

MFSURP

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ABSTRACT. In this set of notes we derive the time-zero prices of various *chooser options*. These are contracts with a fixed maturity date T and a chooser date τ satisfying $0 \leq \tau \leq T$, for which an agent is allowed to choose at time τ the underlying security that determines the structure of the payoff at time T .

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1. INTRODUCTION

1.1. Arbitrage-free pricing. (FIXME: add some background information on arbitrage-free pricing, aka summarize what you learned in 21-270. We can also add this to an appendix.)

1.2. The Blachelier Model. In this paper we work within the context of the *Blachelier model*, where the stock prices $\{S_t\}_{t \geq 0}$ evolves according to

$$S_t = e^{rt} \left(S_0 + \kappa^{-rt} W_t + \kappa r \int_0^t e^{-rs} W_s ds \right), \quad (1.1)$$

where $S_0 > 0$ denotes the initial stock price at time 0 and $\{W_t\}_{t \geq 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. We note to the reader that in the special case when $r = 0$, (1.1) reduces to

$$S_t = S_0 + \kappa W_t. \quad (1.2)$$

(FIXME: explain to the reader why the Blachelier model is interesting).

1.3. Notation and conventions. For a random variable X we use the notation X^+ to denote the random variable $\max(X, 0)$. We note that by definition, we have

$$X = X^+ - (-X)^+, \quad (1.3)$$

from which the *put-call parity* can be derived.

(FIXME: some common notation that needs to be explained:

- (1) the notation $X \sim N(\mu, \sigma^2)$
- (2) ...

)

Key words and phrases. Blachelier model, Chooser options, Exotic options.

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2. ARBITRAGE-FREE PRICING UNDER THE BLACHELIER MODEL

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when $r = 0$.

2.0.1. *European Call*. We first consider a European call where the payoff at time T is given by

$$V_T = (S_T - K)^+ \quad (2.1)$$

for a fixed strike price K . We note that under $\tilde{\mathbb{P}}$, $W_T \sim N(0, T)$, therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.2)$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \quad (2.3)$$

Recall that if we have a random variable X with probability density function f_X under a probability measure \mathbb{P} , then the “law of the unconscious statistician” tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \quad (2.4)$$

In our setting, we have

$$g(S_T) = (S_T - K)^+, \quad (2.5)$$

and the distribution of S_T under $\tilde{\mathbb{P}}$ as a random variable is given in (2.2). Therefore, the time-zero price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x - K)^+ \psi(x) dx, \quad (2.6)$$

where

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right), \quad \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad (2.7)$$

and

$$\mu = S_0, \quad \nu = \kappa\sqrt{T}. \quad (2.8)$$

To compute (2.6), we first note that since $(x - K)^+ = 0$ for $x \leq K$, the domain of integration is the set $\{x \mid x \geq K\}$. Now we use the change of variables

$$y = -\frac{x - \mu}{\nu} \iff x = \mu - \nu y, \quad (2.9)$$

and we note that since $\nu > 0$,

$$x \geq K \iff \frac{x - \mu}{\nu} \geq \frac{K - \mu}{\nu} \iff y \leq \frac{\mu - K}{\nu} =: d_-. \quad (2.10)$$

Then by performing a change of variables, (2.6) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(-y) dy = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(y) dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y) dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K - \mu) \varphi(y) dy}_{:=II}. \quad (2.11)$$

We define the cumulative distribution function of a standard normal random variable X under \mathbb{P} via

$$\varphi(x) = \mathbb{P}[X \leq x] = \mathbb{E}[\mathbb{1}_{X \leq x}] = \int_{-\infty}^x \varphi(y) dy. \quad (2.12)$$

With this notation in hand, we can write

$$II = (K - \mu) \int_{-\infty}^{d_-} \varphi(y) dy = (K - \mu) \varphi(d_-), \quad (2.13)$$

and

$$I = \nu \int_{-\infty}^{d_-} y \varphi(y) dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \rightarrow -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \quad (2.14)$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K - \mu) \varphi(d_-). \quad (2.15)$$

To compute the price of a put, one can use put-call parity (FIXME: derive put-call parity somewhere and reference it).

2.0.2. *Arithmetic Asian calls.* Next we consider an *arithmetic Asian call* where the payoff at time T is given by

$$V_T = (A_T - K)^T, \quad A_T = \frac{1}{T} \int_0^T S_t dt = S_0 + \frac{\kappa}{T} \int_0^T W_t dt. \quad (2.16)$$

Using tools from Stochastic calculus, one can show that under the risk neutral measure $\tilde{\mathbb{P}}$,

$$\int_0^T W_t dt \sim N(0, T^3/3). \quad (2.17)$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.18)$$

Comparing this to (2.2), we see that A_T has a similar distribution, the only difference is that the variance of A_T is smaller by a factor of $1/3$, so the standard deviation of A_T is smaller by a factor of $1/\sqrt{3}$. By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{3}\sqrt{2\pi}} e^{-3d_-^2/2} + (K - \mu)\varphi(\sqrt{3}d_-), \quad (2.19)$$

where

$$\mu = S_0, \quad \nu = \kappa\sqrt{T}, \quad d_- = \frac{\mu - K}{\nu}. \quad (2.20)$$

We note that since $\sqrt{3} > 1$, we see from (2.19) that the price of an Asian option is higher than the price of a European call. This should be expected as one is paying a premium for a less volatile product.

3. CHOOSER OPTIONS

In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date T and a strike price K , and an agent is allowed to decide on a choosing date $\tau < T$ to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+, \quad P_T = (K - A_T)^+, \quad (3.1)$$

where A_T is defined via (2.16). Here, we assume the agent chooses optimally with no outside information. At time τ , the agent will choose the option of higher value between the put and call, therefore the value of this contract at time τ is

$$V_\tau = \max(C_\tau, P_\tau). \quad (3.2)$$

The time-zero price of this contract is then

$$V_0 = \tilde{\mathbb{E}}[V_\tau]. \quad (3.3)$$

In the next subsection, we simplify the expression for V_τ via the method of replication.

3.1. Replication. We first note that by properties of the max function (FIXME: maybe add this in intro and reference?), we can write

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau). \quad (3.4)$$

By (1.3), we have

$$P_T - C_T = (K - S_T)^+ - (S_T - K)^+ = K - A_T. \quad (3.5)$$

Combining (3.2) and (3.5) then gives us

$$V_\tau = C_\tau + \max(0, K - A_T). \quad (3.6)$$

Next, we identify the time- τ prices of contracts paying $P_T - C_T$ and $K - A_T$ at time T .

To replicate a security with payoff $P_T - C_T$, we consider a portfolio that longs a put and shorts a call at time 0, both with strike T (FIXME: what's the maturity?). To replicate a security with payoff $K - A_T$, we consider a portfolio investing Ke^{-rT} into the money account at time 0 and shorting a contract (FIXME: what kind of contract?) to receive A_T at time T .

Since both portfolios have the same payoff at time T by (3.5), they have the same price for all times t where $0 \leq t \leq T$ (FIXME: in the intro explain why this is true in terms of arbitrage).

For any t satisfying $0 \leq t \leq T$, we define P_t to be the time- t value of a put with payoff P_T at time T , C_t to be the time- t value of a call with payoff C_T at time t , A_t to be the time- t value of an asian option paying A_T at time t . (FIXME: maybe move to intro?)

Using this notation, at time τ the value of the first portfolio is $P_\tau - C_\tau$. Also, at time τ the second portfolio has $Ke^{-rT+r\tau}$ in the bank and is shorting a contract which pays A_T at T , therefore the time- τ value of the second portfolio is $Ke^{r(\tau-T)} - A_\tau$. By replication, the time τ prices of the portfolios are equal, therefore we have

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - A_\tau. \quad (3.7)$$

Substituting this result back into (3.2), the value of the original chooser contract at τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - A_\tau). \quad (3.8)$$

Our next goal is to find an explicit formula for A_τ .

For simplicity, we define U_τ to be the time τ price of a contract with payoff Y_T at time T , where Y_T is defined via

$$Y_T = \int_0^T S_t dt. \quad (3.9)$$

Once U_τ is determined, then we can recover A_τ as $A_\tau = \tau U_\tau$.

Note that (3.9) can be split into two parts,

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt. \quad (3.10)$$

Observe that the integral from 0 to τ is known at time τ as each price S_t will be known by the time τ . So we can treat this integral as a constant and now try to replicate the integral from time τ to T .

3.2. Replicating Asian options when $r > 0$. We begin our replicating strategy by buying x shares of stock at time τ . For all times t where $\tau \leq t \leq T$, we will continuously sell off stock at the rate α_t and invest the revenue. With this strategy, at time T , the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt \quad (3.11)$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt = \int_\tau^T S_t dt. \quad (3.12)$$

Solving for α_t , we find that

$$\alpha_t = e^{r(t-T)} \quad (3.13)$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \quad (3.14)$$

This tells us that the cost at time τ to receive the stock from times τ to T continuously is xS_τ . This gives us

$$U_\tau = \int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)}) \quad (3.15)$$

Recall that w_τ is the price at time τ to receive A_T , equivalent to $\frac{Y_T}{T}$, at time T . Thus, the price at τ to receive just A_T is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T} \quad (3.16)$$

Returning to 1.6 Put-Call Parity (FIXME: fix reference), we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T}. \quad (3.17)$$

Substituting this into the chooser option from 1.7, the value of V_τ is

$$V_\tau = C_\tau + \max(0, K e^{r(\tau-T)} - \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T}) \quad (3.18)$$

3.3. Replicating Asian options when $r = 0$. The above formula breaks when $r = 0$ since we divide by r (FIXME: this is a little confusing since the interest rate was never specified. maybe specify that you are assuming $r > 0$ initially and now you're considering $r = 0$ as a special case. If you choose to do this I recommend breaking this section into different subsections). To fix this, we return to our replicating strategy for w_τ accounting for this special case.

Define U_τ and Y_T the same way as above. Again, split the integral Y_T such that

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt \quad (3.19)$$

We now replicate the integral from time τ to T for the special case. We follow the same replicating strategy as before. Purchase x shares of stock. For all times t where $\tau \leq t \leq T$, we continuously sell off at the rate α_t and invest the revenue. By time T , the bank will have

$$\int_\tau^T \alpha_t S_t dt \quad (3.20)$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t dt = \int_\tau^T S_t dt \quad (3.21)$$

Solving for α_t , we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \quad (3.22)$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T - \tau), \quad w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T} \quad (3.23)$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T}. \quad (3.24)$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T}) \quad (3.25)$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \quad (3.26)$$

where $S_0 > 0$ and $\{W_t\}_{t \geq 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t) dt + (S_0 + \kappa W_\tau)(T - \tau)}{T}) \quad (3.27)$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left(K - S_0 - \frac{\kappa(T - \tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t dt \right)^+. \quad (3.28)$$

Then by the risk-neutral pricing formula and the linearity of expectation, the time-zero price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}} \left[\left(K - S_0 - \frac{\kappa(T - \tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t dt \right)^+ \right]. \quad (3.29)$$

Let X be the random variable defined via

$$X = \frac{\kappa(T - \tau)}{T} W_\tau + \frac{\kappa}{T} \int_0^\tau W_t dt. \quad (3.30)$$

It follows that the mean and variance of X can be computed as (IMPORTANT FIXME: add details of this computation!)

$$\mu = \mathbb{E}(X) = 0 \quad (3.31)$$

$$\sigma = \text{Var}(X) = \tau \left(\frac{\kappa(T - \tau)}{T} \right)^2 + \frac{\tau^3}{3} \left(\frac{\kappa}{T} \right)^2 + \tau^2 \frac{\kappa^2(T - \tau)}{T^2}. \quad (3.32)$$

$$\nu = \sqrt{\sigma} \quad (3.33)$$

$$(3.34)$$

(FIXME: typically σ denotes the standard deviation, so this isn't very consistent with standard notation)

Using methods of stochastic calculus (FIXME: we need to find a reference to cite), X is normally distributed with mean μ and variance σ^2 . Therefore we can define the probability density function as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (3.35)$$

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right) \quad (3.36)$$

Now we can substitute back into our equation from (3.29) and use the definition of expectation on a continuous random variable to get

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (K - S_0 - x)^+ \psi(x) dx. \quad (3.37)$$

To integrate the second term in V_0 we will let

$$z = \frac{x - \mu}{\sigma}. \quad (3.38)$$

it follows that

$$x = z\nu + \mu \quad (3.39)$$

$$dx = \nu dz. \quad (3.40)$$

We note that

$$K - S_0 - x \geq 0 \iff x \leq K - S_0 \iff \frac{x - \mu}{\sigma} \leq \frac{K - S_0 - \mu}{\nu}. \quad (3.41)$$

We define d_- via

$$d_- = \frac{K - S_0 - \mu}{\nu} \quad (3.42)$$

so by (3.41), we have

$$K - S_0 - x \geq 0 \iff z \leq d. \quad (3.43)$$

Now using (3.39) and (3.43) we can rearrange (3.37) as:

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \left(\int_{-\infty}^{d_-} (K - S_0 - z\nu - \mu) \varphi(z) dz \right). \quad (3.44)$$

Simplifying this expression we get the form

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \nu \int_{-\infty}^{d_-} z \varphi(z) \nu dz + (K - S_0 - \mu) \int_{-\infty}^{d_-} \varphi(z) dz \quad (3.45)$$

(FIXME: continue with this computation, write down an explicit formula for V_0 , and also the price for the variant.)

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APPENDIX A. VARIOUS REPLICATING STRATEGIES

A.1. Replicating European options.

A.2. Replicating Asian options. CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA

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