Solved selected problems of Special Relativity - Morin

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Chapter 2 - Kinematics, Part 2

Proof. **2.1** By checking the Excercise 1 of Section 1.5 we have that the velocity v and the factor γ for the different frame combinations are

	AB	AC	AD	BC	BD	CD
v	$5c/13$	$4c/5$	$c/5$	$3c/5$	$c/5$	$5c/7$
γ	$13/12$	$5/3$	$5/2\sqrt{6}$	$5/4$	$5/2\sqrt{6}$	$7/2\sqrt{6}$

Now let us verify the values of the table we have using the Lorentz Transformations

$$\begin{aligned}
\Delta x &= \gamma(\Delta x' + v\Delta t') \\
\Delta t &= \gamma(\Delta t' + v\Delta x'/c^2)
\end{aligned}$$

From frame A with Δt_A and Δx_A we calculate Δt_B and Δx_B for frame B as follows

$$\begin{aligned}
\Delta x_B &= \gamma_{AB}(\Delta x_A + v_{AB}\Delta t_A) \\
&= \frac{13}{12}(-L + \frac{5c}{13}\frac{5L}{c}) \\
&= \frac{13}{12}(\frac{12}{13}L) = L \\
\Delta t_B &= \gamma_{AB}(\Delta t_A + v_{AB}\Delta x_A/c^2) \\
&= \frac{13}{12}(\frac{5L}{c} - \frac{5c}{13}\frac{L}{c^2}) \\
&= \frac{13}{12}(\frac{60}{13}\frac{L}{c}) = \frac{5L}{c}
\end{aligned}$$

From frame A with Δt_A and Δx_A we calculate Δt_C and Δx_C for frame C as follows

$$\begin{aligned}
\Delta x_C &= \gamma_{AC}(\Delta x_A + v_{AC}\Delta t_A) \\
&= \frac{5}{3}(-L + \frac{4c}{5}\frac{5L}{c}) \\
&= \frac{5}{3}(3L) = 5L \\
\Delta t_C &= \gamma_{AC}(\Delta t_A + v_{AC}\Delta x_A/c^2) \\
&= \frac{5}{3}(\frac{5L}{c} - \frac{4c}{5}\frac{L}{c^2}) \\
&= \frac{5}{3}(\frac{21}{3}\frac{L}{c}) = \frac{7L}{c}
\end{aligned}$$

From frame A with Δt_A and Δx_A we calculate Δt_D and Δx_D for frame D as follows

$$\begin{aligned}
\Delta x_D &= \gamma_{AD}(\Delta x_A + v_{AD}\Delta t_A) \\
&= \frac{5}{2\sqrt{6}}(-L + \frac{c}{5}\frac{5L}{c}) \\
&= \frac{5}{2\sqrt{6}} \cdot 0 = 0 \\
\Delta t_D &= \gamma_{AD}(\Delta t_A + v_{AD}\Delta x_A/c^2) \\
&= \frac{5}{2\sqrt{6}}(\frac{5L}{c} - \frac{c}{5}\frac{L}{c^2}) \\
&= \frac{5}{2\sqrt{6}}(\frac{24}{5}\frac{L}{c}) = \frac{2\sqrt{6}L}{c}
\end{aligned}$$

From frame B with Δt_B and Δx_B we calculate Δt_C and Δx_C for frame C as follows

$$\begin{aligned}
\Delta x_C &= \gamma_{BC}(\Delta x_B + v_{BC}\Delta t_B) \\
&= \frac{5}{4}(L + \frac{3c}{5}\frac{5L}{c}) \\
&= \frac{5}{4}(4L) = 5L \\
\Delta t_C &= \gamma_{BC}(\Delta t_B + v_{BC}\Delta x_B/c^2) \\
&= \frac{5}{4}(\frac{5L}{c} + \frac{3c}{5}\frac{L}{c^2}) \\
&= \frac{5}{4}(\frac{28}{5}\frac{L}{c}) = \frac{7L}{c}
\end{aligned}$$

From frame B with Δt_B and Δx_B we calculate Δt_D and Δx_D for frame D taking into account that in this case B moves to the left so the sign of the second term is negative then

$$\begin{aligned}
\Delta x_D &= \gamma_{BD}(\Delta x_B - v_{BD}\Delta t_B) \\
&= \frac{5}{2\sqrt{6}}(L - \frac{c}{5}\frac{5L}{c}) \\
&= \frac{5}{2\sqrt{6}} \cdot 0 = 0 \\
\Delta t_D &= \gamma_{BD}(\Delta t_B - v_{BD}\Delta x_B/c^2) \\
&= \frac{5}{2\sqrt{6}}(\frac{5L}{c} - \frac{c}{5}\frac{L}{c^2}) \\
&= \frac{5}{2\sqrt{6}}(\frac{24}{5}\frac{L}{c}) = \frac{2\sqrt{6}L}{c}
\end{aligned}$$

From frame C with Δt_C and Δx_C we calculate Δt_D and Δx_D for frame D taking into account that in this case C moves to the left so the sign of the second term is negative then

$$\begin{aligned}
\Delta x_D &= \gamma_{CD}(\Delta x_C - v_{CD}\Delta t_C) \\
&= \frac{7}{2\sqrt{6}}(5L - \frac{5c}{7}\frac{7L}{c}) \\
&= \frac{7}{2\sqrt{6}} \cdot 0 = 0 \\
\Delta t_D &= \gamma_{CD}(\Delta t_C - v_{CD}\Delta x_C/c^2) \\
&= \frac{7}{2\sqrt{6}}(\frac{7L}{c} - \frac{5c}{7}\frac{5L}{c^2}) \\
&= \frac{7}{2\sqrt{6}}(\frac{24}{7}\frac{L}{c}) = \frac{2\sqrt{6}L}{c}
\end{aligned}$$

□

Proof. **2.2** Since the photon is moving vertically and you are moving to the left with a speed v with respect to the ground then from your frame and using the velocity addition formula the photon is moving with the following velocities

$$u_x = \frac{0+v}{1+0} = v \qquad\qquad u_y = \frac{c}{\gamma_v(1+0)} = \frac{c}{\gamma_v}$$

If we want that the photon travels 45° rightward and upward from your frame then the velocities u_x and u_y must be equal, then

$$\begin{aligned} v &= \frac{c}{\gamma_v} \\ \frac{v^2}{c^2} &= 1 - \frac{v^2}{c^2} \\ v^2 &= \frac{c^2}{2} \end{aligned}$$

Therefore $v = \frac{1}{\sqrt{2}}c$ □

Proof. **2.3** Let us have a frame S' which is travelling along the dotted line with the speed $v\cos\theta$ measured from the lab frame (S) in the particle's opposite x direction. Then to change from S' to S we need to use the following Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}\cos^2\theta}}$$

Now, seen from the frame S' the particles are actually travelling upward and downward. So seen from the lab's frame (S) we have that

$$v\sin\theta = u_y = \frac{u'_y}{\gamma(1+0)} = \frac{u'_y}{\gamma}$$

Therefore in S' each particle moves with velocity

$$u'_y = \gamma v \sin\theta = \frac{v\sin\theta}{\sqrt{1 - \frac{v^2}{c^2}\cos^2\theta}}$$

Finally, seen from the particle's frame the other particle is travelling in the opposite direction along the y direction therefore

$$
\begin{aligned}
V &= \frac{u'_y + u'_y}{1 + u'^2_y/c^2} \\
&= \frac{\frac{2v\sin\theta}{\sqrt{1-\frac{v^2}{c^2}\cos^2\theta}}}{1 + \frac{v^2\sin^2\theta}{c^2 - v^2\cos^2\theta}} \\
&= \frac{\frac{2v\sin\theta\sqrt{1-\frac{v^2}{c^2}\cos^2\theta}}{1-\frac{v^2}{c^2}\cos^2\theta}}{\frac{c^2 - v^2\cos^2\theta + v^2\sin^2\theta}{c^2(1-\frac{v^2}{c^2}\cos^2\theta)}} \\
&= \frac{2v\sin\theta\sqrt{1-\frac{v^2}{c^2}\cos^2\theta}}{1-\frac{v^2}{c^2}\cos 2\theta}
\end{aligned}
$$

□

Proof. **2.6**

(a) From A's frame the ground is moving in westward as B so by using the velocity addition formula we have that A sees B travelling with a velocity

$$V = \frac{2v}{1 + v^2/c^2}$$

Also from A's point of view the train is length contracted by $2L/\gamma_V$. Where γ_V is

$$
\begin{aligned}
\gamma_V &= \frac{c}{\sqrt{c^2 - V^2}} = \frac{c}{\sqrt{c^2 - (\frac{2v}{1+v^2/c^2})^2}} \\
&= \frac{c}{\sqrt{c^2 - \frac{(2vc^2)^2}{(c^2+v^2)^2}}} = \frac{1}{\sqrt{1 - \frac{(2vc)^2}{(c^2+v^2)^2}}} \\
&= \frac{1}{\sqrt{\frac{(c^2+v^2)^2 - (2vc)^2}{(c^2+v^2)^2}}} = \frac{c^2 + v^2}{\sqrt{((c^2)^2 + 2c^2v^2 + (v^2)^2)^2 - (2vc)^2}} \\
&= \frac{c^2 + v^2}{c^2 - v^2}
\end{aligned}
$$

so the time it takes for the trains to pass is $(L + (2L/\gamma_V))/V$

$$
\begin{aligned}
\Delta t_A &= \frac{L + 2L(\frac{1-v^2/c^2}{1+v^2/c^2})}{\frac{2v}{1+v^2/c^2}} \\
\Delta t_A &= \frac{L((1 + v^2/c^2) + 2(1 - v^2/c^2))}{2v} \\
\Delta t_A &= \frac{L(3 - v^2/c^2)}{2v}
\end{aligned}
$$

(b) From B's frame the ground is moving in eastward as A so by using the velocity addition formula we have that B sees A travelling with a velocity

$$V = \frac{2v}{1 + v^2/c^2}$$

Also from B's point of view the train is length contracted by L/γ_V. Where γ_V as before is

$$\gamma_V = \frac{c^2 + v^2}{c^2 - v^2}$$

so the time it takes for the trains to pass is $(2L + (L/\gamma_V))/V$

$$\Delta t_B = \frac{2L + L(\frac{1-v^2/c^2}{1+v^2/c^2})}{\frac{2v}{1+v^2/c^2}}$$

$$\Delta t_B = \frac{L(2(1 + v^2/c^2) + (1 - v^2/c^2))}{2v}$$

$$\Delta t_B = \frac{L(3 + v^2/c^2)}{2v}$$

(c) In the ground frame we see both trains approaching with velocity $2v$ but also we see them length contracted by L/γ_v and $2L/\gamma_v$ respectively then the time it takes for the trains to pass is given by

$$\Delta t_g = \frac{L/\gamma_v + 2L/\gamma_v}{2v}$$

$$\Delta t_g = \frac{3L\sqrt{1 - v^2/c^2}}{2v}$$

(d) Let us now check that all the invariant intervals are the same. In A's frame we have that

$$\begin{aligned} c^2\Delta t_A^2 - \Delta x^2 &= \frac{c^2L^2(3 - v^2/c^2)^2}{(2v)^2} - L^2 \\ &= L^2(\frac{9c^2 - 6v^2 + v^4/c^2}{(2v)^2} - 1) \\ &= \frac{L^2}{4v^2}(9c^2 - 10v^2 + v^4/c^2) \end{aligned}$$

In B's frame we have that

$$\begin{aligned} c^2\Delta t_B^2 - \Delta x^2 &= \frac{c^2L^2(3 + v^2/c^2)^2}{(2v)^2} - 4L^2 \\ &= L^2(\frac{9c^2 + 6v^2 + v^4/c^2}{(2v)^2} - 4) \\ &= \frac{L^2}{4v^2}(9c^2 - 10v^2 + v^4/c^2) \end{aligned}$$

And finally in the ground's frame we have to determine first Δx since both trains travel with the same velocity in opposite direction the back of each train should meet in the middle of the entire length measured from the ground i.e. $(3L/\gamma_v)/2$ but also we need to subtract the length of train A since we measure starting from the moment both fronts meet. Therefore

$$
\begin{aligned}
c^2\Delta t_g^2 - \Delta x^2 &= \frac{c^2 3^2 L^2(1-v^2/c^2)}{(2v)^2} - \left(\frac{L}{2}\sqrt{1-\frac{v^2}{c^2}}\right)^2 \\
&= L^2\left(\frac{9c^2-9v^2}{(2v)^2} - (\frac{1}{4}-\frac{1}{4}\frac{v^2}{c^2})\right) \\
&= \frac{L^2}{4v^2}(9c^2-10v^2+\frac{v^4}{c^2})
\end{aligned}
$$

□

Proof. **2.8** We know that the first event happens at $(x,t)=(0,0)$ and the the next event happens at $(x,t)=(2,1)$. Let us have another frame S' where the worldline x' passes through the point $(x,t)=(2,1)$ which implies that the two events happen simultaneously in S'. The angle between x and x' is given by

$$\tan\theta = 1/2 = \beta = v/c$$

Therefore our new frame S' must be travelling with a velocity $v=c/2$.

□

Proof. **2.9** All the points in the hyperbola must satisfy that $c^2t^2-x^2=1$ and because of the invariant interval the points should also satisfy that $c^2t'^2-x'^2=1$. This means that the point where the ct' line encounters the hyperbola should also satisfy this constraint that we know is a point where $x'=0$ so this is a point where $ct'=1$ (a unit value).

On the other hand, we have that $\tan\theta_1=\beta=x/ct$ therefore $x=\beta\cdot ct$ and replacing this value in the hyperbola equation we get that

$$
\begin{aligned}
c^2t^2-\beta^2c^2t^2 &= 1 \\
ct\sqrt{1-\beta^2} &= 1
\end{aligned}
$$

Also, we have that the point where the hyperbola joins the line ct' (a unit value on ct') is at a distance $\sqrt{c^2t^2+x^2}=\sqrt{c^2t^2+(\beta ct)^2}$ from the origin, then we have that

$$\sqrt{c^2t^2+(\beta ct)^2} = ct\sqrt{1+\beta^2} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

Therefore this is the length on the paper that one unit on the ct' axis has and also since we are taking that one unit on the ct has a length of 1 in the paper this is also the ratio between one unit in the ct' axis and one unit on the ct axis. □

Proof. **2.13** An observer on the initial inertial frame will see d as the string length which is the length contracted proper distance d' seen from the spaceships i.e.

$$d = d'/\gamma$$

Then, this means that the proper length of the string we see from a spaceship is $d' = \gamma d$. Since $\gamma = 1/\sqrt{1 - v^2/c^2}$ when the velocity increase due to the acceleration, γ increases, therefore, d' measured from a spaceship increases which means that the string ends up breaking. $\square$

Proof. **2.14** Let us have two Lorentz transformations where in the first one we transform x and ct to x' and ct' assuming $v_1 = \tanh \phi_1$ and in the second one we transform x' and ct' to x'' and ct'' assuming $v_2 = \tanh \phi_2$ then we have the following set of equations.

$$\begin{aligned} x &= x' \cosh \phi_1 + ct' \sinh \phi_1 \\ ct &= x' \sinh \phi_1 + ct' \cosh \phi_1 \end{aligned}$$

And for the second transformation, we have that

$$\begin{aligned} x' &= x'' \cosh \phi_2 + ct'' \sinh \phi_2 \\ ct' &= x'' \sinh \phi_2 + ct'' \cosh \phi_2 \end{aligned}$$

Then if we want to transform directly from x to x'' we replace variables such that we have

$$\begin{aligned} x &= (x'' \cosh \phi_2 + ct'' \sinh \phi_2) \cosh \phi_1 + (x'' \sinh \phi_2 + ct'' \cosh \phi_2) \sinh \phi_1 \\ x &= x'' \cosh \phi_2 \cosh \phi_1 + ct'' \sinh \phi_2 \cosh \phi_1 + x'' \sinh \phi_2 \sinh \phi_1 + ct'' \cosh \phi_2 \sinh \phi_1 \\ x &= x''(\cosh \phi_2 \cosh \phi_1 + \sinh \phi_2 \sinh \phi_1) + ct''(\sinh \phi_2 \cosh \phi_1 + \cosh \phi_2 \sinh \phi_1) \\ x &= x'' \cosh(\phi_1 + \phi_2) + ct'' \sinh(\phi_1 + \phi_2) \end{aligned}$$

Where we used that $\cosh(\phi_1 + \phi_2) = \cosh \phi_2 \cosh \phi_1 + \sinh \phi_2 \sinh \phi_1$ and that $\sinh(\phi_1 + \phi_2) = \sinh \phi_2 \cosh \phi_1 + \cosh \phi_2 \sinh \phi_1$

In the same way, we calculate the same to transform from ct to ct'' as follows

$$\begin{aligned} ct &= (x'' \cosh \phi_2 + ct'' \sinh \phi_2) \sinh \phi_1 + (x'' \sinh \phi_2 + ct'' \cosh \phi_2) \cosh \phi_1 \\ ct &= x'' \cosh \phi_2 \sinh \phi_1 + ct'' \sinh \phi_2 \sinh \phi_1 + x'' \sinh \phi_2 \cosh \phi_1 + ct'' \cosh \phi_2 \cosh \phi_1 \\ ct &= x''(\cosh \phi_2 \sinh \phi_1 + \sinh \phi_2 \cosh \phi_1) + ct''(\sinh \phi_2 \sinh \phi_1 + \cosh \phi_2 \cosh \phi_1) \\ ct &= x'' \sinh(\phi_1 + \phi_2) + ct'' \cosh(\phi_1 + \phi_2) \end{aligned}$$

Therefore we see that applying twice the Lorentz transformation gives us the same result as applying one Lorentz transformation with a velocity $v = \tanh(\phi_1 + \phi_2)$. $\square$

Proof. **2.15** From the equation 2.53 we know that the velocity in terms of the time t' of the spaceship is given by

$$\beta(t') = \tanh(at'/c)$$

And we know that in the lab frame we see a clock in the spaceship running slow so we have that

$$\begin{aligned}
dt = \gamma dt' &= \frac{dt'}{\sqrt{1-\beta^2}} \\
\int_0^t dt &= \int_0^{t'} \frac{dt'}{\sqrt{1-\beta^2}} \\
t &= \int_0^{t'} \cosh(at'/c)\ dt' \\
t &= \frac{c}{a}\sinh\left(\frac{at'}{c}\right)
\end{aligned}$$

Where we used that $1/\sqrt{1-\tanh(x)^2} = \cosh(x)$. $\square$

Proof. **2.16** Let us have two events which are the photon passing through the origin of S' and another one which is the photon passing through a point A at a distance $\Delta x'$ from the origin. Between this two events $\Delta t'$ time passes and we know the velocity of the photon must be c in S' so we have that

$$\frac{\Delta x'}{\Delta t'} = c$$

Now by the Lorentz Transformations we can compute Δx and Δt for the S frame but we are interested in the velocity of the photon in the S frame so we actually want to calculate $\frac{\Delta x}{\Delta t}$ then we have that

$$\begin{aligned}
\frac{\Delta x}{\Delta t} &= \frac{\gamma(\Delta x' + v\Delta t')}{\gamma(\Delta t' + v\Delta x'/c^2)} \\
&= \frac{\Delta x'(1 + v\frac{\Delta t'}{\Delta x'})}{\Delta t'(1 + \frac{v\Delta x'}{c^2\Delta t'})} \\
&= \frac{\Delta x'(1 + \frac{v}{c})}{\Delta t'(1 + \frac{v}{c})} \\
&= \frac{\Delta x'}{\Delta t'} = c
\end{aligned}$$

□

Proof. **2.17** Let us suppose we have a set of transformations from S' to S as follows

$$\begin{aligned}
\Delta x &= \gamma_{v_1}(\Delta x' + v_1\Delta t') \\
\Delta t &= \gamma_{v_1}(\Delta t' + v_1\Delta x'/c^2)
\end{aligned}$$

And another set from S'' to S'

$$\begin{aligned}
\Delta x' &= \gamma_{v_2}(\Delta x'' + v_2\Delta t'') \\
\Delta t' &= \gamma_{v_2}(\Delta t'' + v_2\Delta x''/c^2)
\end{aligned}$$

We want to find a set of transformations from S'' to S then by replacing values we have that

$$\begin{aligned}
\Delta x &= \gamma_{v_1}(\gamma_{v_2}(\Delta x'' + v_2\Delta t'') + v_1(\gamma_{v_2}(\Delta t'' + v_2\Delta x''/c^2))) \\
&= \gamma_{v_1}\gamma_{v_2}(\Delta x'' + v_2\Delta t'' + v_1\Delta t'' + v_1v_2\Delta x''/c^2)) \\
&= \gamma_{v_1}\gamma_{v_2}(\Delta x''(1 + \frac{v_1v_2}{c^2}) + \Delta t''(v_1 + v_2)) \\
&= \gamma_{v_1}\gamma_{v_2}(1 + \frac{v_1v_2}{c^2})(\Delta x'' + \frac{v_1 + v_2}{1 + v_1v_2/c^2}\Delta t'')
\end{aligned}$$

Let us now verify that if $v_3 = \frac{v_1+v_2}{1+v_1v_2/c^2}$ then $\gamma_{v_3} = \gamma_{v_1}\gamma_{v_2}(1+\frac{v_1v_2}{c^2})$ as follows

$$\begin{aligned}
\gamma_{v_3} &= \frac{1+\frac{v_1v_2}{c^2}}{\sqrt{1-\frac{v_1^2}{c^2}}\sqrt{1-\frac{v_2^2}{c^2}}} \\
&= \sqrt{\frac{(1+\frac{v_1v_2}{c^2})^2}{(1-\frac{v_1^2}{c^2})(1-\frac{v_2^2}{c^2})}} \\
&= \sqrt{\frac{1+\frac{2v_1v_2}{c^2}+\frac{v_1^2v_2^2}{c^4}}{1-\frac{v_1^2}{c^2}-\frac{v_2^2}{c^2}+\frac{v_2^2v_1^2}{c^4}}} \\
&= \sqrt{\frac{c^2+2v_1v_2+\frac{v_1^2v_2^2}{c^2}}{c^2-v_1^2-v_2^2+\frac{v_2^2v_1^2}{c^2}}} \\
&= \sqrt{\frac{1}{\frac{c^2-v_1^2+2v_1v_2-2v_1v_2-v_2^2+\frac{v_2^2v_1^2}{c^2}}{c^2+2v_1v_2+\frac{v_1^2v_2^2}{c^2}}}} \\
&= \sqrt{\frac{1}{1-\frac{v_1^2+2v_1v_2+v_2^2}{c^2+2v_1v_2+\frac{v_1^2v_2^2}{c^2}}}} \\
&= \sqrt{\frac{1}{1-\frac{(v_1+v_2)^2}{c^2(1+2v_1v_2/c^2+v_1^2v_2^2/c^4)}}} \\
&= \sqrt{\frac{1}{1-\frac{(v_1+v_2)^2}{c^2(1+v_1v_2/c^2)^2}}} = \frac{1}{\sqrt{1-\frac{v_3^2}{c^2}}}
\end{aligned}$$

What's left is finding the transformation for Δt so by replacing we have that

$$\begin{aligned}
\Delta t &= \gamma_{v_1}(\gamma_{v_2}(\Delta t'' + v_2\Delta x''/c^2) + v_1(\gamma_{v_2}(\Delta x'' + v_2\Delta t''))/c^2) \\
&= \gamma_{v_1}\gamma_{v_2}(\Delta t'' + v_2\Delta x''/c^2 + v_1\Delta x''/c^2 + v_1v_2\Delta t''/c^2) \\
&= \gamma_{v_1}\gamma_{v_2}(\Delta t''(1+v_1v_2/c^2) + (v_2+v_1)/c^2\Delta x'') \\
&= \gamma_{v_1}\gamma_{v_2}(1+v_1v_2/c^2)(\Delta t'' + \frac{v_1+v_2}{1+v_1v_2/c^2}\frac{\Delta x''}{c^2}) \\
&= \gamma_{v_3}(\Delta t'' + \frac{v_3}{c^2}\Delta x'')
\end{aligned}$$

$\square$

Proof. **2.18**

(a) We want to compute the distance between the two events from the ground using the Lorentz Transformations then if L' is the distance seen from the gound frame we have that

$$L' = \gamma(L + vt)$$

and since we have that $t = 0$ then we have that $L' = \gamma L$

Now in the ground frame we have by the Lorentz Transformations that

$$t' = \gamma(t + vL/c^2)$$

and since the measurement in the train frame happen simultaneously we have that $t = 0$ and therefore $t' = \gamma vL/c^2$

(b) About the time measured between the two events on the ground frame let us assume that both events have a clock attached to it and both happen when $t = 0$ in the train frame but since this two events are not simultaneous in the ground frame, because of the rear clock ahead effect we have that when the left event happen at $t' = 0$ in the ground frame the right clock is showing $-Lv/c^2$ time, so the events will happen with a time interval of Lv/c^2 and also we see this clock running slower by a factor γ then the real time between events is $\gamma Lv/c^2$.

About the length we see from the ground frame we have that the length of the train is length contracted and as we know the two events are not simultaneous on the ground frame then we need to add the distance that accounts for this, so we have that

$$\begin{aligned}
L' &= L/\gamma + \gamma Lv^2/c^2 \\
&= \gamma L(\frac{1}{\gamma^2} + \frac{v^2}{c^2}) \\
&= \gamma L((1 - \frac{v^2}{c^2}) + \frac{v^2}{c^2}) \\
&= \gamma L
\end{aligned}$$

For the train frame, let us suppose that there are two people on the ground and the train sees them at each event, simultaneous for the train then the distance the train sees between the two people is $L = L'/\gamma$ because of the length contraction which means that $L' = \gamma L$

Now about the time we have that in the train frame both events happen simultaneously at $t = 0$ but if we see the watch of the person on the right (the ground is moving leftward from the train perspective) at the moment the two events happen in the train frame we see that the watch is showing Lv/c^2 because of the rear clock ahead effect and in addition to that we see the watch moving slower because of the time dilation effect therefore the time between the two events in the ground frame as seen from the train frame is $\gamma Lv/c^2$.

$\square$

Proof. **2.19** Let S' be the frame in which the clock moves vertically, so seen from our frame, S, the frame S' is moving horizontally and therefore from the velocity addition formula in the y direction we have that

$$u_y = \frac{u}{\gamma_v(1+0)} = \frac{u}{\gamma_v}$$

Where we used that $u'_y = u$ and that $u'_x = 0$. Then we have that

$$\begin{aligned} u_y = \frac{\Delta y}{\Delta t} &= \frac{u}{\gamma_v} \\ \Delta t &= \gamma_v \frac{\Delta y}{u} \\ \Delta t &= \gamma_v \gamma_u \frac{\Delta y'}{u} \\ \Delta t &= \gamma_v \gamma_u \Delta t' \end{aligned}$$

Here we used that $\Delta y = \Delta y' \gamma_u$ because of the length contraction and that $\Delta t' = \Delta y'/u$. Therefore the time we see is slowed by a factor $\gamma_v \gamma_u$. $\square$

Proof. **2.20** Let S' be the train frame, and let S be the ground frame. Then we can calculate the x and y photon's velocities seen from S using the velocity addition formulas as follows

$$u_x = \frac{c/\sqrt{2} - c/\sqrt{2}}{1 - c^2/2c^2} = 0$$

Here we used that $u'_x = -c/\sqrt{2}$ because the photon is moving leftward and $v = c/\sqrt{2}$ because S' is moving rightward. And for the y component we have that

$$\begin{aligned} u_y &= \frac{c/\sqrt{2}}{\gamma_{c/\sqrt{2}}(1 - c^2/2c^2)} \\ &= \frac{2c}{\sqrt{2}\gamma_{c/\sqrt{2}}} \\ &= \frac{2c\sqrt{1 - c^2/2c^2}}{\sqrt{2}} \\ &= \frac{2c}{\sqrt{2}\sqrt{2}} = c \end{aligned}$$

Finally, these results make sense because given that the train is travelling the opposite way we see that the photon is travelling only vertically i.e. $u_x = 0$. Also, the velocity we see in the vertical direction is c which is coherent with the fact that the velocity of the light should be the same in any reference frame. $\square$

Proof. **2.22**

(i) For the person A standing on the ground the velocity between the train and the person is $c/5$ since we already have both velocities measured in the ground frame. Since we see the length of the train length contracted then the distance between the two events is L/γ. Then the time between the two events is

$$\Delta t_A = \frac{5L}{c\gamma_{3c/5}} = \frac{4L}{c}$$

Now the train B will travel for a time $4L/c$ with a velocity of $3c/5$ so it will cover a distance of $12L/5$, but this measurement correspond to the distance travelled by the front of the train so we need to add the length of the train (taking into account the length contraction effect) i.e. $12L/5 + 4L/5 = 16L/5$. Therefore the invariant interval has a value of

$$c^2\Delta t_A^2 - \Delta x_A^2 = (4L)^2 - \frac{(16L)^2}{5^2} = \frac{144}{25}L^2$$

(ii) For the train B the ground is travelling to the left (opposite to C) so by using the velocity addition formula we can calculate the velocity at which C is moving seen from B i.e.

$$V = \frac{4c/5 - 3c/5}{1 - (3 \cdot 4)/5^2} = \frac{5}{13}c$$

From here we can calculate the time between the two events as follows

$$\Delta t_B = \frac{L}{\frac{5}{13}c} = \frac{13L}{5c}$$

Therefore the invariant interval is

$$\begin{aligned} c^2\Delta t_B^2 - \Delta x_B^2 &= (\frac{13L}{5})^2 - L^2 \\ &= L^2(\frac{169}{25} - 1) \\ &= \frac{144L^2}{25} \end{aligned}$$

(iii) For the person C the ground is travelling to the left (opposite to B) so by using the velocity addition formula we can calculate the velocity at which B is moving seen from C i.e.

$$V = \frac{3c/5 - 4c/5}{1 - (3 \cdot 4)/5^2} = -\frac{5}{13}c$$

From here we can calculate the time between the two events as follows

$$\Delta t_C = \frac{L/\gamma_V}{\frac{5}{13}c} = \frac{12L}{5c}$$

Where we used that $\gamma_V = \gamma_{5c/13} = 13/12$. Therefore the invariant interval is

$$c^2\Delta t_C^2 - \Delta x_C^2 = (\frac{12L}{5})^2 - 0$$
$$= L^2(\frac{144}{25})$$

In this invariant interval the distance between the two events is 0 since both happen at C from C's frame.

□

Proof. **2.25**

(a) In the barn frame the events happen at a distance $\Delta x = L$. To compute the time between events we must take into account that the first event (seen from the barn frame) happens at the back of the pole and the other event happens at the front of the pole, therefore we subract the distance of the barn L from the distance of the pole which is length contracted to L/γ then we calculate the following

$$\begin{aligned}\Delta t &= \frac{L - L/\gamma_{3/5}}{3c/5} \\ &= \frac{L/5}{3c/5} \\ &= \frac{L}{3c}\end{aligned}$$

(b) In the Minkowsky diagram we see that E_2 being "the front of the pole passing the right end of the barn" is happening before E_1 being "the back of the pole passing the left end of the barn" seen from the pole frame, then we can find a frame where the $\Delta x'$ axis passes through E_2 with coordinates $(L, L/3)$, meaning that E_1 and E_2 would happen simultaneously from this frame, then our new $\Delta x'$ axis must have an inclination of

$$\tan\theta = \frac{c\Delta t}{\Delta x} = \frac{L/3}{L} = \frac{1}{3}$$

The axis in the pole frame has an inclination of $\theta = \arctan(3/5) = 30.96°$ which is bigger than the inclination we get for the axis in this new frame $\theta = \arctan(1/3) = 18.43°$ as we wanted.

□

Proof. **2.30**

(a) If we imagine a stationary observer C between A and B, C will see the flashes from A with a frequency given by the equation we have

$$f_C = \sqrt{\frac{1+\beta}{1-\beta}}\ f_A$$

where in this case f_A is the frequency A says he is sending the flashes from its frame. But since C is re-sending the flashes instantaneously then B will see the flashes with a frequency of

$$f_B = \sqrt{\frac{1+\beta}{1-\beta}}\ f_C$$

Therefore in terms of f_A we have that the frequency that B sees is given by

$$f_B = \frac{1+\beta}{1-\beta}\ f_A$$

(b) In this case B sees A travelling at a velocity V given by the relativistic sum of velocities i.e.

$$V = \frac{2v}{1+v^2/c^2}$$

Then from the frequency equation which applies in this case too, we have that

$$f_B = \sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}\ f_A$$

$$f_B = \sqrt{\frac{1+\frac{2v}{c(1+v^2/c^2)}}{1-\frac{2v}{c(1+v^2/c^2)}}}\ f_A$$

$$f_B = \sqrt{\frac{\frac{c(1+v^2/c^2+2v/c)}{c(1+v^2/c^2)}}{\frac{c(1+v^2/c^2-2v/c)}{c(1+v^2/c^2)}}}\ f_A$$

$$f_B = \sqrt{\frac{(1+v/c)^2}{(1-v/c)^2}}\ f_A$$

$$f_B = \frac{1+\beta}{1-\beta}\ f_A$$

Which is the same result we obtained the other way.

□

Proof. **2.34**

(a) From the time equation we got from (2.15) and the velocity equation (2.53) we have that

$$t' = \frac{c}{a}\sinh^{-1}\left(\frac{at}{c}\right)$$
$$v(t') = c\tanh(\frac{at'}{c})$$

By replacing variables we have that

$$v(t) = c\tanh(\sinh^{-1}\left(\frac{at}{c}\right))$$
$$v(t) = \frac{at}{\sqrt{(at/c)^2+1}}$$

Since we know $v(t) = dL/dt$ we can integrate to obtain L as a function of t as follows

$$L = c\int_0^t \frac{at/c}{\sqrt{(at/c)^2+1}}dt$$
$$L = c\left[\frac{c}{a}\sqrt{\left(\frac{at}{c}\right)^2+1} - \frac{c}{a}\right]$$
$$\left(\frac{aL}{c^2}+1\right)^2 - 1 = \left(\frac{at}{c}\right)^2$$
$$\left(\frac{at}{c}\right)^2 = \frac{a^2L^2}{c^4} + \frac{2aL}{c^2}$$
$$t^2 = \frac{c^2}{a^2}\left(\frac{a^2L^2}{c^4} + \frac{2aL}{c^2}\right)$$
$$t = \sqrt{\frac{L^2}{c^2} + \frac{2L}{a}}$$
$$t = \frac{L}{c}\sqrt{1 + \frac{2c^2/a}{L}}$$

If $L \to 0$ then L^2/c^2 is negligible so we have that

$$t = \sqrt{\frac{2L}{a}}$$
$$L = \frac{1}{2}at^2$$

Where we have here the known formula for L used in the non-relativistic theory.
If $L \to \infty$ we have that $\frac{2c^2/a}{L}$ is negligible then we get

$$t = \frac{L}{c}$$

(b) From part (a) we have an expression of t' in terms of t so we can replace this value with the expression we found in terms of L as follows

$$t' = \frac{c}{a}\sinh^{-1}\left(\frac{at}{c}\right)$$
$$t' = \frac{c}{a}\sinh^{-1}\left(\frac{a}{c}\sqrt{\frac{L^2}{c^2}+\frac{2L}{a}}\right)$$

If $L \to 0$ then L^2/c^2 is negligible so we have that

$$t' = \frac{c}{a}\sinh^{-1}\left(\sqrt{\frac{2La}{c^2}}\right)$$
$$t' = \frac{c}{a}\sqrt{\frac{2La}{c^2}}$$
$$t' = \sqrt{\frac{2L}{a}}$$
$$L = \frac{1}{2}at'^2$$

Here we used the fact that $\sinh^{-1}(x) \approx x$ when x is small, and we obtained the formula for L used in non-relativistic theory again.
If $L \to \infty$ we have that $\frac{2c^2/a}{L}$ is negligible then we get that

$$t' = \frac{c}{a}\sinh^{-1}\left(\frac{aL}{c^2}\right)$$

$\square$