Introduction to Mobile Robotics

Sheet #1

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Problem 1

(a1)

$$A = \begin{bmatrix} 0.25 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$$

Let x be a 2x1 vector. Denoting its entries by x_1 and x_2:

$$\begin{aligned} x^t Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.25 & 0.1 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.25x_1 + 0.1x_2 \\ 0.2x_1 + 0.5x_2 \end{bmatrix} \\ &= 0.25x_1^2 + 0.2x_1x_2 + 0.1x_1x_2 + 0.5x_2^2 \\ &= 0.25x_1^2 + 0.3x_1x_2 + 0.5x_2^2 = f(x_1, x_2) \end{aligned}$$

Considering the function $f(x_1, x_2)$ we can calculate the partial derivatives and equal them to 0:

$$\begin{cases} \frac{\partial f}{\partial x_1} = 0.5x_1 + 0.3x_2 = 0 \\ \frac{\partial f}{\partial x_2} = 0.3x_1 + x_2 = 0 \end{cases}$$

From which we get that the following points are critical:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

We now calculate the Hessian matrix to analyze the critical point $(0, 0)$.

$$H_f(x_1, x_2) = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 1 \end{bmatrix}$$

The determinant is equal to $0.5 - 0.09 = 0.41$ which is greater than zero meaning $(0, 0)$ is a local extreme (which in this case is also global since we only have one critical point). To confirm whether it is a minimum or a maximum we can calculate eigenvalues which can be easily calculated:

$$\begin{vmatrix} 0.5 - \lambda & 0.3 \\ 0.3 & 1 - \lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 1.5\lambda + 0.41 = 0$$
$$\implies \begin{cases} \lambda_1 = 0.36 \\ \lambda_2 = 1.14 \end{cases}$$

This means the hessian matrix is positive definite which implies the point $(0, 0)$ being a global minimum of f.
So, $f(x_1, x_2) > 0 \; \forall (x_1, x_2) \neq (0, 0)$ making A a positive definite matrix.

Problem 1 continued on next page. . .

(a2)

$$B = \begin{bmatrix} 0.25 & -0.3 \\ -0.3 & 0.5 \end{bmatrix}$$

Since $B = B^T$, the B matrix is symmetric. This means that if the eigenvalues are positive then the matrix is positive definite. *Note: This is proved below*
We calculate the eigenvalues:

$$\begin{vmatrix} 0.25-\lambda & -0.3 \\ -0.3 & 0.5-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 0.75\lambda + 0.035 = 0$$
$$\implies \begin{cases} \lambda_1 = 0.7 \\ \lambda_2 = 0.05 \end{cases}$$

Meaning B is a positive definite matrix.
Proof: If B is symmetric, then we know through the spectral theorem that there is an orthogonal matrix Q such that $A = Q^T \Lambda Q$ where $\Lambda = diag(\lambda_1, \lambda_2)$. If $x = (x_1, x_2)$ is different than zero, then $Qx = z$ is also different than zero. This proves that:

$$x^T Bx = x^T(Q^T \Lambda Q)x = (x^T Q^T)\Lambda(Qx) = z^T \Lambda z = \sum_{i=1}^{2} \lambda_i z_i^2 > 0$$

Which is always true since z is never zero and B has positive eigenvalues.
(b)

$$C = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

C will not be positive definite if either of the eigenvalues are ≤ 0.
We define matrix $C_1 = C + \mu I$, with $\mu \in \mathbb{R}$ and calculate the eigenvalues:

$$\begin{vmatrix} -3+\mu-\lambda & 0 \\ 0 & 1+\mu-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 + (2-2\mu)\lambda - 3 + -2\mu + \mu^2 = 0$$
$$\implies \begin{cases} \lambda_1 = \mu - 3 \\ \lambda_2 = \mu + 1 \end{cases}$$

So, the largest value for μ for which C_1 is not positive definite is $\mu = 3$.
(c)

```python
def is_orthogonal(A):
    # It checks if the matrix is orthogonal
    # A: matrix to check
    # return: True if orthogonal, False otherwise
    return np.allclose(np.dot(A, A.T), np.eye(A.shape[0]))
```

(d) Output of command *python3 1c.py* is *D is orthogonal: True* meaning D is orthogonal.

Problem 2

(a)
The pose of the robot is given by matrix T. Matrix $T1$ is the matrix that gives the pose of the robot w.r.t. the global coordinate system and the coordinates of l w.r.t the global frame is equal to $T_1 l$ so, with $t_1 = (x_1, y_1)$:

$$\begin{bmatrix} l_x^{GF} \\ l_y^{GF} \end{bmatrix} = \begin{bmatrix} R(\theta_1) & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} l_x \\ l_y \end{bmatrix} = \begin{bmatrix} l_x R(\theta_1) + t_1 l_y \\ l_y \end{bmatrix}$$

(b)
We can apply T_1^{-1} to get the coordinates in the robot frame:

$$\begin{bmatrix} l_x^{RF} \\ l_y^{RF} \end{bmatrix} = T_1^{-1} \begin{bmatrix} l_x^{GF} \\ l_y^{GF} \end{bmatrix} = \begin{bmatrix} R(\theta_1)^T & -R(\theta_1)^T t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} l_x R(\theta_1) + t_1 l_y \\ l_y \end{bmatrix}$$
$$= \begin{bmatrix} R(\theta_1)^T l_x R(\theta_1) + R(\theta_1)^T t_1 l_y - R(\theta_1)^T t_1 l_y \\ l_y \end{bmatrix} = \begin{bmatrix} l_x \\ l_y \end{bmatrix}$$

Since $R(\theta_1)^T R(\theta_1) = I$.
(c)
Since $T_{12} = T_2 T_1$:

$$T_{12} = \begin{bmatrix} R(\theta_2) & t_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_1) & t_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(\theta_2)R(\theta_1) & R(\theta_2)t_1 + t_2 \\ 0 & 1 \end{bmatrix}$$

(d)
Since matrix T_{12} represents the pose in position x_2 w.r.t. x_1, if we want to find out the landmark position in the robot's frame we need to do $T_{12}^T l$:

$$\begin{bmatrix} l_x^{RF} \\ l_y^{RF} \end{bmatrix} = \begin{bmatrix} R(\theta_2)R(\theta_1) & 0 \\ R(\theta_2)t_1 + t_2 & 1 \end{bmatrix} \begin{bmatrix} l_x \\ l_y \end{bmatrix} = \begin{bmatrix} R(\theta_2)R(\theta_1) l_x \\ (R(\theta_2)t_1 + t_2) l_x + l_y \end{bmatrix}$$