MFSURP

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Abstract. This paper concerns ...

Contents

1. Introduction

1.1. Arbitrage-free pricing.

1.2. The Blachelier Model.

In this paper we work within the context of the *Blachelier model*, where the stock prices $\{S_t\}_{t\geqslant 0}$ evolves according to

$$S_t = e^{rt}\left(S_0 + \kappa^{-rt}W_t + \kappa r\int_0^t e^{-rs}W_s\,ds\right). \tag{1.1}$$

where $S_0 > 0$ and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. We note to the reader that when $r = 0$, this reduces to

$$S_t = S_0 + \kappa W_t. \tag{1.2}$$

1.3. Notation and conventions.

In this paper we often use the notation f^+ to denote $\max(f, 0)$. We note that by definition, we have

$$f = f^+ - (-f)^+. \tag{1.3}$$

2. Asian options

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when $r = 0$.

2.0.1. *European Call.* We first consider a European call where the payoff at time T is given by

$$V_T = (S_T - K)^+. \tag{2.1}$$

We note that under $\tilde{\mathbb{P}}$, $W_T \sim N(0, T)$, therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.2}$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \tag{2.3}$$

Recall that if we have a random variable X with probability density function f_X under a probability measure $\mathbb{P}$, then the "law of the unconscious statistician" tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx. \tag{2.4}$$

Key words and phrases. Blachelier model, Chooser options, ...

J. Chen, L. Jiang, F. Sacco, A. Zhang were supported by MFSURP ...

In our setting, we have

$$g(S_T) = (S_T - K)^+, \tag{2.5}$$

and the distribution of S_T under $\tilde{\mathbb{P}}$ as a random variable is given in (2.2). Therefore

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x-K)^+ \psi(x)\, dx, \tag{2.6}$$

where

$$\psi(x) = \frac{1}{\nu}\varphi\left(\frac{x-\mu}{\nu}\right),\ \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \tag{2.7}$$

and

$$\mu = S_0,\ \nu = \kappa\sqrt{T}. \tag{2.8}$$

To compute (2.6), we first note that since $(x-K)^+ = 0$ for $x \leqslant K$, the domain of integration is the set $\{x \mid x \geqslant K\}$. Now we use the change of variables

$$y = -\frac{x-\mu}{\nu} \iff x = \mu - \nu y \tag{2.9}$$

and we note that since $\nu > 0$,

$$x \geqslant K \iff \frac{x-\mu}{\nu} \geqslant \frac{K-\mu}{\nu} \iff y \leqslant \frac{\mu - K}{\nu} =: d_-. \tag{2.10}$$

Then by performing a change of variables, (2.6) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu)\varphi(-y)\, dy = \int_{-\infty}^{d_-} (\nu y + K - \mu)\varphi(y)\, dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y)\, dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K-\mu)\varphi(y)\, dy}_{:=II}. \tag{2.11}$$

We define the cumulative distribution function of a standard normal random variable X under $\mathbb{P}$ via

$$\Phi(x) = \mathbb{P}[X \leqslant x] = \mathbb{E}[\mathbb{1}_{X \leqslant x}] = \int_{-\infty}^{x} \varphi(y)\, dy. \tag{2.12}$$

With this notation in hand, we can write

$$II = (K-\mu)\int_{-\infty}^{d_-} \varphi(y)\, dy = (K-\mu)\Phi(d_-), \tag{2.13}$$

and

$$I = \nu \int_{-\infty}^{d_-} y\varphi(y)\, dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \to -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \tag{2.14}$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K-\mu)\Phi(d_-). \tag{2.15}$$

To compute the price of a put, one can use put-call parity.

2.0.2. *Arithmetic Asian call.* Consider an arithmetic Asian call where the payoff at time T is given by

$$V_T = (A_T - K)^T,\ A_T = \frac{1}{T}\int_0^T S_t\, dt = S_0 + \frac{\kappa}{T}\int_0^T W_t\, dt. \tag{2.16}$$

Using Ito's formula, one can show that under the risk neutral measure $\tilde{\mathbb{P}}$,

$$\int_0^T W_t\, dt \sim N(0, T^3/3). \tag{2.17}$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.18}$$

Comparing this to (2.2), we see that A_T has a similar distribution, the only difference is that the variance of A_T is smaller by a factor of $1/3$, so the standard deviation of A_T is smaller by a factor of $1/\sqrt{3}$. By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{3}\sqrt{2\pi}} e^{-3d_-^2/2} + (K-\mu)\Phi(\sqrt{3}d_-), \tag{2.19}$$

where

$$\mu = S_0,\ \nu = \kappa\sqrt{T},\ d_- = \frac{\mu - K}{\nu}. \tag{2.20}$$

We note that since $\sqrt{3} > 1$, we see from (2.19) that the price of an Asian option is higher than the price of a European call. This should be expected as one is paying a premium for a less volatile product.

3. Chooser options

In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date T and a strike price K, and an agent is allowed to decide on a choosing date $\tau < T$ to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+,\ P_T = (K - A_T)^+, \tag{3.1}$$

where A_T is defined via (2.16). Here, we assume the agent chooses optimally with no outside information. At time τ, the agent will choose the option of higher value between the put and call, therefore the value of this contract at time τ is

$$V_\tau = \max(C_\tau, P_\tau). \tag{3.2}$$

By properties of the max function (FIXME: maybe add this in intro and reference?)

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau). \tag{3.3}$$

Note that by (1.3), we have

$$P_T - C_T = (K - S_T)^+ - (S_T - K)^+ = K - A_T. \tag{3.4}$$

Next we want to identify the time-τ prices of contracts paying $P_T - C_T$ and $K - A_T$ at time T. To compute these arbitrage prices, we use the method of replication.

We can replicate the LHS of the above equation by going long a put and short a call at time 0, both with strike T. We also replicate the RHS by investing Ke^{-rT} into the bank at time 0 and shorting a contract to receive A_T at time T. Since both portfolios are of equal price at time T, they have equal price at all times t where $0 \le t \le T$. At time τ, the left portfolio is the value of the put minus the call at time τ. The right portfolio now has $Ke^{-rT+r\tau}$ in the bank and is short a contract which pays A_T at T. Define P_τ and C_τ to be the respective values of a put and call with maturity T at time τ. Let A_τ be the value of the contract at time τ to receive A_T at time T. By replication, our portfolios give us the following equation

$$P_\tau - C_\tau = Ke^{r(\tau - T)} - A_\tau. \tag{3.5}$$

Substituting this result back into (3.2), the value of the contract V at τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - A_\tau). \tag{3.6}$$

Our next goal is to find an explicit formula for A_τ.

For simplicity, we define U_τ to be the time τ price of a contract with payoff Y_T at time T, where Y_T is defined via

$$Y_T = \int_0^T S_t\, dt. \tag{3.7}$$

Once U_tau is determined, then we can recover A_τ as $A_\tau = \tau U_\tau$.

Note that (3.7) can be split into two parts,

$$Y_T = \int_0^\tau S_t\, dt + \int_\tau^T S_t\, dt. \tag{3.8}$$

Observe that the integral from 0 to τ is known at time τ as each price S_t will be known by the time τ. So we can treat this integral as a constant and now try to replicate the integral from time τ to T.

We begin our replicating strategy by buying x shares of stock at time τ. For all times t where $\tau \le t \le T$, we will continuously sell off stock at the rate α_t and invest the revenue. With this strategy, at time T, the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)}\, dt \tag{3.9}$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_{\tau}^{T} \alpha_t S_t e^{r(T-t)}\, dt = \int_{\tau}^{T} S_t\, dt. \tag{3.10}$$

Solving for α_t, we find that

$$\alpha_t = e^{r(t-T)} \tag{3.11}$$

Thus, the amount of shares our strategy started with was

$$x = \int_{\tau}^{T} e^{r(t-T)}\, dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \tag{3.12}$$

This tells us that the cost at time τ to receive the stock from times τ to T continuously is xS_τ. This gives us

$$U_\tau = \int_0^{\tau} S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right) \tag{3.13}$$

Recall that w_τ is the price at time τ to receive A_T, equivalent to $\dfrac{Y_T}{T}$, at time T. Thus, the price at τ to receive just A_T is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^{\tau} S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T} \tag{3.14}$$

Returning to 1.6 Put-Call Parity, we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^{\tau} S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}. \tag{3.15}$$

Substituting this into the chooser option from 1.7, the value of V_τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^{\tau} S_t\, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}) \tag{3.16}$$

(insert simplification for when r != 0)

The above formula breaks when $r = 0$ since we divide by r. To fix this, we return to our replicating strategy for w_τ accounting for this special case.

Define U_τ and Y_T the same way as above. Again, split the integral Y_T such that

$$Y_T = \int_0^{\tau} S_t\, dt + \int_{\tau}^{T} S_t\, dt \tag{3.17}$$

We now replicate the integral from time τ to T for the special case. We follow the same replicating strategy as before. Purchase x shares of stock. For all times t where $\tau \le t \le T$, we continuously sell off at the rate α_t and invest the revenue. By time T, the bank will have

$$\int_{\tau}^{T} \alpha_t S_t\, dt \tag{3.18}$$

We finish the replication by setting this equal to the value we're replicating

$$\int_{\tau}^{T} \alpha_t S_t\, dt = \int_{\tau}^{T} S_t\, dt \tag{3.19}$$

Solving for α_t, we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_{\tau}^{T} dt = T - \tau \tag{3.20}$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T-\tau),\ w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T} \tag{3.21}$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T}. \tag{3.22}$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t\ dt + S_\tau(T-\tau)}{T}) \tag{3.23}$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \tag{3.24}$$

where $S_0 > 0$ and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t)\ dt + (S_0 + \kappa W_\tau)(T-\tau)}{T}) \tag{3.25}$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+. \tag{3.26}$$

Then by the risk-neutral pricing formula, the time-zero price of this contract is given by

$$V_0 = \tilde{\mathbb{E}}(C_\tau) + \tilde{\mathbb{E}}\left(\left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+\right). \tag{3.27}$$

Let

$$X = \tilde{\mathbb{E}}(C_\tau) + \tilde{\mathbb{E}}\left(\left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+\right). \tag{3.28}$$

It follows that the mean and variance of X can be computed as

$$\mu = \mathbb{E}(X) = 0 \tag{3.29}$$

$$\sigma = Var(X) = \tau(\frac{\kappa(T-\tau)}{T})^2 + \frac{\tau^3}{3}(\frac{\kappa}{T})^2 + \tau^2\frac{\kappa^2(T-\tau)}{T^2}. \tag{3.30}$$

$$\nu = \sqrt{\sigma} \tag{3.31}$$

$$\tag{3.32}$$

We can show that X is normally distributed, but this proof has been ejected as it is beyond the scope of this research. Therefore we can define the probability density function as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{3.33}$$

$$\psi(x) = \frac{1}{\nu}\phi(\frac{x-\mu}{\nu}) \tag{3.34}$$

Now we can substitute back into our equation from (3.27) and use the definition of expectation on a continuous random variable to get

$$V_0 = \tilde{\mathbb{E}}(C_\tau) + \left(\int_{-\infty}^{\infty} (K - S_0 - X)\, \psi(X)\, dX \right)^+ . \tag{3.35}$$

To integrate the second term in V_0 we will let

$$Z = \frac{X - \mu}{\sigma}. \tag{3.36}$$

it follows that

$$X = Z\nu + \mu \tag{3.37}$$

$$dX = \nu dZ. \tag{3.38}$$

Note that the second term can only be positive when

$$X \leq K - S_0. \tag{3.39}$$

Which is equivalent to

$$\frac{X - \mu}{\sigma} \leq \frac{K - S_0 - \mu}{\nu} \tag{3.40}$$

let

$$d_= \frac{K - S_0 - \mu}{\nu}. \tag{3.41}$$

then (3.40) is

$$Z \leq d_. \tag{3.42}$$

Now using (3.37) and (3.42) we can rearrange (3.35) as:

$$V_0 = \tilde{\mathbb{E}}(C_\tau) + \left(\int_{-\infty}^{d_-} (K - S_0 - Z\nu - \mu)\, \frac{1}{\nu}\phi(Z)\nu\, dZ \right). \tag{3.43}$$

Simplifying this expression we get the form

$$V_0 = \tilde{\mathbb{E}}(C_\tau) + (K - S_0 - \mu) \int_{-\infty}^{d_-} \phi(Z)\, dZ - \nu \int_{-\infty}^{d_-} Z\phi(Z)\nu\, dZ. \tag{3.44}$$

Acknowledgement. The authors would like to thank Prof. Hrusa for . . .

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