

Álgebra Lineal

Ejercicios Tema 1: Números complejos

Francisco Javier Mercader Martínez

1) Calcula las siguientes operaciones de números complejos.

a) $(-2 + j) + \left(-\frac{1}{2} - 3j\right)$

$$(-2 + j) + \left(-\frac{1}{2} - 3j\right) = -\frac{5}{2} - 2j$$

b) $(-2 - 3j) - \left(-3 + \frac{1}{2}j\right)$

$$(-2 - 3j) - \left(-3 + \frac{1}{2}j\right) = 1 - \frac{7}{2}j$$

c) $-2 \cdot (-1 - j) - 3 \cdot (-2 + j) + 2 \cdot (1 - 2j)$

$$-2 \cdot (-1 - j) - 3 \cdot (-2 + j) + 2 \cdot (1 - 2j) = 2 + 2j + 6 - 3j + 2 - 4j = 10 - 5j$$

d) $j \cdot (-1 + j)$

$$j \cdot (-1 + j) = -j - 1$$

e) $(-2 - j) \cdot \left(-3 + \frac{1}{2}j\right)$

$$(-2 - j) \cdot \left(-3 + \frac{1}{2}j\right) = 6 - j + 3j + \frac{1}{2} = \frac{13}{2} + 2j$$

f) $\frac{1}{-1 - 2j}$

$$\frac{1}{-1 - 2j} = \frac{1}{-1 - 2j} \cdot \frac{-1 + 2j}{-1 + 2j} = \frac{-1 + 2j}{5} = -\frac{1}{5} + \frac{2}{5}j$$

g) $\frac{-j}{2 - 3j}$

$$\frac{-j}{2 - 3j} = \frac{-j}{2 - 3j} \cdot \frac{2 + 3j}{2 + 3j} = \frac{-j \cdot (2 + 3j)}{13} = \frac{-2j + 3}{13} = \frac{3}{13} - \frac{2}{13}j$$

h) $\frac{-1 - j}{-2 + j}$

$$\frac{-1 - j}{-2 + j} = \frac{-1 - j}{-2 + j} \cdot \frac{-2 - j}{-2 - j} = \frac{(-1 - j) \cdot (-2 - j)}{5} = \frac{1 + 3j}{5} = \frac{1}{5} + \frac{3}{5}j$$

i) $\frac{1 - j}{j} - \frac{j}{1 - j}$

$$\frac{1 - j}{j} - \frac{j}{1 - j} = -1 - j + \frac{1}{2} - \frac{1}{2}j = -\frac{1}{2} - \frac{3}{2}j$$

$$\frac{1 - j}{j} = \frac{1 - j}{j} \cdot \frac{-j}{-j} = \frac{(1 - j) \cdot (-j)}{1} = -1 - j$$

$$\frac{j}{1 - j} = \frac{j}{1 - j} \cdot \frac{1 + j}{1 + j} = \frac{j \cdot (1 + j)}{2} = \frac{-1 + j}{2} = -\frac{1}{2} + \frac{1}{2}j$$

2) Obtén las formas polares y trigonométricas de los siguientes números complejos:

a) $1 + j$

$$\begin{cases} |z| = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} \end{cases} \longrightarrow \sqrt{2} \cdot e^{\frac{\pi}{4}j}$$

b) $-j$

$$\begin{cases} |z| = \sqrt{0^2 + (-1)^2} = 1 \\ \theta = \arctan(-j) = -\frac{\pi}{2} \equiv \frac{3\pi}{2} \end{cases} \longrightarrow 1 \cdot e^{\frac{3\pi}{2}j}$$

c) $-1 + \sqrt{3}j$

$$\begin{cases} |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \\ \theta = \arctan\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3} \end{cases} \longrightarrow 2e^{\frac{2\pi}{3}j}$$

d) $2\sqrt{3} - 2j$

$$\begin{cases} |z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4 \\ \theta = \arctan\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6} \equiv \frac{11\pi}{6} \end{cases} \longrightarrow 4e^{\frac{11\pi}{6}j}$$

e) $-1 - j$

$$\begin{cases} |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \\ \theta = \arctan\left(\frac{-1}{-1}\right) = -\frac{3\pi}{4} \equiv \frac{5\pi}{4} \end{cases} \longrightarrow \sqrt{2}e^{\frac{5\pi}{4}j}$$

f) $-2 + j$

$$\begin{cases} |z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \\ \theta = \arctan\left(-\frac{1}{2}\right) \simeq 2.6779 \end{cases} \longrightarrow \sqrt{5}e^{2.6779j}$$

3) Obtén la forma binómica de los siguientes números complejos:

a) $2\frac{\pi}{3}$

$$2 \cdot \left(\cos\left(\frac{\pi}{3}\right) + j \sin\left(\frac{\pi}{3}\right) \right) = 1 + \sqrt{3}j$$

b) 1_{π}

$$\cos(\pi) + j \sin(\pi) = -1$$

c) $3\frac{5\pi}{4}$

$$3 \cdot \left(\cos\left(\frac{5\pi}{4}\right) + j \sin\left(\frac{5\pi}{4}\right) \right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}j$$

4) Calcula las siguientes operaciones de números complejos, expresando el resultado en forma exponencial:

a) $2\frac{5\pi}{3} \cdot 3\frac{\pi}{2}$

$$2\frac{5\pi}{3} \cdot 3\frac{\pi}{2} = (1 - \sqrt{3}j) \cdot 3j = 3\sqrt{3} + 3j$$

$$2\frac{5\pi}{3} = 2 \cdot \left(\cos\left(\frac{5\pi}{3}\right) + j \sin\left(\frac{5\pi}{3}\right) \right) = 1 - \sqrt{3}j$$

$$3\frac{\pi}{2} = 3 \cdot \left(\cos\left(\frac{\pi}{2}\right) + j \sin\left(\frac{\pi}{2}\right) \right) = 3j$$

b) $1\frac{\pi}{4} \cdot 2\frac{7\pi}{3}$

$$1\frac{\pi}{4} \cdot 2\frac{7\pi}{3} = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j \right) \cdot (1 + \sqrt{3}j) = \frac{\sqrt{6} + \sqrt{2}}{2} + \frac{\sqrt{6} - \sqrt{2}}{2}j$$

$$1_{\frac{7\pi}{4}} = \cos\left(\frac{7\pi}{4}\right) + j \sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j$$

$$2_{\frac{7\pi}{3}} = 2 \cdot \left(\cos\left(\frac{7\pi}{3}\right) + j \sin\left(\frac{7\pi}{3}\right)\right) = 1 + \sqrt{3}j$$

c) $2\left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2}\right) \cdot 3_{\frac{11\pi}{6}}$

$$2\left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2}\right) \cdot 3_{\frac{11\pi}{6}} = 2j \cdot \left(\frac{2\sqrt{3}}{2} - \frac{3}{2}j\right) = 3 + 3\sqrt{3}j$$

$$2\left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2}\right) = 2j$$

$$3_{\frac{11\pi}{6}} = 3 \cdot \left(\cos \frac{11\pi}{6} + j \sin \frac{11\pi}{6}\right) = \frac{2\sqrt{3}}{2} - \frac{3}{2}j$$

d) $\frac{4_{\frac{5\pi}{2}}}{2_{\frac{2\pi}{3}}}$

$$\frac{4_{\frac{5\pi}{2}}}{2_{\frac{2\pi}{3}}} = \frac{4j}{-1 + \sqrt{3}j} = \frac{4j}{-1 + \sqrt{3}j} \cdot \frac{-1 - \sqrt{3}j}{-1 - \sqrt{3}j} = \frac{4j \cdot (-1 - \sqrt{3}j)}{4} = \frac{4\sqrt{3} - 4j}{4} = \sqrt{3} - j$$

$$4_{\frac{5\pi}{2}} = 4 \cdot \left(\cos \frac{5\pi}{2} + j \sin \frac{5\pi}{2}\right) = 4j$$

$$2_{\frac{2\pi}{3}} = 2 \cdot \left(\cos \frac{2\pi}{3} + j \sin \frac{2\pi}{3}\right) = -1 + \sqrt{3}j$$

e) $\frac{1_{\pi}}{2_{\frac{7\pi}{6}}}$

$$\frac{1_{\pi}}{2_{\frac{7\pi}{6}}} = \frac{-1}{-\sqrt{3} - j} = \frac{-1}{-\sqrt{3} - j} \cdot \frac{-\sqrt{3} + j}{-\sqrt{3} + j} = \frac{(-1) \cdot (-\sqrt{3} + j)}{4} = \frac{\sqrt{3} - j}{4} = \frac{\sqrt{3}}{4} - \frac{1}{4}j$$

$$1_{\pi} = \cos(\pi) + j \sin(\pi) = -1$$

$$2_{\frac{7\pi}{6}} = 2 \cdot \left(\cos \frac{7\pi}{6} + j \sin \frac{7\pi}{6}\right) = -\sqrt{3} - j$$

f) $\frac{12\left(\cos \frac{5\pi}{4} + j \sin \frac{5\pi}{4}\right)}{8_{\frac{5\pi}{4}}}$

$$\frac{12 \cdot \left(\cos \frac{5\pi}{4} + j \sin \frac{5\pi}{4}\right)}{8_{\frac{5\pi}{4}}} = \frac{12 \cdot \left(\cos \frac{5\pi}{4} + j \sin \frac{5\pi}{4}\right)}{8 \cdot \left(\cos \frac{5\pi}{4} + j \sin \frac{5\pi}{4}\right)} = \frac{12}{8} = \frac{3}{2}$$

5) Calcula las siguientes potencias de números complejos:

a) $(1 - \sqrt{3}j)^6$

$$(1 - \sqrt{3}j)^6 = 64$$

b) $(1 - j)^8$

$$(1 - j)^8 = 16$$

c) $(-\sqrt{3} + j)^{10}$

$$(-\sqrt{3} + j)^{10} = 512 + 886.81j$$

d) $\left(\frac{1-j}{1+j}\right)^5$

$$\left(\frac{1-j}{1+j}\right)^5 = (*) = (-j)^5 = -j$$

$$(*) = \frac{1-j}{1+j} = \frac{1-j}{1+j} \cdot \frac{1-j}{1-j} = \frac{-2j}{2} = -j$$

6) Expresa los siguientes números complejos en forma binómica y en forma exponencial:

a) $(1+j)^3$

$$(1+j)^3 = -2 + 2j$$

$$\begin{cases} |z| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2} \\ \theta = \arctan(-2 + 2j) = \frac{3\pi}{4} \end{cases} \longrightarrow 2\sqrt{2} \cdot e^{\frac{3\pi}{4}j}$$

b) $j^5 + j^{16}$

$$j^5 + j^{16} = 1 + j$$

$$\begin{cases} |z| = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \theta = \arctan(1 + j) = \frac{\pi}{4} \end{cases} \longrightarrow \sqrt{2} \cdot e^{\frac{\pi}{4}j}$$

c) $1 + 3e^{j\pi}$

$$1 + 3e^{j\pi} = 1 + 3 \cdot (\cos \pi + j \sin \pi) = 1 - 3 = -2$$

$$\begin{cases} |z| = \sqrt{(-2)^2 + 0^2} = 2 \\ \theta = \arctan(-2) = \pi \end{cases} \longrightarrow 2e^{\pi j}$$

d) $\frac{2+3j}{3-4j}$

$$\frac{2+3j}{3-4j} = \frac{2+3j}{3-4j} \cdot \frac{3+4j}{3+4j} = \frac{-6+17j}{25} = -\frac{6}{25} + \frac{17}{25}j$$

$$\begin{cases} |z| = \sqrt{\left(-\frac{6}{25}\right)^2 + \left(\frac{17}{25}\right)^2} = \frac{\sqrt{13}}{5} \\ \theta = \arctan\left(-\frac{6}{25} + \frac{17}{25}j\right) = 1.91 \end{cases} \longrightarrow \frac{\sqrt{13}}{5} \cdot e^{1.91j}$$

e) $2\frac{3\pi}{2} + j$

$$2\frac{3\pi}{2} + j = 2 \cdot \left(\cos \frac{3\pi}{2} + j \sin \frac{3\pi}{2} \right) + j = -2j + j = -j$$

$$\begin{cases} |z| = \sqrt{0^2 + (-1)^2} = 1 \\ \theta = \arctan(-j) = -\frac{\pi}{2} \equiv \frac{3\pi}{2} \end{cases} \longrightarrow 1 \cdot e^{\frac{3\pi}{2}j}$$

f) $\frac{1}{j}$

$$\frac{1}{j} = \frac{1}{j} \cdot \frac{-j}{-j} = -j$$

$$\begin{cases} |z| = \sqrt{0^2 + (-1)^2} = 1 \\ \theta = \arctan(-j) = -\frac{\pi}{2} \equiv \frac{3\pi}{2} \end{cases} \longrightarrow 1 \cdot e^{\frac{3\pi}{2}j}$$

7) Calcula el módulo y el argumento principal de los siguientes números complejos:

a) $2j$

$$\begin{cases} |z| = \sqrt{0^2 + 2^2} = 2 \\ \theta = \arctan(2j) = \frac{\pi}{2} \end{cases}$$

b) $1-j$

$$\begin{cases} |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \\ \theta = \arctan(1-j) = -\frac{\pi}{4} \equiv \frac{7\pi}{4} \end{cases}$$

c) -1

$$\begin{cases} |z| = \sqrt{(-1)^2 + 0^2} = 1 \\ \theta = \arctan(-1) = \pi \end{cases}$$

d) $\frac{1+j}{1-j}$

$$\frac{1+j}{1-j} = \frac{1+j}{1-j} \cdot \frac{1+j}{1+j} = \frac{2j}{2} = j$$

$$\begin{cases} |z| = \sqrt{0^2 + 1^2} = 1 \\ \theta = \arctan(j) = \frac{\pi}{2} \end{cases}$$

e) $\frac{1}{j\pi}$

$$\frac{1}{j\pi} = \frac{1}{j\pi} \cdot \frac{-j}{-j} = -\frac{j}{\pi}$$

$$\begin{cases} |z| = \sqrt{0^2 + \left(-\frac{1}{\pi}\right)^2} = \frac{1}{\pi} \\ \theta = \arctan\left(-\frac{j}{\pi}\right) = -\frac{\pi}{2} \equiv \frac{3\pi}{2} \end{cases}$$

f) $-3 + j\sqrt{3}$

$$\begin{cases} |z| = \sqrt{(-3)^2 + (\sqrt{3})^2} = 2\sqrt{3} \\ \theta = \arctan(-3 + \sqrt{3}j) = \frac{5\pi}{6} \end{cases}$$

8) Calcula las raíces cúbicas de los números -8 y $1+j$

- $\sqrt[3]{-8} = -2$
- $\sqrt[3]{1+j} =$

9) Representa gráficamente los siguientes subconjuntos del plano complejo:

a) Números complejos cuyo módulo es igual a 1.

b) $\{z^k \in C : z = e^{\frac{2\pi j}{8}}, 1 \leq k \leq 8\}$

c) $\{z^k \in C : z = e^{-\frac{2\pi j}{8}}, 1 \leq k \leq 8\}$

d) $\{z \in C : z - \bar{z} = j\}$

10) Resuelve las siguientes ecuaciones algebraicas y expresa el resultado en forma binómica:

a) $2 + 4j + 10e^{\frac{\pi}{3}j} = ze^{\frac{\pi}{3}j}$

b) $5_{-\frac{\pi}{6}} + z + 4 + \sqrt{2}j = 6e^{-\frac{\pi}{4}j}z$

c) $2_{\frac{\pi}{3}} + j + ze^{-\frac{\pi}{3}j} = 0$

d) $z + 4jz = 1$

11) Dados los números coomplejos $z_1 = -1 - j$, $z_2 = 2_{\frac{\pi}{3}}$, $z_3 = 3e^{100\pi j}$, representa gráficamente los números $z_1, z_2, z_3, z_1 + z_2 + z_3, z_1 \cdot z_2, z_1 \cdot z_2 \cdot z_3$.

12) Consideremos la función $f(t) = e^{-t+j_{10}t}, 0 \leq t \leq 2\pi$. Se pide:

a) Dibuja de manera aproximada el conjunto $\{(\text{Ref}(t), \text{Im}f(t)), 0 \leq t \leq 2\pi\}$

b) Dibuja de manera aproximada las funciones $\text{Ref}(t), \text{Im}f(t)$, y $|f(t)|$