Solved selected problems of Symmetry in Mechanics by Stephanie Singer

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Chapter 2 - Phase Spaces of Mechanical Systems are Symplectic Manifolds

Solution. **Exercise 12** Let $\boldsymbol{v} \in \mathbb{R}^2$ and let us set our coordinate system such that $\boldsymbol{v}$ is in the r direction then we can write $\boldsymbol{v}$ as

$$\boldsymbol{v} = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}$$

Then we want to find $\boldsymbol{w}$ where

$$\boldsymbol{w} = \begin{pmatrix} r_2 \\ p_2 \end{pmatrix}$$

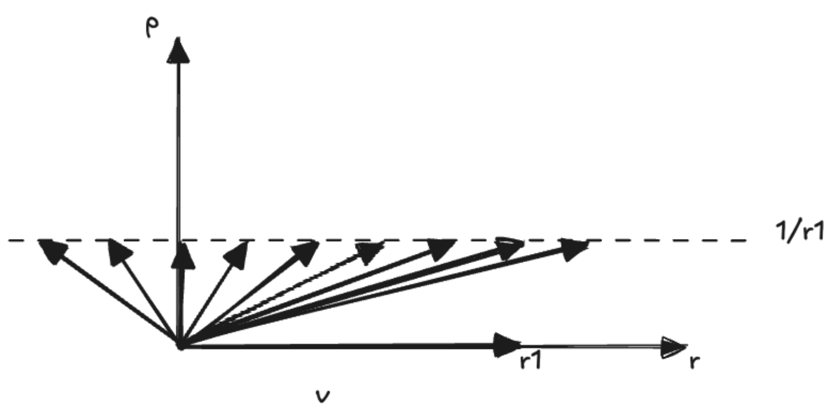
Such that

$$r_1 p_2 - r_2 p_1 = r_1 p_2 = 1$$

This implies that $p_2 = 1/r_1$ then $\boldsymbol{w}$ is of the form

$$\boldsymbol{w} = \begin{pmatrix} r_2 \\ 1/r_1 \end{pmatrix}$$

Below we show a set of the possible $\boldsymbol{w}$'s



□

Solution. **Exercise 13** Let

$$x_1 = \begin{pmatrix} r_1 \\ p_1 \end{pmatrix} \quad x_2 = \begin{pmatrix} r_2 \\ p_2 \end{pmatrix}$$

Also, let B be a 2×2 matrix

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

We know that

$$A(x_1, x_2) = \det \begin{pmatrix} r_1 & r_2 \\ p_1 & p_2 \end{pmatrix}$$

So, we want to know the necessary and sufficient condition such that

$$\det(Bx_1, Bx_2) = \det(x_1, x_2)$$

Hence, given that

$$Bx_1 = \begin{pmatrix} b1r_1 + b_2p_1 \\ b_3r_1 + b_4p_1 \end{pmatrix} \quad Bx_2 = \begin{pmatrix} b1r_2 + b_2p_2 \\ b_3r_2 + b_4p_2 \end{pmatrix}$$

We want that

$$\begin{aligned}
(b_1r_1 + b_2p_1)(b_3r_2 + b_4p_2) - (b_3r_1 + b_4p_1)(b_1r_2 + b_2p_2) &= r_1p_2 - r_2p_1 \\
(b_1r_1 + b_2p_1)b_3r_2 + (b_1r_1 + b_2p_1)b_4p_2 - & \\
-(b_3r_1 + b_4p_1)b_1r_2 - (b_3r_1 + b_4p_1)b_2p_2 &= r_1p_2 - r_2p_1 \\
b_1r_1b_3r_2 + b_2p_1b_3r_2 + b_1r_1b_4p_2 + b_2p_1b_4p_2 - & \\
-b_3r_1b_1r_2 - b_4p_1b_1r_2 - b_3r_1b_2p_2 - b_4p_1b_2p_2 &= r_1p_2 - r_2p_1 \\
b_2p_1b_3r_2 + b_1r_1b_4p_2 - b_4p_1b_1r_2 - b_3r_1b_2p_2 &= r_1p_2 - r_2p_1 \\
r_1p_2(b_1b_4 - b_3b_2) - r_2p_1(b_1b_4 - b_3b_2) &= r_1p_2 - r_2p_1
\end{aligned}$$

Then for this equality to be true we need that $b_1b_4 - b_3b_2 = 1$ i.e.

$$\det(B) = 1$$

□

Solution. **Exercise 14**

1. We see that

$$\boldsymbol{v} \times \boldsymbol{w} = (v_2w_3 - v_3w_2)\boldsymbol{i} + (v_3w_1 - v_1w_3)\boldsymbol{j} + (v_1w_2 - v_2w_1)\boldsymbol{k}$$

Hence

$$\begin{aligned}\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)\\ &= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1\\ &= (u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2) - (u_1v_3w_2 + u_2v_1w_3 + u_3v_2w_1)\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}\det\begin{pmatrix} u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\end{pmatrix} &= (u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2)\\ &\quad - (u_3v_2w_1 + u_2v_1w_3 + u_1v_3w_2)\end{aligned}$$

Therefore

$$\det\begin{pmatrix} u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\end{pmatrix} = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$$

2. We know that $\boldsymbol{v} \times \boldsymbol{w}$ is a vector perpendicular to both vectors with a magnitude given by the area spanned by these vectors. So we can write that

$$\boldsymbol{v} \times \boldsymbol{w} = A\boldsymbol{n}$$

Where A is the area spanned by these vectors.

On the other hand, the dot product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is the projection of $\boldsymbol{u}$ over $\boldsymbol{v} \times \boldsymbol{w}$ times the magnitude of the vector $\boldsymbol{v} \times \boldsymbol{w}$ i.e. the height of the parallelepiped times the area spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$.

Therefore $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ represents the signed volume of the parallelepiped.

3. Given that $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is the signed volume of the parallelepiped formed by these vectors then the products $\boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u})$ and $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$ must be the same volume since the vectors involved are the same.

Therefore must be that $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.

4. ($\Rightarrow$) Let M be a 3×3 matrix with determinant 1 and f be a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that f takes $\boldsymbol{v}$ to $M\boldsymbol{v}$.

   Let us take the edges of the unit cube in $\mathbb{R}^3$ as $\boldsymbol{e_1} = (1, 0, 0)$, $\boldsymbol{e_2} = (0, 1, 0)$ and $\boldsymbol{e_3} = (0, 0, 1)$ then f takes these unit vectors to $M\boldsymbol{e_1}, M\boldsymbol{e_2}$ and $M\boldsymbol{e_3}$ so the parallelepiped spanned by these vectors has a volume of $\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, M\boldsymbol{e_3})$ but we see that

$$\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, M\boldsymbol{e_3}) = \det(M) = 1$$

   Therefore the volume of the parallelepiped spanned by $M\boldsymbol{e_1}, M\boldsymbol{e_2}$ and $M\boldsymbol{e_3}$ is 1 and so the unit cube is sent to a parallelepiped of signed volume 1.

   ($\Leftarrow$) Let f be a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that f takes $\boldsymbol{v}$ to $M\boldsymbol{v}$ and f takes the unit cube in the domain to a parallelepiped of signed volume 1.

   Let us take the edges of the unit cube in $\mathbb{R}^3$ as $\boldsymbol{e_1} = (1, 0, 0)$, $\boldsymbol{e_2} = (0, 1, 0)$ and $\boldsymbol{e_3} = (0, 0, 1)$ then f takes these unit vectors to $M\boldsymbol{e_1}, M\boldsymbol{e_2}$ and $M\boldsymbol{e_3}$. The parallelepiped spanned by these vectors has a volume of 1 by definition i.e. $\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, M\boldsymbol{e_3}) = 1$ but we see that

$$\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, M\boldsymbol{e_3}) = \det(M) = 1$$

   Therefore M is a 3×3 matrix with determinant 1.

$\square$

Solution. **Exercise 15**

($\Rightarrow$) Let M be a $n \times n$ matrix with determinant 1 and f be a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that f takes $\boldsymbol{v}$ to $M\boldsymbol{v}$.

Let us take the edges of the unit cube in $\mathbb{R}^n$ as $\boldsymbol{e_1} = (1, 0, 0, ..., 0)$, $\boldsymbol{e_2} = (0, 1, 0, ..., 0)$, ..., $\boldsymbol{e_n} = (0, 0, 0, ..., 1)$ then f takes these unit vectors to $M\boldsymbol{e_1}, M\boldsymbol{e_2}$, ..., $M\boldsymbol{e_n}$ so the parallelepiped spanned by these vectors has a volume of $\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, ..., M\boldsymbol{e_n})$ but we see that

$$\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, ..., M\boldsymbol{e_n}) = \det(M) = 1$$

Therefore the volume of the parallelepiped spanned by $M\boldsymbol{e_1}, M\boldsymbol{e_2}$ ..., $M\boldsymbol{e_n}$ is 1 and so the unit cube is sent to a parallelepiped of signed volume 1.

($\Leftarrow$) Let f be a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that f takes $\boldsymbol{v}$ to $M\boldsymbol{v}$ and f takes the unit cube in the domain to a parallelepiped of signed volume 1.

Let us take the edges of the unit cube in $\mathbb{R}^n$ as $\boldsymbol{e_1} = (1, 0, 0, ..., 0)$, $\boldsymbol{e_2} = (0, 1, 0, ..., 0)$, ..., $\boldsymbol{e_n} = (0, 0, 0, ..., 1)$ then f takes these unit vectors to $M\boldsymbol{e_1}, M\boldsymbol{e_2}$, ..., $M\boldsymbol{e_n}$. The parallelepiped spanned by these vectors has a volume of 1 by definition i.e. $\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, ..., M\boldsymbol{e_n}) = 1$ but we see that

$$\det(M\boldsymbol{e_1}, M\boldsymbol{e_2}, ..., M\boldsymbol{e_n}) = \det(M) = 1$$

Therefore M is a $n \times n$ matrix with determinant 1. $\square$

Solution. **Exercise 16** Assume that

$$a_r \frac{\partial}{\partial r} + a_p \frac{\partial}{\partial p} = 0$$

Then for all f must be that

$$a_r \frac{\partial f}{\partial r} + a_p \frac{\partial f}{\partial p} = 0$$

Suppose we take $f(r, p) = r$ then we have that $a_r = 0$.
In the same way, if we take $f(r, p) = p$ we get that $a_p = 0$.

Hence since $a_r \frac{\partial f}{\partial r} + a_p \frac{\partial f}{\partial p} = 0$ needs to work for all f must be that $a_r = a_p = 0$ and therefore $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial p}$ are linearly independent. □

Solution. **Exercise 17**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} \int_{\boldsymbol{q}}^{\boldsymbol{q}+t\boldsymbol{v}} \alpha &= \frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} \int_{(r_0,p_0)}^{(r_0,p_0)+t(v_1,v_2)} \alpha_1 \ dr + \alpha_2 \ dp \\
&= \frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} [\alpha_1 r + \alpha_2 p]_{(r_0,p_0)}^{(r_0,p_0)+t(v_1,v_2)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} [(\alpha_1(r_0 + tv_1) + \alpha_2(p_0 + tv_2)) - (\alpha_1 r_0 + \alpha_2 p_0)] \\
&= \frac{\mathrm{d}}{\mathrm{d}t} |_{t=0} [\alpha_1 v_1 t + \alpha_2 v_2 t] \\
&= \alpha_1 v_1 + \alpha_2 v_2
\end{aligned}$$

□

Solution. **Exercise 18** Let F be an antisymmetric bilinear form then $F(\boldsymbol{v}, \boldsymbol{w})$ can be written as the following matrix operation

$$\begin{aligned} F(\boldsymbol{v}, \boldsymbol{w}) &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= w_1(v_1 f_{11} + v_2 f_{21}) + w_2(v_1 f_{12} + v_2 f_{22}) \end{aligned}$$

But also since F is an antisymmetric bilinear form we know that $F(\boldsymbol{v}, \boldsymbol{w}) = -F(\boldsymbol{w}, \boldsymbol{v})$ then must be that

$$\begin{aligned} w_1(v_1 f_{11} + v_2 f_{21}) + w_2(v_1 f_{12} + v_2 f_{22}) &= \\ -v_1(w_1 f_{11} + w_2 f_{21}) - v_2(w_1 f_{12} + w_2 f_{22}) \\ v_1 w_1 f_{11} + v_2 w_1 f_{21} + v_1 w_2 f_{12} + v_2 w_2 f_{22} &= \\ -v_1 w_1 f_{11} - v_1 w_2 f_{21} - v_2 w_1 f_{12} - v_2 w_2 f_{22} \\ 2 v_1 w_1 f_{11} + v_2 w_1 (f_{21} + f_{12}) + v_1 w_2 (f_{12} + f_{21}) + 2 v_2 w_2 f_{22} &= 0 \\ 2 v_1 w_1 f_{11} + (f_{12} + f_{21})(v_1 w_2 + v_2 w_1) + 2 v_2 w_2 f_{22} &= 0 \end{aligned}$$

But if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not null then this equation can only be true if

$$f_{11} = 0 \qquad f_{12} + f_{21} = 0 \qquad f_{22} = 0$$

Or

$$f_{11} = 0 \qquad f_{21} = -f_{12} \qquad f_{22} = 0$$

This implies that $F(\boldsymbol{v}, \boldsymbol{w})$ must be

$$F(\boldsymbol{v}, \boldsymbol{w}) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 0 & f_{12} \\ -f_{12} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Therefore F has a correspondence to a antisymmetric 2×2 matrix. □

Solution. **Exercise 19** Let α, β be covectors and $\boldsymbol{v}, \boldsymbol{w}$ be vectors then the wedge product of α and β is given by

$$(\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w}) = (\alpha \boldsymbol{v})(\beta \boldsymbol{w}) - (\alpha \boldsymbol{w})(\beta \boldsymbol{v})$$

On the other hand, we see that

$$\begin{aligned} -(\beta \wedge \alpha)(\boldsymbol{v}, \boldsymbol{w}) &= -((\beta \boldsymbol{v})(\alpha \boldsymbol{w}) - (\beta \boldsymbol{w})(\alpha \boldsymbol{v})) \\ &= (\beta \boldsymbol{w})(\alpha \boldsymbol{v}) - (\beta \boldsymbol{v})(\alpha \boldsymbol{w}) \\ &= (\alpha \boldsymbol{v})(\beta \boldsymbol{w}) - (\alpha \boldsymbol{w})(\beta \boldsymbol{v}) \end{aligned}$$

Where we used commutativity of scalars in the last step, so

$$(\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w}) = -(\beta \wedge \alpha)(\boldsymbol{v}, \boldsymbol{w})$$

Therefore the wedge product is itself antisymmetric. □

Solution. **Exercise 20** Let F be an antisymmetric bilinear form then $F(\boldsymbol{v}, \boldsymbol{w})$ can be written as

$$\begin{aligned} F(\boldsymbol{v}, \boldsymbol{w}) &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} 0 & f_{12} \\ -f_{12} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= f_{12} w_2 v_1 - f_{12} w_1 v_2 \end{aligned}$$

If we take two covectors

$$\alpha = (0 \quad f_{12}) \quad \text{and} \quad \beta = (-1 \quad 0)$$

Then the wedge product of this covectors become

$$\begin{aligned} (\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w}) &= (\alpha \boldsymbol{v})(\beta \boldsymbol{w}) - (\alpha \boldsymbol{w})(\beta \boldsymbol{v}) \\ &= (0 \quad f_{12}) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (-1 \quad 0) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \\ &\quad - (0 \quad f_{12}) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (-1 \quad 0) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (f_{12} v_2)(-w_1) - (f_{12} w_2)(-v_1) \\ &= f_{12} w_2 v_1 - f_{12} w_1 v_2 \end{aligned}$$

Therefore we see that $F(\boldsymbol{v}, \boldsymbol{w}) = (\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w})$ for α, β defined as mentioned. $\square$

Solution. **Exercise 21** Let F be an antisymmetric bilinear form on $\mathbb{R}^3$ then $F(\boldsymbol{v}, \boldsymbol{w})$ can be written as

$$
\begin{aligned}
F(\boldsymbol{v}, \boldsymbol{w}) &= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 0 & f_{12} & f_{13} \\ -f_{12} & 0 & f_{23} \\ -f_{13} & -f_{23} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\
&= w_1(-f_{12}v_2 - f_{13}v_3) + w_2(f_{12}v_1 - f_{23}v_3) + w_3(f_{13}v_1 + f_{23}v_2) \\
&= f_{12}(w_2v_1 - w_1v_2) + f_{13}(w_3v_1 - w_1v_3) + f_{23}(w_3v_2 - w_2v_3)
\end{aligned}
$$

If we take two covectors

$$
\begin{aligned}
\alpha &= (0 \quad f_{13} \quad f_{23}) \\
\beta &= (-1 \quad 0 \quad f_{23}/f_{12})
\end{aligned}
$$

Then the wedge product of this covectors become

$$
\begin{aligned}
(\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w}) &= (\alpha \boldsymbol{v})(\beta \boldsymbol{w}) - (\alpha \boldsymbol{w})(\beta \boldsymbol{v}) \\
&= (0 \quad f_{12} \quad f_{13}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} (-1 \quad 0 \quad f_{23}/f_{12}) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \\
&\quad - (0 \quad f_{12} \quad f_{13}) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} (-1 \quad 0 \quad f_{23}/f_{12}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
&= (f_{12}v_2 + f_{13}v_3)\left(-w_1 + \frac{f_{23}}{f_{12}}w_3\right) - (f_{12}w_2 + f_{13}w_3)\left(-v_1 + \frac{f_{23}}{f_{12}}v_3\right) \\
&= -f_{12}v_2w_1 - f_{13}v_3w_1 + f_{23}v_2w_3 + f_{13}\frac{f_{23}}{f_{12}}v_3w_3 \\
&\quad + f_{12}v_1w_2 + f_{13}v_1w_3 - f_{23}v_3w_2 - f_{13}\frac{f_{23}}{f_{12}}v_3w_3 \\
&= f_{12}(v_1w_2 - v_2w_1) + f_{13}(v_1w_3 - v_3w_1) + f_{23}(v_2w_3 - v_3w_2)
\end{aligned}
$$

Therefore we see that $F(\boldsymbol{v}, \boldsymbol{w}) = (\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w})$ for α, β defined as mentioned.

$\square$

Solution. **Exercise 22** Let us consider a 4×4 antisymmetric matrix

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

This matrix will be of rank 4 if the only solution to the following system of equations is $\alpha = \beta = \delta = \epsilon = 0$

$$\alpha(0, -a, -b, -c) + \beta(a, 0, -d, -e) + \delta(b, d, 0, -f) + \epsilon(c, e, f, 0) = (0, 0, 0, 0)$$

If we let $a = b = c = d = e = f = 1$ we get a matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

and hence the followeing system of equations

$$\begin{aligned} \beta + \delta + \epsilon &= 0 \\ -\alpha + \delta + \epsilon &= 0 \\ -\alpha - \beta + \epsilon &= 0 \\ -\alpha - \beta - \delta &= 0 \end{aligned}$$

Which has no solutions other than $\alpha = \beta = \delta = \epsilon = 0$. Therefore A is a rank 4 matrix and hence an invertible matrix which shows it cannot be written as a wedge product of two covectors.

Let now F be an antisymmetric bilinear form on $\mathbb{R}^4$ then $F(\boldsymbol{v}, \boldsymbol{w})$ can be written as

$$\begin{aligned} F(\boldsymbol{v}, \boldsymbol{w}) = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} & \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \\ &= w_1(-av_2 - bv_3 - cv_4) + w_2(av_1 - dv_3 - ev_4) + w_3(bv_1 + dv_2 - fv_4) \\ &\quad + w_4(cv_1 + ev_2 + fv_3) \\ &= a(w_2v_1 - w_1v_2) + b(w_3v_1 - w_1v_3) + c(w_4v_1 - w_1v_4) + d(w_3v_2 - w_2v_3) \\ &\quad + e(w_4v_2 - w_2v_4) + f(w_4v_3 - w_3v_4) \end{aligned}$$

If we take two covectors

$$\begin{aligned} \alpha &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \beta &= \begin{pmatrix} 0 & 1 \end{pmatrix} \end{aligned}$$

Then we see that

$$\begin{aligned} (\alpha \wedge \beta)\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= w_2v_1 - w_1v_2 \end{aligned}$$

Then we can write that

$$\begin{aligned} F(\boldsymbol{v}, \boldsymbol{w}) = a&\left[(\alpha \wedge \beta)\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right)\right] + b\left[(\alpha \wedge \beta)\left(\begin{pmatrix} v_1 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_3 \end{pmatrix}\right)\right] \\ &+ c\left[(\alpha \wedge \beta)\left(\begin{pmatrix} v_1 \\ v_4 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_4 \end{pmatrix}\right)\right] + d\left[(\alpha \wedge \beta)\left(\begin{pmatrix} v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_2 \\ w_3 \end{pmatrix}\right)\right] \\ &+ e\left[(\alpha \wedge \beta)\left(\begin{pmatrix} v_2 \\ v_4 \end{pmatrix}, \begin{pmatrix} w_2 \\ w_4 \end{pmatrix}\right)\right] + f\left[(\alpha \wedge \beta)\left(\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}, \begin{pmatrix} w_3 \\ w_4 \end{pmatrix}\right)\right] \end{aligned}$$

Therefore we can write every antisymmetric bilinear form on $\mathbb{R}^4$ as a finite linear combination of wedge products of covectors. $\square$

Solution. **Exercise 23** Let us define

$$
\begin{aligned}
dr_1 &= (1\ 0\ 0\ 0\ 0\ 0) \qquad & dp_1 &= (0\ 0\ 0\ 1\ 0\ 0)\\
dr_2 &= (0\ 1\ 0\ 0\ 0\ 0) \qquad & dp_2 &= (0\ 0\ 0\ 0\ 1\ 0)\\
dr_3 &= (0\ 0\ 1\ 0\ 0\ 0) \qquad & dp_3 &= (0\ 0\ 0\ 0\ 0\ 1)
\end{aligned}
$$

Also, let us take $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^6$ with the following components

$$
\begin{aligned}
\boldsymbol{v} &= (v_{rx}\ v_{ry}\ v_{rz}\ v_{px}\ v_{py}\ v_{pz})\\
\boldsymbol{w} &= (w_{rx}\ w_{ry}\ w_{rz}\ w_{px}\ w_{py}\ w_{pz})
\end{aligned}
$$

Then $(dr_1 \wedge dp_1)(\boldsymbol{v}, \boldsymbol{w})$ is given by

$$
\begin{aligned}
(dr_1 \wedge dp_1)(\boldsymbol{v}, \boldsymbol{w}) &= (dr_1\boldsymbol{v})(dp_1\boldsymbol{w}) - (dr_1\boldsymbol{w})(dp_1\boldsymbol{v})\\
&= v_{rx}w_{px} - w_{rx}v_{px}
\end{aligned}
$$

Hence in the same way we have that

$$
\begin{aligned}
(dr_2 \wedge dp_2)(\boldsymbol{v}, \boldsymbol{w}) &= v_{ry}w_{py} - w_{ry}v_{py}\\
(dr_3 \wedge dp_3)(\boldsymbol{v}, \boldsymbol{w}) &= v_{rz}w_{pz} - w_{rz}v_{pz}
\end{aligned}
$$

Then

$$
\begin{aligned}
&(dr_1 \wedge dp_1 + dr_2 \wedge dp_2 + dr_3 \wedge dp_3)(\boldsymbol{v}, \boldsymbol{w}) =\\
&\quad = v_{rx}w_{px} - w_{rx}v_{px} + v_{ry}w_{py} - w_{ry}v_{py} + v_{rz}w_{pz} - w_{rz}v_{pz}\\
&\quad = (-v_{px}\ \ -v_{py}\ \ -v_{pz}\ v_{rx}\ v_{ry}\ v_{rz}) \begin{pmatrix} w_{rx}\\ w_{ry}\\ w_{rz}\\ w_{px}\\ w_{py}\\ w_{pz} \end{pmatrix}
\end{aligned}
$$

Therefore the antisymmetric 6×6 matrix corresponding to the bilinear form mentioned above is

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
-1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
$$

□

Solution. **Exercise 24** Let F be an antisymmetric bilinear form on $\mathbb{R}^n$ then $F(\boldsymbol{v},\boldsymbol{w})$ can be written as the following matrix operation

$$\begin{aligned}F(\boldsymbol{v},\boldsymbol{w}) &= \begin{pmatrix} v_1 & v_2 & ... & v_n \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & ... & f_{1n} \\ f_{21} & f_{22} & ... & f_{2n} \\ \vdots & \vdots & ... & \vdots \\ f_{n1} & f_{n2} & ... & f_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \\ &= w_1(v_1 f_{11} + v_2 f_{21} + ... + v_n f_{n1}) + w_2(v_1 f_{12} + v_2 f_{22} + ... + v_n f_{n2}) + \\ &+ ... + w_n(v_1 f_{1n} + v_2 f_{2n} + ... + v_n f_{nn})\end{aligned}$$

But also since F is an antisymmetric bilinear form we know that $F(\boldsymbol{v},\boldsymbol{w}) = -F(\boldsymbol{w},\boldsymbol{v})$ then must be that

$$\begin{aligned}& w_1(v_1 f_{11} + v_2 f_{21} + ... + v_n f_{n1}) + w_2(v_1 f_{12} + v_2 f_{22} + ... + v_n f_{n2}) + ... \\ & + w_n(v_1 f_{1n} + v_2 f_{2n} + ... + v_n f_{nn}) = -v_1(w_1 f_{11} + w_2 f_{21} + ... + w_n f_{n1}) \\ & - v_2(w_1 f_{12} + w_2 f_{22} + ... + w_n f_{n2}) - ... - v_n(w_1 f_{1n} + w_2 f_{2n} + ... + w_n f_{nn})\end{aligned}$$

Hence

$$\begin{aligned}& 2w_1 v_1 f_{11} + w_1 v_2 (f_{21} + f_{12}) + ... + w_1 v_n (f_{n1} + f_{1n}) \\ & + w_2 v_1 (f_{12} + f_{21}) + 2 w_2 v_2 f_{22} + ... + w_2 v_n (f_{n2} + f_{2n}) \\ & + ... + w_n v_1 (f_{1n} + f_{n1}) + w_n v_2 (f_{2n} + f_{n2}) + ... + 2 w_n v_n f_{nn} \\ & = 0\end{aligned}$$

But if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not null then this equation can only be true if

$$f_{ij} = \begin{cases} 0 & \text{when } i = j \\ -f_{ji} & \text{when } i \neq j \end{cases}$$

This implies that $F(\boldsymbol{v},\boldsymbol{w})$ must be

$$F(\boldsymbol{v},\boldsymbol{w}) = \begin{pmatrix} v_1 & v_2 & ... & v_n \end{pmatrix} \begin{pmatrix} 0 & f_{12} & ... & f_{1n} \\ -f_{12} & 0 & ... & f_{2n} \\ \vdots & \vdots & ... & \vdots \\ -f_{1n} & -f_{2n} & ... & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

Therefore F has a correspondence to a antisymmetric $n \times n$ matrix. $\square$

Solution. **Exercise 25** Let α be a covector and β an antisymmetric bilinear form on $\mathbb{R}^n$ then $\alpha \wedge \beta$ is given by

$$(\alpha \wedge \beta)(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \alpha(\boldsymbol{u})\beta(\boldsymbol{v}, \boldsymbol{w}) + \alpha(\boldsymbol{v})\beta(\boldsymbol{w}, \boldsymbol{u}) + \alpha(\boldsymbol{w})\beta(\boldsymbol{u}, \boldsymbol{v})$$

Also, we have that

$$\begin{aligned}
-(\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) &= -(\alpha(\boldsymbol{v})\beta(\boldsymbol{u}, \boldsymbol{w}) + \alpha(\boldsymbol{u})\beta(\boldsymbol{w}, \boldsymbol{v}) + \alpha(\boldsymbol{w})\beta(\boldsymbol{v}, \boldsymbol{u})) \\
&= \alpha(\boldsymbol{v})(-\beta(\boldsymbol{u}, \boldsymbol{w})) + \alpha(\boldsymbol{u})(-\beta(\boldsymbol{w}, \boldsymbol{v})) + \alpha(\boldsymbol{w})(-\beta(\boldsymbol{v}, \boldsymbol{u})) \\
&= \alpha(\boldsymbol{v})\beta(\boldsymbol{w}, \boldsymbol{u}) + \alpha(\boldsymbol{u})\beta(\boldsymbol{v}, \boldsymbol{w}) + \alpha(\boldsymbol{w})\beta(\boldsymbol{u}, \boldsymbol{v}) \\
&= (\alpha \wedge \beta)(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})
\end{aligned}$$

Where we used that β is an antisymmetric bilinear form and hence $\beta(\boldsymbol{u}, \boldsymbol{v}) = -\beta(\boldsymbol{v}, \boldsymbol{u})$.
Finally, we see that

$$\begin{aligned}
(\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) &= \alpha(\boldsymbol{v})\beta(\boldsymbol{w}, \boldsymbol{u}) + \alpha(\boldsymbol{w})\beta(\boldsymbol{u}, \boldsymbol{v}) + \alpha(\boldsymbol{u})\beta(\boldsymbol{v}, \boldsymbol{w}) \\
&= \alpha(\boldsymbol{u})\beta(\boldsymbol{v}, \boldsymbol{w}) + \alpha(\boldsymbol{v})\beta(\boldsymbol{w}, \boldsymbol{u}) + \alpha(\boldsymbol{w})\beta(\boldsymbol{u}, \boldsymbol{v}) \\
&= (\alpha \wedge \beta)(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})
\end{aligned}$$

Where we only re-ordered the terms. Therefore

$$(\alpha \wedge \beta)(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -(\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) = (\alpha \wedge \beta)(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u})$$

So $\alpha \wedge \beta$ is alternating. $\square$

Solution. **Exercise 26** By definition $\frac{\partial}{\partial\theta}$ is $\frac{\partial}{\partial\theta}(\rho\cos\theta, \rho\sin\theta)^T$ hence

$$\frac{\partial}{\partial\theta} = \begin{pmatrix} \rho\,\frac{\partial}{\partial\theta}(\cos\theta) \\ \rho\,\frac{\partial}{\partial\theta}(\sin\theta) \end{pmatrix} = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \end{pmatrix} = \begin{pmatrix} -p \\ r \end{pmatrix}$$

If we recall that

$$\frac{\partial}{\partial r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{\partial}{\partial p} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can write that

$$\frac{\partial}{\partial\theta} = -p\frac{\partial}{\partial r} + r\frac{\partial}{\partial p}$$

Now, let us define $\frac{\partial}{\partial\rho}$ as $\frac{\partial}{\partial\rho}(\rho\cos\theta, \rho\sin\theta)^T$ then we have that

$$\frac{\partial}{\partial\rho} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} r/\rho \\ p/\rho \end{pmatrix} = \begin{pmatrix} \frac{r}{\sqrt{r^2+p^2}} \\ \frac{p}{\sqrt{r^2+p^2}} \end{pmatrix} = \frac{1}{\sqrt{r^2+p^2}}\left(r\frac{\partial}{\partial r} + p\frac{\partial}{\partial p}\right)$$

Where we used that $\rho = \sqrt{r^2+p^2}$.

Finally, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function then $\frac{\partial f}{\partial\theta}$ is given by

$$\begin{aligned} \frac{\partial f}{\partial\theta} &= -p\frac{\partial f}{\partial r} + r\frac{\partial f}{\partial p} \\ &= -\rho\sin\theta\frac{\partial f}{\partial r} + \rho\cos\theta\frac{\partial f}{\partial p} \\ &= \rho(\cos\theta\frac{\partial f}{\partial p} - \sin\theta\frac{\partial f}{\partial r}) \end{aligned}$$

On the other hand, $\frac{\partial f}{\partial\rho}$ is

$$\begin{aligned} \frac{\partial f}{\partial\rho} &= \frac{1}{\sqrt{r^2+p^2}}\left(r\frac{\partial f}{\partial r} + p\frac{\partial f}{\partial p}\right) \\ &= \frac{1}{\rho}\left(\rho\cos\theta\frac{\partial f}{\partial r} + \rho\sin\theta\frac{\partial f}{\partial p}\right) \\ &= \cos\theta\frac{\partial f}{\partial r} + \sin\theta\frac{\partial f}{\partial p} \end{aligned}$$

$\square$

Solution. **Exercise 27** Let $f : \mathbb{R}^n \to \mathbb{R}$ then the gradient one-form of f i.e. df is by definition

$$df = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right)$$

But also J_f by definition is the matrix of all possible first partial derivatives with one column for each independent (domain) variable and one row for each dependent (range) variable. Therefore

$$J_f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right) = df$$

If we let $n = 1$ then $f : \mathbb{R} \to \mathbb{R}$ which implies that

$$J_f = \left(\frac{\partial f}{\partial x_1} \right) = f'$$

□

Solution. **Exercise 29** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ then f can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} f_x(x, y, z) \\ f_y(x, y, z) \\ f_z(x, y, z) \end{pmatrix}$$

So the Jacobian, J_f is given by

$$J_f = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{pmatrix}$$

Hence

$$\operatorname{tr} J_f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

On the other hand if we treat f as a vector field $f = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$ then we can compute its divergence as

$$\boldsymbol{\nabla} \cdot f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (f_x, f_y, f_z) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Therefore $\boldsymbol{\nabla} \cdot f = \operatorname{tr} J_f$ □

Solution. **Exercise 30** Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that parametrizes the plane by polar coordinates defined as

$$\begin{pmatrix} \theta \\ \rho \end{pmatrix} \to \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix}$$

Then

$$\frac{\partial \Psi}{\partial \theta} = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \end{pmatrix} \qquad \frac{\partial \Psi}{\partial \rho} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

So the Jacobian of Ψ is

$$J_\Psi(\theta, \rho) = \begin{pmatrix} -\rho\sin\theta & \cos\theta \\ \rho\cos\theta & \sin\theta \end{pmatrix}$$

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary function then via the chain rule we have that

$$\begin{gathered} J_{f\circ\Psi}(\theta, \rho) = J_f(\Psi(\theta, \rho))J_\Psi(\theta, \rho) \\ \begin{pmatrix} \frac{\partial f\circ\Psi}{\partial \theta} & \frac{\partial f\circ\Psi}{\partial \rho} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial p} \end{pmatrix} \cdot \begin{pmatrix} -\rho\sin\theta & \cos\theta \\ \rho\cos\theta & \sin\theta \end{pmatrix} \end{gathered}$$

Then we have that

$$\begin{aligned} \frac{\partial f\circ\Psi}{\partial \theta} &= -\rho\sin\theta\frac{\partial f}{\partial r} + \rho\cos\theta\frac{\partial f}{\partial p} = -p\frac{\partial f}{\partial r} + r\frac{\partial f}{\partial p} \\ \frac{\partial f\circ\Psi}{\partial \rho} &= \cos\theta\frac{\partial f}{\partial r} + \sin\theta\frac{\partial f}{\partial p} = \frac{1}{\sqrt{r^2+p^2}}\left(r\frac{\partial f}{\partial r} + p\frac{\partial f}{\partial p}\right) \end{aligned}$$

Where we used that $(\cos\theta,\ \sin\theta)^T = (r/\rho,\ p/\rho)^T$. Since in the calculations we used an arbitrary function f then we can write that

$$\begin{aligned} \frac{\partial}{\partial \theta} &= -p\frac{\partial}{\partial r} + r\frac{\partial}{\partial p} \\ \frac{\partial}{\partial \rho} &= \frac{1}{\sqrt{r^2+p^2}}\left(r\frac{\partial}{\partial r} + p\frac{\partial}{\partial p}\right) \end{aligned}$$

Which is the same result we got in Exercise 26. □

Solution. **Exercise 31** Let

$$\omega = dp_1 \wedge dr_1 + dp_2 \wedge dr_2 + ... + dp_n \wedge dr_n$$

We know that $(dp_1 \wedge dr_1)(\boldsymbol{v}, \boldsymbol{w})$ is the signed area of the parallelogram spanned by the projection of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ over the r_1-p_1 plane.

Therefore $\omega(\boldsymbol{v}, \boldsymbol{w})$ is the sum of the signed areas of the parallelograms spanned by the projections of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ over the r_i-p_i planes. $\square$

Solution. **Exercise 32** Let

$$T^*S^2 = \{(\boldsymbol{r}, \boldsymbol{p}) \in \mathbb{R}^3 \times (\mathbb{R}^3)^* : |\boldsymbol{r}| = 1 \text{ and } \boldsymbol{pr} = 0\}$$

Hence elements in T^*S^2 needs to satisfy

$$r_1^2 + r_2^2 + r_3^2 = 1 \qquad p_1r_1 + p_2r_2 + p_3r_3 = 0$$

So we define a function $F : \mathbb{R}^6 \to \mathbb{R}^2$ such that for any $\boldsymbol{c} \in \mathbb{R}^6$ we have that $F(\boldsymbol{c}) = 0$ as

$$F\begin{pmatrix}\boldsymbol{r}\\ \boldsymbol{p}\end{pmatrix} = \begin{pmatrix} r_1^2 + r_2^2 + r_3^2 - 1 \\ p_1r_1 + p_2r_2 + p_3r_3 \end{pmatrix}$$

Then the Jacobian of this function is

$$J_F(\boldsymbol{r}, \boldsymbol{p}) = \begin{pmatrix} 2r_1 & 2r_2 & 2r_3 & 0 & 0 & 0 \\ p_1 & p_2 & p_3 & r_1 & r_2 & r_3 \end{pmatrix}$$

Since not all r_i can be 0 at the same time since we are considering a sphere then both rows are linearly independent so the rank of this Jacobian is 2 (full rank) and hence it is onto.
Then the Implicit function theorem guarantees that in some neighbourhood of $\boldsymbol{c}$ in which $F = 0$ the equation for F implicitly expresses one of the variables in terms of the others.
Suppose $r_1 \neq 0$ then we can write explicitly the equations as follows

$$\begin{aligned} r_1 &= \sqrt{1 - r_2^2 - r_3^2} \\ p_1 &= \frac{-p_2r_2 - p_3r_3}{r_1} \end{aligned}$$

The same can be done assuming that $r_2 \neq 0$ or $r_3 \neq 0$.
Therefore, we can parametrize the function F by a function g as follows

$$g : \begin{pmatrix} r_2 \\ r_3 \\ p_2 \\ p_3 \end{pmatrix} \to \begin{pmatrix} \sqrt{1 - r_2^2 - r_3^2} \\ r_2 \\ r_3 \\ \frac{-p_2r_2 - p_3r_3}{\sqrt{1 - r_2^2 - r_3^2}} \\ p_2 \\ p_3 \end{pmatrix}$$

Hence T^*S^2 can be parametrized by four real parameters. $\square$

Solution. **Exercise 33** Let f be the distance function in Cartesian coordinates i.e.

$$f\begin{pmatrix} r \\ p \end{pmatrix} = \sqrt{r^2 + p^2}$$

Then $\Gamma^* f = f \circ \Gamma$ is given by

$$\begin{aligned}
f \circ \Gamma \begin{pmatrix} \rho \\ \theta \end{pmatrix} &= f \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix} \\
&= \sqrt{(\rho\cos\theta)^2 + (\rho\sin\theta)^2} \\
&= \sqrt{\rho^2(\cos^2\theta + \sin^2\theta)} \\
&= \rho
\end{aligned}$$

Therefore $f \circ \Gamma$ is the function

$$\begin{pmatrix} \rho \\ \theta \end{pmatrix} \to \rho$$

□

Solution. **Exercise 34** We know $p = \rho\sin\theta$ then

$$\begin{aligned}
dp &= d(\rho\sin\theta) \\
&= \begin{pmatrix} \frac{\partial}{\partial\rho}\rho\sin\theta & \frac{\partial}{\partial\theta}\rho\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta & \rho\cos\theta \end{pmatrix} \\
&= \sin\theta\ d\rho + \rho\cos\theta\ d\theta
\end{aligned}$$

□

Solution. **Exercise 35** We compute $dr \cdot J_\Gamma$ as follows

$$\begin{aligned}
dr \cdot J_\Gamma &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\rho\sin\theta \end{pmatrix} \\
&= \cos\theta d\rho - \rho\sin\theta d\theta
\end{aligned}$$

□

Solution. **Exercise 36** Let us define new coordinates by

$$\begin{pmatrix} r \\ p \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$$

Where M is a constant 2×2 invertible matrix

1. From the definition we know that r and p can be written as

   $$r = ax + by \quad p = cx + dy$$

   Where a, b, c and d are the coefficients of M but we see that

   $$\frac{\partial r}{\partial x} = a \quad \frac{\partial r}{\partial y} = b \quad \frac{\partial p}{\partial x} = c \quad \frac{\partial p}{\partial y} = d$$

   Hence must be that

   $$M = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \end{pmatrix}$$

   Also, since M is invertible we can also write that

   $$\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} r \\ p \end{pmatrix}$$

   Then using the pushforward method we see that

   $$\frac{\partial}{\partial r} = M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{\partial}{\partial p} = M^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   So in the same way we also have that

   $$\frac{\partial}{\partial x} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{\partial}{\partial y} = M \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2. We know that $dx(\frac{\partial}{\partial x}) = 1$ and $dx(\frac{\partial}{\partial y}) = 0$ hence

$$dx\left(M\begin{pmatrix}1\\0\end{pmatrix}\right) = 1 \quad dx\left(M\begin{pmatrix}0\\1\end{pmatrix}\right) = 0$$

Then dx takes the first column of M to 1 and the second column to 0, then dx must be the first row of M^{-1} i.e.

$$dx = \begin{pmatrix}1 & 0\end{pmatrix} M^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\end{pmatrix} = \frac{1}{ad-bc}(ddr - bdp)$$

Where we used that

$$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

In the same way, we know that $dy(\frac{\partial}{\partial x}) = 0$ and $dy(\frac{\partial}{\partial y}) = 1$ hence

$$dy\left(M\begin{pmatrix}1\\0\end{pmatrix}\right) = 0 \quad dy\left(M\begin{pmatrix}0\\1\end{pmatrix}\right) = 1$$

Then must be that

$$dy = \begin{pmatrix}0 & 1\end{pmatrix} M^{-1} = \frac{1}{ad-bc}(adp - cdr)$$

3. Finally since we know that

$$r = ax + by \qquad p = cx + dy$$

Then must be that

$$dr = adx + bdy \qquad dp = cdx + ddy$$

Then solving for dx gives us

$$\begin{aligned}
dx &= \frac{1}{a}(dr - bdy) \\
dx &= \frac{1}{a}\left(dr - \frac{b}{d}(dp - cdx)\right) \\
dx &= \frac{1}{a}dr - \frac{b}{ad}dp + \frac{bc}{ad}dx \\
dx\left(1 - \frac{bc}{ad}\right) &= \frac{1}{a}dr - \frac{b}{ad}dp \\
dx &= \frac{ad}{ad - bc}\frac{1}{a}dr - \frac{ad}{ad - bc}\frac{b}{ad}dp \\
dx &= \frac{1}{ad - bc}(ddr - bdp)
\end{aligned}$$

And hence dy is

$$\begin{aligned}
dy &= \frac{1}{d}(dp - cdx) \\
dy &= \frac{1}{d}(dp - \frac{c}{ad - bc}(ddr - bdp)) \\
dy &= \frac{1}{d}dp - \frac{c}{ad - bc}dr + \frac{cb}{d(ad - bc)}dp \\
dy &= dp\left(\frac{ad - bc + cb}{d(ad - bc)}\right) - \frac{c}{ad - bc}dr \\
dy &= \frac{1}{ad - bc}(adp - cdr)
\end{aligned}$$

$\square$