

# PRICING ARITHMETIC ASIAN OPTIONS UNDER THE BACHELIER MODEL

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ABSTRACT. In this set of notes we derive the time-zero prices of various *chooser options* under the continuous Bachelier model. These are contracts with a fixed maturity date  $T$  and a chooser date  $\tau$  satisfying  $0 \leq \tau \leq T$ , for which an agent is allowed to choose at time  $\tau$  the underlying security that determines the structure of the payoff at time  $T$ .

This paper is still in progress, the following are yet to be added:

- Go through all the fixme's and edit appropriately
- Add github link to appendix - explain the code
- Finish up 3.57 onward
- Bachelier approximation where asian option is the same price and  $1/\sqrt{3}$  volatility as a european option proof
- Greeks + calculations

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## 1. INTRODUCTION

In April 2020, the price of oil futures went negative. The often used Black-Scholes model, however, is unable to model assets with negative prices, due to its assumption that asset price follows a log-normal distribution. This

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reignited interest for the scarcely-used Bachelier model, a similar mathematical model where asset prices follow a normal distribution, with the advantage of being able to handle negative prices (which was considered a limitation at its inception).

## 2. THE BACHELIER MODEL

In this paper we work within the context of the *Bachelier model*, where the stock prices  $\{S_t\}_{t \geq 0}$  evolves according to

$$S_t = e^{rt} \left( S_0 + \kappa^{-rt} W_t + \kappa r \int_0^t e^{-rs} W_s ds \right), \quad (2.1)$$

where  $S_0 > 0$  denotes the initial stock price at time 0,  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure  $\tilde{\mathbb{P}}$ ,  $r$  is the interest rate, and  $\kappa$  is a measure of volatility. We note to the reader that in the special case when  $r = 0$ , (2.1) reduces to

$$S_t = S_0 + \kappa W_t. \quad (2.2)$$

**2.1. European Call when  $r = 0$ .** We first consider a European call where the payoff at time  $T$  is given by

$$V_T = (S_T - K)^+ \quad (2.3)$$

for a fixed strike price  $K$ . We note that under  $\tilde{\mathbb{P}}$ ,  $W_T \sim N(0, T)$ , therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.4)$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \quad (2.5)$$

Recall that if we have a random variable  $X$  with probability density function  $f_X$  under a probability measure  $\mathbb{P}$ , then the “law of the unconscious statistician” tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \quad (2.6)$$

In our setting, we have

$$g(S_T) = (S_T - K)^+, \quad (2.7)$$

and the distribution of  $S_T$  under  $\tilde{\mathbb{P}}$  as a random variable is given in (2.4). Therefore, the time-zero price  $V_0$  is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x - K)^+ \psi(x) dx, \quad (2.8)$$

where

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right), \quad \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad (2.9)$$

and

$$\mu = S_0, \quad \nu = \kappa \sqrt{T}. \quad (2.10)$$

To compute (2.8), we first note that since  $(x - K)^+ = 0$  for  $x \leq K$ , the domain of integration is the set  $\{x \mid x \geq K\}$ . Now we use the change of variables

$$y = -\frac{x - \mu}{\nu} \iff x = \mu - \nu y, \quad (2.11)$$

and we note that since  $\nu > 0$ ,

$$x \geq K \iff \frac{x - \mu}{\nu} \geq \frac{K - \mu}{\nu} \iff y \leq \frac{\mu - K}{\nu} =: d_-. \quad (2.12)$$

Then by performing a change of variables, (2.8) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(-y) dy = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(y) dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y) dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K - \mu) \varphi(y) dy}_{:=II}. \quad (2.13)$$

We define the cumulative distribution function of a standard normal random variable  $X$  under  $\mathbb{P}$  via

$$\varphi(x) = \mathbb{P}[X \leq x] = \mathbb{E}[\mathbb{1}_{X \leq x}] = \int_{-\infty}^x \varphi(y) dy. \quad (2.14)$$

With this notation in hand, we can write

$$II = (K - \mu) \int_{-\infty}^{d_-} \varphi(y) dy = (K - \mu) \varphi(d_-), \quad (2.15)$$

and

$$I = \nu \int_{-\infty}^{d_-} y \varphi(y) dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \rightarrow -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \quad (2.16)$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K - \mu) \varphi(d_-). \quad (2.17)$$

where

$$\mu = S_0, \quad \nu = \kappa \sqrt{T}, \quad d_- = \frac{\mu - K}{\nu}. \quad (2.18)$$

**2.2. European Put when  $r = 0$ .** To compute the price of a put, one can use put-call parity. (FIXME: derive put-call parity in appendix and reference it). (FOLLOWUP: it has been derived, I do not know how to reference it, though...)

**2.3. Arithmetic Asian Call.** Next we consider an *arithmetic Asian call* where the payoff at time  $T$  is given by

$$V_T = (A_T - K)^T, \quad A_T = \frac{1}{T} \int_0^T S_t dt = S_0 + \frac{\kappa}{T} \int_0^T W_t dt. \quad (2.19)$$

Using tools from stochastic calculus (FIXME: what tools) (FOLLOWUP: not a proper reference, but <https://math.stackexchange.com/questions/1336471/variance-of-an-integral-of-brownian-motion> shows a proof with integration by parts and Fubini), one can show that under the risk neutral measure  $\tilde{\mathbb{P}}$ ,

$$\int_0^T W_t dt \sim N(0, T^3/3). \quad (2.20)$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.21)$$

Comparing this to (2.4), we see that  $A_T$  has a similar distribution, the only difference is that the variance of  $A_T$  is smaller by a factor of  $1/3$ , so the standard deviation of  $A_T$  is smaller by a factor of  $1/\sqrt{3}$ . By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{6\pi}} e^{-3d_-^2/2} + (K - \mu) \varphi(\sqrt{3}d_-), \quad (2.22)$$

where

$$\mu = S_0, \quad \nu = \kappa \sqrt{T}, \quad d_- = \frac{\mu - K}{\nu}. \quad (2.23)$$

We note that since  $\sqrt{3} > 1$ , we see from (2.22) that the price of an Asian call is higher than the price of a European call. This should be expected as the taker is paying a premium for a less volatile product.

**2.4. Arithmetic Asian Put.** FIXME: derive Put-Call parity in appendix and reference it. FOLLOWUP: just need to reference

### 3. CHOOSER PRICING UNDER THE BACHELIER MODEL

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when  $r = 0$ .

**3.1. Properties of a Chooser.** In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date  $T$  and a strike price  $K$ , and an agent is allowed to decide on a choosing date  $\tau < T$  to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+, P_T = (K - A_T)^+, \quad (3.1)$$

where  $A_T$  is defined via (2.19). Here, we assume the agent chooses optimally with no outside information. At time  $\tau$ , the agent will choose the option of higher value between the put and call, therefore the value of this contract at time  $\tau$  is

$$V_\tau = \max(C_\tau, P_\tau). \quad (3.2)$$

The time-zero price of this contract is then

$$V_0 = e^{-rT} \tilde{\mathbb{E}}[V_\tau]. \quad (3.3)$$

In the next subsection, we simplify the expression for  $V_\tau$  via the method of replication.

**3.2. Replication.** We first note that by properties of the max function (FIXME: add this to appendix and reference), we can write

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau) \quad (3.4)$$

where  $P_\tau$  and  $C_\tau$  are an Asian put and call, respectively. By (A.1), we have

$$P_T - C_T = (K - A_T)^+ - (A_T - K)^+ = K - A_T. \quad (3.5)$$

Next, we identify the time- $\tau$  prices of contracts paying  $P_T - C_T$  and  $K - A_T$  at time  $T$ .

To replicate a security with payoff  $P_T - C_T$ , we consider a portfolio that longs a put and shorts a call at time 0, both with maturity  $T$  and strike  $K$  (FIXME: what's the maturity?). To replicate a security with payoff  $K - A_T$ , we consider a portfolio investing  $Ke^{-rT}$  into the money account at time 0 and shorting a contract (FIXME: what kind of contract?) (FOLLOWUP I'm tempted to say ZCB but I'm not sure) to receive  $A_T$  at time  $T$ .

Since both portfolios have the same payoff at time  $T$  by (3.5), they have the same price for all times  $t$  where  $0 \leq t \leq T$  (FIXME: reference appendix some sorta arbitrage).

For any  $t$  satisfying  $0 \leq t \leq T$ , we define  $P_t$  to be the time- $t$  value of a put with payoff  $P_T$  at time  $T$ ,  $C_t$  to be the time- $t$  value of a call with payoff  $C_T$  at time  $t$ ,  $w_t$  to be the time- $t$  value of an asian option paying  $A_T$  at time  $t$ . (FIXME: Move logic to appendix and reference it)

Using this notation, at time  $\tau$  the value of the first portfolio is  $P_\tau - C_\tau$ . Also, at time  $\tau$  the second portfolio has  $Ke^{-rT+r\tau}$  in the bank and is shorting a contract which pays  $A_T$  at  $T$ , therefore the time- $\tau$  value of the second portfolio is  $Ke^{r(\tau-T)} - A_\tau$ . We denote the value of a contract at time  $\tau$  which pays  $A_T$  at time  $T$  as  $w_\tau$ . By replication, the time  $\tau$  prices of the portfolios are equal, therefore we have

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - w_\tau. \quad (3.6)$$

Substituting this result back into (3.2), the value of the original chooser contract at  $\tau$  is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - w_\tau). \quad (3.7)$$

Our next goal is to find an explicit formula for  $w_\tau$ .

For simplicity, we define  $U_\tau$  to be the time  $\tau$  price of a contract with payoff  $Y_T$  at time  $T$ , where  $Y_T$  is defined via

$$Y_T = \int_0^T S_t dt. \quad (3.8)$$

Once  $U_\tau$  is determined, then we can recover  $w_\tau$  as  $w_\tau = \frac{U_\tau}{T}$ .

Note that (3.8) can be split into two parts,

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt. \quad (3.9)$$

Observe that the integral from 0 to  $\tau$  is known at time  $\tau$  as each price  $S_t$  will be known by the time  $\tau$ . So we can treat this integral as a constant and now try to replicate the integral from time  $\tau$  to  $T$ .

**3.3. Replicating the Asian chooser when  $r > 0$ .** We begin our replicating strategy by buying  $x$  shares of stock at time  $\tau$ . For all times  $t$  where  $\tau \leq t \leq T$ , we will continuously sell off stock at the rate  $\alpha_t$  and invest the revenue. With this strategy, at time  $T$ , the bank has

$$\int_{\tau}^T \alpha_t S_t e^{r(T-t)} dt \quad (3.10)$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_{\tau}^T \alpha_t S_t e^{r(T-t)} dt = \int_{\tau}^T S_t dt. \quad (3.11)$$

Solving for  $\alpha_t$ , we find that

$$\alpha_t = e^{r(t-T)} \quad (3.12)$$

Thus, the amount of shares our strategy started with was

$$x = \int_{\tau}^T e^{r(t-T)} dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \quad (3.13)$$

This tells us that the cost at time  $\tau$  to receive the stock from times  $\tau$  to  $T$  continuously is  $xS_{\tau}$ . This gives us

$$U_{\tau} = \int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)}) \quad (3.14)$$

Recall that  $w_{\tau}$  is the price at time  $\tau$  to receive  $A_T$ , equivalent to  $\frac{Y_T}{T}$ , at time  $T$ . Thus, the price at  $\tau$  to receive just  $A_T$  is

$$w_{\tau} = \frac{U_{\tau}}{T} = \frac{\int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)})}{T} \quad (3.15)$$

Returning to (3.6), we can write out the equation as

$$P_{\tau} - C_{\tau} = K e^{r(\tau-T)} - \frac{\int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)})}{T}. \quad (3.16)$$

Substituting this into the chooser option from 1.7, the value of  $V_{\tau}$  is

$$V_{\tau} = C_{\tau} + \max(0, K e^{r(\tau-T)} - \frac{\int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)})}{T}) \quad (3.17)$$

**3.4. Replicating Asian options when  $r = 0$ .** We now consider the case when  $r = 0$ . Observe we cannot plug  $r = 0$  into the formula we got for  $r > 0$  since we divide by  $r$ . However, we can apply a similar replication argument as before. (FIXME: this is a little confusing since the interest rate was never specified. maybe specify that you are assuming  $r > 0$  initially and now you're considering  $r = 0$  as a special case. If you choose to do this I recommend breaking this section into different subsections). To fix this, we return to our replicating strategy for  $w_{\tau}$  accounting for this special case.

Define  $U_{\tau}$  and  $Y_T$  the same way as above. Again, split the integral  $Y_T$  such that

$$Y_T = \int_0^{\tau} S_t dt + \int_{\tau}^T S_t dt \quad (3.18)$$

We now replicate the integral from time  $\tau$  to  $T$  for the special case. We follow the same replicating strategy as before. Purchase  $x$  shares of stock. For all times  $t$  where  $\tau \leq t \leq T$ , we continuously sell off at the rate  $\alpha_t$  and invest the revenue. By time  $T$ , the bank will have

$$\int_{\tau}^T \alpha_t S_t dt \quad (3.19)$$

We finish the replication by setting this equal to the value we're replicating

$$\int_{\tau}^T \alpha_t S_t dt = \int_{\tau}^T S_t dt \quad (3.20)$$

Solving for  $\alpha_t$ , we see that when  $r = 0$  that  $\alpha_t = 1$ . Thus, the number of shares the strategy started with was

$$\int_{\tau}^T dt = T - \tau \quad (3.21)$$

Similar to the  $r \neq 0$  case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T - \tau), \quad w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T} \quad (3.22)$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = K - \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T}. \quad (3.23)$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T}) \quad (3.24)$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \quad (3.25)$$

where  $S_0 > 0$  and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t) dt + (S_0 + \kappa W_\tau)(T - \tau)}{T}) \quad (3.26)$$

Simplifying, we find that

$$V_\tau = C_\tau + \left( K - S_0 - \frac{\kappa(T - \tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t dt \right)^+ \quad (3.27)$$

Then by the risk-neutral pricing formula and the linearity of expectation, the time-zero price  $V_0$  is given by

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}} \left[ \left( K - S_0 - \frac{\kappa(T - \tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t dt \right)^+ \right] \quad (3.28)$$

Let  $X$  be the random variable defined via

$$X = \frac{\kappa(T - \tau)}{T} W_\tau + \frac{\kappa}{T} \int_0^\tau W_t dt. \quad (3.29)$$

We now calculate the mean and variance of random variable  $X$ . We define  $X$  as the sum of random variables

$$Y = \frac{\kappa(T - \tau)}{T} W_\tau \quad (3.30)$$

$$Z = \frac{\kappa}{T} \int_0^\tau W_t dt. \quad (3.31)$$

Recall from the intro (add details on brownian motion in intro later) that the mean of the brownian motions in both  $Y$  and  $Z$  are 0, thus the means of both  $Y$  and  $Z$  are 0. We now calculate the variance of  $X$  as the sum of two random variables

$$\text{Var}(X) = \text{Var}(Y + Z) \quad (3.32)$$

It is known that

$$\text{Var}(Y + Z) = \text{Var}(Y) + \text{Var}(Z) + 2\text{Cov}(YZ). \quad (3.33)$$

Recall from the intro (add later) the variances of brownian motion. It follows that

$$\text{Var}(Y) = \tau \left( \frac{\kappa(T - \tau)}{T} \right)^2 \quad (3.34)$$

$$\text{Var}(Z) = \frac{\tau^3}{3} \left( \frac{\kappa}{T} \right)^2 \quad (3.35)$$

To calculate the covariance term, we expand it out in terms of expected value. Recall that the expected values of the brownian motions are 0.

$$\text{Cov}(YZ) = \mathbb{E}(YZ) - \mathbb{E}(Y)\mathbb{E}(Z) = \mathbb{E}(YZ) \quad (3.36)$$

We can now rewrite the covariance as

$$\text{Cov}(YZ) = \mathbb{E}(W_\tau \int_0^\tau W_t dt) \frac{\kappa^2(T - \tau)}{T^2} \quad (3.37)$$

For simplicity, let  $\alpha = \frac{\kappa^2(T - \tau)}{T^2}$ . By a property of integrals and expected value, we can move the integral outside the expected value as such (probably need to fix this).

$$\alpha \mathbb{E}(W_\tau \int_0^\tau W_t dt) = \alpha \int_0^\tau \mathbb{E}(w_\tau w_t) dt \quad (3.38)$$

Observe that  $t \leq \tau$ . Thus, we can further simplify down to

$$\alpha \int_0^\tau \mathbb{E}((w_\tau + w_t - w_t)w_t) dt = \alpha \int_0^\tau \mathbb{E}(w_t^2 + (w_\tau - w_t)w_t) dt \quad (3.39)$$

We can expand the expected value by linearity of expectations. Recall (another brownian motion thing for intro ig) that the expected value of  $(w_\tau - w_t)w_t$  is 0 and that the expected value of  $w_t^2$  is  $t$ . Thus, we have

$$Cov(XY) = \alpha \int_0^\tau t dt = \alpha \frac{\tau^2}{2} \quad (3.40)$$

It follows that the mean and variance of  $X$  can be computed as (IMPORTANT FIXME: add details of this computation!)

$$\mu = \mathbb{E}(X) = 0 \quad (3.41)$$

$$\sigma^2 = Var(X) = \tau \left( \frac{\kappa(T - \tau)}{T} \right)^2 + \frac{\tau^3}{3} \left( \frac{\kappa}{T} \right)^2 + \tau^2 \frac{\kappa^2(T - \tau)}{T^2}. \quad (3.42)$$

$$\nu = \sigma \quad (3.43)$$

(FIXME: typically  $\sigma$  denotes the standard deviation, so this isn't very consistent with standard notation)

Using methods of stochastic calculus (FIXME: we need to find a reference to cite),  $X$  is normally distributed with mean  $\mu$  and variance  $\sigma^2$ . Therefore we can define the probability density function as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (3.44)$$

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right) \quad (3.45)$$

Now we can substitute back into our equation from (4.12) and use the definition of expectation on a continuous random variable to get

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (K - S_0 - x)^+ \psi(x) dx. \quad (3.46)$$

To integrate the second term in  $V_0$  we will let

$$z = \frac{x - \mu}{\sigma}. \quad (3.47)$$

it follows that

$$x = z\nu + \mu \quad (3.48)$$

$$dx = \nu dz \quad (3.49)$$

We note that

$$K - S_0 - x \geq 0 \iff x \leq K - S_0 \iff \frac{x - \mu}{\sigma} \leq \frac{K - S_0 - \mu}{\nu} \quad (3.50)$$

and define  $d_-$  via

$$d_- = \frac{K - S_0 - \mu}{\nu} \quad (3.51)$$

so by (3.50), we have

$$K - S_0 - x \geq 0 \iff z \leq d_- \quad (3.52)$$

Now using (3.48) and (3.52) we can rearrange (3.46) as:

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \left( \int_{-\infty}^{d_-} (K - S_0 - z\nu - \mu) \varphi(z) dz \right). \quad (3.53)$$

Simplifying this expression we get the form

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \nu \int_{-\infty}^{d_-} z \varphi(z) \nu dz + (K - S_0 - \mu) \int_{-\infty}^{d_-} \varphi(z) dz \quad (3.54)$$

Excluding the call, we can easily simplify the latter of the terms by defining the cumulative distribution function

$$\Phi(x) = \int_{-\infty}^x \varphi(y) dy. \quad (3.55)$$

We resolve the former term by first substituting in (3.44)

$$\frac{-\nu}{\sqrt{2\pi}} \int_{-\infty}^{d_-} y e^{-y^2} = \frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} \quad (3.56)$$

Thus, we find that

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} + (K - S_0 - \mu) \Phi(d_-), \quad (3.57)$$

which through (F.1), gives us the final equation.

$$V_0 = C_0 + \frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} + (K - S_0 - \mu) \Phi(d_-), \quad (3.58)$$

#### 4. CHOOSER OPTION VARIANTS

**4.1. Tail Chooser.** We will now consider a variant of the Asian chooser we looked at earlier. We assume all conditions remain the same, except we now define  $A_{\tau,T}$  as

$$A_{\tau,T} = \int_{\tau}^T S_t dt \quad (4.1)$$

where  $\tau$  is the choice date, and  $T$  is the time of maturity.

**4.2. Asian Tail Choosers when  $r = 0$ .** To price this option, we slightly modify the replication strategy from before. Let  $Y_T = \int_{\tau}^T S_t dt$  and  $U_\tau$  be the price at  $\tau$  to receive  $Y_T$  at time  $T$ .

We proceed with the replication of  $Y_T$ . Suppose an agent purchases  $x$  shares at time  $\tau$ , and chooses to sell them off continuously at rate  $\alpha_t$  at time  $t$ . At time  $T$ , the agent's portfolio is worth

$$\int_{\tau}^T \alpha_t S_t e^{r(T-t)} dt \quad (4.2)$$

Since we assume here that  $r = 0$ , this reduces to

$$\int_{\tau}^T \alpha_t S_t dt \quad (4.3)$$

To complete the replication, we set this equal to the value we are trying to reproduce:

$$\int_{\tau}^T \alpha_t S_t dt = \int_{\tau}^T S_t dt \quad (4.4)$$

It follows that

$$\alpha_t = 1 \quad (4.5)$$

for all  $t$  where  $\tau \leq t \leq T$ . Thus,

$$x = \int_{\tau}^T \alpha_t dt = T - \tau. \quad (4.6)$$

It then follows that  $U_\tau = (T - \tau)S_\tau$ . Observe that

$$A_{\tau,T} = \frac{U_\tau}{T - \tau}. \quad (4.7)$$

Again using the notation  $w_\tau$  as the price needed at time  $\tau$  to receive  $A_{\tau,T}$  at time  $T$ , it follows that

$$w_\tau = \frac{(T - \tau)S_\tau}{T - \tau} = S_\tau. \quad (4.8)$$



Referring back to (3.6) and using  $r = 0$ , we have

$$P_\tau - C_\tau = K - S_\tau. \quad (4.9)$$

Using (3.4), the price of the tail chooser option with choice date  $\tau$ , which we write as  $V_\tau$ , is

$$V_\tau = C_\tau + (K - S_\tau)^+. \quad (4.10)$$

Recall in the Bachelier model that the stock evolves according to  $S_t = S_0 + \kappa W_t$  when  $r = 0$ ,  $S_0 > 0$ , and  $W_t$  is a brownian motion under the risk-neutral measure. Then,

$$V_\tau = C_\tau + (K - (S_0 + \kappa W_\tau))^+. \quad (4.11)$$

Applying the risk-neutral pricing formula and linearity of expectations, we have

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}}((K - S_0 - \kappa W_\tau)^+). \quad (4.12)$$

To simplify the above, define random variable  $X$  and function  $g(X)$  as

$$X = \kappa W_\tau, \quad g(X) = (k - S_0 - X)^+ \quad (4.13)$$

Applying the law of the unconscious statistician, we can express  $V_0$  as

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (k - S_0 - X)^+ \psi(x) dx \quad (4.14)$$

where

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right) \quad (4.15)$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (4.16)$$

Let  $y = \frac{x - \mu}{\nu}$ . Observe that

$$y = \frac{x - \mu}{\nu} \implies x = y\nu + \mu \implies dx = \nu dy \quad (4.17)$$

We can now take the positive part of the integral from (4.14)

$$k - S_0 - x \geq 0 \implies -x \geq S_0 - k \quad (4.18)$$

Adding  $\mu$  and dividing by  $\nu$  on both sides,

$$-y = \frac{-x + \mu}{\nu} \geq \frac{S_0 - k + \mu}{\nu} \quad (4.19)$$

It follows that

$$y \leq \frac{k - S_0 - \mu}{\nu} = d_- \quad (4.20)$$

We now evaluate 4.14 using 4.17

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{d_-} (k - S_0 - y\nu - \mu) \left(\frac{1}{\nu} \varphi(y)\right) (-\nu) dy \quad (4.21)$$

Simplifying and splitting the integral, we have

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \int_{-\infty}^{d_-} (k - S_0 - \mu) \varphi(y) dy + \int_{-\infty}^{d_-} y\nu \varphi(y) dy \quad (4.22)$$

Define the CDF the same as (3.55).

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - (k - S_0 - \mu) \Phi(d_-) + \int_{-\infty}^{d_-} y\nu \varphi(y) dy \quad (4.23)$$

The remaining integral term can be simplified through (4.16).

$$\nu \int_{-\infty}^{d_-} y \varphi(y) dy = \nu \int_{-\infty}^{d_-} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = -\frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} \quad (4.24)$$

The final form for the time 0 price of the tail chooser is then

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - (k - S_0 - \mu) \Phi(d_-) - \frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} \quad (4.25)$$

## 5. APPROXIMATING ARITHMETIC ASIAN UNDER BLACK-SCHOLES MODEL

(FIXME: find reference for why this is an acceptable approximation, flesh out reasoning) In the Black-Scholes Model, the asset price  $S_t$  for  $0 \leq t \leq T$  evolves according to

$$S_t = S_0 e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t}. \quad (5.1)$$

Recall the notation

$$A_T = \frac{1}{T} \int_0^T S_t dt. \quad (5.2)$$

Note that the integrand  $S_t$  is a lognormal random variable. However, the integral of  $S_t$  will not be lognormal (cite somewhere). Thus, we model  $A_t$  with a log-normal  $Y_t$  with the same mean and variance, which leaves us with equations

$$\tilde{\mathbb{E}}[A_T] = \tilde{\mathbb{E}}[Y_T] \quad (5.3)$$

$$\tilde{\mathbb{E}}[A_T^2] = \tilde{\mathbb{E}}[Y_T^2] \quad (5.4)$$

$$Y_t = Y_0 e^{\Gamma W_t + rt - \frac{1}{2}\Gamma^2 t} \quad (5.5)$$

Starting with the RHS of (5.3)

$$\tilde{\mathbb{E}}[A_T] = \frac{1}{T} \int_0^T \tilde{\mathbb{E}}[S_t] dt = \frac{1}{T} \int_0^T \tilde{\mathbb{E}}[S_0 e^{\sigma W_t + rt - \frac{1}{2}\sigma^2 t}] dt = \frac{S_0}{T} \int_0^T \tilde{\mathbb{E}}[e^{\sigma W_t} e^{rt - \frac{1}{2}\sigma^2 t}] dt \quad (5.6)$$

We can factor out the constant part of the expected value and are left to evaluate

$$\tilde{\mathbb{E}}[A_T] = \frac{S_0}{T} \int_0^T e^{rt - \frac{1}{2}\sigma^2 t} \tilde{\mathbb{E}}[e^{\sigma W_t}] dt \quad (5.7)$$

Using Moment Generating Function  $e^{yX}$ , we know if  $X$  is a normally distributed random variable with mean  $\mu$  and variance  $\sigma^2$  then

$$\tilde{\mathbb{E}}[e^{yX}] = e^{\mu y + \frac{1}{2}\sigma^2 y^2} \quad (5.8)$$

Recalling that the Brownian motion  $W_t$  is of mean  $\mu$  and variance  $\sigma^2$ ,

$$\tilde{\mathbb{E}}[e^{\sigma W_t}] = e^{\frac{1}{2}\sigma^2 t}, \quad (5.9)$$

which we can plug in to (5.7) to get

$$\tilde{\mathbb{E}}[A_T] = \frac{S_0}{T} \int_0^T e^{rt - \frac{1}{2}\sigma^2 t} \tilde{\mathbb{E}}[e^{\sigma W_t}] dt = \frac{S_0}{T} \int_0^T e^{\frac{1}{2}\sigma^2 t} e^{rt - \frac{1}{2}\sigma^2 t} dt = \frac{S_0}{T} \int_0^T e^{rt} dt = \frac{S_0}{rT} (e^{rT} - 1) \quad (5.10)$$

We can then simplify the LHS of (5.3) using the moment generating function (5.8) to get

$$\tilde{\mathbb{E}}[Y_T] = \tilde{\mathbb{E}}[Y_0 e^{\Gamma W_t + rt - \frac{1}{2}\Gamma^2 t}] = \tilde{\mathbb{E}}[Y_0] \tilde{\mathbb{E}}[e^{rt - \frac{1}{2}\Gamma^2 t}] \tilde{\mathbb{E}}[\Gamma W_t] = Y_0 e^{rt - \frac{1}{2}\Gamma^2 t} e^{\frac{1}{2}\Gamma^2 t} = Y_0 e^{rt} \quad (5.11)$$

So, according to (5.3), we have

$$Y_0 = \frac{S_0}{rT} (1 - e^{-rT}) \quad (5.12)$$

We next simplify the second equation. Moving onto the LHS of (5.4),

$$\tilde{\mathbb{E}}[A_T^2] = \frac{1}{T^2} \left( \int_0^T \tilde{\mathbb{E}}[S_t] dt \right)^2 \quad (5.13)$$

Observe that the square of an integral can be rewritten as a double integral as follows

$$\frac{1}{T^2} \left( \int_0^T \tilde{\mathbb{E}}[S_t] dt \right)^2 = \frac{1}{T^2} \int_0^T \int_0^T \tilde{\mathbb{E}}[S_s] \tilde{\mathbb{E}}[S_t] ds dt = \frac{1}{T^2} \int_0^T \int_0^T \tilde{\mathbb{E}}[S_s S_t] ds dt \quad (5.14)$$

WLOG, assume that  $0 \leq s \leq t$ . Integrating under these conditions yields us half of the desired area. Through a symmetry argument, we can conclude that

$$\frac{1}{T^2} \int_0^T \int_0^T \tilde{\mathbb{E}}[S_s S_t] ds dt = \frac{2}{T^2} \int_0^T \int_0^t \tilde{\mathbb{E}}[S_s S_t] ds dt \quad (5.15)$$

We now deal with the expression inside the expected value. We can expand  $S_s$  and  $S_t$  and separate out the Brownian motions

$$S_s S_t = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)+\sigma(W_t+W_s)} = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{\sigma W_t} e^{\sigma W_s} \quad (5.16)$$

Observe that  $W_t = W_s + (W_t - W_s)$ . Using this, we write

$$S_s S_t = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{\sigma W_t} e^{\sigma W_s} = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{\sigma W_s} e^{\sigma(W_t-W_s)} e^{\sigma W_s} = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{2\sigma W_s} e^{\sigma(W_t-W_s)} \quad (5.17)$$

This is of interest because the Brownian motions  $W_s$  and  $W_t - W_s$  are independent (reference appendix). Observe that  $W_s \sim N(0, s)$  and  $W_t - W_s \sim N(0, t-s)$  (ref?). Due to independence, we can write out the expected value as follows

$$\tilde{\mathbb{E}}[S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{2\sigma W_s} e^{\sigma(W_t-W_s)}] = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} \tilde{\mathbb{E}}[e^{2\sigma W_s}] \tilde{\mathbb{E}}[e^{\sigma(W_t-W_s)}] \quad (5.18)$$

Using (5.8) again, observe that

$$\tilde{\mathbb{E}}[e^{2\sigma W_s}] = e^{2\sigma^2 s} \quad (5.19)$$

$$\tilde{\mathbb{E}}[e^{\sigma(W_t-W_s)}] = e^{\frac{1}{2}\sigma^2(t-s)} \quad (5.20)$$

Substituting, we find that

$$\tilde{\mathbb{E}}[S_s S_t] = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} \tilde{\mathbb{E}}[e^{2\sigma W_s}] \tilde{\mathbb{E}}[e^{\sigma(W_t-W_s)}] = S_0^2 e^{r(t+s)+\sigma^2 s} \quad (5.21)$$

We now return to the main integral using this new result

$$\frac{2}{T^2} \int_0^T \int_0^t S_0^2 e^{r(t+s)+\sigma^2 s} ds dt = \frac{2}{T^2} \int_0^T S_0^2 e^{rt} \int_0^t e^{(r+\sigma^2)s} ds dt \quad (5.22)$$

We begin by evaluating the inside integral from (5.22)

$$\frac{2}{T^2} \int_0^T S_0^2 e^{rt} \int_0^t e^{(r+\sigma^2)s} ds dt = \frac{2}{T^2} \int_0^T \frac{S_0^2 e^{rt}}{r+\sigma^2} (e^{(r+\sigma^2)t} - 1) dt = \frac{2S_0^2}{T^2(r+\sigma^2)} \int_0^T (e^{(2r+\sigma^2)t} - e^{rt}) dt \quad (5.23)$$

Evaluating the next integral, we see that

$$\frac{2S_0^2}{T^2(r+\sigma^2)} \int_0^T (e^{(2r+\sigma^2)t} - e^{rt}) dt = \frac{2S_0^2}{T^2(r+\sigma^2)} \left[ \frac{1}{2r+\sigma^2} (e^{(2r+\sigma^2)T} - 1) - \frac{1}{r} (e^{rT} - 1) \right] \quad (5.24)$$

Thus,

$$\tilde{\mathbb{E}}[A_T^2] = \frac{2S_0^2}{T^2(r+\sigma^2)} \left[ \frac{1}{2r+\sigma^2} (e^{(2r+\sigma^2)T} - 1) - \frac{1}{r} (e^{rT} - 1) \right] \quad (5.25)$$

Continuing on to the RHS of (5.4), we can apply the moment generating function again with the information that  $W_t \sim N(0, t)$  to get

$$\tilde{\mathbb{E}}[Y_T^2] = Y_0^2 e^{(2r-\Gamma^2)T} \tilde{\mathbb{E}}[e^{2\Gamma W_T}] = Y_0^2 e^{(2r+\Gamma^2)T} \quad (5.26)$$

Equating  $\tilde{\mathbb{E}}[A_T^2]$  and  $\tilde{\mathbb{E}}[Y_T^2]$  in (5.4) and recalling the calibration (5.12) we found from (5.3), it follows that

$$\Gamma^2 = \frac{1}{T} \left( \ln \left( \frac{2S_0^2}{Y_0^2 T^2 (r+\sigma^2)} \right) + \ln \left( \frac{1}{2r+\sigma^2} (e^{2rT+\sigma^2 T} - 1) - \frac{1}{r} (e^{rT} - 1) \right) - 2rT \right) \quad (5.27)$$

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#### APPENDIX A. NOTATION AND CONVENTIONS

For a random variable  $X$  we use the notation  $X^+$  to denote the random variable  $\max(X, 0)$ . We note that by definition, we have

$$X = X^+ - (-X)^+, \quad (A.1)$$

from which the *put-call parity* can be derived.

For a normal random variable  $X$  we use the notation  $X \sim N(\mu, \sigma^2)$  to denote that it has mean  $\mu$  and variance  $\sigma^2$ .

## APPENDIX B. ARBITRAGE-FREE PRICING

Before defining arbitrage-free pricing, we first must define arbitrage. An arbitrage strategy has three properties.

- (1) the agent's initial capital is zero.
- (2) the agent has zero percent chance of losing money.
- (3) the agent has a strictly positive probability of profit.

Under this definition of arbitrage, the arbitrage-free price of an asset is the price where an arbitrage strategy is not possible.

**B.1. Arbitrage-free Market.** In this paper we work under the assumption that the market is arbitrage-free. As such, we claim that if the values of two portfolios are equal at time  $T > 0$ , then for all times  $\tau$  where  $0 \leq \tau \leq T$ , the values of both portfolios are equal.

We prove by contrapositive. Assume that at time  $T > 0$  the prices of two portfolios are equal, and that at time  $\tau$  where  $0 \leq \tau \leq T$  that one portfolio is worth more than the other. Let  $P_1$  be the value of portfolio 1 and  $P_2$  be the value of portfolio 2. Thus, without loss of generality, at time  $\tau$ , let  $P_1 > P_2$ . At time  $\tau$ , we buy portfolio 2 and sell portfolio 1. We can pocket the difference  $P_1 - P_2$ . At time  $T$ , we can then sell portfolio 2 to pay off the time  $T$  cost of portfolio 1. Thus, there exists an arbitrage strategy, which is a contradiction.

It follows that under an arbitrage-free model, if two portfolios have equal value at time  $T$ , they must have equal value at all times from 0 to  $T$ .

## APPENDIX C. PUT-CALL PARITY

An important result used repeatedly throughout this paper is put-call parity.

For some asset of price  $S_T$  at time  $T$ , define the European put and call of strike  $K$  as

$$P_T^E = (K - S_T)^+ \quad (\text{C.1})$$

$$C_T^E = (S_T - K)^+. \quad (\text{C.2})$$

By (A.1),

$$P_T^E - C_T^E = K - S_T. \quad (\text{C.3})$$

We can replicate the LHS portfolio by going long a put and short a call at time  $\tau$ . The RHS can be replicated by investing  $Ke^{r(\tau-T)}$  into a risk-free return and shorting the asset at  $T = \tau$ . Under the assumption of the arbitrage-free market, it follows that

$$P_\tau^E - C_\tau^E = Ke^{r(\tau-T)} - S_\tau. \quad (\text{C.4})$$

We apply a similar argument towards Asian puts and calls. Again define an asset with price  $S_T$  at time  $T$ . Define the Asian put and call with strike price  $K$  respectively as

$$P_T^A = (K - \int_0^T S_t dt)^+ \quad (\text{C.5})$$

$$C_T^A = (\int_0^T S_t dt - K)^+. \quad (\text{C.6})$$

By (A.1),

$$P_T^A - C_T^A = K - \int_0^T S_t dt. \quad (\text{C.7})$$

We replicate the LHS by going long the put and short the call at time  $\tau$ . The RHS can be replicated by investing  $Ke^{r(\tau-T)}$  and shorting an option at  $w_\tau$  which pays  $A_\tau$  at time  $T$ , all at time  $\tau$ . Thus,

$$P_\tau^A - C_\tau^A = Ke^{r(\tau-T)} - w_\tau. \quad (\text{C.8})$$

The calculation of  $w_\tau$  is demonstrated in the above sections.

## APPENDIX D. MAX TRANSFORMATION

We derive the following property of the max function:

$$\max(a, b) = a + \max(0, b - a). \quad (\text{D.1})$$

The proof follows through casework. First consider when  $a > b$ , it follows that

$$\max(a, b) = a \quad (\text{D.2})$$

$$a + \max(0, b - a) = a + 0 = a. \quad (\text{D.3})$$

The second case we consider is  $b \geq a$ , then

$$\max(a, b) = b \quad (\text{D.4})$$

$$a + \max(0, b - a) = a + (b - a) = b. \quad (\text{D.5})$$

The proof is now complete.

## APPENDIX E. MOMENT GENERATING FUNCTIONS

We define  $m_X(y)$  to be moment generating function on random variable  $X$  such that

$$m(y) = \tilde{\mathbb{E}}[e^{yX}] \quad (\text{E.1})$$

where  $y \in \mathbb{R}$ . When  $X \sim N(\mu, \sigma^2)$ , we have

$$m(y) = \tilde{\mathbb{E}}[e^{yX}] = \exp(\mu y + \frac{1}{2}\sigma^2 y^2) \quad (\text{E.2})$$

## APPENDIX F. VARIOUS REPLICATING STRATEGIES

**F.1. Replicating Puts and Calls.** The put and call options are two often used contracts which give the buyer the right, but not obligation, to respectively sell or buy an underlying security.

To "price back" a put or call from time  $t$  to time 0, where  $t \geq 0$ , we simply buy a put or call respectively at time 0. It thus follows that

$$\tilde{\mathbb{E}}[C_t] = C_0 \quad (\text{F.1})$$

$$\tilde{\mathbb{E}}[P_t] = P_0 \quad (\text{F.2})$$

**F.2. Replicating European options.** FIXME

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