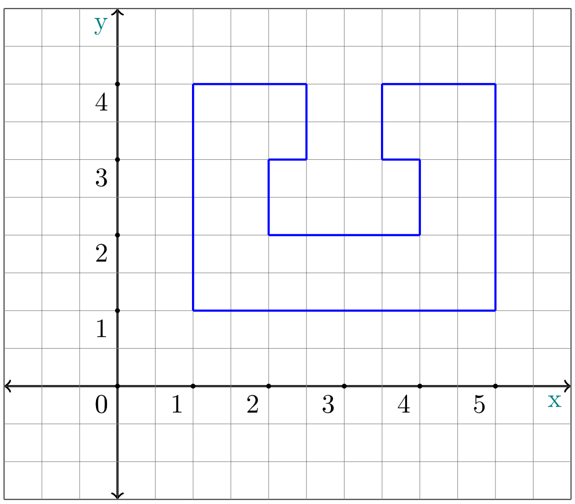
Técnicas de simulación por computadora

Grupo O

2 de junio de 2018

Dada la figura y ecuación diferencial:



$$\nabla(10\nabla T) = -100$$

Mallado de la figura y establecimiento de contornos para condiciones de Dirichlet y Neumann:

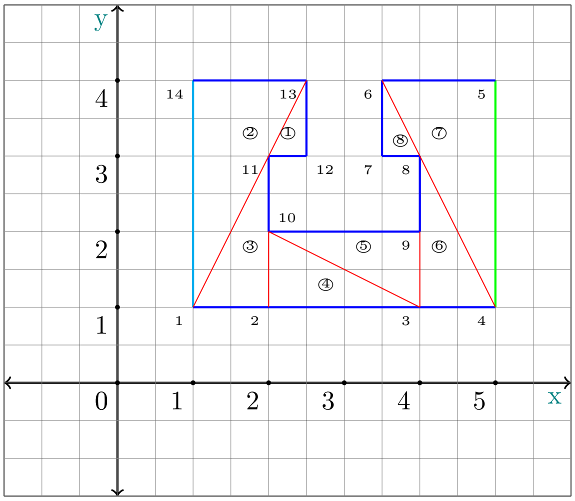


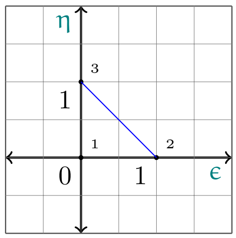
Tabla de conectividad:

elemento	1	2	3
①	11	12	13
②	1	13	14
③	1	2	11
④	2	3	10
⑤	10	3	9
⑥	3	4	8
⑦	6	4	5
⑧	6	7	8

Condiciones a utilizar:

$$\Gamma_D =$$
$$\Gamma_N =$$

Aproximación en el plano isoparamétrico:



$$
\begin{aligned}
T &\approx f_1T_1 + f_2T_2 + f_3T_3 \\
&\approx N_1T_1 + N_2T_2 + N_3T_3 \\
&\approx \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \\
&\approx \underline{\underline{N}}\vec{T}
\end{aligned}
$$

con $N_1 = 1 - \epsilon - \eta$, $N_2 = \epsilon$, $N_3 = \eta$
$x \approx (x_2 - x_1)\epsilon + (x_3 - x_1)\eta + x_1$
$y \approx (y_2 - y_1)\epsilon + (y_3 - y_1)\eta + y_1$

Sustitución de la función por su aproximación:

$$
\begin{aligned}
&\nabla(10\nabla T) = -100 \\
&\nabla(10\nabla \underline{\underline{N}}\vec{T}) \approx -100 \\
&\nabla(10\nabla \underline{\underline{N}}\vec{T}) + 100 \neq 0 \\
&\nabla(10\nabla \underline{\underline{N}}\vec{T}) + 100 = \mathcal{R}
\end{aligned}
$$

W.R.M.:

$$
\begin{aligned}
&\int_\Omega \underline{\underline{W}}\mathcal{R}\, d\Omega = 0 \\
&\int_A \underline{\underline{W}}\mathcal{R}\, dA = 0 \\
&\int_A \underline{\underline{W}}(\nabla(10\nabla \underline{\underline{N}}\vec{T}) + 100)\, dA = 0
\end{aligned}
$$

Galerkin:

$$
\begin{aligned}
&\underline{\underline{W}} = \underline{\underline{N}}^T \\
&\int_A \underline{\underline{N}}^T(\nabla(10\nabla \underline{\underline{N}}\vec{T}) + 100)\, dA = 0 \\
&\int_A \underline{\underline{N}}^T(\nabla(10\nabla \underline{\underline{N}}\vec{T}))\, dA + \int_A \underline{\underline{N}}^T(100)\, dA = 0
\end{aligned}
$$

Integración por partes:

$$
\begin{aligned}
&\text{sea } U = \underline{\underline{N}}^T,\ dU = \nabla \underline{\underline{N}}^T \\
&\text{sea } dV = \nabla(10\nabla \underline{\underline{N}}\vec{T}),\ V = 10\nabla(\underline{\underline{N}}\vec{T})
\end{aligned}
$$

$$
\int_A \underline{\underline{N}}^T(\nabla(10\nabla \underline{\underline{N}}\vec{T}))\, dA = [\underline{\underline{N}}^T 10\nabla(\underline{\underline{N}}\vec{T})]_{\Gamma_N} - \int_A \nabla \underline{\underline{N}}^t 10\nabla(\underline{\underline{N}}\vec{T})\, dA
$$

Trabajando en la ecuación:

$$
\begin{aligned}
-\int_A \nabla \underline{\underline{N}}^T 10 \nabla (\underline{\underline{N}}\vec{T})\, dA + \int_A \underline{\underline{N}}^T (100)\, dA &= 0 \\
\int_A \nabla \underline{\underline{N}}^T 10 \nabla (\underline{\underline{N}}\vec{T})\, dA &= \int_A \underline{\underline{N}}^T (100)\, dA \\
10 \int_A \nabla \underline{\underline{N}}^T \nabla (\underline{\underline{N}})\, dA (\vec{T}) &= 100 \int_A \underline{\underline{N}}^T\, dA
\end{aligned}
$$

Lado derecho:

$$
\begin{aligned}
100 \int_A \underline{\underline{N}}^T\, dA &= 100 \int_A \begin{bmatrix} 1-\epsilon-\eta \\ \epsilon \\ \eta \end{bmatrix} dA \\
&= 100 \int_A \begin{bmatrix} 1-\epsilon-\eta \\ \epsilon \\ \eta \end{bmatrix} dx\, dy
\end{aligned}
$$

Llevando la integral al plano isoparamétrico con un jacobiano:

$$
\begin{aligned}
dx\, dy &= D\, d\epsilon\, d\eta \\
J &= \begin{bmatrix} \frac{\delta x}{\delta \epsilon} & \frac{\delta x}{\delta \eta} \\ \frac{\delta y}{\delta \epsilon} & \frac{\delta y}{\delta \eta} \end{bmatrix} \\
&= \begin{bmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{bmatrix} \\
D &= |J| \\
&= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)
\end{aligned}
$$

Sustituyendo en la integral:

$$
\begin{aligned}
100 \int_A \begin{bmatrix} 1-\epsilon-\eta \\ \epsilon \\ \eta \end{bmatrix} dA &= 100 \int_0^1 \int_0^1 \begin{bmatrix} 1-\epsilon-\eta \\ \epsilon \\ \eta \end{bmatrix} \begin{vmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{vmatrix} d\epsilon\, d\eta \\
&= 100 \begin{vmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{vmatrix} \int_0^1 \int_0^1 \begin{bmatrix} 1-\epsilon-\eta \\ \epsilon \\ \eta \end{bmatrix} d\epsilon\, d\eta \\
&= 100 \begin{vmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{vmatrix} \begin{bmatrix} \int_0^1 \int_0^1 (1-\epsilon-\eta)\, d\epsilon\, d\eta \\ \int_0^1 \int_0^1 \epsilon\, d\epsilon\, d\eta \\ \int_0^1 \int_0^1 \eta\, d\epsilon\, d\eta \end{bmatrix} \\
&= 100 \begin{vmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{vmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}
\end{aligned}
$$

Trabajando lado izquierdo de la ecuación:

$$
\begin{aligned}
10\int_A \nabla\underline{\underline{N}}^T\nabla(\underline{\underline{N}})\,dA(\vec{T}) &= 10\int_c^d\int_a^b \nabla\underline{\underline{N}}^T\nabla(\underline{\underline{N}})\,dx\,dy(\vec{T}) \\
\text{con } \nabla_x\underline{\underline{N}} &= (\nabla_\epsilon\underline{\underline{x}})^{-1}\nabla_\epsilon\underline{\underline{N}} \\
&= \begin{bmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{bmatrix}^{-1}\begin{bmatrix}-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix} \\
&= \frac{1}{\begin{vmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{vmatrix}}\begin{bmatrix}(y_3-y_1) & (y_1-y_2)\\(x_1-x_3) & (x_2-x_1)\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix} \\
&= \frac{1}{\begin{vmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{vmatrix}}\begin{bmatrix}(y_2-y_3) & (y_3-y_1) & (y_1-y_2)\\(x_3-x_2) & (x_1-x_3) & (x_2-x_1)\end{bmatrix} \\
\nabla\underline{\underline{N}}^T &= \frac{1}{\begin{vmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{vmatrix}}\begin{bmatrix}(y_2-y_3) & (x_3-x_2)\\(y_3-y_1) & (x_1-x_3)\\(y_1-y_2) & (x_2-x_1)\end{bmatrix}
\end{aligned}
$$

Sustituyendo en la integral:

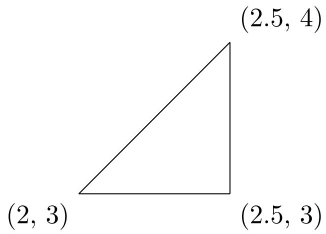
$$
\begin{aligned}
&= 10\int_c^d\int_a^b \frac{1}{\begin{vmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{vmatrix}^2}\begin{bmatrix}(y_2-y_3) & (x_3-x_2)\\(y_3-y_1) & (x_1-x_3)\\(y_1-y_2) & (x_2-x_1)\end{bmatrix}\begin{bmatrix}(y_2-y_3) & (y_3-y_1) & (y_1-y_2)\\(x_3-x_2) & (x_1-x_3) & (x_2-x_1)\end{bmatrix}dx\,dy(\vec{T}) \\
&= 10\frac{(x\Big|_a^b)(y\Big|_c^d)}{\begin{vmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{vmatrix}^2}\begin{bmatrix}(y_2-y_3) & (x_3-x_2)\\(y_3-y_1) & (x_1-x_3)\\(y_1-y_2) & (x_2-x_1)\end{bmatrix}\begin{bmatrix}(y_2-y_3) & (y_3-y_1) & (y_1-y_2)\\(x_3-x_2) & (x_1-x_3) & (x_2-x_1)\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}
\end{aligned}
$$

Sistema local final:

$$
\frac{10(x\Big|_a^b)(y\Big|_c^d)}{\begin{vmatrix}(x_2-x_1) & (y_2-y_1)\\(x_3-x_1) & (y_3-y_1)\end{vmatrix}^2}\begin{bmatrix}(y_2-y_3) & (x_3-x_2)\\(y_3-y_1) & (x_1-x_3)\\(y_1-y_2) & (x_2-x_1)\end{bmatrix}\begin{bmatrix}(y_2-y_3) & (y_3-y_1) & (y_1-y_2)\\(x_3-x_2) & (x_1-x_3) & (x_2-x_1)\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix} = 100\begin{vmatrix}(x_2-x_1) & (x_3-x_1)\\(y_2-y_1) & (y_3-y_1)\end{vmatrix}\begin{bmatrix}0\\1/2\\1/2\end{bmatrix}
$$

Calculando elementos:

1.

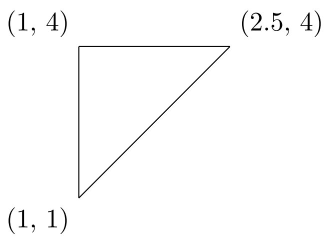


$$\begin{array}{lll} x_1 = 2 & y_1 = 3 & a = 2 \\ x_2 = 2{,}5 & y_2 = 3 & b = 2{,}5 \\ x_3 = 2{,}5 & y_3 = 4 & c = 3 \\ & & d = 4 \end{array}$$

$$\frac{10(0{,}5)(1)}{\begin{vmatrix}0{,}5 & 0\\ 0{,}5 & 1\end{vmatrix}^2}\begin{bmatrix}-1 & 0\\ 1 & -0{,}5\\ 0 & 0{,}5\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 0 & -0{,}5 & 0{,}5\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} = 100\begin{vmatrix}0{,}5 & 0{,}5\\ 0 & 1\end{vmatrix}\begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}$$

$$\begin{bmatrix}20 & -20 & 0\\ -20 & 25 & -5\\ 0 & -5 & 5\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} = \begin{bmatrix}0\\ 25\\ 25\end{bmatrix}$$

2.

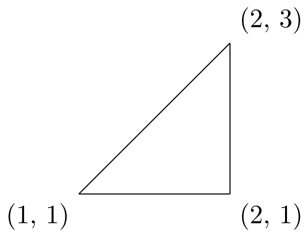


$$\begin{array}{lll} x_1 = 1 & y_1 = 1 & a = 1 \\ x_2 = 2{,}5 & y_2 = 4 & b = 2{,}5 \\ x_3 = 1 & y_3 = 4 & c = 1 \\ & & d = 4 \end{array}$$

$$\frac{10(1{,}5)(3)}{\begin{vmatrix}1{,}5 & 3\\ 0 & 3\end{vmatrix}^2}\begin{bmatrix}0 & -1{,}5\\ 3 & 0\\ -3 & 1{,}5\end{bmatrix}\begin{bmatrix}0 & 3 & -3\\ -1{,}5 & 0 & 1{,}5\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} = 100\begin{vmatrix}1{,}5 & 0\\ 3 & 3\end{vmatrix}\begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & -5\\ 0 & 20 & -20\\ -5 & -20 & 25\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} = \begin{bmatrix}0\\ 255\\ 255\end{bmatrix}$$

3.

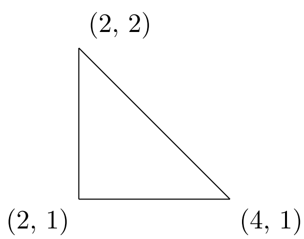


$$\begin{array}{lll} x_1 = 1 & y_1 = 1 & a = 1 \\ x_2 = 2 & y_2 = 1 & b = 2 \\ x_3 = 2 & y_3 = 3 & c = 1 \\ & & d = 3 \end{array}$$

$$\frac{10(1)(2)}{\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}^2} \begin{bmatrix} -2 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = 100 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \begin{bmatrix} 0 \\ 0{,}5 \\ 0{,}5 \end{bmatrix}$$

$$\begin{bmatrix} 20 & -20 & 0 \\ -20 & 25 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix}$$

4.

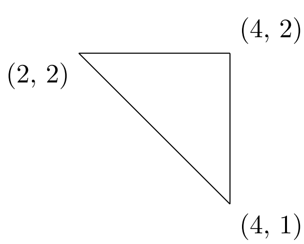


$$\begin{array}{lll} x_1 = 2 & y_1 = 1 & a = 2 \\ x_2 = 3 & y_2 = 1 & b = 4 \\ x_3 = 2 & y_3 = 2 & c = 1 \\ & & d = 2 \end{array}$$

$$\frac{10(2)(1)}{\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}^2} \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = 100 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \begin{bmatrix} 0 \\ 0{,}5 \\ 0{,}5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & -5 & -20 \\ -5 & 5 & 0 \\ -20 & 0 & 20 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix}$$

5.

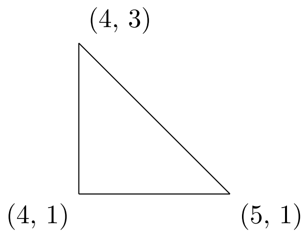


$$\begin{array}{lll} x_1 = 2 & y_1 = 2 & a = 2 \\ x_2 = 4 & y_2 = 1 & b = 4 \\ x_3 = 4 & y_3 = 2 & c = 1 \\ & & d = 2 \end{array}$$

$$\frac{10(2)(1)}{\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix}^2} \begin{bmatrix} -1 & 0 \\ 0 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = 100 \begin{vmatrix} 2 & 2 \\ -1 & 0 \end{vmatrix} \begin{bmatrix} 0 \\ 0{,}5 \\ 0{,}5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 25 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix}$$

6.

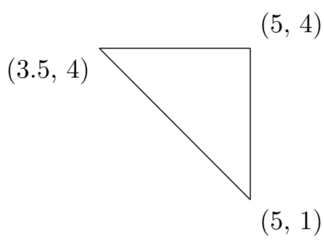


$$\begin{matrix} x_1 = 4 & y_1 = 1 & a = 4 \\ x_2 = 5 & y_2 = 1 & b = 5 \\ x_3 = 4 & y_3 = 3 & c = 1 \\ & & d = 3 \end{matrix}$$

$$\frac{10(1)(2)}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}^2} \begin{bmatrix} -2 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = 100 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \begin{bmatrix} 0 \\ 0,5 \\ 0,5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & -20 & -5 \\ -20 & 20 & 0 \\ -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix}$$

7.

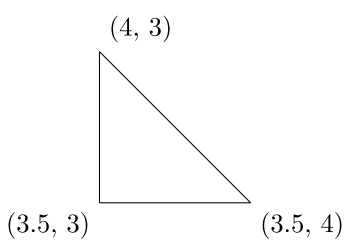


$$\begin{matrix} x_1 = 3,5 & y_1 = 4 & a = 3,5 \\ x_2 = 5 & y_2 = 1 & b = 5 \\ x_3 = 5 & y_3 = 4 & c = 1 \\ & & d = 4 \end{matrix}$$

$$\frac{10(1,5)(3)}{\begin{vmatrix} 1,5 & -3 \\ 1,5 & 0 \end{vmatrix}^2} \begin{bmatrix} -3 & 0 \\ 0 & -1,5 \\ -3 & 1,5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 \\ 0 & -1,5 & 1,5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = 100 \begin{vmatrix} 1,5 & 1,5 \\ -3 & 0 \end{vmatrix} \begin{bmatrix} 0 \\ 0,5 \\ 0,5 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & -20 \\ 0 & 5 & -5 \\ -20 & -5 & 25 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 225 \\ 225 \end{bmatrix}$$

8.



$$\begin{matrix} x_1 = 3,5 & y_1 = 4 & a = 3,5 \\ x_2 = 3,5 & y_2 = 3 & b = 4 \\ x_3 = 4 & y_3 = 3 & c = 3 \\ & & d = 4 \end{matrix}$$

$$\frac{10(0,5)(1)}{\begin{vmatrix} 0 & -1 \\ 0,5 & -1 \end{vmatrix}^2} \begin{bmatrix} 0 & 0,5 \\ -1 & -0,5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0-5 & -0,5 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = 100 \begin{vmatrix} 0 & 0,5 \\ -1 & -1 \end{vmatrix} \begin{bmatrix} 0 \\ 0,5 \\ 0,5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -5 & 0 \\ -5 & 25 & 20 \\ 0 & -20 & 20 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 25 \\ 25 \end{bmatrix}$$