Introduction to Mobile Robotics

Probabilistic Robotics

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Probabilistic Robotics

Key idea:

Explicit representation of uncertainty
(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

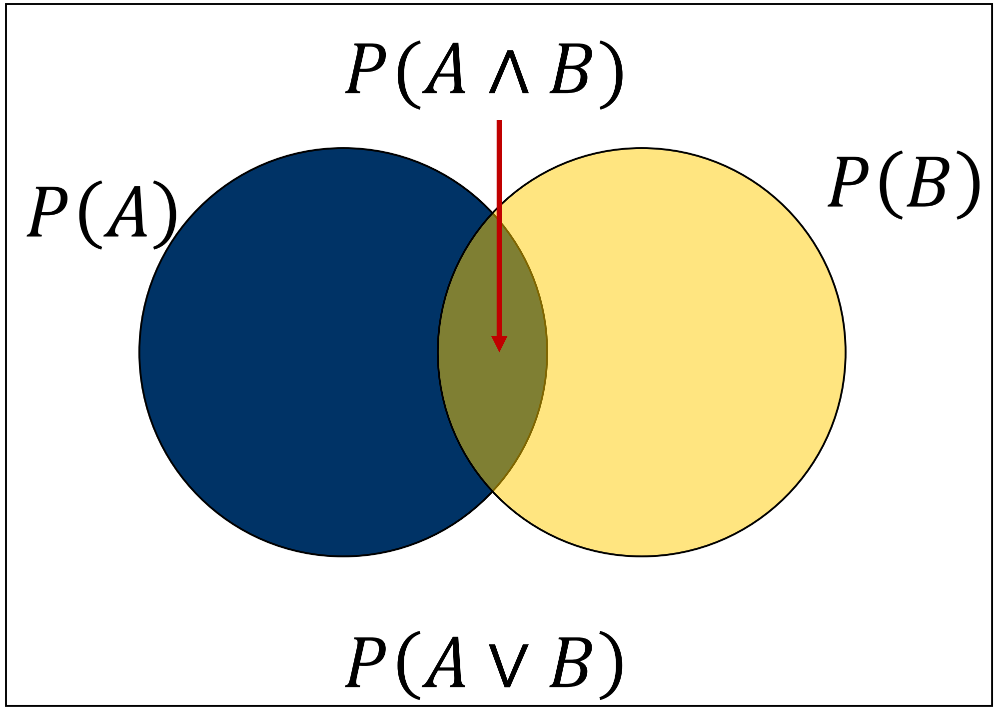
Axioms of Probability Theory

P(A) denotes probability that proposition A is true.

- $0 \leq P(A) \leq 1$
- $P(True) = 1$ $\qquad\qquad$ $P(False) = 0$
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

A Closer Look at Axiom 3

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$



Using the Axioms

$$
\begin{aligned}
P(A \lor \neg A) &= P(A) + P(\neg A) - P(A \land \neg A) \\
P(True) &= P(A) + P(\neg A) - P(False) \\
1 &= P(A) + P(\neg A) - 0 \\
P(\neg A) &= 1 - P(A)
\end{aligned}
$$

Discrete Random Variables

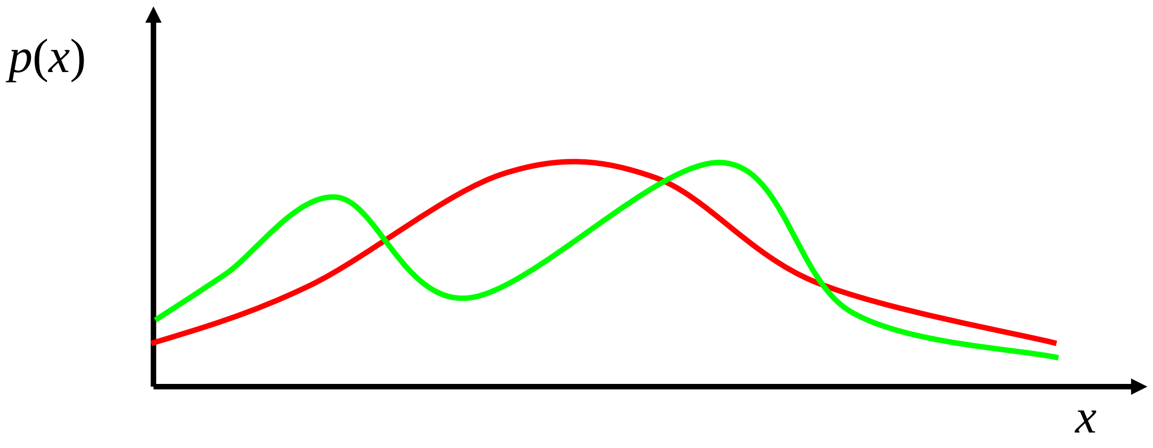
- X denotes a random variable
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable X takes on value x_i
- $P(\cdot)$ is called probability mass function
- E.g., $P(Room) =< 0.7, 0.2, 0.08, 0.02 >$

Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$ or $p(x)$ is a probability density function

$$P(x \in [a,b]) = \int_a^b p(x)dx$$

- E.g.



“Probability Sums up to One”

Discrete case

$$\sum_{x} P(x) = 1$$

Continuous case

$$\int_{X} P(x)dx = 1$$

Joint and Conditional Probability

- $P(X{=}x \text{ and } Y{=}y) = P(x,y)$

- If X and Y are independent then

$$P(x,y) = P(x)\,P(y)$$

- $P(x \mid y)$ is the probability of x given y

$$P(x \mid y) = P(x,y) \,/\, P(y)$$

$$P(x,y) = P(x \mid y)\,P(y)$$

- If X and Y are independent then

$$P(x \mid y) = P(x)$$

Law of Total Probability

Discrete case

$$P(x) = \sum_y P(\, x \mid y \,)P(y)$$

$P(x, y)$

Continuous case

$$p(x) = \int p(\, x \mid y \,)p(y)dy$$

Marginalization

Discrete case

$P(x) = \sum_y P(x, y)$

"1"

x \ y	1	2	3
1	✓	✓	✓
2			

Continuous case

$p(x) = \int p(x, y)\, dy$

Bayes Formula

$$P(x,y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\boxed{P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)}}$$

$$\frac{\text{Likelihood * Prior}}{\text{Evidence}}$$

Posterior

Normalization

$$P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)}$$

- At the same time: $P(y) = \sum_x P(y \mid x)P(x)$

$$P(x \mid y) = \frac{P(y \mid x)P(x)}{\sum_x P(y \mid x)P(x)}$$

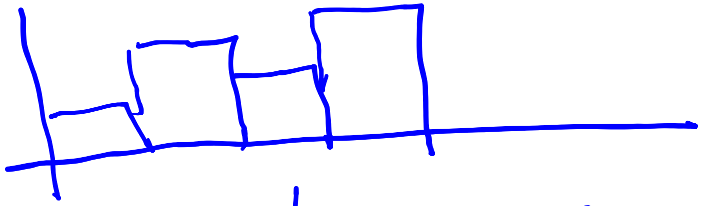
- $P(y)$ is independent of x and thus constant for all x

$$P(x \mid y) = \eta P(y \mid x)P(x)$$

Normalization (1)

$\sum_y p(x|y) \neq 1$ in most cases

$$\sum_x p(x) = 1$$



$$p(y|x) = 0.3 \quad \forall x$$

$$\sum_x p(x|y) = 1$$

$$= \sum_x \frac{p(y|x)\,p(x)}{p(y)}$$

$$\frac{1}{\eta} = p(y) = \boxed{\sum_x p(y|x)\,p(x)}$$

$$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)}$$

Normalization (2)

Bayes Rule with Background Knowledge

$$P(x \mid y, a) = \frac{P(y \mid x, a)P(x \mid a)}{P(y \mid a)}$$

$y_1, y_2, \ldots$

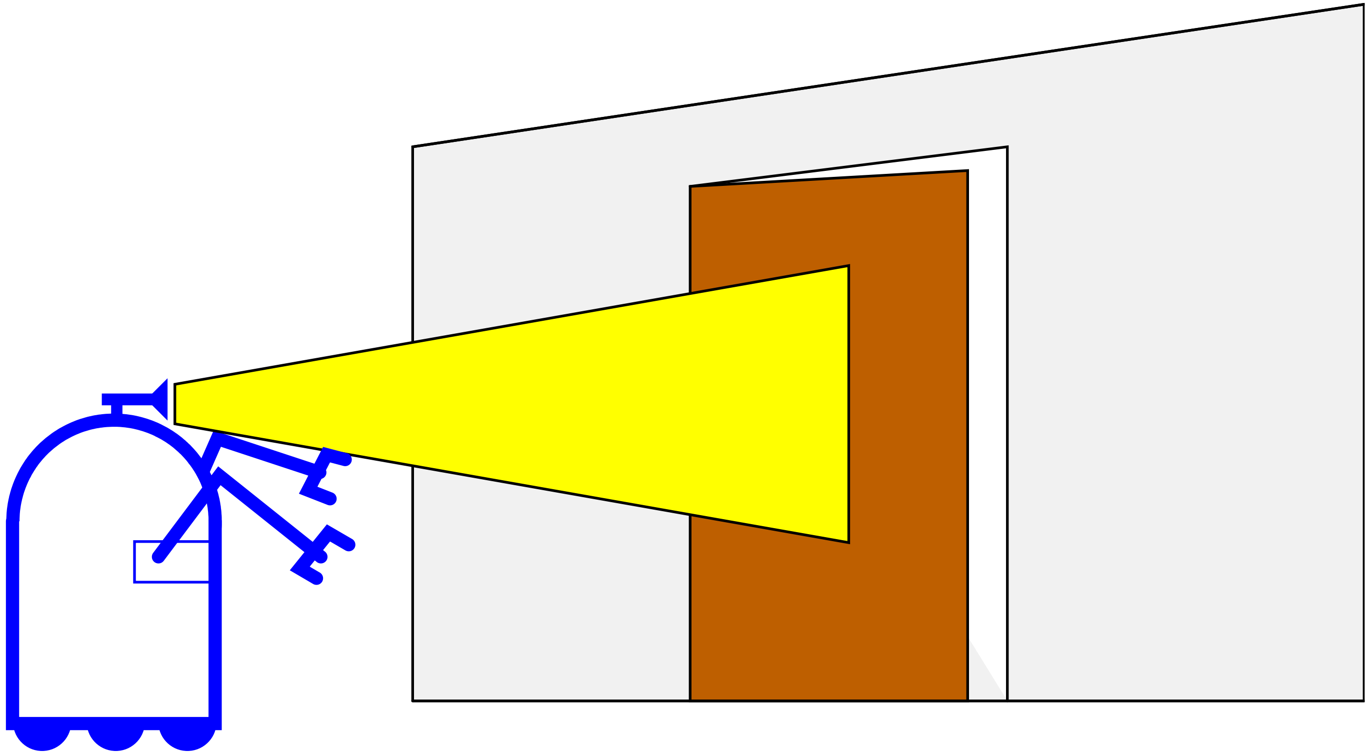
$P(x \mid a, y, y, \ldots)$

Conditional Independence

- $P(x, y \mid z) = P(x \mid z)P(y \mid z)$
- Equivalent to $P(x \mid z) = P(x \mid z, y)$ and $P(y \mid z) = P(y \mid z, x)$
- But this does not necessarily mean $P(x, y) = P(x)P(y)$
- Marginal independence does not mean independence

Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is $P(open \mid z)$?



Causal vs. Diagnostic Reasoning

- $P(open \mid z)$ is diagnostic
- $P(z \mid open)$ is causal
- In some situations, causal knowledge is easier to obtain

count frequencies!

- Bayes rule allows us to use causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

Example

- $P(z \mid open) = 0.6$ $\qquad P(z \mid \neg open) = 0.3$

- $P(open) = P(\neg open) = 0.5$

- $P(open \mid z) = \frac{P(z|open)P(open)}{P(z)} = \frac{0.6 * 0.5}{0.6 * 0.5 + 0.3 * 0.5} = \frac{2}{3}$

- z raises the probability that the door is open

Combining Evidence

- Suppose our robot obtains another observation z_2
- How can we integrate this new information?
- More generally, how can we estimate $P(x \mid z_1, \dots, z_n)$?

$$P(x \mid y, a) = \frac{P(y \mid x, a)\, P(x \mid a)}{P(y \mid a)}$$

Recursive Bayesian Updating

$$P(x \mid \underbrace{z_1, \ldots,}_{a} z_n) = \frac{P(z_n \mid x, \overbrace{z_1, \ldots, z_{n-1}}^{a})\, P(x \mid \overbrace{z_1, \ldots, z_{n-1}}^{a})}{P(z_n \mid \underbrace{z_1, \ldots, z_{n-1}}_{a})}$$

Markov assumption:

z_n is independent of $z_1, \ldots, z_{n-1}$ given we know x

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

$$= \alpha P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})$$

$P(z_{n-1} \mid x)$

$$= \alpha P(x) \prod_{i=1 \ldots n} P(z_i \mid x)$$

Example: Second Measurement

- $P(z_2 \mid open) = 0.25$ $P(z_2 \mid \neg open) = 0.3$
- $P(open \mid z_1) = \frac{2}{3}$

$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)P(open \mid z_1)}{P(z_2 \mid open)P(open \mid z_1) + P(z_2 \mid \neg open)P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{4} \cdot \frac{2}{3}}{\frac{1}{4} \cdot \frac{2}{3} + \frac{3}{10} \cdot \frac{1}{3}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{10}} = \frac{\frac{1}{6}}{\frac{4}{15}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open

Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world

- How can we **incorporate** such **actions**?

Typical Actions

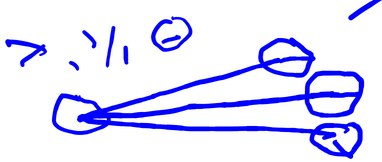
- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time** …

- Actions are **never carried out with absolute certainty**
- In contrast to measurements, **actions generally increase the uncertainty**

Modeling Actions

- To incorporate the outcome of an action *u* into the current "belief", we use the conditional pdf

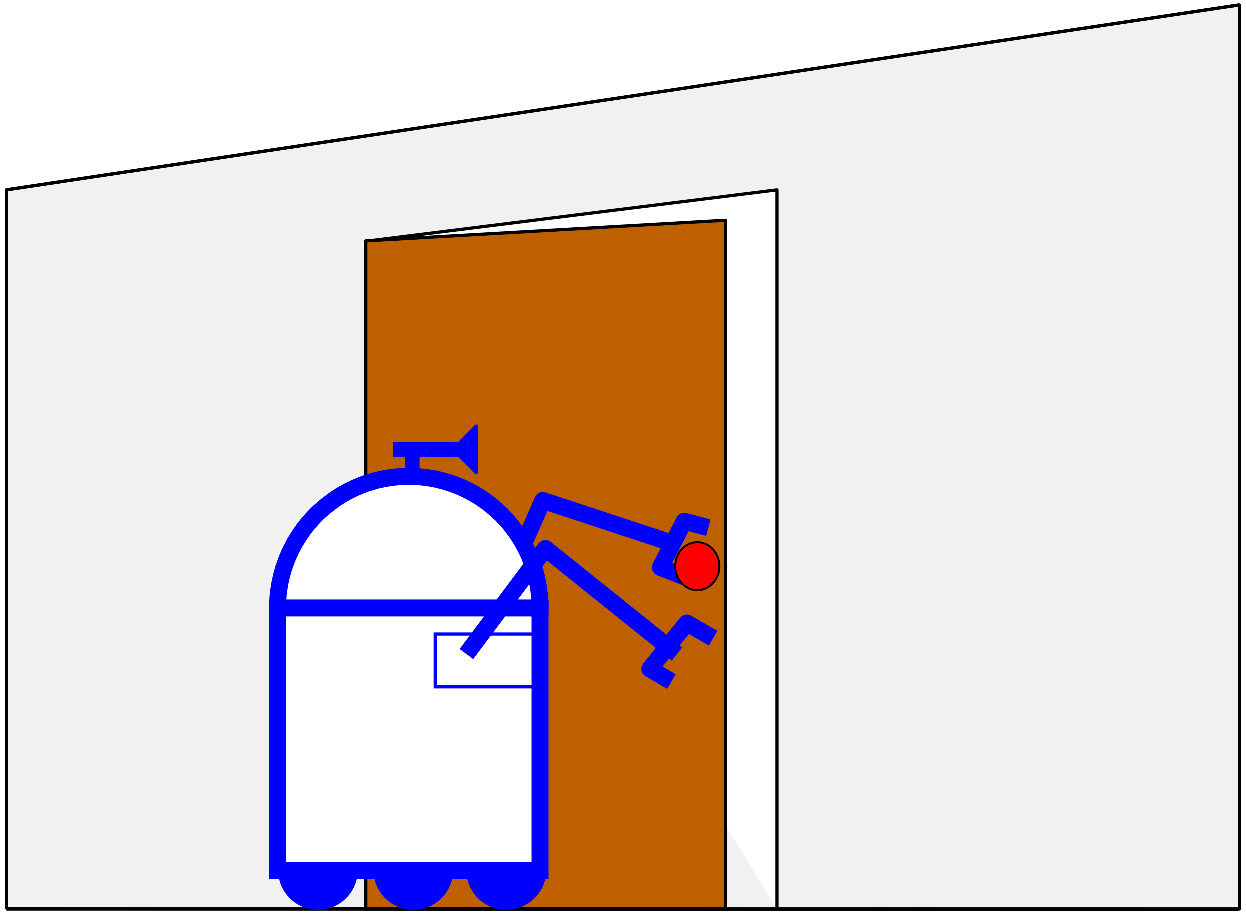
1m forward



$$\boxed{P(x \mid u, x')}$$

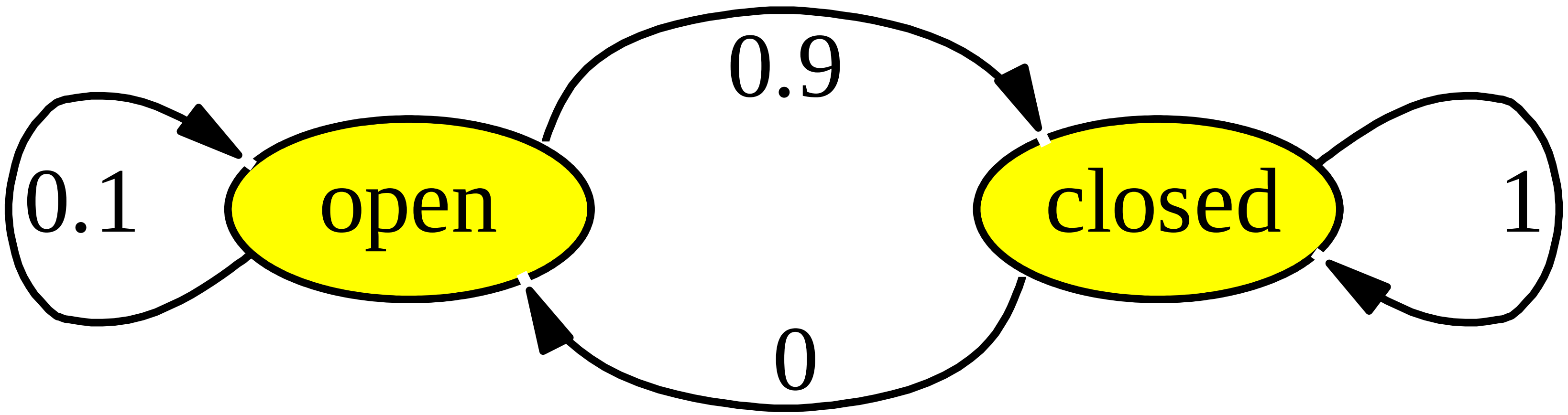
- This term specifies the pdf that **executing *u* changes the state from x' to x.**

Example: Closing the door



State Transitions

$P(x \mid u, x')$ for u = "close door":



If the door is open, the action "close door" succeeds in 90% of all cases

Integrating the Outcome of Actions

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')\, P(x' \mid \cancel{u})\, dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')\, P(x' \mid \cancel{u})$$

We will make an independence assumption to get rid of the u in the second factor in the sum.

Example: The Resulting Belief

$$P(closed \mid u) = \sum P(closed \mid u, x')P(x')$$

$$= P(closed \mid u, open)P(open) + P(closed \mid u, closed)P(closed)$$

$$= \frac{9}{10} \cdot \frac{5}{8} + \frac{1}{1} \cdot \frac{3}{8} = \frac{15}{16}$$

$$P(open \mid u) = \sum P(open \mid u, x')P(x')$$

$$= P(open \mid u, open)P(open) + P(open \mid u, closed)P(closed)$$

$$= \frac{1}{10} \cdot \frac{5}{8} + \frac{0}{1} \cdot \frac{3}{8} = \frac{1}{16}$$

$$= 1 - P(closed \mid u)$$

Bayes Filters: Framework

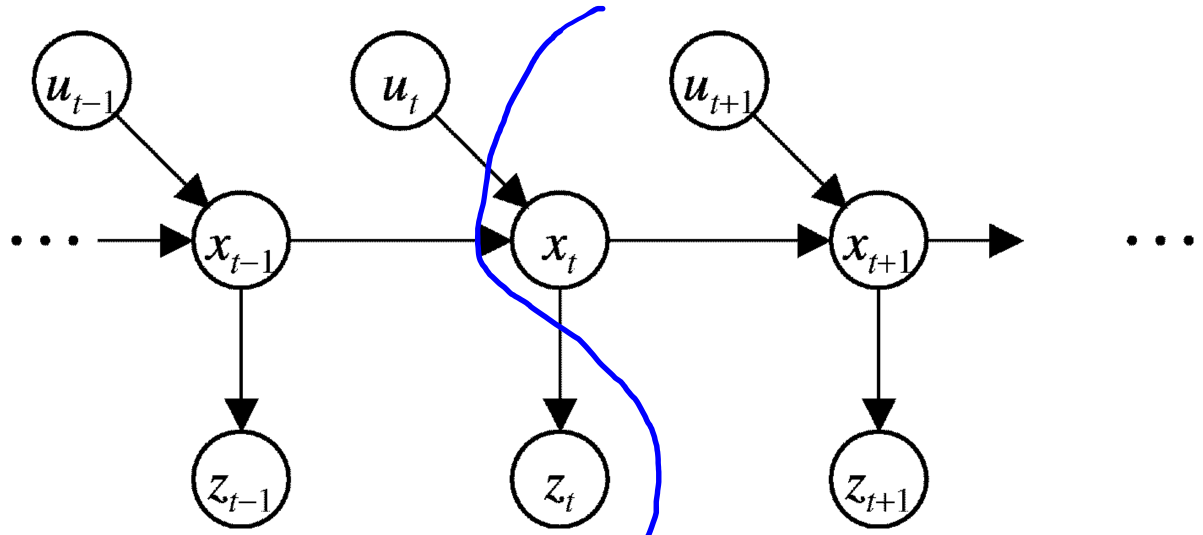
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_1, \dots, u_t, z_t\}$$

  - Sensor model $P(z \mid x)$
  - Action model $P(x \mid u, x')$
  - Prior probability of the system state $P(x)$
- **Wanted:**
  - Estimate of the state X of a dynamical system
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_1, \dots, u_t, z_t)$$

Markov Assumption



$$P(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = P(z_t \mid x_t)$$
$$P(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = P(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters Derivation (1)

Bayes Filters Derivation (2)

Bayes Filters Derivation (3)

Bayes Filters Derivation (4)

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1, \dots, u_t, z_t)$$

Bayes $$= \eta P(z_t \mid x_t, u_1, z_1, \dots, u_t) P(x_t \mid u_1, z_1, \dots, u_t)$$

Markov $$= \eta P(z_t \mid x_t) P(x_t \mid u_1, z_1, \dots, u_t)$$

Total prob. $$= \eta P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \dots, u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \dots, u_t) dx_{t-1}$$

Markov $$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \dots, u_t) dx_{t-1}$$

Markov $$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \dots, z_{t-1}) dx_{t-1}$$

$$\boxed{= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}}$$

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

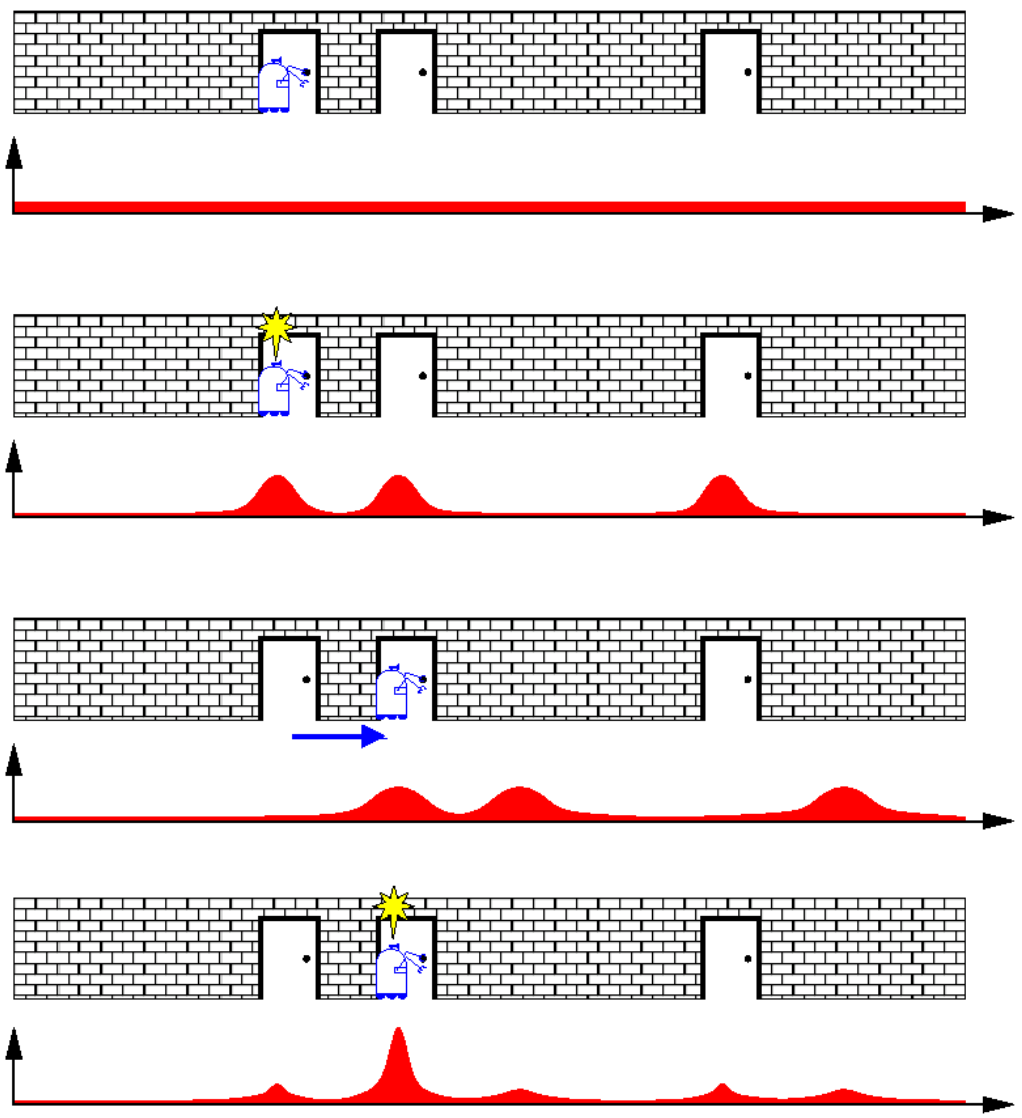
```
Algorithm Bayes_filter(Bel(x), d):
η=0
If d is a perceptual data item z then
    For all x do
        Bel'(x) = P(z | x) Bel(x)
        η = η + Bel'(x)
    For all x do
        Bel'(x) = η^{-1} Bel'(x)
Else if d is an action data item u then
    For all x do
        Bel'(x) = ∫ P(x | u, x') Bel(x') dx'
Return Bel'(x)
```

Bayes Filters are Familiar!

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

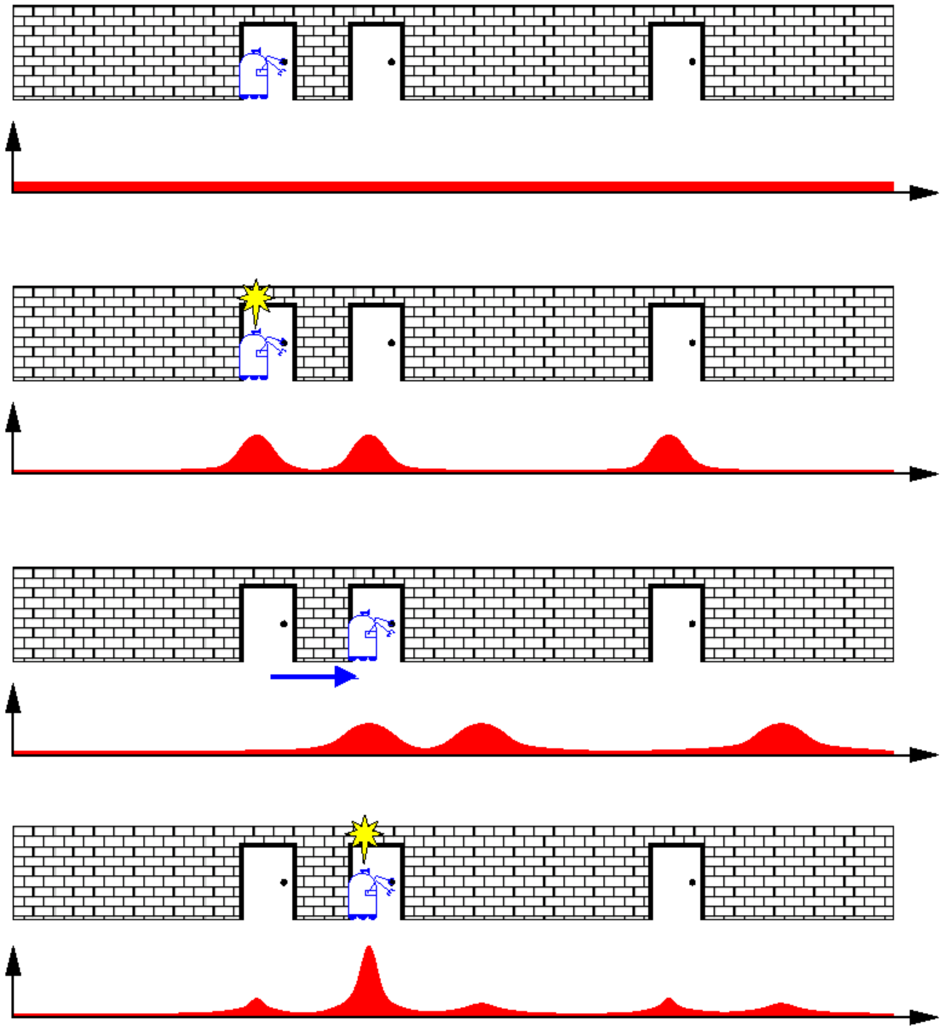
- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

Probabilistic Localization



Probabilistic Localization

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$



Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.