Solved selected problems of Real Analysis - Carothers

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Chapter 4 - Continuous Functions

Proof. **7**

(a) Let $(a, \infty) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(a, \infty)] = \{x : x \in M \text{ and } f(x) > a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv). In the same way, let $(-\infty, a) \subset \mathbb{R}$ which is an open set, then we see that

$$f^{-1}[(-\infty, a)] = \{x : x \in M \text{ and } f(x) < a\}$$

is also an open set because f is continuous and Theorem 5.1 part (iv).

(b) We proved the more general result in part (c) which also applies in this case.

(c) Let V be an open set of $\mathbb{R}$ then since the collection of open intervals with rational endpoints is a base for $\mathbb{R}$ we can write V as

$$V = \bigcup_{\alpha} (p_\alpha, q_\alpha)$$

where $p_\alpha, q_\alpha \in \mathbb{Q}$ so we have that

$$f^{-1}[V] = \bigcup_{\alpha} f^{-1}[(p_\alpha, q_\alpha)]$$

then we can write that

$$f^{-1}[(p_\alpha, q_\alpha)] = f^{-1}[(p_\alpha, \infty)] \cap f^{-1}[(-\infty, q_\alpha)]$$

Also, we know that

$$f^{-1}[(p_\alpha, \infty)] = \{x : f(x) > p_\alpha\} \text{ and } f^{-1}[(-\infty, q_\alpha)] = \{x : f(x) < q_\alpha\}$$

and we know both of them are open sets so $f^{-1}[(p_\alpha, q_\alpha)]$ is the intersection of a finite number of open sets then it is also an open set. Finally, since $f^{-1}[V]$ is the union of open sets it's also an open set. Therefore f is continuous.

$\square$

Proof. **10** Let $\epsilon > 0$ and let us take $\delta = 1$ no matter the value of ϵ then

$$B_\delta(2) = \{x \in A : d(2, x) < 1\} = \{2\}$$

So we have that $f(B_\delta(2)) = \{f(2)\}$ and certainly it must happen that $\{f(2)\} \subset B_\epsilon(f(2))$ because $f(2) \in B_\epsilon(f(2))$. Therefore f is continuous at 2. $\square$

Proof. **11**

(a) Let $x \in A \cup B$, then $x \in A$, $x \in B$ or both of them, also let $\epsilon > 0$ then we know there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ because f is continuous at x by the definition.

(b) Let $A = (0, 1)$ and $B = [1, 2)$ also let $f : A \to \mathbb{R}$ be defined as $f(x) = x$ and $f : B \to \mathbb{R}$ as $f(x) = x + 1$ then we see that $f : A \cup B \to \mathbb{R}$ is not continuous at $x = 1$. Therefore the statement is false.

$\square$

Proof. **14** Given that a continuous function on $\mathbb{R}$ is completely determined by its values on $\mathbb{Q}$. For each $q \in \mathbb{Q}$ we have that $f(q)$ has a cardinality of $\mathfrak{c}$ since for each real number $x \in \mathbb{R}$ we can find an f such that $x = f(q)$. So the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ has a cardinality of $\mathfrak{c}^{|\mathbb{Q}|}$ and doing some cardinality algebra we get that

$$\mathfrak{c}^{|\mathbb{Q}|} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$$

Therefore there are $\mathfrak{c}$ continuous function $f : \mathbb{R} \to \mathbb{R}$. $\square$

Proof. **17** Let $x \in M$, and let us also define $(x_n) \subset D$ such that $x_n \to x$ which we know it exists because D is dense. Then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ also we know that $f(x_n) = g(x_n)$ for every $x_n \in (x_n)$. Finally, since sequences have unique limits it must happen that $f(x) = g(x)$ as we wanted to show.

In the same way, suppose we define $(x_n) \subset D$ such that $x_n \to x$ where $x \in M$. Then $f(x_n) \to f(x)$ also $(f(x_n)) \subset f(D)$. So we have a sequence $(f(x_n))$ for any $f(x)$ and we know that every $y \in N$ has the form $y = f(x)$ because f is onto. Therefore $f(D)$ is dense in N. $\square$

Proof. **22** Let $n, m \in \mathbb{N}$ we want to show that $d(E(n), E(m)) = d(n, m)$ this means that $\|E(n) - E(m)\|_1 = |n - m|$. Let us suppose that $n > m$ then $n = m + b$ where $b \in \mathbb{N}$ and so $|n - m| = b$. Then we have that

$$\|E(n) - E(m)\|_1 = \sum_{i=1}^{\infty} |E_i(n) - E_i(m)|$$

where $E_i(n)$ is the value of the *ith* element in the sequence, the same for $E_i(m)$. If $i \in \{1, 2, ..., m\}$ we have that $|E_i(n) - E_i(m)| = |1 - 1| = 0$ and for $i \in \{n + 1, n + 2, ...\}$ we have that $|E_i(n) - E_i(m)| = |0 - 0| = 0$ so we can write the following

$$\|E(n) - E(m)\|_1 = \sum_{i=m}^{n} |E_i(n) - 0| = \sum_{i=m}^{n} 1 = n - m = b$$

Therefore $\|E(n) - E(m)\|_1 = |n - m|$ as we wanted, in the case of $m \geq n$ the proof is analogous because we are taking the absolute value inside the sum. □

Proof. **23** Let $S : c_0 \to c_0$ be defined as $S(x_0, x_1, ...) = (0, x_0, x_1, ...)$ such that S shifts the entries forward and puts 0 in the empty slot. We want to prove that $d(S(x), S(y)) = d(x, y)$ where $x = (x_0, x_1, ...)$ and $y = (y_0, y_1, ...)$ then

$$\|x - y\|_\infty = \sup\{|x_0 - y_0|, |x_1 - y_1|, ...\}$$

And $\sup\{|x_0 - y_0|, |x_1 - y_1|, ...\} \geq 0$ because both x and y are sequences that tend to 0. Let us suppose that $\sup_n |x_n - y_n| = 0$ then we see that

$$\|S(x) - S(y)\|_\infty = \sup\{|0 - 0|, |x_0 - y_0|, |x_1 - y_1|, ...\} = 0$$

Now let us suppose that $\|x - y\|_\infty > 0$ then we have that

$$\sup\{|x_0 - y_0|, |x_1 - y_1|, ...\} = \sup\{|0 - 0|, |x_0 - y_0|, |x_1 - y_1|, ...\}$$

because $\|S(x) - S(y)\|_\infty$ cannot be 0. Therefore we have that

$$\|S(x) - S(y)\|_\infty = \|x - y\|_\infty$$

as we wanted.

□