

# MFSURP

JESSICA CHEN, LINXUAN JIANG, FRANK SACCO, AND ALBERT ZHANG

ABSTRACT. This paper concerns ...

## CONTENTS

1. Introduction	1
2. Chooser options under the Blachelier Model	1
Acknowledgement	4

## 1. INTRODUCTION

### 2. CHOOSER OPTIONS UNDER THE BLACHELIER MODEL

We consider the Blachelier model with where the stock prices  $\{S_t\}_{t \geq 0}$  evolves according to

$$S_t = e^{rt} \left( S_0 + \kappa^{-rt} W_t + \kappa r \int_0^t e^{-rs} W_s ds \right). \quad (2.1)$$

where  $S_0 > 0$  and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure  $\tilde{\mathbb{P}}$ . When  $r = 0$ , this reduces to

$$S_t = S_0 + \kappa W_t. \quad (2.2)$$

According to

We consider a contract with maturity  $T$  and strike price  $K$ , that allows an agent to decide on a choosing date  $\tau < T$  to choose the underlying derivative. Many results are known when an agent is allowed to choose between a European call and a European put. In this section we consider a contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+, P_T = (K - A_T)^+, A_T = \frac{1}{T} \int_0^T S_t dt. \quad (2.3)$$

Here, we assume the agent chooses optimally with no outside information. At time  $\tau$ , the agent will choose the option of higher value between the put and call, therefore the value of this contract at time  $\tau$  is

$$V_\tau = \max(C_\tau, P_\tau). \quad (2.4)$$

By properties of the max function (FIXME: maybe add this in intro and reference?)

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau) \quad (2.5)$$

which we want to evaluate through Put-Call Parity.

Recall that Put-Call Parity tells us

$$P_T - C_T = K - A_T. \quad (2.6)$$

We can replicate the LHS of the above equation by going long a put and short a call at time 0, both with strike  $T$ . We also replicate the RHS by investing  $Ke^{-rT}$  into the bank at time 0 and shorting a contract to receive  $A_T$  at time  $T$ . Since both portfolios are of equal price at time  $T$ , they have equal price at all times  $t$  where  $0 \leq t \leq T$ . At time  $\tau$ , the left portfolio is the value of the put minus the call at time  $\tau$ . The right portfolio now has  $Ke^{-rT+r\tau}$  in the bank and is short a contract which pays  $A_T$  at  $T$ . Define  $P_\tau$  and  $C_\tau$  to be the respective values of a put and call with maturity  $T$  at time  $\tau$ . Let  $w_\tau$  be the value of the contract at time  $\tau$  to receive  $A_T$  at time  $T$ . By replication, our portfolios give us the following equation

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$$P_\tau - C_\tau = Ke^{r(\tau-T)} - w_\tau. \quad (2.7)$$

Substituting this result back into 1.4, the value of the contract  $V$  at  $\tau$  is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - w_\tau). \quad (2.8)$$

Now we want an expression for the price at time  $\tau$  to receive  $A_T$  at time  $T$ .

For simplicity, define  $U_\tau$  to be the price at time  $\tau$  to receive  $Y_T$  at time  $T$ , where

$$Y_T = \int_0^T S_t dt \quad (2.9)$$

Note that this integral can be split into

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt. \quad (2.10)$$

Observe that the integral from 0 to  $\tau$  is known at time  $\tau$ . We can treat this integral as a constant and now try to replicate the integral from time  $\tau$  to  $T$ .

We begin our replicating strategy by buying  $x$  shares of stock at time  $\tau$ . For all times  $t$  where  $\tau \leq t \leq T$ , we will continuously sell off stock at the rate  $\alpha_t$  and invest the revenue. With this strategy, at time  $T$ , the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt \quad (2.11)$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt = \int_\tau^T S_t dt. \quad (2.12)$$

Solving for  $\alpha_t$ , we find that

$$\alpha_t = e^{r(t-T)} \quad (2.13)$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \quad (2.14)$$

This tells us that the cost at time  $\tau$  to receive the stock from times  $\tau$  to  $T$  continuously is  $xS_\tau$ . This gives us

$$U_\tau = \int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)}) \quad (2.15)$$

Recall that  $w_\tau$  is the price at time  $\tau$  to receive  $A_T$ , equivalent to  $\frac{Y_T}{T}$ , at time  $T$ . Thus, the price at  $\tau$  to receive just  $A_T$  is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T} \quad (2.16)$$

Returning to 1.6 Put-Call Parity, we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T}. \quad (2.17)$$

Substituting this into the chooser option from 1.7, the value of  $V_\tau$  is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T}) \quad (2.18)$$

(insert simplification for when  $r \neq 0$ )

The above formula breaks when  $r = 0$  since we divide by  $r$ . To fix this, we return to our replicating strategy for  $w_\tau$  accounting for this special case.

Define  $U_\tau$  and  $Y_T$  the same way as above. Again, split the integral  $Y_T$  such that

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt \quad (2.19)$$

We now replicate the integral from time  $\tau$  to  $T$  for the special case. We follow the same replicating strategy as before. Purchase  $x$  shares of stock. For all times  $t$  where  $\tau \leq t \leq T$ , we continuously sell off at the rate  $\alpha_t$  and invest the revenue. By time  $T$ , the bank will have

$$\int_\tau^T \alpha_t S_t dt \quad (2.20)$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t dt = \int_\tau^T S_t dt \quad (2.21)$$

Solving for  $\alpha_t$ , we see that when  $r = 0$  that  $\alpha_t = 1$ . Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \quad (2.22)$$

Similar to the  $r \neq 0$  case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T - \tau), \quad w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T} \quad (2.23)$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T}. \quad (2.24)$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t dt + S_\tau(T - \tau)}{T}) \quad (2.25)$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \quad (2.26)$$

where  $S_0 > 0$  and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t) dt + (S_0 + \kappa W_\tau)(T - \tau)}{T}) \quad (2.27)$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left( K - S_0 - \frac{\kappa(T - \tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t dt \right)^+. \quad (2.28)$$

Then by the risk-neutral pricing formula, the time-zero price of this contract is given by

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CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
Email address, J. Chen: [jschen2@andrew.cmu.edu](mailto:jschen2@andrew.cmu.edu)

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
Email address, L. Jiang: [linxuanj@andrew.cmu.edu](mailto:linxuanj@andrew.cmu.edu)

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
Email address, F. Sacco: [fsacco@andrew.cmu.edu](mailto:fsacco@andrew.cmu.edu)

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
Email address, A. Zhang: [albertzh@andrew.cmu.edu](mailto:albertzh@andrew.cmu.edu)