Linear Programming

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1 Optimization Problems $\rightarrow$ LP

- Optimize [min/max] an objective function $\rightarrow$ linear function
- subject to constraints $\rightarrow$ linear equalities and inequalities

2 Diet Minimization Problem

2.1 Problem

- **Given:**
  - n foods $F_1, \ldots, F_n$
  - m nutrients $N_1, \ldots, N_m$
  - Unit of F_j
    - Cost c_j
    - Provides a_{ij} units of N_i
  - Need $\geq b_i$ units of N_i to survive
- **Want:** How much each F_j to buy to minimize cost and satisfy nutritional requirements.

2.2 Define aspects

2.2.1 Variables

- $x_1, \ldots, x_n$ where x_j = amount of F_j to buy.

2.2.2 Objective Function

- Minimize $c_1x_1 + \ldots + c_nx_n$ where c_j is the cost associated with each F_j

2.2.3 Constraints

- Let $a_{i1}x_1 + \ldots + a_{in}x_n \geq b_i$, $1 \leq i \leq m$

  where $a_{i1}x_1$ means the amount of N_i, nutrient i we get from F_1, food 1. We require that the total nutrient across all foods is at least as much as our basic requirements of N_i, which is b_i.

- We also need $x_1, x_2, \ldots, x_n \geq 0$, we cannot consume a negative amount of food

2.3 Input and Output

- **Input:**
  - $c_1, \ldots, c_n \rightarrow \vec{c}$
  - $b_1, \ldots, b_m \rightarrow \vec{b}$
  - a_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq n \rightarrow A_{m \times n}$
- **Output:** Values for $x_1, \ldots, x_n$ in $\mathbb{R}$ that minimizes objective function and satisfy all constraints

  Note that there is no longer a brute force solution for these problems, since you cannot check every real number

3 Profit Maximization Problem

3.1 Problem

- **Given:**
  - n products $P_1, \ldots, P_n$
  - m resources $R_1, \ldots, R_m$
  - Unit of P_j
    - Profit c_j
    - Requires a_{ij} units of R_i
  - Have $\leq b_i$ units of R_i
- **Want:** How much each P_j to make to maximize profit and satisfy requirements.

3.2 Define aspects

3.2.1 Variables

- $x_1, \ldots, x_n$ where x_j = amount of P_j to make.

3.2.2 Objective Function

- Maximize $c_1 x_1 + \ldots + c_n x_n$ where c_j is the cost associated with each P_j

3.2.3 Constraints

- Let $a_{i1} x_1 + \ldots + a_{in} x_n \leq b_i$, $1 \leq i \leq m$

  where $a_{i1} x_1$ means the amount of R_i, resource i we need to make P_1, product 1. We require that the total amount of resource be less than the amount of resources we can afford to use, b_i.
- We also need $x_1, x_2, \ldots, x_n \geq 0$, we cannot make a negative amount of product.

3.3 Input and Output

- **Input:**
  - $c_1, \ldots, c_n \to \vec{c}$
  - $b_1, \ldots, b_m \to \vec{b}$
  - a_{ij} where $1 \le i \le m$ and $1 \le j \le n \to A_{m \times n}$

  Note we can have that all inputs be negative, zero, or positive to give more flexibility to the problem

- **Output:** Values for $x_1, \ldots, x_n$ in $\mathbb{R}$ that maximizes objective function and satisfy all constraints

  If there are no values of $x_1, \ldots, x_n$ that satisfy the constraints, then the problem is infeasible $\to$ **over-constrained problem**

  If there are values of $x_1, \ldots, x_n$ that satisfy the contraints but we cannot find a min/max then the problem is feasible, but unbounded $\to$ **Unbounded Problem**

  Otherwise, we have a solution.

4 Optimization Problem From Geometric Perspective

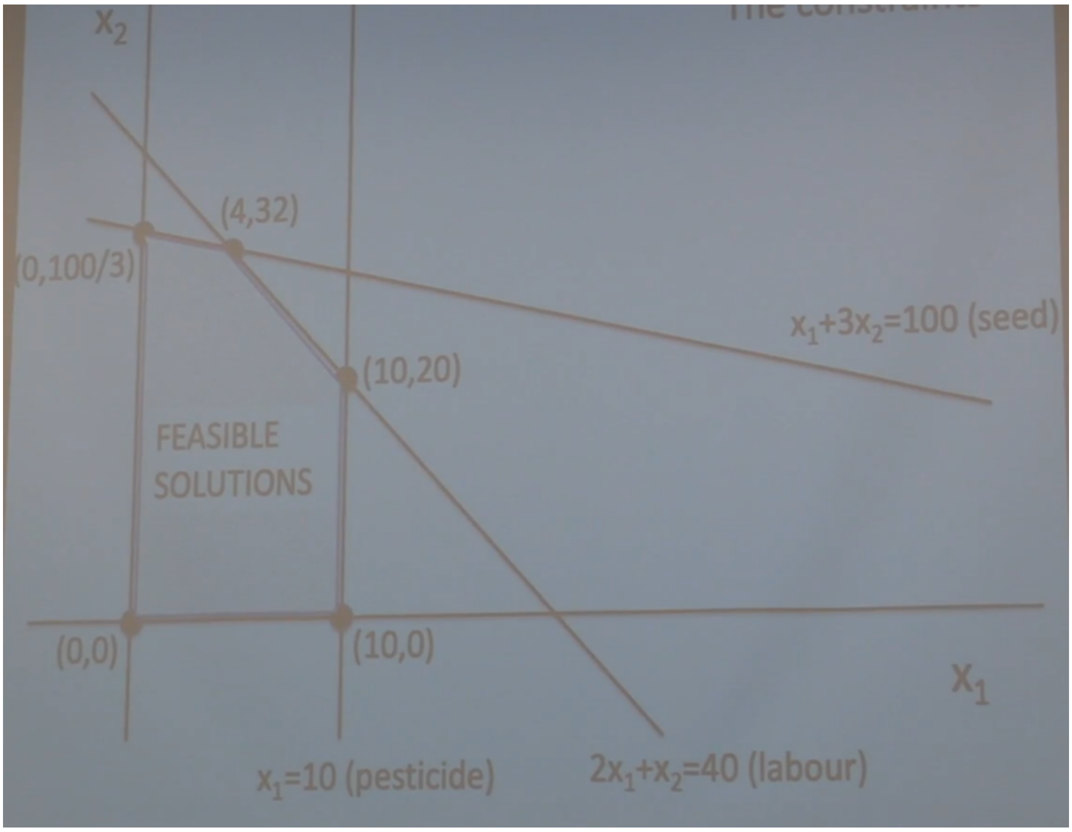
4.1 Problem

- Farmer can raise 2 kinds of corn
  - Corn 1: \$3 per hectare
  - Corn 2: \$2 per hectare
- Available resources
  - Labor: 40 hours
  - Seed: 100kg
  - Pesticide: 10 bags
- Resource requirements
  - Corn 1: Need 2 hours, 1kg of seed, 1 bag of pesticide per hectare
  - Corn 2: Need 1 hour, 3kg of seed, and 0 bags of pesticide per hectare

4.2 Aspects

- Variables
  - x_1, x_2, number of hectares of C1/C2 to plant
- Objective Function
  - maximize $3x_1 + 2x_2$ total amount of money (profit?)
- Constraints
  - $2x_1 + x_2 \le 40$ labor constraint
  - $x_1 + 3x_2 \le 100$ seed constraint
  - $x_1 \le 10$ pesticide constraint
  - $x_1, x_2 \ge 0$ cannot plant negative amount of corn

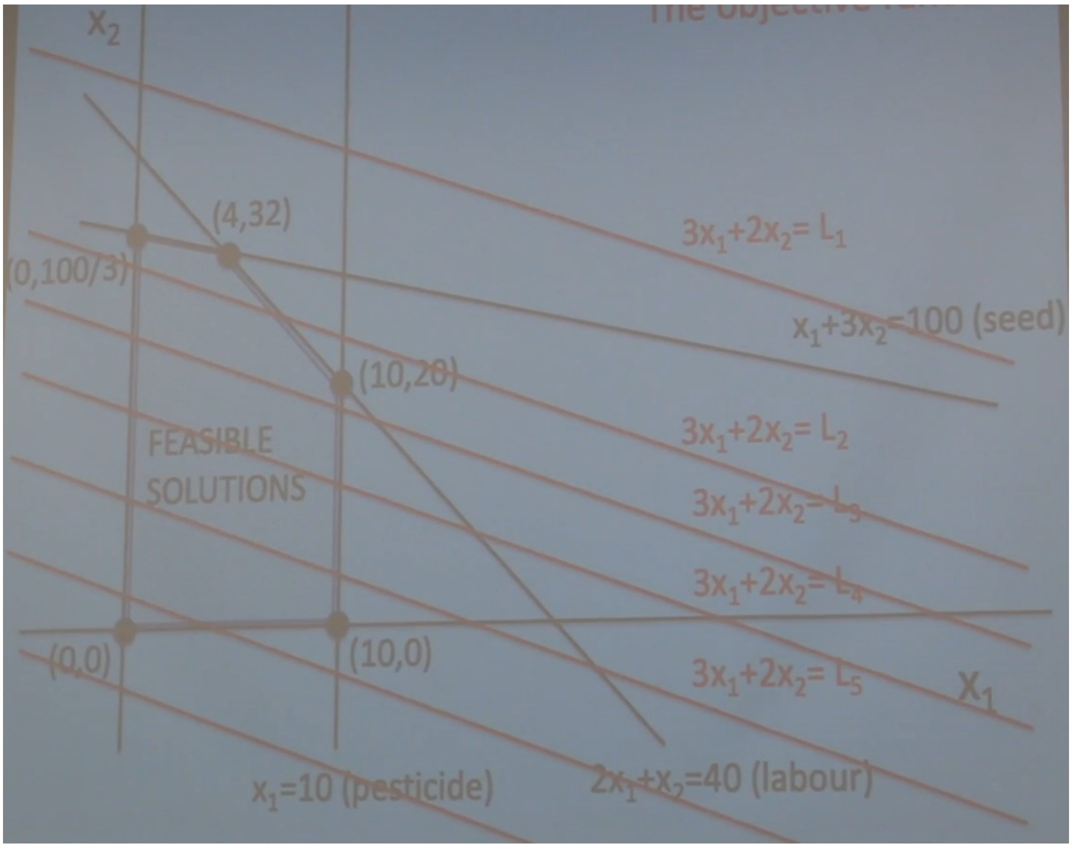
4.3 Geometric Interpretation



- Our set of feasible solutions is a **convex polygon**, which means the line between any 2 points within the polygon never cross outside of the polygon region. (Every angle is below or equal to 90 degrees)
- Also, the polygon has maximum amount of sides equal to the number of constraints

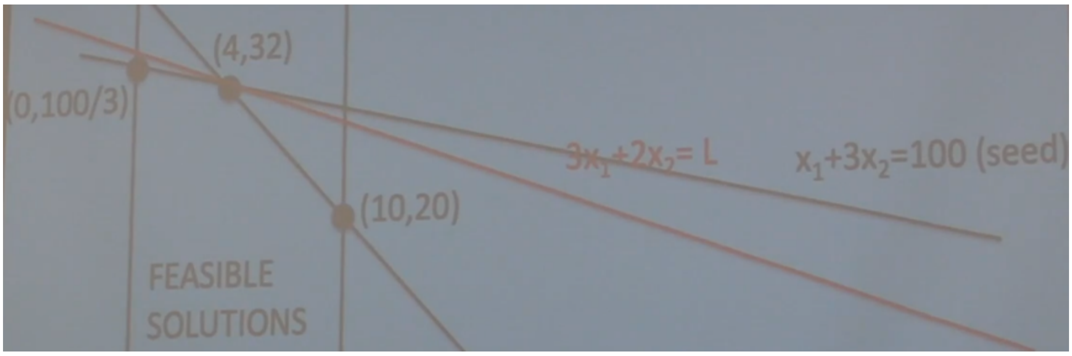
  Note we could have irrelevant constraints (covered by other constraints)

Now consider our objective function



Imagine the line $3x_1 + 2x_2 = L$ sweeping upwards (since we want to maximize L), then we want the L value that is the highest, but such that the line still intersects with the set of feasible solutions.

This will be our line where $x_1 = 4$ and $x_2 = 32$ and L is maximized and equal to 76.



Fact: If an optimal solution exists, then there is an optimal solution that is a basic feasible solution (a vertex/corner of the feasible region)

5 Linear Algebra Interpretation

5.1 Visualization

- We usually want to max/minimize something like

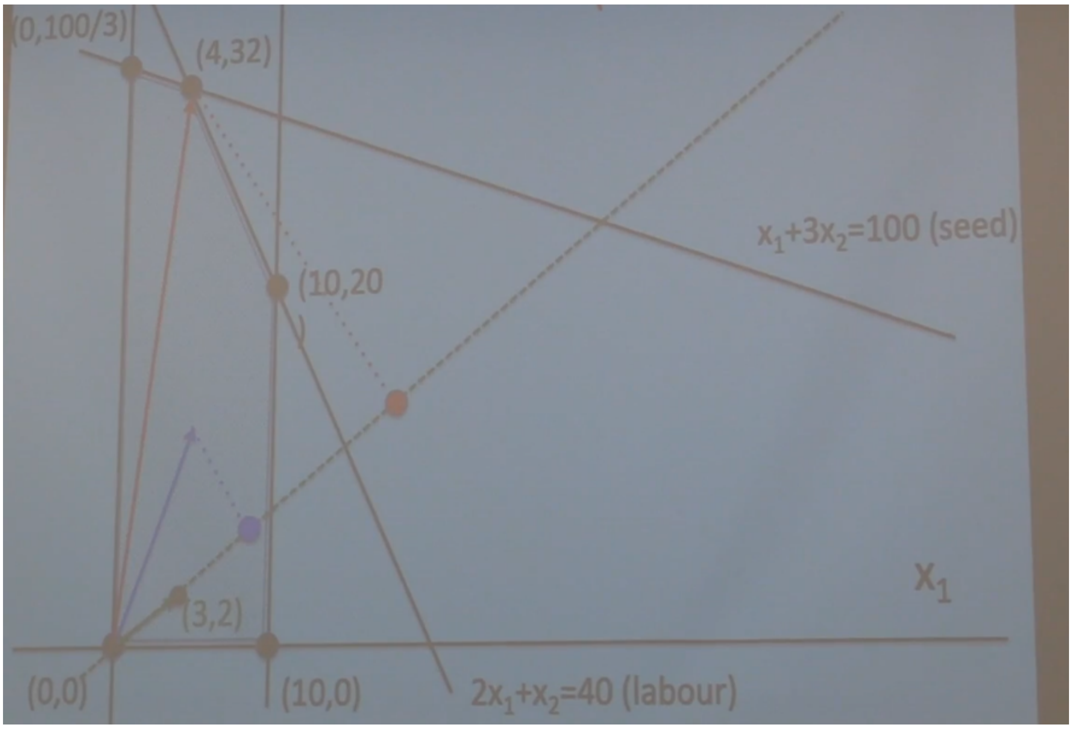
$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = \vec{c} \cdot \vec{x}$$

  Consider that

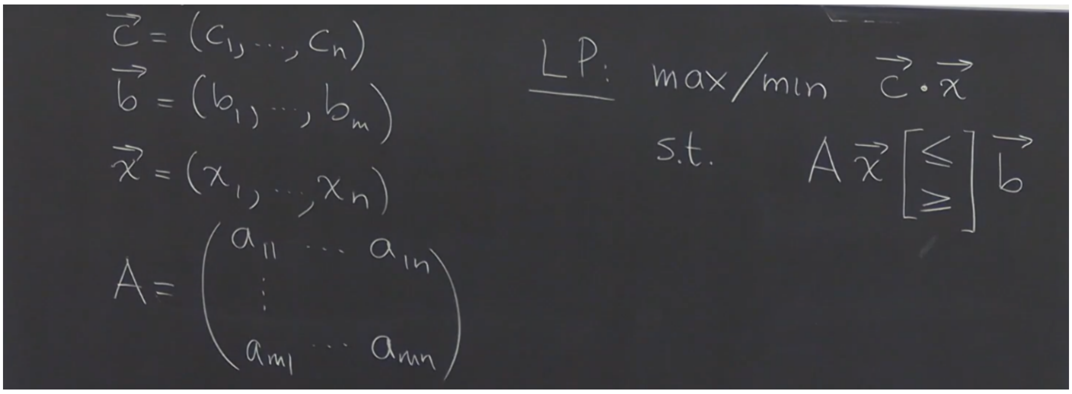
$$\frac{\vec{c} \cdot \vec{x}}{|c|} = \text{length of proj of } \vec{x} \text{ in the direction of } \vec{c}$$

  Then we only need to minimize $\vec{c} \cdot \vec{x}$ to minimize $\frac{\vec{c} \cdot \vec{x}}{|c|}$

  Consider the vector $(3, 2) = \vec{c}$, then we only need to maximize the projection, which happens to be onto $(4, 32)$.

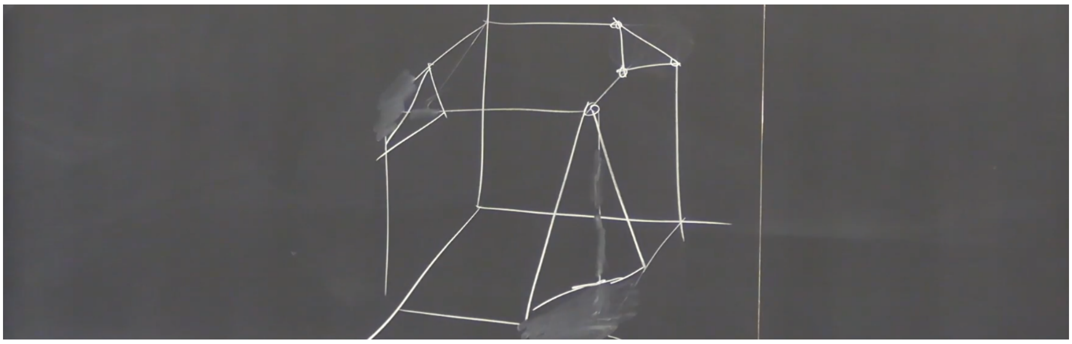


5.2 In formal Notation



6 In 3-Dimensional

- We have a 3-d space, where constraints are planes.



- Once again, we have the property that an optimal solution exists on some vertex.