MFSURP

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Abstract. In this set of notes we derive the time-zero prices of various *chooser options*. These are contracts with a fixed maturity date T and a chooser date τ satisfying $0 \leqslant \tau \leqslant T$, for which an agent is allowed to choose at time τ the underlying security that determines the structure of the payoff at time T.

Contents

1. Introduction

1.1. **Arbitrage-free pricing.** (FIXME: add some background information on arbitrage-free pricing, aka summarize what you learned in 21-270. We can also add this to an appendix.)

1.2. **The Blachelier Model.** In this paper we work within the context of the *Blachelier model*, where the stock prices $\{S_t\}_{t\geqslant 0}$ evolves according to

$$S_t = e^{rt}\left(S_0 + \kappa^{-rt}W_t + \kappa r\int_0^t e^{-rs}W_s\,ds\right), \tag{1.1}$$

where $S_0 > 0$ denotes the initial stock price at time 0 and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. We note to the reader that in the special case when $r = 0$, (1.1) reduces to

$$S_t = S_0 + \kappa W_t. \tag{1.2}$$

(FIXME: explain to the reader why the Blachlier model is interesting).

1.3. **Notation and conventions.** For a random variable X we use the notation X^+ to denote the random variable $\max(X, 0)$. We note that by definition, we have

$$X = X^+ - (-X)^+, \tag{1.3}$$

from which the *put-call parity* can be derived.

(FIXME: some common notation that needs to be explained:

(1) the notation $X \sim N(\mu, \sigma^2)$
(2) ...

)

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2. Arbitrage-free pricing under the Blachelier model

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when $r = 0$.

2.0.1. *European Call.* We first consider a European call where the payoff at time T is given by

$$V_T = (S_T - K)^+ \tag{2.1}$$

for a fixed strike price K. We note that under $\tilde{\mathbb{P}}$, $W_T \sim N(0, T)$, therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.2}$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \tag{2.3}$$

Recall that if we have a random variable X with probability density function f_X under a probability measure $\mathbb{P}$, then the "law of the unconscious statistician" tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx. \tag{2.4}$$

In our setting, we have

$$g(S_T) = (S_T - K)^+, \tag{2.5}$$

and the distribution of S_T under $\tilde{\mathbb{P}}$ as a random variable is given in (2.2). Therefore, the time-zero price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x - K)^+ \psi(x)\, dx, \tag{2.6}$$

where

$$\psi(x) = \frac{1}{\nu}\varphi\left(\frac{x-\mu}{\nu}\right),\ \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \tag{2.7}$$

and

$$\mu = S_0,\ \nu = \kappa\sqrt{T}. \tag{2.8}$$

To compute (2.6), we first note that since $(x-K)^+ = 0$ for $x \leqslant K$, the domain of integration is the set $\{x \mid x \geqslant K\}$. Now we use the change of variables

$$y = -\frac{x-\mu}{\nu} \iff x = \mu - \nu y, \tag{2.9}$$

and we note that since $\nu > 0$,

$$x \geqslant K \iff \frac{x-\mu}{\nu} \geqslant \frac{K-\mu}{\nu} \iff y \leqslant \frac{\mu - K}{\nu} =: d_-. \tag{2.10}$$

Then by performing a change of variables, (2.6) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu)\varphi(-y)\, dy = \int_{-\infty}^{d_-} (\nu y + K - \mu)\varphi(y)\, dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y)\, dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K-\mu)\varphi(y)\, dy}_{:=II}. \tag{2.11}$$

We define the cumulative distribution function of a standard normal random variable X under $\mathbb{P}$ via

$$\varphi(x) = \mathbb{P}[X \leqslant x] = \mathbb{E}[\mathbb{1}_{X \leqslant x}] = \int_{-\infty}^{x} \varphi(y)\, dy. \tag{2.12}$$

With this notation in hand, we can write

$$II = (K - \mu)\int_{-\infty}^{d_-} \varphi(y)\, dy = (K-\mu)\varphi(d_-), \tag{2.13}$$

and

$$I = \nu \int_{-\infty}^{d_-} y\varphi(y)\, dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \to -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \tag{2.14}$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K - \mu)\varphi(d_-). \tag{2.15}$$

To compute the price of a put, one can use put-call parity (FIXME: derive put-call parity somewhere and reference it).

2.0.2. *Arithmetic Asian calls.* Next we consider an *arithmetic Asian call* where the payoff at time T is given by

$$V_T = (A_T - K)^T,\ A_T = \frac{1}{T}\int_0^T S_t\,dt = S_0 + \frac{\kappa}{T}\int_0^T W_t\,dt. \tag{2.16}$$

Using tools from Stochastic calculus, one can show that under the risk neutral measure $\tilde{\mathbb{P}}$,

$$\int_0^T W_t\,dt \sim N(0, T^3/3). \tag{2.17}$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.18}$$

Comparing this to (2.2), we see that A_T has a similar distribution, the only difference is that the variance of A_T is smaller by a factor of $1/3$, so the standard deviation of A_T is smaller by a factor of $1/\sqrt{3}$. By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{3}\sqrt{2\pi}}e^{-3d_-^2/2} + (K-\mu)\varphi(\sqrt{3}d_-), \tag{2.19}$$

where

$$\mu = S_0,\ \nu = \kappa\sqrt{T},\ d_- = \frac{\mu - K}{\nu}. \tag{2.20}$$

We note that since $\sqrt{3} > 1$, we see from (2.19) that the price of an Asian option is higher than the price of a European call. This should be expected as one is paying a premium for a less volatile product.

3. Chooser options

In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date T and a strike price K, and an agent is allowed to decide on a choosing date $\tau < T$ to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+,\ P_T = (K - A_T)^+, \tag{3.1}$$

where A_T is defined via (2.16). Here, we assume the agent chooses optimally with no outside information. At time τ, the agent will choose the option of higher value between the put and call, therefore the value of this contract at time τ is

$$V_\tau = \max(C_\tau, P_\tau). \tag{3.2}$$

The time-zero price of this contract is then

$$V_0 = \tilde{\mathbb{E}}[V_\tau]. \tag{3.3}$$

In the next subsection, we simplify the expression for V_τ via the method of replication.

3.1. **Replication.** We first note that by properties of the max function (FIXME: maybe add this in intro and reference?), we can write

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau). \tag{3.4}$$

By (1.3), we have

$$P_T - C_T = (K - S_T)^+ - (S_T - K)^+ = K - A_T. \tag{3.5}$$

Combining (3.2) and (3.5) then gives us

$$V_\tau = C_\tau + \max(0, K - A_T). \tag{3.6}$$

Next, we identify the time-τ prices of contracts paying $P_T - C_T$ and $K - A_T$ at time T.

To replicate a security with payoff $P_T - C_T$, we consider a portfolio that longs a put and shorts a call at time 0, both with strike T (FIXME: what's the maturity?). To replicate a security with payoff $K - A_T$, we consider a portfolio investing Ke^{-rT} into the money account at time 0 and shorting a contract (FIXME: what kind of contract?) to receive A_T at time T.

Since both portfolios have the same payoff at time T by (3.5), they have the same price for all times t where $0 \leq t \leq T$ (FIXME: in the intro explain why this is true in terms of arbitrage).

For any t satisfying $0 \leqslant t \leqslant T$, we define P_t to be the time-t value of a put with payoff P_T at time T, C_t to be the time-t value of a call with payoff C_T at time t, A_t to be the time-t value of an asian option paying A_T at time t. (FIXME: maybe move to intro?)

Using this notation, at time τ the value of the first portfolio is $P_\tau - C_\tau$. Also, at time τ the second portfolio has $Ke^{-rT+r\tau}$ in the bank and is shorting a contract which pays A_T at T, therefore the time-τ value of the second portfolio is $Ke^{r(\tau-T)} - A_\tau$. By replication, the time τ prices of the portfolios are equal, therefore we have

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - A_\tau. \tag{3.7}$$

Substituting this result back into (3.2), the value of the original chooser contract at τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - A_\tau). \tag{3.8}$$

Our next goal is to find an explicit formula for A_τ.

For simplicity, we define U_τ to be the time τ price of a contract with payoff Y_T at time T, where Y_T is defined via

$$Y_T = \int_0^T S_t \, dt. \tag{3.9}$$

Once U_tau is determined, then we can recover A_τ as $A_\tau = \tau U_\tau$.

Note that (3.9) can be split into two parts,

$$Y_T = \int_0^\tau S_t \, dt + \int_\tau^T S_t \, dt. \tag{3.10}$$

Observe that the integral from 0 to τ is known at time τ as each price S_t will be known by the time τ. So we can treat this integral as a constant and now try to replicate the integral from time τ to T.

3.2. **Replicating Asian options when $r > 0$.** We begin our replicating strategy by buying x shares of stock at time τ. For all times t where $\tau \leq t \leq T$, we will continuously sell off stock at the rate α_t and invest the revenue. With this strategy, at time T, the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt \tag{3.11}$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt = \int_\tau^T S_t \, dt. \tag{3.12}$$

Solving for α_t, we find that

$$\alpha_t = e^{r(t-T)} \tag{3.13}$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} \, dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \tag{3.14}$$

This tells us that the cost at time τ to receive the stock from times τ to T continuously is xS_τ. This gives us

$$U_\tau = \int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right) \tag{3.15}$$

Recall that w_τ is the price at time τ to receive A_T, equivalent to $\frac{Y_T}{T}$, at time T. Thus, the price at τ to receive just A_T is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T} \tag{3.16}$$

Returning to 1.6 Put-Call Parity (FIXME: fix reference), we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}. \tag{3.17}$$

Substituting this into the chooser option from 1.7, the value of V_τ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t\ dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}) \tag{3.18}$$

3.3. **Replicating Asian options when** $r = 0$**.** The above formula breaks when $r = 0$ since we divide by r (FIXME: this is a little confusing since the interest rate was never specified. maybe specify that you are assuming $r > 0$ initially and now you're considering $r = 0$ as a special case. If you choose to do this I recommend breaking this section into different subsections). To fix this, we return to our replicating strategy for w_τ accounting for this special case.

Define U_τ and Y_T the same way as above. Again, split the integral Y_T such that

$$Y_T = \int_0^\tau S_t\ dt + \int_\tau^T S_t\ dt \tag{3.19}$$

We now replicate the integral from time τ to T for the special case. We follow the same replicating strategy as before. Purchase x shares of stock. For all times t where $\tau \leq t \leq T$, we continuously sell off at the rate α_t and invest the revenue. By time T, the bank will have

$$\int_\tau^T \alpha_t S_t\ dt \tag{3.20}$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t\ dt = \int_\tau^T S_t\ dt \tag{3.21}$$

Solving for α_t, we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \tag{3.22}$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T-\tau),\ w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T} \tag{3.23}$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T}. \tag{3.24}$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t\ dt + S_\tau(T-\tau)}{T}) \tag{3.25}$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \tag{3.26}$$

where $S_0 > 0$ and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t)\ dt + (S_0 + \kappa W_\tau)(T-\tau)}{T}) \tag{3.27}$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+. \tag{3.28}$$

Then by the risk-neutral pricing formula and the linearity of expection, the time-zero price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}}\left[\left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+\right]. \tag{3.29}$$

Let X be the random variable defined via

$$X = \frac{\kappa(T-\tau)}{T} W_\tau + \frac{\kappa}{T}\int_0^\tau W_t\ dt. \tag{3.30}$$

It follows that the mean and variance of X can be computed as (IMPORTANT FIXME: add details of this computation!)

$$\mu = \mathbb{E}(X) = 0 \tag{3.31}$$

$$\sigma = Var(X) = \tau(\frac{\kappa(T-\tau)}{T})^2 + \frac{\tau^3}{3}(\frac{\kappa}{T})^2 + \tau^2\frac{\kappa^2(T-\tau)}{T^2}. \tag{3.32}$$

$$\nu = \sqrt{\sigma} \tag{3.33}$$

$$\tag{3.34}$$

(FIXME: typically σ denotes the standard deviation, so this isn't very consistent with standard notation)

Using methods of stochastic calculus (FIXME: we need to find a reference to cite), X is normally distributed with mean μ and variance σ^2. Therefore we can define the probability density function as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{3.35}$$

$$\psi(x) = \frac{1}{\nu}\varphi(\frac{x-\mu}{\nu}) \tag{3.36}$$

Now we can substitute back into our equation from (3.29) and use the definition of expectation on a continuous random variable to get

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty}(K - S_0 - x)^+\,\psi(x)\,dx. \tag{3.37}$$

To integrate the second term in V_0 we will let

$$z = \frac{x-\mu}{\sigma}. \tag{3.38}$$

it follows that

$$x = z\nu + \mu \tag{3.39}$$

$$dx = \nu dz. \tag{3.40}$$

We note that

$$K - S_0 - x \geqslant 0 \iff x \leq K - S_0 \iff \frac{x-\mu}{\sigma} \leq \frac{K-S_0-\mu}{\nu}. \tag{3.41}$$

We define d_- via

$$d_- = \frac{K-S_0-\mu}{\nu} \tag{3.42}$$

so by (3.41), we have

$$K - S_0 - x \geqslant 0 \iff z \leq d_. \tag{3.43}$$

Now using (3.39) and (3.43) we can rearrange (3.37) as:

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \left(\int_{-\infty}^{d_-}(K - S_0 - z\nu - \mu)\,\varphi(z)\,dz\right). \tag{3.44}$$

Simplifying this expression we get the form

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \nu\int_{-\infty}^{d_-} z\varphi(z)\nu\,dz + (K - S_0 - \mu)\int_{-\infty}^{d_-}\varphi(z)\,dz \tag{3.45}$$

(FIXME: continue with this computation, write down an explicit formula for V_0, and also the price for the variant.)

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Appendix A. Various replicating strategies

A.1. **Replicating European options.**

A.2. **Replicating Asian options.** Carnegie Mellon University, Pittsburgh, PA 15213, USA
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