Señales y Sistemas

Problemas Unidad 4

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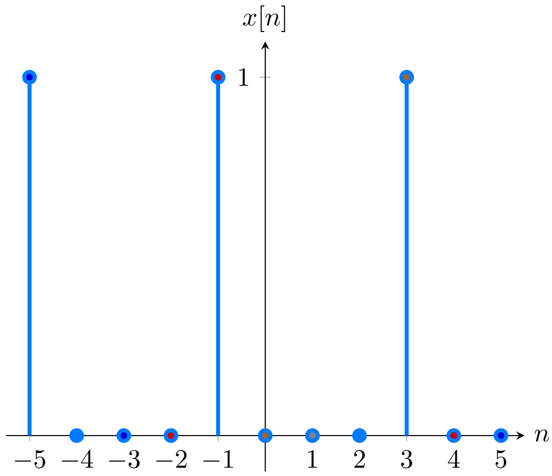
1) Calcular los coeficientes del desarrollo en series de Fourier

$$x[n] = \sum_{m=-\infty}^{\infty} \delta[n+1-4m]$$



Periodo: $N = 4 \implies \omega_0 = \dfrac{2\pi}{N} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

$$a_k = \frac{1}{n} \cdot \sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{4}\sum_{n=-1}^{2} \delta[n+1] \cdot e^{-jk\frac{\pi}{2}n} = \frac{1}{4}\sum_{n=-1}^{2} \delta[n+1]e^{jk\frac{\pi}{2}} = \frac{e^{jk\frac{\pi}{2}}}{4} \underbrace{\sum_{n=-1}^{2} \delta[n+1]}_{1} = \frac{1}{4}e^{jk\frac{\pi}{2}}$$

$$= \frac{1}{4} \cdot \left(e^{j\frac{\pi}{2}}\right)^k = \frac{1}{4} \cdot \left[\cancelto{0}{\cos\left(\frac{\pi}{2}\right)} + j \cdot \sin\left(\frac{\pi}{2}\right)\right]^k = \frac{1}{4} \cdot j^k = \frac{j^k}{4}$$

2) Calcular los coeficientes del desarrollo en series de Fourier

$$x[n] = \sum_{m=-\infty}^{\infty} \prod\left(\frac{n-mN}{2N_1+1}\right)$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] \cdot e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] \cdot e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} \left(e^{-jk\frac{2\pi}{N}}\right)^n = \frac{1}{N} \cdot \frac{e^{jk\frac{2\pi}{N}N_1} - e^{-jk\frac{2\pi}{N}N_2}}{1-e^{-jk\frac{2\pi}{N}}}$$

$$= \frac{1}{N} \cdot \frac{\left(e^{-jk\frac{\pi}{N}} \cdot \left(e^{jk\frac{2\pi}{N}N_1} \cdot e^{jk\frac{\pi}{N}} - e^{-jk\frac{2\pi}{N}N_1} \cdot e^{-jk\frac{2\pi}{N}} e^{jk\frac{\pi}{N}}\right)\right)}{e^{-jk\frac{\pi}{N}}\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}\right)} = \frac{1}{N} \cdot \frac{e^{jk\frac{\pi}{N}(2N_1+1)} - e^{-jk\frac{\pi}{N}(2N_1+1)}}{e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}}$$

$$= \frac{1}{N} \cdot \frac{\sin\left(k\frac{\pi}{N}(2N_1+1)\right)}{\sin\left(k\frac{\pi}{N}\right)}$$

3) Obtener el espectro de la secuencia discreta

$$x[n] = a^n u[n], \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} (a \cdot e^{-j\omega})^n = \frac{1-(a\cdot e^{-j\omega})^{\infty+1}}{1-a\cdot e^{-j\omega}} = \frac{1}{1-a\cdot e^{-j\omega}}$$

4) Obtener el espectro de la secuencia discreta

$$x[n] = \prod\left(\frac{n}{2N_1+1}\right)$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{e^{j\omega N_1} - e^{-j\omega(N_1+1)}}{1-e^{-j\omega}}$$

5) Obtener el espectro de la secuencia discreta

$$x[n] = \cos\left(\frac{2\pi}{5}n\right)$$

$$x[n] = \cos\left(\frac{2\pi}{5}n\right) \implies \frac{e^{\frac{2\pi}{5}jn} + e^{-\frac{2\pi}{5}jn}}{2} \implies \begin{cases} a_1 = \frac{1}{2} \\ a_{-1} = \frac{1}{2} \end{cases}$$

$$X(e^{j\omega}) = \sum_k 2\pi a_k \cdot \delta(\omega - k\omega_0) = 2\pi \cdot \frac{1}{2}^{\delta}\left(\omega - \frac{2\pi}{5}\right) + 2\pi \cdot \frac{1}{2} \cdot \delta\left(\omega + \frac{2\pi}{5}\right) = \pi \cdot \left(\delta\left(\omega - \frac{2\pi}{5}\right) + \delta\left(\omega + \frac{2\pi}{5}\right)\right)$$

6) Obtener el espectro de la secuencia discreta

$$x[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$

$$X(e^{j\omega}) = \sum_k 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0) = \sum_k 2\pi \cdot \frac{1}{N} \cdot \delta\left(\omega - k\frac{2\pi}{N}\right)$$

7) Obtener el espectro aplicando propiedades

$$x[n] = u[n]$$

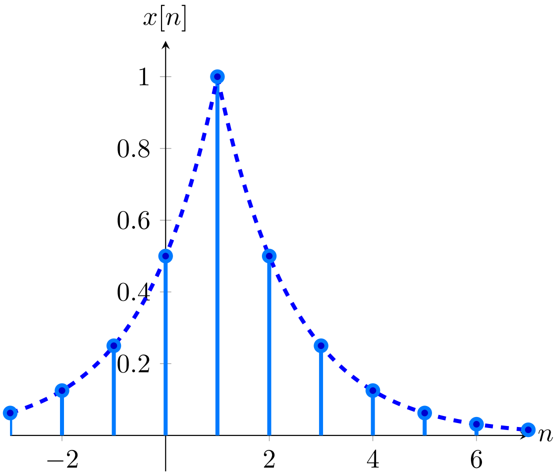
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} e^{-j\omega n} = \frac{1-e^{-j\omega(\infty+1)}}{1-e^{-j\omega}} = \frac{1}{1-e^{-j\omega}}$$

$$z[n] = \delta[n] \to Z(e^{j\omega}) = 1 \qquad X(e^{j\omega}) = \frac{Z(e^{j\omega})}{1-e^{-j\omega}} + \pi \cdot Z(e^{j\cdot 0}) \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$x[n] = \sum_{m=-\infty}^{\infty} z[m] = u[n] \qquad X(e^{j\omega}) = \frac{1}{1+e^{-j\omega}} + \pi \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

8) Obtener el espectro aplicando propiedades

$$x[n] = \left(\frac{1}{2}\right)^{|n-1|}$$



$$\begin{aligned}X(e^{j\omega}) &= \left(\frac{1}{2}\right) \cdot \left(\left(\frac{1}{2}\right)^{-n} \cdot u[-n]\right) + \left(\left(\frac{1}{2}\right)^{n-1} \cdot u[n-1]\right) \\ &= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2} \cdot e^{j\omega}} + \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \cdot e^{-j\omega}\end{aligned}$$

9) Obtener el espectro aplicando propiedades

$$x[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[-n]$$

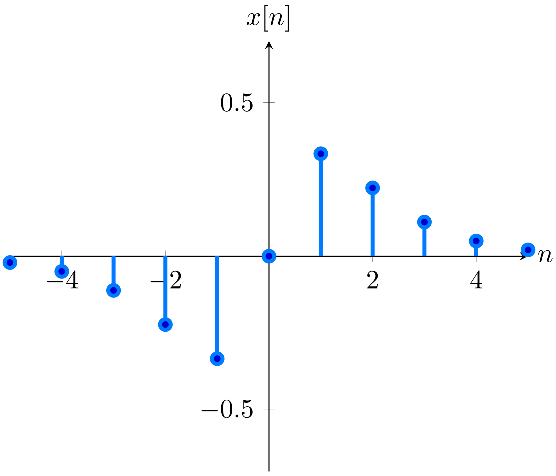
$$x[n] = \frac{1}{2j} \cdot \left(2^n \cdot u[-n] \cdot e^{j\frac{\pi}{4}n}\right) - \left(2^n \cdot u[-n] \cdot e^{-j\frac{\pi}{4}n}\right)$$

$$x_1[n] = 2^n \cdot u[-n] \implies X_1(e^{j\omega}) = \frac{1}{1 - 2e^{j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{2j} \cdot \left(X_1\left(e^{j\left(\omega - \frac{\pi}{4}\right)}\right) - X_1\left(e^{j\left(\omega + \frac{\pi}{4}\right)}\right)\right) = \frac{1}{2j} \cdot \left(\frac{1}{1 - \frac{1}{2} \cdot e^{j\left(\omega - \frac{\pi}{4}\right)}} - \frac{1}{1 - \frac{1}{2}e^{j\left(\omega + \frac{\pi}{4}\right)}}\right)$$

10) Obtener el espectro aplicando propiedades

$$x[n] = n\left(\frac{1}{3}\right)^{|n|}$$



$$\begin{aligned}x[n] &= \begin{cases} n\left(\frac{1}{3}\right)^n, & n \geq 0 \\ n\left(\frac{1}{3}\right)^{-n} & n < 0 \end{cases} \\ &= \begin{cases} n\left(\frac{1}{3}\right)^n, & n \geq 0 \\ n \cdot 3^n, & n < 0 \end{cases}\end{aligned}$$

$$x[n] = x_1[n] + x_2[n]$$

donde:

- $x_1[n] = n\left(\frac{1}{3}\right)^n u[n] \implies X_1(e^{j\omega}) = \frac{\frac{1}{3}e^{j\omega}}{\left(1 - \frac{1}{3}e^{j\omega}\right)^2}$
- $x_2[n] = n \cdot 3^n u[-n-1] = -n \cdot 3^n u[n+1] = -x_1[-n] \implies X_2(e^{j\omega}) = -X_1(e^{-j\omega}) = -\frac{\frac{1}{3}e^{-j\omega}}{\left(1 - \frac{1}{3}e^{-j\omega}\right)^2}$

$$X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega}) = \frac{\frac{1}{3}e^{j\omega}}{\left(1 - \frac{1}{3}e^{j\omega}\right)^2} - \frac{\frac{1}{3}e^{-j\omega}}{\left(1 - \frac{1}{3}e^{-j\omega}\right)^2}$$

11) Ejemplo de sistema LTI discreto causal descrito mediante una ecuación en diferencias lineal con coeficientes constantes:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \implies Y(e^{j\omega}) - \frac{3}{4}Y(e^{j\omega}) \cdot e^{-j\omega} + \frac{1}{8}Y(e^{j\omega}) \cdot e^{-2j\omega} = 2 \cdot X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \{e^{j\omega} = z\} = \frac{2}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$z^2 \cdot \left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = z^2 - \frac{3}{4}z + \frac{1}{8} = 0 \implies z = \frac{\frac{3}{4} \pm \sqrt{\left(-\frac{3}{4}\right)^2 - 4 \cdot 1 \cdot \frac{1}{8}}}{2 \cdot 1} = \frac{\frac{3}{4} \pm \sqrt{\frac{1}{16}}}{2} = \begin{cases} \frac{\frac{3}{4} + \frac{1}{4}}{2} = \frac{1}{2} \\ \frac{\frac{3}{4} - \frac{1}{4}}{2} = \frac{1}{4} \end{cases}$$

$$H(e^{j\omega}) = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - \frac{1}{4}z^{-1}} = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

$$A = H(z) \cdot \left(1 - \frac{1}{2}z^{-1}\right)\Bigg|_{z=\frac{1}{2}} = \frac{2}{1 - \frac{1}{4}z^{-1}}\Bigg|_{z=\frac{1}{2}} = 4$$

$$B = H(z) \cdot \left(1 - \frac{1}{4}z^{-1}\right)\Bigg|_{z=\frac{1}{4}} = \frac{2}{1 - \frac{1}{2}z^{-1}}\Bigg|_{z=\frac{1}{4}} = -2$$

$$h[n] = 4 \cdot \left(\frac{1}{2}\right)^n u[n] - 2 \cdot \left(\frac{1}{4}\right)^n u[n] = 2 \cdot \left(\frac{1}{2}\right)^n \left(2 - \left(\frac{1}{2}\right)^n\right) u[n]$$

12) Calcular la DFT de $x[n] = \cos\left(\frac{\pi}{2}n\right)$, $0 \le n \le 3$

$$a_k = \frac{1}{N} \cdot \sum_{n=0}^{N-1} \cos\left(\frac{\pi}{2}n\right) e^{-j\omega kn} = \frac{1}{4}\sum_{n=0}^{3} \cos\left(\frac{\pi}{2}n\right) e^{-j\frac{2\pi}{4}kn} = \frac{1}{4}\sum_{n=0}^{3} \cos\left(\frac{\pi}{2}n\right) e^{-j\frac{\pi}{2}kn}$$

$$X(e^{j\omega}) = \sum_{n=0}^{3} \cos\left(\frac{\pi}{2}n\right) e^{-j\omega kn} = \sum_{n=0}^{3} \cos\left(\frac{\pi}{2}n\right) e^{-j\frac{2\pi}{4}kn} = \sum_{n=0}^{3} \cos\left(\frac{\pi}{2}n\right) e^{-j\frac{\pi}{2}kn} = 1 + 0 + (-1)\cdot(-1)^k + 0$$

$$\implies \begin{cases} 0, & \text{si } k = 0, 2 \\ 2, & \text{si } k = 1, 3 \end{cases}$$

13) Calcular la DFT de $x[n] = 2^n$, $0 \le n \le 3$

$$X[k] = \sum_{n=0}^{3} 2^n e^{-j\frac{2\pi}{4}kn} = \sum_{n=0}^{3} 2^n e^{-j\frac{\pi}{2}kn} = \sum_{n=0}^{3} \left(2 \cdot \left(e^{-j\frac{\pi}{2}}\right)^k\right)^n = \sum_{n=0}^{3} (2\cdot(-j)^k)^n = \begin{cases} k = 0 : 1 + 2 + 4 + 8 = 15 \\ k = 1 : 1 - 2j - 4 + 8j = -3 + 6j \\ k = 2 : 1 - 2 + 4 - 8 = -5 \\ k = 3 : 1 + 2j - 4 - 8j = -3 - 6j \end{cases}$$

14) Calcular la DFT de $x[n] = \prod\left(\frac{n-2}{5}\right)$

$$x[k] = \sum_{n=0}^{4} e^{-j\frac{2\pi}{5}kn} = \sum_{n=0}^{4} \left(\left(e^{-j\frac{2\pi}{5}}\right)^k\right)^n = \begin{cases} k = 0 : 5 \\ k = 1 : \left(e^{-\frac{2\pi}{5}}\right)^n \to e^{-j\frac{2\pi}{5}} = 0 \\ k = 2 : 0 \\ k = 3 : 0 \\ k = 4 : 0 \end{cases}$$

15) Calcular $z[n] = x[n] \circledast x[n]$, siendo $x[n] = \prod\left(\frac{n-1}{3}\right)$

$$Z[k] = X[k] \cdot X[k] = 3 \cdot \delta[k] \cdot 3 \cdot \delta[k] = 9\delta[k]$$

$$z[n] = \frac{1}{3}\sum_{k=0}^{2} 9\delta[k] = \frac{9}{3} \cdot 1 = 3$$

$$z[n] = 3\prod\left(\frac{n-1}{3}\right)$$