Solved selected problems of Special Relativity - Morin

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Chapter 4 - 4-vectors

Solution. **4.1** The 4-vectors of B and C seen from A's frame are $(\gamma_u, \gamma_u u)$ and $(\gamma_v, -\gamma_v v)$ respectively. Let w be the velocity of B seen from C then the 4-vector of B seen from C is $(\gamma_w, \gamma_w w)$ and the 4-vector of C in its own frame is $(1, 0)$. By using the inner-product invariance, we have that

$$\begin{aligned}(\gamma_w, \gamma_w w) \cdot (1, 0) = (\gamma_u, \gamma_u u) \cdot (\gamma_v, -\gamma_v v)\\ \gamma_w = \gamma_u \gamma_v (1 + uv)\end{aligned}$$

By replacing the value of γ_u, γ_v and γ_w, we get that

$$\begin{aligned}\frac{1}{\sqrt{1-w^2}} &= \frac{(1+uv)}{\sqrt{1-u^2}\sqrt{1-v^2}}\\ \sqrt{1-w^2} &= \frac{\sqrt{1-u^2}\sqrt{1-v^2}}{1+uv}\\ 1-w^2 &= \frac{(1-u^2)(1-v^2)}{(1+uv)^2}\\ 1-w^2 &= \frac{1+2uv+(uv)^2-v^2-u^2-2uv}{(1+uv)^2}\\ 1-w^2 &= \frac{(1+uv)^2-(u+v)^2}{(1+uv)^2}\\ w^2 &= \frac{(u+v)^2}{(1+uv)^2}\\ w &= \frac{u+v}{1+uv}\end{aligned}$$

Which is the velocity addition equation we were looking for. $\square$

Solution. **4.2** The 4-vectors of the particles seen from the lab frame are $(\gamma_v, \gamma_v v\cos\theta, \gamma_v v\sin\theta)$ and $(\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta)$ respectively.

Let w be the velocity of one particle seen from the other particle then the 4-vector is $(\gamma_w, \gamma_w w)$ and the 4-vector of the particle in its frame is $(1, 0)$. By using the inner-product invariance, we have that

$$\begin{aligned}
(\gamma_w, \gamma_w w) \cdot (1, 0) &= (\gamma_v, \gamma_v v\cos\theta, \gamma_v v\sin\theta)(\gamma_v, \gamma_v v\cos\theta, -\gamma_v v\sin\theta) \\
\gamma_w &= \gamma_v^2 - (\gamma_v v\cos\theta)^2 + (\gamma_v v\sin\theta)^2 \\
\gamma_w &= \gamma_v^2(1 - v^2(\cos^2\theta - \sin^2\theta)) \\
\gamma_w &= \gamma_v^2(1 - v^2\cos(2\theta))
\end{aligned}$$

Where we used that $\cos^2\theta - \sin^2\theta = \cos 2\theta$. By replacing the value of γ_w and γ_v, we get that

$$\begin{aligned}
\frac{1}{\sqrt{1-w^2}} &= \frac{1 - v^2\cos(2\theta)}{1 - v^2} \\
1 - w^2 &= \frac{(1-v^2)^2}{(1 - v^2\cos(2\theta))^2} \\
w^2 &= 1 - \frac{(1-v^2)^2}{(1 - v^2\cos(2\theta))^2} \\
w &= \sqrt{1 - \frac{(1-v^2)^2}{(1 - v^2\cos(2\theta))^2}}
\end{aligned}$$

Which is the same velocity we got in problem 2.3. $\square$

Solution. **4.4**

(a) Given that $v(t') = \tanh(at')$ where t' is the proper time on the spaceship and the spaceship is moving in the x direction the velocity 4-vector must be

$$\begin{aligned}
V &= (\gamma,\ \gamma\tanh(at'),\ 0,\ 0)\\
V &= \left(\frac{1}{\sqrt{1-\tanh^2(at')}},\ \frac{\tanh(at')}{\sqrt{1-\tanh^2(at')}},\ 0,\ 0\right)\\
V &= (\cosh(at'),\ \sinh(at'),\ 0,\ 0)
\end{aligned}$$

Also, we know that $A = dV/dt'$ hence

$$A = (a\sinh(at'),\ a\cosh(at'),\ 0,\ 0)$$

(b) In the spaceship frame we know that $v' = 0$ hence

$$V' = (1,\ 0,\ 0,\ 0)$$

since $\sinh(0) = 0$ and $\cosh(0) = 1$ we have that

$$A' = (0,\ a,\ 0,\ 0)$$

(c) We will verify first that V' transforms into V by the Lorentz transformation

$$\begin{aligned}
V_0 &= \gamma(V_0' + vV_1') = \gamma\\
V_1 &= \gamma(V_1' + vV_0') = \gamma v = \gamma\tanh(at')\\
V_2 &= V_2' = 0\\
V_3 &= V_3' = 0
\end{aligned}$$

Now we will check that V transforms into V' taking into account that S is moving backward with respect to S' hence

$$\begin{aligned}
V_0' &= \gamma(V_0 - vV_1) = \gamma(\gamma - \gamma v^2) = 1\\
V_1' &= \gamma(V_1 - vV_0) = \gamma(\gamma v - v\gamma) = 0\\
V_2' &= V_2 = 0\\
V_3' &= V_3 = 0
\end{aligned}$$

We will verify now that A' transforms into A by the Lorentz transformation

$$\begin{aligned}
A_0 &= \gamma(A_0' + vA_1') = \gamma v a = a\frac{\tanh(at')}{\sqrt{1-\tanh^2(at')}} = a\sinh(at')\\
A_1 &= \gamma(A_1' + vA_0') = \gamma a = \frac{a}{\sqrt{1-\tanh^2(at')}} = a\cosh(at')\\
A_2 &= A_2' = 0\\
A_3 &= A_3' = 0
\end{aligned}$$

Finally, we will check that A transforms into A' taking into account that S is moving backward with respect to S' hence

$$A'_0 = \gamma(A_0 - vA_1) = \gamma a(\sinh(at') - \tanh(at')\cosh(at')) = 0$$
$$A'_1 = \gamma(A_1 - vA_0) = \gamma a(\cosh(at') - v\sinh(at')) = \frac{\gamma a}{\cosh(at')} = a$$
$$A'_2 = A_2 = 0$$
$$A'_3 = A_3 = 0$$

□

Solution. **4.6** Let $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ then the derivative is

$$\frac{dv}{dt} = \frac{2v_x\dot{v_x} + 2v_z\dot{v_z} + 2v_z\dot{v_z}}{2\sqrt{v_x^2 + v_y^2 + v_z^2}}$$
$$\frac{dv}{dt} = \frac{v_x\dot{v_x} + v_z\dot{v_z} + v_z\dot{v_z}}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

Since at the moment in question, we have that $v_y = v_z = 0$ then

$$\frac{dv}{dt} = \frac{v_x\dot{v_x}}{\sqrt{v_x^2}} = \dot{v_x} = a_x$$

□

Solution. **4.7** From the equation we got in Problem 4.2 we have that one particle sees the other with a velocity w given by

$$w = \sqrt{1 - \frac{(1-v^2)^2}{(1-v^2\cos(2\theta))^2}}$$

We want that $w = v$ hence the angle θ between them must be

$$\begin{aligned}
v &= \sqrt{1 - \frac{(1-v^2)^2}{(1-v^2\cos(2\theta))^2}} \\
1 - v^2 &= \frac{(1-v^2)^2}{(1-v^2\cos(2\theta))^2} \\
(1-v^2\cos(2\theta))^2 &= 1 - v^2 \\
\cos(2\theta) &= \frac{1-\sqrt{1-v^2}}{v^2} \\
\theta &= \frac{1}{2}\arccos\left(\frac{1-\sqrt{1-v^2}}{v^2}\right)
\end{aligned}$$

Finally, if $v \to 0$ we have that

$$\lim_{v\to 0} \frac{1-\sqrt{1-v^2}}{v^2} = \frac{1}{2}$$

Which implies that

$$\theta = \frac{1}{2}\arccos\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

And if $v \approx 1$ (i.e. $v \approx c$) we have that

$$\theta = \frac{1}{2}\arccos(1) = 0$$

□