A Generalized Method for Generating N-fold Random Joint Distributions from Observations

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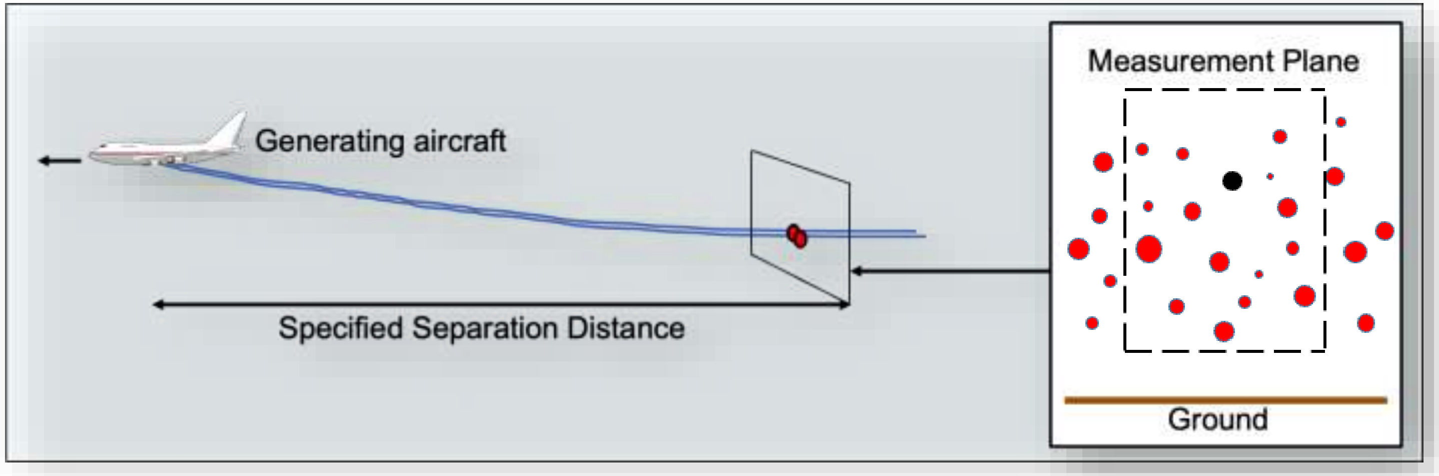
Outline

- **Motivation / problem**
- **Existing approaches**
- **Method & implementations**
- **Practical aspects and areas for improvement**
- **Summary**

Motivation

- **Planes encountering wake vortices can experience loss of control / potentially fatal outcomes**
- **FAA investigating using physics-based models to establish safe separations**
  - **Modeling performed on a site-by-site basis for top NAS airports**
  - **Monte Carlo techniques used to assess risk (~10 million runs)**
  - **Requires joint probability distributions of: landing weight, landing speed, winds (headwind, crosswind), temperature (density), stability (dϴ/dz), turbulence (Eddy Dissipation Rate (EDR))**



Graphic courtesy of NorthWest Research Association

EDR = Eddy Dissipation Rate
FAA = Federal Aviation Association
NAS = National Airspace System

Base Datasets for EDR Calculations

US Department of Energy Atmospheric Radiation Measurement (ARM) sites

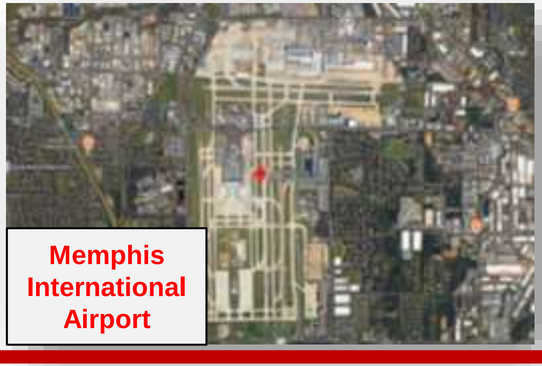
- Cape Cod
  - July 2012 – June 2013
  - Lidar observations
    - Vertical stares
    - Wind profiles
  - Surface observations
  - 8.6 K EDR values (1 / hour)

- Southern Great Plains (SGP) network
  - Lidar observations: 2010-present
    - Vertical stares: 1 Hz, from 105 m
    - Wind profiles: 15 min resolution, every 25 m from 90 m
  - Meteorological tower observations: 2015-present
    - Sonic anemometer winds: 10 Hz at 4, 25, and 60 m
  - Surface observations: 1993-present
    - 1 min resolution
  - 16.3 K EDR values (2 years @ 1 / hour)
  - 182.2 K EDR values (2 years @ 1 / 5 min)

Kansas
Southern Great Plains Site
Oklahoma

NWRA NorthWest Research Associates Since 1984

- NASA Memphis Dataset
  - May 2013 – March 2015
  - Lidar observations
  - Met tower
  - Surface observations
  - 175.4 K observation times (limited to aircraft landings)
  - 60.9 K EDR values



EDR = Eddy Dissipation Rate

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Existing Approaches

Oversample The Observations

- Easy to implement
- Method lacks robustness, especially if number of desired samples >> number of observations
  - 10M samples from 150K observations nominally results in each observation being repeated 67 times
  - Does not allow for realistic though unobserved scenarios

Treat Variables Independently

- Would potentially result in over-representation of unrealistic combinations of parameters
  - Approach ignores meteorological dependencies and interrelationships
  - E.g., high dissipation rate (turbulence) values are much less likely during very stable conditions

Sample from Idealized / Fitted Distributions

- Can efficiently address the robustness issue
- May be difficult to find the appropriate ideal distributions
- May lose desired small-scale distribution characteristics

A robust method which preserves the observed joint relationships between variables is needed

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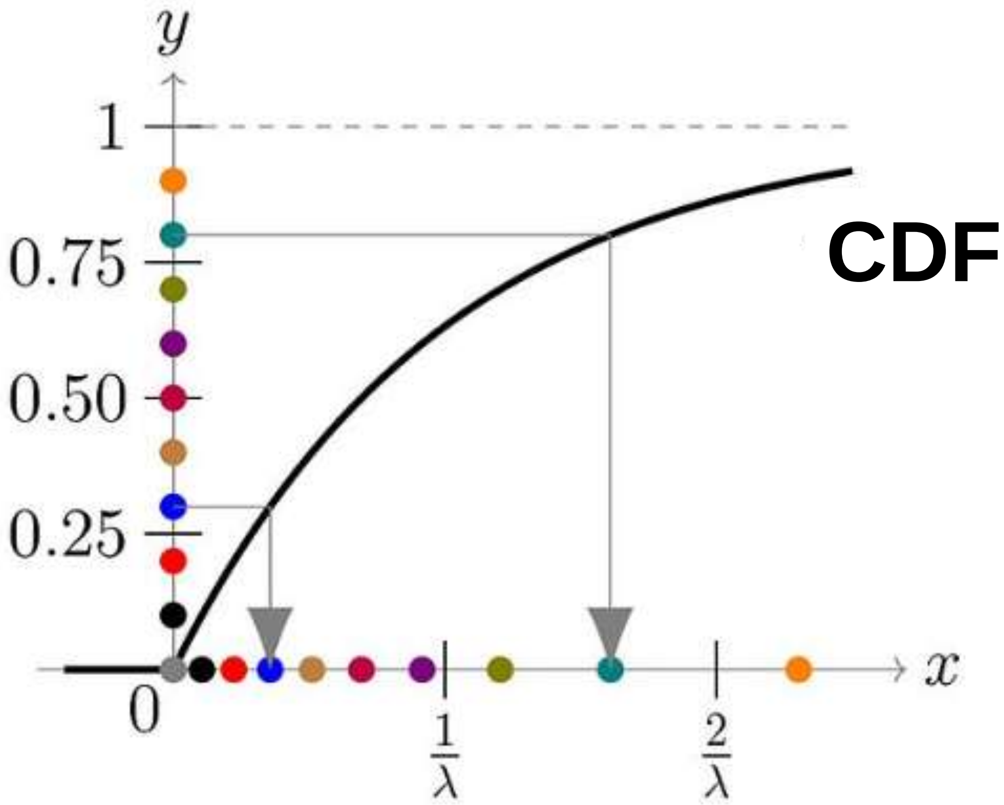
Basis of Method: Inverse Transform Sampling

- **Works backwards from a flat sampling of cumulative distribution function (CDF) values to the variable source values**
- **1-Dimensional illustration:**
  - **Compute the CDF of the observations, which has a range [0, 1]**
  - **Generate a set of random numbers of the desired sample size from a uniform distribution over the range [0, 1]**
  - **These values can be then mapped to their corresponding data values, yielding a realistic random sample of the original distribution**

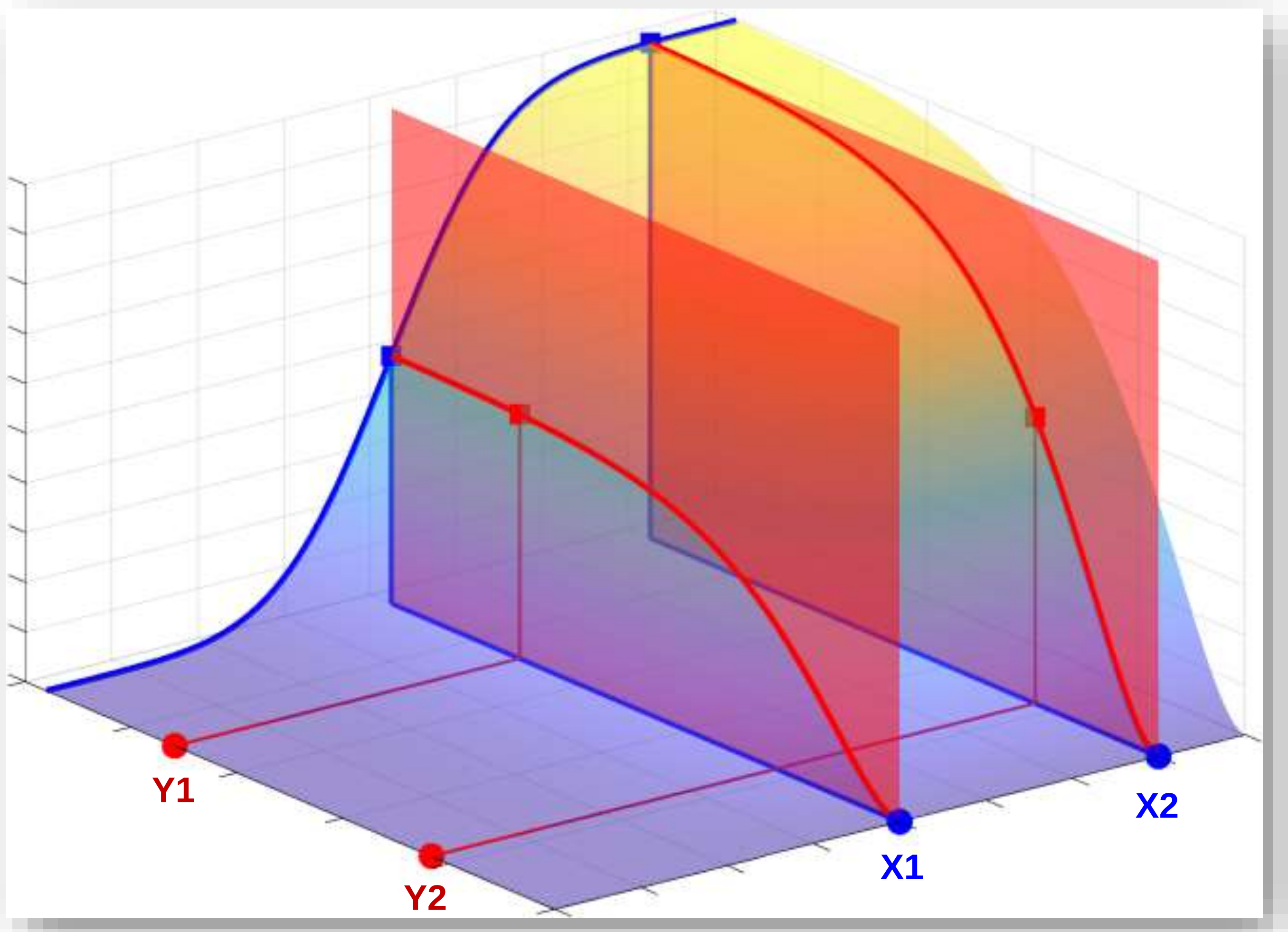


CDF = Cumulative Distribution Function

Extension of Method to Two Dimensions

- **Apply the 1D method to one of the marginal distributions to yield random values for that variable (X)**
- **For each selected random value of X:**
  - **Take a slice of the 2D CDF at that value to yield a CDF of Y**
  - **Apply the 1D method to yield random values of Y for each value of X**
- **This results in random values of X and Y that preserve the original joint relationship**

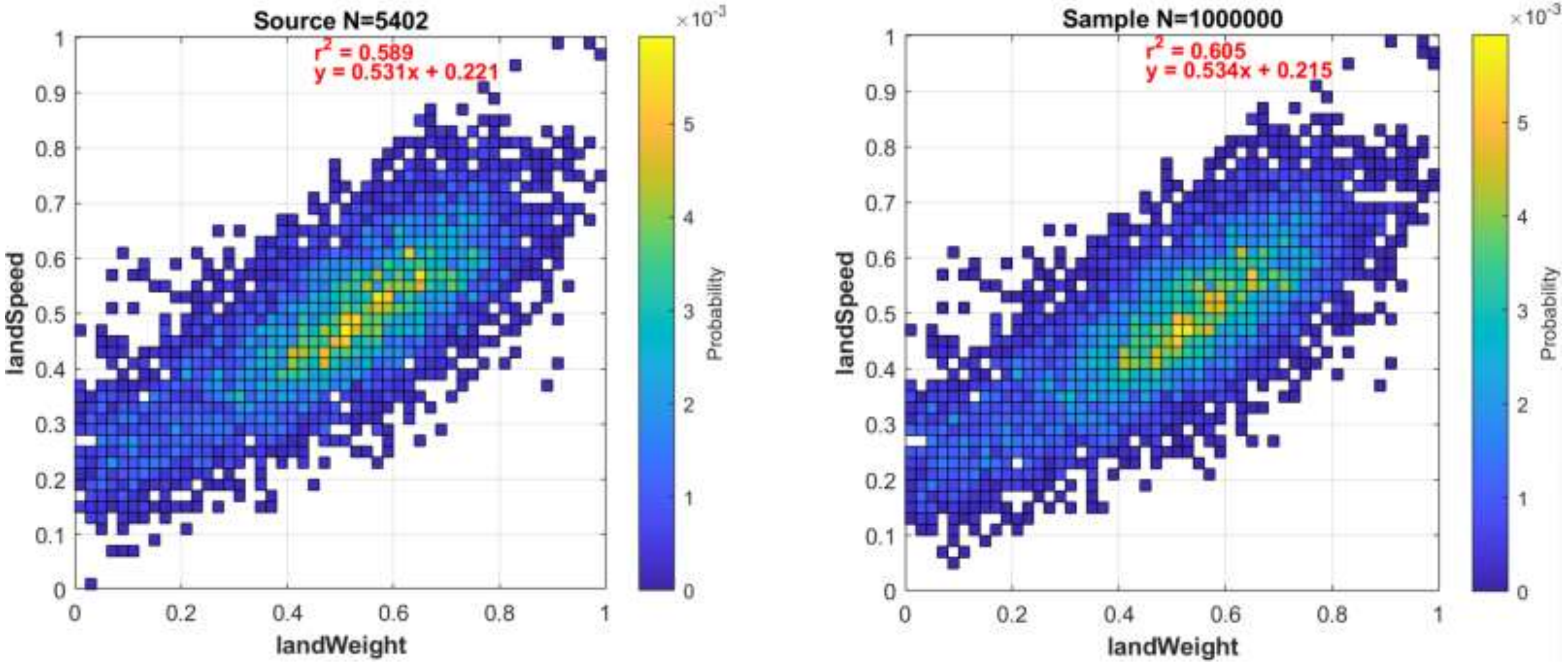
Illustration for Two Random Samples



CDF = Cumulative Distribution Function

Two-Dimensional Application

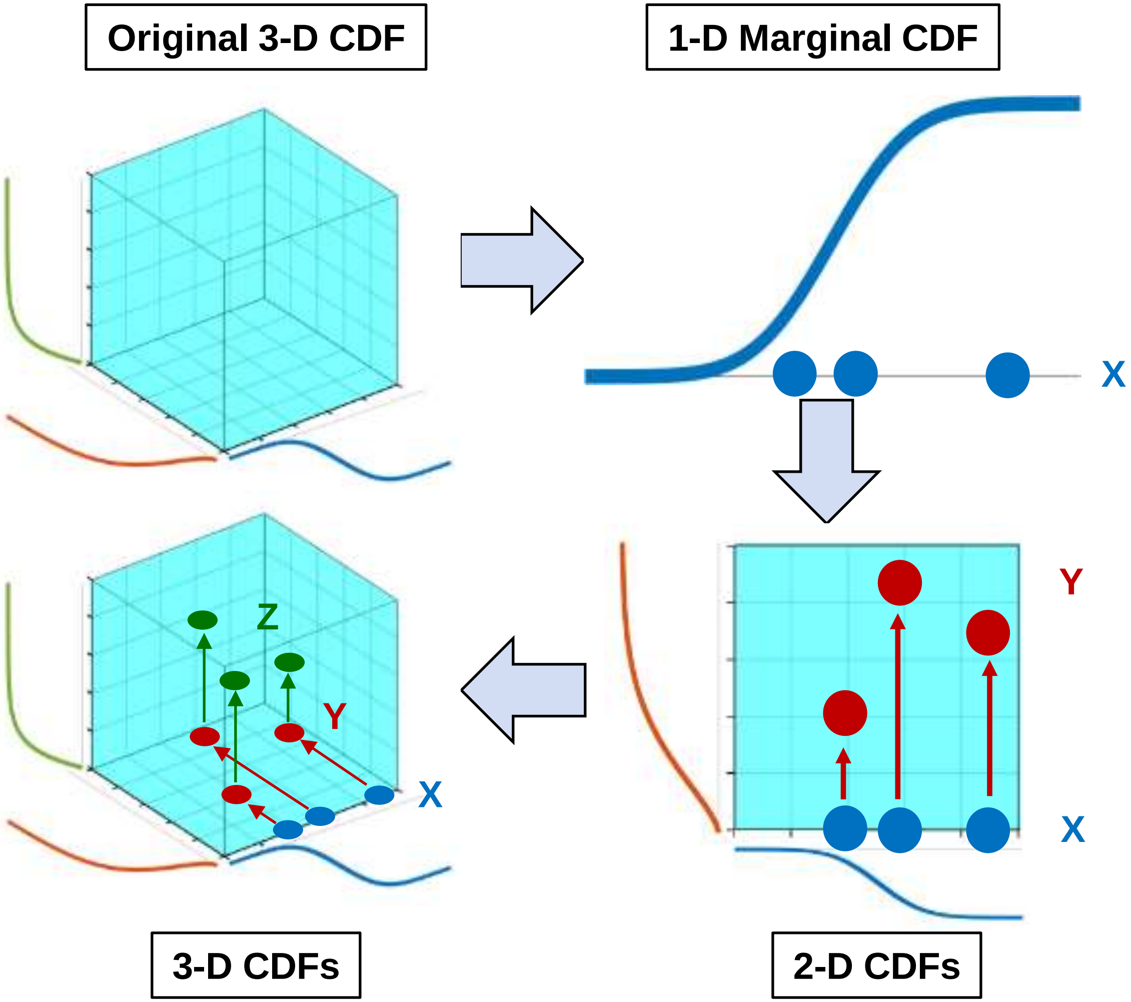
- **Modeling task required random distributions of landing weight and landing speed for selected aircraft**
  - **Joint relationship is required**
- **Data is sampled on a discrete basis, by binning across the available ranges**
  - **~100 bins provide adequate resolution**
- **Generated joint distribution of 1M pairs closely matches the characteristics of the source distribution**



(Data is scaled for display purposes)

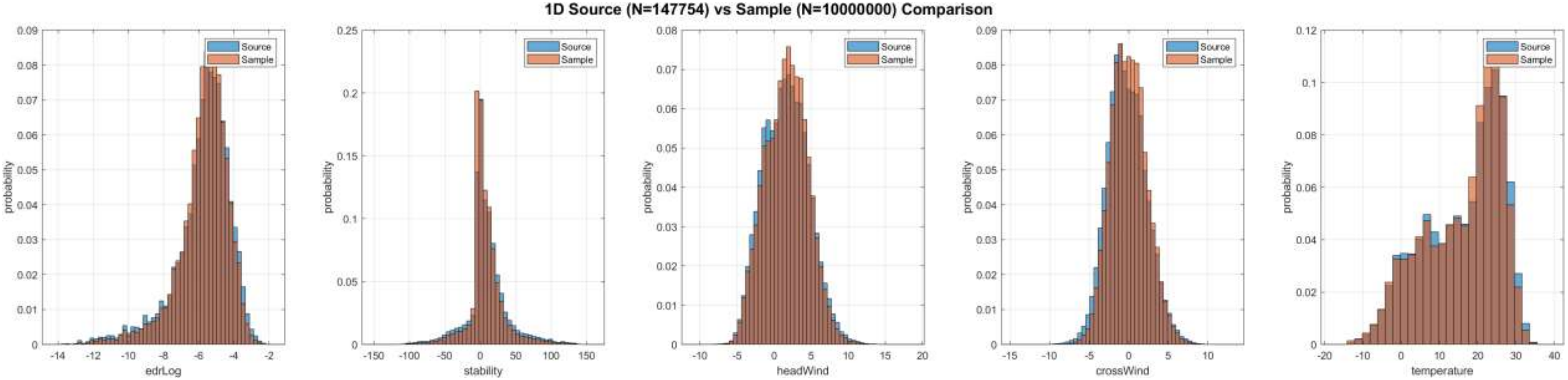
Extension to Three Dimensions and Beyond

- Generate matrix of random index values from a uniform distribution of size (N desired samples) x (M variables)
- Compute marginal CDF along 1 of the dimensions, determine random sample for variable 1
- Expand the CDF to 2 dimensions, employ slices at each value of variable 1 to determine random sample for variable 2
- Expand the CDF to 3 dimensions, employ slices at each combination of variable 1 and variable 2 values to determine random sample for variable 3
- Extend as needed to desired number of dimensions



CDF = Cumulative Distribution Function
PDF = Probability Density Function

Fidelity of Individual Aspect of Distributions


1D Source (N=147754) vs Sample (N=10000000) Comparison

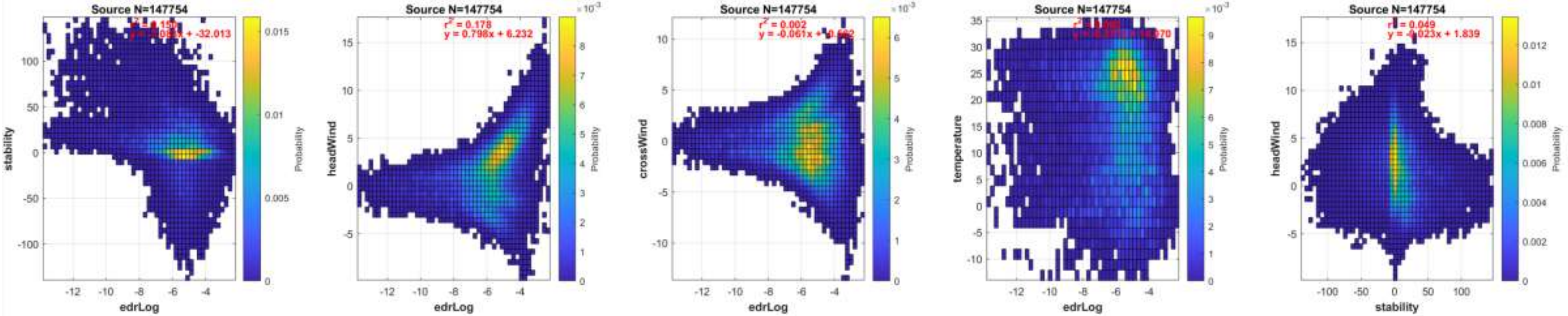
Marginal distributions show generally very good agreement, some discrepancies are noted, especially sample over-emphasis near the distribution peaks

Fidelity of Joint Aspect of Distributions

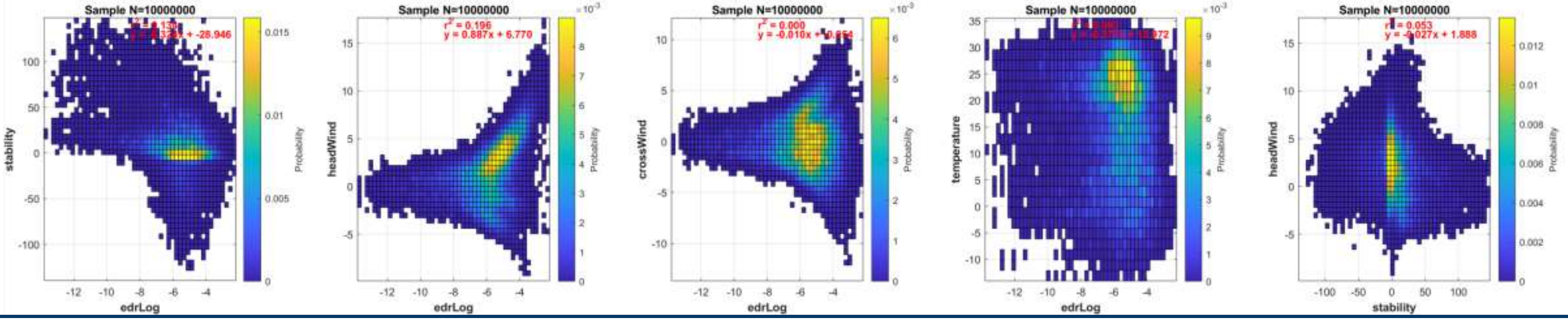
Visualized through examining the 10 2-way combinations of the 5 environmental variables

Generally, very good agreement seen across all variable combinations

Source (N=147754)



Sample (N=10M)



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Operational Aspects and Improvements

- **Specs**
  - 10M joint 5-fold environmental and 2-fold operational distributions
  - Largest CDF matrix (100 x 100 x 100 x 100 x 50): 37.3 GB
  - ~20 min clock (wall) time
    - Intel Core i9-11950H @ 2.60GHz (8 cores), 64 GB RAM

- **Potential Improvements**
  - Parallelization and other efficiency upgrades
    - Exploit MATLAB “big data / tall array” functionality
  - Implement quantitative assessments of closeness
  - Reduce binning artifacts; enable continuous samples

Public Availability

- **Tool is being made available for public use**
- **Is currently in the approval process**
- **Will be found on the Matlab File Exchange / github**
  - **Check for “N-Dimensional Joint Distribution Simulator”**
  - **Or email cde@ll.mit.edu or frankr@ll.mit.edu to get the link when ready**

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Summary

- In response to Monte Carlo aircraft wake modeling needs, a generalized tool was developed to provide large joint N-variable random distributions from observations

- The basis of the tool is inverse transform sampling, whereby samples from a uniform distribution are used to index into an empirical CDF to yield realistic distributions of the base variable(s)

- The tool was successfully used to generate 10M operational (2 variables) and environmental (5 variables) joint random samples whose distributions matched closely those of the originating observations

- The tool has been packaged for general use and is publicly available

Extra Slides

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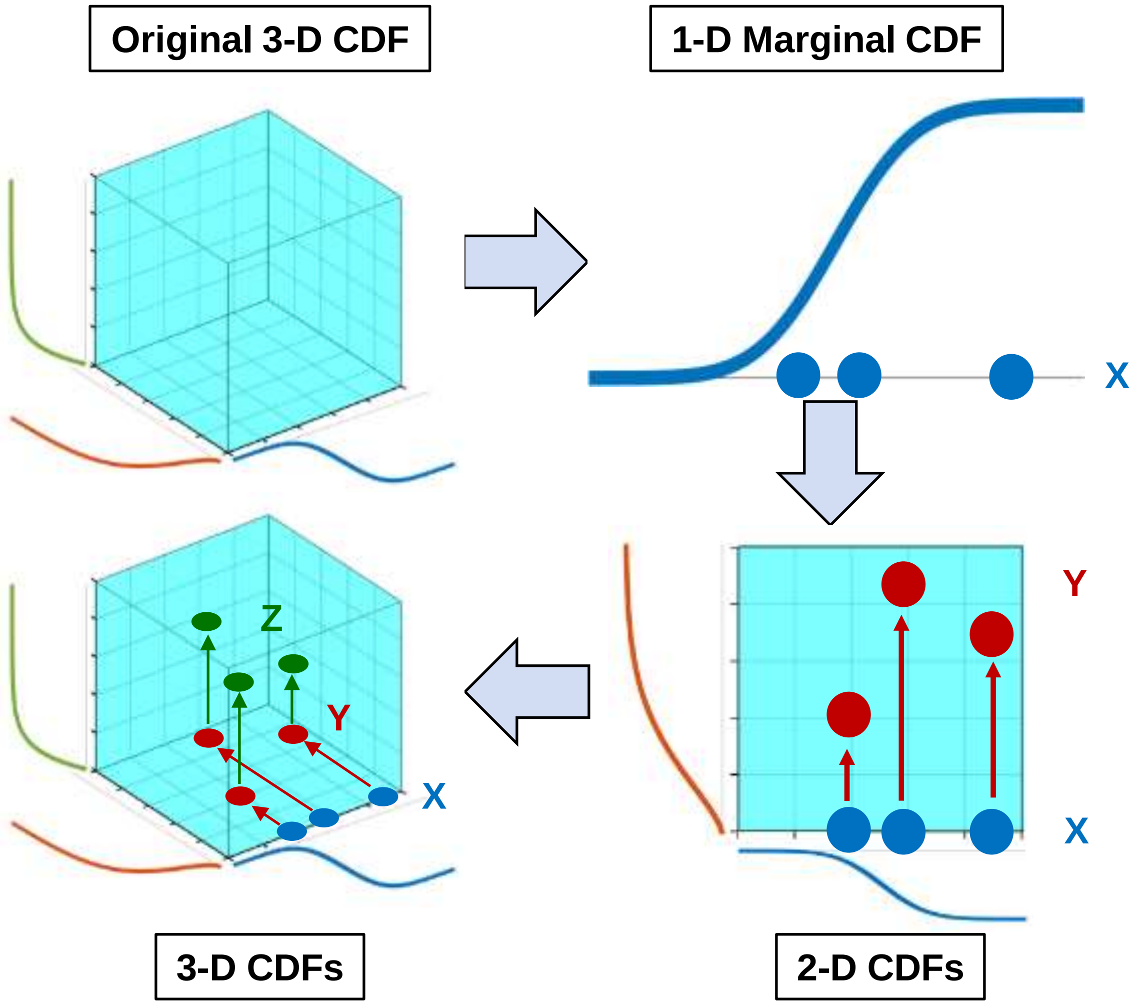
Extension to Three Dimensions and Beyond (detail)

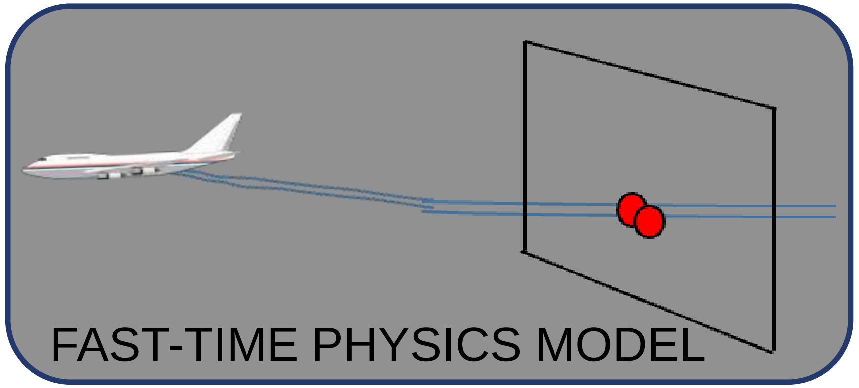
- Generate matrix of random values R from a uniform distribution of range [0,1] and size (N desired samples) x (M variables)
- Compute marginal CDF for 1 of the dimensions. Use random indices for the 1st variable of R (R1) to find corresponding random values X.
- Determine CDF slices across the remaining (2nd & 3rd) dimensions of the original PDF for each of the values in X. Use the corresponding random indices in R2 to determine Y.
- Repeat with the PDF/CDF across the last (3rd) dimension of the original PDF for each of the value pairs (X,Y). Use the corresponding random indices in R3 to generate Z
- The set {X,Y,Z} comprise the final joint distribution of random values
- Method is extensible to M>3 variables



CDF = Cumulative Distribution Function
PDF = Probability Density Function

5-Dimensional Implementation: Problem (detail)

- **Wake behavior modeling task required joint random distributions of turbulence dissipation rate, stability, headwind and crosswind (on arrival), and temperature**
- **Binning/resolution adjustable to the requirements of each variable**
  - **100 bins for all except temperature (50)**
  - **Resolution was deemed adequate for this task**
- **Turbulence has an exponential distribution, sampling is done on a log basis for improved resolution**
- **Generated joint distribution of 10M instances closely matches the characteristics of the source distribution when viewed across all 2-way combinations**



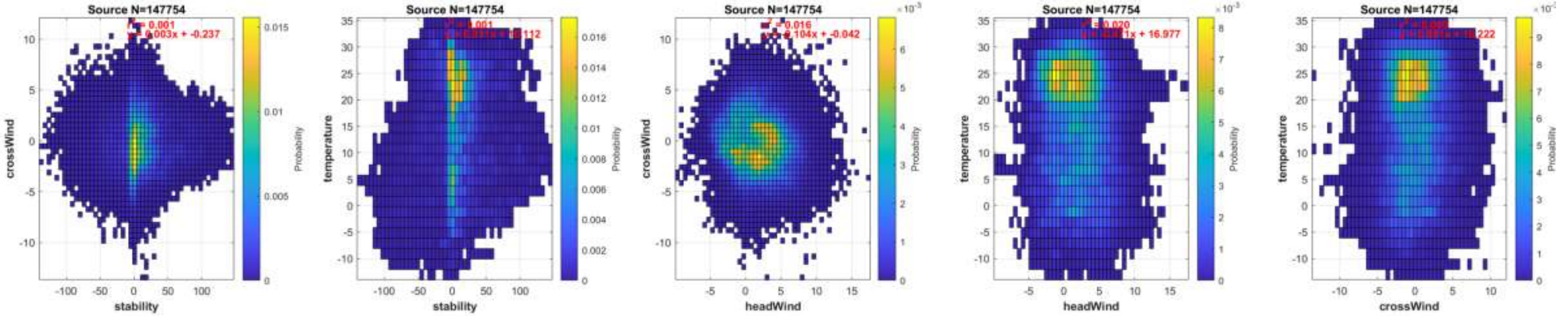
Variable	N Bins	Resolution	Units
landing weight	100	1010	lbs
landing speed	100	0.23	kts
eddy dissipation rate	100	1.29E-03	m^2 / s^3
stability	100	2.88	K / km
headwind, crosswind	100	0.27	kts
temperature	50	0.99	°C

Fidelity of Joint Aspect of Distributions (2/2)

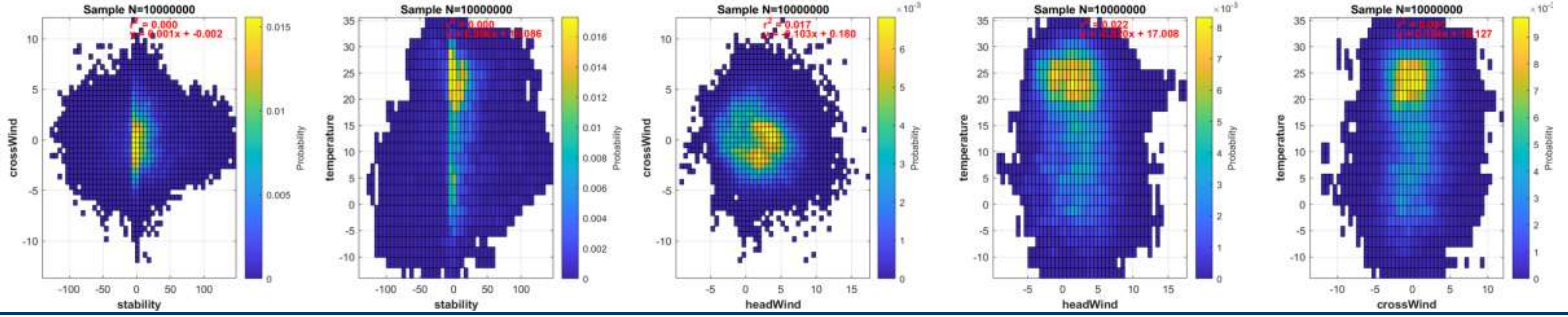
Visualized through examining the 10 2-way combinations of the 5 environmental variables

Generally, very good agreement seen across all variable combinations

Source (N=147754)



Sample (N=10M)


Sample N=10000000
r^2 = 0.000
y = 0.001x + -0.002
crossWind
stability
Probability
Sample N=10000000
r^2 = 0.000
temperature
stability
Sample N=10000000
r^2 = 0.017
y = 0.103x + 0.180
crossWind
headWind
$\times 10^{-3}$
Sample N=10000000
r^2 = 0.022
0x + 17.008
temperature
headWind
$\times 10^{-3}$
Sample N=10000000
temperature
crossWind
$\times 10^{-3}$