# Intervalos de confianza de nivel $1-\alpha$ para $\theta$

PIVOTE → es una fórmula de depende de $x_1, \dots, x_n$ y de $\theta$,
→ la distribución debe ser conocida y no puede depender de ningún parámetro desconocido, ni $\theta$.

1. Sea $X_1, \dots, X_n$ una muestra aleatoria con densidad dada por

$$f_X(x) = \frac{2x}{\theta^2} I_{(0,\theta]}(x) \quad \theta > 0.$$

$x < \theta$

$\frac{x}{\theta} < 1$

*a*) Verificar que $Y = -4 \ln\left(\frac{X}{\theta}\right)$ tiene distribución exponencial de parámetro $\lambda = 1/2$. ¿Qué distribución tiene $\sum_{i=1}^{n} Y_i$?

*b*) Hallar un intervalo de confianza de 90 % para $\theta$ si $\prod_{i=1}^{n \,(15)} X_i = 10$.

a) $F_Y(t) = P(Y \le t) = P\Big(\underbrace{-4 \underbrace{\ln\left(\tfrac{X}{\theta}\right)}_{<0}}_{>0} \le t\Big) \underset{t>0}{=} P\left(X \ge \theta e^{-\frac{t}{4}}\right) =$

Si $t<0$ $\; F_Y(t) = P(\emptyset) = 0$

$= 1 - F_X(\theta e^{-t/4})$ $\quad$ Derivamos

$f_Y(t) = -f_X(\underbrace{\theta e^{-t/4}}_{x}) \cdot \theta e^{-t/4}\left(-\frac{1}{4}\right) = + \frac{2\,\theta e^{-t/4}}{4\theta^2}\,\theta e^{-t/4} \cdot 1$

$= \frac{1}{2} e^{-t/2}$

$\cdot I_{(0,\theta]}(\theta e^{-t/4}) \equiv 1$ $\qquad e^{\overbrace{-t/4}^{<0}} < 1$ vale

$0 < \theta e^{-t/4} < \theta \iff$ (vale)

$$f_Y(t) = \begin{cases} \frac{1}{2} e^{-t/2} & t>0 \\ 0 & t \le 0 \end{cases}$$

$\Rightarrow Y \sim \mathcal{E}(1/2)$

$Y_i \sim \mathcal{E}(\lambda = \frac{1}{2})$ indep $\Rightarrow$ $\underline{\sum_{i=1}^{n} Y_i} \sim \Gamma(n, \lambda = \frac{1}{2}) = \underline{\Gamma(n, \frac{1}{2})} = \underline{\chi^2_{2n}}$

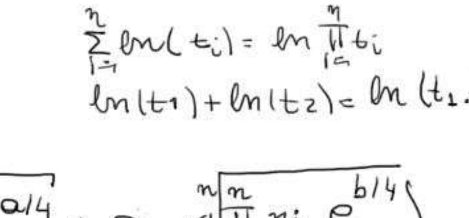
$\mathcal{E}(1/2) \sim \Gamma(1, \frac{1}{2})$

Prop $\quad Z \sim \Gamma(\alpha, \lambda) \Rightarrow aZ \sim \Gamma(\alpha, \frac{\lambda}{a})$

Recordar $\quad \chi^2_1 = \Gamma(\frac{1}{2}, \frac{1}{2})$,

$\chi^2_n = \Gamma(\frac{n}{2}, \frac{1}{2})$

$\chi^2_{2n} = \Gamma(\frac{2n}{2}, \frac{1}{2}) = \Gamma(n, \frac{1}{2})$




b) $\sum_{i=1}^{n} Y_i = \underbrace{\sum_{i=1}^{n} -4 \ln\left(\frac{X_i}{\theta}\right)}_{\text{PIVOTE}} \sim \chi^2_{2n}$ $\quad$ $n$ dato

$1-\alpha = P\Big(\underbrace{\chi^2_{2n,1-\frac{\alpha}{2}}}_{a} \le -4 \ln \prod_{i=1}^{n}\left(\frac{X_i}{\theta}\right) \le \underbrace{\chi^2_{2n,\frac{\alpha}{2}}}_{b}\Big)$ $\quad$ despejar $\theta$

$\sum_{i=1}^{n} \ln(t_i) = \ln \prod_{i=1}^{n} t_i$

$\ln(t_1) + \ln(t_2) = \ln(t_1 \cdot t_2)$

$1-\alpha = P\Big(e^{-a/4} \ge \underbrace{\prod_{i=1}^{n}\left(\frac{x_i}{\theta}\right)}_{\prod_{i=1}^{n} x_i/\theta^n} \ge e^{-b/4}\Big)$

$1-\alpha = P\left(\frac{1}{e^{-a/4}} \le \frac{\theta^n}{\prod_{i=1}^{n} x_i} \le \frac{1}{e^{-b/4}}\right) = P\Big(\sqrt[n]{\underbrace{\prod_{i=1}^{n} x_i}_{10}\, e^{a/4}} \le \theta \le \sqrt[n]{\underbrace{\prod_{i=1}^{n} x_i}_{10}\, e^{b/4}}\Big)$

$\frac{1}{x}$

$IC(\theta) = \Big[\sqrt[n]{\underbrace{\prod_{i=1}^{n} x_i}_{10}\, e^{a/4}}, \sqrt[n]{\prod_{i=1}^{n} x_i\, e^{b/4}}\Big] = [1.5867, 2.4182]$

$n = 15$ $\qquad a = 18{,}49266$

$b = 43{,}77297$

2. Se quiere realizar una encuesta de opinión para conocer la proporción de personas que están, al día de hoy, a favor de un determinado candidato. ¿Qué tamaño de muestra se deberá tomar para asegurar un nivel de confianza del 99 % con un error muestral del 3 %?

Hallar $n$

$$X_i = \begin{cases} 1 & \text{si el i-ésimo indiv está a favor de tal candidato} \\ 0 & \text{c.c} \end{cases}$$

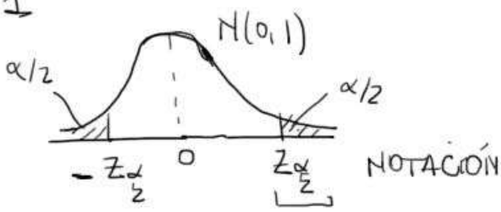
$$T = \underbrace{\sqrt{n}\,\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}}_{\text{PIVOTE}} \xrightarrow[n\to+\infty]{\mathcal{D}} \underline{N(0,1)}$$

$$\hat{p} = \overline{X}_n \xrightarrow[n\to\infty]{} p \quad \text{en } \underline{\text{proba}}$$

$$p = P(\text{"éxito"})$$

¿cómo despejar $p$ si usamos este PIVOTE?

$$\tilde{T} = \sqrt{n}\,\frac{\overline{X}_n - p}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} = \underbrace{\sqrt{n}\,\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}}_{\xrightarrow[\text{TCL}]{\mathcal{D}} N(0,1)} \cdot \underbrace{\frac{\sqrt{p(1-p)}}{\sqrt{\overline{X}_n(1-\overline{X}_n)}}}_{\xrightarrow[n\to\infty]{\text{prob}} 1} \xrightarrow{\mathcal{D}} N(0,1)$$




Prop (Slutsky)

$$\left.\begin{array}{l} Y_n \xrightarrow{\mathcal{D}} Y \\ U_n \xrightarrow{P} a \in \mathbb{R} \end{array}\right\} \Rightarrow U_n Y_n \xrightarrow{\mathcal{D}} aY$$

$$\lim_{n\to\infty} P\left( -Z_{\frac{\alpha}{2}} \le \sqrt{n}\,\frac{\overline{X}_n - p}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \le Z_{\frac{\alpha}{2}} \right) = 1-\alpha$$

$X_1, \dots, X_n$

despejar $p$

$$-Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \le \overline{X}_n - p \le Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}$$

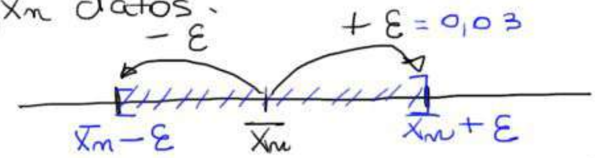
$$-a \le b \le a$$
$$a \le -b \le -a$$

$$-Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \le p - \overline{X}_n \le \le Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}$$

$$\le p \le$$

$$IC^{(\alpha)}(p) = \left[ \overline{X}_n - Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}},\ \overline{X}_n + \underbrace{Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}}_{\varepsilon = \text{error muestral}} \right] \subset [0,1]$$

$n, X_1, \dots, X_n$ datos.




$$\text{"Long } IC(p) = 2\varepsilon = 2 Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}\text{"}$$

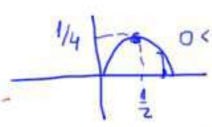
depende de $X_1, \dots, X_n$

$E(\text{Long } IC(p))$ en gral se trabaja con la $E(\text{long } IC(p))$

$$\boxed{\varepsilon \le 0{,}03}$$

$$Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \overset{?}{\le} 0{,}03$$

$f(x) = x(1-x)$

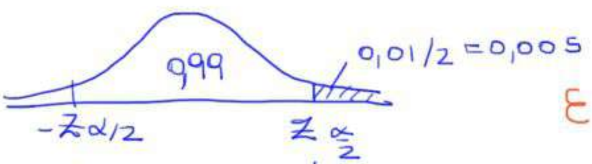



$Z_{\frac{\alpha}{2}}$ $\quad 1-\alpha = 0{,}99$

$$Z_{\frac{\alpha}{2}}\,\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \le Z_{\frac{\alpha}{2}}\,\frac{\sqrt{1/4}}{\sqrt{n}}$$

$x(1-x) \le 1/4$

$0 \le \overline{X}_n < 1$




$$\varepsilon \le \frac{Z_{\frac{\alpha}{2}}}{2\sqrt{n}} \le 0{,}03 \iff n \ge \left(\frac{Z_{\alpha/2}}{2 \cdot 0{,}03}\right)^2$$

0,995

$n \ge \dots$

$$\Phi(Z_{\alpha/2}) = 0.995 \qquad \text{qnorm}(0.995) = Z_{\alpha/2}$$

3. Sea $X_1, \ldots, X_n$ una muestra aleatoria con distribución exponencial de parámetro $\lambda$.

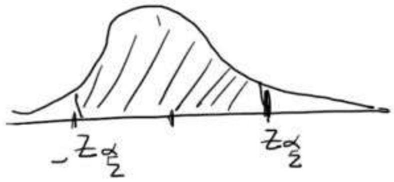
*a)* Hallar un intervalo de confianza de nivel exacto $1-\alpha$ para $\lambda$. Sug: ver clases teóricas 2020 y en las notas de Bianco-Martinez.

*b)* Hallar dos intervalos de confianza de nivel asintótico $1-\alpha$ para $\lambda$. Sug: Usar LGN+Slutsky en uno de ellos.

*c)* Hallar dos intervalos de confianza de nivel asintótico $1-\alpha$ para $1/\lambda$. ¿Cuál de los dos intervalos elegiría si lo que importa es la precisión? ¿Hay algo raro en todo esto?

$E(X_i) = \frac{1}{\lambda} \qquad V(X_i) = \frac{1}{\lambda^2}$

TCL $\quad \underbrace{\sqrt{n}\,\frac{(\overline{X}_n - 1/\lambda)}{1/\lambda}}_{\text{PIVOTE}} \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1)$




$$\sqrt{n}\,(\lambda \overline{X}_n - 1) \overset{(a)}{\approx} N(0,1)$$

$$1-\alpha \overset{(a)}{\cong} P\left(-z_{\frac{\alpha}{2}} \le \sqrt{n}\,(\lambda \overline{X}_n - 1) \le z_{\frac{\alpha}{2}}\right)$$

$$1-\alpha \overset{(a)}{\cong} P\left(\frac{1}{\overline{X}_n}\left(1 - \frac{z_{\alpha/2}}{\sqrt{n}}\right) \le \lambda \le \frac{1}{\overline{X}_n}\left(1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right)\right)$$

$$\overset{(a)}{\cong} P\Bigg(\underbrace{\frac{\overline{X}_n}{1 + \frac{z_{\alpha/2}}{\sqrt{n}}} \le \frac{1}{\lambda} \le \frac{\overline{X}_n}{1 - \frac{z_{\alpha/2}}{\sqrt{n}}}}\Bigg) \qquad IC\left(\tfrac{1}{\lambda}\right) = IC1$$

$f(x) = \frac{1}{x}$

2º Intervalo

$\overline{X}_n \longrightarrow E(X_1) = \frac{1}{\lambda}$

TCL $\quad \sqrt{n}\,\frac{(\overline{X}_n - 1/\lambda)}{1/\lambda} \overset{(a)}{\approx} N(0,1)$

$$\tilde{T} = \sqrt{n}\,\frac{(\overline{X}_n - 1/\lambda)}{\overline{X}_n} = \underbrace{\sqrt{n}\,\frac{(\overline{X}_n - 1/\lambda)}{1/\lambda}}_{\xrightarrow[n\to\infty]{\mathcal{D}} N(0,1)} \;\underbrace{\frac{1/\lambda}{\overline{X}_n}}_{\xrightarrow[n\to\infty]{} 1} \xrightarrow{\mathcal{D}} N(0,1)$$

$$1-\alpha \overset{a}{\cong} P\left(-z_{\frac{\alpha}{2}} \le \sqrt{n}\,\frac{\overline{X}_n - 1/\lambda}{\overline{X}_n} \le z_{\frac{\alpha}{2}}\right)$$

$$\vdots$$

$$1-\alpha \overset{a}{\cong} P\left(\frac{1}{\overline{X}_n\left(1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right)} \le \lambda \le \frac{1}{\overline{X}_n\left(1 - \frac{z_{\alpha/2}}{\sqrt{n}}\right)}\right)$$

$$1-\alpha \overset{a}{\cong} P\Bigg(\underbrace{\overline{X}_n\left(1 - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) \le \frac{1}{\lambda} \le \overline{X}_n\left(1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right)}\Bigg) \qquad IC2$$

$$E(\text{long } IC1) = \underbrace{E(\overline{X}_n)}_{1/\lambda}\left(\frac{1}{1 - \frac{z_{\alpha/2}}{\sqrt{n}}} - \frac{1}{1 + \frac{z_{\alpha/2}}{\sqrt{n}}}\right) =$$

$$= \frac{2\,z_{\alpha/2}}{\sqrt{n}\,\lambda}\;\frac{1}{\left(1 - \frac{z^2_{\alpha/2}}{n}\right)} > 1$$

$$E(\text{long } IC2) = E\left(2\,\overline{X}_n\,\frac{z_{\alpha/2}}{\sqrt{n}}\right) = 2\,\frac{1}{\lambda}\,\frac{z_{\alpha/2}}{\sqrt{n}}$$

$$E(\text{long } IC1) \ge E(\text{long } IC2) \quad \ldots$$