Solved selected problems of Real Analysis - Carothers

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Chapter 4 - Open Sets and Closed Sets

Proof. **1** Let $U = (a, b) \times (c, d)$ and let $(x, y) \in U$ we want to check that if $(x', y') \in B_\epsilon((x, y))$ then $(x', y') \in U$. Since $(x', y') \in B_\epsilon((x, y))$ we have that

$$d_\infty((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\} < \epsilon$$

Then if $\max\{d(x, x'), d(y, y')\} = d(x, x')$ this means that $d(y, y') \leq d(x, x') < \epsilon$. Then $x' \in B_\epsilon(x)$ and since (a, b) is an open set in $\mathbb{R}$ this means that $x' \in (a, b)$, the same can be shown for y' such that $y' \in (c, d)$. Therefore $(x', y') \in U$.

Generalizing, let $U = A \times B$ and let $(a, b) \in U$ we want to check that if $(a', b') \in B_\epsilon((a, b))$ then $(a', b') \in U$. So in the same way since $(a', b') \in B_\epsilon((a, b))$ we have that

$$d_\infty((a, b), (a', b')) = \max\{d(a, a'), d(b, b')\} < \epsilon$$

Then if $\max\{d(a, a'), d(b, b')\} = d(a, a')$ this means that $d(b, b') \leq d(a, a') < \epsilon$. Then $a' \in B_\epsilon(a)$ and since A is an open set in $\mathbb{R}$ this means that $a' \in A$, the same can be shown for b' such that $b' \in B$. Therefore $(a', b') \in U$.

Let now $U = A \times B$ where A and B are closed sets in $\mathbb{R}$, we want to prove that U is also closed in $\mathbb{R}^2$. We see that $(\mathbb{R} \setminus A) \times \mathbb{R}$ and $\mathbb{R} \times (\mathbb{R} \setminus B)$ are open sets because $\mathbb{R} \setminus A$, $\mathbb{R} \setminus B$ and $\mathbb{R}$ are open sets. Also, we know that the union of open sets is also an open set so $\mathbb{R} \times \mathbb{R} \setminus A \times B$ is also an open set which means that $A \times B$ must be a closed set. $\square$

Proof. **3**

($\rightarrow$) Let $x \in U$ where U is an open set of (M, d) and let (x_n) be a sequence that converges to x since it is an open set we know that $x_n \in U$ for all but finitely many n, i.e. there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(x_n, x) < \epsilon$ for some $\epsilon > 0$ which means that $d(x_n, x) \to 0$ but since the metric d and ρ are equivalent then if $d(x_n, x) \to 0$ we have that $\rho(x_n, x) \to 0$. Therefore either ρ or d generate the same set U.

($\leftarrow$) Let U be an open set that is generated either by d and by ρ then if $x \in U$ and we have a sequence (x_n) that converge to x we know that $x_n \in U$ for all but finitely many n i.e. there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(x_n, x) < \epsilon$ and $\rho(x_n, x) < \epsilon$ because they both generate U this means that $d(x_n, x) \to 0$ and that $\rho(x_n, x) \to 0$ which implies that they are equivalent.

$\square$

Proof. **6** An example of an infinite closed set in $\mathbb{R}$ containing only irrationals is the set of all the square roots of the prime numbers, i.e.

$$F = \{\sqrt{2}, \sqrt{3}, \sqrt{5}, ...\sqrt{p_n}, \sqrt{p_{n+1}}, ...\}$$

So the complement of this set is the set

$$F^c = (-\infty, \sqrt{2}) \ \cup \ (\sqrt{2}, \sqrt{3}) \ \cup \ ... \ \cup \ (\sqrt{p_n}, \sqrt{p_{n+1}}) \ \cup \ ...$$

Where the intervals are open intervals of $\mathbb{R}$ and they are open sets plus the union of open sets is open, therefore F^c is open and F by definition is closed.

Let us suppose that we have a set $F \subset \mathbb{R}$ that is an open set consisting entirely of irrationals we want to arrive at a contradiction. Let us grab an element $x \in F$ where by definition is irrational, then the ball around x is defined as $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$, but we know that $\mathbb{Q}$ is a dense set in $\mathbb{R}$ so there is an element $q \in \mathbb{Q}$ such that $q \in (x - \epsilon, x + \epsilon)$ so we have a contradiction and $B_\epsilon(x) \not\subset F$. Therefore there is no open set consisting entirely of irrationals. $\square$

Proof. **7** Let F be an open set in $\mathbb{R}$ then for each $x \in F$ we have that there is $B_\epsilon(x) = (x-\epsilon, x+\epsilon)$ where $B_\epsilon(x) \subset F$. Also, we know that $\mathbb{Q}$ is dense in $\mathbb{R}$ so there is $a_x, b_x \in \mathbb{Q}$ such that $a_x \in (x-\epsilon, x)$ and $b_x \in (x, x+\epsilon)$ so we have that $x \in (a_x, b_x)$ then we can write that

$$F = \bigcup_{x \in F} (a_x, b_x)$$

Finally, we see that each interval is in $\mathbb{Q} \times \mathbb{Q}$ and we know that $\mathbb{Q} \times \mathbb{Q}$ is equivalent to $\mathbb{N}$, therefore since the intervals involved are a subset of $\mathbb{Q} \times \mathbb{Q}$ they are countable.

From what we proved we see that each open set F is a countable union of intervals with rational endpoints, this suggests an injective function that sends an open set F to $F \cap \mathbb{Q}$ where $F \cap \mathbb{Q} \in \mathcal{P}(\mathbb{Q})$ so we can construct an injective map

$$f : \mathcal{U} \to \mathcal{P}(\mathbb{Q})$$

Also notice that $\mathcal{P}(\mathbb{Q})$ is equivalent to $\mathbb{R}$, so we can construct an injective map that sends $x \in \mathbb{R}$ to $(-\infty, x) \in \mathcal{U}$ i.e. we have a map $g : \mathcal{P}(\mathbb{Q}) \to \mathcal{U}$, therefore using Bernstein's Theorem we get that there is a bijective map $h : \mathcal{U} \to \mathcal{P}(\mathbb{Q})$ implying that

$$\text{card}(\mathcal{U}) = \text{card}(\mathbb{R})$$

□

Proof. **11** Let (x_n) be a sequence of sequences from $E = \{e^{(k)} : k \geq 1\}$ then $d(x_n, x_m) = 2$ if $x_n \neq x_m$ and $d(x_n, x_m) = 0$ if $x_n = x_m$. This means that (x_n) converges to some $x \in l_1$ if eventually $x = x_n$ but then $x \in E$. Therefore this implies that E is a closed set of l_1 □

Proof. **13** Let $(x^{(n)})$ be a sequence of sequences from c_0 that converge to $x \in l_\infty$ we want to prove that also $x \in c_0$. Since $(x^{(n)})$ converges to $x \in l_\infty$ then for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $\|x^{(n)} - x\|_\infty < \epsilon$. Then we have that

$$\|x^{(n)} - x\|_\infty = \sup_{k \in \mathbb{N}} |x_k^{(n)} - x_k| < \epsilon$$

So we get that

$$\begin{aligned} |x_k| &= |x_k - x_k^{(n)} + x_k^{(n)}| \\ &\leq |x_k - x_k^{(n)}| + |x_k^{(n)}| \\ &\leq \sup_{k \in \mathbb{N}} |x_k^{(n)} - x_k| + |x_k^{(n)}| \\ &< \epsilon + |x_k^{(n)}| \end{aligned}$$

And since $x^{(n)} \in c_0$ then $|x_k^{(n)}| \to 0$ when $k \to \infty$. Therefore $|x_k| < \epsilon$ implying that $|x_k| \to 0$ and that $x \in c_0$. □

Proof. **15** Let $A = \{y \in M : d(x,y) \leq r\}$ be the closed ball around x, we want to show that $M \setminus A$ is an open set which implies that A is a closed set. If $M \setminus A$ is an open set then for every $z \in M \setminus A$ there is an open ball $B_t(z)$ such that $B_t(z) \subset M \setminus A$.

We have that $d(z,x) > r$ which implies that $d(z,x) - r > 0$ so let us define $t = d(z,x) - r$ then we have found $t > 0$ such that $B_t(z) \subset M \setminus A$ as we wanted. Therefore $B_t(z)$ is an open ball and $M \setminus A$ is an open set, which implies that A is a closed set.

Now let's see that A is not necessarily equal to the closure of the open ball $B_r(x)$. Let's define a metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

Then the open ball $B_1(x)$ with this metric is given by

$$B_1(x) = \{y \in M : d(x,y) < 1\} = \{x\}$$

So now we claim that $\mathrm{cl}(B_1(x)) = \{x\}$ we see this is true because let $y \in M$ such that $y \neq x$ so with this metric $d(x,y) = 1$ then there is an open ball $B_{1/2}(y) \subset M \setminus B_1(x)$ implying that $M \setminus B_1(x)$ is open and $\{x\}$ is closed. Which is different from the closed ball

$$A = \{y \in M : d(x,y) \leq 1\} = M$$

□

Proof. **16** Let $A = \{x \in V : \|x\| < 1\}$ and $B = \{x \in V : \|x\| \leq 1\}$. We know $x \in \bar{A}$ if there is a sequence $(x_n) \subset A$ such that $x_n \to x$. Then suppose $x \in V$ such that $\|x\| = 1$ and let us define a sequence (x_n) that converge to x as

$$x_n = \frac{n-1}{n} x$$

We see that $\|\frac{n-1}{n}x\| = |\frac{n-1}{n}|\|x\| = |\frac{n-1}{n}| \cdot 1 < 1$ then $(x_n) \subset A$ and this implies that $x \in \bar{A}$. Therefore B is always the closure of A. □

Proof. **17**

($\rightarrow$) If A is an open set, since $\mathring{A}$ is the largest open set contained in A then $\mathring{A} = A$.

($\leftarrow$) If $\mathring{A} = A$ since $\mathring{A}$ is an open set then A is open.

($\rightarrow$) If A is closed, since $\bar{A}$ is the smallest closed set that contains A then $\bar{A} = A$.

($\leftarrow$) If $\bar{A} = A$ since $\bar{A}$ is a closed set then A is closed.

□

Proof. **18** Since E is a nonempty bounded subset of $\mathbb{R}$ then there is a non-decreasing sequence $(x_n) \subset E$ where $\lim_{n\to\infty} x_n = \sup E$ therefore $\sup E \in \bar{E}$. In the same way, we know there is a non-increasing sequence $(x_n) \subset E$ where $\lim_{n\to\infty} x_n = \inf E$ therefore $\inf E \in \bar{E}$. $\square$

Proof. **20** Since $A \subset B$ and $B \subset \bar{B}$ then $A \subset \bar{B}$. Now let

$$C = \{F : F \text{ is closed set and } A \subset F\}$$

We know that

$$\bar{A} = \bigcap \{F : F \text{ is closed set and } A \subset F\} = \bigcap C$$

Then this means that $\bar{B} \in C$. Therefore since $\bar{A}$ is the intersection of C we see that $\bar{A} \subset \bar{B}$.

Let us now see why $\bar{A} \subset \bar{B}$ does not imply that $A \subset B$ by checking the following example. Let us define $A = (0, 1]$ and $B = \mathbb{Q}$ then $\bar{A} = [0, 1]$ and $\bar{B} = \mathbb{R}$ so we see that $\bar{A} \subset \bar{B}$ but $A \not\subset B$. $\square$

Proof. **24** Let $A \subset M$ so $A^c = M \setminus A$. Let us also define

$$U = \bigcup \{F : F \text{ is open and } F \subset M \setminus A\} = \text{int}(A^c)$$

So by definition, U is an open set then $U^c = M \setminus U$ is closed and $A \subset U^c$ because of the definition of U also we see that U^c must be the smallest closed set containing A again because of how we defined U. Therefore

$$\text{cl}(A) = (\text{int}(A^c))^c$$

Let us now define

$$I = \bigcap \{F : F \text{ is closed and } M \setminus A \subset F\} = \text{cl}(A^c)$$

So we see that I is a closed set then $I^c = M \setminus I$ is open and $I^c \subseteq A$ because of the definition of I. Also, we see that I^c must be the largest open set contained in A because of how we defined I. Therefore

$$\text{int}(A) = (\text{cl}(A^c))^c$$

$\square$

Proof. **26**

($\rightarrow$) Let $d(x, A) = 0$ then this means that $\inf\{d(x, a) : a \in A\} = 0$ for which we have two cases. If $x \in A$ then we have that

$$\min\{d(x, a) : a \in A\} = \inf\{d(x, a) : a \in A\} = d(x, x) = 0$$

and we have that $x \in \bar{A}$ since $A \subset \bar{A}$.

If $x \notin A$ and we know that $\inf\{d(x, a) : a \in A\} = 0$ then it is possible to form a sequence $(x_n) \subset A$ such that $x_n \rightarrow x$ i.e. $d(x_n, x) \rightarrow 0$ which implie that $x \in \bar{A}$.

($\leftarrow$) If $x \in \bar{A}$ then there is a sequence $(x_n) \subset A$ such that $x_n \rightarrow x$ which implies that $d(x, x_n) \rightarrow 0$ and since by definition of the metrics $d(x, a) \geq 0$ for any $a \in A$ then $\inf\{d(x, a) : a \in A\} = 0$. Therefore $d(x, A) = 0$.

□

Proof. **28** Let $D = \{x \in M : d(x, A) < \epsilon\}$ and let us define $\epsilon' = \epsilon - d(x, A)$ where we see that $\epsilon' > 0$. We want to prove that $B_{\epsilon'}(x) \subset D$ where we know that $B_{\epsilon'}(x) = \{y \in M : d(y, x) < \epsilon'\}$ then we have that

$$\begin{aligned} d(y, x) &< \epsilon - d(x, A) \\ d(y, A) &\leq d(y, x) + d(x, A) < \epsilon \end{aligned}$$

Then this implies that $B_{\epsilon'}(x) \subset D$ and therefore D is an open set.

Let now $F = \{x \in M : d(x, A) \leq \epsilon\}$ and let us suppose that there is a sequence $(x_n) \subset F$ such that $x_n \rightarrow x$ where $x \in M$ then this implies that there is $N \in \mathbb{N}$ such that when $n \geq N$ we have that $d(x_n, x) < \epsilon'$ for some $\epsilon' > 0$. Also from problem 27 we have that

$$|d(x, A) - d(x_n, A)| \leq d(x_n, x) < \epsilon'$$

And from the triangle inequality, we see that

$$\begin{aligned} d(x, A) = |d(x, A) - d(x_n, A) + d(x_n, A)| &\leq \\ \leq |d(x, A) - d(x_n, A)| + |d(x_n, A)| \end{aligned}$$

Then

$$d(x, A) \leq \epsilon' + \epsilon$$

In particular, let us take $\epsilon' = (d(x, A) - \epsilon)/2$ then we have that

$$\begin{aligned} d(x, A) &\leq \frac{d(x, A)}{2} + \frac{\epsilon}{2} \\ d(x, A) &\leq \epsilon \end{aligned}$$

Therefore $x \in F$ which implies that F is a closed set.

Finally, if $x \in A$ we have that $d(x, A) = d(x, x) = 0 < \epsilon$ which implies that $A \subset D$ and $A \subset F$. □

Proof. **29**

(i) From the hint we have, we see that each set $\{x \in M : d(x, A) < 1/n\}$ is an open set. Let's see that

$$\bigcap_{n=1}^{\infty} \{x \in M : d(x, A) < 1/n\} = \{x \in M : d(x, A) = 0\}$$

So, we need to prove that $d(x, A) = 0$ if and only if for all n it holds that $d(x, A) < 1/n$.

($\rightarrow$) If $d(x, A) = 0$ then $d(x, A) = 0 < 1/n$ for all $n \in \mathbb{N}$.

($\leftarrow$) On the other hand, if $d(x, A) < 1/n$ for all n then let us suppose $d(x, A) > 0$ we want to arrive to a contradiction. We know that there is $n \in \mathbb{N}$ such that $n > 1/d(x, A)$ then $d(x, A) > 1/n$ which is a contradiction. Therefore it must be that $d(x, A) = 0$.

Also we know that $d(x, A) = 0$ if and only if $x \in \bar{A}$ so we have that

$$\{x \in M : d(x, A) = 0\} = \bar{A}$$

Where we know that $\bar{A}$ is closed. Therefore every closed set in M is the intersection of countably many open sets.

(ii) Now let's see that $\{x \in M : d(x, A) \geq 1/n\}$ is the complement of the set $\{x \in M : d(x, A) < 1/n\}$ which implies that $\{x \in M : d(x, A) \geq 1/n\}$ is a closed set. Then because of what we saw in part (i) we have that

$$\bigcup_{n=1}^{\infty} \{x \in M : d(x, A) \geq 1/n\} = \bar{A}^c$$

And we know that $\bar{A}^c$ is open, therefore every open set in M is the intersection of countably many closed sets. □

Proof. **33** Let $(B_\epsilon(x) \setminus \{x\}) \cap A = \{x_1, x_2, ..., x_n\}$ i.e. $B_\epsilon(x) \setminus \{x\}$ has finitely many points of A for all $\epsilon > 0$ we want to arrive to a contradiction.

Let us take $x_m \in \{x_1, x_2, ..., x_n\}$ such that

$$d(x_m, x) = \min\{d(x_1, x), d(x_2, x), ..., d(x_n, x)\}$$

Since $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ for all $\epsilon > 0$ in particular let us take $\epsilon' = d(x, x_m)$ then we see that $(B_{\epsilon'}(x) \setminus \{x\}) \cap A = \emptyset$ which is a contradiction and therefore $(B_\epsilon(x) \setminus \{x\}) \cap A$ hast infinitely many points. □

Proof. **34**

($\rightarrow$) Let x be a limit point of A then let us construct a sequence $(x_n) \subset ((B_\epsilon(x)\setminus\{x\})\cap A)$ which we know it exists because $(B_\epsilon(x)\setminus\{x\})\cap A \neq \emptyset$. We can construct the sequence by taking $x_n \in (B_{1/n}(x) \setminus \{x\}) \cap A)$. We see that for each x_n we have that $0 < d(x_n, x) < 1/n$. Therefore $x_n \to x$ and by definition of (x_n) we know that $(x_n) \subset A$ and $x_n \neq x$ for all n.

($\leftarrow$) Let $(x_n) \subset A$ such that $x_n \to x$ and $x_n \neq x$ for all n. Then this implies that given some $\epsilon > 0$ for $n \geq N$ we have that $d(x_n, x) < \epsilon$ where $N \in \mathbb{N}$. So we have that $x_n \in ((B_\epsilon(x) \setminus \{x\}) \cap A)$ for all $n \geq N$ by the definition of an open ball. Therefore since ϵ is arbitrary we have that $B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ i.e. x is a limit point of A.

$\square$

Proof. **41**

(a) Let $x \in \text{bdry}(A)$ then we know that

$$B_\epsilon(x) \cap A \neq \emptyset \quad \text{and} \quad B_\epsilon(x) \cap A^c \neq \emptyset$$

for every $\epsilon > 0$. And since $(A^c)^c = A$ we also have that

$$B_\epsilon(x) \cap (A^c)^c \neq \emptyset$$

This also implies that $x \in \text{bdry}(A^c)$. In the same way, we can show that if $x \in \text{bdry}(A^c)$ then also $x \in \text{bdry}(A)$. And since x is arbitrary we get that $\text{bdry}(A) = \text{bdry}(A^c)$.

(b) Let $x \in \text{bdry}(A)$ so we know that $B_\epsilon(x)\cap A \neq \emptyset$ and that $B_\epsilon(x)\cap A^c \neq \emptyset$ for every $\epsilon > 0$. This implies that $x \in \text{cl}(A)$ too then we have that

$$\text{bdry}(A) \subset \text{cl}(A)$$

We also have that $\text{int}(A) \subseteq A \subset \text{cl}(A)$ therefore we have that

$$(\text{bdry}(A) \cup \text{int}(A)) \subset \text{cl}(A)$$

Now, let $x \in \text{cl}(A)$ and also it could happen that $x \in \text{int}(A)$ or $x \notin \text{int}(A)$. We are interested in the case where $x \notin \text{int}(A)$. Suppose $x \notin \text{bdry}(A)$ we want to arrive to a contradiction then $B_\epsilon(x) \cap A = \emptyset$ or $B_\epsilon(x) \cap A^c = \emptyset$. If $B_\epsilon(x) \cap A = \emptyset$ then this implies that $x \notin \text{cl}(A)$ because of Proposition 4.10 which is a contradiction. So it must be that $B_\epsilon(x) \cap A^c = \emptyset$ then $B_\epsilon(x) \subset A$ but since $B_\epsilon(x)$ is an open set this implies that $B_\epsilon(x) \subset \text{int}(A)$ which is another contradiction to the fact that $x \notin \text{int}(A)$. Then it must be that $x \in \text{bdry}(A)$ and

$$\text{cl}(A) \subset (\text{bdry}(A) \cup \text{int}(A))$$

Therefore

$$\text{cl}(A) = (\text{bdry}(A) \cup \text{int}(A))$$

(c) Let $x \in M$ we want to prove that then $x \in \operatorname{int}(A) \cup \operatorname{bdry}(A) \cup \operatorname{int}(A^c)$. Suppose that $x \in \operatorname{bdry}(A)$ then by definition $x \in M$ so let us suppose that $x \notin \operatorname{bdry}(A)$ then $B_\epsilon(x) \cap A = \emptyset$ or $B_\epsilon(x) \cap A^c = \emptyset$.

If $B_\epsilon(x) \cap A = \emptyset$ then $B_\epsilon(x) \subset A^c$ and since $B_\epsilon(x)$ is an open set this implies that $B_\epsilon(x) \subset \operatorname{int}(A^c)$.

And if $B_\epsilon(x) \cap A^c = \emptyset$ then $B_\epsilon(x) \subset A$ and since $B_\epsilon(x)$ is an open set this implies that $B_\epsilon(x) \subset \operatorname{int}(A^c)$.

Then in the first case $x \in \operatorname{int}(A^c)$ and in the second case $x \in \operatorname{int}(A)$ therefore

$$M = \operatorname{int}(A) \cup \operatorname{bdry}(A) \cup \operatorname{int}(A^c)$$

□

Proof. **42** Let E be a nonempty bounded subset of $\mathbb{R}$ and let $M = \sup E$ then we see that $B_\epsilon(M) = (M - \epsilon, M + \epsilon)$ and by definition we know that $x \leq \sup E$ for all $x \in E$ then there is $y \in E$ such that $M - \epsilon < y \leq M$ so it must happen that $B_\epsilon(M) \cap E \neq \emptyset$. Also, we see that $M < M + \epsilon$ so $M + \epsilon \notin E$ then it must happen that $M + \epsilon \in E^c$ then $B_\epsilon(M) \cap E^c \neq \emptyset$. Therefore since ϵ was arbitrary we have that $\sup E \in \operatorname{bdry}(E)$ as we wanted.

In the same way let $m = \inf E$ we see that $B_\epsilon(m) = (m - \epsilon, m + \epsilon)$ and by definition we know that $\inf E \leq x$ for all $x \in E$ then there is $y \in E$ such that $m \leq y < m + \epsilon$ so it must happen that $B_\epsilon(m) \cap E \neq \emptyset$. Also, we see that $m - \epsilon < m$ so $m - \epsilon \notin E$ then it must happen that $m - \epsilon \in E^c$ then $B_\epsilon(m) \cap E^c \neq \emptyset$. Therefore since this is valid for all $\epsilon > 0$ we have that $\inf E \in \operatorname{bdry}(E)$ as we wanted. □

Proof. **43** We want to prove that $\text{bdry}(A)$ is a closed set. But first, we will check that $\text{int}(A)$, $\text{int}(A^c)$ and $\text{bdry}(A)$ are mutually disjoint.

Since $\text{int}(A) \subseteq A$ and $\text{int}(A^c) \subseteq A^c$ and also A is disjoint from A^c we have that $\text{int}(A)$ is disjoint from $\text{int}(A^c)$.

Let $x \in \text{int}(A)$ then there is a ball $B_\epsilon(x) \subset \text{int}(A)$ for some $\epsilon > 0$. Let us also suppose that $x \in \text{bdry}(A)$ we want to arrive to a contradiction then by definition we know that $B_{\epsilon'}(x) \cap A \neq \emptyset$ and $B_{\epsilon'}(x) \cap A^c \neq \emptyset$ for every $\epsilon' > 0$ but we showed that there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset \text{int}(A)$ then $B_\epsilon(x) \cap A^c = \emptyset$ which is a contradiction. Therefore $\text{int}(A)$ is disjoint from $\text{bdry}(A)$.

Finally, let $x \in \text{int}(A^c)$ then there is a ball $B_\epsilon(x) \subset \text{int}(A^c)$ for some $\epsilon > 0$. Let us also suppose that $x \in \text{bdry}(A)$ we want to arrive to a contradiction then by definition we know that $B_{\epsilon'}(x) \cap A \neq \emptyset$ and $B_{\epsilon'}(x) \cap A^c \neq \emptyset$ for every $\epsilon' > 0$ but we showed that there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset \text{int}(A^c)$ then $B_\epsilon(x) \cap A = \emptyset$ which is a contradiction. Therefore $\text{int}(A^c)$ is disjoint from $\text{bdry}(A)$.

Now let's prove that $\text{bdry}(A)$ is a closed set. From the problem 41(c) we have that $M = \text{int}(A) \cup \text{bdry}(A) \cup \text{int}(A^c)$ and since we proved that $\text{int}(A)$, $\text{int}(A^c)$ and $\text{bdry}(A)$ are mutually disjoint we have that $M \setminus \text{bdry}(A) = \text{int}(A) \cup \text{int}(A^c)$ and since $\text{int}(A)$, $\text{int}(A^c)$ and the union of open sets is open then $M \setminus \text{bdry}(A)$ is an open set. Therefore $\text{bdry}(A)$ is a closed set.

Finally, we want to prove that $\text{bdry}(A) = \text{cl}(A) \setminus \text{int}(A)$ then from the problem 41(b) we have that $\text{cl}(A) = \text{int}(A) \cup \text{bdry}(A)$ also since $\text{int}(A)$ and $\text{bdry}(A)$ are mutually disjoint we have that $\text{cl}(A) \setminus \text{int}(A) = \text{bdry}(A)$ $\square$

Proof. **46**

(a) ($\rightarrow$) If A is dense in M then $\bar{A} = M$ also we know that if $x \in \bar{A}$ i.e. $x \in M$ then there is a sequence $(x_n) \subset A$ such that $x_n \to x$. So every point in M is the limit of a sequence from A.

($\leftarrow$) If every point in M is the limit of a sequence from A then there is a sequence $(x_n) \subset A$ such that $x_n \to x$ also, we know that this implies $x \in \bar{A}$. Therefore $\bar{A} = M$ and A is dense in M.

(b) ($\rightarrow$) If A is dense in M then $\bar{A} = M$ so if $x \in \bar{A}$ i.e. $x \in M$ we know from Proposition 4.10 that $B_\epsilon(x) \cap A \neq \emptyset$ for every $\epsilon > 0$ as we wanted.

($\leftarrow$) If $B_\epsilon(x) \cap A \neq \emptyset$ for every $x \in M$ and every $\epsilon > 0$ again from Proposition 4.10 we have that x must be in $\bar{A}$. Therefore $\bar{A} = M$ and A is dense in M.

(c) ($\rightarrow$) If A is dense in M then $\bar{A} = M$. Let $x \in U$ also we have that $x \in \bar{A}$ then $x \in \text{int}(A)$ or $x \in \text{bdry}(A)$ since they are disjoint. If $x \in \text{int}(A)$ then $x \in A$ and $U \cap A \neq \emptyset$. If $x \in \text{bdry}(A)$ since U is an open neighborhood of x there is some $\epsilon > 0$ for which $B_\epsilon(x) \subseteq U$ and by definition of boundary $B_\epsilon(x) \cap A \neq \emptyset$. Therefore $U \cap A \neq \emptyset$ for every nonempty open set U.

($\leftarrow$) Let us suppose $\bar{A} \neq M$ and let $x \notin \bar{A}$ (so $x \in \bar{A}^c \neq \emptyset$) then there is an $\epsilon > 0$ for which an open ball $B_\epsilon(x) \cap A = \emptyset$ but we know that $U \cap A \neq \emptyset$ for every nonempty open set U so this cannot happen since $B_\epsilon(x)$ is a nonempty open set. Therefore it must happen that $\bar{A} = M$.

(d) ($\rightarrow$) Suppose $x \in \text{int}(A^c)$ i.e. $\text{int}(A^c) \neq \emptyset$ we want to arrive to a contradiction. We know by part (c) that $U \cap A \neq \emptyset$ for every nonempty open set U, but $\text{int}(A^c)$ is open by definition of interior so $\text{int}(A^c) \cap A \neq \emptyset$ which is a contradiction. Therefore it must happen that $\text{int}(A^c) = \emptyset$.

($\leftarrow$) Let $\text{int}(A^c) = \emptyset$ we want to show that $\bar{A} = M$. We know that $\bar{A} = (\text{int}(A^c))^c$ therefore we have that $\bar{A} = \emptyset^c = M$.

□

Proof. **48** An example of countable dense set in $\mathbb{R}$ is $\mathbb{Q}$, in the same way for $\mathbb{R}^2$ we can take $\mathbb{Q} \times \mathbb{Q}$ since a cartesian product of countable sets is also countable and for $\mathbb{R}^n$ we take $\mathbb{Q}^n$. □

Proof. **51** Let M be a separable metric space and let C be the set that contains all the isolated points of M. Let $x \in C$ i.e. x is an isolated point of M then we have that $B_\epsilon(x) \cap M = \{x\}$ for some $\epsilon > 0$. Since M is a separable metric space then there is a countable dense subset A such that $M = \bar{A}$. So also, $x \in \bar{A}$ which means that either $x \in \text{int}(A)$ or $x \in \text{bdry}(A)$. If $x \in \text{bdry}(A)$ then it must happen that $B_{\epsilon'}(x) \cap A \neq \emptyset$ for every $\epsilon' > 0$ but we showed that $B_\epsilon(x) \cap M = \{x\}$ for some $\epsilon > 0$ so it must happen that $x \in A$. And if $x \in \text{int}(A)$ then $x \in A$. Then in any case we have that $x \in A$ which means that $C \subseteq A$ but A is a countable set, therefore C must be countable too. □

Proof. **52** Let M be a separable metric space and let W be a collection of disjoint open sets in M. Let $U \in W$ where U is an open set. Since M is a separable metric space then there is a countable dense subset A such that $M = \bar{A}$ and we showed that for any nonempty open set U it must happen that $U \cap A \neq \emptyset$ then we can build a map $f : W \to A$ such that for any $U \in W$ we assign a value $x \in (U \cap A)$. Finally, we need to check that this mapping is one to one, suppose there is $U, V \in W$ such that $f(U) = f(V)$ then $x \in U \cap A$ and $x \in V \cap A$ but we know that U and V are disjoint sets then it must happen that $U = V$. Therefore since A is countable and we have a map between every element in W to a value $x \in A$ then W should be at most countable. □

Proof. **61** We are asked to prove (ii) and (iii) from Proposition 4.13.

(ii) We want to prove that a set $F \subset A$ is closed in (A, d) if and only if $F = A \cap C$ where C is closed in (M, d).

($\rightarrow$) Suppose F is closed in (A, d) and let us suppose that $C = \mathrm{cl}_M(F)$ then we see that $F \subseteq A \cap C$. Now let $x \in A \cap C$ then $x \in A$ and $x \in \mathrm{cl}_M(F)$ so there is a sequence $(x_n) \subset F$ such that $x_n \rightarrow x$ where $x \in (M, d)$ but also since F is a closed set it must happen that $x \in F$ i.e. $A \cap C \subseteq F$. Therefore joining both inclusions we have that $F = A \cap C$.

($\leftarrow$) Suppose $F = A \cap C$ where C is closed in (M, d). Let $(x_n) \subset F$ such that $x_n \rightarrow x$ and $x \in A$ we want to prove that $x \in F$. Since $(x_n) \subset F$ then $(x_n) \subset C$ and since C is a closed set then it must happen that $x \in C$ then $x \in F$. Therefore F is closed in (A, d).

(iii) We want to prove that $\mathrm{cl}_A(E) = A \cap \mathrm{cl}_M(E)$ for any subset E of A.

Let $x \in \mathrm{cl}_A(E)$ then $B_\epsilon^A(x) \cap E \neq \emptyset$ for every $\epsilon > 0$. But we also know that $B_\epsilon^A(x) = A \cap B_\epsilon^M(x)$ then we have that $A \cap B_\epsilon^M(x) \cap E \neq \emptyset$ but since $E \subset A$ then $A \cap E = E$ so we have that $B_\epsilon^M(x) \cap E \neq \emptyset$ for every $\epsilon > 0$ which implies that $x \in \mathrm{cl}_M(E)$ then $\mathrm{cl}_A(E) \subseteq A \cap \mathrm{cl}_M(E)$.

Let $x \in A \cap \mathrm{cl}_M(E)$ then since $x \in \mathrm{cl}_M(E)$ we have that $B_\epsilon^M(x) \cap E \neq \emptyset$ for every $\epsilon > 0$. Since also $x \in A$ and $E \subset A$ then $B_\epsilon^M(x) \cap A \cap E \neq \emptyset$ hence $B_\epsilon^A(x) \cap E \neq \emptyset$ for every $\epsilon > 0$. Therefore $x \in \mathrm{cl}_A(E)$ i.e. $A \cap \mathrm{cl}_M(E) \subseteq \mathrm{cl}_A(E)$.

Finally, by joining both inclusions we have that $\mathrm{cl}_A(E) = A \cap \mathrm{cl}_M(E)$.

$\square$

Proof. **62**

($\rightarrow$) Let G be open in A then by Proposition 4.13 (i) we know that there is an open set U in M such that $G = A \cap U$. So let $x \in G$, then $x \in A$ and A is open in M so there is $\epsilon_A > 0$ such that $B^M_{\epsilon_A}(x) \subset A$. Also, $x \in U$ and U is open in M so there is $\epsilon_U > 0$ such that $B^M_{\epsilon_U}(x) \subset U$. Let us take $\epsilon = \min(\epsilon_A, \epsilon_U)$ so we have a ball $B^M_\epsilon(x) \subset A$ and $B^M_\epsilon(x) \subset U$ but also since $A \subset M$ and $U \subset M$ it happens that $B^M_\epsilon(x) \subset M$. Therefore G is open in M.

($\leftarrow$) Let G be open in M and let $x \in G$ then there is a ball $B^M_\epsilon(x) \subset M$ for some $\epsilon > 0$. Also, we know that $B^A_\epsilon(x) = A \cap B^M_\epsilon(x)$ which is nonempty since $G \subset A$. Therefore we have a ball $B^A_\epsilon(x) \subset A$ for $\epsilon > 0$ i.e. G is open in A.

Now let us prove that if A is closed in (M, d) and $G \subset A$ then G is closed in A if and only if G is closed in M.

($\rightarrow$) Let G be closed in A and let $x \in M$ such that for any $\epsilon > 0$ we have that $B^M_\epsilon(x) \cap G \neq \emptyset$, we want to prove that $x \in G$. Since $G \subset A$ then $G \cap A = G$ so we can write that $B^M_\epsilon(x) \cap A \cap G \neq \emptyset$. Also, let us suppose that $x \notin A$, we want to arrive to a contradiction then $x \in A^c$ which is an open set because A is a closed set then there is a ball $B^M_{\epsilon'}(x) \subset A^c$ for some $\epsilon' > 0$, but we said that $B^M_\epsilon(x) \cap A \cap G \neq \emptyset$ for any $\epsilon > 0$ then we have a contradiction and $x \in A$. So since x in A we have that $B^A_\epsilon(x) = A \cap B^M_\epsilon(x)$ then we get that $B^A_\epsilon(x) \cap G \neq \emptyset$ for all $\epsilon > 0$ therefore $x \in G$ because G is closed in A which implies that G is closed in M.

($\leftarrow$) Let G be closed in M and let $x \in A$ such that for any $\epsilon > 0$ we have that $B^A_\epsilon(x) \cap G \neq \emptyset$, we want to prove that $x \in G$. Since x in A then we know that $B^A_\epsilon(x) = A \cap B^M_\epsilon(x)$ so we get that $A \cap B^M_\epsilon(x) \cap G \neq \emptyset$ also, since $G \subset A$ then $G \cap A = G$ then we have that $B^M_\epsilon(x) \cap G \neq \emptyset$ for any $\epsilon > 0$ therefore $x \in G$ because G is closed in M. Hence G is closed in A.

$\square$

Proof. **64** Let $E = A = \mathbb{Q}$ then we see that $\text{int}_A(E) = \text{int}_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q}$ but $\text{int}_{\mathbb{R}}(E) = \text{int}_{\mathbb{R}}(\mathbb{Q}) = \emptyset$. $\square$

Proof. **69**

($\rightarrow$) Let (U_n) be a countable open base for M and let us take an element from each U_n such that for each $n \in \mathbb{N}$ we have $x_n \in U_n$, then we can construct a set $D = \{x_n : n \in \mathbb{N}\}$ which is countable. Also, let U be a nonempty open set of M then it can be written as a union of U_n which implies that $U \cap D \neq \emptyset$. Therefore D is a countable dense set of M i.e. M is separable.

($\leftarrow$) Let M be a separable metric space, let F be an open set in M and let $\{x_n\}$ be a countable dense set of M. Let us take $x \in F$ then we have a ball $B_\epsilon(x) \subset F$ for some $\epsilon > 0$. Also, we can have $B_q(x) \subset B_\epsilon(x) \subset F$ such that $q \in \mathbb{Q}$ and $0 < q < \epsilon$ because $\mathbb{Q}$ is dense. Since $\{x_n\}$ is dense then there is some $x_n \in B_q(x)$ and which implies that $x \in B_q(x_n)$ so we can write that

$$F = \bigcup_n B_q(x_n)$$

Therefore F can be written as a union of countable open sets i.e. M has a countable open base. □