

# MFSURP

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ABSTRACT. In this set of notes we derive the time-zero prices of various *chooser options*. These are contracts with a fixed maturity date  $T$  and a chooser date  $\tau$  satisfying  $0 \leq \tau \leq T$ , for which an agent is allowed to choose at time  $\tau$  the underlying security that determines the structure of the payoff at time  $T$ .

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## 1. INTRODUCTION

**1.1. Arbitrage-free pricing.** (FIXME: add some background information on arbitrage-free pricing, aka summarize what you learned in 21-270. We can also add this to an appendix.) Before defining arbitrage-free pricing, we first must define arbitrage. An arbitrage strategy has three properties.

- (1) the agent's initial capital is zero.
- (2) the agent has zero percent chance of losing money.
- (3) the agent has a strictly positive probability of profit.

Under this definition of arbitrage, the arbitrage-free price of an asset is the price where an arbitrage strategy is not possible.

**1.2. Arbitrage-free Market.** In this paper we work under the assumption that the market is arbitrage free. As such, we claim that if the values of two portfolios are equal at time  $T > 0$ , then for all times  $\tau$  where  $0 \leq \tau \leq T$ , the values of both portfolios are equal.

We prove by contrapositive. Assume that at time  $T > 0$  the prices of two portfolios are equal, and that at time  $\tau$  where  $0 \leq \tau \leq T$  that one portfolio is worth more than the other. Let  $P_1$  be the value of portfolio 1 and  $P_2$  be the value of portfolio 2. WLOG, at time  $\tau$ , let  $P_1 > P_2$ . At time  $\tau$ , we buy portfolio 2 and sell portfolio 1. We can pocket the difference  $P_1 - P_2$ . At time  $T$ , we can then sell portfolio 2 and use that money to pay off what we owe from portfolio 1. Thus, there exists an arbitrage strategy, a strategy with zero initial capital and a strictly positive probability of profit.

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*Key words and phrases.* Blachelier model, Chooser options, Exotic options.

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By contrapositive, it thus follows that under an arbitrage-free model, if two portfolios have equal value at time  $T$ , they must have equal value at all times from 0 to  $T$ .

**1.3. The Blachelier Model.** In this paper we work within the context of the *Blachelier model*, where the stock prices  $\{S_t\}_{t \geq 0}$  evolves according to

$$S_t = e^{rt} \left( S_0 + \kappa^{-rt} W_t + \kappa r \int_0^t e^{-rs} W_s ds \right), \quad (1.1)$$

where  $S_0 > 0$  denotes the initial stock price at time 0 and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure  $\tilde{\mathbb{P}}$ . We note to the reader that in the special case when  $r = 0$ , (1.1) reduces to

$$S_t = S_0 + \kappa W_t. \quad (1.2)$$

(FIXME: explain to the reader why the Blachlier model is interesting). Recently, in April 2020, oil prices finished in the negatives. The often used Black-Scholes model, however, is unable to handle negative prices, which sparked interest in the Bachelier model, a mathematically similar model with the advantage of being able to handle negative prices.

**1.4. Notation and conventions.** For a random variable  $X$  we use the notation  $X^+$  to denote the random variable  $\max(X, 0)$ . We note that by definition, we have

$$X = X^+ - (-X)^+, \quad (1.3)$$

from which the *put-call parity* can be derived.

(FIXME: some common notation that needs to be explained:

- (1) the notation  $X \sim N(\mu, \sigma^2)$
- (2) ...

)

## 2. ARBITRAGE-FREE PRICING UNDER THE BLACHELIER MODEL

In this section we derive the arbitrage-free prices of some contracts in the simplest setting when  $r = 0$ .

**2.0.1. European Call.** We first consider a European call where the payoff at time  $T$  is given by

$$V_T = (S_T - K)^+ \quad (2.1)$$

for a fixed strike price  $K$ . We note that under  $\tilde{\mathbb{P}}$ ,  $W_T \sim N(0, T)$ , therefore

$$S_T \sim N(S_0, \kappa^2 T) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.2)$$

According to the risk neutral pricing formula, the time-0 price of this security is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+]. \quad (2.3)$$

Recall that if we have a random variable  $X$  with probability density function  $f_X$  under a probability measure  $\mathbb{P}$ , then the “law of the unconscious statistician” tells us that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \quad (2.4)$$

In our setting, we have

$$g(S_T) = (S_T - K)^+, \quad (2.5)$$

and the distribution of  $S_T$  under  $\tilde{\mathbb{P}}$  as a random variable is given in (2.2). Therefore, the time-zero price  $V_0$  is given by

$$V_0 = \tilde{\mathbb{E}}[(S_T - K)^+] = \tilde{\mathbb{E}}[g(S_T)] = \int_{-\infty}^{\infty} (x - K)^+ \psi(x) dx, \quad (2.6)$$

where

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right), \quad \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad (2.7)$$

and

$$\mu = S_0, \quad \nu = \kappa \sqrt{T}. \quad (2.8)$$

To compute (2.6), we first note that since  $(x - K)^+ = 0$  for  $x \leq K$ , the domain of integration is the set  $\{x \mid x \geq K\}$ . Now we use the change of variables

$$y = -\frac{x - \mu}{\nu} \iff x = \mu - \nu y, \quad (2.9)$$

and we note that since  $\nu > 0$ ,

$$x \geq K \iff \frac{x - \mu}{\nu} \geq \frac{K - \mu}{\nu} \iff y \leq \frac{\mu - K}{\nu} =: d_-. \quad (2.10)$$

Then by performing a change of variables, (2.6) becomes

$$V_0 = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(-y) dy = \int_{-\infty}^{d_-} (\nu y + K - \mu) \varphi(y) dy = \underbrace{\int_{-\infty}^{d_-} \nu y \varphi(y) dy}_{:=I} + \underbrace{\int_{-\infty}^{d_-} (K - \mu) \varphi(y) dy}_{:=II}. \quad (2.11)$$

We define the cumulative distribution function of a standard normal random variable  $X$  under  $\mathbb{P}$  via

$$\varphi(x) = \mathbb{P}[X \leq x] = \mathbb{E}[\mathbb{1}_{X \leq x}] = \int_{-\infty}^x \varphi(y) dy. \quad (2.12)$$

With this notation in hand, we can write

$$II = (K - \mu) \int_{-\infty}^{d_-} \varphi(y) dy = (K - \mu) \varphi(d_-), \quad (2.13)$$

and

$$I = \nu \int_{-\infty}^{d_-} y \varphi(y) dy = \frac{\nu}{\sqrt{2\pi}} \lim_{t \rightarrow -\infty} (e^{-t^2/2} - e^{-d_-^2/2}) = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2}. \quad (2.14)$$

Therefore

$$V_0 = -\frac{\nu}{\sqrt{2\pi}} e^{-d_-^2/2} + (K - \mu) \varphi(d_-). \quad (2.15)$$

To compute the price of a put, one can use put-call parity (FIXME: derive put-call parity somewhere and reference it).

2.0.2. *Arithmetic Asian calls.* Next we consider an *arithmetic Asian call* where the payoff at time  $T$  is given by

$$V_T = (A_T - K)^+, \quad A_T = \frac{1}{T} \int_0^T S_t dt = S_0 + \frac{\kappa}{T} \int_0^T W_t dt. \quad (2.16)$$

Using tools from stochastic calculus, one can show that under the risk neutral measure  $\tilde{\mathbb{P}}$ ,

$$\int_0^T W_t dt \sim N(0, T^3/3). \quad (2.17)$$

Therefore we have

$$A_T \sim N(S_0, \kappa^2 T/3) \text{ under the risk neutral measure } \tilde{\mathbb{P}}. \quad (2.18)$$

Comparing this to (2.2), we see that  $A_T$  has a similar distribution, the only difference is that the variance of  $A_T$  is smaller by a factor of  $1/3$ , so the standard deviation of  $A_T$  is smaller by a factor of  $1/\sqrt{3}$ . By performing the exact same set of calculation as before, the time-0 price of an Asian option is

$$V_0 = -\frac{\nu}{\sqrt{3}\sqrt{2\pi}} e^{-3d_-^2/2} + (K - \mu) \varphi(\sqrt{3}d_-), \quad (2.19)$$

where

$$\mu = S_0, \quad \nu = \kappa\sqrt{T}, \quad d_- = \frac{\mu - K}{\nu}. \quad (2.20)$$

We note that since  $\sqrt{3} > 1$ , we see from (2.19) that the price of an Asian option is higher than the price of a European call. This should be expected as one is paying a premium for a less volatile product.

### 3. CHOOSER OPTIONS

In this section we consider a more complicated type of financial contracts known as *chooser options*. These are contracts with a fixed maturity date  $T$  and a strike price  $K$ , and an agent is allowed to decide on a choosing date  $\tau < T$  to choose the underlying derivative in the contract. Many results are known when an agent is allowed to choose between a European call and a European put; we are interested in a variant of this type of contract that allows an agent to decide between two securities that pays

$$C_T = (A_T - K)^+, \quad P_T = (K - A_T)^+, \quad (3.1)$$

where  $A_T$  is defined via (2.16). Here, we assume the agent chooses optimally with no outside information. At time  $\tau$ , the agent will choose the option of higher value between the put and call, therefore the value of this contract at time  $\tau$  is

$$V_\tau = \max(C_\tau, P_\tau). \quad (3.2)$$

The time-zero price of this contract is then

$$V_0 = \tilde{\mathbb{E}}[V_\tau]. \quad (3.3)$$

In the next subsection, we simplify the expression for  $V_\tau$  via the method of replication.

**3.1. Replication.** We first note that by properties of the max function (FIXME: maybe add this in intro and reference?), we can write

$$V_\tau = C_\tau + \max(0, P_\tau - C_\tau). \quad (3.4)$$

By (1.3), we have

$$P_T - C_T = (K - S_T)^+ - (S_T - K)^+ = K - A_T. \quad (3.5)$$

Combining (3.2) and (3.5) then gives us

$$V_\tau = C_\tau + \max(0, K - A_T). \quad (3.6)$$

Next, we identify the time- $\tau$  prices of contracts paying  $P_T - C_T$  and  $K - A_T$  at time  $T$ .

To replicate a security with payoff  $P_T - C_T$ , we consider a portfolio that longs a put and shorts a call at time 0, both with maturity  $T$  and strike  $K$  (FIXME: what's the maturity?). To replicate a security with payoff  $K - A_T$ , we consider a portfolio investing  $Ke^{-rT}$  into the money account at time 0 and shorting a contract (FIXME: what kind of contract?) (FOLLOWUP I'm tempted to say ZCB but I'm not sure) to receive  $A_T$  at time  $T$ .

Since both portfolios have the same payoff at time  $T$  by (3.5), they have the same price for all times  $t$  where  $0 \leq t \leq T$  (FIXME: in the intro explain why this is true in terms of arbitrage). (FOLLOWUP added to section 1.2 in intro, idk how to ref)

For any  $t$  satisfying  $0 \leq t \leq T$ , we define  $P_t$  to be the time- $t$  value of a put with payoff  $P_T$  at time  $T$ ,  $C_t$  to be the time- $t$  value of a call with payoff  $C_T$  at time  $t$ ,  $A_t$  to be the time- $t$  value of an asian option paying  $A_T$  at time  $t$ . (FIXME: maybe move to intro?)

Using this notation, at time  $\tau$  the value of the first portfolio is  $P_\tau - C_\tau$ . Also, at time  $\tau$  the second portfolio has  $Ke^{-rT+r\tau}$  in the bank and is shorting a contract which pays  $A_T$  at  $T$ , therefore the time- $\tau$  value of the second portfolio is  $Ke^{r(\tau-T)} - A_\tau$ . We denote the value of a contract at time  $\tau$  which pays  $A_T$  at time  $T$  as  $w_\tau$ . By replication, the time  $\tau$  prices of the portfolios are equal, therefore we have

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - w_\tau. \quad (3.7)$$

Substituting this result back into (3.2), the value of the original chooser contract at  $\tau$  is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - w_\tau). \quad (3.8)$$

Our next goal is to find an explicit formula for  $w_\tau$ .

For simplicity, we define  $U_\tau$  to be the time  $\tau$  price of a contract with payoff  $Y_T$  at time  $T$ , where  $Y_T$  is defined via

$$Y_T = \int_0^T S_t dt. \quad (3.9)$$

Once  $U_\tau$  is determined, then we can recover  $w_\tau$  as  $w_\tau = \frac{U_\tau}{T}$ .

Note that (3.9) can be split into two parts,

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt. \quad (3.10)$$

Observe that the integral from 0 to  $\tau$  is known at time  $\tau$  as each price  $S_t$  will be known by the time  $\tau$ . So we can treat this integral as a constant and now try to replicate the integral from time  $\tau$  to  $T$ .

**3.2. Replicating Asian options when  $r > 0$ .** We begin our replicating strategy by buying  $x$  shares of stock at time  $\tau$ . For all times  $t$  where  $\tau \leq t \leq T$ , we will continuously sell off stock at the rate  $\alpha_t$  and invest the revenue. With this strategy, at time  $T$ , the bank has

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt \quad (3.11)$$

To finish the replication, we want our replicating portfolio to be equal to the integral we are replicating:

$$\int_\tau^T \alpha_t S_t e^{r(T-t)} dt = \int_\tau^T S_t dt. \quad (3.12)$$

Solving for  $\alpha_t$ , we find that

$$\alpha_t = e^{r(t-T)} \quad (3.13)$$

Thus, the amount of shares our strategy started with was

$$x = \int_\tau^T e^{r(t-T)} dt = \frac{1}{r} - \frac{e^{r(\tau-T)}}{r}. \quad (3.14)$$

This tells us that the cost at time  $\tau$  to receive the stock from times  $\tau$  to  $T$  continuously is  $xS_\tau$ . This gives us

$$U_\tau = \int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)}) \quad (3.15)$$

Recall that  $w_\tau$  is the price at time  $\tau$  to receive  $A_T$ , equivalent to  $\frac{Y_T}{T}$ , at time  $T$ . Thus, the price at  $\tau$  to receive just  $A_T$  is

$$w_\tau = \frac{U_\tau}{T} = \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T} \quad (3.16)$$

Returning to (3.7) (FIXME: fix reference), we can write out the equation as

$$P_\tau - C_\tau = Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T}. \quad (3.17)$$

Substituting this into the chooser option from 1.7, the value of  $V_\tau$  is

$$V_\tau = C_\tau + \max(0, Ke^{r(\tau-T)} - \frac{\int_0^\tau S_t dt + \frac{S_\tau}{r} (1 - e^{r(\tau-T)})}{T}) \quad (3.18)$$

**3.3. Replicating Asian options when  $r = 0$ .** We now consider the case when  $r = 0$ . Observe we cannot plug  $r = 0$  into the formula we got for  $r > 0$  since we divide by  $r$ . However, we can apply a similar replication argument as before. (FIXME: this is a little confusing since the interest rate was never specified. maybe specify that you are assuming  $r > 0$  initially and now you're considering  $r = 0$  as a special case. If you choose to do this I recommend breaking this section into different subsections). To fix this, we return to our replicating strategy for  $w_\tau$  accounting for this special case.

Define  $U_\tau$  and  $Y_T$  the same way as above. Again, split the integral  $Y_T$  such that

$$Y_T = \int_0^\tau S_t dt + \int_\tau^T S_t dt \quad (3.19)$$

We now replicate the integral from time  $\tau$  to  $T$  for the special case. We follow the same replicating strategy as before. Purchase  $x$  shares of stock. For all times  $t$  where  $\tau \leq t \leq T$ , we continuously sell off at the rate  $\alpha_t$  and invest the revenue. By time  $T$ , the bank will have

$$\int_\tau^T \alpha_t S_t dt \quad (3.20)$$

We finish the replication by setting this equal to the value we're replicating

$$\int_{\tau}^T \alpha_t S_t dt = \int_{\tau}^T S_t dt \quad (3.21)$$

Solving for  $\alpha_t$ , we see that when  $r = 0$  that  $\alpha_t = 1$ . Thus, the number of shares the strategy started with was

$$\int_{\tau}^T dt = T - \tau \quad (3.22)$$

Similar to the  $r \neq 0$  case, it then follows that

$$U_{\tau} = \int_0^{\tau} S_t dt + S_{\tau}(T - \tau), \quad w_{\tau} = \frac{U_{\tau}}{T} = \frac{\int_0^{\tau} S_t dt + S_{\tau}(T - \tau)}{T} \quad (3.23)$$

Thus, Put-Call Parity in the special case tells us that

$$P_{\tau} - C_{\tau} = K - \frac{U_{\tau}}{T} = \frac{\int_0^{\tau} S_t dt + S_{\tau}(T - \tau)}{T}. \quad (3.24)$$

Substituting this result into the chooser option formula, we have

$$V_{\tau} = C_{\tau} + \max(0, K - \frac{\int_0^{\tau} S_t dt + S_{\tau}(T - \tau)}{T}) \quad (3.25)$$

Note that when the interest rate is 0, the stock prices evolve according to

$$S_t = S_0 + \kappa W_t \quad (3.26)$$

where  $S_0 > 0$  and  $\{W_t\}_{t \geq 0}$  is a Brownian motion under the risk neutral measure. We can now rewrite our chooser option formula as

$$V_{\tau} = C_{\tau} + \max(0, K - \frac{\int_0^{\tau} (S_0 + \kappa W_t) dt + (S_0 + \kappa W_{\tau})(T - \tau)}{T}) \quad (3.27)$$

So in conclusion, we find that

$$V_{\tau} = C_{\tau} + \left( K - S_0 - \frac{\kappa(T - \tau)}{T} W_{\tau} - \frac{\kappa}{T} \int_0^{\tau} W_t dt \right)^+. \quad (3.28)$$

Then by the risk-neutral pricing formula and the linearity of expectation, the time-zero price  $V_0$  is given by

$$V_0 = \tilde{\mathbb{E}}[C_{\tau}] + \tilde{\mathbb{E}} \left[ \left( K - S_0 - \frac{\kappa(T - \tau)}{T} W_{\tau} - \frac{\kappa}{T} \int_0^{\tau} W_t dt \right)^+ \right]. \quad (3.29)$$

Let  $X$  be the random variable defined via

$$X = \frac{\kappa(T - \tau)}{T} W_{\tau} + \frac{\kappa}{T} \int_0^{\tau} W_t dt. \quad (3.30)$$

We now calculate the mean and variance of random variable  $X$ . We define  $X$  as the sum of random variables

$$Y = \frac{\kappa(T - \tau)}{T} W_{\tau} \quad (3.31)$$

$$Z = \frac{\kappa}{T} \int_0^{\tau} W_t dt. \quad (3.32)$$

Recall from the intro (add details on brownian motion in intro later) that the mean of the brownian motions in both  $Y$  and  $Z$  are 0, thus the means of both  $Y$  and  $Z$  are 0. We now calculate the variance of  $X$  as the sum of two random variables

$$\text{Var}(X) = \text{Var}(Y + Z) \quad (3.33)$$

It is known that

$$\text{Var}(Y + Z) = \text{Var}(Y) + \text{Var}(Z) + 2\text{Cov}(YZ). \quad (3.34)$$

Recall from the intro (add later) the variances of brownian motion. It follows that

$$\text{Var}(Y) = \tau \left( \frac{\kappa(T - \tau)}{T} \right)^2 \quad (3.35)$$

$$\text{Var}(Z) = \frac{\tau^3}{3} \left( \frac{\kappa}{T} \right)^2 \quad (3.36)$$

To calculate the covariance term, we expand it out in terms of expected value. Recall that the expected values of the brownian motions are 0.

$$Cov(XY) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY) \quad (3.37)$$

We can now rewrite the covariance as

$$Cov(XY) = \mathbb{E}(W_\tau \int_0^\tau W_t dt) \frac{\kappa^2(T - \tau)}{T^2} \quad (3.38)$$

For simplicity, let  $\alpha = \frac{\kappa^2(T - \tau)}{T^2}$ . By a property of integrals and expected value, we can move the integral outside the expected value as such (probably need to fix this).

$$\alpha \mathbb{E}(W_\tau \int_0^\tau W_t dt) = \alpha \int_0^\tau \mathbb{E}(w_\tau w_t) dt \quad (3.39)$$

Observe that  $t \leq \tau$ . Thus, we can further simplify down to

$$\alpha \int_0^\tau \mathbb{E}((w_\tau + w_t - w_t)w_t) dt = \alpha \int_0^\tau \mathbb{E}(w_t^2 + (w_\tau - w_t)w_t) dt \quad (3.40)$$

We can expand the expected value by linearity of expectations. Recall (another brownian motion thing for intro ig) that the expected value of  $(w_\tau - w_t)w_t$  is 0 and that the expected value of  $w_t^2$  is  $t$ . Thus, we have

$$Cov(XY) = \alpha \int_0^\tau t dt = \alpha \frac{\tau^2}{2} \quad (3.41)$$

It follows that the mean and variance of  $X$  can be computed as (IMPORTANT FIXME: add details of this computation!)

$$\mu = \mathbb{E}(X) = 0 \quad (3.42)$$

$$\sigma^2 = Var(X) = \tau \left( \frac{\kappa(T - \tau)}{T} \right)^2 + \frac{\tau^3}{3} \left( \frac{\kappa}{T} \right)^2 + \tau^2 \frac{\kappa^2(T - \tau)}{T^2}. \quad (3.43)$$

$$\nu = \sigma \quad (3.44)$$

(FIXME: typically  $\sigma$  denotes the standard deviation, so this isn't very consistent with standard notation)

Using methods of stochastic calculus (FIXME: we need to find a reference to cite),  $X$  is normally distributed with mean  $\mu$  and variance  $\sigma^2$ . Therefore we can define the probability density function as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (3.45)$$

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right) \quad (3.46)$$

Now we can substitute back into our equation from (4.12) and use the definition of expectation on a continuous random variable to get

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (K - S_0 - x)^+ \psi(x) dx. \quad (3.47)$$

To integrate the second term in  $V_0$  we will let

$$z = \frac{x - \mu}{\sigma}. \quad (3.48)$$

it follows that

$$x = z\nu + \mu \quad (3.49)$$

$$dx = \nu dz. \quad (3.50)$$

We note that

$$K - S_0 - x \geq 0 \iff x \leq K - S_0 \iff \frac{x - \mu}{\sigma} \leq \frac{K - S_0 - \mu}{\nu}. \quad (3.51)$$

We define  $d_-$  via

$$d_- = \frac{K - S_0 - \mu}{\nu} \quad (3.52)$$

so by (3.51), we have

$$K - S_0 - x \geq 0 \iff z \leq d. \quad (3.53)$$

Now using (3.49) and (3.53) we can rearrange (3.47) as:

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \left( \int_{-\infty}^{d_-} (K - S_0 - z\nu - \mu) \varphi(z) dz \right). \quad (3.54)$$

Simplifying this expression we get the form

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \nu \int_{-\infty}^{d_-} z \varphi(z) \nu dz + (K - S_0 - \mu) \int_{-\infty}^{d_-} \varphi(z) dz \quad (3.55)$$

Excluding the call, we can easily simplify the latter of the terms by defining the cumulative distribution function

$$\Phi(x) = \int_{-\infty}^x \varphi(y) dy. \quad (3.56)$$

We resolve the former term by first substituting in (3.45)

$$\frac{-\nu}{\sqrt{2\pi}} \int_{-\inf}^{d_-} y e^{-y^2} = \frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} \quad (3.57)$$

Thus, our final simplified form is

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \frac{\nu}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} + (K - S_0 - \mu) \Phi(d_-) \quad (3.58)$$

(FIXME: continue with this computation, write down an explicit formula for  $V_0$ , and also the price for the variant.) (FOLLOWUP still have to write up variant)

#### 4. CHOOSER OPTION VARIANTS

**4.1. Asian Tail Choosers when  $r = 0$ .** We will now consider a variant of the Asian chooser we looked at earlier. We assume all conditions remain the same, except we now define  $A_{\tau,T}$  as

$$A_{\tau,T} = \int_{\tau}^T S_t dt \quad (4.1)$$

where  $\tau$  is the choice date, and  $T$  is the time of maturity.

To price this option, we slightly modify the replication strategy from before. Let  $Y_T = \int_{\tau}^T S_t dt$  and  $U_\tau$  be the price at  $\tau$  to receive  $Y_T$  at time  $T$ .

We proceed with the replication of  $Y_T$ . Suppose an agent purchases  $x$  shares at time  $\tau$ , and chooses to sell them off continuously at rate  $\alpha_t$  at time  $t$ . At time  $T$ , the agent's portfolio is worth

$$\int_{\tau}^T \alpha_t S_t e^{r(T-t)} dt \quad (4.2)$$

Since we assume here that  $r = 0$ , this reduces to

$$\int_{\tau}^T \alpha_t S_t dt \quad (4.3)$$

To complete the replication, we set this equal to the value we are trying to reproduce:

$$\int_{\tau}^T \alpha_t S_t dt = \int_{\tau}^T S_t dt \quad (4.4)$$

It follows that

$$\alpha_t = 1 \quad (4.5)$$

for all  $t$  where  $\tau \leq t \leq T$ . Thus,

$$x = \int_{\tau}^T \alpha_t dt = T - \tau. \quad (4.6)$$

It then follows that  $U_\tau = (T - \tau)S_\tau$ . Observe that

$$A_{\tau,T} = \frac{U_\tau}{T - \tau}. \quad (4.7)$$



Again using the notation  $w_\tau$  as the price needed at time  $\tau$  to receive  $A_{\tau,T}$  at time  $T$ , it follows that

$$w_\tau = \frac{(T - \tau)S_\tau}{T - \tau} = S_\tau. \quad (4.8)$$

Referring back to (3.7) and using  $r = 0$ , we have

$$P_\tau - C_\tau = K - S_\tau. \quad (4.9)$$

Using (3.4), the price of the tail chooser option with choice date  $\tau$ , which we write as  $V_\tau$ , is

$$V_\tau = C_\tau + (K - S_\tau)^+. \quad (4.10)$$

Recall in the Bachelier model that the stock evolves according to  $S_t = S_0 + \kappa W_t$  when  $r = 0$ ,  $S_0 > 0$ , and  $W_t$  is a brownian motion under the risk-neutral measure. Then,

$$V_\tau = C_\tau + (K - (S_0 + \kappa W_\tau))^+. \quad (4.11)$$

Applying the risk-neutral pricing formula and linearity of expectations, we have

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \tilde{\mathbb{E}}((K - S_0 - \kappa W_\tau)^+). \quad (4.12)$$

To simplify the above, define random variable  $X$  and function  $g(X)$  as

$$X = \kappa W_\tau, \quad g(X) = (k - S_0 - X)^+ \quad (4.13)$$

Applying the law of the unconscious statistician, we can express  $V_0$  as

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{\infty} (k - S_0 - X)^+ \psi(x) dx \quad (4.14)$$

where

$$\psi(x) = \frac{1}{\nu} \varphi\left(\frac{x - \mu}{\nu}\right) \quad (4.15)$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (4.16)$$

Let  $y = \frac{x - \mu}{\nu}$ . Observe that

$$y = \frac{x - \mu}{\nu} \implies x = y\nu + \mu \implies dx = \nu dy \quad (4.17)$$

We can now take the positive part of the integral from (4.14)

$$k - S_0 - x \geq 0 \implies -x \geq S_0 - k \quad (4.18)$$

Adding  $\mu$  and dividing by  $\nu$  on both sides,

$$-y = \frac{-x + \mu}{\nu} \geq \frac{S_0 - k + \mu}{\nu} \quad (4.19)$$

It follows that

$$y \leq \frac{k - S_0 - \mu}{\nu} = d_- \quad (4.20)$$

We now evaluate 4.14 using 4.17

$$V_0 = \tilde{\mathbb{E}}[C_\tau] + \int_{-\infty}^{d_-} (k - S_0 - y\nu - \mu) \left(\frac{1}{\nu} \varphi(y)\right) (-\nu) dy \quad (4.21)$$

Simplifying and splitting the integral, we have

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - \int_{-\infty}^{d_-} (k - S_0 - \mu) \varphi(y) dy + \int_{-\infty}^{d_-} y\nu \varphi(y) dy \quad (4.22)$$

Define the CDF the same as (3.56).

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - (k - S_0 - \mu) \Phi(d_-) + \int_{-\infty}^{d_-} y\nu \varphi(y) dy \quad (4.23)$$

The remaining integral term can be simplified through (4.16).

$$\nu \int_{-\infty}^{d_-} y \varphi(y) dy = \nu \int_{-\infty}^{d_-} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = -\frac{\nu}{2\pi} e^{-\frac{d_-^2}{2}} \quad (4.24)$$

The final form for the time 0 price of the tail chooser is then

$$V_0 = \tilde{\mathbb{E}}[C_\tau] - (k - S_0 - \mu)\Phi(d_-) - \frac{\nu}{2\pi} e^{\frac{-d_-^2}{2}} \quad (4.25)$$

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## APPENDIX A. VARIOUS REPLICATING STRATEGIES

### A.1. Replicating European options.

#### A.2. Replicating Asian options. CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA

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