) - d(x,y)$ is a lower bound for $d(x,a)$ hence we have that

$$\begin{aligned} d(y,A) - d(x,y) &\leq \inf\{d(x,a) : a \in A\} = d(x,A) \\ d(y,A) - d(x,A) &\leq d(x,y) \\ -d(x,y) &\leq d(x,A) - d(y,A) \end{aligned}$$

Hence this implies that $|d(x,A) - d(y,A)| \leq d(x,y)$ as we wanted to show.

Now we will prove that the map $x \to d(x,A)$ is a uniformly continuous map in M. Given some $\epsilon > 0$, let us take $\delta = \epsilon$ then when $d(x,y) < \delta = \epsilon$ from what we proved earlier we have that

$$|d(x,A) - d(y,A)| \leq d(x,y) < \delta = \epsilon$$

Therefore the map $x \to d(x,A)$ is uniformly continuous. $\square$

Proof. **48** Let $f : (M,d) \to (N,\rho)$ be a uniformly continuous map and let $(x_n) \subseteq M$ be a Cauchy sequence. We want to prove that $f((x_n))$ is also a Cauchy sequence.

Since f is uniformly continuous given some $\epsilon > 0$ there is some $\delta > 0$ (which depends on ϵ and/or f) such that $\rho(f(x_n), f(x_m)) < \epsilon$ whenever $x_n, x_m \in (x_n)$ satisfy $d(x_n, x_m) < \delta$ but since (x_n) is Cauchy there is $N \in \mathbb{N}$ where this will be satisfied for every $n, m \geq N$ hence we have that $\rho(f(x_n), f(x_m)) < \epsilon$ is also satisfied for every $n, m \geq N$ which implies that $f((x_n))$ is also a Cauchy sequence. $\square$

Proof. **49** Let $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ be two uniformly continuous maps hence for every $\epsilon > 0$ there is $\delta_f > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in M$ satisfy $d(x, y) < \delta_f$ and there is $\delta_g > 0$ such that $|g(x) - g(y)| < \epsilon$ whenever $x, y \in M$ satisfy $d(x, y) < \delta_g$. We want to prove that $f + g : M \to \mathbb{R}$ is also a uniformly continuous map. Let us take $\delta = \min(\delta_f, \delta_g)$ and let's notice that $|(f(x) + g(x)) - (f(y) + g(y))| = |f(x) - f(y) + g(x) - g(y)|$ and by triangle inequality, we have that

$$|f(x) - f(y) + g(x) - g(y)| < |f(x) - g(x)| + |g(x) - g(y)|$$

If $x, y \in M$ satisfy that $d(x, y) < \delta$ then $d(x, y) < \delta_g$ and $d(x, y) < \delta_f$ so we can claim that

$$|f(x) - f(y) + g(x) - g(y)| < |f(x) - g(x)| + |g(x) - g(y)| < 2\epsilon$$

This implies that $f + g$ is uniformly continuous too.

Lastly, we will give a counterexample of why the product of a uniformly continuous map is not a uniformly continuous map. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$ we see that taking $\delta = \epsilon$ we have that $|x-y| = |f(x)-f(y)| < \delta = \epsilon$ hence f is a uniformly continuous map.
Now, if we take $h(x) = f(x)f(x) = x^2$ we see that only taking $\delta = \epsilon/|x + y|$ when $|x - y| < \delta$ we have that

$$|x - y| < \frac{\epsilon}{|x + y|}$$
$$|x^2 - y^2| = |x - y||x + y| < \epsilon$$

Therefore h is not a uniformly continuous map. □

Proof. **51** Let $(x_n) \subseteq (0, 1)$ be a Cauchy sequence that converges to 0 since f is uniformly continuous we know that $f((x_n))$ is also a Cauchy sequence in $\mathbb{R}$ hence $f((x_n))$ converges to some $L \in \mathbb{R}$ this implies that $\lim_{x\to 0^+} f(x) = L$.

Let us suppose now that (x_n) and (y_n) are two Cauchy sequences converging to 0 since f is uniformly continuous we know that $f((x_n))$ and $f((y_n))$ are also Cauchy sequences in $\mathbb{R}$ we want to show that both of them converge to the same limit. Let us build another sequence (z_n) such that $z_{2n} = x_n$ and $z_{2n+1} = y_n$ where (z_n) is also a Cauchy sequence that converges to 0 hence $f((z_n))$ is a convergent Cauchy sequence. We see that $f((x_n))$ and $f((y_n))$ are subsequences of the convergent sequence $f((z_n))$ therefore they must converge to the same limit i.e. $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$.

What we did for 0 can also be applied to 1 hence both limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ exists so we have extended f to a continuous function $f : [0, 1] \to \mathbb{R}$ and since $[0, 1]$ is a compact set then by the Corollary 8.5. we have that f is bounded. □

Proof. **53** We know that given some $\epsilon > 0$ there is $R > 0$ such that $|f(x)| < \epsilon$ whenever $|x| > R$.

Let us consider the closed interval $[-R, R]$ we know that this closed interval is compact in $\mathbb{R}$ and also f is continuous here hence because of Theorem 8.15 we have that f is uniformly continuous in $[-R, R]$. This implies that given $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in [-R, R]$ satisfy that $|x - y| < \delta$.

Let us preserve the $\delta > 0$ we found and let us take $x, y \in \mathbb{R}$ such that $|x - y| < \delta$. Also, let $|x| > R$ and $|y| > R$ then we have that $|f(x)| < \epsilon$ and $|f(y)| < \epsilon$ since $f(x) \to 0$ therefore $|f(x) - f(y)| \leq |f(x)| + |f(y)| < 2\epsilon$.

Finally, we want to check the case where we still have that $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ but now $|x| < R$ and $|y| > R$ then

$$|f(x) - f(y)| \leq |f(x) - f(R)| + |f(R) - f(y)| < \epsilon + 2\epsilon = 3\epsilon$$

Where we used that $|f(x) - f(R)| < \epsilon$ since $x, R \in [-R, R]$ and $|f(R) - f(y)| \leq |f(R)| + |f(y)| < 2\epsilon$ since $y, R \in [R, \infty)$.

Therefore f is uniformly continuous in every condition as we wanted. $\square$

Proof. **56**

($\Rightarrow$) Let $f : (M, d) \to (N, \rho)$ be uniformly continuous and let (x_n) and (y_n) be two sequences of M such that $d(x_n, y_n) \to 0$.

Since f is uniformly continuous we have that given $\epsilon > 0$ there is $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever $x, y \in M$ satisfy $d(x, y) < \delta$. Let us grab this $\delta > 0$ then we know there is x_n and y_n such that $d(x_n, y_n) < \delta$ since $d(x_n, y_n) \to 0$ but this implies that $\rho(f(x_n), f(y_n)) < \epsilon$ which implies that $\rho(f(x_n), f(y_n)) \to 0$.

($\Leftarrow$) Let us suppose $f : (M, d) \to (N, \rho)$ is not a uniformly continuous map we want to arrive at a contradiction. If f is not a uniformly continuous map then there is some $\epsilon_0 > 0$ where it doesn't matter which $\delta > 0$ we take we can always find $x, y \in M$ such that $d(x, y) < \delta$ but we always have that $\rho(f(x), f(y)) > \epsilon_0$.

Let us take two sequences (x_n) and (y_n) of M such that $d(x_n, y_n) \to 0$ then we know that $\rho(f(x_n), f(y_n)) \to 0$ which implies that if we take $\epsilon_0 > 0$ there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\rho(f(x_n), f(y_n)) < \epsilon_0$ so there must be some $\delta_0 > 0$ such that $d(x_n, y_n) < \delta_0$ which is a contradiction to the first statement. Therefore f is a uniformly continuous map. $\square$

Proof. **58** Let $f : \mathbb{R} \to \mathbb{R}$ be some function where we know that for any $x \in \mathbb{R}$ we have that $|f'(x)| \leq K$ where $K > 0$ since f' is bounded.

On the other hand, from the mean value theorem, let us take an interval $(x, y) \subset \mathbb{R}$ where there must be $c \in (x, y)$ such that $f'(c) = (f(y) - f(x))/(y - x)$ then it must also happen that

$$|f'(c)| = \left|\frac{f(y) - f(x)}{y - x}\right| \leq K$$

which implies that

$$|f(x) - f(y)| \leq K|x - y|$$

Therefore f is Lipschitz of order 1. □

Proof. **61** Let $f : (M, d) \to (N, \rho)$ be a uniform homeomorphism we want to prove that if M is complete then N is also complete.

Given that f is one-to-one and onto let $(f(x_n)) \subseteq N$ be a Cauchy sequence. Since f^{-1} is a uniformly continuous map it sends Cauchy sequences to Cauchy sequences hence we know that $(f^{-1}(f(x_n))) = (x_n) \subseteq M$ must be a Cauchy sequence and since M is complete we have that (x_n) must converge to some $x \in M$. On the other hand, since f is a uniform homeomorphism and x_n converges to x then $(f(x_n)) \subseteq N$ converges to $f(x)$.

Therefore a Cauchy sequence $(f(x_n)) \subseteq N$ converges to $f(x) \in N$ which implies that N is a complete space. □

Proof. **62** Let $i : (M, d) \to (M, \rho)$ be the identity map between (M, d) and (M, ρ), we want to prove it's uniformly continuous knowing that there are constants $0 < c, C < \infty$ such that $c\rho(x, y) \leq d(x, y) \leq C\rho(x, y)$ for every point $x, y \in M$. Let $\epsilon > 0$, let us take $\delta = c\epsilon$ and let us suppose that $d(x, y) < \delta = c\epsilon$ then because $c\rho(x, y) \leq d(x, y)$ we have that $\rho(x, y) < \epsilon$ as we wanted. Therefore i is a uniformly continuous map.

On the other hand, let $i^{-1} : (M, \rho) \to (M, d)$ be the identity map between (M, ρ) and (M, d), we want to prove it's uniformly continuous knowing that there are constants $0 < c, C < \infty$ such that $c\rho(x, y) \leq d(x, y) \leq C\rho(x, y)$ for every point $x, y \in M$. Let $\epsilon > 0$, let us take $\delta = C\epsilon$ and let us suppose that $\rho(x, y) < \delta = C\epsilon$ then because $d(x, y) \leq C\rho(x, y)$ we have that $d(x, y) < \epsilon$ as we wanted. Therefore i^{-1} is a uniformly continuous map.

Finally, since the identity map $i : (M, d) \to (M, \rho)$ is a uniform homeomorphism then d and ρ are uniformly equivalent. □

Proof. **65** Let $F : [0, 1] \to \mathbb{R}$ be defined as $F(0) = f(0+)$, $F(1) = f(1-)$ and $F(x) = f(x)$. We know that F is continuous on $(0, 1)$ since f is continuous there, we want to prove that F is also continuous at 0 and 1. By definition, we know that $f(0+) = \lim_{x\to 0+} f(x)$ hence $F(0) = \lim_{x\to 0} F(x)$ which implies that F is continuous at 0. In the same way, we can get that F is continuous at 1.

Therefore, F is continuous on $[0, 1]$ and since $[0, 1]$ is compact because of Theorem 8.15. we have that F is also uniformly continuous. □

Proof. **77** Let $k \geq 1$ and let us define $f : l_\infty \to \mathbb{R}$ by $f(x) = x_k$. We want to show first that f is linear.
Let us consider $f(x + y)$ where $x, y \in l_\infty$ and $x + y \in l_\infty$ then we have that $f(x+y) = (x+k)_k = x_k + y_k = f(x)+f(y)$ where we used that the sum of two sequences is applied coordinate by coordinate.
On the other hand, let us consider $f(\alpha x)$ where $\alpha \in \mathbb{R}$ then $f(\alpha x) = \alpha x_k = \alpha f(x)$, thus we have proven that f is linear. Finally, we want to prove there is a constant $C < \infty$ such that $\|f(x)\|_1 \leq C\|x\|_\infty$ for every $x \in l_\infty$. Let us take $C = 1$ then we have that $|f(x)| = |x_k| \leq \sup_n |x_n|$ which is true because of the supremum definition.
Now let us suppose we take $C < 1$ and let us build a sequence $(x_n) \in l_\infty$ such that $x_n = 2$ for all $n \in \mathbb{N}$ then $\sup_n |x_n| = 2$ then for any $k \geq 1$ happens that $|x_k| = 2 > C \cdot 2 = C \sup_n |x_n|$.

Therefore we have that

$$\|f\| = \inf\{C : \|f(x)\|_1 \leq C\|x\|_\infty\} = 1$$

as we wanted. □

Proof. **80** Let $I(f) = \int_a^b f(t)dt$ we know $I(f)$ is linear and monotone. We want to prove it's continuous, for this, we want to find a constant $C < \infty$ such that $\|I(f)\| \leq C\|f\|$ for every $f \in C[a, b]$.
Let us note that

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt \leq \int_a^b \max_{x\in[a,b]} |f(x)|dt$$

Hence we have that

$$\left|\int_a^b f(t)dt\right| \leq (b-a) \max_{x\in[a,b]} |f(x)|$$

so if we take $C = b - a$ we are done. Therefore $I(f)$ is continuous.

Finally, we want to find

$$\|I\| = \inf\{C : \|I(f)\| \leq C\|f\| \text{ for all } f \in C[a, b]\}$$

Let us consider $f(x) = 1$ for all $x \in [a, b]$ then we have that

$$\left|\int_a^b 1 \cdot dt\right| = (b-a) = (b-a) \max_{x\in[a,b]} |1|$$

Since we are looking for the infimum value of C such that $\|I(f)\| \leq C\|f\|$ for all $f \in C[a, b]$ therefore it must happen because of what we showed for $f(x) = 1$ that $\|I\| = (b - a)$. $\square$

Proof. **85** First, we want to check $S = \{x \in V : \|x\| = 1\}$ is compact in $(V, \|\cdot\|)$. There is a correspondence between a set of scalars $\alpha_1, ..., \alpha_n \in \mathbb{R}^n$ and $x \in V$ since $x = \sum_{i=1}^n \alpha_i x_i$ so $\|x\| = \sum_{i=1}^n |\alpha_i| = 1$ implies that it is enough to show that $B = \{\alpha \in \mathbb{R}^n : \sum_{i=1}^n |\alpha_i| = 1\}$ is compact in $\mathbb{R}^n$ which we know it is since B is a closed ball on $\mathbb{R}^n$ and they are compact.

Lastly, we want to prove that if $|||x||| \geq c$ whenever $\|x\| = 1$ and this minimum c is actually attained then it must be that $c > 0$. Since $|||\cdot|||$ is a norm by definition we have that $|||x||| \geq 0$ for any $x \in V$ so must be at least that $c \geq 0$ but also by definition $|||x||| = 0$ if and only if $x = 0$ which we know cannot be since $\|x\| = 1$ therefore must be that $c > 0$. $\square$

Proof. **86** Let V be an n-dimensional vector space with basis $x_1, ..., x_n$ then if $x \in V$ we can write $x = \sum_{i=1}^{n} \alpha_i x_i$ for some $\alpha_1, ..., \alpha_n \in \mathbb{R}^n$ so we can define $T : V \to \mathbb{R}^n$ such that $T(x) = (\alpha_1, ..., \alpha_n)$ i.e. T is a map that sends some $x \in V$ to an n-tuple in $\mathbb{R}^n$. Let us show now T is linear. Let $c \in \mathbb{R}$ be some constant then we see that

$$T(cx) = (c\alpha_1, ..., c\alpha_n) = c(\alpha_1, ..., \alpha_n)$$

Also, let $y \in V$ we have that

$$T(x + y) = (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n) = (\alpha_1, ..., \alpha_n) + (\beta_1, ..., \beta_n)$$

where $(\beta_1, ..., \beta_n)$ is the n-tuple of y. Hence T is linear.

On the other hand, let $y \in \mathbb{R}^n$ and let us define $|||y||| = \|T^{-1}(y)\|$ we want to prove this is a norm.

Since T is a bijection we have that T^{-1} is a function so for every $y \in \mathbb{R}^n$ there is $x \in V$ such that $|||y||| = \|T^{-1}(y)\| = \|x\|$ and hence $0 \leq |||y||| < \infty$.

Let $|||y||| = 0$ then $\|T^{-1}(y)\| = \|x\| = 0$ which implies that $x = 0$ then $y = T(x) = T(0) = 0$.

If $y = 0$ then we have that $x = T^{-1}(y) = T^{-1}(0) = 0$ hence this implies that $\|x\| = 0$ so $\|T^{-1}(y)\| = |||y||| = 0$.

Let us consider $|||\beta y||| = \|\beta T^{-1}(y)\| = \|\beta x\|$ for some scalar β then we have that

$$\|\beta x\| = |\beta|\|x\| = |\beta|\|T^{-1}(y)\| = |\beta||||y|||$$

Finally, let $y, z \in \mathbb{R}^n$ then we have that

$$\begin{aligned} |||y + z||| &= \|T^{-1}(y + z)\| = \|T^{-1}(y) + T^{-1}(z)\| = \|x + w\| \\ &\leq \|x\| + \|w\| = \|T^{-1}(y)\| + \|T^{-1}(z)\| = |||y||| + |||z||| \end{aligned}$$

where we used that T^{-1} is also linear.

Therefore $||| \cdot |||$ as defined above is also a norm on $\mathbb{R}^n$ and T is linearly isometric. $\square$

Proof. **87** Let V and W be two normed n-dimensional vector spaces and let $T : V \to W$ be a linear isomorphism which exists since V and W are finite-dimensional. By Corollary 8.23 we know that T is uniformly continuous since V is finite-dimensional. In the same way $T^{-1} : W \to V$ since W is finite-dimensional we have that T^{-1} is also uniformly continuous. Finally, T is a bijection so T is a uniform homeomorphism between V and W. $\square$

Proof. **88** Let V be a normed finite-dimensional vector space with a basis $v_1, ..., v_n$. Let $x \in V$ then $x = \sum_{i=1}^{n} \alpha_i v_i$. Also, let us define a norm for V as $\|x\|_1 = \sum_{i=1}^{n} |\alpha_i|$ which we know is a norm because this was shown in Theorem 8.22 proof. Finally, let $(x_n) \subseteq V$ be a Cauchy sequence with respect to a norm $\|\cdot\|$ we want to prove $x_n \to x$ where $x \in V$.

Since any two norms on a finite-dimensional vector space are equivalent then there is $0 < c, C < \infty$ such that

$$c\|x_n - x_m\|_1 \leq \|x_n - x_m\| \leq C\|x_n - x_m\|_1$$

Where $x_m = \sum_{i=1}^{n} \alpha_{i,m} v_i \in V$, but also since (x_n) is Cauchy with respect to $\|\cdot\|$ then given $\epsilon > 0$ there is $N \in \mathbb{N}$ such that when $n, m \geq N$ we see that

$$c \sum_{i=1}^{n} |\alpha_{i,n} - \alpha_{i,m}| \leq \|x_n - x_m\| < \epsilon$$

If we take $\epsilon' = \epsilon/c$ we see that $\sum_{i=1}^{n} |\alpha_{i,n} - \alpha_{i,m}| < \epsilon'$ which implies that a sequence $(\alpha_n) \subseteq \mathbb{R}^n$ is also Cauchy with respect to $\|\cdot\|_1$ and we know that $\mathbb{R}^n$ is complete so there is some $\beta \in \mathbb{R}^n$ such that $\alpha_n \to \beta$ then given some $\epsilon' = \epsilon/C > 0$ there must be some $N \in \mathbb{N}$ such that when $n \geq N$ we have that

$$\sum_{i=1}^{n} |\alpha_{i,n} - \beta_i| < \epsilon'$$
$$C \sum_{i=1}^{n} |\alpha_{i,n} - \beta_i| < \epsilon$$

Also, there must be some $x \in V$ such that $x = \sum_{i=1}^{n} \beta_i v_i$ hence by using the equivalence between metrics, we have that

$$\|x_i - x\| \leq C \sum_{i=1}^{n} |\alpha_{i,n} - \beta_i| < \epsilon$$

Therefore we have shown that (x_n) converges to $x \in V$ and thus V is complete. $\square$