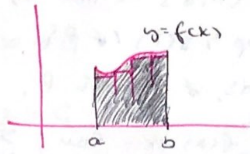
Tema 5. Integrales. BLOQUE III

5.1 La integral de Riemann

Si $f:[a,b] \to \mathbb{R}$ cont (y positiva), se define

"$\int_b^a f(x)\,dx = $ Área"



¿Cómo se puede calcular explícitamente esas áreas?

Método de Riemann

Paso 1 → Tomar partición de $[a,b]$

$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$

Solo consideramos particiones "equiespaciadas":

$n = \Delta t = \frac{b-a}{n}$ $\begin{cases} t_0 = a + j\cdot\Delta t, & j = 1\ldots n \\ I_0 = [t_{j-1}, t_j] \end{cases}$

Paso 2 → Aproximar, por defecto y por exceso, usando rectángulos R_j con base I_j.

- Suma inferior de Riemann

$$s(P) := \sum_{j=1}^{n} \text{Área}(R_j) = \sum_{j=1}^{n} \underbrace{\Delta t}_{\text{base}} \cdot m_j \quad , \text{ donde } m_j = \min_{x\in I_j} f(x)$$

- Suma superior de Riemann

$$S(P) := \sum_{j=1}^{n} \text{Área}(\tilde{R}_j) = \sum_{j=1}^{n} \Delta t \cdot M_j \quad , \text{ donde } M_j = \max_{x\in I_j} f(x)$$

Paso 3 → Definir el candidato a área como

$$\int_a^b f(x)\,dx := \sup\{s(P) / \forall P\} = \inf\{S(P) / \forall P\}$$

o bien $\int_a^b f(x)\,dx := \lim_{\Delta t\to 0} s(P_{\Delta t}) = \lim_{\Delta t \to 0} S(P_{\Delta t})$

Def. $f:[a,b]\to\mathbb{R}$ es integrable de Riemann en $[a,b]$, $f\in R[a,b]$ si se tienen las igualdades en (1, ó 2)

Ejemplo.

$$f(x) = \begin{cases} 1, & x\in \mathbb{Q}\cap[0,1] \\ 0, & x\in (\underbrace{\mathbb{R}-\mathbb{Q}}_{\mathbb{I}\ (\text{irracionales})} \cap [0,1] \end{cases}$$

$\to m_j = \min_{x\in I_j} f(x) = 0 \to s(P) = 0,\ S(P) = 1 \ /\forall P$

$\to M_j = \max_{x\in I_j} f(x) = 1$ $\Rightarrow f$ no es integrable Riemann

Teorema 1.

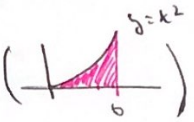
Si $f:[a,b]\to\mathbb{R}$ es continua $\Rightarrow f\in R([a,b])$, es decir, se cumple

(1) $\sup\{s(P) / \forall P\} = \inf\{S(P) / \forall P\}$

Además, $\forall$ partición P, y todo punto $c_i\in I_i$, se tiene

(2) $\int_a^b f(x)\,dx = \lim_{\Delta t\to 0}\sum_{i=1}^{n} f(c_i)\cdot\Delta t$

Ejemplos.

① A = Área () Método de Riemann

$t_i = 0 + j\cdot\frac{b}{n}$, $j=0,\dots,n$, $\Delta t=\frac{b}{n}$

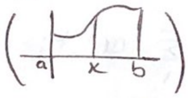
$S(P) = \sum_{i=1}^{n} M_i\cdot\Delta t = \sum_{i=1}^{n}(t_i)^2\cdot\Delta t = \sum_{i=1}^{n}\left(i\frac{b}{n}\right)^2\cdot\frac{b}{n} = \left(\frac{b}{n}\right)^3\sum_{i=1}^{n} i^2 =$

$= \frac{b^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$ $\lim_{\Delta t\to 0} S(P) = \lim_{n\to\infty} b^3\cdot\frac{n(n+1)(2n+1)}{6n^2}$

$\int_0^b x^2\,dx = \lim_{\Delta t\to 0} S(P) = \frac{b^3}{3}$ /

Teorema 2. Teorema fundamental del cálculo.

(I) Si $f:[a,b]\to\mathbb{R}$ es continua

$\to F(x) := \int_a^x f(t)\,dt$ = Área ()

cumple que $F\in D(a,b)$ y $F'(x)=f(x)$

* $F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

(II) Regla de Barrow

Si $f:[a,b]\to\mathbb{R}$ derivable y $f'\in C[a,b]$

$\Longrightarrow \int_a^b f'(t)\,dt = f(b)-f(a)$

* $\int_a^x f'(t)\,dt = f(x)+C$

Def. Dada $f\in C([a,b])$, decimos que

$F(x)$ es una primitiva (o antiderivada) de $f(x)$

si $F'(x) = f(x)$, $\forall x\in(a,b)$

El conjunto de todas las primitivas de f se suele llamar integral indefinida de f, que se denota

"$\int f(x)\,dx$" $:= \{F / F' = f\}$

5.2 Cálculo de primitivas

Lista natural de primitivas inmediatas

- $\int x^{\alpha}\,dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C$
- $\int \dfrac{dx}{x} = \ln|x| + C$
- $\int e^{x}\,dx = e^{x} + C$
- $\int a^{x}\,dx = \dfrac{a^{x}}{\ln a} + C$
- $\int \operatorname{sen} x\,dx = -\cos x + C$
- $\int \cos x\,dx = \operatorname{sen} x + C$
- $\int \dfrac{dx}{1+x^2} = \arctan x + C$
- $\int \dfrac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

→ ¿Qué ocurre si piden $\int f(x)\,dx$, y $f(x)$ no está en la lista?

① Usar técnicas de integración

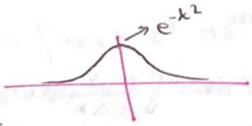
a C.V. (cambio de variables)
b Partes
c Racionales
d otras...

② Hay que usar métodos numéricos para aproximar el área:

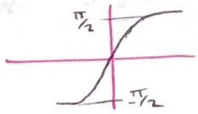
$$\int_a^x f(t)\,dt = \text{Área}$$

Ejemplos de funciones $f(x)$ cuya primitiva no se puede expresar mediante funciones elementales.

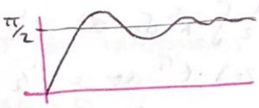
- $\int e^{-x^2}\,dx \longrightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} = \text{Área}$



pero $\int_a^x e^{-t^2}\,dt =$ " erf (x) "



- $\int_0^x \dfrac{\operatorname{sen} t}{t}\,dt = \text{Área} \longrightarrow$



- $\int \operatorname{sen}(x^2)\,dx \longrightarrow \int_0^{\infty} \operatorname{sen}(x^2)\,dx = \sqrt{\pi/8}$
- $\int x^{\alpha} e^{-x}\,dx$ donde $\alpha \notin \mathbb{N}$ $\qquad \Gamma(\alpha) = \int_0^{\infty} x^{\alpha} e^{-x}\,\dfrac{dx}{x} \longrightarrow$ formación Gamma de Euler

Repaso de las técnicas de integración

I) Cambio de variables

$$\int f(g(x))\cdot g'(x)\,dx = \int f(t)\,dt$$

$\left.\begin{array}{l} g(x)=t \\ g'(x)\,dx = dt \end{array}\right\}$

① $\int e^{-3x}\,dx = \int e^t \cdot \frac{dt}{-3} = -\frac{1}{3}\int e^t\,dt =$

$= -\frac{1}{3}e^t + C_1 = -\frac{1}{3}e^{-3x} + C_1$

$\left.\begin{array}{l} -3x = t \\ -3\,dx = dt \\ dx = \frac{dt}{-3} \end{array}\right\}$

② $\int \frac{dx}{5x+6} = \frac{1}{5}\int \frac{dt}{t} = \frac{1}{5}\ln|t| + C_1 = \frac{1}{5}\ln(5x+6) + C_1$

$\left.\begin{array}{l} 5x+6 = t \\ 5\,dx = dt \end{array}\right\}$

③ $\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx = \int \frac{x}{1+t^2}\cdot\frac{dt}{2x} = \frac{1}{2}\arctan(t) + C_1$

$x^2 = t$ $\qquad\qquad\qquad\qquad = \frac{1}{2}\arctan(x^2) + C_1$

$2x\,dx = dt \to dx = \frac{dt}{2x}$

④ $\int (3x^2-1)^{10}x\,dx = \int t^{10}\cdot x\,\frac{dt}{6x} = \frac{1}{6}\cdot\frac{t^{11}}{11} + C_1 = \frac{1}{66}\cdot(3x^2-1)^{11} + C_1$

$3x^2-1 = t$

$6x\,dx = dt$

⑤ $\int \tan x\,dx = \int \frac{\operatorname{sen} x}{\cos x}\,dx = \int \frac{\operatorname{sen} x}{t}\cdot\frac{dt}{-\operatorname{sen} x} = -\ln|t| + C_1 = -\ln|\cos x| + C_1$

$\cos x = t$

$-\operatorname{sen} x\,dx = dt$

$\int x^5\sqrt{1-x^2}\,dx = \int x^5\sqrt{t}\,\frac{dt}{-2x} = -\frac{1}{2}\int x^4\sqrt{t}\,dt = -\frac{1}{2}\int (1-t)^2\sqrt{t}\,dt =$

$= -\frac{1}{2}\int (1+t^2-2t)\cdot t^{1/2}\,dt = -\frac{1}{2}\left(\frac{t^{3/2}}{3/2} + \frac{t^{7/2}}{7/2}\right) + C_1$

$= -\left(\frac{(1-x^2)^{3/2}}{3} + \frac{(1-x^2)^{7/2}}{7} - \frac{2}{5}(1-x^2)^{5/2}\right) + C_1$

- Cambio de variables en integrales definidas.

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt$$

$g(x) = t$ $\qquad x=a \to t = g(a)$

$g'(x)\,dx = dt$ $\qquad x=b \to t = g(b)$

$\left\{\begin{array}{l} \operatorname{sen} x + 8 = t \\ \cos x\,dx = dt \end{array}\right.$

$\int_0^{\pi/2} \frac{\cos x\,dx}{(\operatorname{sen} x+8)^2} = \int_8^9 \frac{dt}{t^2} = \int_8^9 t^2\,dt = \left[\frac{-1}{t}\right]_8^9 = \frac{1}{8} - \frac{1}{9}$

$\left\{\begin{array}{l} x=0 \to t = \operatorname{sen} 0 + 8 = 8 \\ x = \frac{\pi}{2} \to t = \operatorname{sen}\frac{\pi}{2} + 8 = 9 \end{array}\right.$

II, Integración por partes

ALPES → sen/cos/tan
↓ arc / log / polinomio / exponente

$$\int_a^b f(x)\cdot g'(x)\,dx = f(x)\,g(x)\Big]_{x=a}^{x=b} - \int_a^b f'(x)\,g(x)\,dx$$

||

$$\int u\,dv = uv - \int v\,du \qquad \begin{cases} u = f(x) \to du = f'(x)\,dx \\ dv = g'(x)\,dx \to v = g(x) \end{cases}$$

① $\int x\cdot \operatorname{sen}(2x)\,dx = \int x\cdot\left(-\frac{\cos(2x)}{2}\right)' dx = \frac{-x\cos(2x)}{2} + \int x'\cdot\frac{\cos(2x)}{2}\,dx =$

$= \frac{-x\cos(2x)}{2} + \frac{1}{2}\int \cos(2x)\,dx = \frac{-x\cos(2x)}{2} + \frac{1}{2}\,\frac{\operatorname{sen}(2x)}{2} + C_1$

② $\int x^2 \cos x\,dx = \int x^2(\operatorname{sen} x)'\,dx = x^2 \operatorname{sen} x - \int x^{2\,\prime} \operatorname{sen} x\,dx =$

$= x^2 \operatorname{sen} x - 2\int x \operatorname{sen} x\,dx = x^2\operatorname{sen} x - 2\left[\int x(-\cos x)'\,dx\right] = *$

$-x\cos x + \int x' \cos x\,dx = -x\cos x + \operatorname{sen} x$

$* \; x^2 \operatorname{sen} x + 2x\cos x - 2\operatorname{sen} x + C_1$

③ $\int x^2 e^{3x}\,dx =$

④ $\int \ln x\,dx = x\ln x - \int x\cdot\frac{dx}{x} = x\ln x - x + C = x(\ln x - 1) + C =$

$u = \ln x \to du = \frac{dx}{x}$ $\qquad = x\ln\left(\frac{x}{e}\right) + C$

$dx = dv \to v = x$

⑤ $\int \operatorname{sen} x\cdot e^x\,dx = \int \operatorname{sen} x\,(e^x)'\,dx = \operatorname{sen} x\, e^x - \int \cos x\, e^x\,dx =$

$= e^x \operatorname{sen} x - \left[\cos x\, e^x - \int(\cos x)'\, e^x\,dx\right] = e^x \operatorname{sen} x - e^x\cos x - \underbrace{\int \operatorname{sen} x\, e^x\,dx}_{I}$

$2I = e^x(\operatorname{sen} x - \cos x)$

$I = \frac{1}{2}\, e^x(\operatorname{sen} x - \cos x) + C_1$

Def. Decimos que $f(x)$, $g(x)$ son ortogonales en $[a,b]$ si

$$\langle f, g\rangle := \int_a^b f(x)\,g(x)\,dx = 0$$

Ejercicio. Probar que $\langle \operatorname{sen}(nx), \operatorname{sen}(mx)\rangle = 0$ si $n \neq m$ en $[0,\pi]$

$I = \int_0^\pi \operatorname{sen}(nx)\operatorname{sen}(mx)\,dx = \int_0^\pi \operatorname{sen}(nx)\cdot\left(\frac{-\cos(mx)}{m}\right)' dx =$

$= \cancel{\left[\frac{-\operatorname{sen}(nx)\cos(mx)}{m}\right]_0^\pi}^{\,0} + \frac{1}{m}\int_0^\pi n\cos(nx)\cos(mx)\,dx = \frac{n}{m}\int_0^\pi \cos(nx)\left(\frac{\operatorname{sen}(mx)}{m}\right)' dx$

$= \frac{n}{m}\left(\left[\cos(nx)\cdot\frac{\operatorname{sen}(mx)}{m}\right]_0^\pi - \int_0^\pi -\operatorname{sen}(nx)\,n\cdot\frac{\operatorname{sen}(mx)}{m}\,dx\right) =$

$= \frac{n^2}{m^2}\int_0^\pi \operatorname{sen}(nx)\operatorname{sen}(mx)\,dx = \frac{n^2}{m^2} I \to \left(1 - \frac{n^2}{m^2}\right) I = 0 <$ o bien $1 - \frac{n^2}{m^2} = 0$ ✗ $n \neq m$; o bien $I = 0$

III, Integración de funciones racionales

Queremos hallar primitivas de

$$I = \int \frac{P(x)}{Q(x)} d(x) \quad \text{donde } P(x), Q(x) = \text{polinomios}$$

↳ se usa la descomposición de $\frac{P(x)}{Q(x)}$ en fracciones simples
esto permite reducir el cálculo de I a 4 pasos sencillos

$$\int \frac{dx}{x-a}, \quad \int \frac{dx}{(x-a)^n}, \quad \int \frac{dx}{x^2+a^2},$$

Paso 1. Suponer $\deg(P) < \deg(Q)$
(si no fuera así dividir P/Q)

Paso 2. Factorizar $Q(x)$ en términos de sus raíces

$$Q(x) = \underbrace{(x-a_1)^{n_1} \cdots (x-a_N)^{n_N}}_{\text{raíces reales } a_j} \cdot \underbrace{((x-b_1)^2+c_1^2)^{m_1} \cdots ((x-b_M)^2+c_M^2)^{m_M}}_{\text{raíces complejas } b_j \pm i c_j}$$

Ruffini

Paso 3. Descomposición en fracciones simples.

$$\frac{P(x)}{Q(x)} = \sum_{a} \left(\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n} \right) + \sum_{b \pm ic} \left(\frac{B_1 x + C_1}{(x-b)^2+c^2} + \cdots + \frac{B_m x + C_m}{((x-b)^2+c^2)^m} \right)$$

para cada raíz real — raíces complejas

Paso 4. Integrar las fracciones simples

$$\int \frac{P(x)}{Q(x)} dx = \sum_{a} \left[\sum_{j=1}^{n} \int \frac{dx}{(x-a)^j} \right] + \cdots$$

Ejemplos.

① $\int \frac{dx}{x(1-x)} \rightarrow \frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x)+Bx}{x(1-x)}$

$$\underline{0}x + 1 = \underline{(B-A)} \cdot x + A \rightarrow \begin{cases} B-A=0 \\ A=1 \end{cases}$$

$$\rightarrow B = A = 1 \rightarrow \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$\Rightarrow \int \frac{dx}{x(1-x)} = \int \frac{dx}{x} + \int \frac{dx}{1-x} = \ln|x| - \ln|1-x| + C_1 = \ln\left|\frac{x}{1-x}\right| + C_1$$

② $\displaystyle\int \frac{3x-2}{(x-1)^2(x+2)}\,dx$

$$\to \frac{3x-2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} = \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)} \to$$

$$\xrightarrow{\dots} = \frac{8/9}{x-1} + \frac{1/3}{(x-1)^2} - \frac{8/9}{x+2}$$

$$\int \frac{3x-2}{(x-1)^2(x+2)}\,dx = \frac{8}{9}\ln|x-1| + \frac{1}{3}\cdot\frac{(x-1)^{-1}}{-1} - \frac{8}{9}\ln|x+2| + C =$$

$$= \frac{8}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3}\cdot\frac{1}{x-1} + C$$

③ $\displaystyle\int \frac{x}{x^3-x^2+4x-4}\,dx = \int \frac{x}{(x-1)(x^2+4)}\,dx$

$$\to \frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{1/5}{x-1} - \frac{1}{5}\cdot\frac{x-4}{x^2+4}$$

$$\int \frac{x}{x^3-x^2+4x-4} = \frac{1}{5}\int\frac{dx}{x-1} - \frac{1}{5}\int\frac{x-4}{x^2+4}\,dx = \frac{1}{5}\ln(x-1) - \frac{1}{5}\left[\frac{1}{2}\underbrace{\int\frac{2x}{x^2+4}\,dx}_{\ln|x^2+4|} - 4\underbrace{\int\frac{dx}{x^2+4}}_{2\arctan\left(\frac{x}{2}\right)}\right] =$$

$$= -\frac{1}{10}\ln|x^2+4| + \frac{2}{5}\arctan\left(\frac{x}{2}\right) + \frac{1}{5}\ln|x-1| + C$$

④ $\displaystyle\int\frac{dx}{(x-1)^2(x^2+3)} \to \frac{1}{(x-1)^2(x^2+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+3} =$

$$= \frac{-1/8}{x-1} + \frac{1/4}{(x-1)^2} + \frac{1}{8}\cdot\frac{x-1}{x^2+3}$$

$$I = -\frac{1}{8}\int\frac{dx}{x-1} + \frac{1}{4}\int(x-1)^{-2}\,dx + \frac{1}{8}\int\frac{x-1}{x^2-3}\,dx = *$$

$$= -\frac{1}{8}\ln|x-1| - \frac{1}{4}\cdot\frac{1}{x-1} + \frac{1}{16}\ln|x^2+3| - \frac{1}{8}\cdot\frac{1}{\sqrt{3}}\cdot\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

$$* = -\frac{1}{8}\ln|x-1| + \frac{1}{4}\cdot\frac{(x-1)^{-1}}{-1} + \frac{1}{8}\cdot\frac{1}{2}\int\frac{2x}{x^2+3}\,dx - \frac{1}{8}\int\frac{dx}{x^2+3}$$

IV, Otros métodos

- Trigonométricos → $\int R(\operatorname{sen} x, \cos x)\, dx = \int R(t)\, dt$

- Cambio de variable universal →

$$\begin{cases} \tan\left(\frac{x}{2}\right) = t \\ dx = \frac{2dt}{1+t^2} \end{cases} \quad \Bigg| \quad \operatorname{sen} x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$-\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} = \frac{x}{2} + \frac{1}{2} \cdot \frac{\operatorname{sen}(2x)}{2} + C_1$$

$$\cos^2 x - \operatorname{sen}^2 x = \cos(2x) = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$$

$$\hookrightarrow \boxed{\cos^2 x = \frac{1+\cos(2x)}{2}} \quad / \quad \boxed{\operatorname{sen}^2 x = \frac{1-\cos(2x)}{2}}$$

$$-\int \operatorname{sen}^2 x \cdot \cos^4 x \, dx$$

$$(\operatorname{sen}(2x) = 2\operatorname{sen} x \cos x)$$

$$\to I = \frac{1}{4}\int \operatorname{sen}^2(2x)\cdot \cos^2 x \, dx = \frac{1}{4}\int \operatorname{sen}^2(2x)\cdot \frac{1+\cos(2x)}{2}\, dx =$$

$$= \frac{1}{8}\int \operatorname{sen}^2(2x)\, dx + \frac{1}{8}\int \operatorname{sen}^2(2x)\cdot \cos(2x)\, dx =$$

$$= \frac{1}{8}\int \frac{1-\cos(4x)}{2}\, dx + \frac{1}{8}\int t^2 \cdot \cancel{\cos(2x)} \frac{dt}{\cancel{\cos(2x)}} =$$

$$\begin{cases} \operatorname{sen}(2x) = t \\ 2\cos(2x) = dt \end{cases}$$

$$= \frac{1}{16}\left(x - \frac{\operatorname{sen}(4x)}{4}\right) + \frac{1}{16}\cdot\frac{t^3}{3} + C_1 = \frac{1}{16}\left(x - \frac{\operatorname{sen}(4x)}{4}\right) + \frac{1}{16}\cdot\frac{\operatorname{sen}^3(2x)}{3} + C_1$$

5.3 Integración numérica

Queremos calcular $I = \int_a^b f(x)\,dx$ = Área, aproximadamente el área con figuras geométricas.

① Sumas de Riemann

$$\Rightarrow E_n = |I - I_n| \leq \underbrace{\frac{M \cdot (b-a)^2}{n}}$$

$\sigma = \left(\frac{1}{n}\right) \rightsquigarrow$ gran error

② Método del trapecio

$$\Rightarrow \text{Área}(T) = \underbrace{\frac{f(a)+f(b)}{2}}_{\text{media de alturas}} \cdot \underbrace{(b-a)}_{\text{base}}$$

$$I_n = \sum_{i=1}^{n} \text{Área}(T_i)$$

$$I = \int_a^b f(x)\,dx$$

$$I_n = \frac{b-a}{n}\left[\frac{f(a)}{2} + f(x_1) + \dots + f(x_{n-1}) + \frac{f(b)}{2}\right]$$

$$\Rightarrow E_n = \frac{M \cdot (b-a)^3}{12 \cdot n^2}$$

Teorema. Error del método del trapecio
Si $f \in D^2([a,b])$ y $|f''(x)| \leq M$, $x \in [a,b]$
$$\longrightarrow |I - I_n| \leq \boxed{\frac{M \cdot (b-a)^3}{12 n^2}}$$

③ Método de Simpson (1750)

Tomamos n = par y aproximamos el área con parábolas en lugar de trapecios.

Lema. Sea $P(x)$ la parábola que pasa por $(a, f(a))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$, $(b, f(b))$

$$\Rightarrow \int_a^b P(x)\,dx = \frac{b-a}{6} \cdot \left(f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Teorema 2. Error del método de Simpson
Si $f \in D^4([a,b])$ y $|f^{IV}(x)| \leq M$ en $[a,b]$
$$\Rightarrow |I - I_n| \leq \boxed{\frac{M \cdot (b-a)^5}{180 \cdot n^4}}$$

Ejemplo. Hallar $\int_0^1 e^{-x^2}\,dx$ con error $<10^{-4}$ (usando Simpson)

$f(x)=e^{-x^2}$, $x\in[0,1]$ Busco n / $E_n=|I-I_n|\le \frac{M(b-a)^5}{180\cdot n^4}<10^{-4}$

$|f^{IV)}(x)|=|4\cdot(4x^4-12x^2+3)e^{-x^2}|\le 4(4x^4+12x^2+3)e^{-x^2}=$

$=4\cdot 19=76 \to M$

Mirando la gráfica observamos que $|f^{IV}(x)|=|f^{IV}(0)|=12$

$\to$ Escojo $12=M$ (porque es mejor que 76)

Busco n / $\frac{12\cdot(1-0)^5}{180\cdot n^4}<10^{4} \to n^4>\frac{12\cdot 10^4}{180}=\frac{1}{9}10^4 \to$

$\to n>\frac{1}{\sqrt[4]{9}}\cdot 10=\frac{1}{\sqrt{3}}\cdot 10=5'77 \to$ tomo $n=6$

5.4 Integrales impropias

Def. Si $f:[a,+\infty)\to\mathbb{R}$ se define

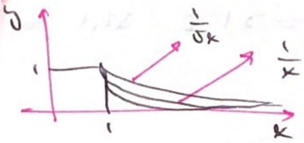
$\int_a^\infty f(x)\,dx=$ Área () $=\lim_{R\to\infty}\int_a^R f(x)\,dx$

Diremos que $f\in\mathcal{R}([a,\infty))$ (integrable de Riemann impropia en $[a,+\infty)$) si existe el límite anterior.

Ejemplo. $\alpha>0$

① $\int_1^\infty \frac{dx}{x^\alpha}=\lim_{R\to\infty}\int_1^R x^{-\alpha}\,dx=\lim_{R\to\infty}\left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_{x=1}^{x=R}=\lim_{R\to\infty}\frac{1}{1-\alpha}\left(\frac{1}{R^{\alpha-1}}-1\right)=$

$=\begin{cases}\frac{1}{\alpha-1}, & \text{si } \alpha>1\\ +\infty, & \text{si } \alpha<1\end{cases}$



si $\alpha=1 \to \int_1^\infty \frac{dx}{x}=\lim_{R\to\infty}\int_1^R\frac{dx}{x}=\lim_{R\to\infty}\ln|x|\Big]_1^R=\lim_{R\to\infty}\ln R-\cancel{\ln 1}=\underline{\underline{\infty}}$

Conclusión

$\boxed{\int_1^\infty \frac{1}{x^\alpha}\,dx<\infty \Rightarrow \alpha>1}$

$\int_0^1 \frac{1}{x^\alpha}\,dx<\infty \Leftrightarrow \alpha<1$

$\int_1^\infty \frac{1}{x^\alpha}\,dx<\infty \Leftrightarrow \alpha>1$

② $\int_0^\infty x e^{-x}\,dx = \int_0^\infty x(-e^{-x})'\,dx = \left[x(-e^{-x})\right]_0^\infty + \int_0^\infty (x)'\, e^{-x}\,dx =$

$= \lim_{R\to\infty}\left[\cancel{-Re^{-R}} + 0\right] + \left[-e^{-x}\right]_0^\infty = -e^{-\infty} + e^{-0} = 1$

$\Rightarrow x e^{-x} \in R[0,+\infty)$

Ver ejer 12.

Practicar.

$\int_0^\infty \frac{dx}{\sqrt{x+100}}$ $\qquad \int_0^\infty \frac{dx}{1+4x^2}$ $\qquad \int_3^\infty \frac{x}{1+4x^2}\,dx$

∞ no integrable — finito integrable — ∞ no integrable

Teorema de comparación.

① Si $|f(x)| \le h(x)$ y $\int_0^\infty h(x)\,dx < \infty \Rightarrow \exists \int_0^\infty f(x)\,dx \in \mathbb{R}$

② Si $|f(x)| \ge h(x) \ge 0$ y $\int_0^\infty h(x)\,dx = \infty \Rightarrow \int_0^\infty f(x)\,dx = \infty$

Ejemplos.

- $\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = -e^{-x}\Big]_1^\infty = -e^{-\infty} + e^{-1} = e^{-1} < \infty$

  $\Rightarrow \int_1^\infty e^{-x^2}\,dx < \infty$

- $\int_0^\infty \frac{dx}{1+x^8} = \int_0^1 \frac{dx}{1+x^8} + \int_1^\infty \frac{dx}{1+x^8} \le \int_0^1 \frac{dx}{1+0} + \int_1^\infty \frac{dx}{x^8} = 1 + \left[\frac{x^{-7}}{-7}\right]_1^\infty = 1 + \frac{1}{7} < \infty$

  $\Rightarrow \frac{1}{1+x^8} \in \mathbb{R}[0,\infty)$

- $\int_0^\infty \frac{x}{\sqrt{1+x^3}}\,dx \ge \int_1^\infty \frac{x}{\sqrt{1+x^3}}\,dx \ge \int_1^\infty \frac{x}{\sqrt{2x^3}}\,dx = \frac{1}{\sqrt{2}}\int_1^\infty \frac{x}{x^{3/2}}\,dx = *$

  $x \ge 1 \to 1 \le x \to 1 \le x^3 \to \frac{1}{1+x^3} \ge \frac{1}{x^3+x^3} = \frac{1}{2x^3}$

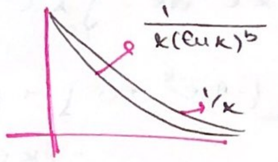
$* = \frac{1}{\sqrt{2}}\int_1^\infty x^{-1/2}\,dx = \frac{1}{\sqrt{2}}\left[\frac{x^{1/2}}{1/2}\right]_1^\infty = \infty$

* Nota: $\int_a^\infty \frac{P(x)}{Q(x)}\,dx < \infty \iff \deg(Q) > \deg(P) + 1$

Ver ejercicio 9.

• $\int_e^\infty \frac{dx}{x(\ln x)^b} = \int_1^\infty \frac{dt}{t^b} < \infty \iff b>1$

$\ln x = t$, $\frac{dx}{x} = dt$ $\left\{ \begin{array}{l} x = e \to t = 1 \\ x = +\infty \to t = \infty \end{array} \right.$



• $\int_1^\infty \frac{\operatorname{sen} x}{x}\, dx = \int_1^\infty \frac{1}{x}(-\cos x)'\, dx = \left[\cancel{\frac{-\cos x}{x}}\right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2}\, dx = ✻$

$\cancel{\frac{-\cos x}{x}} \to 0$

Se intenta usar Tma comp. ①

$|f(x)| = \left|\frac{\operatorname{sen} x}{x}\right| \le \frac{1}{x} = h(x)$

si $x \in [1, \infty)$ y $\int_1^\infty \frac{dx}{x} = +\infty$

$\cancel{\Rightarrow \int_1^\infty \frac{\operatorname{sen} x}{x}\, dx = \infty}$

✻ Ahora $|f_1(x)| = \left|\frac{-\cos x}{x^2}\right| \le \frac{1}{x^2} = h(x)$, y $\int_1^\infty \frac{1}{x^2}\, dx < \infty$

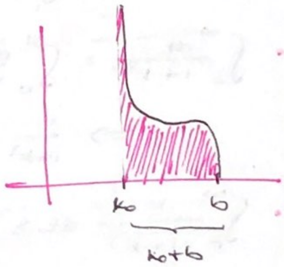
Tma comp. ①

$\Longrightarrow \int f_1(x)\, dx$ es finita.

Def 2. (Integral impropia en un punto x_0)

Sea $f: (x_0, b] \xrightarrow{\text{cont}} \mathbb{R}$ tal que $\lim_{x \to x_0^+} f(x) = +\infty$

Se define $\int_{x_0}^b f(x)\, dx = \lim_{\varepsilon \to 0^+} \int_{x_0+\varepsilon}^b f(x)\, dx$



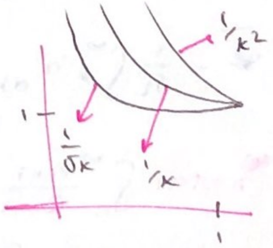
Ejemplo.

1. $f(x) = \frac{1}{x^\alpha}$, $x \in (0, 1]$

$\alpha = 1 \to \int_0^1 \frac{dx}{x} = \ln x\Big]_0^1 = \ln 1 - \ln 0 = \ln 1 + \infty = \infty$

$\hookrightarrow \frac{1}{x} \notin \mathcal{R}([0,1])$

$\alpha \neq 1 \to \int_0^1 \frac{dx}{x^\alpha} = \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_0^1 = \frac{1 - 0^{1-\alpha}}{1-\alpha} =$



$= \begin{cases} \frac{1}{1-\alpha}, & 0 < \alpha < 1 \\ +\infty, & \alpha > 1 \end{cases}$

$\Rightarrow \frac{1}{x^\alpha} \in \mathbb{R}([0,1]) \iff \alpha < 1$

5.5 Ecuaciones diferenciales

Una ecuación diferencial es una expresión

$$\begin{cases} x'(t) = F(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

en la que:

$x(t) \to$ función incógnita

$(t_0, x_0) \to$ dato inicial

$F \to$ función conocida (suele provenir de una ley física)

Método de separación de variables

$$\begin{cases} x'(t) = 3 \cdot t \cdot x(t)^2 \\ x(0) = 1 \end{cases}$$

Paso 1 → escribir fórmula simplificada

$$\frac{dx}{dt} = 3 \cdot t \cdot x^2$$

Paso 2 → separar variables t y x

$$\frac{dx}{x^2} = 3t \, dt$$

Paso 3 → integrar

$$\int \frac{dx}{x^2} = \int 3t \, dt \to \frac{x^{-1}}{-1} = 3t^2 + C_1 \to x = \frac{-1}{3t^2 + C_1}$$

Paso 4 → sacar constante con el dato inicial

$$1 = \frac{-1}{3 \cdot 0 + C_1} \to \boxed{C_1 = -1} \to x(t) = \frac{-1}{3t^2 + (-1)} = \boxed{\frac{1}{1 - 3t^2}}$$