Introduction to Mobile Robotics

Bayes Filter – Extended Kalman Filter

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Bayes Filter Reminder

$$\boxed{Bel(x_t) = \eta \, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}}$$

- Prediction

$$\overline{Bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}$$

- Correction

$$Bel(x_t) = \eta \, p(z_t \mid x_t) \, \overline{Bel}(x_t)$$

Discrete Kalman Filter

Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + e_t}$$

with a measurement

$$\boxed{z_t = C_t x_t + d_t}$$

Components of a Kalman Filter

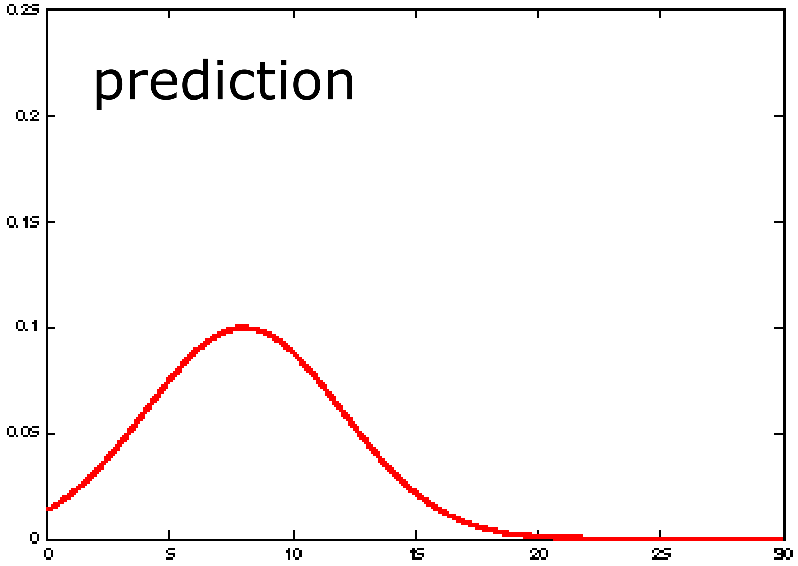
$\boxed{A_t}$ Matrix $(n \times n)$ that describes how the state evolves from t-1 to t without controls or noise.

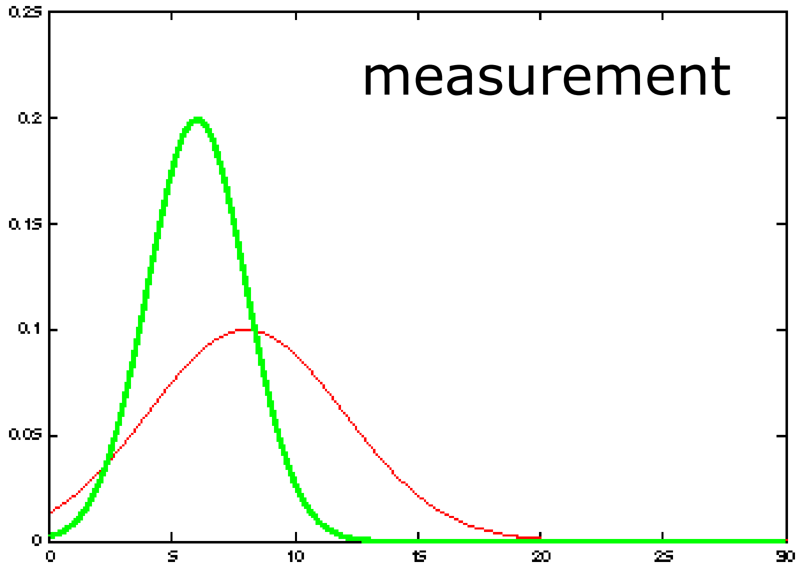
$\boxed{B_t}$ Matrix $(n \times l)$ that describes how the control u_t changes the state from t-1 to t.

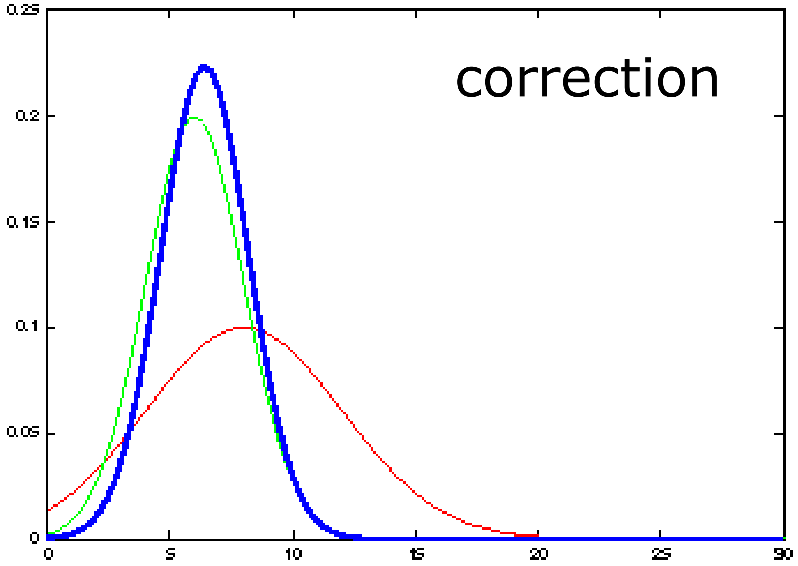
$\boxed{C_t}$ Matrix $(k \times n)$ that describes how to map the state x_t to an observation z_t.

$\boxed{e_t}$ $\boxed{d_t}$ Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance Q_t and R_t respectively.

Kalman Filter Updates in 1D

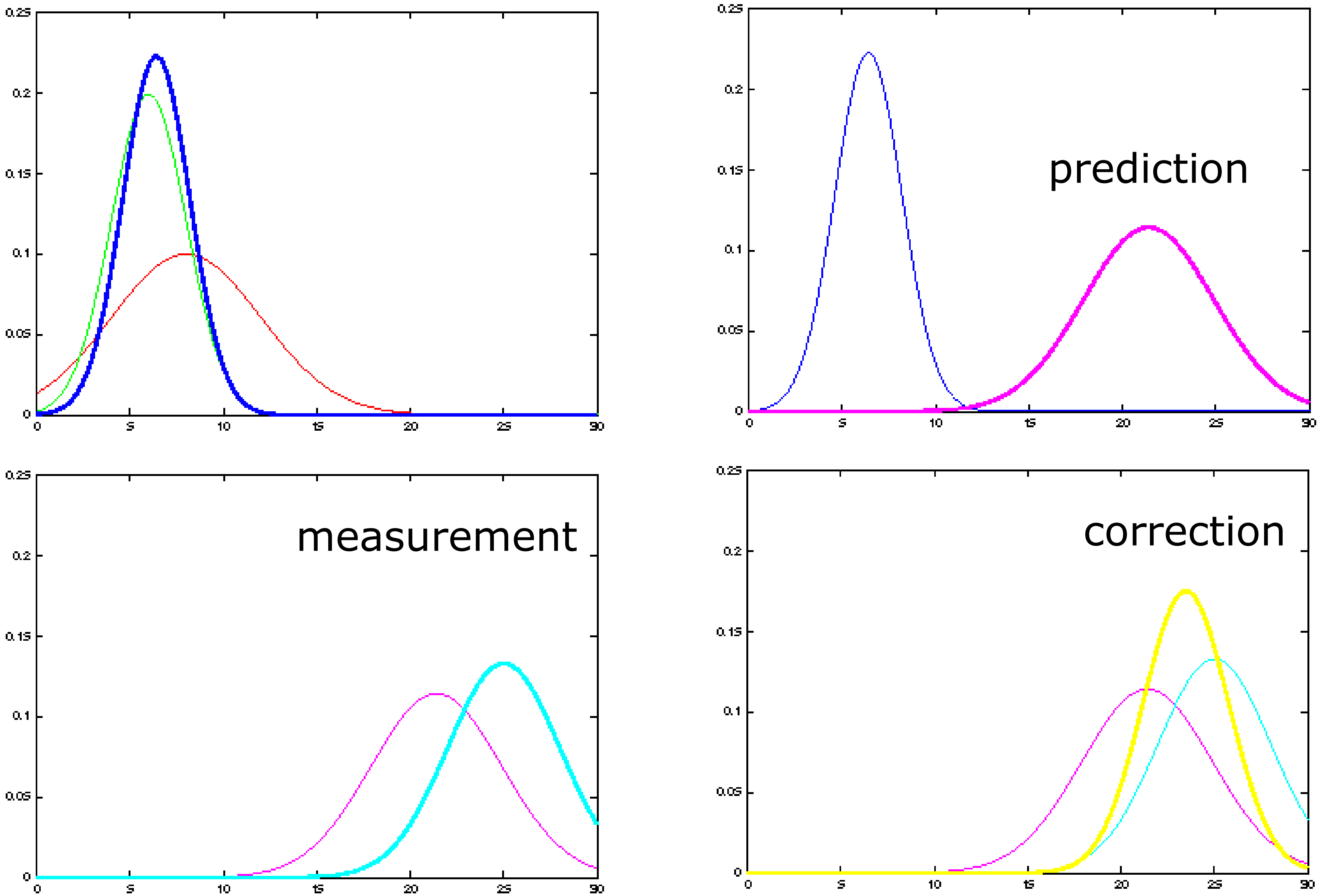






It's a weighted mean!

Kalman Filter Update Example



Kalman Filter Algorithm

1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

$$\boxed{\cancel{x_t = A_t x_{t-1} + B_t u_t + e_t}}$$

$$\Downarrow$$

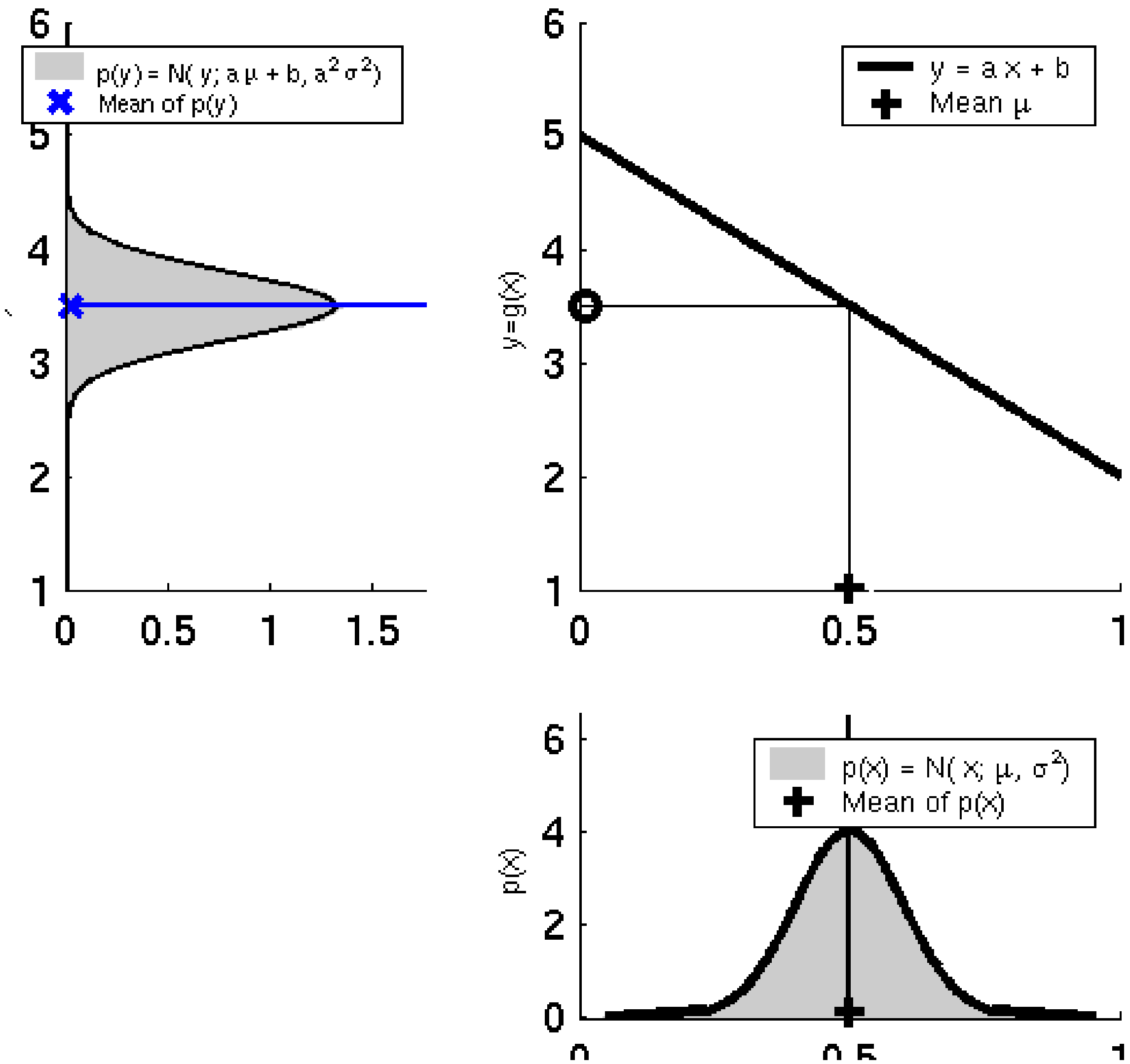
$$\boxed{x_t = g(u_t, x_{t-1})}$$

$$\boxed{\cancel{z_t = C_t x_t + d_t}}$$

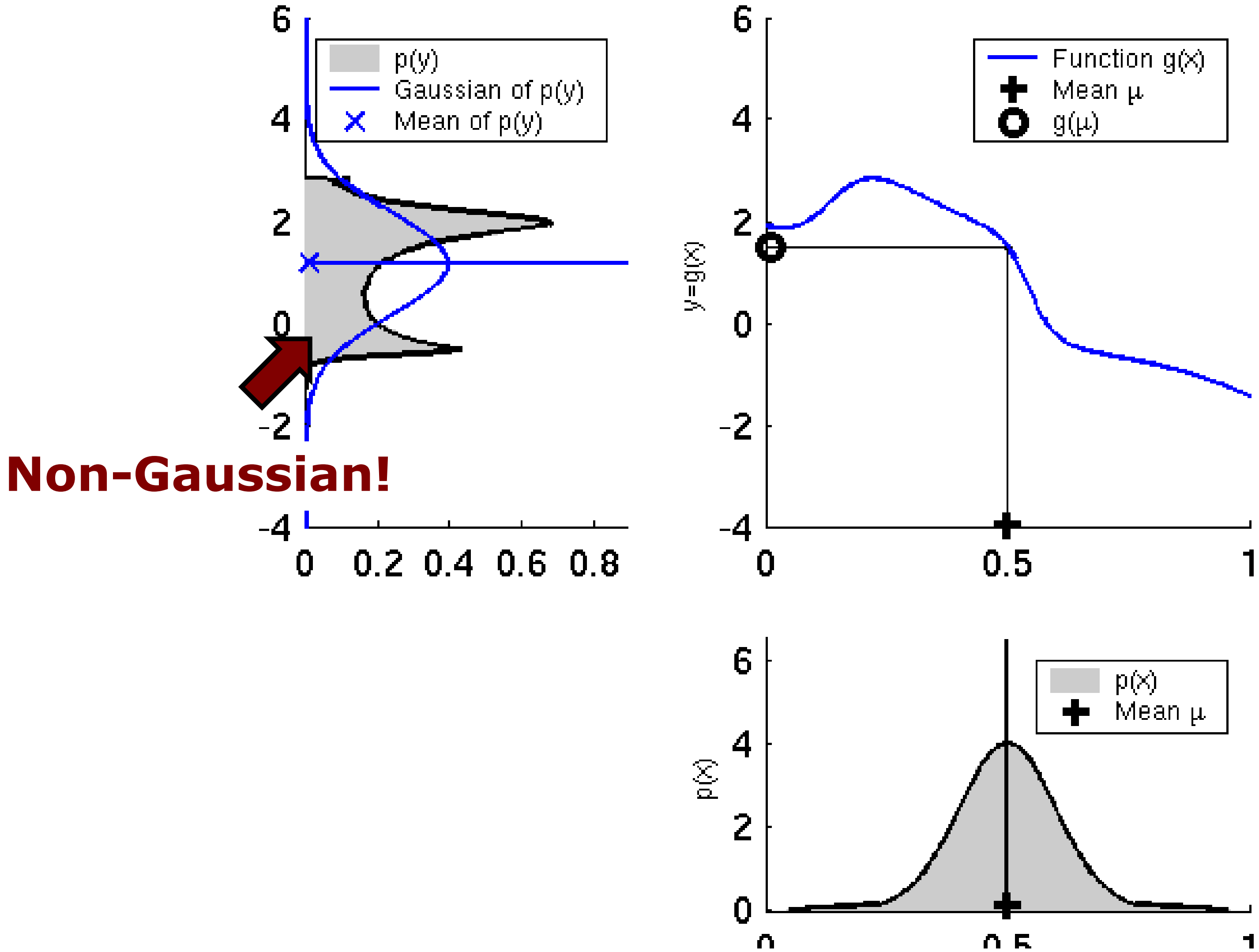
$$\Downarrow$$

$$\boxed{z_t = h(x_t)}$$

Linearity Assumption Revisited



Non-Linear Function



Non-Gaussian Distributions

- The non-linear functions lead to non-Gaussian distributions
- Kalman filter is not applicable anymore!

What can be done to resolve this?

Non-Gaussian Distributions

- The non-linear functions lead to non-Gaussian distributions
- Kalman filter is not applicable anymore!

What can be done to resolve this?

Local linearization!

EKF Linearization: First Order Taylor Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

Jacobian matrices

Reminder: Jacobian Matrix

- It, in general, is a $n \times m$ **non-square matrix**

- Given a vector-valued function

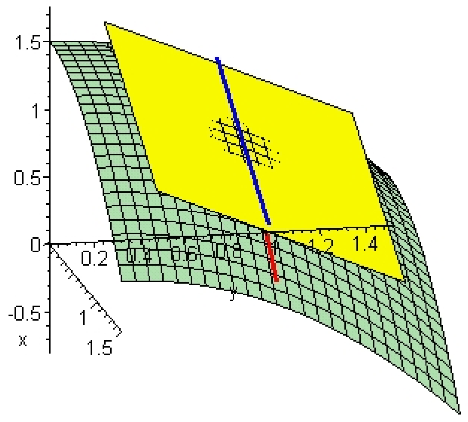
$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

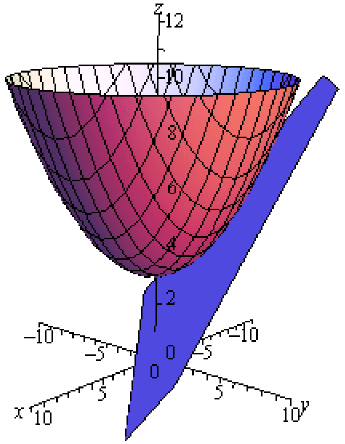
- The **Jacobian matrix** is defined as

$$\mathbf{F_x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Reminder: Jacobian Matrix

- It is the orientation of the tangent plane to the vector-valued function at a given point





- Generalizes the gradient of a scalar valued function

EKF Linearization: First Order Taylor Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

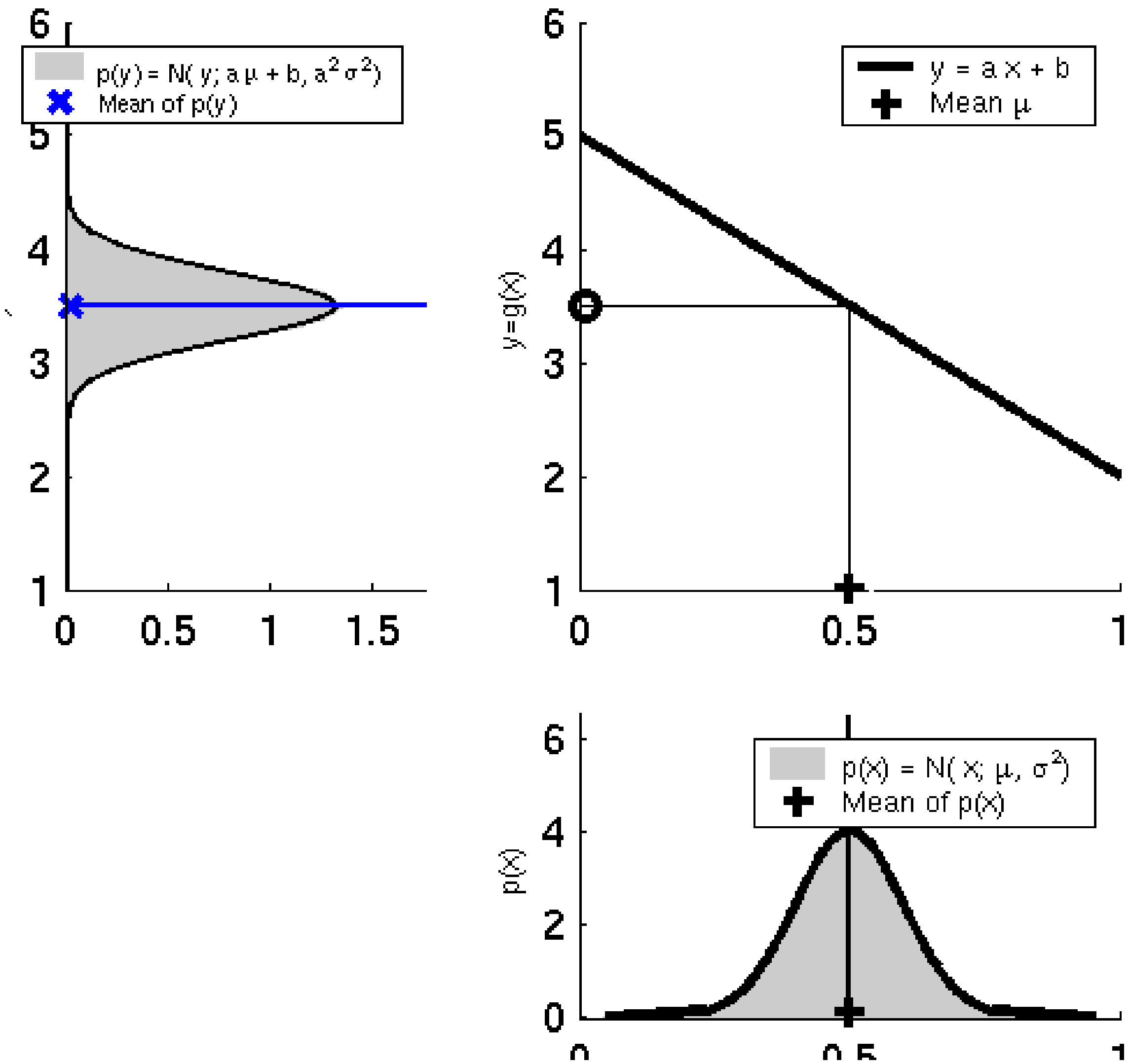
- Correction:

Linear function!

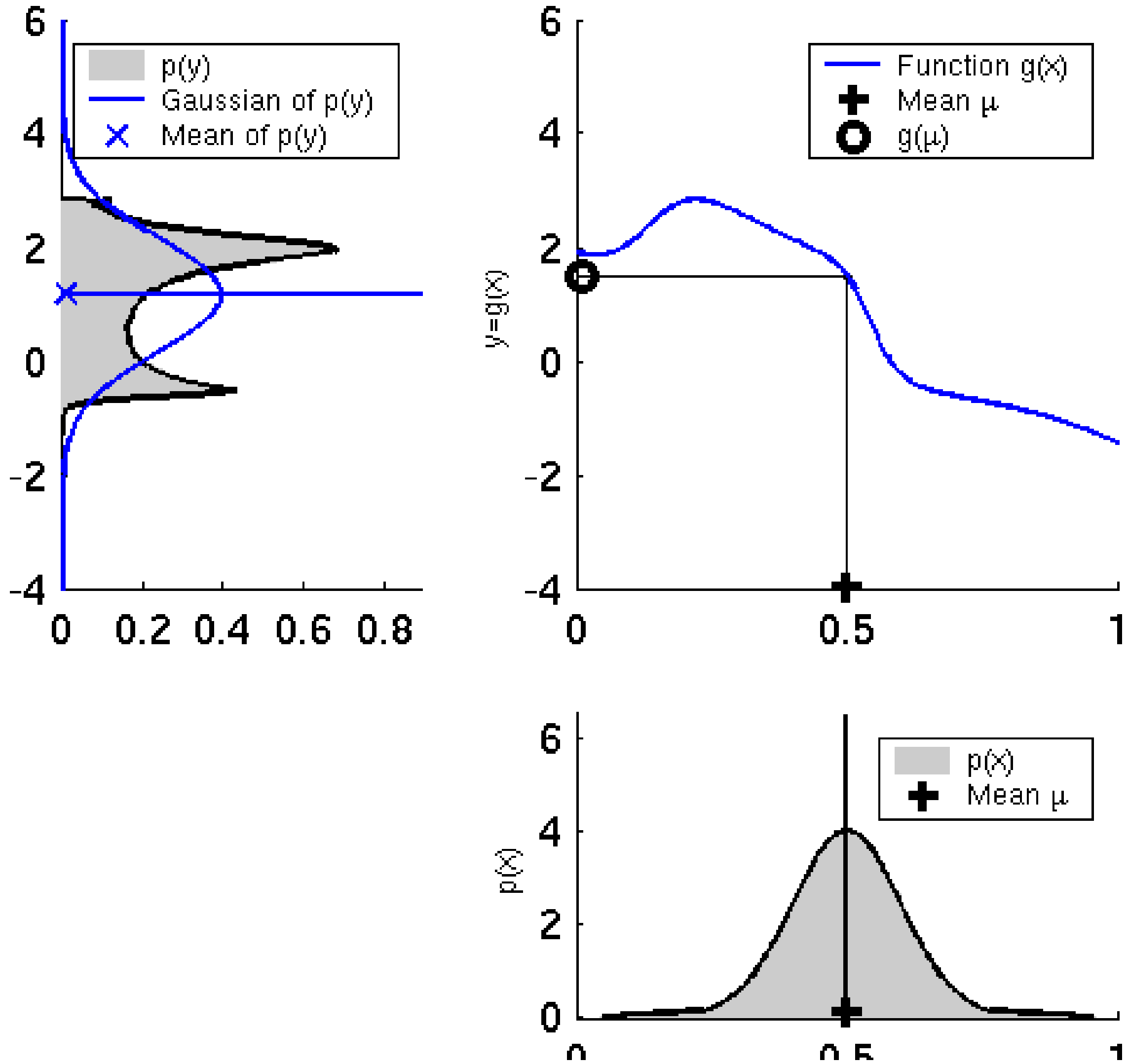
$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

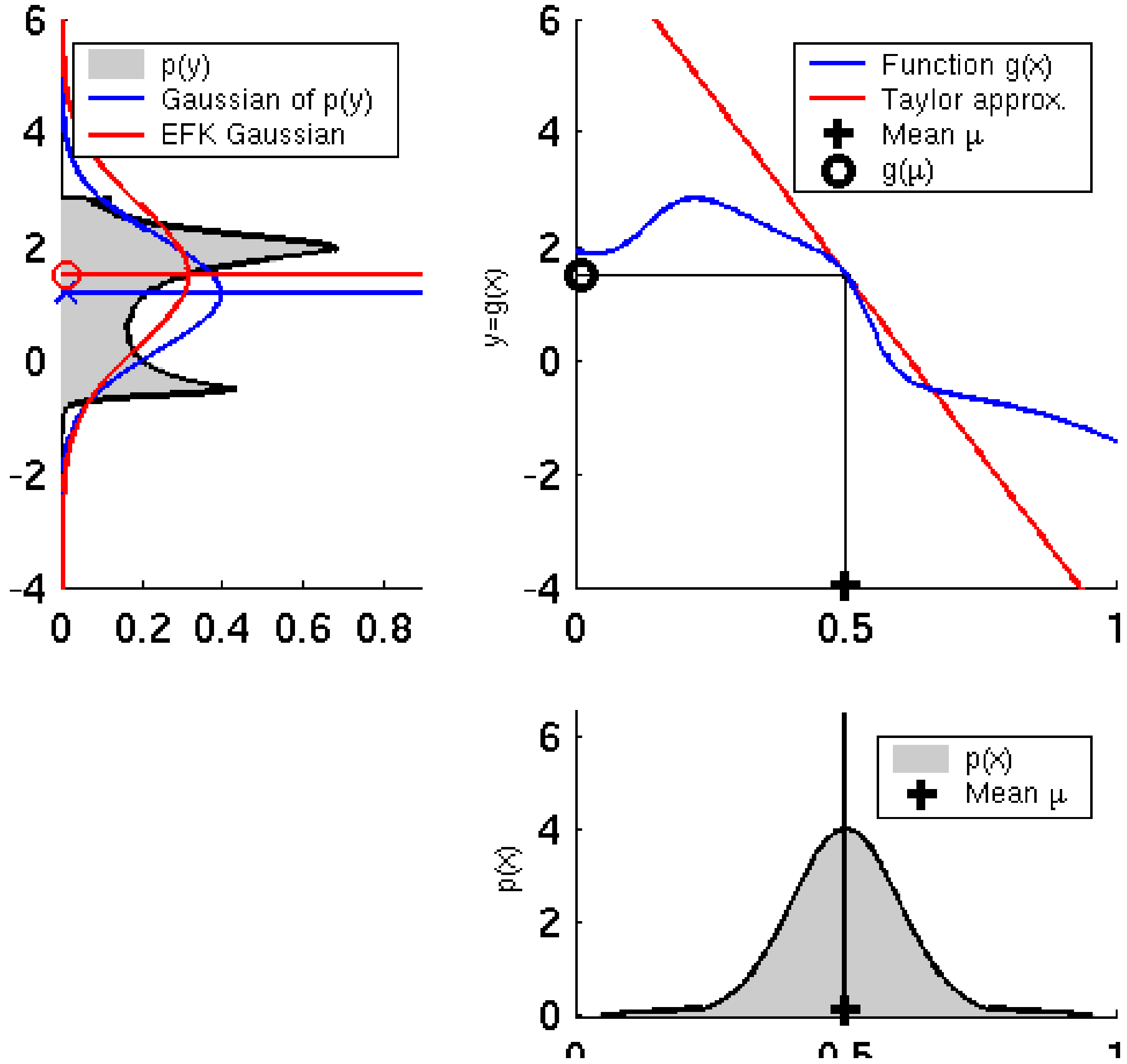
Linearity Assumption Revisited



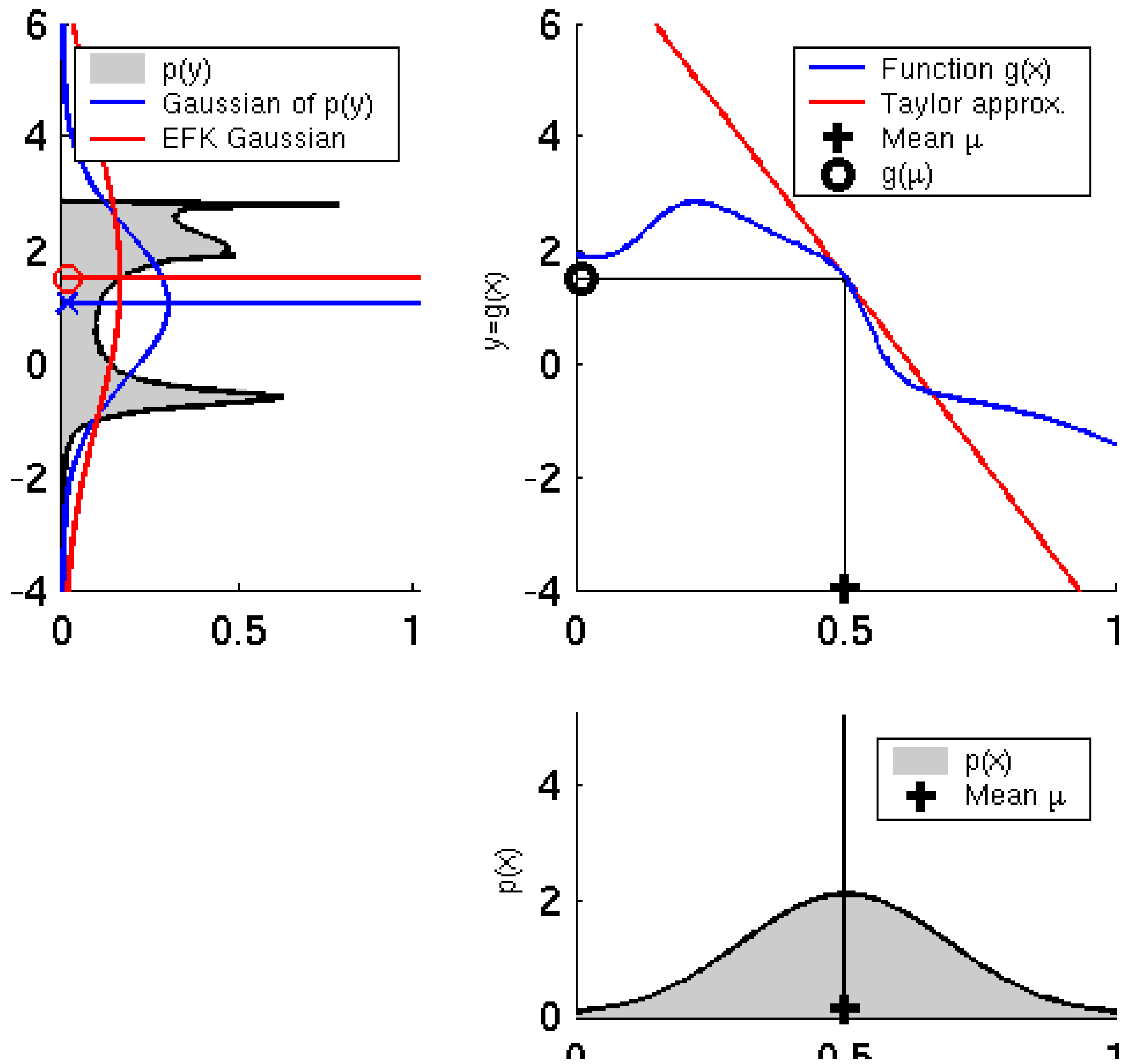
Non-Linear Function



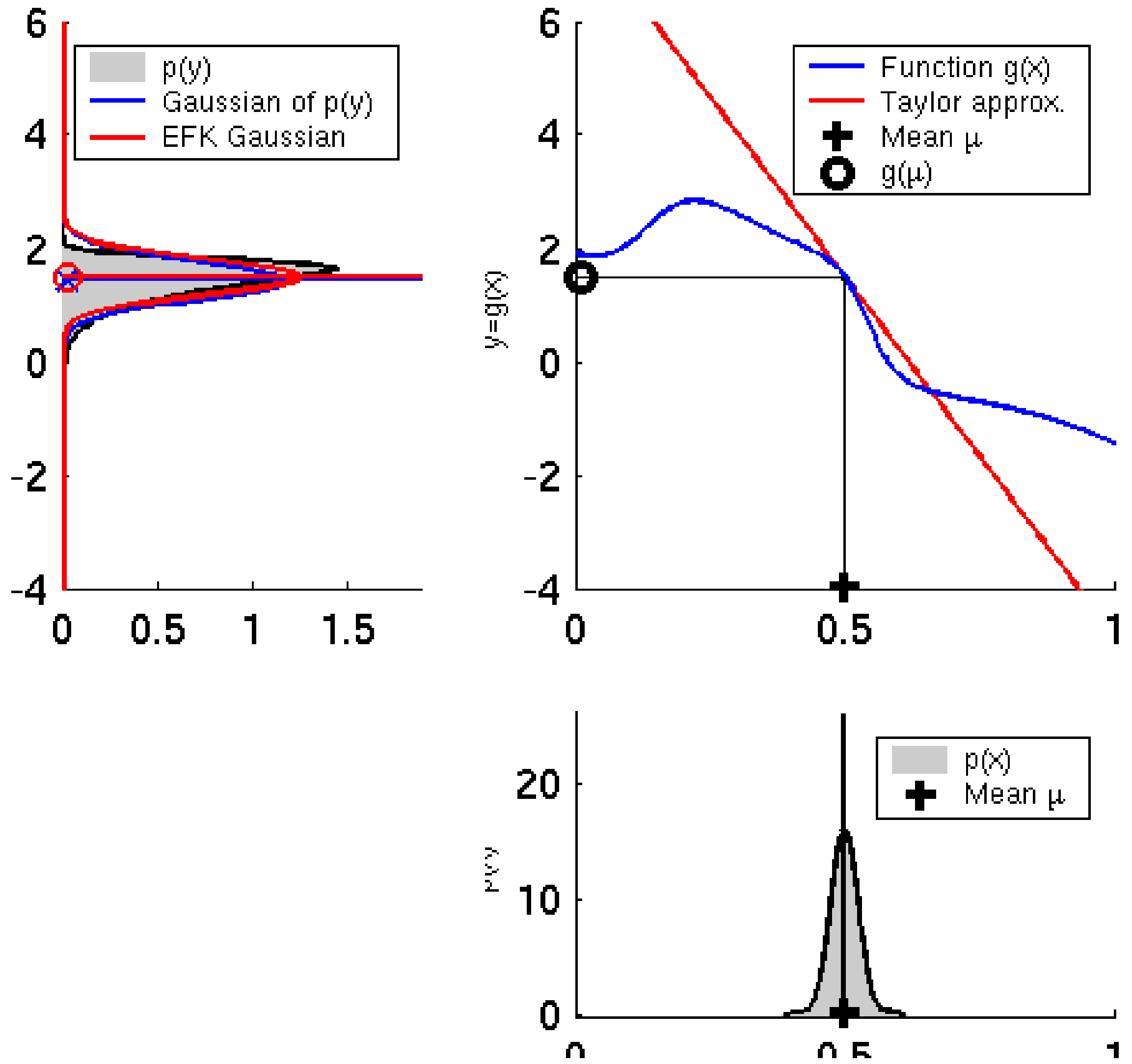
EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)



EKF Algorithm

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t}$$

$$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

1. **Extended_Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

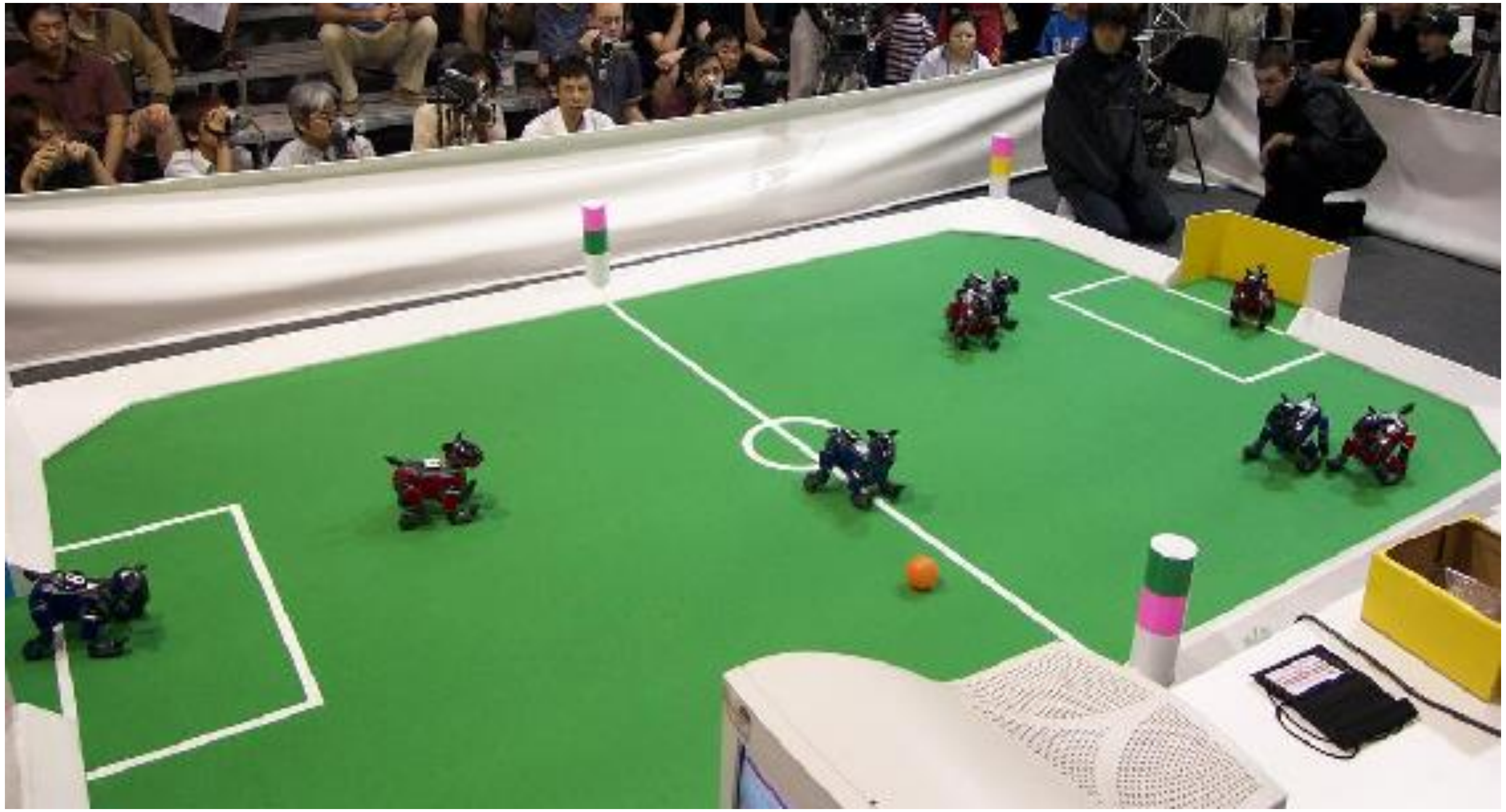
2. Prediction:
3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ $\longleftarrow$ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + Q_t$ $\longleftarrow$ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + R_t)^{-1}$ $\longleftarrow$ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ $\longleftarrow$ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ $\longleftarrow$ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

Example: EKF Localization

- EKF localization with landmarks (point features)



EKF_localization($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Prediction:

$$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial \mu_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$$

Jacobian of g w.r.t location

$$V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$

Jacobian of g w.r.t control

$$Q_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$

Motion noise

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

Predicted mean

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t Q_t V_t^T$$

Predicted covariance (V maps Q into state space)

EKF_localization$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m)$:

Correction:

$$\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

Predicted measurement mean (depends on observation type)

$$H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$$

Jacobian of *h* w.r.t location

$$R_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

$$S_t = H_t \bar{\Sigma}_t H_t^T + R_t$$

Innovation covariance

$$K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$$

Kalman gain

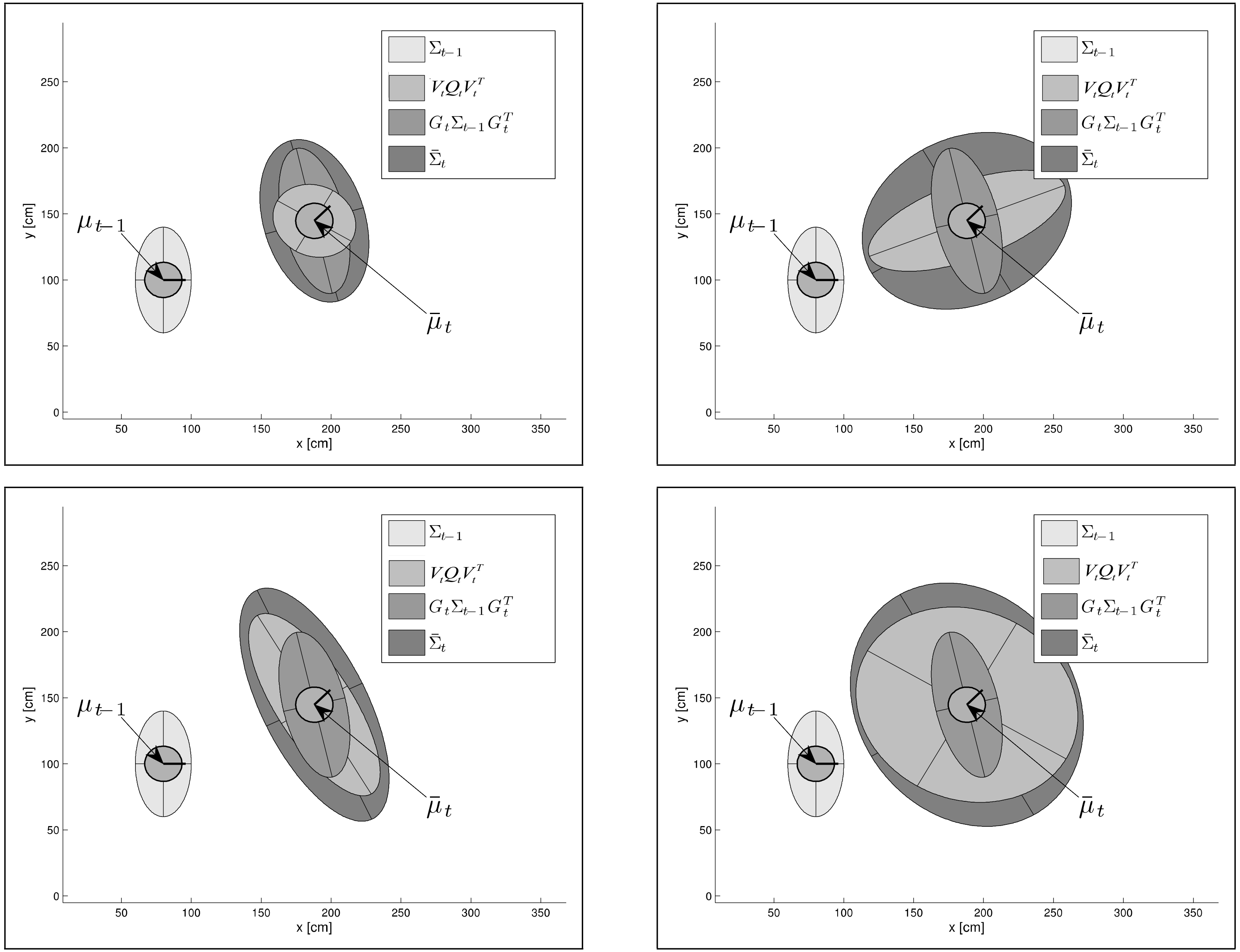
$$\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$$

Updated mean

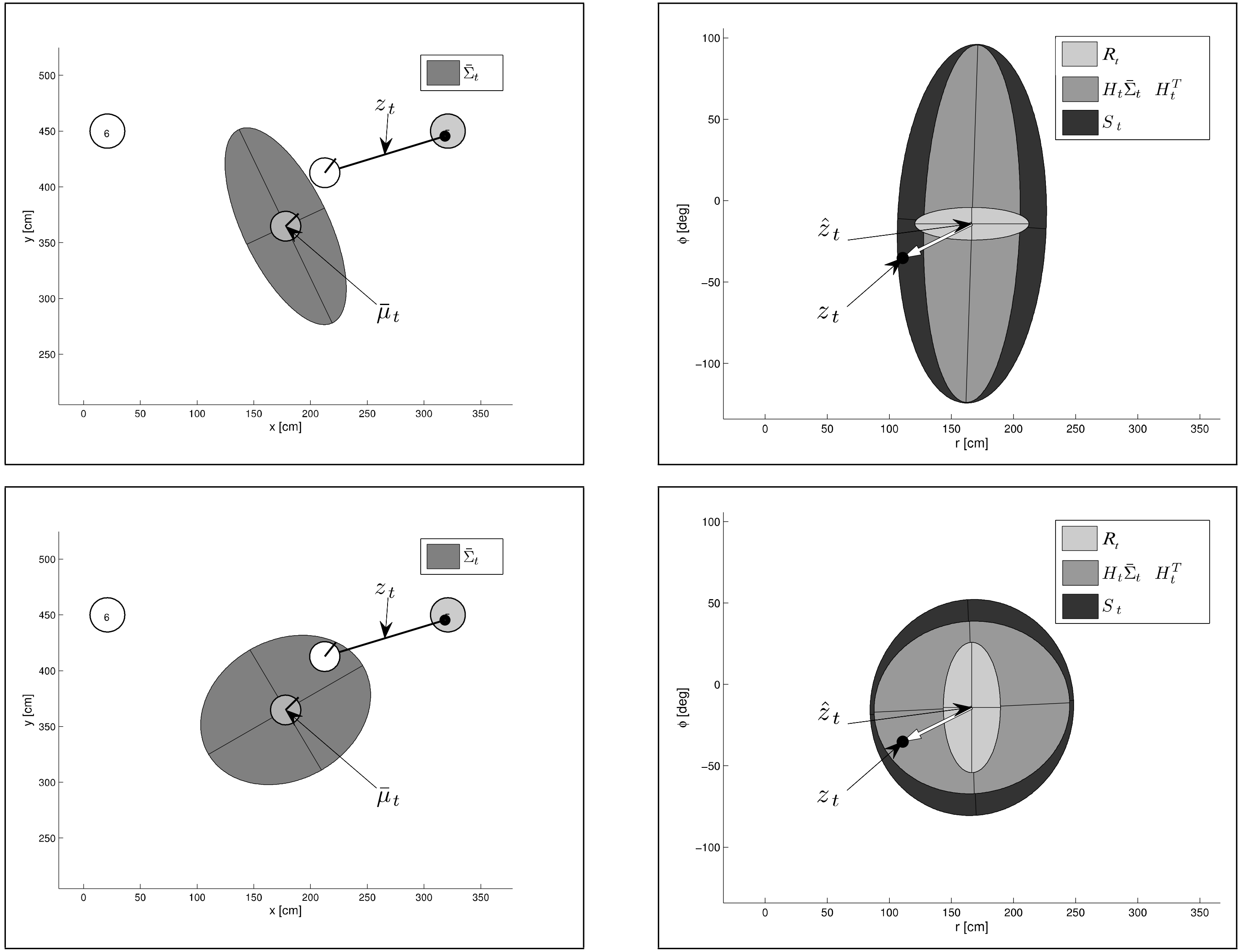
$$\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$$

Updated covariance

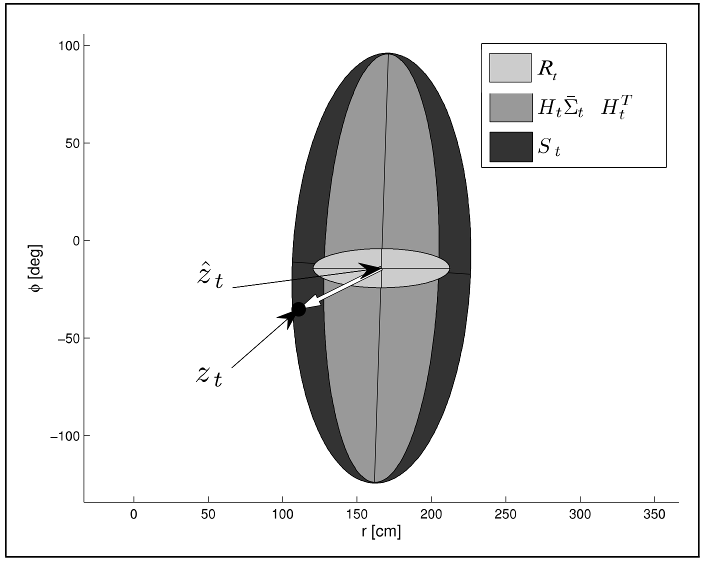
EKF Prediction Step Examples

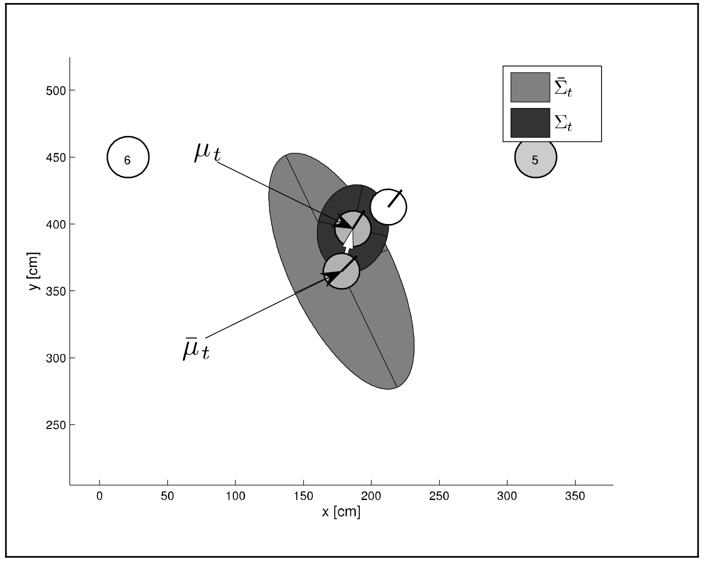


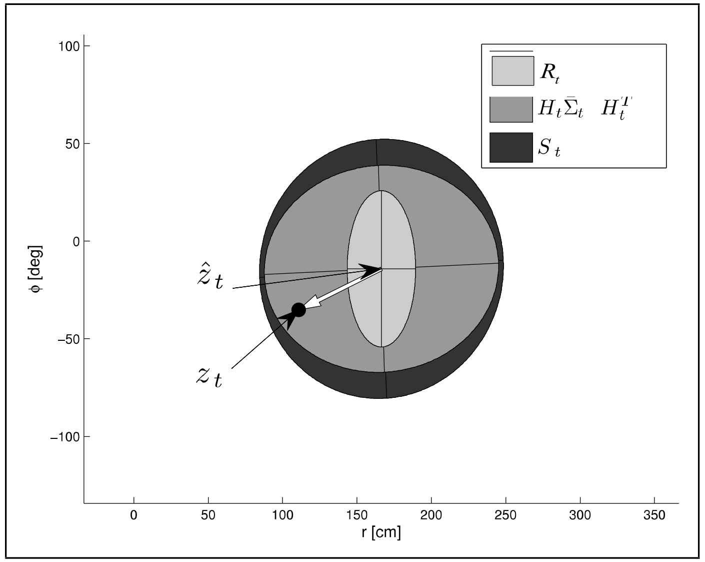
EKF Observation Prediction Step

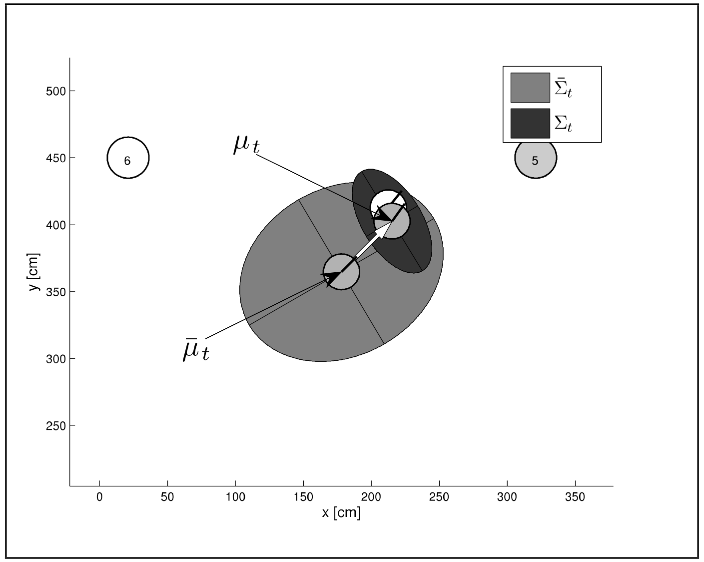


EKF Correction Step

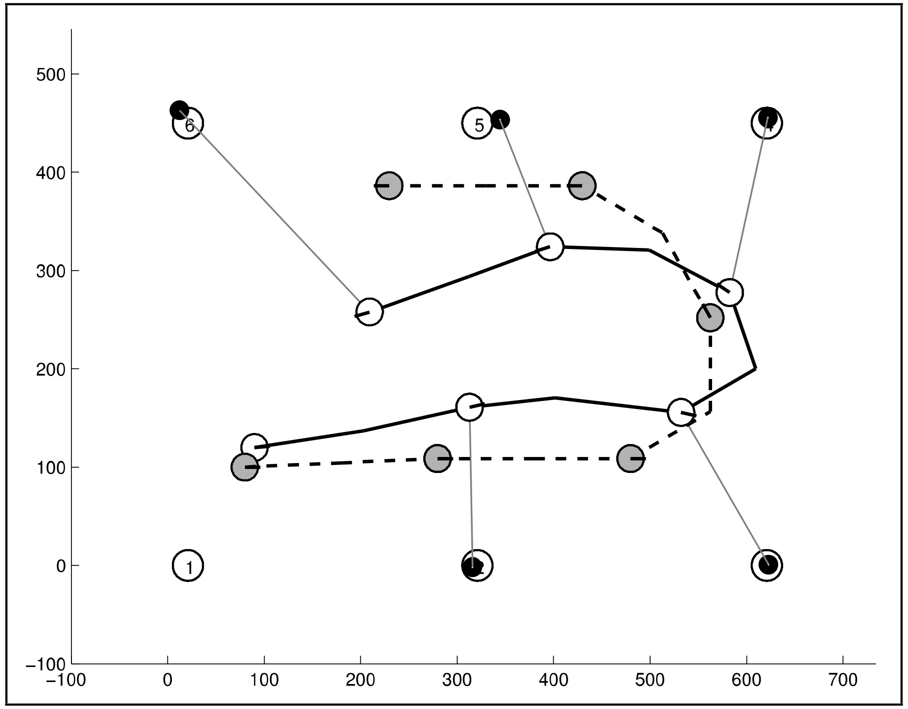


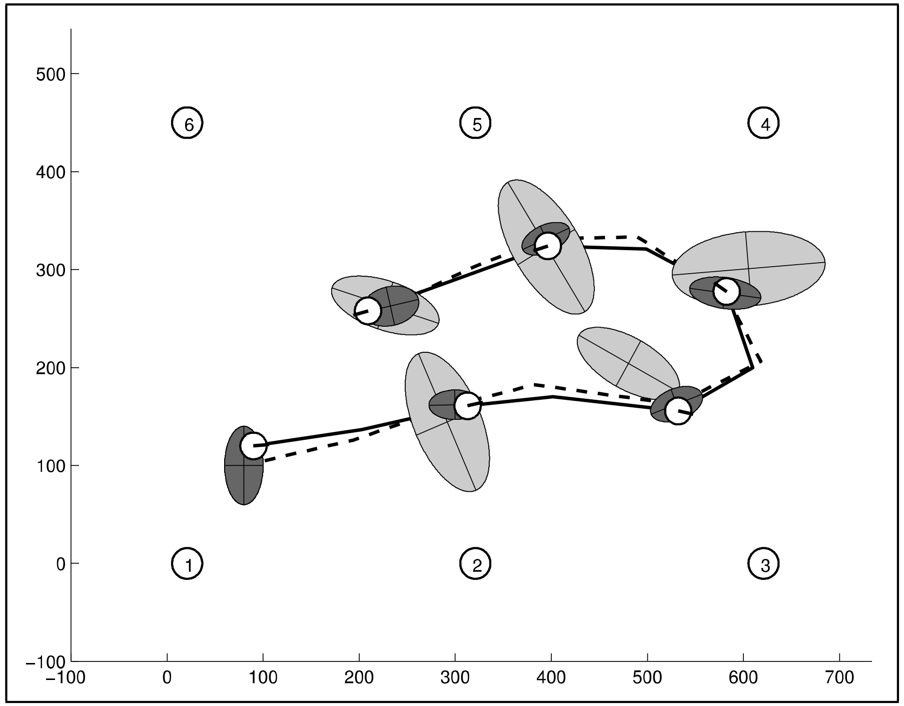




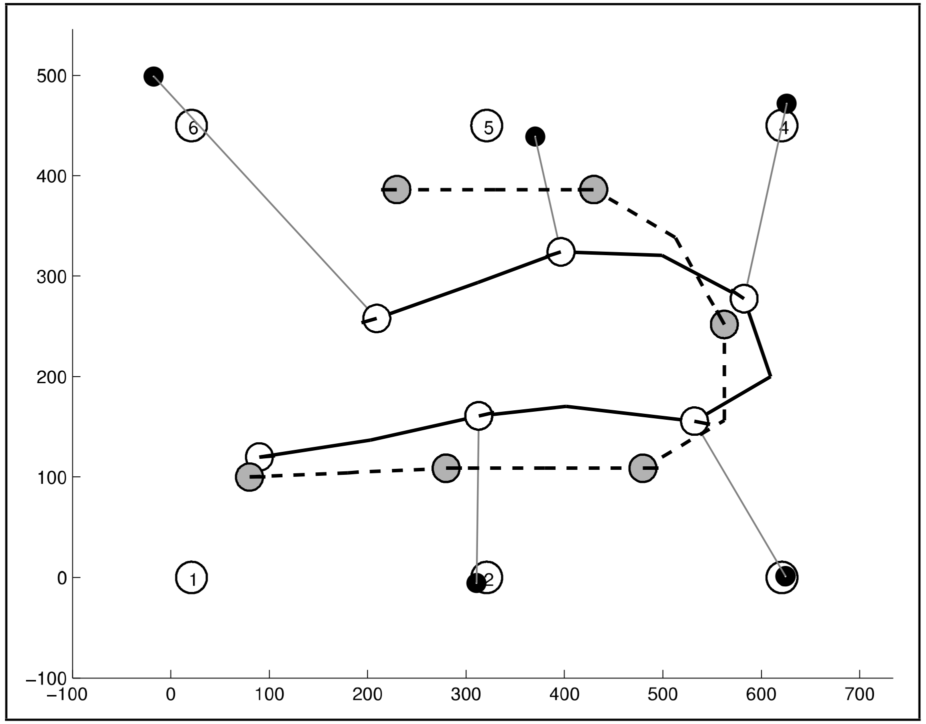


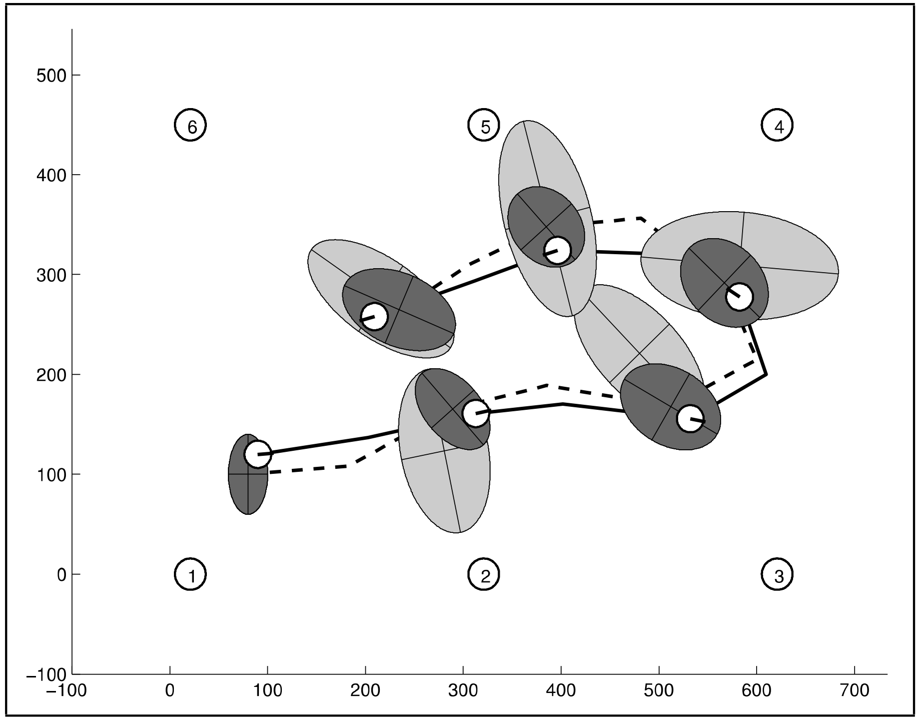
Estimation Sequence (1)





Estimation Sequence (2)





Extended Kalman Filter Summary

- The EKF is an ad-hoc solution to deal with non-linearities
- It performs local linearization in each step
- It works well in practice for moderate non-linearities (example: landmark localization)
- There exist better ways for dealing with non-linearities such as the unscented Kalman filter, called UKF
- Unlike the KF, the EKF in general is not an optimal estimator
- It is optimal if the measurement and the motion models are both linear, in which case the EKF reduces to the KF.