PRICING ARITHMETIC ASIAN OPTIONS UNDER THE BACHELIER MODEL

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ABSTRACT. In this set of notes we derive the time-zero prices of various *chooser options* under the continuous Bachelier model. These are contracts with a fixed maturity date T and a chooser date τ satisfying $0 \leqslant \tau \leqslant T$, for which an agent is allowed to choose at time τ the underlying security that determines the structure of the payoff at time T.

This paper is still in progress, the following are yet to be added:

- Go through all the fixme's and edit appropriately
- Add github link to appendix - explain the code
- Finish up 3.57 onward
- Bachelier approximation where asian option is the same price and 1/sqrt(3) volatility as a european option proof
- Greeks + calculations

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1. INTRODUCTION

In April 2020, the price of oil futures went negative. The often used Black-Scholes model, however, is unable to model assets with negative prices, due to its assumption that asset price follows a log-normal distribution. This reignited interest for the scarcely-used Bachelier model, a similar mathematical model where asset prices follow a

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normal distribution, with the advantage of being able to handle negative prices (which was considered a limitation at its inception).

2. The Bachelier Model

In this paper we work within the context of the *Bachelier model*, where the stock prices $\{S_t\}_{t\geqslant 0}$ evolves according to

$$S_t = e^{rt}\left(S_0 + \kappa^{-rt}W_t + \kappa r\int_0^t e^{-rs}W_s\,ds\right), \tag{2.1}$$

where $S_0 > 0$ denotes the initial stock price at time 0, $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$, r is the interest rate, and κ is a measure of volatility. We note to the reader that in the special case when $r = 0$, (2.1) reduces to

$$S_t = S_0 + \kappa W_t. \tag{2.2}$$

2.1. European Call when r = 0.

We first consider a European call where the payoff at time T is given by

$$V_T = (S_T - K)^+ \tag{2.3}$$

for a fixed strike price K. We note that under $\tilde{\mathbb{P}}$, $W_T \sim N(0,T)$, therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.4}$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \tag{2.5}$$

Recall that if we have a random variable X with probability density function f_X under a probability measure $\mathbb{P}$, then the "law of the unconscious statistician" tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx. \tag{2.6}$$

In our setting, we have

$$g(S_T) = (S_T - K)^+, \tag{2.7}$$

and the distribution of S_T under $\tilde{\mathbb{P}}$ as a random variable is given in (2.4). Therefore, the time-zero price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x-K)^+\psi(x)\,dx, \tag{2.8}$$

where

$$\psi(x) = \frac{1}{\nu}\varphi\left(\frac{x-\mu}{\nu}\right),\ \varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}, \tag{2.9}$$

and

$$\mu = S_0,\ \nu = \kappa\sqrt{T}. \tag{2.10}$$

To compute (2.8), we first note that since $(x-K)^+ = 0$ for $x \leqslant K$, the domain of integration is the set $\{x \mid x \geqslant K\}$. Now we use the change of variables

$$y = -\frac{x-\mu}{\nu} \iff x = \mu - \nu y, \tag{2.11}$$

and we note that since $\nu > 0$,

$$x \geqslant K \iff \frac{x-\mu}{\nu} \geqslant \frac{K-\mu}{\nu} \iff y \leqslant \frac{\mu - K}{\nu} =: d_-. \tag{2.12}$$

Then by performing a change of variables, (2.8) becomes

$$V_0 = \int_{-\infty}^{d_-}(\nu y + K - \mu)\varphi(-y)\,dy = \int_{-\infty}^{d_-}(\nu y + K - \mu)\varphi(y)\,dy = \underbrace{\int_{-\infty}^{d_-}\nu y\varphi(y)\,dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-}(K-\mu)\varphi(y)\,dy}_{:=II}. \tag{2.13}$$

We define the cumulative distribution function of a standard normal random variable X under $\mathbb{P}$ via

$$\varphi(x) = \mathbb{P}[X \leqslant x] = \mathbb{E}[\mathbb{1}_{X\leqslant x}] = \int_{-\infty}^{x}\varphi(y)\,dy. \tag{2.14}$$

With this notation in hand, we can write

$$II = (K-\mu)\int_{-\infty}^{d_-} \varphi(y)\,dy = (K-\mu)\varphi(d_-), \tag{2.15}$$

and

$$I = \nu\int_{-\infty}^{d_-} y\varphi(y)\,dy = \frac{\nu}{\sqrt{2\pi}}\lim_{t\to-\infty}(e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}}e^{-d_-^2/2}. \tag{2.16}$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}}e^{-d_-^2/2} + (K-\mu)\varphi(d_-). \tag{2.17}$$

where

$$\mu = S_0,\ \nu = \kappa\sqrt{T},\ d_- = \frac{\mu-K}{\nu}. \tag{2.18}$$

2.2. **European Put when r = 0.** To compute the price of a put, one can use put-call parity (C.8).
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2.3. **Arithmetic Asian Call.** Next we consider an *arithmetic Asian call* where the payoff at time T is given by

$$V_T = (A_T - K)^T,\ A_T = \frac{1}{T}\int_0^T S_t\,dt = S_0 + \frac{\kappa}{T}\int_0^T W_t\,dt. \tag{2.19}$$

Using tools from stochastic calculus (FIXME: what tools) (FOLLOWUP: not a proper reference, but https://math.stackexchange.com/questions/1336471/variance-of-an-integral-of-brownian-motion shows a proof with integration by parts and Fubini), one can show that under the risk neutral measure $\tilde{\mathbb{P}}$,

$$\int_0^T W_t\,dt \sim N(0, T^3/3). \tag{2.20}$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.21}$$

Comparing this to (2.4), we see that A_T has a similar distribution, the only difference is that the variance of A_T is smaller by a factor of $1/3$, so the standard deviation of A_T is smaller by a factor of $1/\sqrt{3}$. By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{6\pi}}e^{-3d_-^2/2} + (K-\mu)\varphi(\sqrt{3}d_-), \tag{2.22}$$

where

$$\mu = S_0,\ \nu = \kappa\sqrt{T},\ d_- = \frac{\mu-K}{\nu}. \tag{2.23}$$

We note that since $\sqrt{3} > 1$, we see from (2.22) that the price of an Asian call is higher than the price of a European call. This should be expected as the taker is paying a premium for a less volatile product.

2.4. **Arithmetic Asian Put.** To compute the price of a put, one can use put-call parity (C.8).
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3. Chooser Pricing under the Bachelier Model

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when $r = 0$.

3.1. **Properties of a Chooser.** In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date T and a strike price K, and an agent is allowed to decide on a choosing date $\tau < T$ to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+,\ P_T = (K - A_T)^+, \tag{3.1}$$

where A_T is defined via (2.19). Here, we assume the agent chooses optimally with no outside information. At time τ, the agent will choose the option of higher value between the put and call, therefore the value of this contract at time τ is

$$V_\tau = \max(C_\tau, P_\tau). \tag{3.2}$$

The time-zero price of this contract is then

$$V_0 = e^{-r\tau}\tilde{\mathbb{E}}[V_\tau]. \tag{3.3}$$

In the next subsection, we simplify the expression for V_τ via the method of replication.

3.2. **Replication.** We first note that by properties of the max function (FIXME: add this to appendix and reference), we can write

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau) \tag{3.4}$$

where P_τ and C_τ are an Asian put and call, respectively. By (A.1), we have

$$P_T - C_T = (K - A_T)^+ - (A_T - K)^+ = K - A_T. \tag{3.5}$$

Next, we identify the time-τ prices of contracts paying $P_T - C_T$ and $K - A_T$ at time T.

To replicate a security with payoff $P_T - C_T$, we consider a portfolio that longs a put and shorts a call at time 0, both with maturity T and strike K. To replicate a security with payoff $K - A_T$, we consider a portfolio investing Ke^{-rT} into the money account at time 0 and shorting a contract (FIXME: what kind of contract?) (FOLLOWUP I'm tempted to say ZCB but I'm not sure) to receive A_T at time T.

Since both portfolios have the same payoff at time T by (3.5), they have the same price for all times t where $0 \le t \le T$ B (FIXME: fix reference style).

For any t satisfying $0 \leqslant t \leqslant T$, we define P_t to be the time-t value of a put with payoff P_T at time T, C_t to be the time-t value of a call with payoff C_T at time t, w_t to be the time-t value of an asian option paying A_T at time t. (FIXME: Move logic to appendix and reference it)

Using this notation, at time τ the value of the first portfolio is $P_\tau - C_\tau$. Also, at time τ the second portfolio has $Ke^{-rT+r\tau}$ in the bank and is shorting a contract which pays A_T at T, therefore the time-τ value of the second portfolio is $Ke^{r(\tau-T)} - A_\tau$. We denote the value of a contract at time τ which pays A_T at time T as w_τ. By replication, the time τ prices of the portfolios are equal, therefore we have

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - w_\tau. \tag{3.6}$$

Substituting this result back into (3.2), the value of the original chooser contract at τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - w_\tau). \tag{3.7}$$

Our next goal is to find an explicit formula for w_τ.

For simplicity, we define U_τ to be the time τ price of a contract with payoff Y_T at time T, where Y_T is defined via

$$Y_T = \int_0^T S_t\, dt. \tag{3.8}$$

Once U_τ is determined, then we can recover w_τ as $w_\tau = \dfrac{U_\tau}{T}$.

Note that (3.8) can be split into two parts,

$$Y_T = \int_0^\tau S_t\, dt + \int_\tau^T S_t\, dt. \tag{3.9}$$

Observe that the integral from 0 to τ is known at time τ as each price S_t will be known by the time τ. So we can treat this integral as a constant and now try to replicate the integral from time τ to T.

3.3. **Replicating the Asian chooser when $r > 0$.** We begin our replicating strategy by buying x shares of stock at time τ. For all times t where $\tau \le t \le T$, we will continuously sell off stock at the rate α_t and invest the revenue. With this strategy, at time T, the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)}\, dt \tag{3.10}$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)}\, dt = \int_\tau^T S_t\, dt. \tag{3.11}$$

Solving for α_t, we find that

$$\alpha_t = e^{r(t-T)} \tag{3.12}$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)}\, dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \tag{3.13}$$

This tells us that the cost at time τ to receive the stock from times τ to T continuously is xS_τ. This gives us

$$U_\tau = \int_0^\tau S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right) \tag{3.14}$$

Recall that w_τ is the price at time τ to receive A_T, equivalent to $\frac{Y_T}{T}$, at time T. Thus, the price at τ to receive just A_T is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T} \tag{3.15}$$

Returning to (3.6), we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}. \tag{3.16}$$

Substituting this into the chooser option from 1.7, the value of V_τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}) \tag{3.17}$$

3.4. **Replicating Asian options when $r = 0$.** We now consider the case when $r = 0$. Observe we cannot plug $r = 0$ into the formula we got for $r > 0$ since we divide by r. However, we can apply a similar replication argument as before. (FIXME: this is a little confusing since the interest rate was never specified. maybe specify that you are assuming $r > 0$ initially and now you're considering $r = 0$ as a special case. If you choose to do this I recommend breaking this section into different subsections). To fix this, we return to our replicating strategy for w_τ accounting for this special case.

Define U_τ and Y_T the same way as above. Again, split the integral Y_T such that

$$Y_T = \int_0^\tau S_t\, dt + \int_\tau^T S_t\, dt \tag{3.18}$$

We now replicate the integral from time τ to T for the special case. We follow the same replicating strategy as before. Purchase x shares of stock. For all times t where $\tau \le t \le T$, we continuously sell off at the rate α_t and invest the revenue. By time T, the bank will have

$$\int_\tau^T \alpha_t S_t\, dt \tag{3.19}$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t\, dt = \int_\tau^T S_t\, dt \tag{3.20}$$

Solving for α_t, we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \tag{3.21}$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau (T-\tau),\ w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T} \tag{3.22}$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = K - \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T}. \tag{3.23}$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t\ dt + S_\tau(T-\tau)}{T}) \tag{3.24}$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \tag{3.25}$$

where $S_0 > 0$ and $\{W_t\}_{t \geqslant 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t)\ dt + (S_0 + \kappa W_\tau)(T-\tau)}{T}) \tag{3.26}$$

Simplifying, we find that

$$V_\tau = C_\tau + \left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+. \tag{3.27}$$

Then by the risk-neutral pricing formula and the linearity of expection, the time-zero price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}}\left[\left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+\right]. \tag{3.28}$$

Let X be the random variable defined via

$$X = \frac{\kappa(T-\tau)}{T} W_\tau + \frac{\kappa}{T}\int_0^\tau W_t\ dt. \tag{3.29}$$

We now calculate the mean and variance of random variable X. We define X as the sum of random variables

$$Y = \frac{\kappa(T-\tau)}{T} W_\tau \tag{3.30}$$

$$Z = \frac{\kappa}{T}\int_0^\tau W_t\ dt. \tag{3.31}$$

Recall from the intro (add details on brownian motion in intro later) that the mean of the brownian motions in both Y and Z are 0, thus the means of both Y and Z are 0. We now calculate the variance of X as the sum of two random variables

$$Var(X) = Var(Y+Z) \tag{3.32}$$

It is known that

$$Var(Y+Z) = Var(Y) + Var(Z) + 2Cov(YZ). \tag{3.33}$$

Recall from the intro (add later) the variances of brownian motion. It follows that

$$Var(Y) = \tau\left(\frac{\kappa(T-\tau)}{T}\right)^2 \tag{3.34}$$

$$Var(Z) = \frac{\tau^3}{3}\left(\frac{\kappa}{T}\right)^2 \tag{3.35}$$

To calculate the covariance term, we expand it out in terms of expected value. Recall that the expected values of the brownian motions are 0.

$$Cov(YZ) = \mathbb{E}(YZ) - \mathbb{E}(Y)\mathbb{E}(Z) = \mathbb{E}(YZ) \tag{3.36}$$

We can now rewrite the covariance as

$$Cov(YZ) = \mathbb{E}(W_\tau \int_0^\tau W_t\ dt)\ \frac{\kappa^2(T-\tau)}{T^2} \tag{3.37}$$

For simplicity, let $\alpha = \dfrac{\kappa^2(T-\tau)}{T^2}$. By a property of integrals and expected value, we can move the integral outside the expected value as such (probably need to fix this).

$$\alpha\mathbb{E}(W_\tau \int_0^\tau W_t \, dt) = \alpha \int_0^\tau \mathbb{E}(w_\tau w_t) dt \tag{3.38}$$

Observe that $t \leq \tau$. Thus, we can further simplify down to

$$\alpha \int_0^\tau \mathbb{E}((w_\tau + w_t - w_t)w_t)dt = \alpha \int_0^\tau \mathbb{E}(w_t^2 + (w_\tau - w_t)w_t)dt \tag{3.39}$$

We can expand the expected value by linearity of expectations. Recall (another brownian motion thing for intro ig) that the expected value of $(w_\tau - w_t)w_t$ is 0 and that the expected value of w_t^2 is t. Thus, we have

$$Cov(XY) = \alpha \int_0^\tau t \, dt = \alpha \frac{\tau^2}{2} \tag{3.40}$$

It follows that the mean and variance of X can be computed as (IMPORTANT FIXME: add details of this computation!)

$$\mu = \mathbb{E}(X) = 0 \tag{3.41}$$

$$\sigma^2 = Var(X) = \tau(\frac{\kappa(T-\tau)}{T})^2 + \frac{\tau^3}{3}(\frac{\kappa}{T})^2 + \tau^2\frac{\kappa^2(T-\tau)}{T^2}. \tag{3.42}$$

$$\nu = \sigma \tag{3.43}$$

(FIXME: typically σ denotes the standard deviation, so this isn't very consistent with standard notation)

Using methods of stochastic calculus (FIXME: we need to find a reference to cite), X is normally distributed with mean μ and variance σ^2. Therefore we can define the probability density function as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{3.44}$$

$$\psi(x) = \frac{1}{\nu}\varphi(\frac{x-\mu}{\nu}) \tag{3.45}$$

Now we can substitute back into our equation from (4.12) and use the definition of expectation on a continuous random variable to get

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (K - S_0 - x)^+ \, \psi(x) \, dx. \tag{3.46}$$

To integrate the second term in V_0 we will let

$$z = \frac{x-\mu}{\sigma}. \tag{3.47}$$

it follows that

$$x = z\nu + \mu \tag{3.48}$$

$$dx = \nu dz \tag{3.49}$$

We note that

$$K - S_0 - x \geqslant 0 \iff x \leq K - S_0 \iff \frac{x-\mu}{\sigma} \leq \frac{K - S_0 - \mu}{\nu} \tag{3.50}$$

and define d_- via

$$d_- = \frac{K - S_0 - \mu}{\nu} \tag{3.51}$$

so by (3.50), we have

$$K - S_0 - x \geqslant 0 \iff z \leq d_. \tag{3.52}$$

Now using (3.48) and (3.52) we can rearrange (3.46) as:

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \left(\int_{-\infty}^{d_-} (K - S_0 - z\nu - \mu) \, \varphi(z) \, dz \right). \tag{3.53}$$

Simplifying this expression we get the form

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \nu \int_{-\infty}^{d_-} z\varphi(z)\nu \, dz + (K - S_0 - \mu) \int_{-\infty}^{d_-} \varphi(z) \, dz \tag{3.54}$$

Excluding the call, we can easily simplify the latter of the terms by defining the cumulative distribution function

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy. \tag{3.55}$$

We resolve the former term by first substituting in (3.44)

$$\frac{-\nu}{\sqrt{2\pi}} \int_{-\infty}^{d_-} ye^{-y^2} = \frac{\nu}{\sqrt{2\pi}} \, e^{\frac{-d_-^2}{2}} \tag{3.56}$$

Thus, we find that

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \frac{\nu}{\sqrt{2\pi}} \, e^{\frac{-d_-^2}{2}} + (K - S_0 - \mu)\Phi(d_-), \tag{3.57}$$

which through (F.1), gives us the final equation.

$$V_0 = C_0 + \frac{\nu}{\sqrt{2\pi}} \, e^{\frac{-d_-^2}{2}} + (K - S_0 - \mu)\Phi(d_-), \tag{3.58}$$

4. Chooser Option Variants

4.1. **Tail Chooser.** We will now consider a variant of the Asian chooser we looked at earlier. We assume all conditions remain the same, except we now define $A_{\tau,T}$ as

$$A_{\tau,T} = \int_{\tau}^{T} S_t \, dt \tag{4.1}$$

where τ is the choice date, and T is the time of maturity.

4.2. **Asian Tail Choosers when $r = 0$.** To price this option, we slightly modify the replication strategy from before. Let $Y_T = \int_\tau^T S_t \, dt$ and U_τ be the price at τ to receive Y_T at time T.

We proceed with the replication of Y_T. Suppose an agent purchases x shares at time τ, and chooses to sell them off continuously at rate α_t at time t. At time T, the agent's portfolio is worth

$$\int_{\tau}^{T} \alpha_t S_t e^{r(T-t)} \, dt \tag{4.2}$$

Since we assume here that $r = 0$, this reduces to

$$\int_{\tau}^{T} \alpha_t S_t \, dt \tag{4.3}$$

To complete the replication, we set this equal to the value we are trying to reproduce:

$$\int_{\tau}^{T} \alpha_t S_t \, dt = \int_{\tau}^{T} S_t \, dt \tag{4.4}$$

It follows that

$$\alpha_t = 1 \tag{4.5}$$

for all t where $\tau \le t \le T$. Thus,

$$x = \int_{\tau}^{T} \alpha_t \, dt = T - \tau. \tag{4.6}$$

It then follows that $U_\tau = (T - \tau)S_\tau$. Observe that

$$A_{\tau,T} = \frac{U_\tau}{T - \tau}. \tag{4.7}$$

Again using the notation w_τ as the price needed at time τ to receive $A_{\tau,T}$ at time T, it follows that

$$w_\tau = \frac{(T - \tau)S_\tau}{T - \tau} = S_\tau. \tag{4.8}$$

Referring back to (3.6) and using $r = 0$, we have

$$P_\tau - C_\tau = K - S_\tau. \tag{4.9}$$

Using (3.4), the price of the tail chooser option with choice date τ, which we write as V_τ, is

$$V_\tau = C_\tau + (K - S_\tau)^+. \tag{4.10}$$

Recall in the Bachelier model that the stock evolves according to $S_t = S_0 + \kappa W_t$ when $r = 0$, $S_0 > 0$, and W_t is a brownian motion under the risk-neutral measure. Then,

$$V_\tau = C_\tau + (K - (S_0 + \kappa W_\tau))^+. \tag{4.11}$$

Applying the risk-neutral pricing formula and linearity of expectations, we have

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}}((K - S_0 - \kappa W_\tau)^+). \tag{4.12}$$

To simplify the above, define random variable X and function $g(X)$ as

$$X = \kappa W_\tau,\ g(X) = (k - S_0 - X)^+ \tag{4.13}$$

Applying the law of the unconscious statistician, we can express V_0 as

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (k - S_0 - X)^+ \psi(x) dx \tag{4.14}$$

where

$$\psi(x) = \frac{1}{\nu}\varphi\left(\frac{x-\mu}{\nu}\right) \tag{4.15}$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{4.16}$$

Let $y = \dfrac{x-\mu}{\nu}$. Observe that

$$y = \frac{x-\mu}{\nu} \implies x = y\nu + \mu \implies dx = \nu\ dy \tag{4.17}$$

We can now take the positive part of the integral from (4.14)

$$k - S_0 - x \geq 0 \implies -x \geq S_0 - k \tag{4.18}$$

Adding μ and dividing by ν on both sides,

$$-y = \frac{-x+\mu}{\nu} \geq \frac{S_0 - k + \mu}{\nu} \tag{4.19}$$

It follows that

$$y \leq \frac{k - S_0 - \mu}{\nu} = d_- \tag{4.20}$$

We now evaluate 4.14 using 4.17

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{d_-} (k - S_0 - y\nu - \mu)(\frac{1}{\nu}\varphi(y))(-\nu)\ dy \tag{4.21}$$

Simplifying and splitting the integral, we have

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \int_{-\infty}^{d_-} (k - S_0 - \mu)\varphi(y)\ dy + \int_{-\infty}^{d_-} y\nu\varphi(y))\ dy \tag{4.22}$$

Define the CDF the same as (3.55).

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - (k - S_0 - \mu)\Phi(d_-) + \int_{-\infty}^{d_-} y\nu\varphi(y)\ dy \tag{4.23}$$

The remaining integral term can be simplified through (4.16).

$$\nu \int_{-\infty}^{d_-} y\varphi(y)\ dy = \nu \int_{-\infty}^{d_-} y\frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\ dy = -\frac{\nu}{\sqrt{2\pi}} e^{\frac{-d_-^2}{2}} \tag{4.24}$$

The final form for the time 0 price of the tail chooser is then

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - (k - S_0 - \mu)\Phi(d_-) - \frac{\nu}{\sqrt{2\pi}} e^{\frac{-d_-^2}{2}} \tag{4.25}$$

5. Approximating Arithmetic Asian under Black-Scholes Model

(FIXME: find reference for why this is an acceptable approximation, flesh out reasoning) In the Black-Scholes Model, the asset price S_t for $0 \le t \le T$ evolves according to

$$S_t = S_0 e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t}. \tag{5.1}$$

Recall the notation

$$A_T = \frac{1}{T}\int_0^T S_t \, dt. \tag{5.2}$$

Note that the integrand S_t is a lognormal random variable. However, the integral of S_t will not be lognormal (cite somewhere). Thus, we model A_t with a log-normal Y_t with the same mean and variance, which leaves us with equations

$$\tilde{\mathbb{E}}[A_T] = \tilde{\mathbb{E}}[Y_T] \tag{5.3}$$

$$\tilde{\mathbb{E}}[A_T^2] = \tilde{\mathbb{E}}[Y_T^2] \tag{5.4}$$

$$Y_t = Y_0 e^{\Gamma W_t + rt - \frac{1}{2}\Gamma^2 t} \tag{5.5}$$

Starting with the RHS of (5.3)

$$\tilde{\mathbb{E}}[A_T] = \frac{1}{T}\int_0^T \tilde{\mathbb{E}}[S_t] \, dt = \frac{1}{T}\int_0^T \tilde{\mathbb{E}}[S_0 e^{\sigma W_t + rt - \frac{1}{2}\sigma^2 t}] \, dt = \frac{S_0}{T}\int_0^T \tilde{\mathbb{E}}[e^{\sigma W_t} e^{rt - \frac{1}{2}\sigma^2 t}] \, dt \tag{5.6}$$

We can factor out the constant part of the expected value and are left to evaluate

$$\tilde{\mathbb{E}}[A_T] = \frac{S_0}{T}\int_0^T e^{rt - \frac{1}{2}\sigma^2 t}\tilde{\mathbb{E}}[e^{\sigma W_t}] \, dt \tag{5.7}$$

Using Moment Generating Function e^{yX}, we know if X is a normally distributed random variable with mean μ and variance σ^2 than

$$\tilde{\mathbb{E}}[e^{yX}] = e^{\mu y + \frac{1}{2}\sigma^2 y^2} \tag{5.8}$$

Recalling that the Brownian motion W_t is of mean μ and variance σ^2,

$$\tilde{\mathbb{E}}[e^{\sigma W_t}] = e^{\frac{1}{2}\sigma^2 t}, \tag{5.9}$$

which we can plug in to (5.7) to get

$$\tilde{\mathbb{E}}[A_T] = \frac{S_0}{T}\int_0^T e^{rt - \frac{1}{2}\sigma^2 t}\tilde{\mathbb{E}}[e^{\sigma W_t}] \, dt = \frac{S_0}{T}\int_0^T e^{\frac{1}{2}\sigma^2 t} e^{rt - \frac{1}{2}\sigma^2 t} \, dt = \frac{S_0}{T}\int_0^T e^{rt} \, dt = \frac{S_0}{rT}(e^{rT} - 1) \tag{5.10}$$

We can then simplify the LHS of (5.3) using the moment generating function (5.8) to get

$$\tilde{\mathbb{E}}[Y_T] = \tilde{\mathbb{E}}[Y_0 e^{\Gamma W_t + rt - \frac{1}{2}\Gamma^2 t}] \ = \tilde{\mathbb{E}}[Y_0]\tilde{\mathbb{E}}[e^{rt - \frac{1}{2}\Gamma^2 t}]\tilde{\mathbb{E}}[\Gamma W_t] = Y_0 e^{rt - \frac{1}{2}\Gamma^2 t} e^{\frac{1}{2}\Gamma^2 t} = Y_0 e^{rT} \tag{5.11}$$

So, according to (5.3), we have

$$Y_0 = \frac{S_0}{rT}(1 - e^{-rT}) \tag{5.12}$$

We next simplify the second equation. Moving onto the LHS of (5.4),

$$\tilde{\mathbb{E}}[A_T^2] = \frac{1}{T^2}\left(\int_0^T \tilde{\mathbb{E}}[S_t] \, dt\right)^2 \tag{5.13}$$

Observe that the square of an integral can be rewritten as a double integral as follows

$$\frac{1}{T^2}\left(\int_0^T \tilde{\mathbb{E}}[S_t] \, dt\right)^2 = \frac{1}{T^2}\int_0^T \int_0^T \tilde{\mathbb{E}}[S_s]\tilde{\mathbb{E}}[S_t] \, ds \, dt = \frac{1}{T^2}\int_0^T \int_0^T \tilde{\mathbb{E}}[S_s S_t] \, ds \, dt \tag{5.14}$$

WLOG, assume that $0 \le s \le t$. Integrating under these conditions yields us half of the desired area. Through a symmetry argument, we can conclude that

$$\frac{1}{T^2}\int_0^T \int_0^T \tilde{\mathbb{E}}[S_s S_t] \, ds \, dt = \frac{2}{T^2}\int_0^T \int_0^t \tilde{\mathbb{E}}[S_s S_t] \, ds \, dt \tag{5.15}$$

We now deal with the expression inside the expected value. We can expand S_s and S_t and separate out the Brownian motions

$$S_sS_t = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)+\sigma(W_t+W_s)} = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{\sigma W_t} e^{\sigma W_s} \tag{5.16}$$

Observe that $W_t = W_s + (W_t - W_s)$. Using this, we write

$$S_sS_t = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{\sigma W_t} e^{\sigma W_s} = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{\sigma W_s} e^{\sigma(W_t-W_s)} e^{\sigma W_s} = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{2\sigma W_s} e^{\sigma(W_t-W_s)} \tag{5.17}$$

This is of interest because the Brownian motions W_s and $W_t - W_s$ are independent (reference appendix). Observe that $W_s \sim N(0, s)$ and $W_t - W_s \sim N(0, t-s)$ (ref?). Due to independence, we can write out the expected value as follows

$$\tilde{\mathbb{E}}[S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} e^{2\sigma W_s} e^{\sigma(W_t-W_s)}] = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} \tilde{\mathbb{E}}[e^{2\sigma W_s}] \tilde{\mathbb{E}}[e^{\sigma(W_t-W_s)}] \tag{5.18}$$

Using (5.8) again, observe that

$$\tilde{\mathbb{E}}[e^{2\sigma W_s}] = e^{2\sigma^2 s} \tag{5.19}$$

$$\tilde{\mathbb{E}}[e^{\sigma(W_t-W_s)}] = e^{\frac{1}{2}\sigma^2(t-s)} \tag{5.20}$$

Substituting, we find that

$$\tilde{\mathbb{E}}[S_sS_t] = S_0^2 e^{(r-\frac{1}{2}\sigma^2)(t+s)} \tilde{\mathbb{E}}[e^{2\sigma W_s}] \tilde{\mathbb{E}}[e^{\sigma(W_t-W_s)}] = S_0^2 e^{r(t+s)+\sigma^2 s} \tag{5.21}$$

We now return to the main integral using this new result

$$\frac{2}{T^2} \int_0^T \int_0^t S_0^2 e^{r(t+s)+\sigma^2 s} \, ds \, dt = \frac{2}{T^2} \int_0^T S_0^2 e^{rt} \int_0^t e^{(r+\sigma^2)s} \, ds \, dt \tag{5.22}$$

We begin by evaluating the inside integral from (5.22)

$$\frac{2}{T^2} \int_0^T S_0^2 e^{rt} \int_0^t e^{(r+\sigma^2)s} \, ds \, dt = \frac{2}{T^2} \int_0^T \frac{S_0^2 e^{rt}}{r+\sigma^2} (e^{(r+\sigma^2)t} - 1) \, dt = \frac{2S_0^2}{T^2(r+\sigma^2)} \int_0^T (e^{(2r+\sigma^2)t} - e^{rt}) \, dt \tag{5.23}$$

Evaluating the next integral, we see that

$$\frac{2S_0^2}{T^2(r+\sigma^2)} \int_0^T (e^{(2r+\sigma^2)t} - e^{rt}) \, dt = \frac{2S_0^2}{T^2(r+\sigma^2)} \left[\frac{1}{2r+\sigma^2} \left(e^{(2r+\sigma^2)t} - 1 \right) - \frac{1}{r} \left(e^{rT} - 1 \right) \right] \tag{5.24}$$

Thus,

$$\tilde{\mathbb{E}}[A_T^2] = \frac{2S_0^2}{T^2(r+\sigma^2)} \left[\frac{1}{2r+\sigma^2} \left(e^{(2r+\sigma^2)t} - 1 \right) - \frac{1}{r} \left(e^{rT} - 1 \right) \right] \tag{5.25}$$

Continuing on to the RHS of (5.4), we can apply the moment generating function again with the information that $W_t \sim N(0, t)$ to get

$$\tilde{\mathbb{E}}[Y_T^2] = Y_0^2 e^{(2r-\Gamma^2)T} \tilde{\mathbb{E}}[e^{2\Gamma W_T}] = Y_0^2 e^{(2r+\Gamma^2)T} \tag{5.26}$$

Equating $\tilde{\mathbb{E}}[A_T^2]$ and $\tilde{\mathbb{E}}[Y_T^2]$ in (5.4) and recalling the calibration (5.12) we found from (5.3), it follows that

$$\Gamma^2 = \frac{1}{T} \left(\ln \left(\frac{2S_0^2}{Y_0^2 T^2 (r+\sigma^2)} \right) + \ln \left(\frac{1}{2r+\sigma^2} \left(e^{2rT+\sigma^2 T} - 1 \right) - \frac{1}{r} (e^{rT} - 1) \right) - 2rT \right) \tag{5.27}$$

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Appendix A. Notation and conventions

For a random variable X we use the notation X^+ to denote the random variable $\max(X, 0)$. We note that by definition, we have

$$X = X^+ - (-X)^+, \tag{A.1}$$

from which the *put-call parity* can be derived.

For a normal random variable X we use the notation $X \sim N(\mu, \sigma^2)$ to denote that it has mean μ and variance σ^2.

Appendix B. Arbitrage-free pricing

Before defining arbitrage-free pricing, we first must define arbitrage. An arbitrage strategy has three properties.

(1) the agent's initial capital is zero.
(2) the agent has zero percent chance of losing money.
(3) the agent has a strictly positive probability of profit.

Under this definition of arbitrage, the arbitrage-free price of an asset is the price where an arbitrage strategy is not possible.

B.1. **Arbitrage-free Market.** In this paper we work under the assumption that the market is arbitrage-free. As such, we claim that if the values of two portfolios are equal at time $T > 0$, then for all times τ where $0 \leq \tau \leq T$, the values of both portfolios are equal.

We prove by contrapositive. Assume that at time $T > 0$ the prices of two portfolios are equal, and that at time τ where $0 \leq \tau \leq T$ that one portfolio is worth more than the other. Let P_1 be the value of portfolio 1 and P_2 be the value of portfolio 2. Thus, without loss of generality, at time τ, let $P_1 > P_2$. At time τ, we buy portfolio 2 and sell portfolio 1. We can pocket the difference $P_1 - P_2$. At time T, we can then sell portfolio 2 to pay off the time T cost of portfolio 1. Thus, there exists an arbitrage strategy, which is a contradiction.

It follows that under an arbitrage-free model, if two portfolios have equal value at time T, they must have equal value at all times from 0 to T.

Appendix C. Put-Call Parity

An important result used repeatedly throughout this paper is put-call parity.

For some asset of price S_T at time T, define the European put and call of strike K as

$$P_T^E = (K - S_T)^+ \tag{C.1}$$
$$C_T^E = (S_T - K)^+. \tag{C.2}$$

By (A.1),

$$P_T^E - C_T^E = K - S_T. \tag{C.3}$$

We can replicate the LHS portfolio by going long a put and short a call at time τ. The RHS can be replicated by investing $Ke^{r(\tau-T)}$ into a risk-free return and shorting the asset at $T = \tau$. Under the assumption of the arbitrage-free market, it follows that

$$P_\tau^E - C_\tau^E = Ke^{r(\tau-T)} - S_\tau. \tag{C.4}$$

We apply a similar argument towards Asian puts and calls. Again define an asset with price S_T at time T. Define the Asian put and call with strike price K respectively as

$$P_T^A = (K - \int_0^T S_t \, dt)^+ \tag{C.5}$$
$$C_T^A = (\int_0^T S_t \, dt - K)^+. \tag{C.6}$$

By (A.1),

$$P_T^A - C_T^A = K - \int_0^T S_t \, dt. \tag{C.7}$$

We replicate the LHS by going long the put and short the call at time τ. The RHS can be replicated by investing $Ke^{r(\tau-T)}$ and shorting an option at w_τ which pays A_T at time T, all at time time τ. Thus,

$$P_\tau^A - C_\tau^A = Ke^{r(\tau-T)} - w_\tau. \tag{C.8}$$

The calculation of w_τ is demonstrated in the above sections.

Appendix D. Max Transformation

We derive the following property of the max function:

$$\max(a,b) = a + \max(0, b-a). \tag{D.1}$$

The proof follows through casework. First consider when $a > b$, it follows that

$$\max(a,b) = a \tag{D.2}$$

$$a + \max(0, b-a) = a + 0 = a. \tag{D.3}$$

The second case we consider is $b \geq a$, then

$$\max(a,b) = b \tag{D.4}$$

$$a + \max(0, b-a) = a + (b-a) = b. \tag{D.5}$$

The proof is now complete.

Appendix E. Moment Generating Functions

We define $m_X(y)$ to be moment generating function on random variable X such that

$$m(y) = \tilde{\mathbb{E}}[e^{yX}] \tag{E.1}$$

where $y \in \mathbb{R}$. When $X \sim N(\mu, \sigma^2)$, we have

$$m(y) = \tilde{\mathbb{E}}[e^{yX}] = \exp(\mu y + \frac{1}{2}\sigma^2 y^2) \tag{E.2}$$

Appendix F. Various replicating strategies

F.1. **Replicating Puts and Calls.** The put and call options are two often used contracts which give the buyer the right, but not obligation, to respectively sell or buy an underlying security.

To ”price back” a put or call from time t to time 0, where $t \geq 0$, we simply buy a put or call respectively at time 0. It thus follows that

$$\tilde{\mathbb{E}}[C_t] = C_0 \tag{F.1}$$

$$\tilde{\mathbb{E}}[P_t] = P_0 \tag{F.2}$$

F.2. **Replicating European options.** FIXME

F.3. **Replicating Asian options.** FIXME

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