1. NÚMEROS REALES

E.1 Sea $\begin{cases} x \leq y \\ u \leq v \end{cases} \Rightarrow$ 1) ¿$x+u \leq y+v$? 2) ¿$xu \leq yv$?

dem. [1] $x \leq y \Rightarrow x+u \leq y+u \Rightarrow x+u \leq y+v$

[2] Sea $\left.\begin{matrix} x=-1 \\ y=1 \end{matrix}\right\}$ Tomando $\begin{cases} u=-2 \\ v=0 \end{cases} \Rightarrow \left.\begin{matrix} xu = 2 \\ yv = 0 \end{matrix}\right\}$ $xu \not\leq yv$ ■

E.2

(i) $|x+1|<3$

$|x+1|<3 \Leftrightarrow -3<x+1<3 \Leftrightarrow -4<x<2 \Leftrightarrow \boxed{x \in (-4,2)}$

(ii) $|x-a|<R$

$-R<x-a<R \Leftrightarrow a-R<x<a+R$ con $a, R \in \mathbb{R}$

(iii) $|x-1| \leq |x+1|$

$|x-1| \leq |x+1| \Leftrightarrow |x-1|^2 \leq |x+1|^2 \Leftrightarrow x^2+1-2x \leq x^2+1+2x \Leftrightarrow -4x \leq 0 \Leftrightarrow \boxed{x \leq 0}$

E.HI $1+2+3+\dots+n = \dfrac{n(n+1)}{2}$

• $n=1 \rightarrow 1 = \dfrac{1(1+1)}{2} = 1$
$n=2 \rightarrow 3 = \dfrac{2(3)}{2} = 3$ } ok

• Supongamos que es cierto para n. ¿$n+1$?

$S_{n+1} = 1+\dots+n+(n+1) = \dfrac{(n+1)(n+2)}{2} \Leftrightarrow S_n + n+1 = \dfrac{n(n+1)}{2} + (n+1) = n(n+1)+2n+2 = (n+1)(n+2) \Leftrightarrow$

$\Leftrightarrow n^2+n+2n+2 = n^2+3n+2 = (n+1)(n+2)$ cqd

E.3 Desigualdad de Bernoulli $(1+x)^n \geq 1+nx$ con $n \in \mathbb{N}$, $x \geq -1$

$P(n) = (1+x)^n$

1) $P(1) = (1+x)^1 = 1+x \geq 1+x$ ✓

2) Supongamos por HI que es cierto $\forall n \in \mathbb{N}$. ¿ $(1+x)^{n+1} \geq 1+x(n+1)$?

3) $P(n+1) = (1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+x(n+1)+nx^2 \geq 1+(n+1)x$ ✓

Propiedades Supremo

- Si $A \subseteq B \Rightarrow \sup A \leq \sup B$, $\inf A \geq \inf B$

Sea $\beta := \sup B \Rightarrow \forall b \in B$, $b \leq \beta$. Luego, $\forall a \in A$, $a \leq b \leq \beta$ y $\alpha := \sup A \Rightarrow a \leq \alpha \Rightarrow \sup A \leq \sup B$

- $A = \{1/n\}_{n \in \mathbb{N}}$ $\sup A = 1$, $\inf A = 0$ $B = \{\sin x \mid x \in [0, 2\pi]\}$ $\sup B = 1$, $\inf B = -1$

$A = (0, +\infty)$ $\sup A = +\infty$, $\inf A = 0$

Parte entera

$\lfloor 3'14 \rfloor = 3$

$\lfloor 3'99 \rfloor = 3$

$\lfloor 3'0 \rfloor = 3$

2. SUCESIONES

- **Límites**

1. $\lim_{n\to\infty} 1/n^2 = 1/\infty = 0$

2. $\lim_{n\to\infty} n^2+3n+2 = \infty$

3. $\lim_{n\to\infty} \frac{n^3+1}{n^2-1} = \lim_{n\to\infty} \frac{1+1/n^3}{1/n - 1/n^3} = \frac{1+0}{0-0} = \frac{1}{0} = \infty$ $* \ \frac{0}{0}$ LH $\to \lim_n \frac{1/n}{1} = 0$

4. $\lim_n \frac{\sqrt{2n^2-1}}{n+2} = \lim_n \frac{\sqrt{2-1/n^2}}{1+2/n} = \sqrt{2}$

5. $\lim_n (\sqrt{n+1}-\sqrt{n}) = \lim_n \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_n 1/\infty = 0$

6. $\lim_n n^{1/n} = \infty^0 = e^\wedge[\lim_n [\frac{1}{n}]\cdot \log n] \underset{*}{=} e^0 = 1$

7. $\lim_n \frac{1^2+\dots+n^2}{n^3} = \frac{\infty}{\infty}$ Stolz $= \lim_n \frac{n^2}{n^3-(n-1)^3} = \lim_n \frac{n^2}{3n^2-3n+1} = 1/3$

8. $\lim_n \frac{\log n}{n} = \frac{\infty}{\infty}$ Stolz $= \lim_n \frac{\log n - \log(n-1)}{n-(n-1)} = \lim_n \log\left(\frac{n}{n-1}\right) = \log\left[\lim\left(\frac{n}{n-1}\right)\right] = \log 1 = 0$

9. $\lim_n \frac{(-1)^n}{n} \to$ TSándwich $\begin{cases} \lim_n -1/n = 0 \\ \lim_n \frac{1}{n} = 0 \end{cases} \to \lim_n \frac{(-1)^n}{n} = 0$

10. $\lim_n \frac{\operatorname{sen} n}{n} \to$ TSándwich $\begin{cases} \lim_n -1/n = 0 \\ \lim_n 1/n = 0 \end{cases} \to \lim_n \frac{\operatorname{sen} n}{n} = 0$

11. $\lim_n \frac{1+n^2}{(1+n)^n} \to$ TSándwich $\to 0 \le \frac{1+n^2}{(1+n)^n} \underset{n\ge 3}{\le} \frac{1+n^2}{(1+n)^3} \to 0 \to \lim_n [-] = 0$

12. $\lim_n \frac{n!}{n^n} \to$ TSándwich $\to 0 \le \frac{n!}{n^n} = \frac{n(n-1)\cdot\dots\cdot 2\cdot 1}{n\cdot n\cdot\dots\cdot n} = \underbrace{\frac{n}{n}\cdot\frac{n-1}{n}\cdot\dots\cdot\frac{2}{n}}_{\le 1}\cdot\frac{1}{n} \le \frac{1}{n} \longrightarrow 0 \to \lim_n \frac{n!}{n^n} = 0$

13. $\lim_n \left[\frac{n}{2n^2+1}+\dots+\frac{n}{2n^2+n}\right] \to$ TSándwich $\to \begin{cases} n\cdot n/(2n^2+n) \to 1/2 \\ n\cdot n/(2n^2+1) \to 1/2 \end{cases} \to \lim_n [-] = 1/2$

14. $\lim_n n^{1/n}$ = CRaíz = $\lim_n \sqrt[n]{n} = \lim_n \frac{n}{n-1} = 1$

15. $\lim_n (1+\frac{1}{2}+\dots+\frac{1}{n})^{1/n}$ = CRaíz = $\lim_n \left(\frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}\right)$ = Stolz = $\lim_n \frac{1/n}{1/n-1} = \lim_n \frac{n-1}{n} = 1$

16. $\lim_n (n!)^{1/n}$ = CRaíz = $\lim_n \frac{n!}{(n-1)!} = \lim_n n = \infty$

• Monotonía

1. $a_n = 1-\frac{1}{n}$ CRECIENTE $a_{n+1} \geq a_n \Leftrightarrow 1-\frac{1}{n+1} \geq 1-\frac{1}{n} \Rightarrow \frac{1}{n} \geq \frac{1}{n+1} \Rightarrow n+1 \geq n$

2. $a_n = (-1)^n$ no monótona

3. $a_n = (-1)^{n+1}/n$ no monótona

• Sucesión recurrente

$$\begin{cases} a_{n+1} = \sqrt{2a_n} \\ a_1 = 1 \end{cases}$$

1) Supongamos que $\exists \lim_n a_n = L \Rightarrow \lim_n a_{n+1} = L = \sqrt{2L} \Rightarrow L^2 - 2L = 0 \Leftrightarrow L = \{0, 2\}$

2) Probar que $a_n \leq 2, \forall n \in \mathbb{N}$ [HI] $n=1: a_1 = 1 \leq 2 \rightsquigarrow a_{n+1} = \sqrt{2a_n} \leq \sqrt{2 \cdot 2} = 2$

3) ¿a_n creciente?

$a_{n+1} \geq a_n \Leftrightarrow \sqrt{2a_n} \geq a_n \Rightarrow 2a_n \geq a_n^2 \Leftrightarrow 2 \geq a_n$

* Luego, $\exists \lim_n a_n = 2$ y a_n es creciente

3. FUNCIONES

• Función inversa

1. $f(x) = x^2$ $\qquad y = x^2 \Rightarrow x = \pm\sqrt{y} = f^{-1}(y)$ $\qquad f^{-1}(y) = \{-\sqrt{y}, \sqrt{y}\}$ con $\mathrm{dom} f^{-1} = [0, +\infty)$

2. $f(x) = e^x$ $\qquad y = e^x \Rightarrow \ln y = x \Rightarrow f^{-1}(y) = \ln y$ $\qquad f^{-1}(y) = \ln y$ con $\mathrm{dom} f^{-1} = [0, +\infty)$

3. $f(x) = \operatorname{sen} x$ $\qquad y = \operatorname{sen} x \Rightarrow x = f^{-1}(y) = \operatorname{arcsen} y$ $\qquad f^{-1}(y) = \operatorname{arcsen} y$ con $\mathrm{dom} f^{-1} = [-1, 1]$

• Límites

1. $\lim_{x\to 0} x \cdot \operatorname{sen}\frac{1}{x}$ = T. Sándwich : $-1 \leq \operatorname{sen}\frac{1}{x} \leq 1 \Rightarrow -x \leq x \operatorname{sen}\frac{1}{x} \leq x \Rightarrow \lim_{x\to 0} x \operatorname{sen}\frac{1}{x} = 0$

2. $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{0}{0} = \lim_{x\to 0} \frac{1+x-1+x}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = 1$

3. $\lim_{x\to -2} \frac{x^3+8}{x^2-4} = \frac{0}{0} = \lim_{x\to -2} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)} = \lim_{x\to -2} \frac{x^2-2x+4}{x-2} = -3$

4. Hallar a, b $\rightarrow \lim_{x\to\infty} [\sqrt{x^2+x+1} - ax - b] = 1$

Sup. $a = 1$

$$\lim_{x\to\infty} (\sqrt{x^2+x+1} - ax) = 1+b \Rightarrow \lim_{x\to\infty} \frac{x^2+x+1-a^2x^2}{\sqrt{x^2+x+1}+ax} = \lim_{x\to\infty} \frac{1+1/x}{\sqrt{1+1/x+1/x^2}+a} = \frac{1}{1+a} = b+1$$

pero si $a = 1 \Rightarrow b = 1/2$

T4

• Definición de derivada.

1. $f(x) = x^2$ $\qquad f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{(a+h)^2 - a^2}{h} = \lim\limits_{h\to 0} \dfrac{h(h+2a)}{h} = 2a$

2. $f(x) = \operatorname{sen} x$ $\qquad f'(a) = \lim\limits_{h\to 0} \dfrac{\operatorname{sen}(h+a) - \operatorname{sen}(a)}{h} = \lim\limits_{h\to 0} \dfrac{\operatorname{sen}a\cos h + \operatorname{sen}h\cos a - \operatorname{sen}a}{h} = \lim\limits_{h\to 0}\left[\operatorname{sen}a\overbrace{\left(\dfrac{\cos h - 1}{h}\right)}^{0} + \cos a \cdot \overbrace{\dfrac{\operatorname{sen}h}{h}}^{1}\right] = \cos a$

3. $f(x) = \begin{cases} x\operatorname{sen}\frac{1}{x} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$ $\qquad \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\operatorname{sen}\frac{1}{h} - 0}{h} = \lim\limits_{h\to 0} \operatorname{sen}\frac{1}{h}$!! $\nexists \Rightarrow$ no derivable en $\underline{x=0}$

• Regla de la Cadena

1. $f(x) = \operatorname{sen}(x^2)$ $\qquad \begin{cases} g(x) = \operatorname{sen}x \\ h(x) = x^2 \end{cases} \Rightarrow f'(x) = g'(h(x))h'(x) = \cos(x^2)2x = 2x\cos(x^2)$

2. $f(x) = (\operatorname{sen}x)^2$ $\qquad \begin{cases} g(x) = x^2 \\ h(x) = \operatorname{sen}x \end{cases} \Rightarrow f'(x) = g'(h(x))h'(x) = 2\operatorname{sen}x\cos x$

• Derivada de la función inversa

1. $f(x) = \operatorname{sen}x$ $\qquad y = \operatorname{sen}x \Leftrightarrow x = \arcsen(y) = f^{-1}(y)$

$$(f^{-1}(y))' = 1/f'(x) = 1/\cos x = \frac{1}{\sqrt{1-\operatorname{sen}^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

• Derivación logarítmica

1. $y = x^{\operatorname{sen}x}$ $\qquad \log y = \operatorname{sen}x\log x \Leftrightarrow \frac{1}{y}\cdot y' = \cos x\log x + \operatorname{sen}x\frac{1}{x} \Leftrightarrow y' = x^{\operatorname{sen}x}\left(\cos x\log x + \frac{1}{x}\operatorname{sen}x\right)$

• Hallar MÁX/mín

1. $f(x) = x^3 - 3x^2 - 9x + 1$ en $I = [-2, 6]$

1) Esbozar la gráfica

2) Ptos. críticos

3) Extremos

2) $f'(x) = 3x^2 - 6x - 9 = 0 \leftrightarrow x = \{3, -1\}$ $\begin{cases} f(-1) = 6 & \text{máx loc.} \\ f(3) = -26 & \text{mín. loc y abs} \end{cases}$

3) $f(-2) = -1$

$f(6) = 55$ máx. abs

2. $j(x) = x + 3/x$, $x \in (0, +\infty)$

$j'(x) = 1 - \frac{3}{x^2} = 0 \leftrightarrow x = \sqrt{3}$ $\{ j(\sqrt{3}) = 2\sqrt{3}$ mín. local y abs

Si $x \in (0, \sqrt{3}) \rightarrow j'(x) < 0 \rightarrow j(x)$ dec.

Si $x \in (\sqrt{3}, +\infty) \rightarrow j'(x) > 0 \rightarrow j(x)$ crece

• Número de raíces

1. $f(x) = 6x^4 - 3x - 1 = 0$

$f'(x) = 24x^3 - 3 = 0 \leftrightarrow x = 1/2 \rightarrow$ Como f', 1 raíz $\rightarrow f$ a lo sumo, 2 raíces

$\left.\begin{matrix} f(0) < 0 \\ f(1) > 0 \end{matrix}\right\} \rightarrow TB \rightarrow \exists c \in (0,1) \mid f(c) = 0$

$\left.\begin{matrix} f(-1) > 0 \\ f(0) < 0 \end{matrix}\right\} \rightarrow TB \rightarrow \exists c' \in (-1,0) \mid f(c') = 0$

• TVM [demos]

1. $\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$

Sea $f(x) = \sqrt{x}$, $a = 64$, $b = 66 \rightarrow$ TVM $\mid \exists c \in (64, 66) \mid f'(c)(66-64) = f(66) - f(64) \leftrightarrow 2f'(c) = \sqrt{66} - 8$

$\frac{1}{9} < 2f'(c) < \frac{1}{8} \leftrightarrow \frac{1}{9} < \frac{1}{\sqrt{c}} < \frac{1}{8} \leftrightarrow 8 < \sqrt{c} < 9$

2. $\log(1+x) < x, \ \forall x > 0$

Sea $f(x) = \log(1+x), \ x>0 \Rightarrow$ TVM, $\exists x_0 \in (0,x) \mid f'(x_0)x = f(x) - f(0) = \log(1+x) \Rightarrow \frac{x}{1+x_0} = \log(1+x) \Rightarrow \frac{x}{1+x_0} < x$

3. $e^x > 1+x, \ \forall x>0$

$j(x) = e^x, \ x>0 \Rightarrow$ TVM : $j'(c)\cdot x = e^x - 1 \Rightarrow e^c . x = e^x - 1$ donde $e^c x > x \Rightarrow e^x - 1 > x \Rightarrow e^x > x+1$

4. $\operatorname{sen} x \leq x, \ \forall x \geq 0$

$h(x) := \operatorname{sen} x - x \Rightarrow h'(x) = \cos x - 1 \leq 0, \ \forall x \in \mathbb{R} \Rightarrow h$ decrece en $\mathbb{R} \Rightarrow h(x) \leq h(0) = 0 \Rightarrow \operatorname{sen} x \leq x, \forall x \geq 0$

5. $1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + x^4/4!, \ \forall x \geq 0$

$j(x) = \cos x - (1 - x^2/2) = \cos x - 1 + x^2/2$

$j'(x) = -\operatorname{sen} x + x \geq 0 \Rightarrow x \geq \operatorname{sen} x$ [Creciente] $\Rightarrow j(x) \geq j(0), \ \forall x \geq 0 \Rightarrow \cos x - 1 + \frac{x^2}{2} \geq \cdots$

?

- L'Hopital

1. $\lim_{x\to 0} \frac{1-\cos x}{x^2/2} = \frac{0}{0}$ LH $= \lim_{x\to 0} \frac{\operatorname{sen} x}{x} = \frac{0}{0}$ LH $= \lim_{x\to 0} \cos x = 1$

2. $\lim_{x\to 0} \frac{e^x - 1 - x}{ax^2} = \frac{0}{0}$ LH $= \lim_{x\to 0} \frac{e^x - 1}{2ax} = \frac{0}{0}$ LH $= \lim_{x\to 0} \frac{e^x}{2a} = (2a)^{-1}$

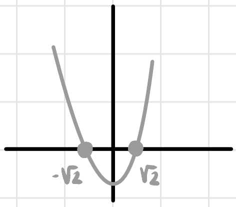
- Método de Newton

1. $f(x) = x^2 - 2 \qquad f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}$$

Observamos

n	x_n	$f(x_n)$
0	1'5	0'25
1	1'41...	$7\cdot 10^{-3}$



$$\left.\begin{array}{l} f(1) = -1 \\ f(2) = 3 \end{array}\right\} \rightarrow \exists\, r \in (1,2) \mid f'(r) = 0$$

2. $f(x) = \tan\left(\frac{x}{4}\right) - 1$

$f(x) = \tan\left(\frac{x}{4}\right) - 1 = 0 \Leftrightarrow \tan\left(\frac{x}{4}\right) = 1 \Leftrightarrow \arctan(1) = \frac{x}{4} \Leftrightarrow x = 4\arctan(1) \Leftrightarrow x = \pi$, luego $\frac{\pi}{4} \in (0,\pi)$

$f'(x) = \left(\cos^2\left(\frac{x}{4}\right)\right)^{-1} \cdot \frac{1}{4} = 1/\left[1+\cos\left(\frac{x}{2}\right)\right]\cdot 2$

$$\boxed{x_{n+1} = x_n - \left(\tan\left(\tfrac{x}{4}\right) - 1\right)2\left(1+\cos\left(\tfrac{x}{2}\right)\right)}$$

$\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

• Sucesión logística $x_{n+1} = r x_n(1-x)$

Si $f(x) = rx(1-x)$ y $x_{n+1} = f(x_n)$

1. Tomamos $x = 1/2 \rightarrow f(1/2) = r/4 \leq 1 \Leftrightarrow r \leq 4$ Entonces, $f : [0,1] \longrightarrow [0,1]$

2. $f(x) = x \Leftrightarrow rx(1-x) = x \Leftrightarrow \begin{cases} x = 0 \\ r(1-x) = 1 \Leftrightarrow x = 1 - 1/r \end{cases}$

3. $L = 0$. $|f'(0)| = r < 1 \rightarrow \lim_n x_n = 0$ [pto. fijo estable]

4. $L = 1 - 1/r$ si $r \in (1,4) \rightarrow |f'(L)| = |r(1-2(1-1/r))| = r\left|\frac{2}{r} - 1\right| = |2-r| \rightarrow |f'(L)| = |2-r| < 1 \Leftrightarrow -1 < 2-r < 1 \Leftrightarrow r \in (1,3)$

• Polinomio de Taylor

1. $f(x) = e^x$ en $x = 0$.

$$\left.\begin{array}{l} f'(x) = e^x \rightarrow f'(0) = 1 \\ f''(x) = e^x \rightarrow f''(0) = 1 \end{array}\right\} \ldots \rightarrow P_n(x;0) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

2. $f(x) = \log(1+x)$ en $x=0$

$f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$f'(x) = \frac{1}{1+x}$, $f'(0) = 1$; $f''(x) = -(1+x)^{-2}$, $f''(0) = -1$; $f'''(x) = 2(1+x)^{-3}$, $f'''(0) = 2$; ... ; $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$$P_n(x;0) = x - \frac{x^2}{2!} + \frac{x^3}{3} + \dots + \frac{(-1)^{n-1}(n-1)!\,x^n}{n!} = x - \frac{x^2}{2!} + \frac{x^3}{3} + \dots + (-1)^{n-1}\cdot\frac{x^n}{n}$$

- **Pol. Taylor en $a=0$**

1. $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + O(|x|^{n+1})$

2. $\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(|x|^{2n+3})$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + O(|x|^{2n+2})$

4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + O(|x|^{n+1})$

- **Pol. Taylor → Límites**

1. $\lim_{x\to 0} \frac{(\operatorname{sen} x - x)^4}{(\log(1+x)-x)^6} = \lim_{x\to 0} \frac{(x - x^3/3! + O(|x|^5) - x)^4}{(x - x^2/2 + O(|x|^3) - x)^6} = \lim_{x\to 0} \frac{x^{12}(-1/3! + O(|x|^2))^4}{x^{12}(-1/2 + O(|x|))^6} = \frac{[-1/3!]^4}{[-1/2]^6} = 4/81$

   - $\operatorname{sen} x = x - \frac{x^3}{3!} + O(|x|^5)$
   - $\log(1+x) = x - \frac{x^2}{2} + O(|x|^3)$

2. $\lim_{x\to 0} \frac{e^x - 1 - x}{\cos(2x) - 1} = \lim_{x\to 0} \frac{1 + x + x^2/2! + O(|x|^3) - 1 - x}{1 - (2x)^2/2! + O(|x|^4) - 1} = \lim_{x\to 0} \frac{x^2/2! + O(|x|^3)}{-(2x)^2/2! + O(|x|^4)} = \lim_{x\to 0} \frac{1/2! + O(|x|^3)}{-4/2! + O(|x|^4)} = -1/4$

   - $e^x = 1 + x + x^2/2! + O(|x|^3)$
   - $\cos(2x) = 1 - (2x)^2/2! + O(|x|^4)$

3. Hallar a, b, c, d → $\lim_{x\to 0} \frac{\log(1+x^2) - a - bx - cx^2 - dx^3}{x^3} = 0$

   - $\log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} + O(|x|^8)$

VER VÍDEO xq NO ENTIENDO

4. Hallar $a,b \mid \exists L := \lim_{x\to 0} \frac{x + a\,\text{sen}x + b\tan x}{x^5} \in \mathbb{R}$

• $\text{sen}x = x - x^3/_{3!} + x^5/_{5} + o(|x|^7)$

• $\tan x = x + x^3/_{3} + \frac{2}{15}x^5 + o(|x|^7)$

$$L = \lim_{x\to 0} \frac{x + a(x - x^3/_{3!} + x^5/_{5!} + o(|x|^7)) + b(x + x^3/_3 + {}^2/_{15}x^5 + o(|x|^7))}{x^5} = \lim_{x\to 0} \frac{x(1+a+b) + x^3(-a/_{3!} + b/_3) + x^5(a/_{5!} + {}^2/_{15}b) + o(|x|^7)}{x^5} \to$$

$\to$ Para que $\exists L \to \begin{cases} 1+a+b=0 \\ -a/_{3!} + b/_3 = 0 \end{cases} \to \begin{cases} b = -1-a \leftrightarrow b = -1 + {}^2/_3 = -1/3 \\ \frac{-a}{3!} - \frac{1-a}{3} = 0 \leftrightarrow -a-2-2a = -3a-2 = 0 \leftrightarrow a = -{}^2/_3 \end{cases} \to \boxed{b = -1/3 \ ; \ a = -2/3}$

$$L = \lim_{x\to 0} \frac{x^5[-1/20]}{x^5} = -1/20$$

5. $\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=1}^{\infty} x^n$, si $|x| < 1$

Si tomo N fijo, $1 + x + x^2 + \dots + x^N = \frac{1-x^{N+1}}{1-x}$

$\lim_{N\to\infty} \left(\frac{1-x^{N+1}}{1-x}\right) =$ Pedir no veo

• Aproximación con márgenes de error.

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + E_n(x)$ para aproximar el valor de e^x con $x=1$ en 5 decimales.

Sea $f(x) = e^x$. Busco $n \mid |e^1 - P_n(1)| < 10^{-6}$ $\quad * \theta \in (0,1)$

Sabemos que $|E_0(1)| \overset{*}{=} \left|\frac{f^{(n+1)}(\theta)}{(n+1)!}(1-0)^{n+1}\right| \leq \frac{e^\theta}{(n+1)!} \leq \frac{e}{(n+1)!} \leq \frac{3}{(n+1)!} \leq 10^{-6} \leadsto 3\cdot 10^6 \leq (n+1)!$

Probando con $n = 9 \to (n+1)! = 10! > 3\cdot 10^6$

Luego, $e = P_9(1) \pm 10^{-6} = (1 + 1 + \frac{1}{2!} + \dots + \frac{1}{9!}) \pm 10^{-6} = 2{,}7182815\dots$ donde $\begin{cases} 2{,}718215 \ \text{[aprox]} \\ 2{,}718215\dots \ \text{[real]} \end{cases}$

2. Hallar $\cos(0{,}5)$ con error $\leq 10^{-8}$

Sea $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + E_{2n}(x)$

- $f(0{,}5) = \cos\frac{1}{2} = P_{2n}(1/2) + E_{2n}(1/2)$. Busco $n \mid |E_{2n}(1/2)| \leq 10^{-8}$

Usando la fórmula del error: $|E_{2n}(0{,}5)| \underset{\theta\in(0,1/2)}{=} \left| \frac{f^{(2n+1)}(\theta)}{(2n+1)!} \cdot 0{,}5^{2n+1} \right| \leq \frac{1}{(2n+1)!} \cdot \frac{1}{2^{2n+1}} \leq 10^{-8}$

- Probando con $n = 4 \Rightarrow 10^8 \leq 9!\, 2^9$

$\Rightarrow$ Si $n = 4 \Rightarrow \cos(1/2) = \left(1 - \frac{(1/2)^2}{2} + \frac{(1/2)^4}{4!} - \frac{(1/2)^6}{6!} + \frac{(1/2)^8}{8!}\right) \pm 10^{-8} \Rightarrow \begin{cases} 0{,}87758256\ldots & [\text{aprox}] \\ 0{,}8775825618\ldots & [\text{real}] \end{cases}$

3. Hallar $\sqrt[3]{1{,}02}$ usando P_3 y estimando el error.

Sea $j(x) = \sqrt[3]{1+x} = P_3(x; a=0) + E_3(x)$

- $j(x) = (1+x)^{1/3} \Rightarrow j(0) = 1$
- $j'(x) = \frac{1}{3}(1+x)^{-2/3} \Rightarrow j'(0) = 1/3$
- $j''(x) = \frac{-2}{9}(1+x)^{-5/3} \Rightarrow j''(0) = -2/9$
- $j'''(x) = \frac{10}{27}(1+x)^{-8/3} \Rightarrow j'''(0) = 10/27$

$P_3(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{10x^3}{81}$

$\Rightarrow P_3(0{,}02) = 1{,}00662716\ldots$

$$|E_3(0{,}02)| = \left| \frac{f^{(iv)}(\theta)}{4!} (0{,}02)^4 \right| = \frac{1 \cdot 2 \cdot 5 \cdot 8}{3^4 \cdot 4!} |x|^4 \leq 6{,}6 \cdot 10^{-9} < 10^{-8}$$

Finalmente, $\sqrt[3]{1{,}02} = P_3(1{,}02) \pm 10^{-8}$

- Series de Taylor. Cálculo práctico.

1. $f(x) = e^x \ln(1+x) \Rightarrow \begin{cases} \text{Hallar serie de orden 3 con máximo de error} \\ \text{Hallar } P_3(x) \text{ en } (a=0) \end{cases}$

$$\begin{cases} e^x = 1 + x + \frac{x^2}{2} + O(x^3) \\ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) \end{cases}$$

- $f(x) = e^x \log(1+x) = (1+x+\frac{x^2}{2}+O(x^3))(x-\frac{x^2}{2}+\frac{x^3}{3}+O(x^4)) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) + x^2 - \frac{x^3}{2} + O(x^4) + \frac{x^3}{2} + O(x^4) + O(x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$

  - Sol.: $P_3(x) = x + \frac{x^2}{2} + \frac{x^3}{3}$

2. $f(x) = \frac{1}{1+x^2}$. Hallar $P_{2n}(x)$

$$\left\{\begin{array}{ll} f(x) = (1+n)^{-1} \Rightarrow f(0) = 1 \\ f'(x) = -(1+n)^{-2} \Rightarrow f'(0) = -1 \\ f''(x) = 2(1+n)^{-3} \Rightarrow f''(0) = 2 \\ f'''(x) = -6(1+n)^{-4} \Rightarrow f'''(0) = -6 \\ f^{IV}(x) = 24(1+n)^{-5} \Rightarrow f^{IV}(0) = 24 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} P_n(x) = 1 - x + x^2 - x^3 + x^4 + \dots + (-1)^n x^n + O(x^{n+1}) \\ \text{Si } x = x^2 \Rightarrow f(x) = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + O(x^{2n+2}) \\ \text{Luego, } P_{2n}(x) = 1 - x^2 + x^n + \dots + (-1)^n x^{2n} \end{array}\right.$$

- Derivación e integración de Series de Taylor.

1. Hallar Serie de Taylor de $f(x) = \arctan(x)$.

$(\arctan(x))' = (1+x^2)^{-1} = 1 - x^2 + x^4 + \dots + (-1)^n x^2 + O(x^{2n+2})$

$\Rightarrow \arctan(x) = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + \dots + (-1)^n x^{2n} + O(x^{2n+2}))\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots (-1)^n \cdot \frac{x^{2n+1}}{2n+1} + O(x^{2n+3}) + C$

- Si probamos en $x=0 \Rightarrow C = 0$

  Sol.: $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots (-1)^n \cdot \frac{x^{2n+1}}{2n+1} + O(x^{2n+3})$

Si $f''(a) = 0 \Rightarrow$ duda.

E1 Estimación de π usando la Serie de Taylor de arctan x.

Sea $\frac{\pi}{4} = \arctan(1/2) + \arctan(1/3)$

$\Rightarrow \frac{\pi}{4} = P_{2n+1}(1/2) + E_{2n+1}(1/2) + P_{2n+1}(1/3) + E_{2n+1}(1/3)$

Se puede probar que $\boxed{|E_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{2n+3}}$

$$\left.\begin{array}{l} |E_{2n+1}(1/2)| \leq \frac{1}{2}\cdot 10^{-9} \\ |E_{2n+1}(1/3)| \leq \frac{1}{2}\cdot 10^{-9} \end{array}\right\} \Rightarrow |E_{2n+1}(1/2)| \leq \frac{|1/2|^{2n+3}}{2n+3} \leq \frac{1}{2}\cdot 10^{-9} \mapsto \dots \mapsto \boxed{n=12}$$

- $\frac{\pi}{4} = \left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} + \dots + (-1)^{12}\cdot\frac{(1/2)^{25}}{25}\right] + \left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} + \dots + (-1)^{12}\frac{(1/3)^{25}}{25}\right] \pm 10^{-9}$

- **Integral Riemann (método)**

- $A = \int_0^b x^2\,dx$ Sea $t_i = \frac{ib}{n}$, $i = 1,\dots,n \Rightarrow \Delta t = \frac{b}{n}$

$S(P) = \sum_{i=1}^n M_i \cdot \Delta t = \sum_{i=1}^n (t_i)^2 \frac{b}{n}$ donde $M_i = f(t_i)$

$$\text{Área} = \lim_{\Delta t \to 0} S(P) = \lim_{n \to \infty} \sum_{i=1}^n \left(i \cdot \frac{b}{n}\right)^2 \frac{b}{n} = \left(\frac{b}{n}\right)^3 \sum_{i=1}^n i^2 = \lim_{n\to\infty} \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{3}$$

- **Primitivas**

1. $f(x) = x^2$ $F(x) = x^3/3$ es primitiva de $f \Rightarrow TFC \Rightarrow \int_0^b x^2 dx = \int_0^b f(x)\,dx = [F(x)]_{x=0}^{x=b} = \left[x^3/3\right]_0^b = b^3/3$

2. $\int e^{-3x} dx = \left(-3x = t\ ;\ dx = \frac{dt}{-3}\right) = \int e^t \frac{dt}{-3} = -\frac{1}{3}e^t + C = -\frac{1}{3}e^{-3x} + C$

3. $\int \frac{dx}{5x+6} = \left(5x+6 = t\ ;\ x = \frac{t-6}{5} \Rightarrow dx = \frac{1}{5}dt\right) = \int \frac{1}{t}\cdot\frac{1}{5}\cdot dt = \frac{1}{5}\log|t| + C = \frac{1}{5}\log|5x+6| + C$

4. $\int \frac{x}{1+x^4}dx = \left(t = x^2\ ;\ dt = d(x^2) = 2x dx \Rightarrow dx = \frac{dt}{2x}\right) = \int \frac{x}{1+t^2}\cdot\frac{dt}{2x} = \int \frac{dt}{2(1+t^2)} = \frac{1}{2}\arctan(t) + C = \frac{1}{2}\arctan(x^2) + C$

5. $\int (3x^2-1)^{10} x\,dx = \left(t = 3x^2-1\ ;\ dt = d(3x^2-1) = 6x dx \Rightarrow dx = \frac{dt}{6x}\right) = \int t^{10}\cdot x \cdot \frac{dt}{6x} = \frac{1}{6}\cdot\frac{t^{11}}{11} + C = \frac{(3x^2-1)^{11}}{66} + C$

6. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \left(t = \cos x\ ;\ dt = -\sin x\,dx\right) = \int \frac{\sin x}{t}\cdot\frac{dt}{-\sin x} = -\log|t| + C = -\log|\cos x| + C$

7. $\int_0^{\pi/2} \frac{\cos x}{(\sin x + 8)^2}dx = \int_0^1 (t+8)^{-2} dt = \left[-\frac{1}{t+8}\right]_0^1 = 1/72$

- $t = \sin x\ ;\ dt = \cos x\,dx \Rightarrow dx = dt/\cos x$
- $x = 0 \Rightarrow \sin(0) = 0 = t$
  $x = \pi/2 \Rightarrow \sin(\pi/2) = 1 = t$

8. $\int x \sin(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

$$\begin{cases} u = x\ ; & du = dx \\ dv = \sin(2x)dx\ ; & v = \int \sin(2x)dx = \frac{1}{2}\cdot -\cos(2x) \end{cases}$$

9. $\int x^2\cos x\,dx = x^2\operatorname{sen}x - 2\int x\operatorname{sen}x\,dx = x^2\operatorname{sen}x - 2\left[-x\cos x + \int\cos x\,dx\right] = x^2\operatorname{sen}x + 2x\cos x - 2\operatorname{sen}x + C$

$\begin{cases} u = x^2;\ du = 2x\,dx \\ dv = \cos x\,dx;\ v = \int\cos x\,dx = \operatorname{sen}x \end{cases}$ $\begin{cases} u = x;\ du = dx \\ dv = \operatorname{sen}x\,dx;\ v = -\cos x \end{cases}$

- **Ortogonales**

1. $\operatorname{sen}(nx)$ y $\operatorname{sen}(mx)$ en $[0,\pi]$ si $n \neq m \in \mathbb{N}$

$I = \int_0^\pi \operatorname{sen}(nx)\operatorname{sen}(mx)\,dx = \int_0^\pi \operatorname{sen}(nx)\cdot\left(-\frac{1}{m}\cos(mx)\right)'dx = -\frac{1}{m}\operatorname{sen}(nx)\cos(mx)\Big]_0^\pi - \int_0^\pi(\operatorname{sen}(nx))'\left[-\frac{1}{m}\cos(mx)\right]dx =$

$= [0-0] + \frac{n}{m}\int_0^\pi\cos(nx)\cos(mx)\,dx = \frac{n}{m}\int_0^\pi\cos(nx)\left[\frac{1}{m}\operatorname{sen}(mx)\right]'dx = \frac{n}{m}\cdot\cos(nx)\frac{1}{m}\operatorname{sen}(mx)\Big]_0^\pi - \frac{n}{m}\int_0^\pi(\cos(nx))'\frac{1}{m}\operatorname{sen}(mx)\,dx =$

$= \frac{n^2}{m^2}\int_0^\pi\operatorname{sen}(nx)\operatorname{sen}(mx)\,dx = \frac{n^2}{m^2}I$

- $I = I\left(\frac{n}{m}\right)^2 \Rightarrow I\left(1-\left(\frac{n}{m}\right)^2\right) = 0 \Rightarrow I = 0$

- **Ejercicios para practicar:**

1. $\int\frac{e^x}{1+e^{2x}}dx = \int\frac{t}{1+t^2}\cdot\frac{dt}{t} = \arctan(t) + C = \arctan(e^x) + C$

- $t = e^x \Rightarrow dt = d(e^x) = e^x\,dx$

2. $\int x^5\sqrt{1-x^2}\,dx = \int(1-t)^2 x\sqrt{t}\,\frac{1}{-2x}dt = -\frac{1}{2}\int(1-t)^2\sqrt{t}\,dt = -\frac{1}{2}\int(1+t^2-2t)\sqrt{t}\,dt = -\frac{1}{2}\int t^{1/2}+t^{5/2}-2t^{3/2}\,dt =$

- $t = 1-x^2 \Rightarrow dt = -2x\,dx$

$= -\frac{1}{2}\left[\frac{t^{3/2}}{3/2}+\frac{t^{7/2}}{7/2}-2\frac{t^{5/2}}{5/2}\right] = -\frac{t}{3}\sqrt{t} - \frac{t^3}{7}\sqrt{t} + \frac{2}{5}t^2\sqrt{t} = t\sqrt{t}\left(-\frac{1}{3}-\frac{t^2}{7}+\frac{2}{5}t\right) + C$

3. $\int\frac{x^3}{(1+x^4)^3}dx = \int\frac{x^3}{t^3}\cdot dt/_{4x^3} = \frac{1}{4}\int t^{-3}dt = \frac{1}{4}\cdot\frac{t^{-2}}{-2} = \frac{-1}{8t^2} + C = -\frac{1}{8(1+x^4)} + C$

- $t = 1+x^4 \Rightarrow dt = 4x^3\,dx$

4. $\int e^{3x}\operatorname{sen}(2x)\,dx = \int e^{3x}\left[-\frac{1}{2}\cos(2x)\right]'dx = -\frac{1}{2}e^{3x}\cos(2x) - \int (e^{3x})'\left(-\frac{1}{2}\cos(2x)\right)dx = -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\int e^{3x}\cos(2x)\,dx =$

$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\int e^{3x}\left[\frac{1}{2}\operatorname{sen}(2x)\right]'dx = -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\operatorname{sen}(2x) - \frac{3}{4}\int (e^{3x})'\operatorname{sen}(2x)\,dx =$

$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\operatorname{sen}(2x) - \frac{9}{4}I = I \Rightarrow \frac{13}{4}I = \frac{1}{2}e^{3x}\left(-\cos(2x) + \frac{3}{2}\operatorname{sen}(2x)\right) \Rightarrow I = \frac{2}{13}e^{3x}\left(\frac{3}{2}\operatorname{sen}(2x) - \cos(2x)\right) + C$

- **Descomposición.**

1. $\int \frac{1}{x(1-x)}\,dx = \int \frac{1}{x}dx + \int \frac{1}{1-x}dx = \log(x) - \log(1-x) = \log\left|\frac{x}{1-x}\right| + C$

$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x)+Bx}{x(1-x)} \Rightarrow \begin{cases} x=0 \Rightarrow 1 = A \\ x=1 \Rightarrow B = 1 \end{cases}$

2. $\int \frac{3x-2}{(x-1)^2(x+2)}\,dx = \frac{8}{9}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx - \frac{8}{9}\int {}^{1}/_{(x+2)}\,dx = \frac{8}{9}\left(\log(x-1) - \log(x+2)\right) - \frac{1}{3}(x-1)^{-1} + C$

$\frac{3x-2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} = \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)} \Rightarrow \begin{cases} x=1 \Rightarrow 1 = 3B \Rightarrow B = 1/3 \\ x=-2 \Rightarrow -8 = 9C \Rightarrow C = -8/9 \end{cases}$ $A = 8/9$

3. $\int \frac{x}{x^3-x^2+4x-4}\,dx = \frac{1}{5}\int \frac{1}{x-1}dx - \frac{1}{5}\int \frac{x-4}{x^2+4}dx = \frac{1}{5}\log(x-1) - \frac{1}{10}\log(x^2+4) + \frac{4}{5}\underset{*}{\int \frac{1}{x^2+4}dx} = \log\left[\frac{(x-1)^2}{x^2+4}\right] + \frac{2}{5}\arctan\left(\frac{x}{2}\right) + C$

$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} \Rightarrow \{ A = 1/5 \; ; \; B = -1/5 \; ; \; C = -4/5$ $\quad * \; x^2+4 = 4\left[\left(\frac{x}{2}\right)^2 + 1\right]$

4. $\int \cos^2 x\,dx = \int \frac{1}{2}(\cos(2x)+1)\,dx = \frac{x}{2} + \frac{1}{4}\operatorname{sen}(2x) + C$

$1 = \cos^2 x + \operatorname{sen}^2 x \; ; \; \cos(2x) = \cos^2 x - \operatorname{sen}^2 x = \cos^2 x + \cos^2 x - 1 = 2\cos^2 x - 1$

5. $\int \operatorname{sen}^2 x \cos^4 x\,dx = \frac{1}{4}\int \operatorname{sen}^2(2x)\cos^2 x\,dx = \frac{1}{8}\int \operatorname{sen}^2(2x)[\cos(2x)+1]\,dx = \frac{1}{8}\int \operatorname{sen}^2(2x)\,dx + \frac{1}{8}\int \operatorname{sen}^2(2x)\cos(2x)\,dx =$

$\operatorname{sen}(2x) = 2\operatorname{sen}x\cos x \Rightarrow \operatorname{sen}x = \operatorname{sen}(2x)/2\cos x$ $\quad = \frac{1}{8}\int \frac{1-\cos(4x)}{2}dx + \frac{1}{8}\int t^2\cos(2x)\frac{1}{2\cos(2x)}dt = \frac{1}{16}\left[x - \frac{1}{4}\operatorname{sen}(4x)\right] + \frac{1}{16}\cdot\frac{t^3}{3} + C$

$\operatorname{sen}^2\theta = \frac{1-\cos(2\theta)}{2}$ $\quad t = \operatorname{sen}(2x) \Rightarrow dt = 2\cos(2x)\,dx$

- Trapecio. $\int_0^1 e^{-x^2}dx$ con error $< 10^{-4}$

$f(x) = e^{-x^2}$; $f'(x) = -2xe^{-x^2}$; $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = e^{-x^2}(4x^2-2)$ $\to \max_{x\in[0,1]} |f''(x)| \leq M = 6$

Busco $n \mid |I - I_n| \leq \frac{1}{12n^2} M(b-a)^3 = \frac{1}{12n^2} 6(1-0) < 10^{-4} \to 10^4/6 < n^2 \to \boxed{n = 41}$

- Simpson. $\int_0^1 e^{-x^2}dx$ con error $< 10^{-4}$

$f^{IV}(x) = 4(4x^4 - 12x^2 + 3)e^{-x^2}$ $\to \max_{x\in[0,1]} |f^{IV}(x)| \leq 12 = M$ (gráfica)

Busco $n \mid |I - I_n| \leq \frac{1}{180n^4} M(b-a)^5 = \frac{1}{15} n^{-4} < 10^{-4} \to 10^4/15 < n^4 \to 5{,}08 < n \to \boxed{n = 6}$

... falta lo del final ...

- Integrales impropias

1. $\int_1^\infty \frac{dx}{x^\alpha} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^\alpha} \overset{\alpha\neq 1}{=} \lim_{R\to\infty} \left[\frac{x^{-\alpha+1}}{1-\alpha}\right]_1^R = \lim_{R\to\infty} \frac{R^{1-\alpha}-1}{1-\alpha} = \begin{cases} \text{Si } \alpha < 1, \to +\infty \\ \text{Si } \alpha > 1, \to 1/_{\alpha - 1} \end{cases}$

Si $\alpha = 1 \to \int_1^\infty \frac{dx}{x} = \log|x|\Big]_1^\infty = \log(\infty) - \cancel{\log(1)}^{0} = +\infty$ $\boxed{\text{Sol.: } \frac{1}{x^\alpha} \in R[1,\infty) \leftrightarrow \alpha > 1}$

2. $\int_0^\infty xe^{-x}dx \overset{\text{IPP}}{=} \int_0^\infty x(-e^{-x})'dx = -xe^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}dx = -xe^{-x}\Big]_0^\infty + [-e^{-x}]_0^\infty = 1$ $\to xe^{-x} \in R[0,\infty)$

3. $\int_1^\infty e^{-x^2}dx \leq \int_1^\infty e^{-x}dx < \infty$
$e^{-x^2} \leq e^{-x}$

4. $\int_0^\infty \frac{x}{\sqrt{1+x^3}}dx$ Supongamos $x^3 \geq 1 \to \frac{x}{\sqrt{1+x^3}} \geq \frac{x}{\sqrt{2x^3}} = \frac{1}{\sqrt{2}} \cdot \frac{x}{x^{3/2}} = \frac{1}{\sqrt{2}} \cdot x^{-1/2} \to \int_0^\infty x/(\sqrt{1+x^3})\,dx \geq \frac{1}{\sqrt{2}} \int_0^\infty \frac{dx}{\sqrt{x}} = \infty$

Hay ⊕ ejemplos

- **Integrales impropias en un punto**

1. $f(x)=\frac{1}{x^\alpha}$, $x\in(0,1]$

$$\int_0^1 \frac{dx}{x^\alpha} = \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-\alpha}dx = \left.\frac{x^{1-\alpha}}{1-\alpha}\right]_\varepsilon^1 = \begin{cases} \text{Si } \alpha<1 \Rightarrow (1-\alpha)^{-1} \\ \text{Si } \alpha\geq 1 \Rightarrow +\infty \end{cases}$$

Series

1. $\sum \frac{1}{n!}$ $\quad a_n=\frac{1}{n!}$

$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{n!}{(n+1)!} = 0 \Rightarrow$ Como $L=0<1 \Rightarrow \sum \frac{1}{n!}$ converge

2. $\sum \frac{x^n}{n!}$ $\quad \lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{x^{n+1}\cdot n!}{x^n(n+1)!} = \lim_n \frac{x}{n+1} = 0 \Rightarrow 0<1 \Rightarrow \sum \frac{x^n}{n!}$ converge, $\forall x\in\mathbb{R}$

3. $\sum \frac{2^n n!}{n^n}$ $\quad \lim_n \frac{2^{n+1}(n+1)!\,n^n}{2^n\,n!\,(n+1)^{n+1}} = \lim_n \frac{2n^n}{(n+1)^n} = 2\lim_n(1^\infty) = 2\cdot e^{\wedge}[\lim_n (n(\frac{n}{n+1}-1))] = 2e^{\wedge}(\lim_n(\frac{-n}{n+1}))] = 2e^{-1}$

Como $L=\frac{2}{e}<1 \Rightarrow$ CONVERGE

4. $\sum \frac{1}{n^\alpha}$ $\quad \lim_n \frac{n^\alpha}{(n+1)^\alpha} = \lim_n\left(\frac{n}{n+1}\right)^\alpha = 1$ [duda]

5. $\sum\left(\frac{n}{2n+1}\right)^n$ $\quad \lim_n \sqrt[n]{a_n} = \lim_n \frac{n}{2n+1} = 1/2 < 1 \Rightarrow \sum$ converge

6. $\sum n\cdot 2^{-n}$ $\quad \lim_n \sqrt[n]{a_n} = \lim_n (n2^{-n})^{1/n} = \lim_n (n^{1/n}\cdot 1/2) = \frac{1}{2} < 1 \Rightarrow \sum$ converge

7. $\sum \frac{1}{n^\alpha}$ $\quad \lim_n (n^{-\alpha})^{1/n} = 1 \Rightarrow$ duda

Criterio integral: $\frac{1}{n^\alpha}=f(n) \Rightarrow f(x)$ es pos y decreciente $\Rightarrow \sum\frac{1}{n^\alpha}$ converge $\Leftrightarrow \int_1^\infty \frac{dx}{x^\alpha}<\infty \Leftrightarrow \alpha>1$

- $\sum_{n=1}^\infty \frac{1}{n^2}$ con error $\leq 10^{-4}$

Hallamos N: $\int_N^\infty \frac{dx}{x^2} = \left[\frac{-1}{x}\right]_N^\infty = \frac{1}{N} \leq 10^{-4} \Leftrightarrow 10^4 \leq N$

8. $\sum_{n=2}^\infty \frac{1}{n\log n}$ $\quad a_n=f(n)$ con $f(x)$ decreciente y pos en $[2,+\infty)$

$\int_2^\infty \frac{1}{x\log x} = \int_2^\infty \frac{dt}{t} = [\log t]_2^\infty = +\infty \Rightarrow \sum$ DIVERGE

$t=\log x \Leftrightarrow dt=\frac{1}{x}dx$

Otro tipo:

1. $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\dots$ $\quad a_n = (-1)^n$ y $\lim_n a_n \neq 0$ $[\nexists \lim_n a_n] \Rightarrow \Sigma$ DIVERGENTE

2. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ DIVERGE xq $\lim_n a_n = 1 \neq 0$

3. $\sum_{n=1}^{\infty} (1+1/n)^n$ DIVERGE xq $\lim_n a_n = e \neq 0$

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ Serie alternada con $a_n = 1/\sqrt{n}$ $\Rightarrow \begin{cases} a_n \text{ decreciente} \\ \lim_n a_n = 0 \end{cases} \Rightarrow$ Crit. Leibniz $\Rightarrow \Sigma$ convergente

Error $\leq 10^{-4} \longrightarrow |S - S_N| \leq a_{N+1} = \frac{1}{\sqrt{n+1}} \leq 10^{-4} \Rightarrow N+1 > 10^8$

2. $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \dots$

$a_n = \frac{1}{n} \Rightarrow$ decreciente y $\lim_n a_n = 0$ $\qquad B_N = \sum_{n=1}^{N} b_n \in \{1, 0, -1\} \Rightarrow$ acotado $\qquad \Rightarrow$ converge