

Solved selected problems of Symmetry in Mechanics by Stephanie Singer

Franco Zacco

Chapter 1 - The Two-Body Problem

Solution. **Exercise 6**

1. Given $\mathbf{f}, \mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^3$ we know that the cross product between $\mathbf{f}(t)$ and $\mathbf{g}(t)$ give us

$$(\mathbf{f}(t) \times \mathbf{g}(t))_1 = (f_2g_3 - f_3g_2)$$

$$(\mathbf{f}(t) \times \mathbf{g}(t))_2 = (f_3g_1 - f_1g_3)$$

$$(\mathbf{f}(t) \times \mathbf{g}(t))_3 = (f_1g_2 - f_2g_1)$$

Also we know that

$$\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t)) = \left(\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_1, \frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_2, \frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_3 \right)$$

Then applying the product rule we get that

$$\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_1 = (f'_2g_3 + f_2g'_3 - f'_3g_2 - f_3g'_2)$$

$$\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_2 = (f'_3g_1 + f_3g'_1 - f'_1g_3 - f_1g'_3)$$

$$\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_3 = (f'_1g_2 + f_1g'_2 - f'_2g_1 - f_2g'_1)$$

On the other hand, we know that

$$(\mathbf{f}'(t) \times \mathbf{g}(t))_1 = (f'_2g_3 - f'_3g_2)$$

$$(\mathbf{f}'(t) \times \mathbf{g}(t))_2 = (f'_3g_1 - f'_1g_3)$$

$$(\mathbf{f}'(t) \times \mathbf{g}(t))_3 = (f'_1g_2 - f'_2g_1)$$

And that

$$(\mathbf{f}(t) \times \mathbf{g}'(t))_1 = (f_2g'_3 - f_3g'_2)$$

$$(\mathbf{f}(t) \times \mathbf{g}'(t))_2 = (f_3g'_1 - f_1g'_3)$$

$$(\mathbf{f}(t) \times \mathbf{g}'(t))_3 = (f_1g'_2 - f_2g'_1)$$

So

$$\begin{aligned}(\mathbf{f}'(t) \times \mathbf{g}(t))_1 + (\mathbf{f}(t) \times \mathbf{g}'(t))_1 &= (f'_2 g_3 - f'_3 g_2) + (f_2 g'_3 - f_3 g'_2) \\(\mathbf{f}'(t) \times \mathbf{g}(t))_2 + (\mathbf{f}(t) \times \mathbf{g}'(t))_2 &= (f'_3 g_1 - f'_1 g_3) + (f_3 g'_1 - f_1 g'_3) \\(\mathbf{f}'(t) \times \mathbf{g}(t))_3 + (\mathbf{f}(t) \times \mathbf{g}'(t))_3 &= (f'_1 g_2 - f'_2 g_1) + (f_1 g'_2 - f_2 g'_1)\end{aligned}$$

Hence, we see that

$$\begin{aligned}\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_1 &= (\mathbf{f}'(t) \times \mathbf{g}(t))_1 + (\mathbf{f}(t) \times \mathbf{g}'(t))_1 \\ \frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_2 &= (\mathbf{f}'(t) \times \mathbf{g}(t))_2 + (\mathbf{f}(t) \times \mathbf{g}'(t))_2 \\ \frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t))_3 &= (\mathbf{f}'(t) \times \mathbf{g}(t))_3 + (\mathbf{f}(t) \times \mathbf{g}'(t))_3\end{aligned}$$

This implies that

$$\frac{d}{dt}(\mathbf{f}(t) \times \mathbf{g}(t)) = \mathbf{f}'(t) \times \mathbf{g}(t) + (\mathbf{f}(t) \times \mathbf{g}'(t))$$

2. The dot product between $\mathbf{f}(t)$ and $\mathbf{g}(t)$ gives us

$$\mathbf{f}(t) \cdot \mathbf{g}(t) = f_1 g_1 + f_2 g_2 + f_3 g_3$$

Hence the derivative with respect to t is

$$\frac{d}{dt}(\mathbf{f}(t) \cdot \mathbf{g}(t)) = f'_1 g_1 + f_1 g'_1 + f'_2 g_2 + f_2 g'_2 + f'_3 g_3 + f_3 g'_3$$

But also we see that

$$\mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t) = (f'_1 g_1 + f'_2 g_2 + f'_3 g_3) + (f_1 g'_1 + f_2 g'_2 + f_3 g'_3)$$

Therefore

$$\frac{d}{dt}(\mathbf{f}(t) \cdot \mathbf{g}(t)) = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$$

3. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and suppose we orientate the axes of \mathbb{R}^3 such that \mathbf{v} and \mathbf{w} lay on the xy plane on this system of coordinates. We also rotate the xy axes such that the x axis match the direction of \mathbf{v} then vectors have the following coordinates

$$\mathbf{v} = \begin{pmatrix} v_x \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} w_x \\ w_y \\ 0 \end{pmatrix}$$

So we see that the area A of the parallelogram spanned by \mathbf{v} and \mathbf{w} is $A = v_x w_y$.

On the other hand, computing the cross product between \mathbf{v} and \mathbf{w} gives us

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= (v_y w_z - v_z w_y)\mathbf{i} + (v_z w_x - v_x w_z)\mathbf{j} + (v_x w_y - v_y w_x)\mathbf{k} \\ &= (v_x w_y)\mathbf{k}\end{aligned}$$

Where \mathbf{i} , \mathbf{j} and \mathbf{k} are unit vectors in the direction of x , y and z respectively.

Therefore $|\mathbf{v} \times \mathbf{w}| = \sqrt{(v_x w_y)^2} = v_x w_y$ which is the area the parallelogram spanned by \mathbf{v} and \mathbf{w} .

□

Solution. **Exercise 7** We know that

$$\begin{aligned} r_x &= \rho \cos \theta \\ r_y &= \rho \sin \theta \end{aligned}$$

And that

$$\begin{aligned} \frac{p_x}{\mu} &= \dot{\rho} \cos \theta - \rho \dot{\theta} \sin \theta \\ \frac{p_y}{\mu} &= \dot{\rho} \sin \theta + \rho \dot{\theta} \cos \theta \end{aligned}$$

Also, the equation of motion states that

$$\frac{d\mathbf{p}}{dt} = -\frac{GM\mu}{|\mathbf{r}|^3} \mathbf{r}$$

Let us compute $\frac{1}{\mu} \frac{d\mathbf{p}}{dt}$ as follows

$$\begin{aligned} \frac{1}{\mu} \frac{dp_x}{dt} &= \frac{d\dot{\rho}}{dt} \cos \theta - 2\dot{\rho}\dot{\theta} \sin \theta - \rho \left(\frac{d\dot{\theta}}{dt} \sin \theta + \dot{\theta}^2 \cos \theta \right) \\ \frac{1}{\mu} \frac{dp_y}{dt} &= \frac{d\dot{\rho}}{dt} \sin \theta + 2\dot{\rho}\dot{\theta} \cos \theta + \rho \left(\frac{d\dot{\theta}}{dt} \cos \theta - \dot{\theta}^2 \sin \theta \right) \end{aligned}$$

On the other hand, we see that $|\mathbf{r}| = \sqrt{\rho^2(\cos^2 \theta + \sin^2 \theta)} = \rho$ hence

$$\begin{aligned} -\frac{GM}{\rho^3} r_x &= -\frac{GM}{\rho^2} \cos \theta \\ -\frac{GM}{\rho^3} r_y &= -\frac{GM}{\rho^2} \sin \theta \end{aligned}$$

Then the equations of motion become

$$\begin{aligned} \frac{d\dot{\rho}}{dt} \cos \theta - 2\dot{\rho}\dot{\theta} \sin \theta - \rho \left(\frac{d\dot{\theta}}{dt} \sin \theta + \dot{\theta}^2 \cos \theta \right) &= -\frac{GM}{\rho^2} \cos \theta \\ \frac{d\dot{\rho}}{dt} \sin \theta + 2\dot{\rho}\dot{\theta} \cos \theta + \rho \left(\frac{d\dot{\theta}}{dt} \cos \theta - \dot{\theta}^2 \sin \theta \right) &= -\frac{GM}{\rho^2} \sin \theta \end{aligned}$$

Multiplying the first equation by $\cos \theta$, the second one by $\sin \theta$ and then summing them we get that

$$\begin{aligned} \frac{d\dot{\rho}}{dt} (\cos^2 \theta + \sin^2 \theta) - \rho \dot{\theta}^2 (\cos^2 \theta + \sin^2 \theta) &= -\frac{GM}{\rho^2} (\cos^2 \theta + \sin^2 \theta) \\ \frac{d\dot{\rho}}{dt} - \rho \dot{\theta}^2 + \frac{GM}{\rho^2} &= 0 \end{aligned}$$

Which is one of the equations we are looking for.

Now, if we multiply the first equation by $\sin \theta$, the second one by $\cos \theta$ and then we subtract the second from the first one we get that

$$\begin{aligned} -2\dot{\rho}\dot{\theta}(\sin^2 \theta + \cos^2 \theta) - \rho \frac{d\dot{\theta}}{dt} (\sin^2 \theta + \cos^2 \theta) &= 0 \\ 2\dot{\rho}\dot{\theta} + \rho \frac{d\dot{\theta}}{dt} &= 0 \end{aligned}$$

Which is another equation that we are looking for.

Finally by definition we know that

$$\frac{d\rho}{dt} = \dot{\rho} \qquad \frac{d\theta}{dt} = \dot{\theta}$$

Which completes the last two equations we are looking for. □

Solution. Exercise 8 We know that $q = 1/\rho$ hence $\rho = 1/q$ so derivating we get that

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{1}{q^2} \frac{dq}{dt} \\ \frac{d\rho}{dt} &= -\frac{1}{q^2} \frac{dq}{d\theta} \frac{d\theta}{dt}\end{aligned}$$

And substituting the equation for $d\theta/dt$ we have from the hint we get that

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{1}{q^2} \frac{dq}{d\theta} \left(q^2 \frac{|\tilde{\mathbf{L}}|}{\mu} \right) \\ &= -\frac{|\tilde{\mathbf{L}}|}{\mu} \frac{dq}{d\theta}\end{aligned}$$

Now we derivate with respect to t again

$$\begin{aligned}\frac{d^2\rho}{dt^2} &= -\frac{|\tilde{\mathbf{L}}|}{\mu} \frac{d}{dt} \left(\frac{dq}{d\theta} \right) \\ &= -\frac{|\tilde{\mathbf{L}}|}{\mu} \frac{d^2q}{d\theta^2} \frac{d\theta}{dt}\end{aligned}$$

Finally, we replace again the value we have for $d\theta/dt$

$$\frac{d^2\rho}{dt^2} = -\frac{|\tilde{\mathbf{L}}|^2}{\mu^2} q^2 \frac{d^2q}{d\theta^2}$$

□

Solution. **Exercise 9** No, Mars will not crash into the sun because for this to happen the ellipse path that Mars is following needs to transform into an hyperbola and we saw that angular momentum is conserved hence, a planet cannot slow down too much, ruling out bounded motion along a parabola or hyperbola. \square

Solution. Exercise 10 We know that the polar coordinate ρ from the sun to the comet Hale-Bopp is

$$\frac{1}{\rho} = \frac{GM\mu^2}{|\tilde{\mathbf{L}}|^2} + A \cos(\theta + \theta_0)$$

Also, we know the angular momentum is given by

$$|\tilde{\mathbf{L}}| = \mu \rho^2 \dot{\theta} = \mu \rho v$$

Where v is the tangential velocity and since the angular momentum is conserved, for the perihelion we have that

$$\frac{|\tilde{\mathbf{L}}|}{\mu} = \rho_p v_p = (1.36732 \times 10^8)(44) = 6016208000 \frac{km^2}{s}$$

Considering we are at the perihelion we can take $\theta + \theta_0 = 2\pi$ and hence A is

$$\begin{aligned} \frac{1}{\rho_p} &= \frac{GM}{\rho_p^2 v_p^2} + A \\ A &= \frac{1}{\rho_p} - \frac{GM}{\rho_p^2 v_p^2} \end{aligned}$$

Then given that $GM = 132712440044 \text{ km}^3/s^2$ we have that

$$\begin{aligned} A &= \frac{1}{1.36732 \times 10^8} - \frac{132712440044}{(6016208000)^2} \\ A &= 3.64695 \times 10^{-9} \text{ km}^{-1} \end{aligned}$$

Now, let us suppose we are at the aphelion then must be that $\cos \theta + \theta_0 = -1$ hence

$$\begin{aligned} \frac{1}{\rho_a} &= \frac{GM\mu^2}{|\tilde{\mathbf{L}}|^2} - A \\ &= \frac{132712440044}{(6016208000)^2} - 3.64695 \times 10^{-9} \\ &= 1.96638 \times 10^{-11} \end{aligned}$$

Then $\rho_a = 50854756626$ and the semi-major axis is given by

$$a = \frac{\rho_a + \rho_p}{2} = 25495744313 \text{ km} \approx 170AU$$

Finally, from Kepler's third law the period of the Hale-Bopp comet is

$$\begin{aligned} \tau^2 &= \frac{4\pi^2 a^3}{GM_s} \\ \tau^2 &= \frac{4\pi^2 (25495744313)^3}{32712440044} \\ \tau &= \sqrt{2.00009 \times 10^{22} \text{ s}^2} \\ \tau &= 141424594179 \text{ s} \approx 4484 \text{ years} \end{aligned}$$

Therefore Hale-Bopp will return in 4484 years. □

Solution. **Exercise 11** We know that

$$\frac{1}{(b^2/a)} = \frac{GM\mu^2}{|\tilde{\mathbf{L}}|}$$

But since we are considering a circular orbit then $b = a$ and $|\tilde{\mathbf{L}}| = \mu\rho v = \mu av$ where v is the velocity of the satellite then

$$\begin{aligned}\frac{1}{a} &= \frac{GM\mu^2}{\mu^2 a^2 v^2} \\ v^2 &= \frac{GM}{a} \\ v &= \sqrt{\frac{GM}{a}}\end{aligned}$$

Now, to put the satellite into a circular orbit that passes over the north and south poles the velocity direction needs to change $\pi/2$ radians. Then the change in velocity Δv is the hypotenuse of a right triangle where each side is v then

$$\Delta v = \sqrt{v^2 + v^2} = \sqrt{2}v$$

□

Solution. Exercise 12 Let $\mathbf{v} \in \mathbb{R}^2$ and let us set our coordinate system such that \mathbf{v} is in the r direction then we can write \mathbf{v} as

$$\mathbf{v} = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}$$

Then we want to find \mathbf{w} where

$$\mathbf{w} = \begin{pmatrix} r_2 \\ p_2 \end{pmatrix}$$

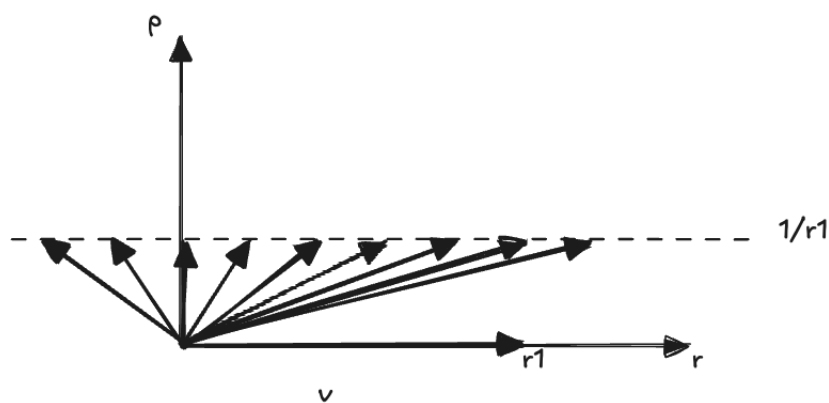
Such that

$$r_1 p_2 - r_2 p_1 = r_1 p_2 = 1$$

This implies that $p_2 = 1/r_1$ then \mathbf{w} is of the form

$$\mathbf{w} = \begin{pmatrix} r_2 \\ 1/r_1 \end{pmatrix}$$

Below we show a set of the possible \mathbf{w} 's



□

Solution. **Exercise 13** Let

$$x_1 = \begin{pmatrix} r_1 \\ p_1 \end{pmatrix} \quad x_2 = \begin{pmatrix} r_2 \\ p_2 \end{pmatrix}$$

Also, let B be a 2×2 matrix

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

We know that

$$A(x_1, x_2) = \det \begin{pmatrix} r_1 & r_2 \\ p_1 & p_2 \end{pmatrix}$$

So, we want to know the necessary and sufficient condition such that

$$\det(Bx_1, Bx_2) = \det(x_1, x_2)$$

Hence, given that

$$Bx_1 = \begin{pmatrix} b_1r_1 + b_2p_1 \\ b_3r_1 + b_4p_1 \end{pmatrix} \quad Bx_2 = \begin{pmatrix} b_1r_2 + b_2p_2 \\ b_3r_2 + b_4p_2 \end{pmatrix}$$

We want that

$$\begin{aligned} (b_1r_1 + b_2p_1)(b_3r_2 + b_4p_2) - (b_3r_1 + b_4p_1)(b_1r_2 + b_2p_2) &= r_1p_2 - r_2p_1 \\ (b_1r_1 + b_2p_1)b_3r_2 + (b_1r_1 + b_2p_1)b_4p_2 - & \\ -(b_3r_1 + b_4p_1)b_1r_2 - (b_3r_1 + b_4p_1)b_2p_2 &= r_1p_2 - r_2p_1 \\ b_1r_1b_3r_2 + b_2p_1b_3r_2 + b_1r_1b_4p_2 + b_2p_1b_4p_2 - & \\ -b_3r_1b_1r_2 - b_4p_1b_1r_2 - b_3r_1b_2p_2 - b_4p_1b_2p_2 &= r_1p_2 - r_2p_1 \\ b_2p_1b_3r_2 + b_1r_1b_4p_2 - b_4p_1b_1r_2 - b_3r_1b_2p_2 &= r_1p_2 - r_2p_1 \\ r_1p_2(b_1b_4 - b_3b_2) - r_2p_1(b_1b_4 - b_3b_2) &= r_1p_2 - r_2p_1 \end{aligned}$$

Then for this equality to be true we need that $b_1b_4 - b_3b_2 = 1$ i.e.

$$\det(B) = 1$$

□