Solved selected problems of Real Analysis - Carothers

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Chapter 8 - Compactness

Proof. **1** If K is a non-empty compact subset of $\mathbb{R}$ then K is bounded and closed therefore the $\sup K \in K$ and $\inf K \in K$. $\square$

Proof. **2** Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ then the complement on $\mathbb{Q}$ is

$$\begin{aligned} E^c = &\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\} \cup \\ &\{x \in \mathbb{Q} : x < 0 \text{ and } x^2 > 3\} \cup \\ &\{x \in \mathbb{Q} : x^2 < 2\} \end{aligned}$$

We see that $\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\} = (\sqrt{3}, \infty) \cap \mathbb{Q}$ where $(\sqrt{3}, \infty)$ and $\mathbb{Q}$ are open sets hence $\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\}$ is open. Also, we see that $\{x \in \mathbb{Q} : x < 0 \text{ and } x^2 > 3\} = (-\infty, -\sqrt{3}) \cap \mathbb{Q}$ and that $\{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ so both $\{x \in \mathbb{Q} : x < 0 \text{ and } x^2 > 3\}$ and $\{x \in \mathbb{Q} : x^2 < 2\}$ are open sets. Therefore since E^c is the union of open sets it's also an open set hence E is closed.

On the other hand, if $x \in E$ then $x \in (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ or $x \in (-\sqrt{2}, -\sqrt{3}) \cap \mathbb{Q}$ hence $-2 < x < 2$ which implies that E is bounded.

Let us call the $\sup E$ (that we know exists) as $\sqrt{3}$ we want to prove that there is a sequence in E that tends to it. Let us form a sequence (x_n) where each element $x_n \in B_{1/n}(\sqrt{3}) = (\sqrt{3} - 1/n, \sqrt{3} + 1/n)$ then we see that $\sqrt{3} - 1/n < x_n < \sqrt{3}$ for every $n \in \mathbb{N}$ which implies that $x_n \to \sqrt{3}$ therefore we have a Cauchy sequence that converges to a point that is not in E hence E is neither complete nor compact. $\square$

Proof. **3** Let A be compact in M then A is totally bounded so given $\epsilon > 0$ there are finitely many sets $A_1, ..., A_n \subset A$ with $\mathrm{diam}(A_i) < \epsilon$ such that $A \subset \bigcup_{i=1}^n A_i$ so let $B = \bigcup_{i=1}^n A_i$ we see that $\mathrm{diam}(B) < \infty$ since every set is of diameter at most ϵ also we have that $\mathrm{diam}(A) \leq \mathrm{diam}(B) < \infty$ which implies that $\mathrm{diam}(A)$ is finite.

On the other hand, we know that $\mathrm{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. Let us define $(x_n) \subseteq A$ and $(y_n) \subseteq A$ where each x_n and y_n is defined such that $\mathrm{diam}(A) - 1/n < d(x_n, y_n) \leq \mathrm{diam}(A)$ which we know it exists because otherwise $\mathrm{diam}(A) - 1/n$ would be an upper bound which is smaller than $\mathrm{diam}(A) = \sup\{d(a,b) : a, b \in A\}$, implying a contradiction. This in turn implies that $d(x_n, y_n) \to \mathrm{diam}(A)$.
Since A is compact from Theorem 8.2 we have that every sequence in A has a subsequence that converges to a point in A hence there is a subsequence $(x_{n_k}) \subset A$ from (x_n) such that $x_{n_k} \to x$ where $x \in A$ also from (y_n) we can select a subsequence $(y_{n_k}) \subset A$ where we took the n_k's from the (x_{n_k}) subsequence this implies that (y_{n_k}) might not converge but we know there is a subsequence $(y_{n_{k_t}})$ that converges to a point $y \in A$ hence we can take $(x_{n_{k_t}})$ from (x_{n_k}) that also converges to $x \in A$. This implies that $d(x_{n_{k_t}}, y_{n_{k_t}}) \to d(x, y)$. Finally, since every subsequence must converge to the same limit as the main sequence therefore we have that $d(x, y) = \mathrm{diam}(A)$. $\square$

Proof. **4** Let A and B be compact in M, we want to show that $A \cup B$ is compact. Let $(x_n) \subseteq A \cup B$ be a sequence then either $(x_n) \subset A$ or $(x_n) \subset B$ or in both for infinitely many points in any case we can take a subsequence (x_{n_k}) that converges to a point in A and/or in B since they are compact. Therefore since (x_n) has a convergent subsequence $(x_{n_k}) \subset A \cup B$ we get from Theorem 8.2 that $A \cup B$ is compact. $\square$

Proof. **6** Let $(a_n) \subset A$ and $(b_n) \subset B$ be sequences, since A is compact then there is $(a_{n_k}) \subset A$ such that it converges to $a \in A$. We can also take a sequence $(b_{n_k}) \subset (b_n) \subset B$ which has a convergent subsequence $(b_{n_{k_t}}) \subset B$ that converges to $b \in B$ since B is compact, hence we can also take $(a_{n_{k_t}}) \subset A$ which still converges to $a \in A$.

On the other hand, let us also define a sequence $(a_n, b_n) \subset A \times B$. We know because of problem 3.46 that the subsequence $(a_{n_{k_t}}, b_{n_{k_t}}) \subset A \times B$ also converges in $A \times B$ because each subsequence converges separately in A and B. Therefore $A \times B$ is compact as well. $\square$

Proof. **8** Let $K = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ since K is a subset of $\mathbb{R}^n$ to show K is compact in $\mathbb{R}^n$ under the Euclidean norm we need to show that K is closed and bounded under the Euclidean norm.

Let $x \in K$ we know that $0 \leq \|x\|_2 \leq \|x\|_1 = 1$ hence K is bounded under the Euclidean norm.

Now let us define $f(x) = \|x\|_1$ we see that $K = f^{-1}(\{1\})$ since $\{1\}$ is a closed set and f is cotinuous in $\mathbb{R}^n$ under the 1-norm we see that K must be closed under the 1-norm. This implies that for some $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\|x_n - x\|_1 < \epsilon$ but also we know that $\|x_n - x\|_2 \leq \|x_n - x\|_1 < \epsilon$ hence K is also closed under the Euclidean norm.

Therefore K is compact in $\mathbb{R}^n$ under the Euclidean norm. $\square$

Proof. **21** Let $f : [a, b] \to \mathbb{R}$ be a continuous function, since $[a, b]$ is a closed and bounded subset of $\mathbb{R}$ we know that $[a, b]$ is compact hence $f([a, b]) \in \mathbb{R}$ is compact because of Theorem 8.4. then $f([a, b])$ is bounded and closed so there is $c, d \in \mathbb{R}$ such that $c \leq f(x) \leq d$ for every $f(x) \in f([a, b])$ or $f([a, b]) \subset [c, d]$ moreover there is $x_1, x_2 \in [a, b]$ such that $f(x_1) = c$ and $f(x_2) = d$.

Let us take $J = [x_1, x_2]$ if $x_1 \leq x_2$ or $J = [x_2, x_1]$ if $x_1 > x_2$ where $J \subset [a, b]$. Since f is continuous and because of the Intermediate Value Theorem we know that f takes any value between $f(x_1)$ and $f(x_2)$ which implies that $[f(x_1), f(x_2)] = [c, d] \subset f([a, b])$. Therefore $f([a, b]) = [c, d]$.

$\square$

Proof. **22** Let $E \subseteq M$ be a closed set (hence compact because of Corollary 8.3) and let us take a convergent sequence $(y_n) \subseteq f(E)$ such that it converges to $y \in N$ we want to prove that also $y \in f(E)$ which would imply that $f(E)$ is a closed set.

By definition, there is $x_n \in E$ such that $f(x_n) = y_n$ hence we can form a sequence $(x_n) \subseteq E$, but E is compact so there is $(x_{n_k}) \subseteq E$ such that $x_{n_k} \to x$ where $x \in E$. Also, f is continuous so $f(x_{n_k}) \to f(x)$ or $y_{n_k} \to f(x)$ but we knew that $y_n \to y$ so by unicity of limits we have that $y = f(x) \in f(E)$. Therefore $f(E)$ is closed and f is a closed map. $\square$

Proof. **23** Let E be a closed set from M since M is compact and $f : M \to N$ is continuous then from proof 22 we know that f is a closed map hence $f(E)$ is closed in N but also we know that $f(E) = (f^{-1})^{-1}(E)$ since f is bijective therefore f^{-1} is continuous and f is a homeomorphism. $\square$

Proof. **25** Let V be a normed vector space and let a function $f : [0,1] \to V$ defined as $f(t) = x + t(y-x)$ where $x \neq y \in V$.

First, we want to prove that f is continuous. Let $\epsilon > 0$ and let $s, t \in [0,1]$ if $|s-t| < \delta$ where $\delta = \epsilon/\|y-x\|$ (we can do this since $x \neq y$) we have that

$$\begin{aligned}
|s-t| &< \frac{\epsilon}{\|y-x\|} \\
\|(s-t)(y-x)\| &< \epsilon \\
\|s(y-x) - t(y-x)\| &< \epsilon \\
\|x + s(y-x) - (x + t(y-x))\| &< \epsilon \\
\|f(s) - f(t)\| &< \epsilon
\end{aligned}$$

Therefore f is continuous.

Now we want to prove that f is one-to-one and onto (i.e. bijective). Suppose $f(t) = f(s)$ for some $t, s \in [0,1]$ hence

$$\begin{aligned}
x + t(y-x) &= x + s(y-x) \\
t(y-x) &= s(y-x) \\
t &= s
\end{aligned}$$

Therefore f is one-to-one.

To prove that f is onto suppose $z \in V$ we want to prove that there is $t \in [0,1]$ such that $f(t) = z$ let us take $t = (z-x)/(y-x)$ hence

$$f(t) = x + \frac{z-x}{y-x}(y-x) = z$$

Therefore f is onto as we wanted.

Finally, since $[0,1]$ is compact in $\mathbb{R}$ because it's closed and bounded and f is continuous and bijective from proof 23 we have that f is a homeomorphism from $[0,1]$ to V. $\square$

Proof. **30** We want to prove first that (a) is equivalent to (b). Let $\mathcal{F}$ be a collection of closed sets in M such that $\bigcap_{i=1}^{n} F_i \neq \emptyset$ for all choices of finitely many sets $F_1, ..., F_n$ let us suppose $\bigcap\{F : F \in \mathcal{F}\} = \emptyset$ we want to arrive at a contradiction.

Now let us define $\mathcal{G} = \{F^c : F \in \mathcal{F}\}$ we see that $(\bigcap\{F : F \in \mathcal{F}\})^c = M$ also from De Morgan's law, we have that $(\bigcap\{F : F \in \mathcal{F}\})^c = \bigcup\{F^c : F \in \mathcal{F}\}$ hence $M \subseteq \bigcup\{G : G \in \mathcal{G}\}$ then from (a) we have that there are finitely many sets $G_1, ..., G_n \in \mathcal{G}$ such that $M \subseteq \bigcup_{i=1}^{n} G_i$ where $G_i = (F_i)^c$ then $(\bigcup_{i=1}^{n}(F_i)^c)^c = \emptyset$ but we know that $(\bigcup_{i=1}^{n}(F_i)^c)^c = \bigcap_{i=1}^{n}((F_i)^c)^c = \bigcap_{i=1}^{n} F_i$ hence $\bigcap_{i=1}^{n} F_i = \emptyset$ but we know that $\bigcap_{i=1}^{n} F_i \neq \emptyset$ then we have a contradiction therefore it must be that $\bigcap\{F : F \in \mathcal{F}\} \neq \emptyset$.

Finally, we want to prove that (b) is equivalent to (a). Let $\mathcal{G}$ be a collection of open sets in M such that $M \subseteq \bigcup\{G : G \in \mathcal{G}\}$ and let us suppose that for every combination of finitely many sets $G_1, ..., G_n \in \mathcal{G}$ we have that $M \not\subseteq \bigcup_{i=1}^{n} G_i$ we want to arrive at a contradiction.

Let us define $\mathcal{F} = \{(G_i)^c : G_i \in \mathcal{G}\}$ for $1 \leq i \leq n$ such that $\bigcap_{i=1}^{n}(G_i)^c \neq \emptyset$ which we know it exists because if $\bigcap_{i=1}^{n}(G_i)^c = \emptyset$ then $\bigcap_{i=1}^{n}(G_i)^c = (\bigcup_{i=1}^{n} G_i)^c = \emptyset$ which implies that $\bigcup_{i=1}^{n} G_i = M$ but we said that $M \not\subseteq \bigcup_{i=1}^{n} G_i$. Then because of (b) we have that $\bigcap\{(G)^c : G \in \mathcal{G}\} \neq \emptyset$ but also from De Morgan's law, we have that $(\bigcap\{(G)^c : G \in \mathcal{G}\})^c = \bigcup\{G : G \in \mathcal{G}\}$ so $M \subseteq (\bigcap\{(G)^c : G \in \mathcal{G}\})^c$ hence it must happen that $\bigcap\{(G)^c : G \in \mathcal{G}\} = \emptyset$ which is a contradiction to what we've got from (b), therefore it must happen that there are finitely many sets $G_1, ..., G_n \in \mathcal{G}$ such that $M \subseteq \bigcup_{i=1}^{n} G_i$. □

Proof. **36** Let us suppose that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} = 0$ we want to arrive at a contradiction. Let us take $(x_n) \subseteq F$ and $(y_n) \subseteq K$ such that $d(x_n, y_n) \to 0$. Since K is compact then (y_n) has a subsequence such that $y_{n_k} \to y$ where $y \in K$. Also, let us take a subsequence $(x_{n_k}) \subseteq (x_n)$ so we have that

$$0 \leq d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y)$$

We see that $d(x_{n_k}, y_{n_k}) \to 0$ since it is a subsequence of $d(x_n, y_n)$ hence both have the same limit and $d(y_{n_k}, y) \to 0$ because K is compact as we just saw therefore $x_{n_k} \to y$ but we know F is closed then $y \in F$ but also $K \cap F = \emptyset$ hence we have a contradiction and must be that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} > 0$.

Finally, let $F = \{(x, y) : y = 0\}$ and $K = \{(x, y) : y = 1/x\}$ we see that both F and K are closed sets and disjoint but $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} = 0$. □

Proof. **44** Let $f : (M, d) \to (N, \rho)$ be a Lipschitz map then there is $K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$ hence given $\epsilon > 0$ there is $\delta = \epsilon/K$ such that when $d(x, y) < \delta = \epsilon/K$ we have that

$$\rho(f(x), f(y)) \leq Kd(x, y) < \epsilon$$

Therefore f is uniformly continuous.

Let us suppose now that f is isometric then we know that $\rho(f(x), f(y)) = d(x, y)$ hence given $\epsilon > 0$ if we take $\delta = \epsilon$ we have that whenever $d(x, y) < \delta = \epsilon$ we get that $\rho(f(x), f(y)) = d(x, y) < \epsilon$. Therefore an isometry is also uniformly continuous. □

Proof. **45** Let $f : \mathbb{N} \to \mathbb{R}$ and if we take $\delta = 1/2$ we have that $|n - m| < 1/2$ for every $n, m \in \mathbb{N}$ hence $n = m$ so $|f(n) - f(m)| < \epsilon$ no mater which $\epsilon > 0$ we take since $f(n) = f(m)$. Therefore f is uniformly continuous. □

Proof. **46** First, we want to prove that $|d(x, z) - d(y, z)| \leq d(x, y)$. From the triangle inequality we know that

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ d(x, z) - d(y, z) &\leq d(x, y) \end{aligned}$$

and that

$$\begin{aligned} d(y, z) &\leq d(y, x) + d(x, z) \\ d(y, z) - d(x, z) &\leq d(x, y) \\ -d(x, y) &\leq d(x, z) - d(y, z) \end{aligned}$$

Hence this implies that $|d(x, z) - d(y, z)| \leq d(x, y)$ as we wanted to show.

Now we will prove that the map $x \to d(x, z)$ for some fixed $z \in M$ is a uniformly continuous map in M. Given some $\epsilon > 0$, let us take $\delta = \epsilon$ then when $d(x, y) < \delta = \epsilon$ from what we proved earlier we have that

$$|d(x, z) - d(y, z)| \leq d(x, y) < \delta = \epsilon$$

Therefore the map $x \to d(x, z)$ is uniformly continuous. □

Proof. **47** First, we want to prove that $|d(x, A) - d(y, A)| \leq d(x, y)$. From the triangle inequality for any $a \in A$ we know that

$$\begin{aligned} d(x, A) = \inf\{d(x, a) : a \in A\} \leq d(x, a) &\leq d(x, y) + d(y, a) \\ d(x, A) - d(x, y) &\leq d(y, a) \end{aligned}$$

So we see that $d(x, A) - d(x, y)$ is a lower bound for $d(y, a)$ hence we have that

$$\begin{aligned} d(x, A) - d(x, y) &\leq \inf\{d(y, a) : a \in A\} = d(y, A) \\ d(x, A) - d(y, A) &\leq d(x, y) \end{aligned}$$

Similarly, we have that

$$\begin{aligned} d(y, A) = \inf\{d(y, a) : a \in A\} \leq d(y, a) &\leq d(y, x) + d(x, a) \\ d(y, A) - d(x, y) &\leq d(x, a) \end{aligned}$$

So we see that $d(y, A) - d(x, y)$ is a lower bound for $d(x, a)$ hence we have that

$$\begin{aligned} d(y, A) - d(x, y) &\leq \inf\{d(x, a) : a \in A\} = d(x, A) \\ d(y, A) - d(x, A) &\leq d(x, y) \\ -d(x, y) &\leq d(x, A) - d(y, A) \end{aligned}$$

Hence this implies that $|d(x, A) - d(y, A)| \leq d(x, y)$ as we wanted to show.

Now we will prove that the map $x \to d(x, A)$ is a uniformly continuous map in M. Given some $\epsilon > 0$, let us take $\delta = \epsilon$ then when $d(x, y) < \delta = \epsilon$ from what we proved earlier we have that

$$|d(x, A) - d(y, A)| \leq d(x, y) < \delta = \epsilon$$

Therefore the map $x \to d(x, A)$ is uniformly continuous. □

Proof. **48** Let $f : (M, d) \to (N, \rho)$ be a uniformly continuous map and let $(x_n) \subseteq M$ be a Cauchy sequence. We want to prove that $f((x_n))$ is also a Cauchy sequence.

Since f is uniformly continuous given some $\epsilon > 0$ there is some $\delta > 0$ (which depends on ϵ and/or f) such that $\rho(f(x_n), f(x_m)) < \epsilon$ whenever $x_n, x_m \in (x_n)$ satisfy $d(x_n, x_m) < \delta$ but since (x_n) is Cauchy there is $N \in \mathbb{N}$ where this will be satisfied for every $n, m \geq N$ hence we have that $\rho(f(x_n), f(x_m)) < \epsilon$ is also satisfied for every $n, m \geq N$ which implies that $f((x_n))$ is also a Cauchy sequence. □