# MFSURP

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ABSTRACT. This paper concerns ...

## CONTENTS



## 1. INTRODUCTION

### 1.1. Arbitrage-free pricing.

### 1.2. The Blachelier Model.

In this paper we work within the context of the *Blachelier model*, where the stock prices $\{S_t\}_{t\geqslant 0}$ evolves according to

$$S_t = e^{rt}\left(S_0 + \kappa^{-rt}W_t + \kappa r\int_0^t e^{-rs}W_s\, ds\right). \tag{1.1}$$

where $S_0 > 0$ and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. We note to the reader that when $r = 0$, this reduces to

$$S_t = S_0 + \kappa W_t. \tag{1.2}$$

### 1.3. Notation and conventions.

In this paper we often use the notation $f^+$ to denote $\max(f, 0)$. We note that by definition, we have

$$f = f^+ - (-f)^+. \tag{1.3}$$

## 2. ASIAN OPTIONS

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when $r = 0$.

2.0.1. *European Call.* We first consider a European call where the payoff at time $T$ is given by

$$V_T = (S_T - K)^+. \tag{2.1}$$

We note that under $\tilde{\mathbb{P}}$, $W_T \sim N(0, T)$, therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.2}$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \tag{2.3}$$

Recall that if we have a random variable $X$ with probability density function $f_X$ under a probability measure $\mathbb{P}$, then the "law of the unconscious statistician" tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx. \tag{2.4}$$

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In our setting, we have

$$g(S_T) = (S_T - K)^+, \tag{2.5}$$

and the distribution of $S_T$ under $\tilde{\mathbb{P}}$ as a random variable is given in (2.2). Therefore

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x-K)^+ \psi(x)\, dx, \tag{2.6}$$

where

$$\psi(x) = \frac{1}{\nu}\varphi\left(\frac{x-\mu}{\nu}\right),\ \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \tag{2.7}$$

and

$$\mu = S_0,\ \nu = \kappa\sqrt{T}. \tag{2.8}$$

To compute (2.6), we first note that since $(x-K)^+ = 0$ for $x \leqslant K$, the domain of integration is the set $\{x \mid x \geqslant K\}$. Now we use the change of variables

$$y = -\frac{x-\mu}{\nu} \iff x = \mu - \nu y \tag{2.9}$$

and we note that since $\nu > 0$,

$$x \geqslant K \iff \frac{x-\mu}{\nu} \geqslant \frac{K-\mu}{\nu} \iff y \leqslant \frac{\mu - K}{\nu} =: d_-. \tag{2.10}$$

Then by performing a change of variables, (2.6) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu)\varphi(-y)\, dy = \int_{-\infty}^{d_-} (\nu y + K - \mu)\varphi(y)\, dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y)\, dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K-\mu)\varphi(y)\, dy}_{:=II}. \tag{2.11}$$

We define the cumulative distribution function of a standard normal random variable $X$ under $\mathbb{P}$ via

$$\Phi(x) = \mathbb{P}[X \leqslant x] = \mathbb{E}[\mathbb{1}_{X \leqslant x}] = \int_{-\infty}^{x} \varphi(y)\, dy. \tag{2.12}$$

With this notation in hand, we can write

$$II = (K-\mu)\int_{-\infty}^{d_-} \varphi(y)\, dy = (K-\mu)\Phi(d_-), \tag{2.13}$$

and

$$I = \nu \int_{-\infty}^{d_-} y\varphi(y)\, dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \to -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \tag{2.14}$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K-\mu)\Phi(d_-). \tag{2.15}$$

To compute the price of a put, one can use put-call parity.

2.0.2. *Arithmetic Asian call.* Consider an arithmetic Asian call where the payoff at time $T$ is given by

$$V_T = (A_T - K)^T,\ A_T = \frac{1}{T}\int_0^T S_t\, dt = S_0 + \frac{\kappa}{T}\int_0^T W_t\, dt. \tag{2.16}$$

Using Ito's formula, one can show that under the risk neutral measure $\tilde{\mathbb{P}}$,

$$\int_0^T W_t\, dt \sim N(0, T^3/3). \tag{2.17}$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \tag{2.18}$$

Comparing this to (2.2), we see that $A_T$ has a similar distribution, the only difference is that the variance of $A_T$ is smaller by a factor of $1/3$, so the standard deviation of $A_T$ is smaller by a factor of $1/\sqrt{3}$. By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{3}\sqrt{2\pi}} e^{-3d_-^2/2} + (K-\mu)\Phi(\sqrt{3}d_-), \tag{2.19}$$

where

$$\mu = S_0,\ \nu = \kappa\sqrt{T},\ d_- = \frac{\mu - K}{\nu}. \tag{2.20}$$

We note that since $\sqrt{3} > 1$, we see from (2.19) that the price of an Asian option is higher than the price of a European call. This should be expected as one is paying a premium for a less volatile product.

## 3. Chooser options

In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date $T$ and a strike price $K$, and an agent is allowed to decide on a choosing date $\tau < T$ to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+,\ P_T = (K - A_T)^+, \tag{3.1}$$

where $A_T$ is defined via (2.16). Here, we assume the agent chooses optimally with no outside information. At time $\tau$, the agent will choose the option of higher value between the put and call, therefore the value of this contract at time $\tau$ is

$$V_\tau = \max(C_\tau, P_\tau). \tag{3.2}$$

By properties of the max function (FIXME: maybe add this in intro and reference?)

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau). \tag{3.3}$$

Note that by (1.3), we have

$$P_T - C_T = (K - S_T)^+ - (S_T - K)^+ = K - A_T. \tag{3.4}$$

Next we want to identify the time-$\tau$ prices of contracts paying $P_T - C_T$ and $K - A_T$ at time $T$. To compute these arbitrage prices, we use the method of replication.

We can replicate the LHS of the above equation by going long a put and short a call at time 0, both with strike $T$. We also replicate the RHS by investing $Ke^{-rT}$ into the bank at time 0 and shorting a contract to receive $A_T$ at time $T$. Since both portfolios are of equal price at time $T$, they have equal price at all times $t$ where $0 \leq t \leq T$. At time $\tau$, the left portfolio is the value of the put minus the call at time $\tau$. The right portfolio now has $Ke^{-rT+r\tau}$ in the bank and is short a contract which pays $A_T$ at $T$. Define $P_\tau$ and $C_\tau$ to be the respective values of a put and call with maturity $T$ at time $\tau$. Let $w_\tau$ be the value of the contract at time $\tau$ to receive $A_T$ at time $T$. By replication, our portfolios give us the following equation

$$P_\tau - C_\tau = Ke^{r(\tau - T)} - w_\tau. \tag{3.5}$$

Substituting this result back into 1.4 (FIXME: fix reference), the value of the contract $V$ at $\tau$ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - w_\tau). \tag{3.6}$$

Now we want an expression for the price at time $\tau$ to receive $A_T$ at time $T$.

For simplicity, define $U_\tau$ to be the price at time $\tau$ to receive $Y_T$ at time $T$, where

$$Y_T = \int_0^T S_t \, dt \tag{3.7}$$

Note that this integral can be split into

$$Y_T = \int_0^\tau S_t \, dt + \int_\tau^T S_t \, dt. \tag{3.8}$$

Observe that the integral from 0 to $\tau$ is known at time $\tau$. We can treat this integral as a constant and now try to replicate the integral from time $\tau$ to $T$.

We begin our replicating strategy by buying $x$ shares of stock at time $\tau$. For all times $t$ where $\tau \leq t \leq T$, we will continuously sell off stock at the rate $\alpha_t$ and invest the revenue. With this strategy, at time T, the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt \tag{3.9}$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt = \int_\tau^T S_t \, dt. \tag{3.10}$$

Solving for $\alpha_t$, we find that

$$\alpha_t = e^{r(t-T)} \tag{3.11}$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} \, dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \tag{3.12}$$

This tells us that the cost at time $\tau$ to receive the stock from times $\tau$ to $T$ continuously is $xS_\tau$. This gives us

$$U_\tau = \int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right) \tag{3.13}$$

Recall that $w_\tau$ is the price at time $\tau$ to receive $A_T$, equivalent to $\dfrac{Y_T}{T}$, at time T. Thus, the price at $\tau$ to receive just $A_T$ is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T} \tag{3.14}$$

Returning to 1.6 Put-Call Parity, we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}. \tag{3.15}$$

Substituting this into the chooser option from 1.7, the value of $V_\tau$ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}) \tag{3.16}$$

(insert simplification for when r != 0)

The above formula breaks when $r = 0$ since we divide by $r$. To fix this, we return to our replicating strategy for $w_\tau$ accounting for this special case.

Define $U_\tau$ and $Y_T$ the same way as above. Again, split the integral $Y_T$ such that

$$Y_T = \int_0^\tau S_t\ dt + \int_\tau^T S_t\ dt \tag{3.17}$$

We now replicate the integral from time $\tau$ to $T$ for the special case. We follow the same replicating strategy as before. Purchase $x$ shares of stock. For all times $t$ where $\tau \leq t \leq T$, we continuously sell off at the rate $\alpha_t$ and invest the revenue. By time $T$, the bank will have

$$\int_\tau^T \alpha_t S_t\ dt \tag{3.18}$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t\ dt = \int_\tau^T S_t\ dt \tag{3.19}$$

Solving for $\alpha_t$, we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \tag{3.20}$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau(T-\tau),\ w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T} \tag{3.21}$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau(T-\tau)}{T}. \tag{3.22}$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t\ dt + S_\tau(T-\tau)}{T}) \tag{3.23}$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \tag{3.24}$$

where $S_0 > 0$ and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t)\ dt + (S_0 + \kappa W_\tau)(T-\tau)}{T}) \tag{3.25}$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left(K - S_0 - \frac{\kappa(T-\tau)}{T} W_\tau - \frac{\kappa}{T}\int_0^\tau W_t\ dt\right)^+. \tag{3.26}$$

Then by the risk-neutral pricing formula, the time-zero price of this contract is given by

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