Determination of the fifth Busy Beaver value

The bbchallenge Collaboration

In particular for this work: Justin Blanchard, Daniel Briggs, Konrad Deka, Nathan Fenner,Yannick Forster, Georgi Georgiev (Skelet), Matthew L. House, Rachel Hunter, Iijil, Maja Kądziołka, Pavel Kropitz, Shawn Ligocki, mxdys, Mateusz Naściszewski, savask, Tristan Stérin, Chris Xu, Jason Yuen, and Théo Zimmermann.

https://bbchallenge.org

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program

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https://prgm.dev

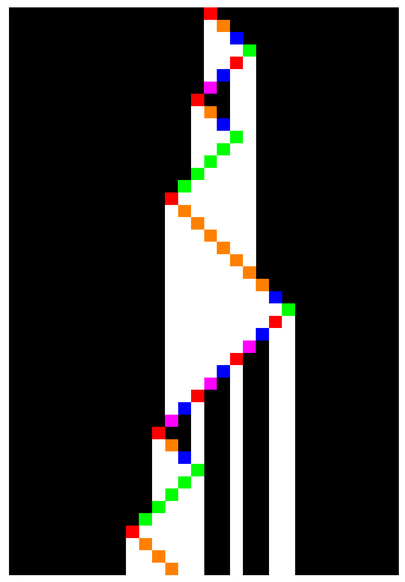
Uncomputable function!

BB(n): "Maximum number of steps performed by a halting 2-symbol n-state Turing machine from all-0 tape" [Radó, 1962]

Landscape of small Busy Beaver values

: already known but we proved it in Coq : new results, also proved in Coq : existence of *Cryptids*

Symbols	2-State	3-State	4-State	5-State	6-State
2	$\mathrm{BB}(2) = 6$ [Radó, 1962]	$\mathrm{BB}(3) = 21$ [Lin, 1963]	$\mathrm{BB}(4) = 107$ [Brady, 1983]	$\mathrm{BB}(5) = 47{,}176{,}870$	$\mathrm{BB}(6) > 2 \uparrow\uparrow\uparrow 5$
3	$\mathrm{BB}(2,3) = 38$ [Lafitte & Papazian, 2007]	$\mathrm{BB}(3,3) > 10^{17}$	$\mathrm{BB}(4,3) > 2 \uparrow\uparrow\uparrow 2^{2^{32}}$	–	–
4	$\mathrm{BB}(2,4) = 3{,}932{,}964$	$\mathrm{BB}(3,4) > 2 \uparrow^{15} 5$	–	–	–
5	$\mathrm{BB}(2,5) > 10 \uparrow\uparrow 4$	–	–	–	–



45-step *space-time diagram* of the 5-state winner

	0	1
A	1RB	1LC
B	1RC	1RB
C	1RD	0LE
D	1LA	1LD
E	—	0LA

5-state Busy Beaver winner

BB(5) = 47,176,870

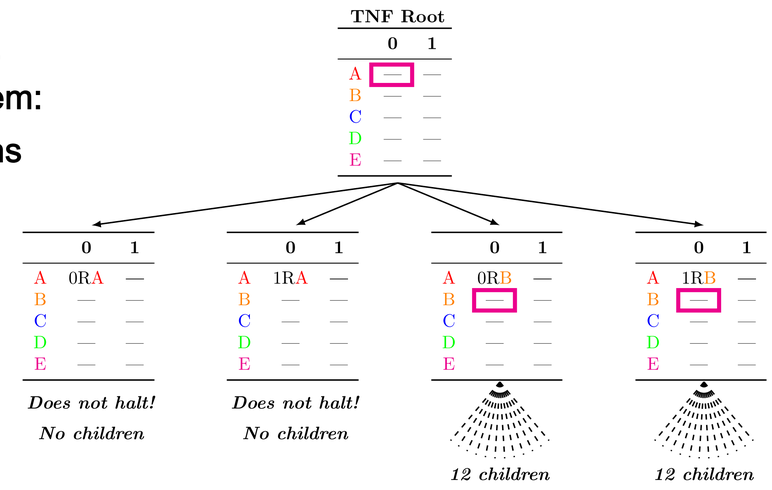
[Marxen & Buntrock, 1989]
[bbchallenge, 2024]

The proof of BB(5) = 47,176,870: Coq-BB5

https://github.com/ccz181078/Coq-BB5

I. Enumerate Turing machines in Tree Normal Form (TNF)

- There are 10,000 billion 5-state Turing machines
- TNF allows to consider "only" 181,385,789 of them:
  - No machine with set unreachable transitions
  - No isomorphic machines
- TNF enumeration is implemented in Coq and proved correct (e.g. no missing machine)
- Each of the 181,385,789 machines are then proven to halt in less than 47,176,870 steps or proven not to halt (harder for 5-state machines!)



II. Apply automated proof strategies: deciders and verifiers

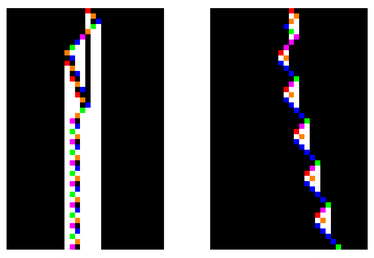
$S(5)$ pipeline		Nonhalt	Halt	Total decided
1. Loops	Decider	126,994,099	48,379,711	175,373,810
2. n-gram Closed Position Set (NGramCPS)	Decider	6,005,142	0	6,005,142
3. Repeated Word List (RepWL)	Decider	6,577	0	6,577
4. Finite Automata Reduction (FAR)	Verifier	23	0	23
5. Weighted Finite Automata Reduction (WFAR)	Verifier	17	0	17
6. Long halters (simulation up to 47,176,870 steps)	Verifier	0	183	183
7. Sporadic machines, individual proofs		13	0	13
8. `1RB`-reduction		24	0	24
Total		133,005,895	48,379,894	181,385,789

Decider : algorithm that takes a Turing machine as input and outputs HALT, NONHALT or UNKNOWN

Verifier : algorithm that verifies an hardcoded certificate of halting/nonhalting for a machine

Deciders and verifiers are proved correct in Coq: i.e. their output can be trusted and used in the proof

Example of decider: **Loops**; surprinsgly two *transcript* repetitions are (almost) enough



Cycler

Translated Cycler

Transcript: sequence of (state, read symbol) seen by the machine

```
A0(0)* B0(1)* B1(0) C0(-1) D1(0) E1(1)* E1(0) E0(-1) A0(-2) B1(-1) C1(-2) C1(-1) C0(0)
D0(1)* E0(0) A0(-1) B1(0) C1(-1) C1(0) C1(1)* C0(2)*
D0(3)* E0(2) A0(1) B1(2) C1(1) C1(2) C1(3)* C0(4)*
```

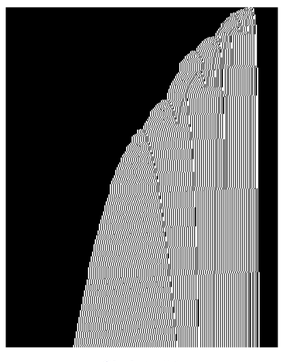
* marks being on an extremity of the tape

III. Individual proofs for the remaining 13 Sporadic Machines

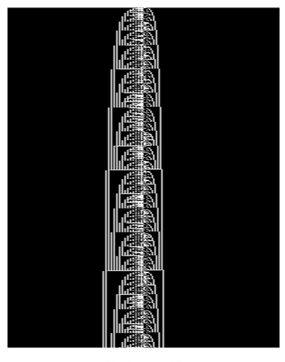
Skelet #1

Enormous loop:
- 10^51 preperiod
- 10^9 period

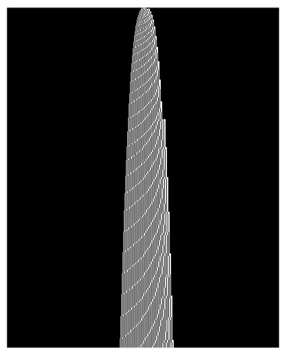
Translated Cycler



Skelet #1



Skelet #10
Double Fibonacci counter



Skelet #17
Obfuscated Gray Code

All Sporadic Machines are proved nonhalting in Coq.

The proof of Skelet #1 is takes 10 minutes to verify on a laptop.

The proof of Skelet #17 is 10,000 lines long. [Xu, 2024] [mxdys, 2024]

Machines named after Georgi Georgiev (Skelet) who first found them in 2003.

Use in Molecular Computing

BB has been used in theoretical work to construct large structures or long-running computations from small molecular programs:

- tile assembly [Rothemund & Winfree, 2000] [Cannon et al., 2021] [Caballero *et al.*, 2024]
- polymer reaction networks [Johnson & Winfree, 2020]
- chemical reaction networks [Alhazov *et al.*, 2025]

Our result gives a tight bound for molecular systems simulating 5-state Turing machines.

Coq and Coq-BB5

Coq is a Proof Assistant and programming language.

- Allows to formally write logical proofs
- Coq "compiles" if and only if the proof is correct
  (caveat: bugs inside Coq itself are possible)
- Coq is **really** suited to reason about programs. Our proof, Coq-BB5, is a collection of algorithms programmed and proved correct in Coq
- Coq-BB5, which enumerates and proves 181,385,789 Turing machines, compiles in 45 minutes on a laptop

bbchallenge.org

Online research community created by Stérin in 2022.
Consists of: website, forum, wiki, Discord chat server.
~1,200 Discord members
~100 active members
~120,000 messages exchanged
~100 messages per day in average
https://discord.gg/wuZhtTvYU3
Most contributors are not academically affiliated.
Some core contributors are anonymous.
Entropically driven.
Proofs become software projects.

Now working on BB(6), BB(3,3), BB(2,5), BB(7), but there are ***Cryptids***: mathematically-hard machines!

Conjecture 1.1 (Antihydra does not halt)**.** Consider the Collatz-like map $H : \mathbb{N} \to \mathbb{N}$ defined by $H(x) = 3\frac{x}{2}$ if x is even and $H(x) = 3\frac{x-1}{2}$ if x is odd. Iterating H from $x = 8$, there are never (strictly) more than twice as many odd numbers as even numbers.

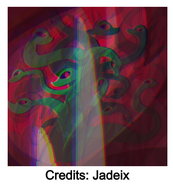
Credits: Lauren

	0	1
A	1RB	1RA
B	0LC	1LE
C	1LD	1LC
D	1LA	0LB
E	1LF	1RE
F	—	0RA

Antihydra

Does Antihydra halt from all-0 tape?

Smallest open problem in mathematics on the Busy Beaver scale.



Credits: Jadeix