# MFSURP

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Abstract. This paper concerns . . .

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## 1. Introduction

## 2. Chooser options under the Blachelier Model

We consider the Blachelier model with where the stock prices $\{S_t\}_{t\geqslant 0}$ evolves according to

$$S_t = e^{rt}\left(S_0 + \kappa^{-rt}W_t + \kappa r\int_0^t e^{-rs}W_s\,ds\right). \tag{2.1}$$

where $S_0 > 0$ and $\{W_t\}_{t\geqslant 0}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. When $r = 0$, this reduces to

$$S_t = S_0 + \kappa W_t. \tag{2.2}$$

According to (2.2), . . .

We consider a contract with maturity $T$ and strike price $K$, that allows an agent to decide on a choosing date $\tau < T$ to choose the underlying derivative. Many results are known when an agent is allowed to choose between a European call and a European put. In this section we consider a contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+,\ P_T = (K - A_T)^+,\ A_T = \frac{1}{T}\int_0^T S_t\,dt. \tag{2.3}$$

Here, we assume the agent chooses optimally with no outside information. At time $\tau$, the agent will choose the option of higher value between the put and call, therefore the value of this contract at time $\tau$ is

$$V_\tau = \max(C_\tau, P_\tau). \tag{2.4}$$

By properties of the max function (FIXME: maybe add this in intro and reference?)

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau) \tag{2.5}$$

which we want to evaluate through Put-Call Parity.

Recall that Put-Call Parity tells us

$$P_T - C_T = K - A_T. \tag{2.6}$$

We can replicate the LHS of the above equation by going long a put and short a call at time 0, both with strike $T$. We also replicate the RHS by investing $Ke^{-rT}$ into the bank at time 0 and shorting a contract to receive $A_T$ at time $T$. Since both portfolios are of equal price at time $T$, they have equal price at all times $t$ where $0 \leq t \leq T$. At time $\tau$, the left portfolio is the value of the put minus the call at time $\tau$. The right portfolio now has $Ke^{-rT+r\tau}$ in the bank and is short a contract which pays $A_T$ at $T$. Define $P_\tau$ and $C_\tau$ to be the respective values of a put and call with maturity $T$ at time $\tau$. Let $w_\tau$ be the value of the contract at time $\tau$ to receive $A_T$ at time $T$. By replication, our portfolios give us the following equation

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$$P_\tau - C_\tau = Ke^{r(\tau - T)} - w_\tau. \tag{2.7}$$

Substituting this result back into 1.4, the value of the contract $V$ at $\tau$ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau - T)} - w_\tau). \tag{2.8}$$

Now we want an expression for the price at time $\tau$ to receive $A_T$ at time $T$.

For simplicity, define $U_\tau$ to be the price at time $\tau$ to receive $Y_T$ at time $T$, where

$$Y_T = \int_0^T S_t \, dt \tag{2.9}$$

Note that this integral can be split into

$$Y_T = \int_0^\tau S_t \, dt + \int_\tau^T S_t \, dt. \tag{2.10}$$

Observe that the integral from 0 to $\tau$ is known at time $\tau$. We can treat this integral as a constant and now try to replicate the integral from time $\tau$ to $T$.

We begin our replicating strategy by buying $x$ shares of stock at time $\tau$. For all times $t$ where $\tau \leq t \leq T$, we will continuously sell off stock at the rate $\alpha_t$ and invest the revenue. With this strategy, at time T, the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt \tag{2.11}$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} \, dt = \int_\tau^T S_t \, dt. \tag{2.12}$$

Solving for $\alpha_t$, we find that

$$\alpha_t = e^{r(t-T)} \tag{2.13}$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} \, dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \tag{2.14}$$

This tells us that the cost at time $\tau$ to receive the stock from times $\tau$ to $T$ continuously is $xS_\tau$. This gives us

$$U_\tau = \int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right) \tag{2.15}$$

Recall that $w_\tau$ is the price at time $\tau$ to receive $A_T$, equivalent to $\frac{Y_T}{T}$, at time T. Thus, the price at $\tau$ to receive just $A_T$ is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T} \tag{2.16}$$

Returning to 1.6 Put-Call Parity, we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}. \tag{2.17}$$

Substituting this into the chooser option from 1.7, the value of $V_\tau$ is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t \, dt + \frac{S_\tau}{r}\left(1 - e^{r(\tau-T)}\right)}{T}) \tag{2.18}$$

(insert simplification for when r != 0)

The above formula breaks when $r = 0$ since we divide by $r$. To fix this, we return to our replicating strategy for $w_\tau$ accounting for this special case.

Define $U_\tau$ and $Y_T$ the same way as above. Again, split the integral $Y_T$ such that

$$Y_T = \int_0^\tau S_t \, dt + \int_\tau^T S_t \, dt \tag{2.19}$$

We now replicate the integral from time $\tau$ to $T$ for the special case. We follow the same replicating strategy as before. Purchase $x$ shares of stock. For all times $t$ where $\tau \leq t \leq T$, we continuously sell off at the rate $\alpha_t$ and invest the revenue. By time $T$, the bank will have

$$\int_\tau^T \alpha_t S_t \, dt \tag{2.20}$$

We finish the replication by setting this equal to the value we're replicating

$$\int_\tau^T \alpha_t S_t \, dt = \int_\tau^T S_t \, dt \tag{2.21}$$

Solving for $\alpha_t$, we see that when $r = 0$ that $\alpha_t = 1$. Thus, the number of shares the strategy started with was

$$\int_\tau^T dt = T - \tau \tag{2.22}$$

Similar to the $r \neq 0$ case, it then follows that

$$U_\tau = \int_0^\tau S_t dt + S_\tau (T - \tau), \; w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau (T-\tau)}{T} \tag{2.23}$$

Thus, Put-Call Parity in the special case tells us that

$$P_\tau - C_\tau = K - \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + S_\tau (T-\tau)}{T}. \tag{2.24}$$

Substituting this result into the chooser option formula, we have

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau S_t \, dt + S_\tau (T-\tau)}{T}) \tag{2.25}$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \tag{2.26}$$

where $S_0 > 0$ and $\{W_t\}_{t \geqslant 0}$ is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_\tau = C_\tau + \max(0, K - \frac{\int_0^\tau (S_0 + \kappa W_t) \; dt + (S_0 + \kappa W_\tau)(T-\tau)}{T}) \tag{2.27}$$

So in conclusion, we find that

$$V_\tau = C_\tau + \left( K - S_0 - \frac{\kappa (T-\tau)}{T} W_\tau - \frac{\kappa}{T} \int_0^\tau W_t \, dt \right)^+ . \tag{2.28}$$

Then by the risk-neutral pricing formula, the time-zero price of this contract is given by

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