Digital Signal Processing

Spectrum estimation

Review

- Sampling
- Fourier analysis
- Digital filtering
- Multirate DSP

Introduction

- Given a wide stationary sense (WSS) random process x[k] the PSD is defined by:

$$S_x(\omega) = \lim_{M\to\infty} E\left\{ \frac{1}{2M+1} \left| \sum_{n=-M}^{M} x[n] e^{-j\omega n} \right|^2 \right\}$$

$$S_x(\omega) = \mathcal{F}\{r_x[k]\}$$

$$= \sum_{k=-\infty}^{\infty} r_x[k] e^{-j\omega k}$$

$$\boxed{\underbrace{r_x[k] = E\{x[n]x^*[n+k]\}}_{\text{ACF of RP } x[n]}}$$

Introduction

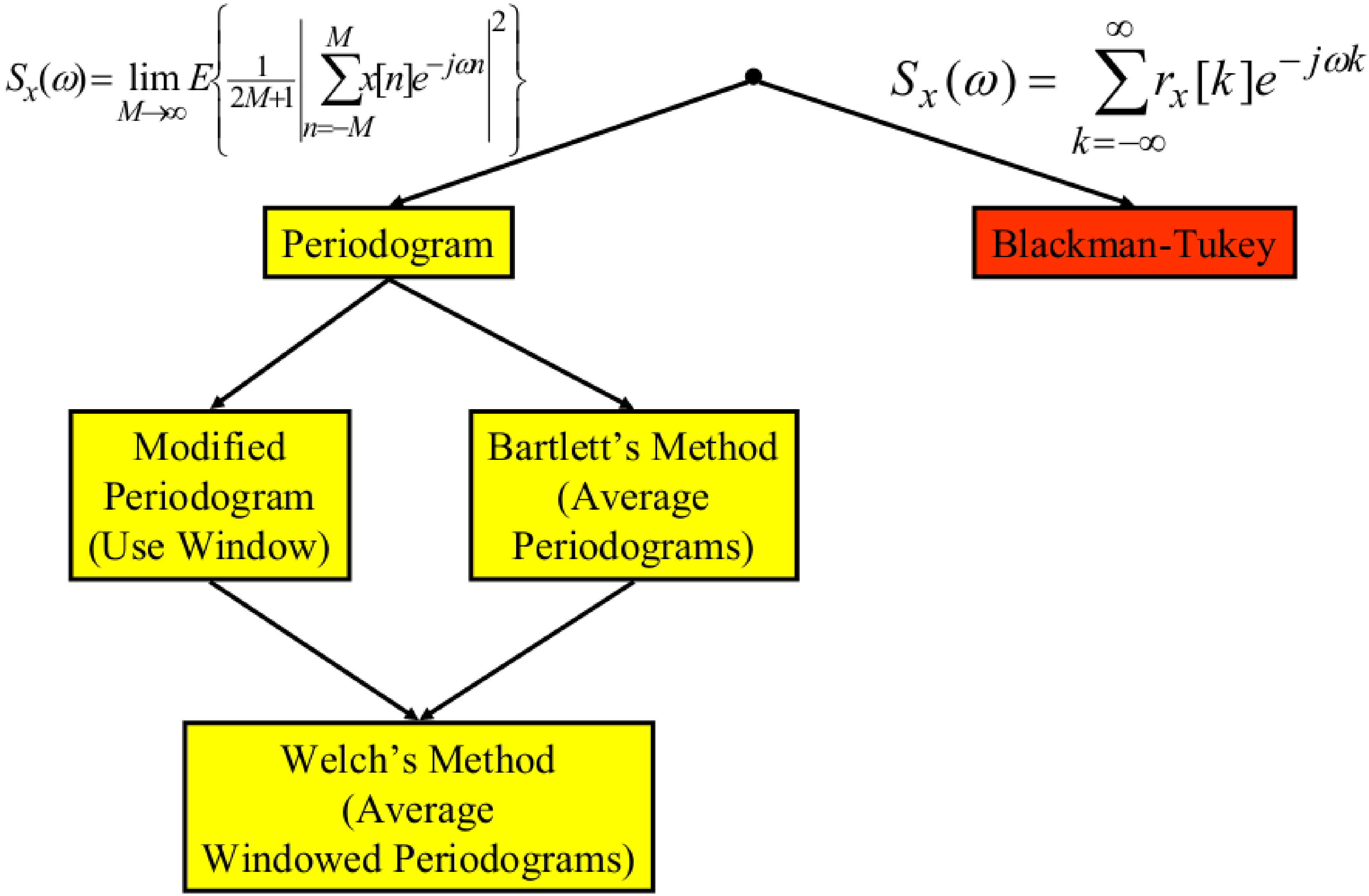
- We have to deal with:

1. Expectation or ensemble averaging BUT in practice we get only ONE realization from the ensemble
2. Fourier transform of infinite length BUT in practice we get only a finite number of samples

Introduction

- Given these limitations, two approaches come up:
  1. Compute the DFT of the signal and then do some form of averaging
  2. Compute and estimate of the ACF using some form of averaging and then compute the DFT

Introduction



Periodogram

$$S_x(\omega) = \lim_{M\to\infty} E\left\{ \frac{1}{2M+1} \left| \sum_{n=-M}^{M} x[n] e^{-j\omega n} \right|^2 \right\}$$

- In practice we have one set of finite-duration data.
- Two Practical Problems:
  1. Can’t do the expected value
  2. Can’t do the limit
- The periodogram is a method that ignores them both !!!

$$\boxed{\hat{S}_{PER}(\omega) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \right|^2}$$

Periodogram

- Desirable estimator features:
  - "correct on average" or unbiased:

$$E\{\hat{A}\} = A$$

  - or at least asymptotically unbiased:

$$\lim_{N\to\infty} E\{\hat{A}\} = A$$

Periodogram

- Desirable estimator features:
  - small fluctuations from estimate to estimate

$$\mathrm{var}\{\hat{A}\} = small$$

  - or at least asymptotically *small*

$$\lim_{N\to\infty} \mathrm{var}\{\hat{A}\} = 0$$

Periodogram

- For spectrum estimation we want:

$$E\{\hat{S}_x(\omega)\} = S_x(\omega)$$

$$\mathrm{var}\{\hat{S}_x(\omega)\} = small$$

Periodogram: bias

$$E\left\{\hat{S}_{PER}(\omega)\right\} = \frac{1}{N} E\left\{\left|\sum_{n=0}^{N-1} x[n] e^{-j\omega n}\right|^2\right\}$$

$$= \frac{1}{N} E\left\{\left[\sum_{n=0}^{N-1} x[n] e^{-j\omega n}\right]\left[\sum_{m=0}^{N-1} x^*[m] e^{j\omega m}\right]\right\}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} r_x[n-m] e^{-j\omega(n-m)}$$

"<u>Sum On Diagonals</u>"

$r_x[n\text{-}m]$ is constant on each diagonal

$$= \sum_{k=-(N-1)}^{N-1} \underbrace{\left(1 - \frac{|k|}{N}\right)}_{w_B[k]} r_x[k] e^{-j\omega k}$$

Bartlett (Triangle) Window

$$= \frac{1}{2\pi} S_x(\omega) \underset{circ}{*} W_B(\omega) \neq S_x(\omega)$$

Periodogram: bias

- The periodogram is biased
- But …

$$\lim_{N\to\infty} E\left\{\hat{S}_{PER}(\omega)\right\} = \lim_{N\to\infty} \sum_{k=-(N-1)}^{N-1} \underbrace{\left(1-\frac{|k|}{N}\right)}_{\to 1} r_x[k]e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} r_x[k]e^{-j\omega k} = S_x(\omega)$$

- It is asymptotically unbiased

Periodogram: variance

- The variance of the Periodogram does NOT (in general) tend to zero as $N \to \infty$
- Under this assumption, complex-valued white Gaussian process w/ zero mean and variance σ^2

$$S(\omega) = \sigma^2, \quad \forall \omega \quad \& \quad r_x[k] = \sigma^2 \delta[k]$$

$$\boxed{\operatorname{var}\left\{\hat{S}_{PER}(\omega)\right\} = \sigma^4}$$

Periodogram summary

- Biased Estimate
- Variance does NOT decrease with increasing N
- Rapid Fluctuations

Modified Periodogram

$$\boxed{\hat{S}_{MP}(\omega) = \frac{1}{NU}\left|\sum_{n=0}^{N-1} x[n]w[n]e^{-j\omega n}\right|^2}$$

$$U = \frac{1}{N}\sum_{n=0}^{N-1}\left|w[n]\right|^2$$

- Has reduced bias but is still biased
- Remains asymptotically unbiased
- Roughly equal periodogram's variance

Bartlett's Method

- Introduces averaging of k blocks
- The signal data of length N is chopped into K non-overlapping blocks of length L

$$N = KL$$

$$\boxed{\begin{aligned}\hat{S}_B(\omega) &= \frac{1}{K}\sum_{i=0}^{K-1}\frac{1}{L}\left|\sum_{n=0}^{L-1} x_i[n]e^{-j\omega n}\right|^2 \\ &= \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x_i[n]e^{-j\omega n}\right|^2\end{aligned}}$$

Bartlett's Method

- Ensemble averaging is implemented as time averaging → ergodicity asumption
- Implicit assumptions:
  - Ergodic process: "A process is ergodic if time averaging of any realization is equivalent to ensemble averaging"

Bartlett's Method

- Averaging k blocks improves the variance

$$\boxed{\operatorname{var}\left\{\hat{S}_B(\omega)\right\} \approx \frac{1}{K} S_x^2(\omega)}$$

- But for a given data length N
  - More Blocks means Shorter Blocks
  - Shorter Blocks means poorer resolution
- Trade off between variance and resolution

Up to now …

- Windowing reduces bias
- Averaging reduces variance
- Then use both ! → Welch’s method

Welch's method

- Averaged and windowed periodogram

$$\boxed{\hat{S}_W(\omega) = \frac{1}{KLU}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x_i[n]w[n]e^{-j\omega n}\right|^2}$$

- Also uses block overlap (50% typically)

Welch's method: variance

- For 50% block overlap

$$\boxed{\operatorname{var}\left\{\hat{S}_W(\omega)\right\} \approx \frac{9}{16}\frac{L}{N} S_x^2(\omega)}$$

- Almost 50% reduction compared to Bartlett's method for the same N and L

$$\boxed{\operatorname{var}\left\{\hat{S}_W(\omega)\right\} \approx \frac{9}{16} \operatorname{var}\left\{\hat{S}_B(\omega)\right\}}$$

Hands on

- Seno + ruido (probar varias distrs)
- Probar ergodicidad estudiando time/ensemble avgs

Next

- Adaptive filtering
- Time-frequency analysis