Solved selected problems of Real Analysis - Carothers

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Chapter 2 - Countable and Uncountable Sets

Proof. **1** We want to prove that the relation "is equivalent to" defines an equivalence relation, then we prove the following.

(i) First, we want to prove that $A \sim A$. Let $f : A \to A$ such that $f(x) = x$ then $A \sim A$.

(ii) If $A \sim B$ then there exists some $f : A \to B$ such that it is an onto and a one-to-one function. This means that it must exist $f^{-1} : B \to A$ which is also an onto and a one-to-one function. Therefore $B \sim A$.

(iii) Finally, if $A \sim B$ and $B \sim C$ we must have two functions $f : A \to B$ and $g : B \to C$ such that both of them are onto and one-to-one. Now, let $h(x) = g(f(x))$ we see that $h : A \to C$ and since both f and g are onto and one-to-one functions then h is also onto and one-to-one. Therefore $A \sim C$.

$\square$

Proof. **2** Let us grab one element from A that we know it exists because A it's an infinite set, and let us call this element a_1 then we can build a subset of A as $\{a_1\}$ which has one element, then the case of $n = 1$ is done. Now let us grab another element from A and let us call it a_2 then we can build another subset of A as $\{a_1, a_2\}$ of size $n = 2$, if we continue this procedure we can build a subset of A of any size such that $n \geq 1$. $\square$

Proof. **3**

Let us have two finite countable sets A_1 and A_2 of sizes n and m respectively, then the set $A_1 \cup A_2$ will have $n + m - k$ elements where k is the number of elements that are in both sets, but then the set $A_1 \cup A_2$ is equivalent to a set $\{1, 2, 3, ..., n+m-k\}$ so $A_1 \cup A_2$ is also a finite countable set. We can continue this procedure for a set of finite countable sets so $A_1 \cup A_2 \cup A_3 ... \cup A_n$ is also a finite countable set.
In the case where $A_1, A_2, ..., A_n$ are infinite countable sets let us call a_{ij} the element of A_i in the position j then we can map the elements in the following way

$$\begin{array}{ccc} 1 \to a_{11} & n+1 \to a_{12} & ... \\ 2 \to a_{21} & n+2 \to a_{22} & ... \\ 3 \to a_{31} & n+3 \to a_{32} & ... \\ ... & ... & \\ n \to a_{n1} & n+n \to a_{n2} & ... \end{array}$$

So we have mapped each element of $\mathbb{N}$ to an element of $A_1 \cup A_2 \cup A_3 ... \cup A_n$ then $A_1 \cup A_2 \cup A_3 ... \cup A_n$ is an infinite countable set.

For the set $A_1 \times A_2 \times ... \times A_n$ we can write it as $\{(a_1, a_2, ..., a_n) \mid a_1 \in A_1$ and $a_2 \in A_2$ and ... and $a_n \in A_n\}$ and we see that this operation between the sets gave us a set of size $m_1 \cdot m_2 \cdot ... \cdot m_n$ where m_1 is the number of elements in A_1, m_2 is the number of elements in A_2 and so on, but then this set is equivalent to the set $\{1, 2, 3, ..., m_1 \cdot m_2 \cdot ... \cdot m_n\}$. Therefore the cartesian product $A_1 \times A_2 \times ... \times A_n$ is also a finite countable set.
In the case where $A_1, A_2, ..., A_n$ are infinite countable sets we know all of them are equivalent to $\mathbb{N}$ and we also we know that $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$ then we can write that $A_1 \times A_2 \times ... \times A_n \sim \mathbb{N} \times \mathbb{N} \times ... \times \mathbb{N} \sim \mathbb{N}$ therefore $A_1 \times A_2 \times ... \times A_n$ is also an infinite countable set. □

Proof. **4** Let us suppose we have a set A which is infinite, then we can select a set of elements from A and call it B such that $B = \{a_1, a_2, ..., \}$ where a_1 is one element from A, a_2 is another element from A and so on. Then B will be equivalent to $\mathbb{N}$ and therefore B is an infinitely countable set. □

Proof. **15** Given that every nonempty open interval in $\mathbb{R}$ has a rational number inside let us grab one rational q_1 from one of the intervals then we grab from another open interval another rational number q_2 and so on, then we can generate a set with these numbers as $B = \{q_1, q_2, ...\}$ so B is equivalent to $\mathbb{N}$ and therefore B is countable and the set of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is too. □

Proof. **19** Let G be the set of all functions $g : A \to \{0,1\}$ and let us define a function f such that $f : P(A) \to G$ and $f(\alpha) = g_\alpha$ where $\alpha \in P(A)$ and $g_\alpha : A \to \{0,1\}$ is defined as

$$g_\alpha(a) = \begin{cases} 1 \text{ if } a \in \alpha \\ 0 \text{ if } a \notin \alpha \end{cases}$$

We want to check that the function f is bijective (i.e. one-to-one and onto). Suppose then that $f(\alpha) = f(\beta)$ where $\alpha, \beta \in P(A)$ then $g_\alpha = g_\beta$ now let us suppose that there is some $a \in A$ such that $g_\alpha(a) = 1$ and $g_\beta(a) = 0$ we want to arrive to a contradiction this means that $a \in \alpha$ but $a \notin \beta$ i.e $\alpha \neq \beta$ but we said that $g_\alpha = g_\beta$ then this is a contradiction and either $g_\beta(a) = 1$ or $g_\alpha(a) = 0$ and therefore $\alpha = \beta$.
Now let us have a function $g \in G$ such that $g : A \to \{0,1\}$. Let us also have a set $\alpha = \{a \in A : g(a) = 1\}$ so we can define $g = g_\alpha$ and we see that $\alpha \in P(A)$. Therefore for any $g \in G$ we can find an $\alpha \in P(A)$ as we wanted. □

Proof. **21** We know that the Cantor set consists of those points in $[0,1]$ having some base 3 decimal representation that excludes the digit 1. Then a ternary decimal of the form $0.a_1a_2a_3...a_n11$ is not in Δ. □

Proof. **22** If $x, y \in \Delta$ then x and y can be written as $x = 0.x_1x_2x_3...$ and $y = 0.y_1y_2y_3...$ where each digit is either 0 or 2.
Since we know that $x < y$ we know there is some nth digit where $x_n = 0$ and $y_n = 2$, let us select this nth index to be the minimum digit where this happen. Then we can construct a number z such that $z_k = x_k = y_k$ where $k \in \{0, 1, 2, ..., n-1\}$ and then $z_n = 1$ therefore we have that $z \notin \Delta$ and $x < z < y$. □

Proof. **26** Let $x, y \in \Delta$ then we can write $x = a_1a_2a_3...$ and $y = b_1b_2b_3...$ where $a_n, b_n \in \{0,2\}$ and $n \in \mathbb{N}$, but since $x < y$ then there must be some digit where $a_k = 0$ and $b_k = 2$ so when we apply the Cantor function we see that $f(a_k) = 0$ and $f(b_k) = 1$ what could happen here is that the binary number formed has two binary representations so if $f(a_m) = 1$ for $m \in \mathbb{N}$ and $m > k$ and a_m is not terminating then $f(x) = f(y)$. Therefore we have that $f(x) \leq f(y)$.
$(\to)$ If $f(x) = f(y)$ then this means that $f(x) = 0.c_1c_2...c_k0\bar{1}$ and $f(y) = 0.c_1c_2...c_k1$ where $c_n \in \{0,1\}$ and $n = \{1, 2, 3, ..., k\}$ but this means that $x = 0.a_1a_2...a_k0\bar{2} = 0.a_1a_2...a_k1$ and $y = 0.a_1a_2...a_k2$ where $a_n \in \{0,2\}$ and $n = \{1, 2, 3, ..., k\}$.
$(\leftarrow)$ Now if $x = 0.a_1a_2...a_k1$ and $y = 0.a_1a_2...a_k2$ we can write $x = 0.a_1a_2...a_k0\bar{2}$ then $f(x) = 0.c_1c_2...c_k0\bar{1} = 0.c_1c_2...c_k1 = f(y)$ □

Proof. **29** Let $f : [0, 1] \to [0, 1]$ be the extended Cantor function.
If $x, y \in \Delta$ and $x < y$ we saw that $f(x) \leq f(y)$ so f is increasing in this case.
If $x \in \Delta,\ y \in [0, 1] \setminus \Delta$ and $x < y$ then given that $f(y)$ is defined as $f(y) = \sup\{f(z) : z \in \Delta,\ z \leq y\}$ then $f(x) \leq f(y)$ so f is increasing.
If $x \in [0, 1] \setminus \Delta,\ y \in \Delta$ and $x < y$ then given that $f(x)$ is defined as $f(x) = \sup\{f(z) : z \in \Delta,\ z \leq y\}$ but also $f(z) \leq f(y)$ then $f(x) \leq f(y)$ therefore f is increasing. □

Proof. **30** At step 1 we discard an interval of length $\alpha/3$ in the step 2 we discard 2 intervals of length $\alpha/3^2$ then the total length discarded in this step is $2\alpha/3^2$, we can continue this procedure so in the *nth* step we discard $2^{n-1}\alpha/3^n$. Now let us sum all the discarded intervals

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{n-1}\alpha}{3^n} &= \frac{\alpha}{3} \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n-1}} \\ &= \frac{\alpha}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\ &= \frac{\alpha}{3} \frac{1}{1 - 2/3} = \alpha\end{aligned}$$

Therefore the generalized Cantor set has a measure of $1 - \alpha$. □

Proof. **32** Let us suppose that there is one open interval where the monotone function f doesn't have continuity points i.e. all points are discontinued (we have uncountable many of them), but we saw in Theorem 2.17 that if f is a monotone function has at most countable many points of discontinuity so we have arrived at a contradiction and therefore f has points of continuity on every open interval. □

Proof. **33** Let us suppose that $\sum_{i=1}^{n} |f(x_i+) - f(x_i-)| > |f(b) - f(a)|$ we want to arrive to a contradiction. Let us suppose f is monotone increasing (the proof should be analogous if f is decreasing) then we can find the biggest $k \in \mathbb{N}$ where we have that $\sum_{i=1}^{k} |f(x_i+) - f(x_i-)| < |f(b) - f(a)|$ but then it must happen that $f(x_{k+1}+) > f(b)$ so from here on f must be decreasing, which is a contradiction because we said that f is monotone increasing. Therefore it must happen that $\sum_{i=1}^{n} |f(x_i+) - f(x_i-)| \leq |f(b) - f(a)|$.

Proving that f has at most countably many jump discontinuities is analogous to prove that $T_n = \{x \in [a,b] : |f(x+) - f(x-)| \geq 1/n\}$ is finite because $T_1 \subset T_2 \subset T_3 \subset ...$ and $\cup_{n=1}^{\infty} T_n$ is the set of all discontinuities, then if all T_n are countable then $\cup_{n=1}^{\infty} T_n$ is countable. Let us suppose T_n is infinite then definitely there is a finite number of points $M = 2n|f(b) - f(a)|$ inside and we know that

$$\sum_{i=1}^{M} \frac{1}{n} \leq \sum_{i=1}^{M} |f(x_i+) - f(x_i-)| \leq |f(b) - f(a)|$$

then

$$2|f(b) - f(a)| = \sum_{i=1}^{2n|f(b)-f(a)|} \frac{1}{n} \leq |f(b) - f(a)|$$

which is not true therefore T_n must be finite. $\square$