MFSURP Notes

Contents

1. Chooser options under the Blachelier Model

We consider the Blachelier model with where the stock prices $\{S_t\}_{t\geq 0}$ evolves according to

$$S_t = e^{rt}\left(S_0 + \kappa^{-rt}W_t + \kappa r\int_0^t e^{-rs}W_s\,ds\right). \tag{1.1}$$

where $S_0 > 0$ and $\{W_t\}_{t\geq 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$.

We consider a contract with maturity T and strike price K, that allows an agent to decide on a choosing date $\tau < T$ to choose the underlying derivative. Many results are known when an agent is allowed to choose between a European call and a European put. In this section we consider a contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+,\ P_T = (K - A_T)^+,\ A_T = \frac{1}{T}\int_0^T S_t\,dt. \tag{1.2}$$

(fill in derivation that gives V_τ, the value of the contract at time τ). Here, we assume the agent chooses optimally with no outside information. At time τ, the agent will choose the option of higher value between the put and call. The value of contract V at time τ is

$$V_\tau = \max(C_\tau, P_\tau). \tag{1.3}$$

We subtract C_τ out of the max function and get

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau) \tag{1.4}$$

which we want to evaluate through Put-Call Parity.

Recall that Put-Call Parity tells us

$$P_T - C_T = K - A_T. \tag{1.5}$$

We can replicate the LHS of the above equation by going long a put and short a call at time 0, both with strike T. We also replicate the RHS by investing Ke^{-rT} into the bank at time 0 and shorting a contract to receive A_T at time T. Since both portfolios are of equal price at time T, they have equal price at all times t where $0 \leq t \leq T$. At time τ, the left portfolio is the value of the put minus the call at time τ. The right portfolio now has $Ke^{-rT+r\tau}$ in the bank and is short a contract which pays A_T at T. Define P_τ and C_τ to be the respective values of a put and call with maturity T at time τ. Let w_τ be the value of the contract at time τ to receive A_T at time T. By replication, our portfolios give us the following equation

$$P_\tau - C_\tau = Ke^{r(\tau - T)} - w_\tau. \tag{1.6}$$

Substituting this result back into 1.4, the value of the contract V at τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - w_\tau). \tag{1.7}$$

Now we want an expression for the price at time τ to receive A_T at time T.

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For simplicity, define U_τ to be the price at time τ to receive Y_T at time T, where

$$Y_T = \int_0^T S_t \, dt \tag{1.8}$$

Note that this integral can be split into

$$Y_T = \int_0^\tau S_t \, dt + \int_\tau^T S_t \, dt. \tag{1.9}$$

Observe that the integral from 0 to τ is known at time τ. We can treat this integral as a constant and now try to replicate the integral from time τ to T.

We begin our replicating strategy by buying x shares of stock at time τ. For all times t where $\tau \leq t \leq T$, we will continuously sell off stock at the rate α_t and invest the revenue. With this strategy, at time T, the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt \tag{1.10}$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt = \int_\tau^T S_t \, dt. \tag{1.11}$$

Solving for α_t, we find that

$$\alpha_t = e^{r(t-T)} \tag{1.12}$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} \, dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \tag{1.13}$$

This tells us that the cost at time τ to receive the stock from times τ to T continuously is xS_τ. This gives us

$$U_\tau = \int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right) \tag{1.14}$$

Recall that w_τ is the price at time τ to receive A_T, equivalent to $\frac{Y_T}{T}$, at time T. Thus, the price at τ to receive just A_T is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T} \tag{1.15}$$

Returning to 1.6 Put-Call Parity, we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}. \tag{1.16}$$

Substituting this into the chooser option from 1.7, the value of V_τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}) \tag{1.17}$$

(insert simplification for when r != 0)

The above formula breaks when $r = 0$ since we divide by r. To fix this, we return to our replicating strategy for w_τ accounting for this special case.

Define U_τ and Y_T the same way as above. Again, split the integral Y_T such that

$$Y_T = \int_0^\tau S_t \, dt + \int_\tau^T S_t \, dt \tag{1.18}$$

We now replicate the integral from time τ to T for the special case. We follow the same replicating strategy as before. Purchase x shares of stock. For all times t where $\tau \leq t \leq T$, we continuously sell off at the rate α_t and invest the revenue. By time T, the bank will have

$$\int_\tau^T \alpha_t S_t \, dt \tag{1.19}$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t \, dt = \int_\tau^T S_t \, dt \tag{1.20}$$

Solving for α_t, we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \tag{1.21}$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T-\tau), \; w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T} \tag{1.22}$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T}. \tag{1.23}$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t \, dt + S_\tau(T-\tau)}{T}) \tag{1.24}$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \tag{1.25}$$

where $S_0 > 0$ and $\{W_t\}_{t \geq 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t) \; dt + (S_0 + \kappa W_\tau)(T-\tau)}{T}) \tag{1.26}$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t \, dt\right)^+. \tag{1.27}$$

Then by the risk-neutral pricing formula, the time-zero price of this contract is given by