1) Calcula las siguientes integrales (utiliza integración por partes):

$$\int \ln(x)\,\mathrm{d}x = \int 1\cdot\ln(x)\,\mathrm{d}x = \left\{\begin{array}{ll} u=\ln(x) & \mathrm{d}u=\dfrac{1}{x}\,\mathrm{d}x \\ \mathrm{d}v = 1\,\mathrm{d}x & v = x \end{array}\right\} = x\ln(x) - \int x\cdot\frac{1}{x}\,\mathrm{d}x = x\ln(x) - \int 1\,\mathrm{d}x =$$

$$\boxed{x\ln(x) - x + \mathrm{C}}$$

2) Calcula las siguientes integrales (utiliza integración por partes):

$$\underbrace{\int \sin(2x)e^{4x}}_{I} = \left\{\begin{array}{ll} u=\sin(2x) & \mathrm{d}u = 2\cos(2x)\,\mathrm{d}x \\ dv = e^{4x}\,\mathrm{d}x & v = \dfrac{e^{4x}}{4} \end{array}\right\} = \frac{1}{4}e^{4x}\sin(2x) - \frac{1}{2}\int e^{4x}\cos(2x)\,\mathrm{d}x$$

$$= \left\{\begin{array}{ll} u=\sin(2x) & \mathrm{d}u = 2\cos(2x)\,\mathrm{d}x \\ dv = e^{4x}\,\mathrm{d}x & v = \dfrac{e^{4x}}{4} \end{array}\right\}$$

$$= -\frac{1}{4}e^{4x}\sin(2x) - \frac{1}{2}\left(\frac{1}{4}e^{4x}\sin(2x) - \frac{1}{2}\int e^{4x}\cos(2x)\,\mathrm{d}x\right)$$

$$= \frac{1}{4}e^{4x}\sin(2x) - \frac{1}{8}e^{4x}\cos(2x) - \frac{1}{4}\underbrace{\int e^{4x}\sin(2x)\,\mathrm{d}x}_{I}$$

Podemos escribir esto como:

$$I = \frac{1}{4}e^{4x}\sin(2x) - \frac{1}{8}e^{4x}\cos(2x) - \frac{1}{4}I \longrightarrow \frac{5}{4}I = \frac{1}{4}e^{4x}\sin(2x) - \frac{1}{8}e^{4x}\cos(2x)$$

$$\boxed{I = \frac{1}{5}e^{4x}\sin(2x) - \frac{1}{10}e^{4x}\cos(2x) + \mathrm{C}}$$

3) Calcula las siguientes integrales (utiliza integración por partes):

$$\int x^2e^{2x}\,\mathrm{d}x = \left\{\begin{array}{ll} u=x^2 & \mathrm{d}u = 2x\,\mathrm{d}x \\ \mathrm{d}v = e^{2x}\,\mathrm{d}x & v = \dfrac{e^{2x}}{2} \end{array}\right\} = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\mathrm{d}x = \left\{\begin{array}{ll} u=x^2 & \mathrm{d}u = 2x\,\mathrm{d}x \\ \mathrm{d}v = e^{2x}\,\mathrm{d}x & v = \dfrac{e^{2x}}{2} \end{array}\right\} =$$

$$\frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{x^2} - \int \frac{1}{2}e^{2x}\,\mathrm{d}x\right) = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\,\mathrm{d}x = \boxed{\frac{1}{2}x^2ex^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + \mathrm{C}}$$

4) Calcular las primitivas de las siguientes fracciones racionales:

$$\int \frac{x^7+x^3}{x^4-1}\,\mathrm{d}x$$

Como el polinomio del numerador es de grado mayor que es del denominador, dividiremos los polinomios y aplicaremos el algoritmo de Euclides: $\dfrac{N(x)}{D(x)} = C(x) + \dfrac{R(x)}{D(x)}$

$$\begin{array}{r|l} x^7 + x^3 & x^4 - 1 \\ \cline{2-2} -x^7 + x^3 & x^3 \\ \cline{1-1} 2x^3 & \end{array} \longrightarrow \frac{x^7+x^3}{x^4-1} = x^3 + \frac{2x^3}{x^4-1}$$

$$\int \frac{x^7+x^3}{x^4-1}\,\mathrm{d}x = \int x^3 + \frac{2x^3}{x^4-1}\,\mathrm{d}x = \int x^3\,\mathrm{d}x + \underbrace{\int \frac{2x^3}{x^4-1}}_{I} = \boxed{\frac{x^4}{4} + \frac{1}{2}\log|x^4-1| + \mathrm{C}}$$

$$I = \int \frac{2x^3}{x^4-1}\,\mathrm{d}x = \frac{1}{2}\int \frac{4x^3}{x^4-1}\,\mathrm{d}x = \left\{ \begin{array}{l} x^4-1=t \\ 4x^3\,\mathrm{d}x = \mathrm{d}t \end{array} \right\} = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\log(t) = \frac{1}{2}\log|x^4-1|$$

5) Calcular las primitivas

$\int \cos(x)^4 \sin(x)^3\,\mathrm{d}x = \int \cos^4(x)\cdot\sin^2(x)\cdot\sin(x)\mathrm{d}x = \int \cos^4(x)(1-\cos^2(x))\sin(x)\mathrm{d}x = -\int \cos^4(x)(1-$ $\cos^2(x))(-\sin(x))\,\mathrm{d}x = \left\{ \begin{array}{l} \cos(x)=t \\ -\sin(x)\,\mathrm{d}x = \mathrm{d}t \end{array} \right\} = -\int t^4\cdot(1-t^2)\,\mathrm{d}t = -\int t^4-t^6\,\mathrm{d}t = -\frac{t^5}{5}+\frac{t^7}{7}+\mathrm{C} =$ $\boxed{-\frac{\cos^5(x)}{5}+\frac{\cos^7(x)}{7}+\mathrm{C}}$

6) Calcular las siguientes integrales definidas:

$$\int_0^1 xe^x\,\mathrm{d}x$$

Calcularemos lo primero la primitiva:

$$\int xe^x\mathrm{d}x = \left\{ \begin{array}{ll} u=x & du = \mathrm{d}x \\ \mathrm{d}v = e^x\,\mathrm{d}x & v=e^x \end{array} \right\} = xe^x - \int e^x\mathrm{d}x = xe^x - e^x$$

Nota: Regla de Barrow

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b)-F(a)$$

Evaluando en la integral definida tenemos que:

$$\int_0^1 xe^x\,\mathrm{d}x = [xe^x-e^x]_0^1 = (\cancel{e}-\cancel{e})-(0-1) = \boxed{1}$$

7) Calcular las siguientes integrales definidas:

$$\int_0^{\frac{\pi}{2}} \sin(x)\,\mathrm{d}x = [-\cos(x)]_0^{\frac{\pi}{2}} = -\cancelto{0}{\cos\left(\frac{\pi}{2}\right)}+1 = \boxed{1}$$

8) Calcula mediante cambio de variable:

$$\int \frac{\mathrm{d}x}{x+\sqrt{x}}$$

$\int \frac{1}{x+\sqrt{x}}\,\mathrm{d}x = \left\{ \begin{array}{l} x=t^2 \\ \mathrm{d}x = 2t\,\mathrm{d}t \end{array} \right\} = \int \frac{1}{t^2+t}\cdot$ $2t\,\mathrm{d}t = \int \frac{2\cancel{t}}{\cancel{t}(t+1)}\,\mathrm{d}t = 2\int \frac{1}{t+1}\,\mathrm{d}t = 2\log|x+$ $1|+\mathrm{C} = \boxed{2\log(\sqrt{x}+1)+\mathrm{C}}$

Nota: Siempre que tengamos una integral en la que aparezcan raices podemos plantearnos realizar el cambio $x = \{\text{lo de dentro de la raíz}\} = t^p$ siendo p el mínimo común múltiplo de las ordenes de todas las raices. Este tipo de integrales son las llamadas integrales binómicas.

9) Calcula mediante cambio de variable:

$$\int \frac{x^2}{\sqrt{x+a}}\,\mathrm{d}x = \left\{\begin{array}{c} x+a=t^2 \\ x=t^2-a \\ \mathrm{d}x = 2t\,\mathrm{d}t \end{array}\right\} = \int \frac{(t^2-a)^2}{\cancel{t}}\cdot 2\cancel{t}\mathrm{d}t = 2\int t^4+a^2-2at^2\,\mathrm{d}t = 2\left[\frac{t^5}{5}+a^2t-2a\cdot\frac{t^3}{3}\right] =$$

$$\boxed{\frac{2}{5}\left(\sqrt{x+a}\right)^5 + 2a^2\sqrt{x+a} - \frac{4}{3}a\left(\sqrt{x+a}\right)^3 + \mathrm{C}}$$

10) Calcular la siguiente integral:

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,\mathrm{d}x = \left\{\begin{array}{c} x=t^6 \\ \mathrm{d}x = 6t^5\,\mathrm{d}t \end{array}\right\} = \int \frac{1}{t^3+t^2}\cdot 6^5\,\mathrm{d}t = \int \frac{6t^{\cancel{5}}}{\cancel{t^2}(t+1)}\,\mathrm{d}t = \int \frac{6t^3}{t+1} = (*)$$

$$\begin{array}{rrrr|l} 6t^3 & & & & t+1 \\ \hline -6t^3 & -6t^2 & & & 6t^2-6t+6 \\ \hline & -6t^2 & & & \\ & 6t^2 & +6t & & \\ \hline & & 6t & & \\ & & -6t & -6 & \\ \hline & & & -6 & \end{array}$$

$$\boxed{\frac{6t^3}{t+1} = 6t^2-6t+6-\frac{6}{t+1}}$$

$$(*) = \int 6t^2-6t+6-\frac{6}{t+1}\,\mathrm{d}t = \frac{6t^3}{3}-\frac{6t^2}{2}+6t-6\ln|t+1|+\mathrm{C} = 2t^3-3t^2+6t-6\ln|t+1|+\mathrm{C}$$

$$= 2(\sqrt[6]{x})^3 - 3(\sqrt[6]{x})^2 + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x}+1| + \mathrm{C}$$

$$= \boxed{2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)+\mathrm{C}}$$

11) Halle las derivadas de cada una de las siguientes funciones:

$$F(x) = \int_a^{x^3} \sin^3(t)\,\mathrm{d}t$$

$$F'(x) = \left(\int_a^{x^3} \sin^3(t)\,\mathrm{d}t\right)' = \sin^3(x^3)\cdot 3x^2 - \sin^3(a)\cdot 0 \longrightarrow \boxed{F'(x) = 3x^2\cdot\sin^3(x^3)}$$

Nota: Dada la función $T(x) = \int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t$, para hallar su derivada respecto de x, debemos aplicar la siguiente propiedad:
$T'(x) = \left(\int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t\right)' = f\left(g(x)\right)\cdot g'(x) - f\left(h(x)\right)\cdot h'(x)$

12) Calcula:

$$\lim_{x\to 0^+} \frac{\int_0^{x^2}\sin(\sqrt{t})\,\mathrm{d}t}{x^3} = \left(\frac{0}{0}\right) = \{\text{L'Hôpital}\} = \lim_{x\to 0^+}\frac{2\cancel{x}\sin(x)}{3x^{\cancel{2}}} = \lim_{x\to 0^+}\frac{2\sin(x)}{3x} = \left(\frac{0}{0}\right)$$

$$= \left\{\begin{array}{c} \text{equivalencias} \\ \sin(x) \rightsquigarrow_0 x \end{array}\right\} = \lim_{x\to 0^+}\frac{2x}{2x} = \boxed{\frac{2}{3}}$$

$$\left(\int_0^{x^2}\sin(\sqrt{t})\,\mathrm{d}t\right)' = \sin(x)\cdot 2x - \cancel{\sin(0)\cdot 0} = 2x\sin(x)$$

13) Calcular las siguientes integrales definidas:

$$\int_1^2 \log(x)\,\mathrm{d}x$$

Empezamos calculando la primitiva, para luego poder sustituirla:

$$\int 1\log(x)\mathrm{d}x = \left\{ \begin{array}{ll} u=\log(x) & \mathrm{d}u=\frac{1}{x}\,\mathrm{d}x \\ \mathrm{d}v = 1\,\mathrm{d}x & v=x \end{array} \right\} = x\log(x) - \int x\cdot\frac{1}{x}\mathrm{d}x = x\log(x) - \int 1\mathrm{d}x = x\log(x)-x$$

Una vez que tenemos la primitiva, evaluamos:

$$\int_1^2 \log(x)\,\mathrm{d}x = [x\log(x)-x]_1^2 = (2\log(2)-2) - (\overset{0}{\cancel{\log(1)}} - 1) = \boxed{2\log(2)-1}$$

14) Calcular las siguientes integrales

$$\int \frac{1}{x\sqrt{1-\ln^2(x)}}\,\mathrm{d}x = \int \frac{1}{x}\cdot\frac{1}{\sqrt{1-\ln^2(x)}}\,\mathrm{d}x = \left\{ \begin{array}{l} \ln(x)=t \\ \frac{1}{x}\,\mathrm{d}x = \mathrm{d}t \end{array} \right\} = \int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t = \arcsin(t) =$$
$$\boxed{\arcsin(\ln(x)) + \mathrm{C}}$$

15) Calcular las siguientes integrales definidas:

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$$

Comenzaremos calculando la primera, para luego poder evaluar:

$$\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \left\{ \begin{array}{c} x+1=t^2 \\ x=t^2-1 \\ \mathrm{d}x = 2t\,\mathrm{d}t \end{array} \right\} = \int \frac{t^2-1}{\cancel{t}}\cdot 2\cancel{t}\,\mathrm{d}t = 2\int t^2-1\,\mathrm{d}t = 2\left[\frac{t^3}{3}-t\right] = \{t=\sqrt{x+1}\} =$$
$$\frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1}$$

Evaluar tenemos que:

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \left[\frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1}\right]_0^3 0 \left(\frac{2}{3}8-4\right) - \left(\frac{2}{3}-2\right) = \frac{16}{3} - \frac{12}{3} - \frac{2}{3} + \frac{6}{3} = \boxed{\frac{8}{3}}$$

16) Halle $F'(x)$ si $F(x) = \int_0^x xf(t)\,\mathrm{d}t$

$$F(x)=\int_0^x xf(t)\,\mathrm{d}t = \underbrace{x}\cdot\underbrace{\int_0^x f(t)\,\mathrm{d}t}$$

$F'(x) = \{\text{derivada de un producto}\} = 1\cdot\int_0^x f(t)\,\mathrm{d}t + x\left(\int_0^x f(t)\,\mathrm{d}t\right)' = \int_0^x f(t)\,\mathrm{d}t + x\left[f(x)\cdot 1 - \cancelto{0}{f(0)\cdot 0}\right] = \int_0^x f(t)\,\mathrm{d}t + x\cdot f(x)$

Por lo tanto: $\boxed{F'(x) = \int_0^x f(t)\,\mathrm{d}t + xf(x)}$

Nota: Dada la función $T(x)=\int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t$, para hallar su derivada respecto de x, debemos aplicar la siguiente propiedad:

$$T'(x) = \left(\int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t\right)' = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$$

17) Calcular las siguientes integrales

$$\int \frac{\sin(\sqrt{x^3})}{\sqrt{x^3}}x^2\,\mathrm{d}x = \frac{1}{3}\int\frac{\sin(\sqrt{x^3})}{\sqrt{x^3}}(3x^2)\mathrm{d}x = \left\{\begin{array}{l} x^3 = t \\ 3x^2\,\mathrm{d}x = \ \mathrm{d}t \end{array}\right\} = \frac{1}{3}\int\frac{\sin(\sqrt{t})}{\sqrt{t}}\mathrm{d}t = \frac{1}{3}\int\frac{1}{\sqrt{t}}\sin(\sqrt{t})\mathrm{d}t =$$

$$\frac{1}{3}\cdot 2\cdot\int\frac{1}{2\sqrt{t}}\sin(\sqrt{t})\,\mathrm{d}t)\left\{\begin{array}{l} \sqrt{t} = u \\ \dfrac{1}{2\sqrt{t}}\,\mathrm{d}t = \ \mathrm{d}u \end{array}\right\} = \frac{2}{3}\int\sin(u)\,\mathrm{d}u = -\frac{2}{3}\cos(u) + \mathrm{C} = \boxed{-\frac{2}{3}\cos(\sqrt{x^3}) + \mathrm{C}}$$

18) Calcular las siguientes integrales:

$$\int\left(\frac{1-x}{1+x}\right)^{\frac{1}{3}}\cdot(1+x)^{-2}\,\mathrm{d}x = -\frac{1}{2}\int\left(\frac{1-x}{1+x}\right)^{\frac{1}{3}}\cdot\frac{-2}{(1+x)^2}\mathrm{d}x = \left\{\begin{array}{l} \dfrac{1-x}{1+x} = t \\ \dfrac{-2}{(1+x)^2}\,\mathrm{d}x = \ \mathrm{d}t \end{array}\right\} = -\frac{1}{2}\int t^{\frac{1}{3}}\,\mathrm{d}t =$$

$$-\frac{1}{2}\cdot\frac{t^{\frac{4}{3}}}{\frac{4}{3}} + \mathrm{C} = \boxed{-\frac{3}{8}\cdot\left(\frac{1-x}{1+x}\right)^{\frac{4}{3}} + \mathrm{C}}$$

$$\left(\frac{1-x}{1+x}\right)' = \frac{(-1)(1+x)-(1-x)}{(1+x)^2} = -\frac{2}{(1+x^2)}$$

19) Calcular las siguientes integrales

$$\int\cos(x)(e^{\sin(x)}-1)\,\mathrm{d}x = \left\{\begin{array}{l} \sin(x) = t \\ \cos(x)\,\mathrm{d}x = \ \mathrm{d}t \end{array}\right\} = \int(e^t-1)\,\mathrm{d}t = e^t - t + \mathrm{C} = \boxed{e^{\sin(x)}\cdot\sin(x) + \mathrm{C}}$$

20) Calcular las siguientes integrales

$$\int\cos^3(x)\,\mathrm{d}x = \int\cos^2(x)\cdot\cos(x)\,\mathrm{d}x = \int(1-\sin^{(}x))\cos(x)\,\mathrm{d}x = \left\{\begin{array}{l} \sin(x) = t \\ \cos(x)\,\mathrm{d}x = \ \mathrm{d}t \end{array}\right\} = \int 1-t^2\,\mathrm{d}t =$$

$$t - \frac{t^3}{3} + \mathrm{C} = \boxed{\sin(x) - \frac{\sin^3(x)}{3} + \mathrm{C}}$$

21) Calcular las siguientes integrales:

$$\int x(a+bx)^{-\frac{3}{2}}\,\mathrm{d}x = \frac{1}{b}\int x(a+bx)^{-\frac{3}{2}}\cdot b\,\mathrm{d}x = \left\{\begin{array}{l} a+bx = t \\ b\,\mathrm{d}x = \ \mathrm{d}t \\ x = \dfrac{1}{b}(t-a) \end{array}\right\} = \frac{1}{b}\int\frac{1}{b}(t-a)\cdot t^{-\frac{3}{2}}\,\mathrm{d}t = \frac{1}{b^2}\int t^{-\frac{1}{2}} -$$

$$at^{-\frac{3}{2}}\,\mathrm{d}t = \frac{1}{b^2}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}} - a\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}}\right] = \frac{1}{b^2}\left(2\sqrt{t} + 2a\frac{1}{\sqrt{t}}\right) = \boxed{\frac{1}{b^2}\left(2\sqrt{a+bx} + \frac{2a}{\sqrt{a+bx}}\right) + \mathrm{C}}$$

22) Calcular las siguientes integrales:

$$\int xe^{-\frac{x^2}{2a}}\,\mathrm{d}x = -a\int -\frac{1}{a}xe^{-\frac{x^2}{2a}}\,\mathrm{d}x = \left\{\begin{array}{l} -\dfrac{x^2}{2a} = t \\ -\dfrac{x}{a}\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = -a\int e^t\,\mathrm{d}t = -ae^t + \mathrm{C} = \boxed{-ae^{-\frac{x^2}{2a}} + \mathrm{C}}$$

23) Calcula mediante cambio de variable:

$$\int_0^{\frac{\pi}{4}} \frac{e^{\tan(x)}}{\cos^2(x)}\,\mathrm{d}x$$

Lo primero que vamos a hacer es calcular la primitiva:

$$\int \frac{e^{\tan(x)}}{\cos^2(x)}\,\mathrm{d}x = \int \frac{1}{\cos^2(x)}\cdot e^{\tan(x)}\,\mathrm{d}x = \left\{\begin{array}{l} \tan(x) = t \\ \dfrac{1}{\cos^2(x)}\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = \int e^t\,\mathrm{d}t = e^t = e^{\tan(x)}$$

Por lo tanto:

$$\int_0^{\frac{\pi}{4}} \frac{e^{\tan(x)}}{\cos^2(x)}\,\mathrm{d}x = \left[e^{\tan(x)}\right]_0^{\frac{\pi}{4}} = e^1 - e^0 = \boxed{e - 1}$$

24) Calcula mediante cambio de variable:

$$\int \frac{\sqrt[4]{x}}{\sqrt{x}+1}\,\mathrm{d}x$$

Estamos ante una integral binómica, donde realizamos un cambio de la forma $x = t^p$, donde p será el mínimo común múltiplo de las ordenes de las raices:

$$\int \frac{\sqrt[4]{x}}{\sqrt{x}+1}\,\mathrm{d}x = \left\{\begin{array}{l} x = t^4 \\ \mathrm{d}x = 4t^3\,\mathrm{d}t \end{array}\right\} = \int \frac{t}{t^2+1}\cdot 4t^3\,\mathrm{d}t = \int \frac{4t^4}{t^2+1}\,\mathrm{d}t = (*)$$

$$\begin{array}{rrr|l} 4t^4 & & & t^2+1 \\ \cline{4-4} -4t^4 & -4t^2 & & 4t^2-4 \\ \hline & -4t^2 & & \\ & 4t^2 & +4 & \\ \cline{2-3} & & 4 & \end{array}$$

$$(*) = \int 4t^2 - 4 + \frac{4}{t^2+1}\,\mathrm{d}t = \frac{4t^3}{3} - 4t\arctan(t) + \mathrm{C}$$

$$= \boxed{\frac{4}{3}\sqrt[4]{x^3} - 4\sqrt[4]{x} + 4\arctan(\sqrt{x}) + \mathrm{C}}$$

25) Calcula mediante cambio de variable:

$$\int \frac{\tan(\ln(x))}{x}\,\mathrm{d}x = \int \frac{1}{x}\cdot\tan(\ln(x))\,\mathrm{d}x = \left\{\begin{array}{l} \ln(x)=t \\ \frac{1}{x}\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = \int \tan(t)\,\mathrm{d}t$$

$$= -\int \frac{-\sin(t)}{\cos(t)}\,\mathrm{d}t = \left\{\begin{array}{l} \cos(t)=u \\ -\sin(t)\,\mathrm{d}t = \mathrm{d}u \end{array}\right\} = -\int \frac{1}{u}\,\mathrm{d}u = -\ln|u| + \mathrm{C}$$

$$= -\ln|\cos(t)| + \mathrm{C} = \boxed{-\ln|\cos(\ln(x))| + \mathrm{C}}$$

26) Halle sin realizar ningún cálculo:

$$\int_{-1}^{1} x^3\sqrt{1-x^2}\,\mathrm{d}x$$

$I = [-1, 1]$

$$\begin{array}{ccc} \vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- & + & -\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv \\ -1 & 0 & 1 \end{array}$$

$f(x) = x^3\sqrt{1-x^2}$

Si estudiamos la simetría de la función a integrar:

$$f(-x) = (-x)^3\sqrt{1-(-x)^2} = -\underbrace{x^3\sqrt{1-x^2}} = -f(x) \text{ Simetría impar}$$

Las imágenes en las $x < 0$ son iguales que las de $x > 0$, pero cambiadas de signo, por lo tanto se compensarían:

$$\boxed{\int_{-1}^{1} x^3\sqrt{1-x^2}\,\mathrm{d}x = 0}$$

27) Calcula

$$\int x^5(1+x^3)^{\frac{1}{3}}\,\mathrm{d}x = \int x^2\cdot x^3(1+x^3)^{\frac{1}{3}}\,\mathrm{d}x = \frac{1}{3}\int 3x^2\cdot x^3(1+x^3)^{\frac{1}{3}}\,\mathrm{d}x = \left\{\begin{array}{l} 1+x^3 = t^3 \\ 3x^2\,\mathrm{d}x = 3t^2\,\mathrm{d}t \\ x^3 = t^3-1 \end{array}\right\}$$

$$= \frac{1}{3}\int (t^3-1)\cdot t\cdot 3t^2\,\mathrm{d}t = \frac{1}{43}\int 3t^6 - 3t^3\,\mathrm{d}t = \frac{1}{3}\left[\frac{3t^7}{7} - \frac{3t^4}{4}\right] + \mathrm{C} = \{t = \sqrt[3]{1+x^3}\}$$

$$= \boxed{\frac{1}{7}\left(\sqrt[3]{1+x^3}\right)^7 - \frac{1}{4}\left(\sqrt[3]{1+x^3}\right)^4 + \mathrm{C}}$$

28) Calcula

$$\frac{1}{\cos(x)}\,\mathrm{d}x = \int \frac{\cos(x)}{\cos^2(x)}\mathrm{d}x = \int \frac{\cos(x)}{1-\sin^2(x)}\mathrm{d}x = \left\{\begin{array}{l} \sin(x)=t \\ \cos(x)\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = \int \frac{1}{1-t^2}\mathrm{d}t = \int \frac{1}{(1-t)(1+t)}\mathrm{d}t =$$

$$\int \frac{-1}{(t-1)(t+1)}\,\mathrm{d}t = (*)$$

$$-\frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1)+B(t-1)}{(t-1)(t+1)} \longrightarrow -1 = A(t+1)+B(t-1)$$

Si $t=1 \longrightarrow A = -\frac{1}{2}$

Si $t=-1 \longrightarrow B = \frac{1}{2}$

$$(*) = \int \frac{-\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}\,\mathrm{d}t = -\frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1| + \mathrm{C} = \boxed{-\frac{1}{2}\ln|sin(x)-1| + \frac{1}{2}\ln|\sin(x)+1| + \mathrm{C}}$$

29) Calcula

$$\int \frac{\sqrt{4+3x}}{4-3x}\,\mathrm{d}x = \frac{1}{3}\int 3\frac{\sqrt{4+2x}}{4-3x}\,\mathrm{d}x = \left\{ \begin{array}{l} 4+3x=t^2 \\ 3\,\mathrm{d}x = 2t\,\mathrm{d}t \\ 3x = t^2-4 \end{array} \right\} = \frac{1}{3}\int \frac{t}{4-(t^2-4)}2t\,\mathrm{d}t = \frac{2}{3}\int \frac{t^2}{8-t^2}\,\mathrm{d}t =$$

$$\frac{2}{3}\int -1 + \frac{8}{8-t^2}\,\mathrm{d}t = -\frac{2}{3}t + \frac{16}{3}\underbrace{\int \frac{-1}{t^2-8}\,\mathrm{d}t}_{I_1} = (**)$$

$$\begin{array}{rr|l} \cancel{t^2} & & -t^2+8 \\ \hline \cancel{-t^2} & +8 & -1 \\ \hline & 8 & \end{array}$$

$$I_1 = \int \frac{-1}{(t-2\sqrt{2})(t+2\sqrt{2})}\,\mathrm{d}t = (*)$$

$$\frac{-1}{(t-2\sqrt{2})(t+2\sqrt{2})} = \frac{A}{t-2\sqrt{2}} + \frac{B}{t+2\sqrt{2}} = \frac{A(t+2\sqrt{2})+B(t-2\sqrt{2})}{(t-2\sqrt{2})(t+2\sqrt{2})}$$

$-1 = A(t+2\sqrt{2}) + B(t-2\sqrt{2})$

Si $t = 2\sqrt{2} \longrightarrow -1 = A\cdot(4\sqrt{2}) \longrightarrow A = -\frac{1}{4\sqrt{2}}$

Si $t = -2\sqrt{2} \longrightarrow -1 = B\cdot(-4\sqrt{2}) \longrightarrow B = \frac{1}{4\sqrt{2}}$

$$(*) = \int \frac{-\frac{1}{4\sqrt{2}}}{t-2\sqrt{2}} + \frac{\frac{1}{4\sqrt{2}}}{t+2\sqrt{2}}\,\mathrm{d}t = -\frac{1}{4\sqrt{2}}\ln\left|t-2\sqrt{2}\right| + \frac{1}{4\sqrt{2}}$$

$$(**) = -\frac{2}{3}t - \frac{4}{3\sqrt{2}}\ln|t-2\sqrt{2}| + \frac{4}{3\sqrt{2}}\ln|t+2\sqrt{2}| + \mathrm{C}$$

$$= \boxed{-\frac{2}{3}\sqrt{4+3x} - \frac{4}{3\sqrt{2}}\ln|\sqrt{4+3x}-2\sqrt{2}| + \frac{4}{3\sqrt{2}}\ln\left|\sqrt{4+3x}+2\sqrt{2}\right| + \mathrm{C}}$$

30) Calcula

$$\int \sin^5(x)\cos^2(x) = \int \sin(x)\cdot\sin^4(x)\cdot\cos^2(x)\,\mathrm{d}x = \int \sin(x)\left(\sin^2(x)\right)^2\cos^2(x)\,\mathrm{d}x = -\int -\sin(x)\cdot(1-$$

$$\cos^2(x))^2\cos^2(x)\,\mathrm{d}x = \left\{ \begin{array}{l} \cos(x) = t \\ -\sin(x)\,\mathrm{d}x = \mathrm{d}t \end{array} \right\} = -\int (1-t^2)^2\cdot t^2\,\mathrm{d}t = -\int (1+t^4-2t^2)\cdot t^2\,\mathrm{d}t =$$

$$-\int t^2+t^6-2t^4\,\mathrm{d}t = -\frac{t^3}{3} - \frac{t^7}{7} + 2\frac{t^5}{5} + \mathrm{C} = \boxed{-\frac{\cos^3(x)}{3} - \frac{\cos^7(x)}{7} + \frac{2}{5}\cos^5(x) + \mathrm{C}}$$

31) Idem con las siguientes funciones (funciones de la forma $f(g(x))g'(x)$; primitivas de funciones de la forma $f(e^x)$):

$$\int x^5 \arctan\frac{x^6+4}{5}\,dx = \frac{5}{6}\int \frac{6}{5}x^5 \arctan\left(\frac{x^6+4}{5}\right)\,dx = \left\{\begin{array}{l} \frac{x^6+4}{5}=t \\ \frac{6}{5}x^5\,dx = dt \end{array}\right\} = \frac{5}{6}\int \arctan(t)\,dt =$$

$$\left\{\begin{array}{ll} u=\arctan(t) & du=\frac{1}{1+t^2}\,dt \\ dv = 1\,dt & v=t \end{array}\right\} = \frac{5}{6}\left(t\cdot\arctan(t) - \underbrace{\int \frac{t}{1+t^2}\,dt}_{I_1}\right) = \frac{5}{6}t\cdot\arctan(t) - \frac{5}{12}\ln(1+$$

$$t^2)+C = \boxed{\frac{5}{6}\cdot\frac{x^6+4}{5}\arctan\left(\frac{x^6+4}{5}\right) - \frac{5}{12}\ln\left(1+\left(\frac{x^6+4}{5}\right)^2\right)+C}$$

$$I_1 = \int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int\frac{2t}{1+t^2}\,dt = \left\{\begin{array}{l} 1+t^2=u \\ 2t\,dt = du \end{array}\right\} = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(1+t^2)$$

32) Idem con las siguientes funciones (funciones de la forma $f(g(x))g'(x)$; primitivas de funciones de la forma $f(e^x)$):

$$\int \frac{1}{a^2e^x+b^2e^{-x}}\,dx = \int\frac{1}{a^2e^x+b^2\frac{1}{e^x}}\,dx = \int\frac{1}{\frac{a^2(e^x)^2+b^2}{e^x}}\,dx = \int\frac{e^x}{a^2(e^x)^2+b^2}\,dx = \left\{\begin{array}{l} e^x=t \\ e^x\,dx = dt \end{array}\right\} =$$

$$\int\frac{1}{a^2t^2+b^2}\,dt = \int\frac{\frac{1}{b^2}}{\frac{a^2t^2}{b^2}+1}\,dt = \frac{1}{b^2}\int\frac{1}{\left(\frac{at}{b}\right)^2+1}\,dt = \frac{1}{ab}\int\frac{\frac{a}{b}}{\left(\frac{at}{b}\right)^2+1}\,dt = \left\{\begin{array}{l} \frac{at}{b}=u \\ \frac{a}{b}\,dt = du \end{array}\right\} =$$

$$\frac{1}{ab}\int\frac{1}{u^2+1}\,du = \frac{1}{ab}\arctan(u)+C = \frac{1}{ab}\arctan\left(\frac{at}{b}\right)+C = \boxed{\frac{1}{ab}\arctan\left(\frac{ae^x}{b}\right)+C}$$

33) Calculas las primitivas de las siguientes fracciones racionales:

$$\int\frac{3x^2+2x+4}{x^3+x^2+x+1}\,dx$$

$$x^3+x^2+x+1=0$$

$$\begin{array}{r|rrrr} & 1 & 1 & 1 & 1 \\ -1 & & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array} \qquad (x^3+x^2+x+1) = (x+1)(x^2+1)$$

$$\int\frac{3x^2+2x+4}{x^3+x^2+x+1}\,dx = \int\frac{3x^2+2x+4}{(x+1)(x^2+1)}\,dx = (*)$$

$$\frac{3x^2+2x+4}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\left.\begin{array}{l} 3x^2+2x+4 = A(x^2+1)+(Bx+C)(x+1) \\ \begin{cases} x=-1 \longrightarrow 2A=5 \longrightarrow A=\dfrac{5}{2} \\ x=0 \longrightarrow 4=\dfrac{5}{2}+\mathrm{C} \longrightarrow C=\dfrac{3}{2} \\ x=1 \longrightarrow 9=5+\left(B+\dfrac{3}{2}\right)\cdot 2 \longrightarrow B+\dfrac{3}{2}=2 \longrightarrow B=\dfrac{1}{2} \end{cases} \end{array}\right\} \frac{3x^2+2x+4}{(x+1)(x^2+1)} = \frac{\frac{5}{2}}{x+1}+\frac{\frac{1}{2}x+\frac{3}{2}}{x^2+1}$$

$$(*) = \int \frac{\frac{5}{2}}{x+1}+\frac{\frac{1}{2}x+\frac{3}{2}}{x^2+1}\,\mathrm{d}x = \frac{5}{2}\underbrace{\int \frac{1}{x+1}\,\mathrm{d}x}_{I_1}+\frac{1}{2}\underbrace{\int \frac{x}{x^2+1}\,\mathrm{d}x}_{I_3}+\frac{3}{2}\underbrace{\int \frac{1}{x^2}\,\mathrm{d}x}_{I_2} = (**)$$

$$I_1 = \int \frac{1}{x+1}\,\mathrm{d}x = \ln|x+1|$$

$$I_2 = \int \frac{1}{x^2}\,\mathrm{d}x = \arctan(x)$$

$$I_3 = \int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x = \left\{\begin{array}{l} x^2+1=t \\ 2x\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|1+x^2|$$

$$(**) = \boxed{\frac{5}{2}\ln|x+1|+\frac{1}{4}\ln|1+x^2|+\frac{3}{2}\arctan(x)+\mathrm{C}}$$

34) Halle las derivadas de cada una de las siguientes funciones:

$$F(x) = \int_y^{\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)} \frac{1}{1+\sin^6(t)+t^2}\,\mathrm{d}t$$

$$F'(x) = \frac{1}{1+\sin^6\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)+\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)^2} \cdot \left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)' - \cancel{\frac{1}{1+\sin^6(y)+y^2}\cdot 0} =$$

$$\frac{1}{1+\sin^6\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)+\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)^2} \cdot \left[\sin^3(x)\cdot 1 - \cancel{\sin^3(1)\cdot 0}\right]$$

$$\boxed{F'(x) = \frac{\sin^3(x)}{1+\sin^6\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)+\left(\int_1^x \sin^3(t)\,\mathrm{d}t\right)^2}}$$

<u>Nota:</u> Dada la función $T(x) = \displaystyle\int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t$, para hallar su derivada respecto de x, debemos aplicar la siguiente propiedad:

$$T'(x) = \left(\int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t\right)' = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$$

35) Si f es continua en $[0,1]$, calcule $\displaystyle\lim_{x\to 0^+} x\int_x^1 \frac{f(t)}{t}\,\mathrm{d}t$.

$$\lim_{x\to 0^+} x\int_x^1 \frac{f(t)}{t}\,\mathrm{d}t = (0\cdot\infty) = \lim_{x\to 0^+}\frac{\int_x^1 \frac{f(t)}{t}\,\mathrm{d}t}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right) = \{\text{L'Hôpital}\} = \lim_{t\to 0^+}\frac{\cancel{\frac{f(1)}{1}\cdot 0} - \frac{f(x)}{x}\cdot 1}{-\frac{1}{x^2}} =$$

$$\lim_{x\to 0^+}\frac{\frac{f(x)}{\cancel{x}}}{\frac{1}{x^{\cancel{2}}}} = \lim_{x\to 0^+} xf(x) = \{\lim_{x\to 0^+} f(x) = f(0)\} = 0\cdot f(0) = \boxed{0}$$

36) Calcular la siguiente integral impropia:

$$\int_1^{+\infty} \frac{1}{x^3}\,\mathrm{d}x$$

$$\int_1^{+\infty} \frac{1}{x^3}\,\mathrm{d}x = \lim_{t\to+\infty} \underbrace{\int_1^t \frac{1}{x^3}\,\mathrm{d}x}_{I_1} = \lim_{t\to+\infty}\left(\cancelto{0}{\frac{-1}{2t^2}} + \frac{1}{2}\right) = \frac{1}{2}$$

$$I_1 = \int_1^t x^{-3}\,\mathrm{d}x = \left[\frac{x^{-2}}{-2}\right]_1^t = \left[\frac{-1}{2x^2}\right]_1^t = \frac{-1}{2t^2} + \frac{1}{2}$$

Por lo tanto tenemos que la integral es convergente y su valor es:

$$\boxed{\int_1^{+\infty} \frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2}}$$

37) Calcular la siguiente integral impropia:

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

Estamos ante una integral impropia de 2ª especie:

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to0^+} \underbrace{\int_t^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x}_{I_1} = \lim_{t\to0^+}(2 - \cancelto{0}{2\sqrt{t}}) = 2$$

$$I_1 = \int_t^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int_t^1 x^{-\frac{1}{2}}\,\mathrm{d}x = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_t^1 = \left[2\sqrt{x}\right]_t^1 = 2 - 2\sqrt{t}$$

Por lo tanto, la integral es convergente, y su valor viene dado por:

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2}$$

38) Calcula:

$$\int_1^{+\infty} \frac{1}{x(x+1)}\,\mathrm{d}x = \lim_{t\to+\infty} \underbrace{\int_1^t \frac{1}{x(x+1)}\,\mathrm{d}x}_{I_1} = (*)$$

$$I_1 = \int_1^t \frac{1}{x(x+1)}\,\mathrm{d}x = \int_1^t \frac{1}{x} - \frac{1}{x+1}\,\mathrm{d}x = [\ln(x) - \ln(x+1)]_1^t = \ln(t) - \ln(t+1) - \cancelto{0}{\ln(1)} + \ln(2)$$

$$= \ln\left(\frac{t}{t+1}\right) + \ln(2)$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$$

$$1 = A(x+1) + Bx \begin{cases} \text{Si } x = 0 \longrightarrow A = 1 \\ \text{Si } x = -1 \longrightarrow B = -1 \end{cases} \qquad \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$(*) = \lim_{t\to+\infty}\left[\ln\left(\frac{t}{t+1}\right) + \ln(2)\right] = \cancelto{0}{\ln(1)} + \ln(2) = \ln(2)$$

$$\lim_{t\to+\infty} \frac{t}{t+1} = \left(\frac{\infty}{\infty}\right) = \{\text{L'Hôpital}\} = \lim_{t\to+\infty} \frac{1}{1} = 1$$

La integral impropia es convergente y su valor es:

$$\boxed{\int_1^{+\infty} \frac{1}{x(x+)}\,\mathrm{d}x = \ln(2)}$$

39) Calcula:

$$\int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,\mathrm{d}x$$

Estamos ante una integral impropia de 2ª especie:

$$\int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,\mathrm{d}x = \lim_{t\to 0^+} \underbrace{\int_t^1 \frac{e^x}{\sqrt{e^x-}}\,\mathrm{d}x}_{I_1} = (*)$$

Vamos a hacer primero la primitiva:

$$\int \frac{e^x}{\sqrt{e^x-1}}\,\mathrm{d}x = \left\{\begin{array}{l} e^x-1=T \\ e^x\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = \int \frac{1}{\sqrt{t}}\,\mathrm{d}t = \int t^{-\frac{1}{2}}\,\mathrm{d}t = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{e^x-1}$$

$$(*) = \lim_{t\to 0^+}\left[2\sqrt{e^x-1}\right]_t^1 = \lim_{t\to 0^+}(2\sqrt{e-1} - \cancelto{0}{2\sqrt{e^t-1}}) = 2\sqrt{e-1}$$

Por lo tanto, la integral es convergente y su valor es:

$$\boxed{\int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,\mathrm{d}x = 2\sqrt{e-1}}$$

40) Calcula:

$$\int_1^1 \ln(x)\,\mathrm{d}x) \lim_{t\to 0^+}\int_t^1 \ln(x)\,\mathrm{d}x = (*)$$

Calcularemos la primitiva:

$$\int \ln(x)\,\mathrm{d}x) \left\{\begin{array}{ll} u=\ln(x) & \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \\ \mathrm{d}v = 1\,\mathrm{d}x & v = x \end{array}\right\} = x\ln(x) - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,\mathrm{d}x = x\ln(x) - x$$

$$(*) = \lim_{t\to 0^+}\left[x\ln(x) - x\right]_t^1 = \lim_{t\to 0^+}(\cancelto{0}{1\cdot\ln(1)} - 1 - \underbrace{t\ln(t)}_{L_1} + \cancelto{0}{t}) = \boxed{-1}$$

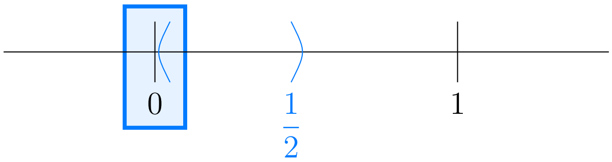
$$L_1 = \lim_{t\to 0^+} t\ln(t) = (0\cdot\infty) = \lim_{t\to 0^+}\frac{\ln(t)}{\frac{1}{t}} = \left(\frac{\infty}{\infty}\right) = \{\text{L'Hôpital}\} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+}\frac{-t^{\cancel{2}}}{\cancel{t}} = \lim_{t\to 0^+}(-t) = 0$$

Por lo tanto es una integral convergente y su valor es:

$$\boxed{\int_0^1 \ln(x)\,\mathrm{d}x = -1}$$

41) Calcula:

$$\int_0^{\frac{1}{e}} \frac{1}{(\ln(x))^2\, x}$$



$$\int_0^{\frac{1}{e}} \frac{1}{(\ln(x))^2 x}\,\mathrm{d}x = \lim_{t\to 0^+} \int_t^{\frac{1}{e}} \frac{1}{(\ln(x))^2 x}\,\mathrm{d}x = (*)$$

Hacemos la primitiva a parte:

$$\int \frac{1}{(\ln(x))^2\cdot x}\,\mathrm{d}x = \int \frac{1}{(\ln(x))^2}\cdot\frac{1}{x}\,\mathrm{d}x = \left\{\begin{array}{l} \ln(x) = u \\ \frac{1}{x}\,\mathrm{d}x = \ \mathrm{d}u \end{array}\right\} = \int \frac{1}{u^2}\,\mathrm{d}u = \int u^{-2}\,\mathrm{d}u = \frac{u^{-1}}{-1} = -\frac{1}{u} = -\frac{1}{\ln(x)}$$

$$(*) = \lim_{t\to 0^+}\left[-\frac{1}{\ln(x)}\right]_t^{\frac{1}{e}} = \lim_{t\to 0^+}\left(-\frac{1}{\cancel{\ln(e)}^{-1}} + \cancelto{0}{\frac{1}{\ln(t)}}\right) = \left\{\ln(e)^{-1} = (-1)\underbrace{\ln(e)}_{1} = -1\right\} = 1$$

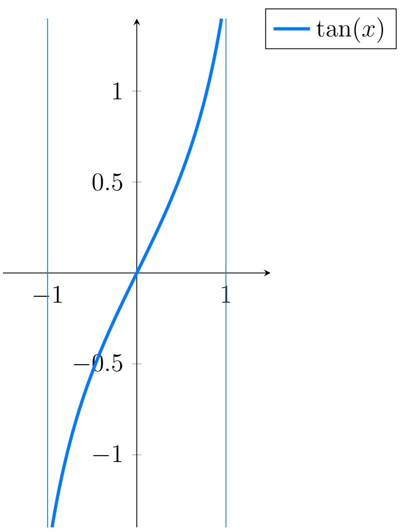
La integral impropia es convergente y su valor es:

$$\boxed{\int_0^1 \frac{1}{(\ln(x))^2 x}\,\mathrm{d}x = 1}$$

42) Calcular la integral:

$$\int_1^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{t\to+\infty} \frac{1}{t}\,\frac{1}{1+x^2}\mathrm{d}x = \lim_{t\to+\infty}\left[\arctan(x)\right]_1^t = \lim_{t\to+\infty}\left[\arctan(t) - \arctan(1)\right] = \underbrace{\arctan(+\infty)}_{\frac{\pi}{2}} -$$

$$\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$



$\tan(0) = 0$

$\tan\left(\frac{\pi}{4}\right) = 1$

$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\tan\left(-\frac{\pi}{4}\right) = -1$

$\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

$\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

$\tan\left(\frac{\pi}{2}^-\right) = +\infty$

$\tan\left(\frac{\pi}{2}^+\right) = -\infty$

43) Calcular la siguiente integral

$$\int \frac{1}{(x-1)(x^2+2)}\, \mathrm{d}x = (*)$$

Aplicaremos para empezar, separación por fracciones simples:

$$\frac{1}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2} = \frac{A(x^2+2)+(x-1)}{(}Bx+C)(x-1)(x^2+2)$$

$$1 = A(x^2+2) + (x-1)(Bx+C)$$

$$\text{Si } x = 1 \longrightarrow 3A = 1 \longrightarrow \boxed{A = \frac{1}{3}}$$

$$\text{Si } x = 0 \longrightarrow 1 = \frac{2}{3} + (-1)C \longrightarrow \boxed{C = -\frac{1}{3}}$$

$$\frac{1}{(x-1)(x^2+2)} = \frac{1}{3}\cdot\frac{1}{x-1} + \frac{-\frac{1}{3}x-\frac{1}{3}}{x^2+2}$$

$$\text{Si } x = -1 \longrightarrow \cancel{1}) = \cancel{1} + (-2)\left(-B-\frac{1}{3}\right) \longrightarrow \boxed{B = -\frac{1}{3}}$$

$$(*) = \int \frac{1}{3}\cdot\frac{1}{x-1} + \frac{-\frac{1}{3}x-\frac{1}{3}}{x^2+2}\,\mathrm{d}x = \frac{1}{3}\underbrace{\int \frac{1}{x-1}\,\mathrm{d}x}_{I_1} - \frac{1}{3}\underbrace{\int \frac{x}{x^2+2}\,\mathrm{d}x}_{I_2} - \frac{1}{3}\underbrace{\int \frac{1}{x^2+2}\,\mathrm{d}x}_{I_3} = (**)$$

$$I_1 = \int \frac{1}{x-1}\,\mathrm{d}x = \log|x-1|$$

$$I_2 = \int \frac{x}{x^2+2}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+2}\,\mathrm{d}x = \left\{\begin{array}{l} x^2+2 = t \\ 2x\,\mathrm{d}x = \mathrm{d}t \end{array}\right\} = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|x^2+2|$$

$$I_3 = \int \frac{1}{x^2+2}\,\mathrm{d}x = \int \frac{\frac{1}{2}}{\frac{x^2}{2}+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,\mathrm{d}x = \frac{\sqrt{2}}{2}\int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,\mathrm{d}x = \left\{\begin{array}{l} \frac{x}{\sqrt{2}} = t \\ \frac{1}{\sqrt{2}}\,\mathrm{d}x = \mathrm{d}t \end{array}\right\}$$

$$= \frac{\sqrt{2}}{2}\int \frac{1}{t^2+1}\,\mathrm{d}t = \frac{\sqrt{2}}{2}\arctan(t) = \frac{\sqrt{2}}{2}\arctan\left(\frac{x}{\sqrt{2}}\right)$$

$$(**) = \boxed{\frac{1}{3}\log|x-1| - \frac{1}{6}\ln|x^2+2| - \frac{\sqrt{2}}{6}\arctan\left(\frac{x}{\sqrt{2}}\right) + \mathrm{C}}$$

44) Calcular la siguiente integral:

$$\int \sin^2(x)\cdot\cos^2(x)\,\mathrm{d}x = \left\{\begin{array}{l} \sin(2x) = 2\sin(x)\cos(x) \\ \sin^2(2x) = 4\sin^2(x)\cos^2(x) \end{array}\right\} = \int \frac{1}{4}\sin^2(2x)\mathrm{d}x = \left\{\sin^2(x) = \frac{1-\cos^2(x)}{2}\right\} =$$

$$\frac{1}{4}\int \frac{1-\cos(4x)}{2}\,\mathrm{d}x = \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + \mathrm{C} = \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + \mathrm{C}}$$

45) Calcular la siguiente integral:

$$\int x^2\sin(3x)\,\mathrm{d}x = \left\{\begin{array}{ll} u = x^2 & \mathrm{d}u = 2x \\ \mathrm{d}v = \sin(3x)\,\mathrm{d}x & v = -\dfrac{\cos(3x)}{3} \end{array}\right\} = -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\int x\cos(3x)\,\mathrm{d}x$$

$$= \left\{\begin{array}{ll} u = x & \mathrm{d}u = \mathrm{d}x \\ \mathrm{d}v = \cos(3x)\,\mathrm{d}x & v = \dfrac{\sin(3x)}{3} \end{array}\right\} = -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\left[x\frac{\sin(3x)}{3} - \int \frac{\sin(3x)}{3}\,\mathrm{d}x\right]$$

$$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) - \frac{2}{9}\int \sin(3x)\,\mathrm{d}x$$

$$= \boxed{-\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + \mathrm{C}}$$

46) Calcular la siguiente integral:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = \int \sqrt{4-\sin(t)}\ \ \mathrm{d}t = \left\{\begin{array}{l} x = 2\sin(t) \\ \mathrm{d}x = 2\cos(t)\,\mathrm{d}t \end{array}\right\} =$$

$$\int \sqrt{4-(2\sin(t))^2}\cdot 2\cos(t)\ \mathrm{d}t = \int \sqrt{4-4\sin^2(t)}\cdot 2\cos(t)\ \mathrm{d}t =$$

$$\int \sqrt{4\underset{\overset{\|}{\cos^2(t)}}{(1-\sin^2(t))}}\cdot 2\cos(t)\,\mathrm{d}t = 4\int \cos^2(t)\,\mathrm{d}t = 4\int \frac{1+\cos(2t)}{2}\,\mathrm{d}t =$$

$$2\left[t + \frac{\sin(2t)}{2}\right] + \mathrm{C} = 2t + 2\sin(t)\cos(t) + \mathrm{C} = (*)$$

Nota:

$\sqrt{a^2-x^2}$

$x = a\sin(t)$

$$x = 2\sin(t) \longrightarrow \boxed{\sin(t) = \frac{x}{2}} \longrightarrow \boxed{t = \arcsin\left(\frac{x}{2}\right)}$$

$$\cos^2(t) + \sin^2(t) = 1 \longrightarrow \cos(t) = \sqrt{1-\sin^2(t)} = \sqrt{1-\left(\frac{x}{2}\right)^2} = \sqrt{1-\frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$$

$$\longrightarrow \boxed{\cos(t) = \frac{1}{2}\sqrt{4-x^2}}$$

$$(*) = 2\arcsin\left(\frac{x}{2}\right) + \cancel{2}\cdot\frac{x}{\cancel{2}}\cdot\frac{1}{2}\sqrt{4-x^2} + \mathrm{C} = \boxed{2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\sqrt{4-x^2} + \mathrm{C}}$$

47) Calcule $f(4)$ si

$$\int_0^{x^2} f(t)\,\mathrm{d}t = x\cos(\pi x)$$

Lo que vamos a hacer es derivar esta ecuación:

$$\left(\int_0^{x^2} f(t)\,\mathrm{d}t\right)' = f(x^2)\cdot 2x - \cancel{f(0)\cdot 0} = 2xf(x^2)$$

$(x\cos(\pi x))' = 1\cos(\pi x) + x\cdot(-\sin(\pi x))\cdot\pi = \cos(\pi x) - x\sin(\pi x)$

Igualando: $2xf(x^2) = \cos(\pi x) - x\sin(\pi x)$

$$\boxed{f(x^2) = \frac{1}{2x}\left[\cos(\pi x) - x\sin(\pi x)\right]}$$

$f(4) = f(2^2) = \{x = 2\} = \frac{1}{4}[\underbrace{\cos(2\pi)}_{1} - \cancel{2\sin(2\pi)}^{0}] \longrightarrow$

$$\boxed{f(4) = \frac{1}{4}}$$

Nota: Dada la función $T(x) = \int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t$, para hallar su derivada respecto de x, debemos aplicar la siguiente propiedad:

$$T'(x) = \left(\int_{h(x)}^{g(x)} f(t)\,\mathrm{d}t\right)' = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$$