Álgebra Lineal

Ejercicios Tema 1: Números complejos

Francisco Javier Mercader Martínez

1) Calcula las siguientes operaciones de números complejos.

a) $(-2+j)+\left(-\frac{1}{2}-3j\right)$

$$(-2+j)+\left(-\frac{1}{2}-3j\right)=-\frac{5}{2-2j}$$

b) $(-2-3j)-\left(-3+\frac{1}{2}j\right)$

$$(-2-3j)-\left(-3+\frac{1}{2}j\right)=1-\frac{7}{2}j$$

c) $-2\cdot(-1-j)-3\cdot(-2+j)+2\cdot(1-2j)$

$$-2\cdot(-1-j)-3\cdot(-2+j)+2\cdot(1-2j)=2+2j+6-3j+2-4j=10-5j$$

d) $j\cdot(-1+j)$

$$j\cdot(-1+j)=-j-1$$

e) $(-2-j)\cdot\left(-3+\frac{1}{2}j\right)$

$$(-2-j)\cdot\left(-3+\frac{1}{2}j\right)=6-j+3j+\frac{1}{2}=\frac{13}{2}+2j$$

f) $\frac{1}{-1-2j}$

$$\frac{1}{-1-2j}=\frac{1}{-1-2j}\cdot\frac{-1+2j}{-1+2j}=\frac{-1+2j}{5}=-\frac{1}{5}+\frac{2}{5}j$$

g) $\frac{-j}{2-3j}$

$$\frac{-j}{2-3j}=\frac{-j}{2-3j}\cdot\frac{2+3j}{2+3j}=\frac{-j\cdot(2-3j)}{13}=\frac{-2j+3}{13}=\frac{3}{13}-\frac{2}{13}j$$

h) $\frac{-1-j}{-2+j}$

$$\frac{-1-j}{-2+j}=\frac{-1-j}{-2+j}\cdot\frac{-2-j}{-2-j}=\frac{(-1-j)\cdot(-2-j)}{5}=\frac{1+3j}{5}=\frac{1}{5}+\frac{1}{3}j$$

i) $\frac{1-j}{j}-\frac{j}{1-j}$

$$\frac{1-j}{j}-\frac{j}{1-j}=-1-j+\frac{1}{2}-\frac{1}{2}j=-\frac{1}{2}-\frac{3}{2}j$$

$$\frac{1-j}{j}=\frac{1-j}{j}\cdot\frac{-j}{-j}=\frac{(1-j)\cdot(-j)}{1}=-1-j$$
$$\frac{j}{1-j}=\frac{j}{1-j}\cdot\frac{1+j}{1+j}=\frac{j\cdot(1+j)}{2}=\frac{-1+j}{2}=-\frac{1}{2}+\frac{1}{2}j$$

2) Obtén las formas polares y trigonométricas de los siguientes números complejos:

a) $1+j$

$$\begin{cases} |z| = \sqrt{1^2+1^2} = \sqrt{2} \\ \theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} \end{cases} \longrightarrow \sqrt{2}\cdot e^{\frac{\pi}{4}j}$$

b) $-j$

$$\begin{cases} |z| = \sqrt{0^2+(-1)^2} = 1 \\ \theta = \arctan(-j) = -\frac{\pi}{2} \equiv \frac{3\pi}{2} \end{cases} \longrightarrow 1\cdot e^{\frac{3\pi}{2}j}$$

c) $-1+\sqrt{3}j$

$$\begin{cases} |z| = \sqrt{(-1)^2+(\sqrt{3})^2} = 2 \\ \theta = \arctan\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3} \end{cases} \longrightarrow 2e^{\frac{2\pi}{3}j}$$

d) $2\sqrt{3}-2j$

$$\begin{cases} |z| = \sqrt{(2\sqrt{3})^2+(-2)^2} = 4 \\ \theta = \arctan\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6} \equiv \frac{11\pi}{6} \end{cases} \longrightarrow 4e^{\frac{11\pi}{6}j}$$

e) $-1-j$

$$\begin{cases} |z| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} \\ \theta = \arctan\left(\frac{-1}{-1}\right) = -\frac{3\pi}{4} \equiv \frac{5\pi}{4} \end{cases} \longrightarrow \sqrt{2}e^{\frac{5\pi}{4}j}$$

f) $-2+j$

$$\begin{cases} |z| = \sqrt{(-2)^2+1^2} = \sqrt{5} \\ \theta = \arctan\left(-\frac{1}{2}\right) \simeq 2.6779 \end{cases} \longrightarrow \sqrt{5}e^{2.6779j}$$

3) Obtén la forma binómica de los siguientes números complejos:

a) $2_{\frac{\pi}{3}}$

$$2\cdot\left(\cos\left(\frac{\pi}{3}\right)+j\sin\left(\frac{\pi}{3}\right)\right) = 1+\sqrt{3}j$$

b) 1_{π}

$$\cos(\pi)+j\sin(\pi) = -1$$

c) $3_{\frac{5\pi}{4}}$

$$3\cdot\left(\cos\left(\frac{5\pi}{4}\right)+j\sin\left(\frac{5\pi}{4}\right)\right) = -\frac{3\sqrt{2}}{2}-\frac{3\sqrt{2}}{2}j$$

4) Calcula las siguientes operaciones de números complejos, expresando el resultado en forma exponencial:

a) $2_{\frac{5\pi}{3}}\cdot 3_{\frac{\pi}{2}}$

$$2_{\frac{5\pi}{3}}\cdot 3_{\frac{\pi}{2}} = (1-\sqrt{3}j)\cdot 3j = 3\sqrt{3}+3j$$

$$2_{\frac{5\pi}{3}} = 2\cdot\left(\cos\left(\frac{5\pi}{3}\right)+j\sin\left(\frac{5\pi}{3}\right)\right) = 1-\sqrt{3}j$$

$$3_{\frac{\pi}{2}} = 3\cdot\left(\cos\left(\frac{\pi}{2}\right)+j\sin\left(\frac{\pi}{2}\right)\right) = 3j$$

b) $1_{\frac{7\pi}{4}}\cdot 2_{\frac{7\pi}{3}}$

$$1_{\frac{7\pi}{4}}\cdot 2_{\frac{7\pi}{3}} = \left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}j\right)\cdot(1+\sqrt{3}j) = \frac{\sqrt{6}+\sqrt{2}}{2}+\frac{\sqrt{6}-\sqrt{2}}{2}j$$

$$1_{\frac{7\pi}{4}} = \cos\left(\frac{7\pi}{4}\right) + j\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j$$
$$2_{\frac{7\pi}{3}} = 2\cdot\left(\cos\left(\frac{7\pi}{3}\right) + j\sin\left(\frac{7\pi}{3}\right)\right) = 1+\sqrt{3}j$$

c) $2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)\cdot 3_{\frac{11\pi}{6}}$

$$2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)\cdot 3_{\frac{11\pi}{6}} = 2j\cdot\left(\frac{2\sqrt{3}}{2} - \frac{3}{2}j\right) = 3+3\sqrt{3}j$$

$$2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 2j$$
$$3_{\frac{11\pi}{6}} = 3\cdot\left(\cos\frac{11\pi}{6} + j\sin\frac{11\pi}{6}\right) = \frac{2\sqrt{3}}{2} - \frac{3}{2}j$$

d) $\dfrac{4_{\frac{5\pi}{2}}}{2_{\frac{2\pi}{3}}}$

$$\frac{4_{\frac{5\pi}{2}}}{2_{\frac{2\pi}{3}}} = \frac{4j}{-1+\sqrt{3}j} = \frac{4j}{-1+\sqrt{3}j}\cdot\frac{-1-\sqrt{3}j}{-1-\sqrt{3}j} = \frac{4j\cdot(-1-\sqrt{3}j)}{4} = \frac{4\sqrt{3}-4j}{4} = \sqrt{3}-j$$

$$4_{\frac{5\pi}{2}} = 4\cdot\left(\cos\frac{5\pi}{2} + j\sin\frac{5\pi}{2}\right) = 4j$$
$$2_{\frac{2\pi}{3}} = 2\cdot\left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right) = -1+\sqrt{3}j$$

e) $\dfrac{1_{\pi}}{2_{\frac{7\pi}{6}}}$

$$\frac{1_{\pi}}{2_{\frac{7\pi}{6}}} = \frac{-1}{-\sqrt{3}-j} = \frac{-1}{-\sqrt{3}-j}\cdot\frac{-\sqrt{3}+j}{-\sqrt{3}+j} = \frac{(-1)\cdot(-\sqrt{3}+j)}{4} = \frac{\sqrt{3}-j}{4} = \frac{\sqrt{3}}{4} - \frac{1}{4}j$$

$$1_{\pi} = \cos(\pi) + j\sin(\pi) = -1$$
$$2_{\frac{7\pi}{6}} = 2\cdot\left(\cos\frac{7\pi}{6} + j\sin\frac{7\pi}{6}\right) = -\sqrt{3}-j$$

f) $\dfrac{12\left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)}{8_{\frac{5\pi}{4}}}$

$$\frac{12\cdot\left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)}{5_{\frac{5\pi}{4}}} = \frac{12\cdot\cancel{\left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)}}{8\cdot\cancel{\left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)}} = \frac{12}{8} = \frac{3}{2}$$

5) Calcula las siguientes potencias de números complejos:

a) $(1-\sqrt{3}j)^6$

$$(1-\sqrt{3}j)^6 = 64$$

b) $(1-j)^8$

$$(1-j)^8 = 16$$

c) $(-\sqrt{3}+j)^{10}$

$$(-\sqrt{3}+j)^{10} = 512 + 886.81j$$

d) $\left(\dfrac{1-j}{1+j}\right)^5$

$$\left(\frac{1-j}{1+j}\right)^5 = (*) = (-j)^5 = -j$$

$$(*) = \frac{1-j}{1+j} = \frac{1-j}{1+j}\cdot\frac{1-j}{1-j} = \frac{-2j}{2} = -j$$

6) Expresa los siguientes números complejos en forma binómica y en forma exponencial:

a) $(1+j)^3$

$(1+j)^3 = -2+2j$

$$\begin{cases} |z| = \sqrt{(-2)^2+2^2} = 2\sqrt{2} \\ \theta = \arctan(-2+2j) = \dfrac{3\pi}{4} \end{cases} \longrightarrow 2\sqrt{2}\cdot e^{\frac{3\pi}{4}j}$$

b) $j^5 + j^{16}$

$j^5 + j^{16} = 1+j$

$$\begin{cases} |z| = \sqrt{1^2+1^2} = \sqrt{2} \\ \theta = \arctan(1+j) = \dfrac{\pi}{4} \end{cases} \longrightarrow \sqrt{2}\cdot e^{\frac{\pi}{4}j}$$

c) $1+3e^{j\pi}$

$1+3e^{j\pi} = 1+3\cdot(\cos\pi + j\sin\pi) = 1-3 = -2$

$$\begin{cases} |z| = \sqrt{(-2)^2+0^2} = 2 \\ \theta = \arctan(-2) = \pi \end{cases} \longrightarrow 2e^{\pi j}$$

d) $\dfrac{2+3j}{3-4j}$

$$\frac{2+3j}{3-4j} = \frac{2+3j}{3-4j}\cdot\frac{3+4j}{3+4j} = \frac{-6+17j}{25} = -\frac{6}{25}+\frac{17}{25}j$$

$$\begin{cases} |z| = \sqrt{\left(-\dfrac{6}{25}\right)+\left(\dfrac{17}{25}\right)} = \dfrac{\sqrt{13}}{5} \\ \theta = \arctan\left(-\dfrac{6}{25}+\dfrac{17}{25}j\right) = 1.91 \end{cases} \longrightarrow \frac{\sqrt{13}}{5}\cdot e^{1.91j}$$

e) $2_{\frac{3\pi}{2}} + j$

$$2_{\frac{3\pi}{2}} + j = 2\cdot\left(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\right) + j = -2j + j = -j$$

$$\begin{cases} |z| = \sqrt{0^2+(-1)^2} = 1 \\ \theta = \arctan(-j) = -\dfrac{\pi}{2} \equiv \dfrac{3\pi}{2} \end{cases} \longrightarrow 1\cdot e^{\frac{3\pi}{2}j}$$

f) $\dfrac{1}{j}$

$$\frac{1}{j} = \frac{1}{j}\cdot\frac{-j}{-j} = -j$$

$$\begin{cases} |z| = \sqrt{0^2+(-1)^2} = 1 \\ \theta = \arctan(-j) = -\dfrac{\pi}{2} \equiv \dfrac{3\pi}{2} \end{cases} \longrightarrow 1\cdot e^{\frac{3\pi}{2}j}$$

7) Calcula el módulo y el argumento principal de los siguientes números complejos:

a) $2j$

$$\begin{cases} |z| = \sqrt{0^2+2^2} = 2 \\ \theta = \arctan(2j) = \dfrac{\pi}{2} \end{cases}$$

b) $1-j$

$$\begin{cases} |z| = \sqrt{1^2+(-1)^2} = \sqrt{2} \\ \theta = \arctan(1-j) = -\dfrac{\pi}{4} \equiv \dfrac{7\pi}{4} \end{cases}$$

c) -1

$$\begin{cases} |z| = \sqrt{(-1)^2+0^2} = 1 \\ \theta = \arctan(-1) = \pi \end{cases}$$

d) $\frac{1+j}{1-j}$

$$\frac{1+j}{1-j} = \frac{1+j}{1-j} \cdot \frac{1+j}{1+j} = \frac{2j}{2} = j$$

$$\begin{cases} |z| = \sqrt{0^2+1^2} = 1 \\ \theta = \arctan(j) = \dfrac{\pi}{2} \end{cases}$$

e) $\frac{1}{j\pi}$

$$\frac{1}{j\pi} = \frac{1}{j\pi} \cdot \frac{-j}{-j} = -\frac{j}{\pi}$$

$$\begin{cases} |z| = \sqrt{0^2 + \left(-\dfrac{1}{\pi}\right)^2} = \dfrac{1}{\pi} \\ \theta = \arctan\left(-\dfrac{j}{\pi}\right) = -\dfrac{\pi}{2} \equiv \dfrac{3\pi}{2} \end{cases}$$

f) $-3+j\sqrt{3}$

$$\begin{cases} |z| = \sqrt{(-3)^2+(\sqrt{3})^2} = 2\sqrt{3} \\ \theta = \arctan(-3+\sqrt{3}j) = \dfrac{5\pi}{6} \end{cases}$$

8) Calcula las raíces cúbicas de los números -8 y $1+j$

- $\sqrt[3]{-8} = -2$
- $\sqrt[3]{1+j}$

La raíz cúbica de un número complejo se calcula expresándolo en su **forma polar** y luego aplicando la fórmula de las raíces de un número complejo.

Paso 1: Convertir $1+j$ a su forma polar

$$\begin{cases} r = |z| = \sqrt{1^2+1^2} = \sqrt{2} \\ \theta = \arctan\left(\dfrac{1}{1}\right) = \dfrac{\pi}{4} \end{cases}$$

Por lo tanto, la forma polar de $1+j$ es:

$$1+j = \sqrt{2}e^{j\frac{\pi}{4}}$$

Paso 2: Calcular la raíz cúbica

La fórmula para las n-ésimas de un número complejo es:

$$z_K = r^{\frac{1}{n}} e^{j\left(\frac{\theta+2k\pi}{n}\right)} \quad k = 0, 1, \ldots, n-1$$

Aquí:

- $n = 3$
- $r = \sqrt{2}$
- $\theta = \dfrac{\pi}{4}$

Por lo tanto:

$$z_k = \left(\sqrt{2}\right)^{1/3} e^{j\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right)}, \quad k = 0, 1, 2)$$

Paso 3: Expresar las raíces

El módulo de las raíces es:

$$r^{\frac{1}{3}} = \left(\sqrt{2}\right)^{\frac{1}{3}}$$

El argumento de las raíces es:

$$\theta_k = \frac{\pi}{4} + \frac{2k\pi}{3}, \quad k = 0, 1, 2$$

Por lo tanto, las raíces son:

$$z_k = \left(\sqrt{2}\right)^{\frac{1}{3}} \left[\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + j\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right], \quad k = 0, 1, 2$$

1) Para $k = 0$:

$$z_0 = \left(\sqrt{2}\right)^{\frac{1}{3}} \left[\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right] = \left(\sqrt{2}\right)^{\frac{1}{3}} \cdot \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j\right) = \frac{j+1}{\sqrt[3]{2}}$$

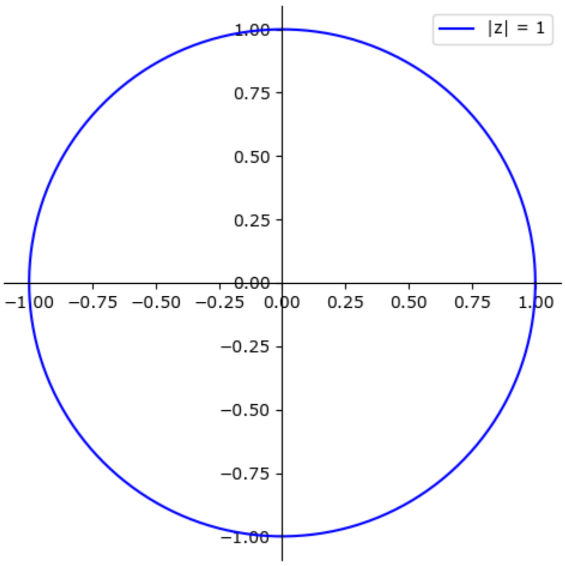
2) Para $k = 1$:

$$z_1 = \left(\sqrt{2}\right)^{\frac{1}{3}} \left[\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) + j\sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right] = \left(\sqrt{2}\right)^{\frac{1}{3}} \cdot \left(-\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}j\right)$$

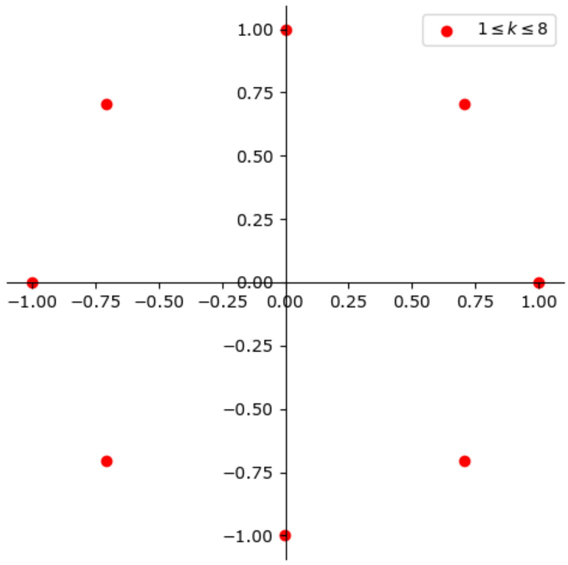
3) Para $k = 2$:

$$z_2 = \left(\sqrt{2}\right)^{\frac{1}{3}} \left[\cos\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) + j\sin\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)\right] = \left(\sqrt{2}\right)^{\frac{1}{3}} \cdot \left(\frac{\sqrt{6}-\sqrt{2}}{4} - \frac{\sqrt{6}+\sqrt{2}}{4}j\right)$$

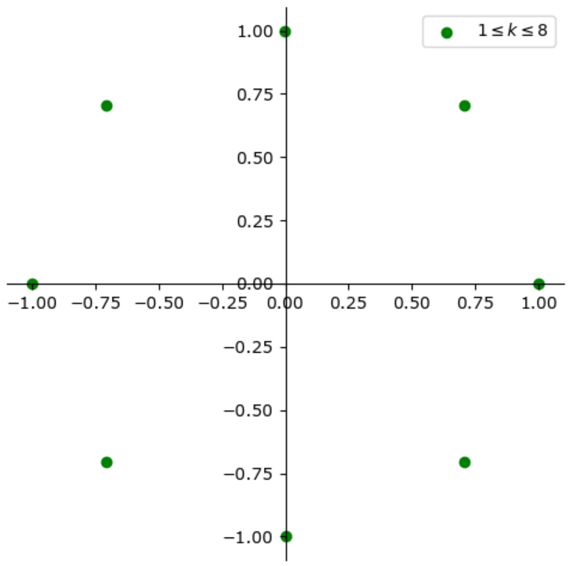
a) Números complejos cuyo móudlo es igual a 1.



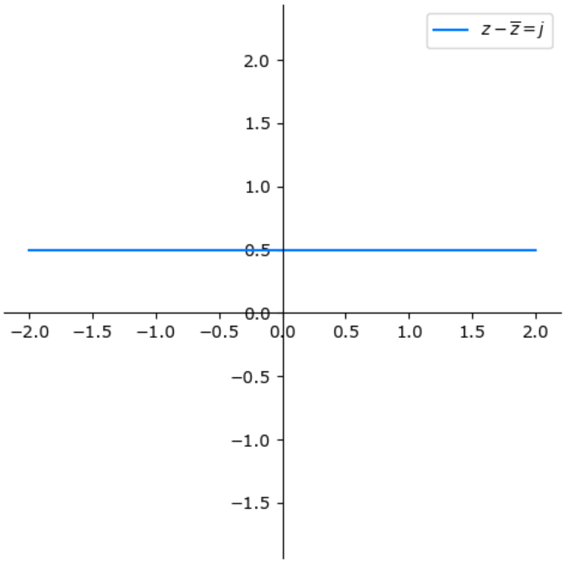
b) $\{z^k \in C : z = e^{\frac{2\pi j}{8}}, 1 \leq k \leq 8\}$



c) $\{z^k \in C : z = e^{-\frac{2\pi j}{8}}, 1 \leq k \leq 8\}$



d) $\{z \in C : z - \overline{z} = j\}$



9) Resuelve las siguientes ecuaciones algebraicas y expresa el resultado en forma binómica:

- **a)** $2 + 4j + 10e^{\frac{\pi}{3}j} = ze^{\frac{\pi}{3}j}$
- **b)** $5_{-\frac{\pi}{6}} + z + 4 + \sqrt{2}j = 6e^{-\frac{\pi}{4}j}z$
- **c)** $2_{\frac{\pi}{3}} + j + ze^{-\frac{\pi}{3}j} = 0$
- **d)** $z + 4jz = 1$

10) Dados los números coomplejos $z_1 = -1 - j, z_2 = 2_{\frac{\pi}{3}}, z_3 = 3e^{100\pi j}$, representa gráficamente los números $z_1, z_2, z_3, z_1 + z_2 + z_3, z_1 \cdot z_2, z_1 \cdot z_2 \cdot z_3$.

11) Consideremos la función $f(t) = e^{-t+j_{10}t}, 0 \leq t \leq 2\pi$. Se pide:

- **a)** Dibuja de manera aproximada el conjunto $\{(\mathrm{Re}f(t), \mathrm{Im}f(t)), 0 \leq t \leq 2\pi\}$
- **b)** Dibuja de manera aproximada las funciones $\mathrm{Re}f(t), \mathrm{Im}f(t)$, y $|f(t)|$