Solved selected problems of Calculus on Manifolds by Michael Spivak

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2 - Differentiation

Basic Definitions

Proof. **2-2.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be independent of the second variable i.e. for each $x \in \mathbb{R}$ we have that $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$.
Then we can take some $y_0 \in \mathbb{R}$ and define a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x, y_0)$, then, for any $y \in \mathbb{R}$ we get that $f(x, y) = f(x, y_0) = g(x)$ because f is independent of the second variable.

On the other hand, suppose we have a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x, y)$ then since this is true for any $y \in \mathbb{R}$ then we can write that

$$f(x, y_1) = g(x) = f(x, y_2)$$

for any $y_1, y_2 \in \mathbb{R}$, which implies that f is independent of the second variable.

The Jacobian of f at (a, b) i.e. $f'(a, b)$ is given by

$$f'(a, b) = (D_1 f(a, b),\ D_2 f(a, b))$$

Since $g(x) = f(x, y)$ then $D_1 f(a, b) = D_1 g(a) = g'(a)$ therefore

$$f'(a, b) = (g'(a),\ 0)$$

□

Proof. **2-3.** A function $f : \mathbb{R}^2 \to \mathbb{R}$ is independent of the first variable if for each $y \in \mathbb{R}$ we have that$f(x_1, y) = f(x_2, y)$ for all $x_1, x_2 \in \mathbb{R}$.

The Jacobian $f'(a, b)$ in this case is

$$f'(a, b) = (D_1 f(a, b),\ D_2 f(a, b)) = (0,\ D_2 f(a, b))$$

In the same way as we did in problem 2-2, we can show that f is independent of the first variable if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(y) = f(x, y)$.
So we get that $D_2 f(a, b) = D_2 g(b) = g'(b)$ and therefore

$$f'(a, b) = (0, g'(b))$$

Finally, the functions that are independent of the first and the second variable are the functions of the form $f(x, y) = c$ where c is a constant. $\square$

Proof. **2-4.** Let us define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} |x| \cdot g\big(\frac{x}{|x|}\big) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Where g is a continuous real-valued function on the unit circle $\{x \in \mathbb{R}^2 \ : \ |x| = 1\}$ such that $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$.

(a) If $x \in \mathbb{R}^2$ and $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = f(tx)$, we want to show that h is differentiable.
Suppose $x \neq 0$, also, let $a \in \mathbb{R}$ and let $\lambda(k) = k|x|g(\frac{x}{|x|})$ then

$$\begin{aligned} \lim_{k\to 0} \frac{|h(a+k) - h(a) - \lambda(k)|}{|k|} &= \lim_{k\to 0} \frac{|f((a+k)x) - f(ax) - \lambda(k)|}{|k|} = \\ &= \lim_{k\to 0} \frac{1}{|k|}\left| |(a+k)x|g\left(\frac{(a+k)x}{|(a+k)x|}\right) - |ax|g\left(\frac{ax}{|ax|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| \end{aligned}$$

Suppose $a + k > 0$, and suppose $a > 0$ then

$$\begin{aligned} &\frac{1}{|k|}\left| |(a+k)x|g\left(\frac{(a+k)x}{|(a+k)x|}\right) - |ax|g\left(\frac{ax}{|ax|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| = \\ &= \frac{1}{|k|}\left| (a+k)|x|g\left(\frac{x}{|x|}\right) - a|x|g\left(\frac{x}{|x|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| \\ &= 0 \end{aligned}$$

If $a < 0$ then

$$|ax|g\left(\frac{ax}{|ax|}\right) = -a|x|g\left(-\frac{ax}{a|x|}\right) = a|x|g\left(\frac{x}{|x|}\right)$$

Where we used that $g(-x) = -g(x)$ and hence

$$\begin{aligned} &\frac{1}{|k|}\left| |(a+k)x|g\left(\frac{(a+k)x}{|(a+k)x|}\right) - |ax|g\left(\frac{ax}{|ax|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| = \\ &= \frac{1}{|k|}\left| (a+k)|x|g\left(\frac{x}{|x|}\right) - a|x|g\left(\frac{x}{|x|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| \\ &= 0 \end{aligned}$$

Now suppose $a + k < 0$ then

$$\begin{aligned} &\frac{1}{|k|}\left| |(a+k)x|g\left(\frac{(a+k)x}{|(a+k)x|}\right) - |ax|g\left(\frac{ax}{|ax|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| = \\ &= \left| -(a+k)|x|g\left(-\frac{(a+k)x}{(a+k)|x|}\right) - a|x|g\left(\frac{x}{|x|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| \\ &= \left| (a+k)|x|g\left(\frac{x}{|x|}\right) - a|x|g\left(\frac{x}{|x|}\right) - k|x|g\left(\frac{x}{|x|}\right)\right| \\ &= 0 \end{aligned}$$

Finally if $a+k=0$

$$\begin{aligned}
&\frac{1}{|k|}\left| |(a+k)x| g\left(\frac{(a+k)x}{|(a+k)x|}\right) - |ax| g\left(\frac{ax}{|ax|}\right) - k|x| g\left(\frac{x}{|x|}\right)\right| = \\
&= \frac{1}{|k|}\left| k|x| g\left(\frac{ax}{|ax|}\right) - k|x| g\left(\frac{x}{|x|}\right)\right| \\
&= 0
\end{aligned}$$

In the case $x=0$ we see that $\lambda(k)=0$ and hence

$$\begin{aligned}
&\frac{1}{|k|}\left| |(a+k)x| g\left(\frac{(a+k)x}{|(a+k)x|}\right) - |ax| g\left(\frac{ax}{|ax|}\right) - k|x| g\left(\frac{x}{|x|}\right)\right| = \\
&= \frac{1}{|k|}\left| 0 - 0 - 0 \right| \\
&= 0
\end{aligned}$$

Therefore h is differentiable.

(b) Let us compute $Df(0,0)$ as follows

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{|f(0+(h,k)) - f(0) - Df(0,0)(h,k)|}{|(h,k)|} \\
&= \lim_{(h,k)\to(0,0)} \frac{1}{|(h,k)|}\left| |(h,k)| g\left(\frac{(h,k)}{|(h,k)|}\right) - Df(0,0)(h,k)\right|
\end{aligned}$$

Let $h=0$ then

$$\begin{aligned}
&\frac{1}{|(0,k)|}\left| |(0,k)| g\left(\frac{(0,k)}{|(0,k)|}\right) - Df(0,0)(0,k)\right| \\
&= \frac{1}{|k|}\left| |k| g\left(0, \frac{k}{|k|}\right) - D_2 f(0,0) k\right| \\
&\le \frac{1}{|k|}\left| |k| g\left(0, \frac{k}{|k|}\right)\right| + \frac{1}{|k|}\left| D_2 f(0,0) k\right| \\
&= \left| g\left(0, \frac{k}{|k|}\right)\right| + \left| D_2 f(0,0)\right|
\end{aligned}$$

We see that $g(0,k/|k|) = g(0,1) = 0$ if $k>0$, if $k<0$ then $g(0,k/|k|) = g(0,-1) = g(-(0,1)) = -g(0,1) = 0$, then we get that

$$0 \le \frac{1}{|(0,k)|}\left| |(0,k)| g\left(\frac{(0,k)}{|(0,k)|}\right) - Df(0,0)(0,k)\right| \le \left| D_2 f(0,0)\right|$$

In the same way, if we let $k = 0$ instead

$$\begin{aligned}
&\frac{1}{|(h,0)|}\left|\,|(h,0)|g\left(\frac{(h,0)}{|(h,0)|}\right) - Df(0,0)(h,0)\right| \\
&= \frac{1}{|h|}\left|\,|h|g\left(\frac{h}{|h|},0\right) - D_1f(0,0)h\right| \\
&\leq \frac{1}{|h|}\left|\,|h|g\left(\frac{h}{|h|},0\right)\right| + \frac{1}{|h|}\left|D_1f(0,0)h\right| \\
&= \left|g\left(\frac{h}{|h|},0\right)\right| + \left|D_1f(0,0)\right|
\end{aligned}$$

Then since $g(0,1) = 0$ we get that

$$0 \leq \frac{1}{|(h,0)|}\left|\,|(h,0)|g\left(\frac{(h,0)}{|(h,0)|}\right) - Df(0,0)(h,0)\right| \leq \left|D_1f(0,0)\right|$$

Therefore must be that $Df(0,0) = 0$ for the limit

$$\lim_{(h,k)\to(0,0)} \frac{|f(0+(h,k)) - f(0) - Df(0,0)(h,k)|}{|(h,k)|}$$

to converge to 0.
Then for any path taken by h and k the limit becomes

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{|f(0+(h,k)) - f(0) - Df(0,0)(h,k)|}{|(h,k)|} \\
&= \lim_{(h,k)\to(0,0)} \frac{1}{|(h,k)|}\left|\,|(h,k)|g\left(\frac{(h,k)}{|(h,k)|}\right) - 0\right| \\
&= \lim_{(h,k)\to(0,0)} \left|g\left(\frac{(h,k)}{|(h,k)|}\right)\right|
\end{aligned}$$

But g is only defined at the unit circle so this limit can only be 0 if $g = 0$ and hence f is not continuous at $(0,0)$.

$\square$

Proof. **2-5.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

We can write that

$$\frac{x|y|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \cdot \frac{x|y|}{x^2+y^2} = |(x,y)| \cdot \frac{x|y|}{|(x,y)|^2}$$

Then defining g as $g(x,y) = x|y|$ we see that

$$g\left(\frac{(x,y)}{|(x,y)|}\right) = \frac{x}{|(x,y)|}\left|\frac{y}{|(x,y)|}\right| = \frac{x|y|}{|(x,y)|^2}$$

Also, we see that $g(0,1) = g(1,0) = 0$ and that

$$g(-x,-y) = -x|-y| = -xy = -x|y| = -g(x,y)$$

Hence we can write $f(x,y)$ as

$$f(x,y) = \begin{cases} |(x,y)| \cdot g\left(\frac{(x,y)}{|(x,y)|}\right) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

Where g is a continuous real-valued function with the same properties described in Problem 2-4.
Therefore we can conclude that f is not differentiable at $(0,0)$. $\square$

Proof. **2-6.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$, we may also define f as

$$f(x,y) = \begin{cases} \frac{\sqrt{x^2+y^2}\sqrt{|xy|}}{\sqrt{x^2+y^2}} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

We see that

$$\frac{\sqrt{x^2+y^2}\sqrt{|xy|}}{\sqrt{x^2+y^2}} = |(x,y)|\frac{\sqrt{|xy|}}{|(x,y)|}$$

We can define a function g as $g(x,y) = \sqrt{|xy|}$ so

$$g\left(\frac{(x,y)}{|(x,y)|}\right) = \sqrt{\left|\frac{x}{|(x,y)|}\frac{y}{|(x,y)|}\right|} = \sqrt{\frac{|xy|}{|(x,y)|^2}} = \frac{\sqrt{|xy|}}{|(x,y)|}$$

Also, we see that $g(0,1) = g(1,0) = 0$ and that

$$g(-x,-y) = \pm\sqrt{|(-x)(-y)|} = \pm\sqrt{|xy|} = \pm g(x,y)$$

Hence we can write $f(x,y)$ as

$$f(x,y) = \begin{cases} |(x,y)| \cdot g\left(\frac{(x,y)}{|(x,y)|}\right) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

Where g is a continuous real-valued function with the same properties described in Problem 2-4.
Therefore we can conclude that f is not differentiable at $(0,0)$. $\square$

Proof. **2-7.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$ we want to show that f is differentiable at 0.
Let us consider the following limit

$$\lim_{h \to 0} \frac{|f(0+h) - f(0) - Df(0)h|}{|h|}$$

We see that

$$\begin{aligned} 0 \leq \frac{|f(0+h) - f(0) - Df(0)h|}{|h|} &\leq \frac{|f(h)| + |f(0)| + |Df(0)h|}{|h|} \\ &\leq \frac{|h|^2 + 0 + |Df(0)||h|}{|h|} \\ &\leq |h| + |Df(0)| \end{aligned}$$

We see that $|h| + |Df(0)| \to 0$ as $h \to 0$ if $Df(0) = 0$.
Therefore f is differnetiable at 0. $\square$

Basic Theorems