

# MFSURP

JESSICA CHEN, LINXUAN JIANG, FRANK SACCO, AND ALBERT ZHANG

ABSTRACT. This paper concerns ...

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## 1. INTRODUCTION

### 1.1. Arbitrage-free pricing.

**1.2. The Blachelier Model.** In this paper we work within the context of the *Blachelier model*, where the stock prices  $\{S_t\}_{t \geq 0}$  evolves according to

$$S_t = e^{rt} \left( S_0 + \kappa^{-rt} W_t + \kappa r \int_0^t e^{-rs} W_s ds \right). \quad (1.1)$$

where  $S_0 > 0$  and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure  $\tilde{\mathbb{P}}$ . We note to the reader that when  $r = 0$ , this reduces to

$$S_t = S_0 + \kappa W_t. \quad (1.2)$$

**1.3. Notation and conventions.** In this paper we often use the notation  $f^+$  to denote  $\max(f, 0)$ . We note that by definition, we have

$$f = f^+ - (-f)^+. \quad (1.3)$$

## 2. ASIAN OPTIONS

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when  $r = 0$ .

**2.0.1. European Call.** We first consider a European call where the payoff at time  $T$  is given by

$$V_T = (S_T - K)^+. \quad (2.1)$$

We note that under  $\tilde{\mathbb{P}}$ ,  $W_T \sim N(0, T)$ , therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.2)$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \quad (2.3)$$

Recall that if we have a random variable  $X$  with probability density function  $f_X$  under a probability measure  $\mathbb{P}$ , then the “law of the unconscious statistician” tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \quad (2.4)$$

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*Key words and phrases.* Blachelier model, Chooser options, ...

J. Chen, L. Jiang, F. Sacco, A. Zhang were supported by MFSURP ...

In our setting, we have

$$g(S_T) = (S_T - K)^+, \quad (2.5)$$

and the distribution of  $S_T$  under  $\tilde{\mathbb{P}}$  as a random variable is given in (2.2). Therefore

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x - K)^+ \psi(x) dx, \quad (2.6)$$

where

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right), \quad \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad (2.7)$$

and

$$\mu = S_0, \quad \nu = \kappa\sqrt{T}. \quad (2.8)$$

To compute (2.6), we first note that since  $(x - K)^+ = 0$  for  $x \leq K$ , the domain of integration is the set  $\{x \mid x \geq K\}$ . Now we use the change of variables

$$y = -\frac{x - \mu}{\nu} \iff x = \mu - \nu y \quad (2.9)$$

and we note that since  $\nu > 0$ ,

$$x \geq K \iff \frac{x - \mu}{\nu} \geq \frac{K - \mu}{\nu} \iff y \leq \frac{\mu - K}{\nu} =: d_-. \quad (2.10)$$

Then by performing a change of variables, (2.6) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(-y) dy = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(y) dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y) dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K - \mu) \varphi(y) dy}_{:=II}. \quad (2.11)$$

We define the cumulative distribution function of a standard normal random variable  $X$  under  $\mathbb{P}$  via

$$\Phi(x) = \mathbb{P}[X \leq x] = \mathbb{E}[\mathbb{1}_{X \leq x}] = \int_{-\infty}^x \varphi(y) dy. \quad (2.12)$$

With this notation in hand, we can write

$$II = (K - \mu) \int_{-\infty}^{d_-} \varphi(y) dy = (K - \mu) \Phi(d_-), \quad (2.13)$$

and

$$I = \nu \int_{-\infty}^{d_-} y \varphi(y) dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \rightarrow -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \quad (2.14)$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K - \mu) \Phi(d_-). \quad (2.15)$$

To compute the price of a put, one can use put-call parity.

**2.0.2. Arithmetic Asian call.** Consider an arithmetic Asian call where the payoff at time  $T$  is given by

$$V_T = (A_T - K)^T, \quad A_T = \frac{1}{T} \int_0^T S_t dt = S_0 + \frac{\kappa}{T} \int_0^T W_t dt. \quad (2.16)$$

Using Ito's formula, one can show that under the risk neutral measure  $\tilde{\mathbb{P}}$ ,

$$\int_0^T W_t dt \sim N(0, T^3/3). \quad (2.17)$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.18)$$

Comparing this to (2.2), we see that  $A_T$  has a similar distribution, the only difference is that the variance of  $A_T$  is smaller by a factor of  $1/3$ , so the standard deviation of  $A_T$  is smaller by a factor of  $1/\sqrt{3}$ . By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{3}\sqrt{2\pi}} e^{-3d_-^2/2} + (K - \mu) \Phi(\sqrt{3}d_-), \quad (2.19)$$

where

$$\mu = S_0, \nu = \kappa\sqrt{T}, d_- = \frac{\mu - K}{\nu}. \quad (2.20)$$

We note that since  $\sqrt{3} > 1$ , we see from (2.19) that the price of an Asian option is higher than the price of a European call. This should be expected as one is paying a premium for a less volatile product.

### 3. CHOOSER OPTIONS

In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date  $T$  and a strike price  $K$ , and an agent is allowed to decide on a choosing date  $\tau < T$  to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+, P_T = (K - A_T)^+, \quad (3.1)$$

where  $A_T$  is defined via (2.16). Here, we assume the agent chooses optimally with no outside information. At time  $\tau$ , the agent will choose the option of higher value between the put and call, therefore the value of this contract at time  $\tau$  is

$$V_\tau = \max(C_\tau, P_\tau). \quad (3.2)$$

By properties of the max function (FIXME: maybe add this in intro and reference?)

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau). \quad (3.3)$$

Note that by (1.3), we have

$$P_T - C_T = (K - S_T)^+ - (S_T - K)^+ = K - A_T. \quad (3.4)$$

Next we want to identify the time- $\tau$  prices of contracts paying  $P_T - C_T$  and  $K - A_T$  at time  $T$ . To compute these arbitrage prices, we use the method of replication.

We can replicate the LHS of the above equation by going long a put and short a call at time 0, both with strike  $T$ . We also replicate the RHS by investing  $Ke^{-rT}$  into the bank at time 0 and shorting a contract to receive  $A_T$  at time  $T$ . Since both portfolios are of equal price at time  $T$ , they have equal price at all times  $t$  where  $0 \leq t \leq T$ . At time  $\tau$ , the left portfolio is the value of the put minus the call at time  $\tau$ . The right portfolio now has  $Ke^{-rT+r\tau}$  in the bank and is short a contract which pays  $A_T$  at  $T$ . Define  $P_\tau$  and  $C_\tau$  to be the respective values of a put and call with maturity  $T$  at time  $\tau$ . Let  $A_\tau$  be the value of the contract at time  $\tau$  to receive  $A_T$  at time  $T$ . By replication, our portfolios give us the following equation

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - A_\tau. \quad (3.5)$$

Substituting this result back into 1.4 (FIXME: fix reference), the value of the contract  $V$  at  $\tau$  is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - A_\tau). \quad (3.6)$$

Our next goal is to find an explicit formula for  $A_\tau$ .

For simplicity, we define  $U_\tau$  to be the time  $\tau$  price of a contract with payoff  $Y_T$  at time  $T$ , where  $Y_T$  is defined via

$$Y_T = \int_0^T S_t dt. \quad (3.7)$$

Once  $U_\tau$  is determined, then we can recover  $A_\tau$  as  $A_\tau = \tau U_\tau$ .

Note that (3.7) can be split into two parts,

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt. \quad (3.8)$$

Observe that the integral from 0 to  $\tau$  is known at time  $\tau$ . We can treat this integral as a constant (FIXME: be more precise about this?) and now try to replicate the integral from time  $\tau$  to  $T$ .

We begin our replicating strategy by buying  $x$  shares of stock at time  $\tau$ . For all times  $t$  where  $\tau \leq t \leq T$ , we will continuously sell off stock at the rate  $\alpha_t$  and invest the revenue. With this strategy, at time  $T$ , the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt \quad (3.9)$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_{\tau}^T \alpha_t S_t e^{r(T-t)} dt = \int_{\tau}^T S_t dt. \quad (3.10)$$

Solving for  $\alpha_t$ , we find that

$$\alpha_t = e^{r(t-T)} \quad (3.11)$$

Thus, the amount of shares our strategy started with was

$$x = \int_{\tau}^T e^{r(t-T)} dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \quad (3.12)$$

This tells us that the cost at time  $\tau$  to receive the stock from times  $\tau$  to  $T$  continuously is  $xS_{\tau}$ . This gives us

$$U_{\tau} = \int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)}) \quad (3.13)$$

Recall that  $w_{\tau}$  is the price at time  $\tau$  to receive  $A_T$ , equivalent to  $\frac{Y_T}{T}$ , at time  $T$ . Thus, the price at  $\tau$  to receive just  $A_T$  is

$$w_{\tau} = \frac{U_{\tau}}{T} = \frac{\int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)})}{T} \quad (3.14)$$

Returning to 1.6 Put-Call Parity, we can write out the equation as

$$P_{\tau} - C_{\tau} = Ke^{r(\tau-T)} - \frac{\int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)})}{T}. \quad (3.15)$$

Substituting this into the chooser option from 1.7, the value of  $V_{\tau}$  is

$$V_{\tau} = C_{\tau} + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^{\tau} S_t dt + \frac{S_{\tau}}{r} (1 - e^{r(\tau-T)})}{T}) \quad (3.16)$$

(insert simplification for when  $r \neq 0$ )

The above formula breaks when  $r = 0$  since we divide by  $r$ . To fix this, we return to our replicating strategy for  $w_{\tau}$  accounting for this special case.

Define  $U_{\tau}$  and  $Y_T$  the same way as above. Again, split the integral  $Y_T$  such that

$$Y_T = \int_0^{\tau} S_t dt + \int_{\tau}^T S_t dt \quad (3.17)$$

We now replicate the integral from time  $\tau$  to  $T$  for the special case. We follow the same replicating strategy as before. Purchase  $x$  shares of stock. For all times  $t$  where  $\tau \leq t \leq T$ , we continuously sell off at the rate  $\alpha_t$  and invest the revenue. By time  $T$ , the bank will have

$$\int_{\tau}^T \alpha_t S_t dt \quad (3.18)$$

We finish the replication by setting this equal to the value we're replicating

$$\int_{\tau}^T \alpha_t S_t dt = \int_{\tau}^T S_t dt \quad (3.19)$$

Solving for  $\alpha_t$ , we see that when  $r = 0$  that  $\alpha_t = 1$ . Thus, the number of shares the strategy started with was

$$\int_{\tau}^T dt = T - \tau \quad (3.20)$$

Similar to the  $r \neq 0$  case, it then follows that

$$U_{\tau} = \int_0^{\tau} S_t dt + S_{\tau}(T - \tau), \quad w_{\tau} = \frac{U_{\tau}}{T} = \frac{\int_0^{\tau} S_t dt + S_{\tau}(T - \tau)}{T} \quad (3.21)$$

Thus, Put-Call Parity in the special case tells us that

$$P_{\tau} - C_{\tau} = K - \frac{U_{\tau}}{T} = \frac{\int_0^{\tau} S_t dt + S_{\tau}(T - \tau)}{T}. \quad (3.22)$$

Substituting this result into the chooser option formula, we have

$$V_{\tau} = C_{\tau} + \max(0, K - \frac{\int_0^{\tau} S_t dt + S_{\tau}(T - \tau)}{T}) \quad (3.23)$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \quad (3.24)$$

where  $S_0 > 0$  and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_{\tau} = C_{\tau} + \max(0, K - \frac{\int_0^{\tau} (S_0 + \kappa W_t) dt + (S_0 + \kappa W_{\tau})(T - \tau)}{T}) \quad (3.25)$$

So in conclusion, we find that

$$V_{\tau} = C_{\tau} + \left( K - S_0 - \frac{\kappa(T - \tau)}{T} W_{\tau} - \frac{\kappa}{T} \int_0^{\tau} W_t dt \right)^+. \quad (3.26)$$

Then by the risk-neutral pricing formula, the time-zero price of this contract is given by

**Acknowledgement.** The authors would like to thank Prof. Hrusa for ...

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
*Email address*, J. Chen: [jschen2@andrew.cmu.edu](mailto:jschen2@andrew.cmu.edu)

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
*Email address*, L. Jiang: [linxuanj@andrew.cmu.edu](mailto:linxuanj@andrew.cmu.edu)

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
*Email address*, F. Sacco: [fsacco@andrew.cmu.edu](mailto:fsacco@andrew.cmu.edu)

CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA  
*Email address*, A. Zhang: [albertzh@andrew.cmu.edu](mailto:albertzh@andrew.cmu.edu)