Solved selected problems of Special Relativity - Morin

Franco Zacco

Chapter 3 - Dynamics

Solution. **3.1** Let the mass in the rest frame have the initial energy E_0 then by conservation of energy each photon must have $E_0/2 = hf/2$ energy after the decay.

Then in the frame where the mass is moving with velocity v, we have to take into account the relativistic Doppler effect hence the energy of the photons is given by

$$\frac{hf}{2}\sqrt{\frac{1+v}{1-v}} \qquad \text{and} \qquad \frac{hf}{2}\sqrt{\frac{1-v}{1+v}}$$

Where we assumed that each photon goes in a different direction. Then the total energy of the photons is

$$\begin{aligned}
E &= \frac{E_0}{2}\sqrt{\frac{1+v}{1-v}} + \frac{E_0}{2}\sqrt{\frac{1-v}{1+v}} \\
&= \frac{E_0}{2}\left((1+v)\sqrt{\frac{1}{1-v^2}} + (1-v)\sqrt{\frac{1}{1-v^2}}\right) \\
&= \frac{E_0}{\sqrt{1-v^2}} = \gamma E_0
\end{aligned}$$

Also since E_0 is the energy of the rest mass and γE_0 is the energy of a moving mass with velocity v then $\gamma E_0 - E_0$ must reduce to $mv^2/2$ in the non-relativistic limit then

$$\begin{aligned}
\frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}} - E_0 &= \frac{mv^2}{2} \\
E_0\left(\left(1 + \frac{(v/c)^2}{2}\right) - 1\right) &= \frac{mv^2}{2} \\
E_0\frac{(v/c)^2}{2} &= \frac{mv^2}{2} \\
E_0 &= mc^2
\end{aligned}$$

Where we approximated by Taylor series $1/\sqrt{1-(v/c)^2}$ as $1 + (v/c)^2/2 + O((v/c)^4)$. Therefore the total energy is given by $E = \gamma mc^2$.

On the other hand, let p_0 be the momentum of the mass in the frame where it is traveling with velocity v then when the decay happens the momentum of the photon must take into account the relativistic Doppler effect and the direction of each photon so we will have opposite signs for each momentum, hence the total momentum will be

$$\begin{aligned} p &= \frac{p_0}{2}\sqrt{\frac{1+v}{1-v}} - \frac{p_0}{2}\sqrt{\frac{1-v}{1+v}} \\ &= \frac{p_0}{2}\left(\frac{1}{\sqrt{1-v^2}}((1+v)-(1-v))\right) \\ &= \gamma v p_0 \end{aligned}$$

By adding the missing c we have that $p = \gamma(v/c)p_0$. Since p must reduce to mv in the non-relativistic limit where $\gamma \approx 1$ we must have that $p_0 = mc$. Therefore the total momentum is given by $p = \gamma mv$. □

Solution. **3.2** Let the particle be traveling at u' velocity with respect to the S' reference frame with an energy $E' = E_{total}$ and a momentum $p' = p_{total}$. Since the velocity of the particle with respect to the CM reference frame is 0 then the CM reference frame S_{CM} must be traveling at the particle velocity u but then seen from the CM reference frame the S' reference frame is moving with a velocity $-u$. Then from the (Energy/Momentum) Lorentz transformation we have that

$$\begin{aligned} E_{CM} &= \gamma_u(E' - up') \\ 0 &= \gamma_u(p' - uE'/c^2) \end{aligned}$$

Therefore from the last equation, we have that the velocity of the frame in which the total momentum is zero is

$$u = c^2\,\frac{p_{total}}{E_{total}}$$

. □

Solution. **3.3** Let the mass resulting from the collision be M' then from the conservation of energy and momentum we have that

$$\begin{aligned} E_{M'} &= E_M + E_m = \gamma M + m \\ p_{M'} &= p_M + p_m = \gamma M v + 0 \end{aligned}$$

Where we dropped the c's. Also, we know that $M'^2 = E_{M'}^2 - p_{M'}^2$ hence

$$\begin{aligned} M' &= \sqrt{(\gamma M + m)^2 - (\gamma M v)^2} \\ &= \sqrt{\gamma^2 M^2 + 2\gamma M m + m^2 - \gamma^2 M^2 v^2} \\ &= \sqrt{\gamma^2 M^2 (1 - v^2) + 2\gamma M m + m^2} \\ &= \sqrt{M^2 + 2\gamma M m + m^2} \end{aligned}$$

Here we used that $\gamma^2(1 - v^2) = 1$. Now if we consider that $m \ll M$ we can drop the term m^2 hence

$$\begin{aligned} M' &\approx \sqrt{M^2 + 2\gamma M m} \\ &\approx M\sqrt{1 + \frac{2\gamma m}{M}} \\ &\approx M + \gamma m \end{aligned}$$

We went from the second to the last step by applying a binomial approximation which we can do because $2\gamma m/M < 1$. $\square$

Solution. **3.4** Let us assume, without loss of generality, that after the decay the particles M_B and M_C have only a momentum p_B and p_C respectively in the x direction.

Let's write the 4-momenta before the decay for the particle M_A (dropping the c's) where we have that

$$P = (M_A, 0, 0, 0)$$

After the decay, we will have that

$$P_B = (\gamma M_B, p_B, 0, 0) \qquad P_C = (\gamma M_C, p_C, 0, 0)$$

Then by the conservation laws of energy and momentum, we get that

$$\begin{aligned} M_A &= \gamma M_B + \gamma M_C \\ p_B &= -p_C \end{aligned}$$

We see that both particles have opposite velocity directions.

On the other hand, we know that

$$p_B^2 = E_B^2 - M_B^2 = E_C^2 - M_C^2 = p_C^2$$

Hence

$$E_B^2 = E_C^2 - M_C^2 + M_B^2 \qquad E_C^2 = E_B^2 - M_B^2 + M_C^2$$

So if we replace $E_C = M_A - \gamma M_B$ and knowing that $E_B = \gamma M_B$ we get that

$$\begin{aligned} E_B^2 &= (M_A - \gamma M_B)^2 - M_C^2 + M_B^2 \\ E_B^2 &= M_A^2 - 2\gamma M_B M_A + (\gamma M_B)^2 - M_C^2 + M_B^2 \\ 0 &= M_A^2 - 2E_B M_A - M_C^2 + M_B^2 \\ E_B &= \frac{M_A^2 + M_B^2 - M_C^2}{2M_A} \end{aligned}$$

In the same way, replacing $E_B = M_A - \gamma M_C$ in the other equation and knowing that $E_C = \gamma M_C$ we get that

$$\begin{aligned} E_C^2 &= (M_A - \gamma M_C)^2 - M_B^2 + M_C^2 \\ E_C &= \frac{M_A^2 + M_C^2 - M_B^2}{2M_A} \end{aligned}$$

Finally, we can determine the momentum of particles M_C and M_B as follows

$$\begin{aligned} p_B = -p_C &= \sqrt{\left(\frac{M_A^2 + M_B^2 - M_C^2}{2M_A}\right)^2 - M_B^2} \\ &= \sqrt{\frac{(M_A^2 + M_B^2 - M_C^2)^2 - 4M_A^2 M_B^2}{4M_A^2}} \\ &= \frac{1}{2M_A}\sqrt{(M_A^2 + M_B^2 - M_C^2)^2 - 4M_A^2 M_B^2} \end{aligned}$$

$\square$

Solution. **3.5** Let a particle with mass m and energy E collide with an identical stationary particle.

From the conservation energy equation, where we dropped the c's, we want to produce N particles that will have E' energy, hence

$$E + m = NE'$$

Analogously from the momentum conservation equation, we would want that

$$p = Np'$$

where p' is the momentum of each produced particle. Also, we know that $E'^2 = m^2 + p'^2$ and $p^2 = E^2 - m^2$ then by squaring the energy conservation equation and replacing the values of E'^2 and p^2 we get that

$$\begin{aligned}
(E+m)^2 &= (NE')^2 \\
(E+m)^2 &= N^2(m^2 + p'^2) \\
(E+m)^2 &= N^2m^2 + p^2 \\
E^2 + 2Em + m^2 &= N^2m^2 + E^2 - m^2 \\
2Em &= N^2m^2 - 2m^2 \\
E &= m\left(\frac{N^2}{2} - 1\right)
\end{aligned}$$

Which gives us the minimum energy that we would need to produce N particles with m (rest) mass. $\square$

Solution. **3.6** Let a particle with mass M decay into several particles where one of them has a mass of m and the sum of the rest sum up to a mass of μ.

From the energy conservation equation, where we dropped the c's we have that

$$\begin{aligned} M &= E_m + E_\mu \\ M - E_m &= E_\mu \end{aligned}$$

Also, from the momentum conservation equation, we see that

$$\begin{aligned} 0 &= p_m + p_\mu \\ -p_m &= p_\mu \end{aligned}$$

By squaring both sides of the energy conservation equation we get that

$$\begin{aligned} (M - E_m)^2 &= E_\mu^2 \\ M^2 - 2ME_m + E_m^2 &= E_\mu^2 \end{aligned}$$

Also we know that $E_\mu^2 - p_\mu^2 = E_{CM}^2$ where E_{CM}^2 is the total energy in the center-of-mass frame for the μ particles and that $p_m^2 = p_\mu^2$ hence

$$\begin{aligned} M^2 - 2ME_m + E_m^2 - p_\mu^2 &= E_\mu^2 - p_\mu^2 \\ M^2 - 2ME_m + E_m^2 - p_m^2 &= E_{CM}^2 \end{aligned}$$

So by replacing $m^2 = E_m^2 - p_m^2$ we get that

$$\begin{aligned} M^2 - 2ME_m + m^2 &= E_{CM}^2 \\ M^2 - 2ME_m &= E_{CM}^2 - m^2 \\ 2ME_m &= m^2 + M^2 - E_{CM}^2 \\ E_m &= \frac{m^2 + M^2 - E_{CM}^2}{2M} \end{aligned}$$

We can maximize E_m by minimizing E_{CM} which is going to happen when all of the μ particles are at rest in the CM frame hence $E_{CM} = \mu$. Therefore E_m is going to be the maximum when

$$E_m = \frac{m^2 + M^2 - \mu^2}{2M}$$

$\square$

Solution. **3.10** Before the collision, we see that

$$P_p = \left(\frac{hc}{\lambda},\ \frac{hc}{\lambda},\ 0,\ 0\right)$$
$$P_m = \left(mc^2,\ 0,\ 0,\ 0\right)$$

Where P_p is the 4-momentum for the photon and P_m is the 4-momentum for the electron. Then after the collision, we have that

$$P'_p = \left(\frac{hc}{\lambda'},\ \frac{hc}{\lambda'}\cos\theta,\ \frac{hc}{\lambda'}\sin\theta,\ 0\right)$$
$$P'_m = \left(E',\ p'_x c,\ p'_y c,\ 0\right)$$

From the conservation of energy and momentum, we know that $P_p + P_m = P'_p + P'_m$ and hence

$$\begin{aligned}
P'^2_m &= (P_p + P_m - P'_p)^2 \\
P'^2_m &= P^2_p + 2P_p(P_m - P'_p) + (P_m - P'_p)^2 \\
P'^2_m &= P^2_p + 2P_pP_m - 2P_pP'_p + P^2_m - 2P_mP'_p + P'^2_p \\
m^2c^4 &= 0 + 2\frac{hc}{\lambda}mc^2 - 2\left(\frac{h^2c^2}{\lambda\lambda'} - \frac{h^2c^2}{\lambda\lambda'}\cos\theta\right) + m^2c^4 - 2\frac{hc}{\lambda'}mc^2 + 0 \\
0 &= \frac{hc}{\lambda}mc^2 - \frac{h^2c^2}{\lambda\lambda'} + \frac{h^2c^2}{\lambda\lambda'}\cos\theta - \frac{hc}{\lambda'}mc^2 \\
0 &= hmc^3\lambda' - h^2c^2 + h^2c^2\cos\theta - hmc^3\lambda \\
\lambda' &= \lambda + \frac{h}{mc} - \frac{h}{mc}\cos\theta
\end{aligned}$$

Therefore the wavelength λ' in terms of λ is given by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

□

Solution. **3.11** Let us compute the increase in speed in S i.e. $a_x dt$ by adding relativistically $a'_x dt'$ to a given speed v hence

$$v + a_x dt = \frac{a'_x dt' + v}{1 + a'_x dt' v}$$

also by time dilation, we know that the observed time in S is slowed down by a factor γ hence $dt = \gamma dt'$ so by replacing we have that

$$\begin{aligned}
\frac{a'_x dt' + v}{1 + a'_x dt' v} &= v + a_x \gamma dt' \\
a'_x dt' + v &= (v + a_x \gamma dt')(1 + a'_x dt' v) \\
a'_x dt' + v &= v + a_x \gamma dt' + a'_x dt' v^2 + \gamma a_x a'_x dt'^2 v \\
a'_x dt' &= a_x \gamma dt' + a'_x dt' v^2 \\
a'_x (1 - v^2) &= a_x \gamma \\
a'_x &= a_x \gamma^3
\end{aligned}$$

Where we dropped terms of order dt'^2 and we used that $\gamma^2 = 1/(1-v^2)$. $\square$

Solution. **3.12** Let us take a small period of time Δt where the velocity of the stick goes from 0 to Δu because of the force applied, then each dumbbell's mass will have a velocity with respect to the stick which can be determined by relativistically adding/subtracting v and Δu since we can assume that the time is so small that the masses still move horizontally hence

$$u = \frac{v \pm \Delta u}{1 \pm v\Delta u}$$

This implies that the momentum for the mass moving in the positive direction (Δu direction) is given by

$$\begin{aligned} p_u &= \gamma_u m u \\ &= \gamma_{\Delta u}\gamma_v(1 + v\Delta u)m\left(\frac{v + \Delta u}{1 + v\Delta u}\right) \\ &= \gamma_{\Delta u}\gamma_v(v + \Delta u)m \end{aligned}$$

Then the momentum for the mass moving in the negative direction is given by $-\gamma_{\Delta u}\gamma_v(v - \Delta u)m$. Where we used that $\gamma_u = \gamma_{\Delta u}\gamma_v(1 \pm v\Delta u)$. So the change in momentum of the system after a period Δt is

$$\begin{aligned} \Delta p &= \gamma_{\Delta u}\gamma_v(v + \Delta u)m - \gamma_{\Delta u}\gamma_v(v - \Delta u)m - 0 \\ &= 2\gamma_{\Delta u}\gamma_v m\Delta u \end{aligned}$$

but since Δu is really small then $\gamma_{\Delta u} \approx 1$ hence $p = 2\gamma_v m\Delta u$. Also we have that $F = \Delta p/\Delta t$ so we get that

$$\begin{aligned} F = \frac{\Delta p}{\Delta t} &= 2\gamma_v m\frac{\Delta u}{\Delta t} \\ &= 2\gamma m a \end{aligned}$$

Where we replaced $a = \Delta u/\Delta t$ as the acceleration we added to the system and $\gamma_v = \gamma$ which implies that the system behaves like a mass $M = 2\gamma m$. $\square$

Solution. **3.13** We know from the relativistic Newton's equation that

$$F = \gamma^3 ma$$

but also we know that F is equal to Hooke's spring force so we have that

$$\begin{aligned}
\gamma^3 ma &= -m\omega^2 x \\
\gamma^3 v \frac{dv}{dx} &= -\omega^2 x
\end{aligned}$$

where we used that $a = vdv/dx$, now by integrating we get that

$$\begin{aligned}
\int \gamma^3 v dv &= \int -\omega^2 x dx \\
\frac{1}{\sqrt{1-v^2}} &= -\omega^2 \frac{x^2}{2} + C \\
\gamma &= -\frac{\omega^2 x^2}{2} + C
\end{aligned}$$

From the initial conditions, when $x = b$ we have that $v = 0$ then $C = 1 + \frac{\omega^2 b^2}{2}$ so replacing and adding the c's to fit the units we have that

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2)$$

Now we need to solve the differential equation we have to determine the period. Here we named $A = \frac{\omega^2}{2c^2}$ for simplicity hence

$$\begin{aligned}
1 + A(b^2 - x^2) &= \frac{1}{\sqrt{1 - (\frac{dx}{dt}/c)^2}} \\
1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2} &= \frac{1}{(1 + A(b^2 - x^2))^2} \\
\left(\frac{dx}{dt}\right)^2 \frac{1}{c^2} &= 1 - \frac{1}{(1 + A(b^2 - x^2))^2} \\
\frac{dx}{dt} &= \frac{c\sqrt{(1 + A(b^2 - x^2))^2 - 1}}{1 + A(b^2 - x^2)}
\end{aligned}$$

Since we know $\gamma = 1 + A(b^2 - x^2)$ by replacing and solving the differential equation we get that

$$\begin{aligned}
\int_0^{T/4} dt &= \int_0^b \frac{\gamma}{c\sqrt{\gamma^2 - 1}} dx \\
\frac{T}{4} &= \frac{1}{c} \int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}} dx \\
T &= \frac{4}{c} \int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}} dx
\end{aligned}$$

□

Solution. **3.15** Let the dustpan have a mass M after some time t in the dustpan frame. Then after dt time the energy would change from γM to $\gamma M + \lambda dx = \gamma M + \lambda v dt$ and the momentum would be $\gamma M v$ also, we know that $M^2 = E^2 - p^2$ hence

$$\begin{aligned}
(M+dM)^2 &= (\gamma M + \lambda v dt)^2 - (\gamma M v)^2 \\
M + dM &= \sqrt{(\gamma M)^2 + 2\gamma M \lambda v dt + (\lambda v dt)^2 - (\gamma M v)^2} \\
M + dM &\approx \sqrt{(\gamma M)^2 + 2\gamma M \lambda v dt - (\gamma M v)^2} \\
M + dM &\approx \sqrt{2\gamma M \lambda v dt + (\gamma M)^2(1 - v^2)} \\
M + dM &\approx M\sqrt{\frac{2\gamma\lambda v dt}{M} + 1}
\end{aligned}$$

Where we dismissed the term involving dt^2. On the other hand, by applying the binomial approximation we have that

$$\begin{aligned}
M + dM &\approx M\left(1 + \frac{\gamma\lambda v dt}{M}\right) \\
dM &\approx \gamma\lambda v dt
\end{aligned}$$

Therefore the rate at which the mass of the dustpan plus the dust inside is increasing is

$$\frac{dM}{dt} \approx \gamma\lambda v$$

□

Solution. **3.16** The energy of the dustpan at the beginning is $\gamma_0 M_0$ in the lab frame hence after a distance of x the energy will be $\gamma M = \gamma_0 M_0 + \lambda x$. Also, we know that the momentum is conserved then at any moment the momentum must be $\gamma_0 M_0 v_0$ hence when a distance x has been covered, we can write that $\gamma M v = \gamma_0 M_0 v_0$ so we have that

$$\begin{aligned}\gamma_0 M_0 + \lambda x &= \frac{\gamma_0 M_0 v_0}{v}\\ v &= \frac{\gamma_0 M_0 v_0}{\gamma_0 M_0 + \lambda x}\end{aligned}$$

Which is the function $v(x)$ we are looking for.

Now from the $v(x)$ equation, we can determine $x(t)$ by integration as follows

$$\begin{aligned}\frac{dx}{dt} &= \frac{\gamma_0 M_0 v_0}{\gamma_0 M_0 + \lambda x}\\ \int (\gamma_0 M_0 + \lambda x) dx &= \int \gamma_0 M_0 v_0 dt\\ \gamma_0 M_0 x + \frac{\lambda x^2}{2} &= \gamma_0 M_0 v_0 t + C\\ \frac{\lambda x^2}{2\gamma_0 M_0} + x &= v_0 t\end{aligned}$$

Where we used that $x = 0$ when $t = 0$ hence the constant of integration $C = 0$. By solving for x we get that

$$\begin{aligned}x &= \frac{-1 \pm \sqrt{1 + \frac{4\lambda v_0 t}{2\gamma_0 M_0}}}{\frac{2\lambda}{2\gamma_0 M_0}}\\ x &= \frac{-\gamma_0 M_0 \pm \gamma_0 M_0 \sqrt{1 + \frac{2\lambda v_0 t}{\gamma_0 M_0}}}{\lambda}\\ x &= \frac{-\gamma_0 M_0 \pm \sqrt{(\gamma_0 M_0)^2 + 2\lambda v_0 \gamma_0 M_0 t}}{\lambda}\end{aligned}$$

Finally, to obtain $v(t) = dx/dt$ we derivate the above equation with respect to t as follows

$$\begin{aligned}v(t) &= \frac{2\lambda v_0 \gamma_0 M_0}{2\lambda \sqrt{(\gamma_0 M_0)^2 + 2\lambda v_0 \gamma_0 M_0 t}}\\ v(t) &= \frac{v_0}{\sqrt{1 + \frac{2\lambda v_0 t}{\gamma_0 M_0}}}\end{aligned}$$

□

Solution. **3.17** From the dustpan frame, the dust is traveling with a velocity v towards the dustpan, if we analyze a small timeframe dt' in the dustpan frame we see that the dust mass colliding with the dustpan must be $\gamma\lambda v dt'$ because of length contraction. So the dustpan momentum is going to change by $-\gamma(\gamma\lambda v dt')v$ were we used a negative sign since the dust is traveling toward the dustpan hence this is the change in momentum $dp = -\gamma^2\lambda v^2 dt'$ which implies that the force exerted on the dustpan is

$$F = \frac{dp}{dt'} = -\gamma^2\lambda v^2$$

If we work now from the lab frame we see that the dustpan is traveling towards the dust with a velocity v so after a time dt in the lab frame the dustpan must have collided (and grabbed) a dust mass dm of $\gamma\lambda dx = \gamma\lambda v dt$ because of what we computed in problem 3.15 which changes the momentum of the dustpan by $dp = -\gamma(\gamma\lambda v dt)v$. Therefore the force exerted on the dustpan is

$$F = \frac{dp}{dt} = -\gamma^2\lambda v^2$$

Which has the same magnitude as the one we computed from the dustpan frame. □

Solution. **3.31** The 4-momentum vector before the decay (where we dropped the c's) is given by

$$P = (\gamma m, \gamma m v, 0, 0)$$

After the decay, the photons will have the following 4-momentum vectors

$$\begin{aligned} P_{ph1} &= (E_1, E_1, 0, 0) \\ P_{ph2} &= \left(E_2, -\frac{E_2}{2}, \frac{\sqrt{3}E_2}{2}, 0\right) \\ P_{ph3} &= \left(E_3, -\frac{E_3}{2}, -\frac{\sqrt{3}E_3}{2}, 0\right) \end{aligned}$$

Where we are considering the x positive direction to the right and the y positive direction upwards. Also, we used that $\sin 30 = 1/2$ and $\cos 30 = \sqrt{3}/2$.

From the conservation of energy and momentum we have that

$$\gamma m = E_1 + E_2 + E_3 \tag{1}$$

$$\gamma m v = E_1 - \frac{E_2}{2} - \frac{E_3}{2} \tag{2}$$

$$0 = \frac{\sqrt{3}E_2}{2} - \frac{\sqrt{3}E_3}{2} \tag{3}$$

From (3) we have that $E_3 = E_2$ hence by replacing in (1) and (2) we get that

$$\gamma m = E_1 + 2E_2 \tag{4}$$

$$\gamma m v = E_1 - E_2 \tag{5}$$

Then by subtracting the first equation from the last equation, we get that

$$\begin{aligned} \gamma m v - \gamma m &= E_1 - E_2 - (E_1 + 2E_2) \\ \gamma m(v-1) &= -3E_2 \\ E_2 &= \frac{\gamma m(1-v)}{3} \end{aligned}$$

We know that $E_3 = E_2$ hence $E_3 = \frac{\gamma m(1-v)}{3}$. Finally by replacing this value in (4) we get that

$$\begin{aligned} \gamma m &= E_1 + \frac{2\gamma m(1-v)}{3} \\ E_1 &= \gamma m\left(1 - \frac{2(1-v)}{3}\right) \end{aligned}$$

□

Solution. **3.32** The 4-momentum vector before the collision (where we dropped the c's) is given by

$$\begin{aligned} P_p &= (E, E, 0, 0) \\ P_M &= (M, 0, 0, 0) \end{aligned}$$

After the collision, the photon and the mass will have the following 4-momentum vectors

$$\begin{aligned} P'_p &= (E', 0, E', 0) \\ P'_M &= (\gamma M, p\cos\theta, -p\sin\theta, 0) \end{aligned}$$

Where we assumed the positive y-axis upwards in the direction of the photon after the collision.

From the conservation of energy and momentum, we have that

$$\begin{aligned} E + M &= E' + \gamma M && \text{(energy conservation)} \\ E &= p\cos\theta && \text{(x-axis momentum conservation)} \\ 0 &= E' - p\sin\theta && \text{(y-axis momentum conservation)} \end{aligned}$$

From the x-axis and y-axis momentum conservation, we have that $E = p\cos\theta$ and $E' = p\sin\theta$ so the total momentum is given by $p^2 = (p\cos\theta)^2 + (p\sin\theta)^2 = E^2 + E'^2$.

Also, we know that $(\gamma M)^2 = p^2 + M^2$ then by replacing p^2 we get that $(\gamma M)^2 = E^2 + E'^2 + M^2$. Finally, by replacing $\gamma M = E + M - E'$ that we got from the energy conervation equation we get that

$$\begin{aligned} (E + M - E')^2 &= E^2 + E'^2 + M^2 \\ (E+M)^2 - 2(E+M)E' + E'^2 &= E^2 + E'^2 + M^2 \\ E' &= -\frac{E^2 + M^2 - (E+M)^2}{2(E+M)} \\ E' &= \frac{-E^2 - M^2 + E^2 + 2EM + M^2}{2(E+M)} \\ E' &= \frac{EM}{E+M} \end{aligned}$$

□

Solution. **3.40** Let us analyze the moment right before the collision then the acceleration of mass at this moment is $a = F/(\gamma^3 m)$ and we know that $a = vdv/dx$ hence the velocity at this point is

$$\begin{aligned}
v\frac{dv}{dx} &= \frac{F}{\gamma^3 m} \\
\int_0^v \gamma^3 v dv &= \frac{F}{m}\int_x^0 dx \\
\frac{1}{\sqrt{1-v^2}} - 1 &= -\frac{F}{m}x \\
\frac{1}{\sqrt{1-v^2}} &= \frac{m - Fx}{m} \\
1 - v^2 &= \left(\frac{m}{m - Fx}\right)^2 \\
v &= \sqrt{1 - \left(\frac{m}{m - Fx}\right)^2}
\end{aligned}$$

Now let M be the joint mass after the collision. So from the conservation of momentum we have that

$$p_M = p_m + p_m = \gamma m v + 0 = \gamma m v$$

And from the conservation of energy we get that

$$E_M = E_m + E_m = \gamma m + m = m(\gamma + 1)$$

But also we know that $M^2 = E_M^2 - p_M^2$ hence

$$\begin{aligned}
M^2 &= m^2(\gamma + 1)^2 - \gamma^2 m^2 v^2 \\
&= m^2(\gamma^2 + 2\gamma + 1 - \gamma^2 v^2) \\
&= m^2(\gamma^2(1 - v^2) + 2\gamma + 1) \\
&= 2m^2(1 + \gamma)
\end{aligned}$$

Now by replacing the value we have for v we get that

$$\begin{aligned}
M^2 &= 2m^2\left(1 + \frac{1}{\sqrt{1 - (1 - (\frac{m}{m-Fx})^2)}}\right) \\
M^2 &= 2m^2\left(1 + \frac{1}{\frac{m}{m-Fx}}\right) \\
M^2 &= 2m^2\left(\frac{m + m - Fx}{m}\right) \\
M^2 &= 2m(2m - Fx) \\
M &= \sqrt{2m(2m - Fx)}
\end{aligned}$$

□

Solution. **3.45** Let l be the maximum distance the string will extend and let us assume the tension of the string T is constant so we have that

$$\begin{aligned}
F\Delta x &= \Delta E \\
-T(l-0) &= m - \gamma m \\
l &= \frac{m(\gamma-1)}{T} = \frac{m}{4T}
\end{aligned}$$

Where we used that $\gamma_{3c/5} = 5/4$. Now we are interested in the distance x at which the masses will meet again. At this point, the rear mass will have an energy of $m + Tx$, and the front mass will have an energy of $m + T(l - x)$. Also we know that $p = \sqrt{E^2 - m^2}$ so the momentum at the meeting point for each mass will be

$$p_{m_r} = \sqrt{(m+Tx)^2 - m^2} \quad \text{and} \quad p_{m_f} = \sqrt{(5m/4 - Tx)^2 - m^2}$$

But $F = dp/dt$ tells us that these magnitudes must be equal, because the same force T (in magnitude, but opposite in direction) acts on the two masses for the same time hence

$$\begin{aligned}
\sqrt{(m+Tx)^2 - m^2} &= \sqrt{\left(\frac{5m}{4} - Tx\right)^2 - m^2} \\
m^2 + 2mTx + (Tx)^2 - m^2 &= \left(\frac{5m}{4}\right)^2 - 2Tx\left(\frac{5m}{4}\right) + (Tx)^2 - m^2 \\
2mTx &= \left(\frac{5m}{4}\right)^2 - \frac{5mTx}{2} - m^2 \\
2mTx + \frac{5mTx}{2} &= m^2\left(\frac{25}{16} - 1\right) \\
Tx\left(\frac{9}{2}\right) &= m\left(\frac{9}{16}\right) \\
x &= \frac{m}{8T}
\end{aligned}$$

Therefore the masses will meet at a distance $x = m/8T$ from the starting point. □